ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 22 (2022) #P3.05 / 443–458
https://doi.org/10.26493/1855-3974.2645.8fc
(Also available at http://amc-journal.eu)

On the essential annihilating-ideal graph of commutative rings*

Mohd Nazim, Nadeem ur Rehman †

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Received 30 May 2021, accepted 8 September 2021, published online 9 June 2022

Abstract

Let R be a commutative ring with unity, $A(R)$ be the set of annihilating-ideals of R and $A^*(R) = A(R) \setminus \{0\}$. In this paper, we introduced and studied the *essential annihilating-ideal graph* of R, denoted by $\mathcal{EG}(R)$, with vertex set $A^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $Ann(I_1 I_2)$ is an essential ideal of R. We prove that $\mathcal{EG}(R)$ is a connected graph with diameter at most three and girth at most four if $\mathcal{EG}(R)$ contains a cycle. Furthermore, the rings R are characterized for which $\mathcal{EG}(R)$ is a star or a complete graph. Finally, we classify all the Artinian rings R for which $\mathcal{EG}(R)$ is isomorphic to some well-known graphs.

Keywords: Annihilating-ideal graph, zero-divisor graph, complete graph, planar graph, genus of a graph.

Math. Subj. Class. (2020): 13A15, 05C10, 05C12, 05C25

1 Introduction

Throughout this paper all rings are commutative rings (not a field) with unit element such that $1 \neq 0$. For a commutative ring R, we use $\mathbb{I}(R)$ to denote the set of ideals of R and $\mathbb{I}^*(R) = \mathbb{I}(R) \setminus \{0\}$. An ideal I of R is said to be *non-trivial* if it is nonzero and proper both. An ideal I of R is said to be *annihilator ideal* if there is a nonzero ideal J of R such that $IJ = 0$. For $X \subseteq R$, we define $annihilator$ of X as $Ann(X) = \{r \in R : rX = 0\}$. We use $A(R)$ to denote the set of annihilator ideas of R and $A^*(R) = A(R) \setminus \{0\}$. We denote the set of zero-divisors, the set of nilpotent elements, the set of maximal ideals, the set of minimal prime ideals, and the set of Jacobson radical of a ring R by $Z(R)$, $Nil(R)$,

*The authors are greatly indebted to the referee for his/her constructive comments and suggestion, which improves the quality of the paper a lot.

†Corresponding author.

E-mail addresses: mnazim1882@gmail.com (Mohd Nazim), nu.rehman.mm@amu.ac.in (Nadeem ur Rehman)

$Max(R)$, $Min(R)$ and $J(R)$, respectively. A nonzero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a nonzero intersection with every nonzero ideal of R. Also, if I is not an essential ideal of R then, it is denoted by $I \nleq_e R$. A ring R is said to be reduced, if it has no nonzero nilpotent element. For a nonzero nilpotent element x of R, we use η to denote the index of nilpotency of x. If S is any subset of R, then S^* denote the set $S \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to see [9].

Let G be a graph with vertex set $V(G)$. The $distance$ between two vertices u and v of G denoted by $d(u,v)$, is the smallest path from u to v. If there is no such path, then $d(u,v) = \infty$. The $diameter$ of G is defined as $diam(G) = \sup\{d(u,v) : u, v \in V(G)\}$. A $cycle$ is a closed path in G. The $girth$ of G denoted by $gr(G)$ is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycle). A graph is said to be *complete* if all its vertices are adjacent to each other. A complete graph with n vertices is denoted by K_n. If G is a graph such that the vertices of G can be partitioned into two nonempty disjoint sets U_1 and U_2 such that vertices u and v are adjacent if and only if $u \in U_1$ and $v \in U_2$, then G is called a *complete bipartite graph.* A complete bipartite graph with disjoint vertex sets of size m and n, respectively, is denoted by $K_{m,n}$. We write $K_{n,\infty}$ (respectively, $K_{\infty,\infty}$) if one (respectively, both) of the disjoint vertex sets is infinite. A complete bipartite graph of the form $K_{1,n}$ is called a *star graph.* A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. The *genus* of a graph G, denoted by $\gamma(G)$, is the minimum integer k such that the graph can be drawn without crossing itself on a sphere with k handles (i.e. an oriented surface of genus k). Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing. For more details on graph theory, we refer to reader to see [21, 22].

The concept of *zero-divisor graph* of a commutative ring R, denoted by $\Gamma(R)$, was introduced by I. Beck [10]. The vertex set of $\Gamma(R)$ is $Z^*(R) = Z(R) \setminus \{0\}$ (set of nonzero zero-divisors of R) and two distinct vertices x and y are adjacent if and only if $xy = 0$, for details see [5, 8, 7]. In [14], Dol$\check{z}$an and Oblak also obtained several interesting results related with zero-divisor graph of rings and semirings. The zero-divisor graph of a noncommutative ring has been introduced and studied by Redmond [18], whereas the same concept for semigroup by Demeyer et al. [13].

In [11], Behboodi et al. generalized the zero-divisor graph to ideals by defining *the annihilating-ideal graph* $AG(R)$, with vertex set is $A^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1 I_2 = 0$. For more details on annihilating-ideal graph, we refer the reader to see [1, 2, 3, 4, 6, 12, 16].

In [17], M. Nikmehr et al. introduced *the essential graph* $EG(R)$ with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $ann_R(xy)$ is an essential ideal of R.

Motivated by [17], we define *the essential annihilating-ideal graph* of R denoted by $\mathcal{EG}(R)$ with vertex set $A^*(R)$ and two distinct vertices I_1 and I_2 adjacent if and only if $Ann(I_1 I_2)$ is an essential ideal of R. In this paper we first prove that $AG(R)$ is a subgraph of $\mathcal{EG}(R)$ and then studied some basic properties of $\mathcal{EG}(R)$ such as connectedness, diameter, girth and shows that $\mathcal{EG}(R)$ is a connected graph with $diam(\mathcal{EG}(R)) \leq 3$ and $gr(\mathcal{EG}(R)) \leq 4$, if $\mathcal{EG}(R)$ contains a cycle. In the third section, we determine some conditions on R under which $\mathcal{EG}(R)$ is a star graph or a complete graph. In the last, we identify all the Artinian rings R for which $\mathcal{EG}(R)$ is isomorphic to some well-known graphs.

2 Basic properties of essential annihilating-ideal graph

We begin this section with the following lemma given by [17].

Lemma 2.1 ([17, Lemma 2.1])**.** *Let R be a commutative ring and I be an ideal of R. Then*

(1) *$I + Ann(I)$ is an essential ideal of R.*

(2) *If $I^2 = (0)$, then $Ann(I)$ is an essential ideal of R.*

(3) *If R contains no proper essential ideals, then $J(R) = (0)$.*

The following lemma is analogue of [17, Lemma 2.2].

Lemma 2.2. *Let R be a commutative ring. Then*

(1) *If I_1 and I_2 are adjacent in $AG(R)$, then I_1 and I_2 are also adjacent in $\mathcal{EG}(R)$.*

(2) *If $I^2 = 0$ for some $I \in A^*(R)$, then I is adjacent to every other vertex in $\mathcal{EG}(R)$.*

Proof. (1) Suppose I_1 and I_2 are adjacent in $AG(R)$, then $I_1I_2 = 0$ and so $Ann(I_1I_2) = R$, is an essential ideal of R. Thus I_1 and I_2 are also adjacent in $\mathcal{EG}(R)$.
(2) Suppose that $I^2 = 0$ for some $I \in A^*(R)$. Then by Lemma 2.1(2), $Ann(I)$ is an essential ideal of R. Since $Ann(I) \subseteq Ann(IJ)$ for every $J \in A^*(R)$, therefore $Ann(IJ)$ is also an essential ideal of R. Thus I is adjacent to every other vertex of $\mathcal{EG}(R)$. $\square$

Let R be a commutative ring. By [11, Theorem 2.1], the annihilating ideal graph $AG(R)$ is a connected graph with $diam(AG(R)) \leq 3$. Moreover, if $AG(R)$ contains a cycle, then $gr(AG(R)) \leq 4$.

