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NON-SINGULAR METHOD OF FUNDAMENTAL SOLUTIONS FOR THREE–DIMENSIONAL ISOTROPIC ELASTICITY PROBLEMS WITH DISPLACEMENT BOUNDARY CONDITIONS

NESINGULARNA METODA FUNDAMENTALNIH REŠITEV ZA DEFORMACIJO TRIDIMENZIJSKIH ELASTIČNIH PROBLEMOV Z DEFORMACIJSKIMI ROBNIMI POGOJI

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The purpose of the present paper is to develop the Non-Singular Method of Fundamental Solutions (NMFS) based on the boundary-distributed source method for three-dimensional elasticity problems with displacement boundary conditions. In the NMFS, the source points and the collocation points coincide and both are positioned on the boundary of the problem domain. In this case, the fundamental solution is singular. In order to remove the singularities of the fundamental solution, the concentrated point sources are replaced by the distributed sources over the sphere around the singularity. The values of the distributed sources are calculated directly in the case of displacement boundary conditions for isotropic problems. The performance of the novel approach is shown on two three-dimensional elastic problems with displacement boundary conditions. The method requires the discretization of the boundary only and shows excellent accuracy. It represents an efficient alternative to the classic numerical methods. The developments lead to the possibility of modelling micromechanical problems without the discretization of the interor of each of the grains, like required in classic numerical methods.

Keywords: linear isotropic elasticity, non-singular method of fundamental solutions, boundary meshless method

Namen članka je razvoj nesingularne metode fundamentalnih rešitev (NMFS) na podlagi robno distribuirane metode izvirov za tridimenzijske probleme linearne elastičnosti z deformacijskimi robnimi pogoji. V NMFS se izvirne in kolokacijske točke skladajo in so pozicionirane na robu obravnavanega območja. V tem primeru je fundamentalna rešitev singularna. Za odstranitev singularnosti fundamentalne rešitve so koncentrirani izviri nadomeščeni s porazdeljenimi izviri po krogli okoli singularnosti. Vrednosti porazdeljenih izvirov so neposredno izračunane pri Dirichletovih robnih pogojih za izotropne probleme. Značilnosti novega načina so prikazane na dveh primerih tridimenzijskih problemov z deformacijskimi robnimi pogoji. Metoda zahteva zgolj diskretizacijo roba in prikazuje odlično natančnost. Pomeni tudi učinkovito alternativo klasičnim numeričnim metodam. Opisani razvoj vodi do možnosti simulacije mikromehanskih problemov brez diskretizacije notranjosti zrn, kot je to potrebno pri klasičnih numeričnih metodah.

Ključne besede: linerna izotropna elastičnost, nesingularna metoda fundamentalnih rešitev, robna brezmrežna metoda

1 INTRODUCTION

The main idea of MFS[1] consists of approximating the solution of the partial differential equation by a linear combination of fundamental solutions, defined in source points. The expansion coefficients are calculated by collocation or a least-squares fit of the boundary conditions. The fundamental solution is usually singular in the source points and this is the reason why the source points are located outside the domain in the MFS. In this case, the original problem is reduced to determining the unknown coefficients of the fundamental solutions and the coordinates of the source points by requiring the approximation to satisfy the boundary conditions and hence solving a non-linear problem. If the source points are a priori fixed, then the coefficients of the MFS approximation are determined by solving a linear problem. The MFS has become very popular in recent years because of its simplicity[2–5] and for 3D problems.[6,7]

In the traditional MFS, a fictitious boundary, positioned outside the problem domain, is required to place the source points. This is very impractical or even impossible, particularly when solving muti-body problems. In recent years, various efforts have been made, with the aim being to remove this barrier in the MFS, so that the source points can be placed on the real boundary directly[8–12] In the present paper, we use a Non-Singular MFS based on[8] to deal with the three-dimensional isotropic elasticity problems with displacement boundary condition. The application of a non-singular method of fundamental solutions (NMFS) in two-dimensional isotropic and anisotropic linear elasticity has been originally developed.[13–15] We respectively used area-distributed sources covering the source points to replace the concentrated point sources. This NMFS approach also does not require any information about the neighboring points for each source point, thus it is a truly a meshfree boundary method. The present develoments are dedicated to enabling NMFS for solving three-dimensional micromechanical elasticity problems. This is of utmost importance in the simulation of an effective Young's modulus and Poisson's ratio for multigrain systems that appear in many engineering systems.

