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Features of growth

Every organism should grow and progress, until its development stops and it starts the path towards its end. But for it to have a long and healthy life, its growth must be moderate. Growing too fast can be very dangerous; uncontrolled growth may cause damage, and even death — for individuals and businesses, and also for journals.

We started this journal in 2008, publishing 20 papers in that first year. The number of papers grew to 35 in 2013 (a 75 percent increase). More than half of the papers for the year 2014 are already on-line, and we have a growing back-log of papers accepted but not yet published. The quality of the research published in these papers and the inclusion of the journal in various databases have made it popular among mathematicians worldwide. The number of submissions to our journal is also growing, with over 15 in each of September and October this year, and correspondingly, both the number and percentage of papers that we have to reject or redirect to other journals are increasing as well.

We would like to shape this journal into a self-consistent form that will attract the best possible papers from a rich and wide range of fields of mathematics, while retaining an expectation that their content combines at least two branches of a discrete nature. To pursue this goal, however, we must carefully control the growth of our journal, with respect to its size and maturity. That explains why we are taking some novel approaches to the journal's production.

For reasons of business viability, we changed the main publisher from a learned society to a university. We decided to apply for support being offered by the Republic of Slovenia to scientific journals, and a visible consequence is the translation of abstracts into the Slovenian language. Next, because we are committed to preserving our policy that neither readers nor authors should pay for access to the journal's papers over the internet, from 2014 we are introducing a 'Creative Commons Copyright' model for our journal. We hereby announce that *Ars Mathematica Contemporanea* will publish four issues per year, from 2015. If you wish to support our journal and help with the long-term preservation of its contents, please subscribe to *Ars Mathematica Contemporanea*, and ask your library to subscribe to the printed edition.

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The excluded minor structure theorem with planarly embedded wall

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Abstract

A graph is "nearly embedded" in a surface if it consists of graph G_0 that is embedded in the surface, together with a bounded number of vortices having no large transactions. It is shown that every large wall (or grid minor) in a nearly embedded graph, many rows of which intersect the embedded subgraph G_0 of the near-embedding, contains a large subwall that is planarly embedded within G_0. This result provides some hidden details needed for a strong version of the Robertson and Seymour's excluded minor theorem as presented in [1].

Keywords: Graph, graph minor, surface, near-embedding, grid minor, excluded minor.

Math. Subj. Class.: 05C10, 05C82

1 Introduction

A graph is a *minor* of another graph if the first can be obtained from a subgraph of the second by contracting edges. One of the highlights of the graph minors theory developed by Robertson and Seymour is the Excluded Minor Theorem (EMT) that describes a rough structure of graphs that do not contain a fixed graph H as a minor. Two versions of EMT appear in [7, 8]; see also [3] and [4].

In [1] and [2] the authors used a strong version of EMT in which it is concluded that every graph without a fixed minor and whose tree-width is large has a tree-like structure, whose pieces are subgraphs that are almost embedded in some surface, and in which one of the pieces contains a large grid minor that is (essentially) embedded in a disk on the surface. Although not explicitly mentioned, this version of EMT follows from the published results of Robertson and Seymour [8] by applying standard techniques of routings on surfaces.

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Experts in this area are familiar with these techniques (that are also present in Robertson and Seymour's work [6]). However, they may be harder to digest for newcomers in the area, and thus deserve to be presented in the written form. The purpose of this note is to provide a proof of an extended version of EMT as stated in [1, Theorem 4.2].

It may be worth mentioning that the proof in [1] does not really need the extended version of the EMT, but the proof in [2] does. Thus, this note may also be viewed as a support for the main proof in [2].

We assume that the reader is familiar with the basic notions of graph theory and in particular with the basic notions related to graph minors; we refer to [3] for all terms and results not explained here.

2 Walls in near-embeddings

In this section, we present our main lemma, which shows that for every large wall (to be defined in the sequel) in a "nearly embedded" graph, a large subwall must be contained in the embedded subgraph of the near-embedding. Let us first introduce the notion of the wall and some of its elementary properties.