In view of part (1) of Lemma 2.2, we have the following result.

Theorem 2.3. *Let R be a commutative ring. Then $\mathcal{EG}(R)$ is connected with $diam(\mathcal{EG}(R)) \leq 3$. Moreover, if $\mathcal{EG}(R)$ contain a cycle, then $gr(\mathcal{EG}(R)) \leq 4$.*

In Lemma 2.2(1), we proved that $AG(R)$ is a spanning subgraph of $\mathcal{EG}(R)$ but this containment may be proper. The following examples shows that $AG(R)$ and $\mathcal{EG}(R)$ are not identical.

Example 2.4.

1. If $R = \mathbb{Z}_{16}$, then $AG(R)$ is P_3 and $\mathcal{EG}(R)$ is K_3.

2. If $R = \mathbb{Z}_{p^5}$, where p is a prime number. Then $AG(R)$ is the following graph and $\mathcal{EG}(R)$ is K_4.

Theorem 2.5. *Let R be a commutative reduced ring. Then $\mathcal{EG}(R) = AG(R)$.*

Proof. Clearly, $AG(R) \subseteq \mathcal{EG}(R)$. We just have to prove that $\mathcal{EG}(R)$ is a subgraph of $AG(R)$. Suppose on contrary that $I_1 \sim I_2$ is an edge of $\mathcal{EG}(R)$ such that $I_1I_2 \neq 0$. Since R is a reduced ring, then $I_1I_2 \cap Ann(I_1I_2) = 0$, which implies that $Ann(I_1I_2)$ is not an essential ideal of R, a contradiction. Thus $I_1I_2 = 0$ and $\mathcal{EG}(R) = AG(R)$. $\square$

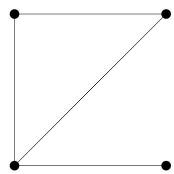

Figure 1: The graph $AG(\mathbb{Z}_{p^5})$.

Theorem 2.6 ([12, Theorem 1.9(3)]). *Let R be a commutative ring with finitely many minimal primes. Then $diam(AG(R)) = 2$ if and only if either R is reduced with exactly two minimal primes and at least three nonzero annihilating-ideals, or R is not reduced, $Z(R)$ is an ideal whose square is not (0) and for each pair of annihilating-ideals I_1 and I_2, $I_1 + I_2$ is an annihilating-ideal.*

Theorem 2.7. *Let R be a commutative ring with $|Min(R)| < \infty$. Then*

(1) *If R is reduced ring, then $diam(\mathcal{EG}(R)) = 2$ if and only if $|Min(R)| = 2$ and R has at least three nonzero annihilating-ideals. Moreover, in this case $gr(\mathcal{EG}(R)) \in \{4, \infty\}$.*

(2) *If R is non-reduced, then $diam(\mathcal{EG}(R)) \leq 2$. Moreover, in this case $gr(\mathcal{EG}(R)) \in \{3, \infty\}$.*

Proof. (1) First part is clear from Theorems 2.5 and 2.6. Now, let $Min(R) = \{P_1, P_2\}$, then $\mathcal{EG}(R)$ is a complete bipartite graph with partitions $V_1 = \{I \in V(\mathcal{EG}) : I \subseteq P_1\}$ and $V_2 = \{I \in V(\mathcal{EG}) : I \subseteq P_2\}$ by [12, Theorem 1.2]. Hence $gr(\mathcal{EG}(R)) \in \{4, \infty\}$.

(2) Since R is a non-reduced ring, then there is $I_1 \in A^*(R)$ such that $I_1^2 = 0$. Thus by Lemma 2.2(2), I_1 is adjacent to every other vertex of $\mathcal{EG}(R)$. Hence $diam(\mathcal{EG}(R)) \leq 2$. Also, if there are $I, J \in V(\mathcal{EG}(R)) \setminus \{I_1\}$ such that $I \sim J$ is an edge of $\mathcal{EG}(R)$, then $I_1 \sim I \sim J \sim I_1$ is a triangle in $\mathcal{EG}(R)$. Thus, $gr(\mathcal{EG}(R)) = 3$, otherwise $gr(\mathcal{EG}(R)) = \infty$. □

3 Completeness of essential annihilating-ideal graph

In this section, we characterize commutative rings R for which $\mathcal{EG}(R)$ is a star graph or a complete graph. We begin with the following lemma.

Lemma 3.1. *Let R be a commutative nonreduced ring. Then*

(1) *For every nilpotent ideal I_1 of R, I_1 is adjacent to every other vertex of $\mathcal{EG}(R)$.*

(2) *The subgraph induced by the nilpotent ideals of R is a complete subgraph of $\mathcal{EG}(R)$.*

Proof. (1) Suppose that I_1 be any nilpotent ideal of R. Let $I_2 \in A^*(R)$. We show that $Ann(I_1 I_2) \leq_e R$. Since $Ann(I_1) \subseteq Ann(I_1 I_2)$, then it is enough to show that $Ann(I_1) \leq_e R$. Suppose on contrary that $Ann(I_1) \not\leq_e R$, then there exists $I_3 \in \mathbb{I}^*(R)$ such that $Ann(I_1) \cap I_3 = 0$, which implies that $rI_1 \neq 0$ for every $r \in I_3^*$. Since $0 \neq rI_1 \subseteq I_3$, then $I_1 \cdot rI_1 = rI_1^2 \neq 0$. Continuing this process, we get $rI_1^n \neq 0$, for every positive integer n, which is a contradiction. This complete the proof.
(2) It is clear from (1). □

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring. Then $\mathcal{EG}(R)$ is a complete graph.*

Proof. Follows from Lemma 3.1. $\square$

Lemma 3.3. *Let R be a commutative decomposable ring. Then $\mathcal{EG}(R)$ is a star graph if and only if $R = F \times D$, where F is a field and D is an integral domain.*

Proof. $(\Rightarrow)$ Suppose that $\mathcal{EG}(R)$ is a star graph and let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. If R_1 and R_2 both are not fields and $I_1 \in \mathbb{I}^*(R_1)$, $I_2 \in \mathbb{I}^*(R_2)$, then $(R_1 \times (0)) \sim ((0) \times R_2) \sim (I_1 \times (0)) \sim ((0) \times I_2) \sim (R_1 \times (0))$ is a cycle of length 4 in $\mathcal{EG}(R)$, a contradiction. Thus, without loss of generality we can assume that R_1 is a field. We claim that R_2 is an integral domain. Suppose on contrary that R_2 is not an integral domain, then there exists $I_3, I_4 \in \mathbb{I}^*(R_1)$ such that $I_3 I_4 = 0$. If $I_3 \neq I_4$, then $(R_1 \times (0)) \sim ((0) \times I_3) \sim ((0) \times I_4) \sim (R_1 \times (0))$ is a triangle in $\mathcal{EG}(R)$, a contradiction. Also, if $I_3 = I_4$, then by Lemma 3.1, $(R_1 \times (0)) \sim ((0) \times I_3) \sim ((0) \times R_2) \sim (R_1 \times (0))$ is a triangle in $\mathcal{EG}(R)$, again a contradiction. This complete the proof.
$(\Leftarrow)$ is clear. $\square$

Theorem 3.4. *Let R be an Artinian commutative ring with atleast two non-trivial ideals. Then $\mathcal{EG}(R)$ is a star graph if and only if $\mathcal{EG}(R) \cong K_2$.*