The rest of the paper is structured as follows. The governing equations are shown in matrix form. The solution procedure is given for MFS and NMFS. A three-dimensional example in two cases, translation and deformation, is given, followed by the conclusions and future research.

2 GOVERNING EQUATIONS

Consider a 3D domain Ω with the boundary Γ filled with isotropic elasticity materials. Let us introduce a 3D Cartesian coordinate system with the orthonormal base vectors $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ and the coordinates p_x, p_y and p_z of the position vector $\mathbf{p}$, i.e., $\mathbf{p} = p_x\,\mathbf{i}_x + p_y\,\mathbf{i}_y + p_z\,\mathbf{i}_z$. To simplify the calculations we shall assume that (i) the solid is free of body forces and (ii) the thermal strains can be neglected. Under these conditions the general equation of elasticity[16] is:

$$C_{\zeta\xi\upsilon\tau}\frac{\partial^2 u_\upsilon(\mathbf{p})}{\partial p_\zeta \partial p_\tau}=0,\quad \zeta,\xi,\upsilon,\tau=x,y,z \tag{1}$$

where u_υ are the displacements, $C_{\zeta\xi\upsilon\tau}$ are the elastic stiffnesses and the components of a fourth rank stiffness tensor:[17]

$$\mathbf{C}=C_{\zeta\xi\upsilon\tau}\begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & C_{xxyz} & C_{xxxz} & C_{xxxy} \\ C_{xxyy} & C_{yyyy} & C_{yyzz} & C_{yyyz} & C_{xzyy} & C_{xyyy} \\ C_{xxzz} & C_{yyzz} & C_{zzzz} & C_{yzzz} & C_{xzzz} & C_{xyzz} \\ C_{xxyz} & C_{yyyz} & C_{yzzz} & C_{yzyz} & C_{xzyz} & C_{xyyz} \\ C_{xxxz} & C_{xzyy} & C_{xzzz} & C_{xzyz} & C_{xzxz} & C_{xyxz} \\ C_{xxxy} & C_{xyyy} & C_{xyzz} & C_{xyyz} & C_{xyxz} & C_{xyxy} \end{bmatrix}= \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \tag{2}$$

In subsequent discussions, it will be convenient to write the equilibrium Equation (1) in matrix form as:

$$\begin{bmatrix} \frac{\partial}{\partial p_x} & 0 & 0 & \frac{\partial}{\partial p_y} & \frac{\partial}{\partial p_z} & 0 \\ 0 & \frac{\partial}{\partial p_y} & 0 & \frac{\partial}{\partial p_x} & 0 & \frac{\partial}{\partial p_z} \\ 0 & 0 & \frac{\partial}{\partial p_z} & 0 & \frac{\partial}{\partial p_x} & \frac{\partial}{\partial p_y} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \cdot \begin{bmatrix} \partial u_x/\partial p_x \\ \partial u_y/\partial p_y \\ \partial u_z/\partial p_z \\ \partial u_y/\partial p_z + \partial u_z/\partial p_y \\ \partial u_x/\partial p_z + \partial u_z/\partial p_x \\ \partial u_x/\partial p_y + \partial u_y/\partial p_x \end{bmatrix} = 0 \tag{3}$$

The stresses $\sigma_{\zeta\xi}$ are related to the strains through the generalized Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \tag{4}$$

where $C_{\zeta\xi\upsilon\tau}$ satisfy the fully symmetrical conditions:

$$C_{\zeta\xi\upsilon\tau}=C_{\xi\zeta\upsilon\tau},\; C_{\zeta\xi\upsilon\tau}=C_{\zeta\xi\tau\upsilon},\; C_{\zeta\xi\upsilon\tau}=C_{\upsilon\tau\zeta\xi} \tag{5}$$

$\boldsymbol{\varepsilon}$ is the strains vector:

$$\boldsymbol{\varepsilon}\equiv\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}=\begin{bmatrix} \partial u_x/\partial p_x \\ \partial u_y/\partial p_y \\ \partial u_z/\partial p_z \\ \partial u_y/\partial p_z+\partial u_z/\partial p_y \\ \partial u_x/\partial p_z+\partial u_z/\partial p_x \\ \partial u_x/\partial p_y+\partial u_y/\partial p_x \end{bmatrix} \tag{6}$$