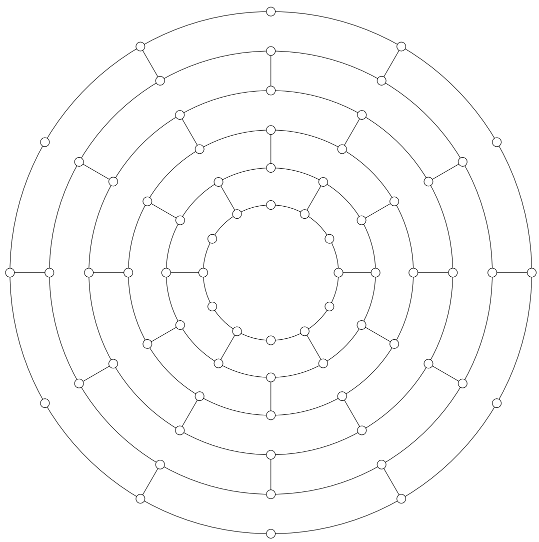

Figure 1: The cylindrical 6-wall Q_6

For an integer $r \geq 3$, we define a *cylindrical r-wall* as a graph that is isomorphic to a subdivision of the graph Q_r defined as follows. We start with vertex set $V = \{(i,j) \mid 1 \leq i \leq r,\ 1 \leq j \leq 2r\}$, and make two vertices (i,j) and (i',j') adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j-1, j+1\}$, where the values $j-1$ and $j+1$ are considered modulo $2r$.

(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

Less formally, Q_r consists of r disjoint cycles $C_1, \ldots, C_r$ of length $2r$ (where $V(C_i) =$

$\{(i, j) \mid 1 \le j \le 2r\}$), called the *meridian cycles* of Q_r. Any two consecutive cycles C_i and C_{i+1} are joined by r edges so that the edges joining C_i and C_{i-1} interlace on C_i with those joining C_i and C_{i+1} for $1 < i < r$. Figure 1 shows the cylindrical 6-wall Q_6.

By deleting the edges joining vertices $(i, 1)$ and $(i, 2r)$ for $i = 1, \ldots, r$, we obtain a subgraph of Q_r. Any graph isomorphic to a subdivision of this graph is called an r-*wall*.

To relate walls and cylindrical walls to $(r \times r)$-grid minors, we state the following easy correspondence:

(a) Every $(4r + 2)$-wall contains a cylindrical r-wall as a subgraph.

(b) Every cylindrical r-wall contains an $(r \times r)$-grid as a minor.

(c) Every $(r \times r)$-grid minor contains an $\lfloor \frac{r-1}{2} \rfloor$-wall as a subgraph.

Lemma 2.1. Suppose that $1 \le i < j \le r$ and let $t = j - i - 1$. Let $S_i \subset C_i$ and $S_j \subseteq C_j$ be paths of length at least $2t - 1$ in the meridian cycles C_i, C_j of Q_r. Then Q_r contains t disjoint paths linking C_i and C_j. Moreover, for each of these paths and for every cycle C_k, $i < k < j$, the intersection of the path with C_k is a connected segment of C_k.

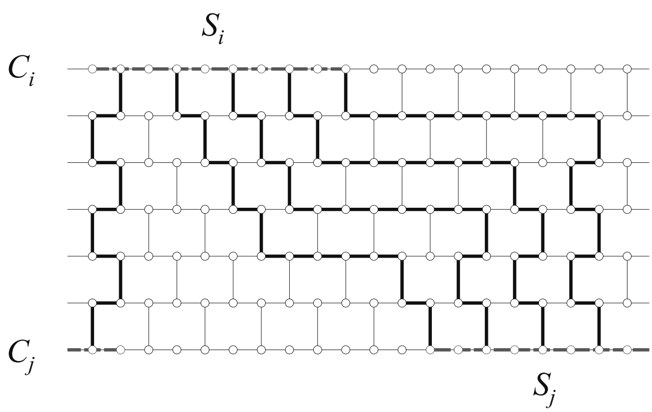

Figure 2: Paths linking S_i and S_j

Proof. The lemma is easy to prove and the idea is illustrated in Figure 2, in which the edges on the left are assumed to be identified with the corresponding edges on the right. The paths are shown by thick lines and the segments S_i and S_j are shown by thick broken lines. □

A *surface* is a compact connected 2-manifold (with or without boundary). The components of the boundary are called the *cuffs*. If a surface S has Euler characteristic c, then the non-negative number $g = 2 - c$ is called the *Euler genus* of S. Note that a surface of Euler genus g contains at most g cuffs.