Proof. $(\Rightarrow)$ Suppose $\mathcal{EG}(R)$ is a star graph. If R is a local ring, then from Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is a star graph, therefore $\mathcal{EG}(R) \cong K_2$. If R is non-local ring, then it is decomposable. Thus by Lemma 3.3, $R = F \times D$, where F is a field and D is an integral domain. Since R is Artinian ring, then D is Artinian and hence is a field. Thus $\mathcal{EG}(R) \cong K_2$.
$(\Leftarrow)$ is evident. $\square$

Theorem 3.5. *Let R be a commutative ring with at least two non-trivial ideals. Then $\mathcal{EG}(R)$ is a star graph if and only if one of the following holds:*

(1) *R has exactly two non-trivial ideals.*

(2) *$R = F \times D$, where F is a field and D is an integral domain which is not a field.*

(3) *R has a minimal ideal I_1 such that I_1 is not an essential ideal of R, $I_1^2 = 0$ and for any nonzero annihilating ideal I_2 of R, $Ann(I_2) = I_1$.*

Proof. $(\Rightarrow)$ Suppose $\mathcal{EG}(R)$ is a star graph. If $|A^*(R)| < \infty$, then from [11, Theorem 1.1], R is an Artinian ring. Thus, by Theorem 3.4, $\mathcal{EG}(R) \cong K_2$ and hence (1) hold.
Now, let $|A^*(R)| = \infty$ and I_1 is adjacent to every other vertex of $\mathcal{EG}(R)$. We show that I_1 is minimal ideal of R. Suppose on contrary that there exists $I_2 \in \mathbb{I}^*(R)$ such that $I_2 \subset I_1$. Let $I_3 \in A^*(R) \setminus \{I_1, I_2\}$, then $Ann(I_1 I_3) \leq_e R$. Since $I_2 I_3 \subseteq I_1 I_3$, then $Ann(I_2 I_3)$ is also essential ideal of R. This implies that I_2 is also adjacent to every other vertex of $\mathcal{EG}(R)$, a contradiction. Now, following two cases occur:
Case I: $I_1^2 \neq 0$. Then $I_1^2 = I_1$, thus by Brauer's Lemma [15, p. 172, Lemma 10.22], R is decomposable. Since $|A^*(R)| = \infty$ and $\mathcal{EG}(R)$ is a star graph. Then from Lemma 3.3, $R = F \times D$, where F is a field and D is an integral domain which is not a field. Hence (2) hold.
Case II: $I_1^2 = 0$. Let $I_2 \in A^*(R) \setminus \{I_1\}$. Then $I_2 \neq Ann(I_2)$, otherwise $I_2^2 = 0$

implies that I_2 is also adjacent to every other vertex of $\mathcal{EG}(R)$, a contradiction. Now, since $I_2 \sim Ann(I_2)$, then $Ann(I_2) = I_1$. If I_1 is an essential ideal of R, then $Ann(I_2)$ is also an essential ideal of R. This shows that I_2 is also adjacent with every other vertex of $\mathcal{EG}(R)$, which is a contradiction to our assumption that $\mathcal{EG}(R)$ is a star graph because we are assuming that I_1 is adjacent with every other vertex of $\mathcal{EG}(R)$ and $I_1 \neq I_2$. Hence I_1 is not an essential ideal of R.

($\Leftarrow$) If R has exactly two non-trivial ideals, then R is Artinian ring with $|A^*(R)| = 2$. Since $\mathcal{EG}(R)$ is connected, therefore $\mathcal{EG}(R) \cong K_2$. If $R = F \times D$, where F is a field and D is an integral domain which is not a field, then from Lemma 3.3, $\mathcal{EG}(R)$ is a star graph. Now, suppose that R has a minimal ideal I_1 such that I_1 is not an essential ideal of R, $I_1^2 = 0$ and for any nonzero annihilating ideal I_2 of R, $Ann(I_2) = I_1$. Let $I_2, I_3 \in A^*(R) \setminus \{I_1\}$ such that $I_2 \sim I_3$ in $\mathcal{EG}(R)$. This implies that $Ann(I_2I_3) \leq_e R$ and $Ann(I_2) = I_1 = Ann(I_3)$. Since $Ann(I_2) = Ann(I_3)$ is not an essential of R, there exists a nonzero ideal I_4 of R such that $Ann(I_2) \cap I_4 = Ann(I_3) \cap I_4 = 0$. This shows that $rI_2 \neq 0$ and $rI_3 \neq 0$ for every $r \in I_4^*$. On the other hand, since $Ann(I_2I_3) \leq_e R$, then $Ann(I_2I_3) \cap I_4 \neq 0$. That is there exists $s \in I_4^*$ such that $sI_2I_3 = 0$. Now, observe that $sI_2 \subseteq I_4^*$ satisfies $sI_2 \subseteq Ann(I_3)$, which implies that $Ann(I_3) \cap I_4 \neq 0$, a contradiction. This complete the proof. □

Theorem 3.6. *Let R be a commutative Artinian ring. Then $\mathcal{EG}(R)$ is a complete graph if and only if one of the following holds:*

(1) *$R = F_1 \times F_2$, where F_1 and F_2 are fields.*

(2) *R is a local ring.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{EG}(R)$ is a complete graph. Since R is Artinian, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is Artinian local ring for each $1 \leq i \leq n$. The following cases occur:

Case I: $n \geq 3$. Then $R_1 \times (0) \times \cdots \times (0)$ and $R_1 \times (0) \times R_3 \times \cdots \times (0)$ are nonzero annihilating ideals of R such that $(R_1 \times (0) \times \cdots \times (0)) \not\sim (R_1 \times (0) \times R_3 \times \cdots \times (0))$ in $\mathcal{EG}(R)$, a contradiction.

Case II: $n = 2$. We show that R_1 and R_2 are fields. Suppose on contrary that R_1 is not a field with non-trivial maximal ideal $\mathfrak{m}$. Then $Ann(((0) \times R_2) \cdot (\mathfrak{m} \times R_2)) = Ann((0) \times R_2) = R_1 \times (0)$, which is not an essential ideal of R. Thus $((0) \times R_2) \not\sim (\mathfrak{m} \times R_2)$ in $\mathcal{EG}(R)$, a contradiction. Hence (2) holds.

Case III: $n = 1$. Then R is Artinian local ring and (1) holds.

($\Leftarrow$) If R is local, then from Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. If $R = F_1 \times F_2$, where F_1 and F_2 are fields, then $\mathcal{EG}(R) \cong K_2$. □

Theorem 3.7. *Let R be a commutative ring with at least one minimal ideal. Then $\mathcal{EG}(R) \cong K_{m,n}$, where $m, n \geq 2$ if and only if $R = D \times S$, where D and S are integral domains which are not fields.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{EG}(R) \cong K_{m,n}$, where $m, n \geq 2$. Let I_1 be minimal ideal of R. If $I_1^2 = 0$, then from Lemma 2.2, I_1 is adjacent to every other vertex, a contradiction. Thus $I_1^2 \neq 0$. Since I_1 is minimal, therefore $I_1^2 = I_1$. Therefore, Brauer's Lemma [15, p. 172, Lemma 10.22], $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. Now, our objective is to show that R_1 and R_2 are integral domains. Suppose on contrary that

R_1 is not an integral domain with nonzero annihilating ideal I_2. As above, $I_2^2 \neq 0$ which implies that $I_2 \notin Ann(I_2)$. Thus $(I_2 \times (0)) \sim ((0) \times R_2) \sim (Ann(I_2) \times (0)) \sim (I_2 \times (0))$ is a triangle in $\mathcal{EG}(R)$, a contradiction. Hence R_1 is an integral domain. Similarly, one can prove that R_2 is an integral domain. Since $m, n \geq 2$, therefore R_1 and R_2 are not fields.
($\Leftarrow$) Suppose that $R = D \times S$, where D and S are integral domains which are not fields. Let $U = \{I_1 \times (0) : I_1 \in \mathbb{I}^*(D)\}$ and $V = \{(0) \times I_2 : I_2 \in \mathbb{I}^*(S)\}$. Then $A^*(R) = U \cup V$ such that no two vertices of U or V are adjacent in $\mathcal{EG}(R)$. Also, every vertex of U is adjacent to every vertex of V in $\mathcal{EG}(R)$. Thus, $\mathcal{EG}(R) \cong K_{m,n}$. Since D and S are not fields, therefore $m, n \geq 2$. □