3 SOLUTION PROCEDURE

The fundamental solution for the isotropic elasticity is given[18] in three dimensions (3D) by:

$$U_{\zeta\xi}(\mathbf{p},\mathbf{s})=\frac{1}{16\pi\mu(1-\nu)r}\cdot \left\{(3-4\nu)\delta_{\zeta\xi}+\frac{(p_\zeta-s_\zeta)(p_\xi-s_\xi)}{r^2}\right\},\quad \zeta,\xi=x,y,z \tag{7}$$

where $U_{\zeta\xi}(\mathbf{p},\mathbf{s})$ represents the displacement in the direction ζ at point $\mathbf{p}$ due to a unit point force acting in the direction ξ at point $\mathbf{s}$. $r = [(p_x - s_x)^2 + (p_y - s_y)^2 + (p_z - s_z)^2]^{1/2}$ is the distance between the point $\mathbf{p}$ and the source point $\mathbf{s}$. Equation (7) is expanded as follows:

$$U_{xx}=\frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)+\frac{(p_x-s_x)^2}{r^2}\right]$$
$$U_{yy}=\frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)+\frac{(p_y-s_y)^2}{r^2}\right]$$
$$U_{zz}=\frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)+\frac{(p_z-s_z)^2}{r^2}\right] \tag{8}$$
$$U_{xy}=U_{yx}=\frac{1}{16\pi\mu(1-\nu)r}\frac{(p_x-s_x)(p_y-s_y)}{r^2}$$
$$U_{xz}=U_{zx}=\frac{1}{16\pi\mu(1-\nu)r}\frac{(p_x-s_x)(p_z-s_z)}{r^2}$$
$$U_{yz}=U_{zy}=\frac{1}{16\pi\mu(1-\nu)r}\frac{(p_y-s_y)(p_z-s_z)}{r^2}$$

It can be shown that the following u_x, u_y and u_z satisfy the governing Equations (3):

$$u_x(\mathbf{p})=U_{xx}(\mathbf{p},\mathbf{s})\alpha+U_{xy}(\mathbf{p},\mathbf{s})\beta+U_{xz}(\mathbf{p},\mathbf{s})\gamma \tag{9}$$
$$u_y(\mathbf{p})=U_{yx}(\mathbf{p},\mathbf{s})\alpha+U_{yy}(\mathbf{p},\mathbf{s})\beta+U_{yz}(\mathbf{p},\mathbf{s})\gamma \tag{10}$$
$$u_z(\mathbf{p})=U_{zx}(\mathbf{p},\mathbf{s})\alpha+U_{zy}(\mathbf{p},\mathbf{s})\beta+U_{zz}(\mathbf{p},\mathbf{s})\gamma \tag{11}$$

where α, β and γ represent arbitrary constants. The fundamental solution $U_{\zeta\xi}(\mathbf{p},\mathbf{s})$ is singular when $\mathbf{p} = \mathbf{s}$. We use the desingularization technique, proposed by Liu[8] for evaluating the singular values. We modify his approach in a sense of preserving the original fundamental solution at all the points except the singularity, and by scaling the singularity with the area of the sphere over which the desingularization integration is performed. This allows us to treat the MFS and the NMFS in formally the same way. The desingularization (transformation of $U_{\zeta\xi}(\mathbf{p},\mathbf{s})$ into $\tilde{U}_{\zeta\xi}(\mathbf{p},\mathbf{s})$) is thus performed in the following way:

$$\tilde{U}_{\zeta\xi}(\mathbf{p},\mathbf{s})=\begin{cases} U_{\zeta\xi}(\mathbf{p},\mathbf{s}) & r>R \\ \dfrac{1}{\pi R^2}\displaystyle\int_{A(\mathbf{s},R)} U_{\zeta\xi}(\mathbf{p},\mathbf{s})\mathrm{d}A & r\le R \end{cases} \tag{12}$$