Disjoint cycles C, C' in a graph embedded in a surface S are *homotopic* if there is a cylinder in S whose boundary components are the cycles C and C'. The cylinder bounded by homotopic cycles C, C' is denoted by $int(C, C')$. Disjoint paths P, Q whose initial vertices lie in the same cuff C and whose terminal vertices lie in the same cuff C' in S

(possibly $C' = C$) are *homotopic* if P and Q together with a segment in C and a segment in C' form a contractible closed curve A in S. The disk bounded by A will be denoted by $int(P, Q)$. The following basic fact about homotopic curves on a surface will be used throughout (cf., e.g., [5, Propositions 4.2.6 and 4.2.7]).

Lemma 2.2. Let S be a surface of Euler genus g. Then every collection of more than $3g$ disjoint non-contractible cycles contains two cycles that are homotopic. Similarly, every collection of more than $3g$ disjoint paths, whose ends are on the same (pair of) cuffs in S, contains two paths that are homotopic.

Let G be a graph and let $W = \{w_0, \ldots, w_n\}$, $n = |W| - 1$, be a linearly ordered subset of its vertices such that w_i precedes w_j in the linear order if and only if $i < j$. The pair (G, W) is called a *vortex* of *length* n, W is the *society* of the vortex and all vertices in W are called *society vertices*. When an explicit reference to the society is not needed, we will as well say that G is a vortex. A collection of disjoint paths $R_1, \ldots, R_k$ in G is called a *transaction* of *order* k in the vortex (G, W) if there exist i, j $(0 \le i \le j \le n)$ such that all paths have their initial vertices in $\{w_i, w_{i+1}, \ldots, w_j\}$ and their endvertices in $W \setminus \{w_i, w_{i+1}, \ldots, w_j\}$.

Let G be a graph that can be expressed as $G = G_0 \cup G_1 \cup \cdots \cup G_v$, where G_0 is embedded in a surface S of Euler genus g with v cuffs $\Omega_1, \ldots, \Omega_v$, and G_i $(i = 1, \ldots, v)$ are pairwise disjoint vortices, whose society is equal to their intersection with G_0 and is contained in the cuff Ω_i, with the order of the society being inherited from the circular order around the cuff. Then we say that G is *near-embedded* in the surface S *with vortices* $G_1, \ldots, G_v$. A subgraph H of a graph G that is near-embedded in S is said to be *planarly embedded* in S if H is contained in the embedded subgraph G_0, and there exists a cycle $C \subseteq G_0$ that is contractible in S and H is contained in the disk on S that is bounded by C. Our main result is the following.

Theorem 2.3. For every non-negative integers g, v, a there exists a positive integer $s = s(g, v, a)$ such that the following holds. Suppose that a graph G is near-embedded in the surface S with vortices $G_1, \ldots, G_v$, such that the maximum order of transactions of the vortices is at most a. Let Q be a cylindrical r-wall contained in G, such that at least $r_0 \ge 3s$ of its meridian cycles have at least one edge contained in G_0. Then $Q \cap G_0$ contains a cylindrical r'-wall that is planarly embedded in S and has $r' \ge r_0/s$.

Proof. Let $C_{p_1}, C_{p_2}, \ldots, C_{p_{r_0}}$ $(p_1 < p_2 < \cdots < p_{r_0})$ be meridian cycles of Q having an edge in G_0. For $i = 1, \ldots, r_0$, let L_i be a maximal segment of C_{p_i} containing an edge in $E(C_{p_i}) \cap E(G_0)$ and such that none of its vertices except possibly the first and the last vertex are on a cuff. It may be that $L_i = C_{p_i}$ if C_{p_i} contains at most one vertex on a cuff; if not, then L_i starts on some cuff and ends on (another or the same) cuff. (We think of the meridian cycles to have the orientation as given by the meridians in the wall.) At least $r_0/(v^2+1)$ of the segments L_i either start and end up on the same cuffs Ω_x and Ω_y (possibly $x = y$), or are all cycles. In each case, we consider their homotopies. By Lemma 2.2, these segments contain a subset of $q \ge r_0/((3g+1)(v^2+1))$ homotopic segments (or cycles). Since we will only be interested in these homotopic segments or cycles, we will assume henceforth that $L_1, \ldots, L_q$ are homotopic.