Lemma 3.8. *Let R be a commutative ring. Then*

(1) *Let $I_1, I_2, I_3 \in A^*(R)$ such that $Ann(I_1) = Ann(I_2)$. Then $I_1 \sim I_3$ is an edge of $\mathcal{EG}(R)$ if and only if $I_2 \sim I_3$ is an edge of $\mathcal{EG}(R)$.*

(2) *Let $I \in A^*(R)$. Then $Ann(I) \leq_e R$ if and only if $Ann(I^n) \leq_e R$ for every $n \geq 2$. In particular, if $Ann(I^3) \leq_e R$, then $Ann(I^n) \leq_e R$ for every $n \geq 1$.*

Proof. (1) ($\Rightarrow$) Suppose that $I_1 \sim I_3$ is an edge of $\mathcal{EG}(R)$, then $Ann(I_1I_3) \leq_e R$. We have to show that $Ann(I_2I_3) \leq_e R$. Suppose on contrary that $Ann(I_2I_3)$ is not an essential ideal of R, then there exits $I_4 \in \mathbb{I}^*(R)$ such that $Ann(I_2I_3) \cap I_4 = 0$. This implies that $rI_2I_3 \neq 0$ for all $r \in I_4^*$. On the other hand, since $Ann(I_1I_3)$ is an essential ideal of R, then $Ann(I_1I_3) \cap I_4 \neq 0$. That is there exists some $s \in I_4^*$ such that $sI_1I_3 = 0$. Now, observe that $sI_3 \subseteq I_4^*$ satisfies $sI_3 \subseteq Ann(I_1) = Ann(I_2)$, which implies that $sI_2I_3 = 0$, a contradiction.
($\Leftarrow$) Using similar argument as above we get the required result.
(2) ($\Rightarrow$) is clear.
($\Leftarrow$) Suppose on contrary that $Ann(I)$ is not an essential ideal of R, then there exists nonzero ideal I_1 of R such that $Ann(I) \cap I_1 = 0$. This implies that $rI \neq 0$ for all $r \in I_1^*$. On the other hand, since $Ann(I^2) \leq_e R$, then $Ann(I^2) \cap I_1 \neq 0$. That is there exists some $s \in I_1^*$ such that $sI^2 = 0$. Now, observe that $r = sI \subseteq I_1^*$ such that $rI = 0$, a contradiction.
For the particular case, we need to show that $Ann(I^2) \leq_e R$. Suppose on contrary that there is some $I_1 \in \mathbb{I}^*(R)$ such that $Ann(I^2) \cap I_1 = 0$, which implies that $rI^2 \neq 0$ for all $r \in I_1^*$. On the other hand, since $Ann(I^3) \leq_e R$, then $Ann(I^3) \cap I_1 \neq 0$. That is there exists some $s \in I_1^*$ such that $sI^3 = 0$. Now, observe that $r = sI^2 \subseteq I_1^*$ such that $rI = 0$, which implies that $Ann(I) \cap I_1 \neq 0$. Since $Ann(I)$ is a subset of $Ann(I^2)$, then $Ann(I^2) \cap I_1 \neq 0$, a contradiction. □

Theorem 3.9. *Let R be a commutative non-reduced ring. Then $\mathcal{EG}(R)$ is a complete graph if and only if $Ann(I) \leq_e R$ for every $I \in A^*(R)$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{EG}(R)$ is a complete graph. We claim that R is indecomposable ring. Suppose on contrary that $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. Since R is non-reduced ring, without loss of generality, we can assume that R_1 is non-reduced ring with nonzero nilpotent element x. Let $I_1 = xR_1$. Then $Ann((I_1 \times R_2) \cdot ((0) \times R_2)) = Ann((0) \times R_2) = R_1 \times (0)$, is not an essential ideal of R, a contradiction to the completeness of $\mathcal{EG}(R)$. Let $I \in A^*(R)$ be arbitrary. If I is nilpotent ideal, then from Lemma 3.1(1), $Ann(I) \leq_e R$. Suppose I is not nilpotent ideal.

Since R is indecomposable, then $I^2 \neq I$, which implies that $Ann(I^3) \leq_e R$. Hence by Lemma 3.8(2), $Ann(I) \leq_e R$.
($\Leftarrow$) is evident. □

4 Essential annihilating-ideal graph as some special type of graphs

In this section, we characterize all the Artinian rings R for which $\mathcal{EG}(R)$ is a tree, a unicycle graph, a split graph, a outerplanar graph, a planar graph and a toroidal graph.

Theorem 4.1. *Let R be a commutative Artinian ring (not a field). Then $\mathcal{EG}(R)$ is a tree if and only if either $R \cong F_1 \times F_2$, where F_1 and F_2 are fields or R is a local ring with at most two non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is a tree. Since R is an Artinain ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is an Artinian local ring. If $n \geq 3$. Consider $I_1 = R_1 \times (0) \times \cdots \times (0)$, $I_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$ and $I_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$. Then $I_1 \sim I_2 \sim I_3 \sim I_1$ is a cycle of in $\mathcal{EG}(\mathcal{R})$, a contradiction.
Suppose $n = 2$, then we show that R_1 and R_2 both are fields. Suppose on contrary that R_1 is not a field with nonzero maximal ideal $\mathfrak{m}$. Consider $J_1 = (0) \times R_2$, $J_2 = \mathfrak{m} \times (0)$, $J_3 = \mathfrak{m} \times R_2$ and $J_4 = R_1 \times (0)$. Then $J_1 \sim J_2 \sim J_3 \sim J_4 \sim J_1$ is a cycle in $\mathcal{EG}(R)$, a contradiction.
If $n = 1$, then R is Artinian local ring. Thus by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is a tree, therefore R has at most two non-trivial ideal.
Converse is clear. □

Theorem 4.2. *Let R be a commutative Artinian ring (not a field). Then $\mathcal{EG}(R)$ is unicycle if and only if either $R \cong F_1 \times F_2 \times F_3$, where F_i is a field for each $1 \leq i \leq 3$ or R is an Artinain local ring with exactly three non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is unicycle. Since R is Artinian ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is Artinian local ring for each $1 \leq i \leq n$. Let $n \geq 4$. Consider $I_1 = R_1 \times (0) \times \cdots \times (0)$, $I_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $I_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ and $J_1 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$, $J_2 = R_1 \times R_2 \times (0) \times \cdots \times (0)$, $J_3 = (0) \times (0) \times (0) \times R_4 \times (0) \times \cdots \times (0)$. Then $I_1 \sim I_2 \sim I_3 \sim I_1$ as well as $J_1 \sim J_2 \sim J_3 \sim J_1$ are two different cycles in $\mathcal{EG}(R)$, a contradiction. Hence $n \leq 3$.
First, let $n = 3$ and suppose on contrary that R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Consider $I_1 = R_1 \times (0) \times (0)$, $I_2 = (0) \times R_2 \times (0)$, $I_3 = (0) \times (0) \times R_3$ and $J_1 = R_1 \times (0) \times (0)$, $J_2 = (0) \times \mathfrak{m} \times (0)$, $J_3 = (0) \times (0) \times R_3$. Then $I_1 \sim I_2 \sim I_3 \sim I_1$ and $J_1 \sim J_2 \sim J_3 \sim J_1$ are two different cycles in $\mathcal{EG}(R)$, a contradiction. Hence R_i is a field for each $1 \leq i \leq 3$.
Now, let $n = 2$. If R_1 and R_2 both are fields then $\mathcal{EG}(R) \cong K_2$, a contradiction. Thus one of R_i say R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then $(R_1 \times (0)) \sim ((0) \times \mathfrak{m}) \sim ((0) \times R_2) \sim (R_1 \times (0))$ as well as $(R_1 \times \mathfrak{m} \sim ((0) \times \mathfrak{m}) \sim ((0) \times R_2) \sim (R_1 \times \mathfrak{m})$ are two different cycles in $\mathcal{EG}(R)$, again a contradiction.
If $n = 1$, then R is an Artinian local ring. Thus, by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is unicycle, R have exactly three non-trivial ideals. □