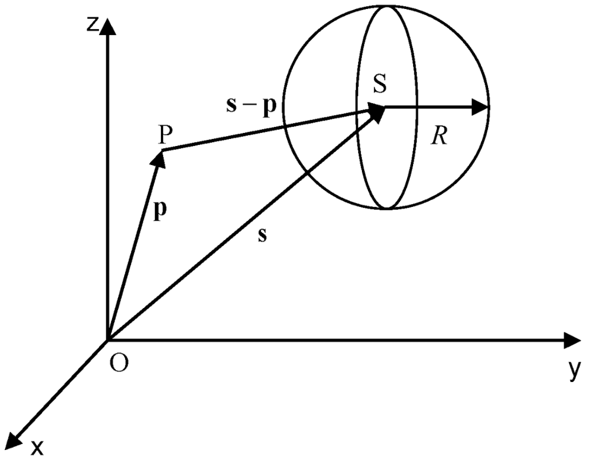

Figure 1: Distributed source on a sphere $A(\mathbf{s},R)$ with radius R
Slika 1: Porazdeljeni izviri na krogli $A(\mathbf{s},R)$ z radijem R

where $A(\mathbf{s},R)$ represents a sphere with radius R, centered around $\mathbf{s}$. The involved integrals can be calculated as follows (by using the integration in polar coordinates $p_x - s_x = r \sin\varphi \cos\theta$, $p_y - s_y = r \sin\varphi \sin\theta$ and $p_z - s_z = r \cos\varphi$, **Figure 1**):

$$\tilde{U}_{xx}(\mathbf{p},\mathbf{p})=\tilde{U}_{yy}(\mathbf{p},\mathbf{p})=\tilde{U}_{zz}(\mathbf{p},\mathbf{p})=\frac{5-6\nu}{16\pi\mu(1-\nu)R}$$
$$\tilde{U}_{xy}(\mathbf{p},\mathbf{p})=\tilde{U}_{yx}(\mathbf{p},\mathbf{p})=0 \tag{13}$$
$$\tilde{U}_{xz}(\mathbf{p},\mathbf{p})=\tilde{U}_{zx}(\mathbf{p},\mathbf{p})=0$$
$$\tilde{U}_{yz}(\mathbf{p},\mathbf{p})=\tilde{U}_{zy}(\mathbf{p},\mathbf{p})=0$$

It can also be shown that the following u_x, u_y and u_z satisfy the governing Equations (3):

$$u_x(\mathbf{p})=\tilde{U}_{xx}(\mathbf{p},\mathbf{s})\alpha+\tilde{U}_{xy}(\mathbf{p},\mathbf{s})\beta+\tilde{U}_{xz}(\mathbf{p},\mathbf{s})\gamma \tag{14}$$
$$u_y(\mathbf{p})=\tilde{U}_{yx}(\mathbf{p},\mathbf{s})\alpha+\tilde{U}_{yy}(\mathbf{p},\mathbf{s})\beta+\tilde{U}_{yz}(\mathbf{p},\mathbf{s})\gamma \tag{15}$$
$$u_z(\mathbf{p})=\tilde{U}_{zx}(\mathbf{p},\mathbf{s})\alpha+\tilde{U}_{zy}(\mathbf{p},\mathbf{s})\beta+\tilde{U}_{zz}(\mathbf{p},\mathbf{s})\gamma \tag{16}$$

The solution of the problem is sought in the form:

$$u_x(\mathbf{p})=\sum_{n=1}^{N}\tilde{U}_{xx}(\mathbf{p},\mathbf{p}_n)\alpha_n+\sum_{n=1}^{N}\tilde{U}_{xy}(\mathbf{p},\mathbf{p}_n)\beta_n+ +\sum_{n=1}^{N}\tilde{U}_{xz}(\mathbf{p},\mathbf{p}_n)\gamma_n \tag{17}$$
$$u_y(\mathbf{p})=\sum_{n=1}^{N}\tilde{U}_{yx}(\mathbf{p},\mathbf{p}_n)\alpha_n+\sum_{n=1}^{N}\tilde{U}_{yy}(\mathbf{p},\mathbf{p}_n)\beta_n+ +\sum_{n=1}^{N}\tilde{U}_{yz}(\mathbf{p},\mathbf{p}_n)\gamma_n \tag{18}$$
$$u_z(\mathbf{p})=\sum_{n=1}^{N}\tilde{U}_{zx}(\mathbf{p},\mathbf{p}_n)\alpha_n+\sum_{n=1}^{N}\tilde{U}_{zy}(\mathbf{p},\mathbf{p}_n)\beta_n+ +\sum_{n=1}^{N}\tilde{U}_{zz}(\mathbf{p},\mathbf{p}_n)\gamma_n \tag{19}$$