Let us first look at the case when $L_1, \ldots, L_q$ are cycles. Since $s = s(g, v, a)$ can be chosen to be arbitrarily large (as long as it only depends on the parameters), we may assume

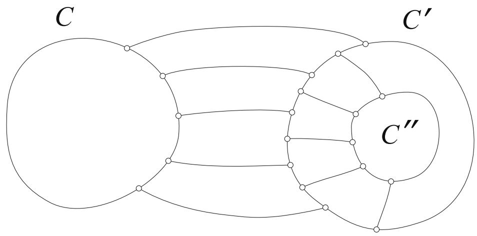

Figure 3: Many contractible cycles

that q is as large as needed in the sequel. If the cycles L_i are pairwise homotopic and non-contractible, then it is easy to see that two of them bound a cylinder in S containing many of these cycles. This cylinder also contains the paths connecting these cycles; thus it contains a large planarly embedded wall and hence also a large planarly embedded cylindrical wall. So, we may assume that the cycles $L_1, \dots, L_q$ are contractible. By Lemma 2.1, Q contains t paths linking any two of these cycles that are t apart in Q, say $C = L_i$ and $C' = L_{i+t+1}$. (Here we take t large enough that the subsequent arguments will work.) Again, many of these paths either reach C' without intersecting any of the cuffs, or many reach the same cuff Ω. A large subset of them is homotopic. In the former case, the paths linking C' with $C'' = L_{i+2t+2}$ can be chosen so that their initial vertices interlace on C' with the end-vertices of the homotopic paths coming from C. This implies that C or C'' lies in the disk bounded by C' (cf. Figure 3). By repeating the argument, we obtain a sequence of nested cycles and interlaced linkages between them. This clearly gives a large subwall, which contains a large cylindrical subwall that is planarly embedded. In the latter case, when the paths from C to C' go through the same cuff Ω_j, we get a contradiction since the vortex on Ω_j does not admit a transaction of large order, and thus too many homotopic paths cannot reach C''.

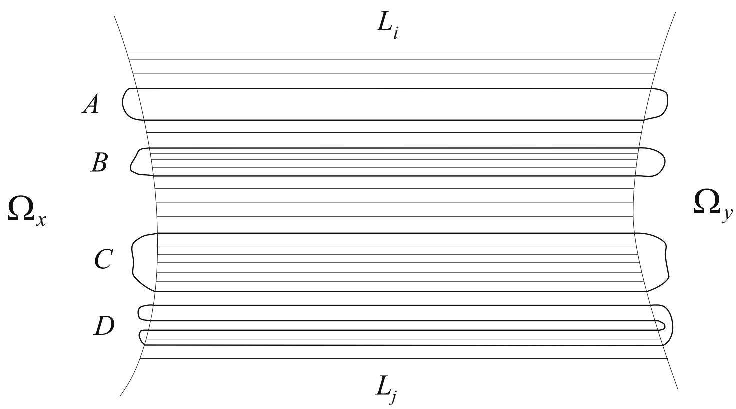

Figure 4: Many homotopic segments joining two cuffs

We get a similar contradiction as in the last case above, when too many homotopic segments L_i start and end up on the same cuffs Ω_x and Ω_y. We shall give details for the case when $x \neq y$, but the same approach works also if $x = y$. (In the case when $x = y$ and the homotopic segments L_i are contractible, the proof is similar to the part of the proof given above.)

Let us consider the "extreme" segments L_i, L_j, whose disk $int(L_i, L_j)$ contains many homotopic segments (cf. Figure 4). Let us enumerate these segments as $L'_1 = L_i, L'_2, \dots, L'_m = L_j$ in the order as they appear inside $int(L_i, L_j)$. Let C'_t (for $1 < t < m$) be the meridian cycle containing the segment L'_t. Since vortices admit no transactions of order more than a, at most $4a$ of the cycles C'_t $(1 < t < m)$ can leave $int(L_i, L_j)$. By adjusting m, we may thus assume that none of them does. In particular, each L'_t has another homotopic segment in $int(L_i, L_j)$. Since there are no transactions of order more than a, there is a large subset of the cycles C'_t that follow each other in $int(L_i, L_j)$ as shown by the thick cycles in Figure 4. Consider four of these meridian cycles A, B, C, D that are pairwise far apart in the wall Q and appear in the order A, B, C, D within $int(L_i, L_j)$. Then A and C are linked in Q by a large collection of disjoint paths by Lemma 2.1. At most $8a$ of these paths can escape intersecting two fixed segments L'_u and L'_v of B or two such segments of D by passing through a vortex. All other paths linking A and C intersect either two segments of B or two segments of D. However, this is a contradiction since the paths linking A and C can be chosen in Q so that each of them intersects each meridian cycle in a connected segment (Lemma 2.1). This completes the proof. □