Theorem 4.3 ([21]). *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4, C_5.*

Theorem 4.4. *Let R be a commutative Artinian non-local ring. Then $\mathcal{EG}(R)$ is split graph if and only if either $R \cong F_1 \times F_2 \times F_3$ or $R \cong F_1 \times F_2$, where F_i is a field for each $1 \leq i \leq 3$.*

Proof. Suppose that $\mathcal{EG}(R)$ is a split graph. Since R is Artinian non-local ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is an Artinian local ring and $n \geq 2$. If $n \geq 4$, then $I_1 = R_1 \times R_2 \times (0) \times \cdots \times (0) \sim J_1 = (0) \times (0) \times R_3 \times R_4 \times (0) \times \cdots \times (0)$ and $I_2 = R_1 \times (0) \times R_3 \times (0) \times \cdots \times (0) \sim J_2 = (0) \times R_2 \times (0) \times R_4 \times (0) \times \cdots \times (0)$ induces $2K_2$ in $\mathcal{EG}(R)$, a contradiction. Hence $n = 2$ or 3. We have following cases:
Case I: If $n = 3$, then we show that each R_i ia a field. Suppose on contrary that R_1 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then $(R_1 \times (0) \times (0)) \sim ((0) \times R_2 \times R_3) \sim (\mathfrak{m} \times (0) \times (0)) \sim ((0) \times R_2 \times (0)) \sim (R_1 \times (0) \times (0))$ is C_4 in $\mathcal{EG}(R)$, a contradiction. Hence R_i is a field for each $1 \leq i \leq 3$.
Case II: Let $n = 2$ and suppose that R_2 is not a field with nonzero maximal ideal $\mathfrak{m}'$. Then $(R_1 \times (0)) \sim ((0) \times R_2) \sim (R_1 \times \mathfrak{m}') \sim ((0) \times \mathfrak{m}') \sim (R_1 \times (0))$ is C_4 in $\mathcal{EG}(R)$, a contradiction. Hence R_1 and R_2 both are fields.
Converse is clear. □

Theorem 4.5 ([22])**.** *A graph G is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.*

Theorem 4.6. *Let R be a commutative Artinian ring. Then $\mathcal{EG}(R)$ is outerplanar if and only if one of the following holds:*

(1) $R = F_1 \times F_2 \times F_3$, *where F_i is a field for each* $1 \leq i \leq 3$.

(2) $R = F_1 \times F_2$, *where F_1 and F_2 are fields.*

(3) $R = F \times R_1$, *where F is a field and $(R_1, \mathfrak{m})$ is a local ring with $\mathfrak{m}$ is the only non-trivial ideal of R_1.*

(4) *R is a local ring with at most three non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is outerplanar. Since R is Artinian ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is Artinian local ring. If $n \geq 4$, then the set $\{I_1 = R_1 \times (0) \times \cdots \times (0), I_2 = (0) \times R_2 \times (0) \times \cdots \times (0), I_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0), I_4 = (0) \times (0) \times (0) \times R_4 \times (0) \times \cdots \times (0)\}$ induces K_4 in $\mathcal{EG}(R)$, a contradiction. Hence $n \leq 3$. The following cases occur:
Case I: $n = 3$. We claim that R_i is a field for each $1 \leq i \leq 3$. Suppose on contrary that R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then the set $\{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0), (0) \times (0) \times R_3, (0) \times R_2 \times R_3\}$ induces a copy of $K_{2,3}$ with partition sets $A = \{(0) \times (0) \times R_3, (0) \times R_2 \times R_3\}$ and $B = \{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0)\}$, a contradiction. Therefore R_i is a field for each $1 \leq i \leq 3$.
Case II: $n = 2$ and let R_i is not a field with nonzero maximal ideal $\mathfrak{m}_i$ for each $i = 1, 2$. Then the set $\{R_1 \times (0), (0) \times R_2, \mathfrak{m}_1 \times (0), (0) \times \mathfrak{m}_2\}$ induces a copy of K_4 in $\mathcal{EG}(R)$, a contradiction. Hence one of R_i (say R_1) must be a field. Let I be a non-trivial ideal of R_2 other than maximal ideal $\mathfrak{m}_2$. Then the set $\{R_1 \times (0), R_1 \times \mathfrak{m}_2, (0) \times R_2, (0) \times \mathfrak{m}_2, (0) \times I\}$ induces a copy of $K_{2,3}$ with partition sets $A = \{R_1 \times (0), R_1 \times \mathfrak{m}_2\}$ and $B = \{(0) \times R_2, (0) \times \mathfrak{m}_2, (0) \times I\}$ in $\mathcal{EG}(R)$, a contradiction. Hence R_2 is a field or

has unique non-trivial ideal.

Case III: $n = 1$, then R is an Artinian local ring. Thus by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $\mathcal{EG}(R)$ is outerplanar, R have at most three non-trivial ideals.

Converse follows from Lemma 3.2, Theorem 4.5, Figures 2 and 3. $\square$

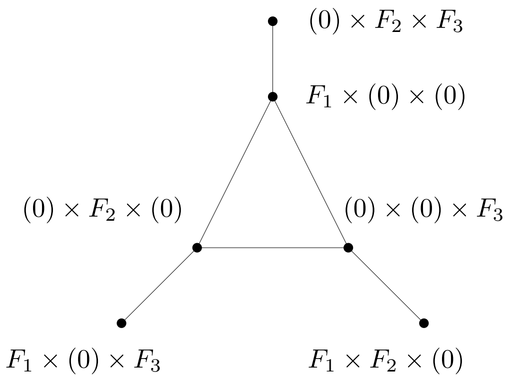

Figure 2: The graph $\mathcal{EG}(F_1 \times F_2 \times F_3)$.

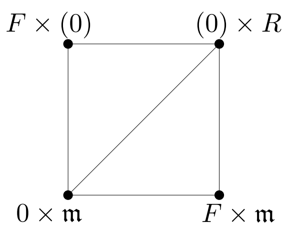

Figure 3: The graph $\mathcal{EG}(F \times R_1)$, where $\mathfrak{m}$ is the only non-trivial ideal of R_1.

Lemma 4.7 ([20, Proposition 2.7])**.** *If $(R, \mathfrak{m})$ is an Artinian local ring and there is an ideal I of R such that $I \neq \mathfrak{m}^i$ for every i, then R has at least three distinct non-trivial ideals J, K and L such that $J, K, L \neq \mathfrak{m}^i$ for each i.*

Theorem 4.8 (Kuratowski's Theorem)**.** *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Lemma 4.9. *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring. Then $\mathcal{EG}(R)$ is planar if and only if R have at most four non-trivial ideals.*

Proof. It is clear from Lemma 3.2 and Theorem 4.8. $\square$

Theorem 4.10. *Let R be a commutative Artinian ring. Then $\mathcal{EG}(R)$ is planar graph if and only if one of the following hold:*

(1) $R = F_1 \times F_2 \times F_3$, *where F_i is a field for each $1 \leq i \leq 3$.*