The coefficients α_n, β_n and γ_n are calculated from a system of $3N$ algebraic equations:

$$\mathbf{Ax} = \mathbf{b} \tag{20}$$

where $\mathbf{A}$ stands for a $3N \times 3N$ matrix with the entries A_{ij}, $\mathbf{x}$ is a $3N \times 1$ vector with the entries x_i, and $\mathbf{b}$ is a $3N \times 1$ vector with entries b_i:

$$A_{ij}=\tilde{U}_{xx}(\mathbf{p}_i,\mathbf{p}_j),\quad A_{i(N+j)}=\tilde{U}_{xy}(\mathbf{p}_i,\mathbf{p}_j)$$
$$A_{i(2N+j)}=\tilde{U}_{xz}(\mathbf{p}_i,\mathbf{p}_j),\quad A_{(N+i)j}=\tilde{U}_{yx}(\mathbf{p}_i,\mathbf{p}_j)$$
$$A_{(N+i)(N+j)}=\tilde{U}_{yy}(\mathbf{p}_i,\mathbf{p}_j),\quad A_{(N+i)(2N+j)}=\tilde{U}_{yz}(\mathbf{p}_i,\mathbf{p}_j) \tag{21}$$
$$A_{(2N+i)j}=\tilde{U}_{zx}(\mathbf{p}_i,\mathbf{p}_j),\quad A_{(2N+i)(N+j)}=\tilde{U}_{zy}(\mathbf{p}_i,\mathbf{p}_j)$$
$$A_{(2N+i)(2N+j)}=\tilde{U}_{zz}(\mathbf{p}_i,\mathbf{p}_j),\quad i,j=1,2,\ldots,N$$

$$x_i=\alpha_i,\quad x_{(N+i)}=\beta_i,\quad x_{(2N+i)}=\gamma_i,\quad i=1,2,\ldots,N \tag{22}$$

$$b_i=u_x(\mathbf{p}_i),\, b_{(N+i)}=u_y(\mathbf{p}_i),\, b_{(2N+i)}=u_z(\mathbf{p}_i),\quad i=1,2,\ldots,N \tag{23}$$

By knowing all the elements A_{ij} and b_i of the system (20), we can determine the values of x_i (i.e., α_n, β_n and γ_n). Afterwards, we can calculate the solution of the governing equation from:

$$u_\zeta(\mathbf{p}) = \sum_{n=1}^{N} \widetilde{U}_{\zeta x}(\mathbf{p},\mathbf{p}_n)\alpha_n + \sum_{n=1}^{N} \widetilde{U}_{\zeta y}(\mathbf{p},\mathbf{p}_n)\beta_n + + \sum_{n=1}^{N} \widetilde{U}_{\zeta z}(\mathbf{p},\mathbf{p}_n)\gamma_n, \quad \zeta = x, y, z \tag{24}$$

where **p** is any point inside the domain or on the boundary.

4 NUMERICAL EXAMPLES

We consider a cube with the side length a = 2 m centered around p_x = 0 m, p_y = 0 m, p_z = 0 m. The elastic media is defined by E = 1 N/m^2, v = 0.3.

4.1 Translation

We consider a solution of the governing equations in this cube subject to the boundary conditions $\overline{u}_x$ = 2 m, $\overline{u}_y$ = 2 m, $\overline{u}_z$ = 2 m. The analytical solution is:

$$u_x = 2 \text{ m}, \quad u_y = 2 \text{ m}, \quad u_z = 2 \text{ m}, \tag{25}$$

A plot of the translation, obtained with the analytical solution and the numerical solutions with MFS and NMFS, is shown in **Figure 2** for the case with 150 nodes (25 nodes on each side of he cube). The distance of the fictitious boundary from the true boundary for the MFS is set R_M = 5d, where d is the smallest distance between two nodes on the boundary. The radius of the sphere for the distributed area source covering each node is set to R = d/3.