(2) *R has at most four non-trivial ideals.*

Proof. Suppose that $\mathcal{EG}(R)$ is a planar graph. If $|A^*(R)| \leq 4$, then (2) holds. Thus, we assume that $|A^*(R)| \geq 5$. Since R is Artinian ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is Artinian local ring. If $n \geq 4$, then the set $\{R_1 \times (0) \times \cdots \times (0), R_1 \times R_2 \times (0) \times \cdots \times (0), (0) \times R_2 \times (0) \times \cdots \times (0)\} \cup \{(0) \times (0) \times R_3 \times R_4 \times (0) \times \cdots \times (0), (0) \times (0) \times R_3 \times (0) \times \cdots \times (0), (0) \times (0) \times (0) \times R_4 \times (0) \times \cdots \times (0)\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$, a contradiction. Hence $n \leq 3$. The following cases occur:

Case I: $n = 3$. We claim that R_i is a field for each $1 \leq i \leq 3$. Suppose on contrary that one of R_i say R_2 is not a field with nonzero maximal ideal $\mathfrak{m}$. Then the set $\{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times R_3, (0) \times (0) \times R_3, (0) \times R_2 \times R_3\}$ induces a copy of $K_{3,3}$ with partition sets $A = \{R_1 \times (0) \times (0), R_1 \times \mathfrak{m} \times (0), (0) \times \mathfrak{m} \times (0)\}$ and $B = \{(0) \times (0) \times R_3, (0) \times \mathfrak{m} \times R_3, (0) \times R_2 \times R_3\}$ in $\mathcal{EG}(R)$, a contradiction. Hence, (1) satisfied.

Case II: $n = 2$. Since $|A^*(R)| \geq 5$, then one of R_i is not a field for some $i = 1, 2$. Suppose that R_1 is not a field with nonzero maximal ideal $\mathfrak{m}_1$. If R_2 is a field, then $|A^*(R)| \geq 5$ shows that R_1 have at least two non-trivial ideals. Let I be a non-trivial ideal of R_1 other than the maximal ideal. Then the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), I \times (0)\} \cup \{(0) \times R_2, \mathfrak{m}_1 \times R_2, I \times R_2\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$, a contradiction.

Now, if R_2 is not a field with nonzero maximal ideal $\mathfrak{m}_2$, then the set $\{R_1 \times (0), (0) \times \mathfrak{m}_2, R_1 \times \mathfrak{m}_2\} \cup \{(0) \times R_2, \mathfrak{m}_1 \times (0), \mathfrak{m}_1 \times R_2\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$, again a contradiction.

Case III: $n = 1$. Then R is an Artinian local ring. Thus, by Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Since $|A^*(R)| \geq 5$, then $\mathcal{EG}(R)$ contains a copy of K_5, which is a contradiction.

Conversely, If R is an Artinian ring with at most four non-trivial ideals, then by Theorem 4.8, $\mathcal{EG}(R)$ is planar. Also, if $R = F_1 \times F_2 \times F_3$, where F_i is a field for each $1 \leq i \leq 3$, then from Figure 2, $\mathcal{EG}(R)$ is planar. □

Lemma 4.11 ([22])**.** $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$, *where* $\lceil x \rceil$ *is the least integer that is greater than or equal to* x. *In particular,* $\gamma(K_n) = 1$ *if* $n = 5, 6, 7$.

Lemma 4.12 ([22])**.** $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$, *where* $\lceil x \rceil$ *is the least integer that is greater than or equal to* x. *In particular,* $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ *if* $n = 3, 4, 5, 6$.

Theorem 4.13. *Let* $(R, \mathfrak{m})$ *be a commutative Artinian local ring. Then* $\gamma(\mathcal{EG}(R)) = 1$ *if and only if* R *have at least five and at most seven non-trivial ideals.*

Proof. Since $(R, \mathfrak{m})$ is an Artinian local ring, then from Lemma 3.2, $\mathcal{EG}(R)$ is a complete graph. Thus, by Lemma 4.11, $5 \leq r \leq 7$, where r is the number of non-trivial ideals of R. □

Theorem 4.14. *Let* R *be a commutative Artinian ring such that* $R = F_1 \times F_2 \times \cdots \times F_n$, *where* $n \geq 4$ *and* F_i *is a field for each* $1 \leq i \leq n$. *Then* $\gamma(\mathcal{EG}(R)) = 1$ *if and only if* $n = 4$.

Proof. Since R is a reduced ring, $\mathcal{EG}(R) = AG(R)$ by Theorem 2.5. Hence the result follows from [19, Theorem 2].

□

Theorem 4.15. *Let R be a commutative Artinian ring such that $R = R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 2$ and each $(R_i, \mathfrak{m}_i)$ is an Artinian local ring with $m_i \neq 0$. Let η_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(\mathcal{EG}(R)) = 1$ if and only if $n = 2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals of R_1 and R_2 respectively.*

Proof. Suppose that $\gamma(\mathcal{EG}(R)) = 1$. If $n \geq 3$, then the set $\{\mathfrak{m}_1^{\eta_1-1} \times (0) \times \cdots \times (0), (0) \times \mathfrak{m}_2^{\eta_2-1} \times (0) \times \cdots \times (0), \mathfrak{m}_1^{\eta_1-1} \times \mathfrak{m}_2^{\eta_2-1} \times (0) \times \cdots \times (0)\} \cup \{(0) \times (0) \times R_3 \times (0) \times \cdots \times (0), (0) \times (0) \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), \mathfrak{m}_1 \times (0) \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times (0) \times \cdots \times (0), \mathfrak{m}_1 \times (0) \times R_3 \times (0) \times \cdots \times (0), (0) \times \mathfrak{m}_2 \times R_3 \times (0) \times \cdots \times (0)\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$. Thus, from Lemma 4.12, $\gamma(\mathcal{EG}(R)) > 1$, a contradiction. Hence $n = 2$.
Suppose I is non-trivial ideal of R_1 such that $I \neq \mathfrak{m}_1$. Then the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), R_1 \times \mathfrak{m}_2, I \times (0), I \times \mathfrak{m}_2\} \cup \{(0) \times R_2, (0) \times \mathfrak{m}_2, \mathfrak{m}_1 \times R_2, \mathfrak{m}_1 \times \mathfrak{m}_2\}$ induces a copy of $K_{4,5}$ in $\mathcal{EG}(R)$. By Lemma 4.12, $\gamma(\mathcal{EG}(R)) > 1$, a contradiction. Hence R_1 has unique non-trivial ideal $\mathfrak{m}_1$. Similarly, we can show that R_2 has unique non-trivial ideal $\mathfrak{m}_2$.
Conversely, let $R = R_1 \times R_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals of R_1 and R_2 respectively, then $|A^*(R)| = 7$. It is easy to see that the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), R_1 \times \mathfrak{m}_2\} \cup \{(0) \times R_2, (0) \times \mathfrak{m}_2, \mathfrak{m}_1 \times R_2\}$ induces a copy of $K_{3,3}$, which implies that $K_{3,3} \leq \mathcal{EG}(R) \leq K_7$. Hence, by Lemma 4.11 and 4.12, $\gamma(\mathcal{EG}(R)) = 1$. □

Theorem 4.16 ([19, Theorem 4])**.** *Let $R = R_1 \times R_2 \times F$ be a commutative ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and F is a field. Let η_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(AG(R)) > 1$.*

Theorem 4.17 ([19, Theorem 5])**.** *Let $R = R_1 \times F_1 \times F_2 \times \cdots \times F_m$ be a commutative ring, where each $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq 0$ and each F_j is a field. Let η_1 be the nilpotency of $\mathfrak{m}_1$ and $m \geq 3$. Then $\gamma(AG(R)) > 1$.*

Theorem 4.18. *Let R be a commutative Artinian ring such that $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$, where each $(R_i, \mathfrak{m}_i)$ is an Artinian local ring with $\mathfrak{m}_i \neq 0$ and each F_j is a field. Let η_i be the nilpotency of $\mathfrak{m}_i$ and $n \geq 2$ or $m \geq 3$. Then $\gamma(\mathcal{EG}(R)) > 1$.*

Proof. Follows from Theorems 4.16 and 4.17. □

Theorem 4.19. *Let R be a commutative Artinian ring such that $R = R_1 \times F_1 \times F_2$, where $(R_1, \mathfrak{m})$ is an Artinian local ring and F_1 and F_2 are fields. Let η be the nilpotency of $\mathfrak{m}$. Then $\gamma(\mathcal{EG}(R)) = 1$ if and only if $\eta = 2$ and $\mathfrak{m}$ is the only non-trivial ideal of R_1.*

Proof. Suppose that $\eta = 2$ and $\mathfrak{m}$ is the only non-trivial ideal of R_1. Then from Figure 5, we get $\gamma(\mathcal{EG}(R)) = 1$, where $a = \mathfrak{m} \times (0) \times (0)$, $b = R_1 \times (0) \times (0)$, $c = \mathfrak{m} \times F_1 \times F_2$, $d = (0) \times F_1 \times F_2$, $e = \mathfrak{m} \times (0) \times F_2$, $f = (0) \times F_1 \times (0)$, $g = R_1 \times F_1 \times (0)$, $h = R_1 \times (0) \times F_2$, $i = (0) \times (0) \times F_2$, $j = \mathfrak{m} \times F_1 \times (0)$.
Conversely, assume that $\gamma(\mathcal{EG}(R)) = 1$. Let J be a non-trivial ideal of R_1 such that $J \neq \mathfrak{m}$. Then the set $\{\mathfrak{m} \times (0) \times (0), \mathfrak{m} \times F_1 \times (0), J \times F_1 \times (0), (0) \times F_1 \times (0)\} \cup \{J \times (0) \times (0), \mathfrak{m} \times (0) \times F_2, J \times (0) \times F_2, (0) \times (0) \times F_2, R_1 \times (0) \times (0)\}$ induces a copy of $K_{4,5}$ in $\mathcal{EG}(R)$, which is a contradiction. Hence $\mathfrak{m}$ is the only non-trivial ideal of R_1. □

Theorem 4.20. *Let R be a commutative Artinian ring such that $R = R_1 \times F$, where $(R_1, \mathfrak{m})$ is an Artinian local ring and F is a field. Let η be the nilpotency of $\mathfrak{m}$. Then $\gamma(\mathcal{EG}(R)) = 1$ if and only if one of the following holds:*

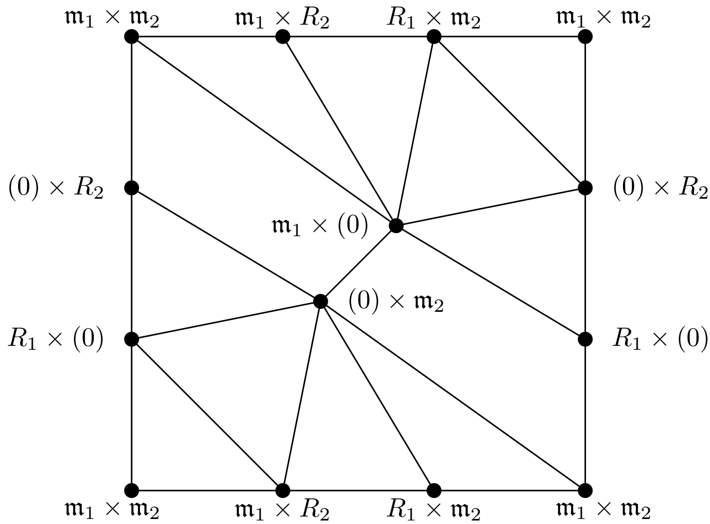

Figure 4: Toroidal embedding of $\mathcal{EG}(R_1 \times R_2)$, where $\mathfrak{m}_i$ is the only non-trivial ideal of R_i for $i = 1, 2$.

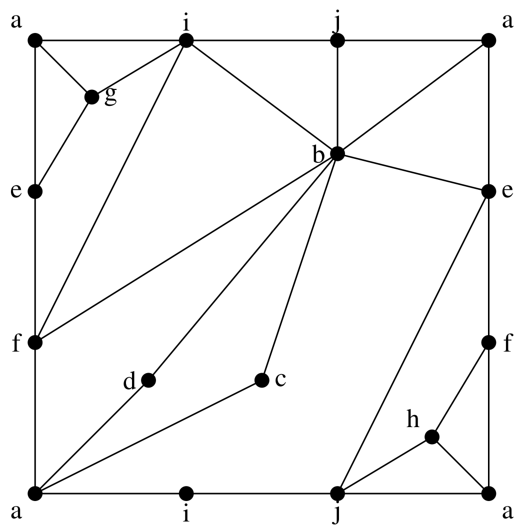

Figure 5: Toroidal embedding of $\mathcal{EG}(R_1 \times F_1 \times F_2)$, where $\mathfrak{m}$ is the only non-trivial ideal of R_1.

(1) $\eta = 3$ *and* $\mathfrak{m}$ *and* $\mathfrak{m}^2$ *are the only non-trivial ideals of* R_1.

(2) $\eta = 4$ *and* $\mathfrak{m}$, $\mathfrak{m}^2$ *and* $\mathfrak{m}^3$ *are the only non-trivial ideals of* R_1.

Proof. Suppose that $\gamma(\mathcal{EG}(R)) = 1$. If $\eta \geq 5$, then the set $\{\mathfrak{m}^{\eta-1} \times (0), \mathfrak{m}^{\eta-2} \times (0), \mathfrak{m}^{\eta-3} \times (0)\} \cup \{R_1 \times (0), \mathfrak{m} \times (0), (0) \times F, \mathfrak{m}^{\eta-1} \times F, \mathfrak{m}^{\eta-2} \times F, \mathfrak{m}^{\eta-3} \times F, \mathfrak{m} \times F\}$

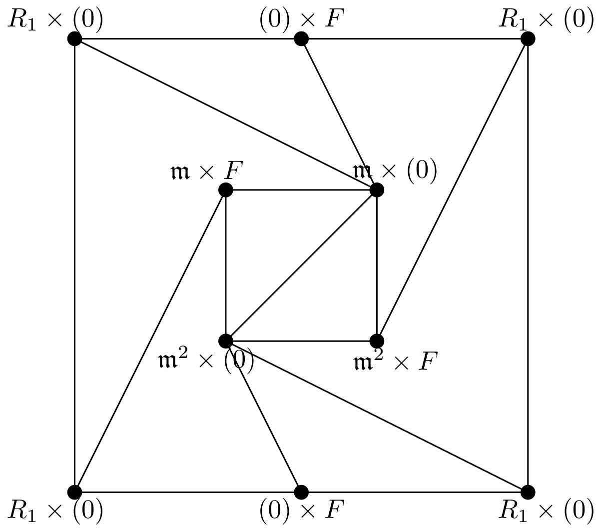

Figure 6: Toroidal embedding of $\mathcal{EG}(R_1 \times F)$, where $\mathfrak{m}$ and $\mathfrak{m}^2$ are only non-trivial ideals of R_1.

induces a copy of $K_{3,7}$. Thus, by Lemma 4.12, $\gamma(\mathcal{EG}(R)) > 1$, a contradiction. Hence $\eta \leq 4$. We have following cases:

Case I: $\eta = 2$. Let J be a non-trivial ideal of R_1 such that $J \neq \mathfrak{m}$. Then by Lemma 4.7, R_1 has at least three non-trivial ideals I_1, I_2 and I_3 such that $I_1, I_2, I_3 \neq \mathfrak{m}$. We can see that the set $\{R_1 \times (0), J \times (0), I_1 \times (0), I_2 \times (0)\} \cup \{(0) \times F, J \times F, I_1 \times F, I_2 \times F, \mathfrak{m} \times F\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$, a contradiction. Hence $\mathfrak{m}$ is the only non-trivial ideal of R_1. It follows from Theorem 4.10 that $\mathcal{EG}(R)$ is a planar graph, a contradiction.