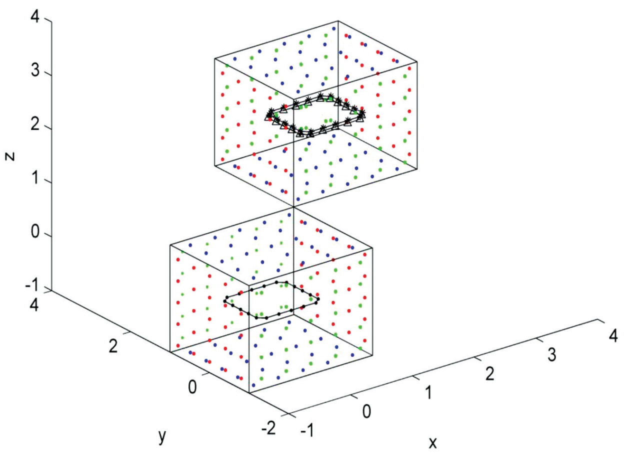

Figure 2: The analytical solution and the numerical solution of MFS and NMFS for the translation case with N = 150, R = d/3, R_M = 5d (•: collocation points, +: analytical solution, x: MFS solution, Δ: NMFS solution)

Slika 2: Analitična in numerična rešitev z MFS in NMFS za translacijski primer z N = 150, R = d/3, R_M = 5d (•: kolokacijske točke, +: analitična rešitev, x: MFS rešitev, Δ: NMFS rešitev)

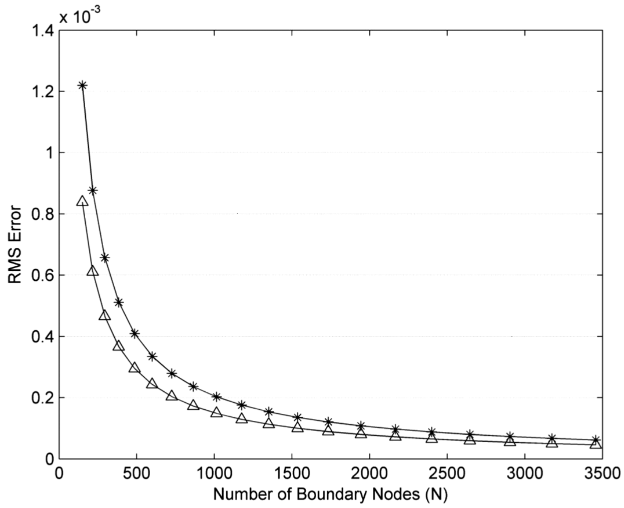

Figure 3: The relationship between the RMS errors and the number of boundary nodes for translation case, calculated by NMFS. R = d/3 (+: e_x, x: e_y, Δ: e_z).

Slika 3: Odvisnost med RMS-napakami in številom robnih točk za translacijski primer, izračunan z NMFS. R = d/3 (+: e_x, x: e_y, Δ: e_z).

The solution of the points on a square with the side length a = 1 m centered around p_x = 0 m, p_y = 0 m, p_z = 0 m on the plane p_z = 0 are computed and compared with the analytical solutions. The root-mean-square (RMS) errors of the numerical solution are defined as:

$$e_\zeta = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(\underline{u}_{\zeta n} - u_{\zeta n})^2}, \quad \zeta = x, y \tag{26}$$

Table 1: RMS errors of NMFS solutions for the translation case with R = d/3

Tabela 1: RMS-napake NMFS-rešitev za translacijski primer z R = d/3

Num. of boundary nodes (N)	e_x(× 10^{-3})	e_y(× 10^{-3})	e_z(× 10^{-3})
150	1.2200	1.2200	0.8390
216	0.8769	0.8769	0.6109
294	0.6570	0.6570	0.4658
384	0.5112	0.5112	0.365
486	0.4091	0.4091	0.2950
600	0.3348	0.3348	0.2428
726	0.2791	0.2791	0.2033
864	0.2363	0.2363	0.1727
1014	0.2026	0.2026	0.1485
1176	0.1757	0.1757	0.1291
1350	0.1538	0.1538	0.1132
1536	0.1357	0.1357	0.1001
1734	0.1207	0.1207	0.0891
1944	0.1080	0.1080	0.0799
2166	0.0973	0.0973	0.0720
2400	0.0880	0.0880	0.0652
2646	0.0800	0.0800	0.0594
2904	0.0731	0.0731	0.0543
3174	0.0670	0.0670	0.0498
3456	0.0617	0.0617	0.0459

where $\underline{u}_{\zeta k}$ and $u_{\zeta k}(\zeta = x, y)$ are the analytical and the numerical solutions, respectively. The number of boundary nodes used is from 150 to 3 456.