Case II: $\eta = 3$. Let I be a non-trivial ideal of R_1 such that $I \neq \mathfrak{m}, \mathfrak{m}^2$. Then by Lemma 4.7, R_1 has at least three non-trivial ideals I_1, I_2 and I_3 such that $I_1, I_2, I_3 \neq \mathfrak{m}, \mathfrak{m}^2$. It is easy to see that the set $\{R_1 \times (0), \mathfrak{m} \times (0), \mathfrak{m}^2 \times (0)\} \cup \{I \times (0), I_1 \times (0), I_2 \times (0), 0 \times F, \mathfrak{m} \times F, \mathfrak{m}^2 \times F, I \times F\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$, a contradiction. Hence $\mathfrak{m}$ and $\mathfrak{m}^2$ are the only non-trivial ideals of R_1.

Case III: $\eta = 4$. Let I be a non-trivial ideal of R_1 such that $I \neq \mathfrak{m}^i$ for each $i = 1, 2, 3$. Then by Lemma 4.7, R_1 has at least three non-trivial ideals I_1, I_2 and I_3 such that $I_1, I_2, I_3 \neq \mathfrak{m}^i$ for each $i = 1, 2, 3$. It is easy to see that the set $\{\mathfrak{m} \times (0), \mathfrak{m}^2 \times (0), \mathfrak{m}^3 \times (0)\} \cup \{R_1 \times (0), I \times (0), I_1 \times (0), I_2 \times (0), \mathfrak{m} \times F, \mathfrak{m}^2 \times (0), \mathfrak{m}^3 \times (0)\}$ induces a copy of $K_{3,7}$ in $\mathcal{EG}(R)$, a contradiction. Hence $\mathfrak{m}$, $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are the only non-trivial ideals of R_1.

Conversely, if $\mathfrak{m}$ and $\mathfrak{m}^2$ are the only non-trivial ideals of R_1, then $|A^*(R)| = 6$ and the set $\{R_1 \times (0), \mathfrak{m} \times (0), \mathfrak{m}^2 \times (0)\} \cup \{(0) \times F, \mathfrak{m} \times F, \mathfrak{m}^2 \times F\}$ induces a copy of $K_{3,3}$ in $\mathcal{EG}(R)$. Thus, $K_{3,3} \leq \mathcal{EG}(R) \leq K_6$, which implies that $\gamma(\mathcal{EG}(R)) = 1$.

Now, if $\mathfrak{m}$, $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are the only non-trivial ideals of R_1. Then from Figure 7, $\gamma(\mathcal{EG}(R)) = 1$. □

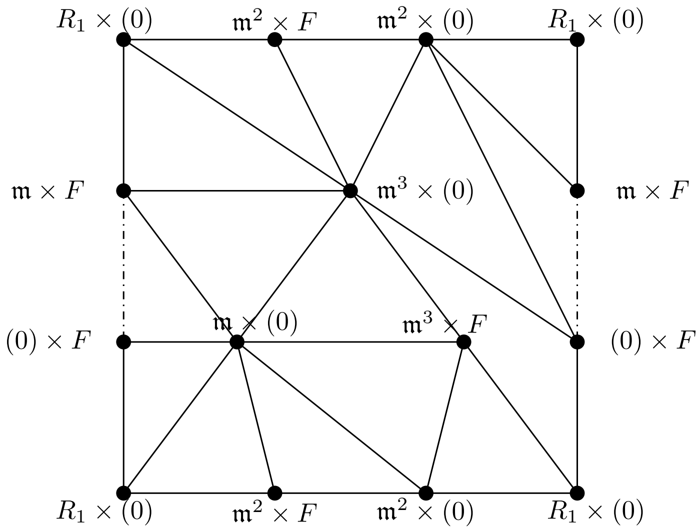

Figure 7: Toroidal embedding of $\mathcal{EG}(R_1 \times F)$, where $\mathfrak{m}$, $\mathfrak{m}^2$ and $\mathfrak{m}^3$ are non-trivial ideals of R_1.

ORCID iDs

Mohd Nazim https://orcid.org/0000-0001-8817-4336
Nadeem ur Rehman https://orcid.org/0000-0003-3955-7941

References

[1] G. Aalipour, S. Akbari, M. Behboodi, R. Nikandish, M. J. Nikmehr and F. Shaveisi, The classification of the annihilating-ideal graphs of commutative rings, *Algebra Colloq.* **21** (2014), 249–256, doi:10.1142/s1005386714000200.

[2] G. Aalipour, S. Akbari, R. Nikandish, M. Nikmehr and F. Shaveisi, On the coloring of the annihilating-ideal graph of a commutative ring, *Discrete Math.* **312** (2012), 2620–2626, doi: 10.1016/j.disc.2011.10.020.

[3] G. Aalipour, S. Akbari, R. Nikandish, M. J. Nikmehr and F. Shaveisi, Minimal prime ideals and cycles in annihilating-ideal graphs, *Rocky Mountain J. Math.* **43** (2013), 1415–1425, doi: 10.1216/rmj-2013-43-5-1415.

[4] M. Ahrari, S. A. S. Sabet and B. Amini, On the girth of the annihilating-ideal graph of a commutative ring, *J. Linear Topol. Algebra* **4** (2015), 209–216.

[5] S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra* **274** (2004), 847–855, doi:10.1016/s0021-8693(03)00435-6.

[6] F. Aliniaeifard, M. Behboodi, E. Mehdi-Nezhad and A. M. Rahimi, On the diameter and girth of an annihilating-ideal graph, 2014, `arXiv:1411.4163v1 [math.CO]`.

[7] D. D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra* **159** (1993), 500–514, doi:10.1006/jabr.1993.1171.

[8] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434–447, doi:10.1006/jabr.1998.7840.

[9] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[10] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208–226, doi:10.1016/0021-8693(88)90202-5.

[11] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.* **10** (2011), 727–739, doi:10.1142/s0219498811004896.

[12] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings II, *J. Algebra Appl.* **10** (2011), 741–753, doi:10.1142/s0219498811004902.

[13] F. R. DeMeyer, T. McKenzie and K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum* **65** (2002), 206–214, doi:10.1007/s002330010128.

[14] D. Dolžan and P. Oblak, The zero-divisor graphs of rings and semirings, *Internat. J. Algebra Comput.* **22** (2012), 1250033, 20, doi:10.1142/s0218196712500336.

[15] T. Y. Lam, *A First Course in Non-Commutative Rings*, Springer-Verlag, New York, 1991, doi:10.1007/978-1-4684-0406-7.

[16] M. J. Nikmehr and S. M. Hosseini, More on the annihilator-ideal graph of a commutative ring, *J. Algebra Appl.* **18** (2019), 1950160, 14, doi:10.1142/s0219498819501603.

[17] M. J. Nikmehr, R. Nikandish and M. Bakhtyiari, On the essential graph of a commutative ring, *J. Algebra Appl.* **16** (2017), 1750132, 14, doi:10.1142/s0219498817501328.

[18] S. P. Redmond, The zero-divisor graph of a non-commutative ring, in: *Internat. J. Commutative Rings*, Nova Sci. Publ., Hauppauge, NY, volume 1, 2002.

[19] K. Selvakumar and P. Subbulakshmi, Classification of rings with toroidal annihilating-ideal graph, *Commun. Comb. Optim.* **3** (2018), 93–119, doi:10.22049/cco.2018.26060.1072.

[20] K. Selvakumar, P. Subbulakshmi and J. Amjadi, On the genus of the graph associated to a commutative ring, *Discrete Math. Algorithms Appl.* **9** (2017), 1750058, 11, doi:10.1142/s1793830917500586.

[21] D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India, New Delhi, 2002.

[22] A. T. White, *Graphs, Groups and Surfaces*, North-Holland Mathematics Studies, No. 8, North-Holland Publishing Co., Amsterdam, 1973.