Figure 3 shows the RMS errors of the results obtained using the NMFS. The errors are already less than 10^{-3} with $N = 216$ and the solution converges to the analytical solution with an increasing number of nodes (**Table 1**). The MFS result is shown in **Table 2** for $R_M = 5d$. Here it should be noted that the MFS solution error is relatively small; however, the convergence is not uniform. This fact is due to the choice of the artificial boundary position, which was for all node arrangements $R_M = 5d$ and thus most probably not optimally varying.

Table 2: RMS errors of MFS solutions for the translation case with $R_M = 5d$
Tabela 2: RMS-napake MFS-rešitev za translacijski primer z $R_M = 5d$

Num. of boundary nodes (N)	$e_x(\times 10^{-14})$	$e_y(\times 10^{-14})$	$e_z(\times 10^{-14})$
150	0.2204	0.2204	0.5548
216	1.7907	1.7907	4.8854
294	0.0364	0.0364	0.0794
384	0.1590	0.1590	0.1617
486	0.0348	0.0348	0.0017
600	0.0058	0.0058	0.0015
726	0.1103	0.1102	0.1631
864	0.0004	0.0004	0.0001
1014	0.0475	0.0445	0.0478
1176	0.0033	0.0050	0.0025
1350	0.0005	0.0003	0.0010
1536	0.0038	0.0295	0.0238
1734	0.0000	0.0000	0.0000
1944	0.0000	0.0000	0.0000
2166	0.0004	0.0004	0.0006
2400	0.0000	0.0000	0.0000
2646	0.0000	0.0000	0.0000
2904	0.0000	0.0000	0.0000
3174	0.0000	0.0001	0.0000
3456	0.0000	0.0000	0.0000

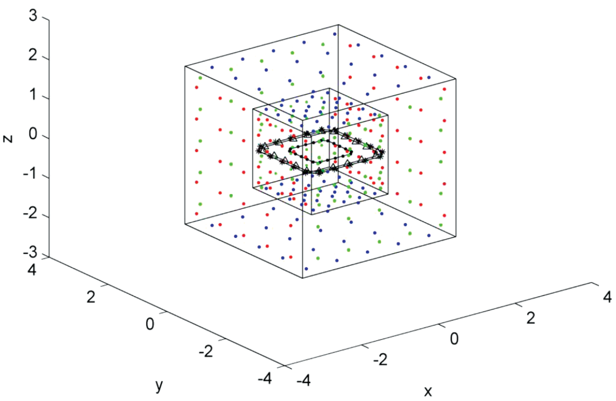

Figure 4: The analytical solution and the numerical solution of MFS and NMFS for the deformation case with $N = 150$, $R = d/3$, $R_M = 5d$ (•: collocation points, +: analytical solution, x: MFS solution, Δ: NMFS solution)
Slika 4: Analitična in numerična rešitev z MFS in NMFS za deformacijski primer z $N = 150$, $R = d/3$, $R_M = 5d$ (•: kolokacijske točke, +: analitična rešitev, x: MFS rešitev, Δ: NMFS rešitev)

4.2 Deformation

We consider a solution of the governing equations in this cube subject to the boundary conditions $\overline{u}_x = p_x$, $\overline{u}_y = p_y$, $\overline{u}_z = p_z$. The analytical solution is:

$$u_x = p_x,\ u_y = p_y,\ u_z = p_z \tag{27}$$

A plot of the deformation, obtained with the analytical solution and the numerical solutions with MFS and NMFS, is shown in **Figure 4** for the case with 150 nodes

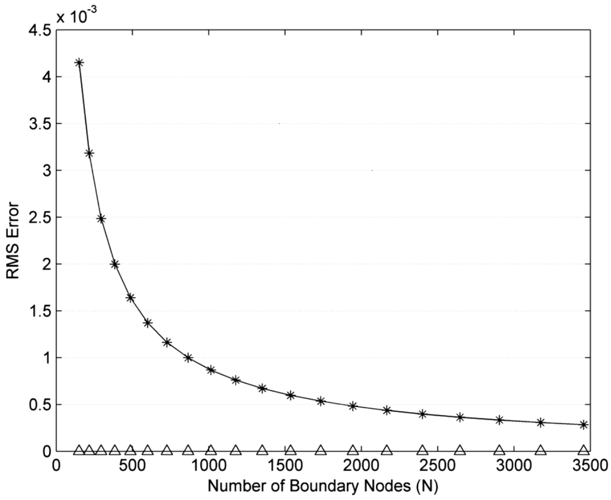

Figure 5: The relationship between the RMS errors and the number of boundary nodes for the deformation case, calculated by NMFS. $R=d/3$ (+: e_x, x: e_y, Δ: e_z).
Slika 5: Odvisnost med RMS-napakami in številom robnih točk za deformacijski primer, izračunan z NMFS. $R = d/3$ (+: e_x, x: e_y, Δ: e_z).

Table 3: RMS errors of the NMFS solutions for the deformation case with $R=d/3$
Tabela 3: Odvisnost med RMS-napakami in številom robnih točk za deformacijski primer, izračunan z NMFS, $R = d/3$

Num. of boundary nodes (N)	$e_x(\times 10^{-3})$	$e_y(\times 10^{-3})$	$e_z(\times 10^{-3})$
150	4.1487	4.1487	0.0000
216	3.1826	3.1826	0.0000
294	2.4837	2.4837	0.0000
384	1.9972	1.9972	0.0000
486	1.6395	1.6395	0.0000
600	1.3703	1.3703	0.0000
726	1.1623	1.1623	0.0000
864	0.9983	0.9983	0.0000
1014	0.8667	0.8667	0.0000
1176	0.7596	0.7596	0.0000
1350	0.6711	0.6711	0.0000
1536	0.5973	0.5973	0.0000
1734	0.5350	0.5350	0.0000
1944	0.4820	0.4820	0.0000
2166	0.4365	0.4365	0.0000
2400	0.3971	0.3971	0.0000
2646	0.3629	0.3629	0.0000
2904	0.3329	0.3329	0.0000
3174	0.3064	0.3064	0.0000
3456	0.2830	0.2830	0.0000

(25 nodes on each side of the cube). The same R and R_M as with example 4.1 are used.

Figure 5 shows the RMS errors of the results obtained using the NMFS and the solution converges to the analytical solution with an increasing number of nodes (**Table 3**). The MFS results are shown in **Table 4** for $R_M = 5d$.

Table 4: RMS errors of the MFS solutions for the deformation case with $R_M = 5d$
Tabela 4: RMS-napake MFS-rešitev za deformacijski primer $R_M = 5d$

Num. of boundary nodes (N)	$e_x(\times 10^{-11})$	$e_y(\times 10^{-11})$	$e_z(\times 10^{-11})$
150	0.1410	0.1410	0.0000
216	0.0256	0.0256	0.0000
294	0.0018	0.0018	0.0000
384	0.0012	0.0012	0.0000
486	0.0014	0.0014	0.0000
600	2.1088	2.1088	0.0000
726	0.0010	0.0010	0.0000
864	0.0001	0.0001	0.0000
1014	0.0000	0.0000	0.0000
1176	0.0000	0.0000	0.0000
1350	0.0010	0.0008	0.0019
1536	0.0000	0.0002	0.0001
1734	0.0000	0.0000	0.0000
1944	0.0002	0.0001	0.0001
2166	0.0005	0.0005	0.0008
2400	0.0000	0.0000	0.0000
2646	0.0000	0.0000	0.0000
2904	0.0000	0.0000	0.0000
3174	0.0000	0.0000	0.0000
3456	0.0000	0.0000	0.0000

5 CONCLUSION

A new, non-singular method of fundamental solutions[13] is extended in the present paper to solve 3D linear elasticity problems. In this approach, the singular values of the fundamental solution are integrated over a small sphere, so that the coefficients in the system of equations can be evaluated analytically and consistently, leading to an extremely simple computer implementation of this method. The method essentially gives similar results as the classic MFS. It has the advantage that the artificial boundary is not present; however, the problems with the traction boundary condition have not yet been solved. The main advantage of the method is that the discretisation is performed only on the boundary of the domain and no polygonisation is needed, like in the finite-element method. The NMFS, presented in this paper, can be adapted or extended to handle many related problems, such as anisotropic elasticity, and multi-body problems, which all represent directions for our further investigations. The advantage of not having to generate the artificial boundary is particularly welcome in these types of problems. The method will be used in the future for the calculation of 3D enginering deformation problems in steel and aluminium alloys, with realistic grain shapes, obtained from microscope images. The developed method is believed to represent the simplest state-of-the-art way to numerically cope with these types of problems.

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