3 The excluded minor structure

In this section, we define some of the structures found in Robertson-Seymour's Excluded Minor Theorem [7] which describes the structure of graphs that do no contain a given graph as a minor. Robertson and Seymour proved a strengthened version of that theorem that gives a more elaborate description of the structure in [8]. Our terminology follows that introduced in [1].

Let G_0 be a graph. Suppose that (G'_1, G'_2) is a separation of G_0 of order $t \leq 3$, i.e., $G_0 = G'_1 \cup G'_2$, where $G'_1 \cap G'_2 = \{v_1, \dots, v_t\} \subset V(G_0)$, $1 \leq t \leq 3$, $V(G'_2) \setminus V(G'_1) \neq \emptyset$. Let us replace G_0 by the graph G', which is obtained from G'_1 by adding all edges $v_i v_j$ $(1 \leq i < j \leq t)$ if they are not already contained in G'_1. We say that G' has been obtained from G_0 by an *elementary reduction*. If $t = 3$, then the 3-cycle $T = v_1 v_2 v_3$ in G' is called the *reduction triangle*. Every graph G'' that can be obtained from G_0 by a sequence of elementary reductions is a *reduction* of G_0.

We say that a graph G_0 can be *embedded* in a surface Σ *up to* 3*-separations* if there is a reduction G'' of G_0 such that G'' has an embedding in Σ in which every reduction triangle bounds a face of length 3 in Σ.

Let H be an r-wall in the graph G_0 and let G'' be a reduction of G_0. We say that the reduction G'' *preserves* H if for every elementary reduction used in obtaining G'' from G_0, at most one vertex of degree 3 in H is deleted. (With the above notation, $G'_2 \setminus G'_1$ contains at most one vertex of degree 3 in H.)

Lemma 3.1. Suppose that G'' is a reduction of the graph G_0 and that G'' preserves an r-wall H in G_0. Then G'' contains an $\lfloor (r+1)/3 \rfloor$-wall, all of whose edges are contained in the union of H and all edges added to G'' when performing elementary reductions.

Proof. Let H' be the subgraph of the r-wall H obtained by taking every third row and every third "column". See Figure 5 in which H' is drawn with thick edges. It is easy to

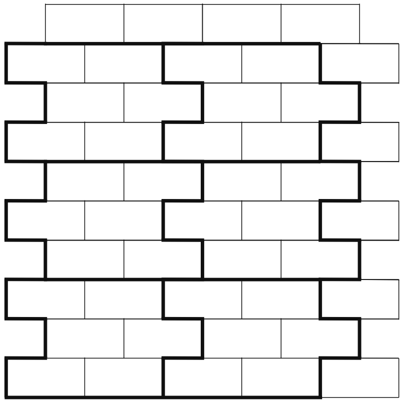

Figure 5: Smaller wall contained in a bigger wall

see that for every elementary reduction we can keep a subgraph homeomorphic to H' by replacing the edges of H' which may have been deleted by adding some of the edges v_iv_j involved in the reduction. The only problem would occur when we lose a vertex of degree 3 and when all vertices v_1, v_2, v_3 involved in the elementary reduction would be of degree 3 in H'. However, this is not possible since G'' preserves H. $\square$

Suppose that for $i = 0, \dots, n$, there exist vertex sets, called *parts*, $X_i \subseteq V(G)$, with the following properties:

(V1) $X_i \cap W = \{w_i, w_{i+1}\}$ for $i = 0, \dots, n$, where $w_{n+1} = w_n$,

(V2) $\bigcup_{0 \le i \le n} X_i = V(G)$,

(V3) every edge of G has both endvertices in some X_i, and

(V4) if $i \le j \le k$, then $X_i \cap X_k \subseteq X_j$.

Then the family $(X_i\,;\, i = 0, \dots, n)$ is called a *vortex decomposition* of the vortex (G, W). For $i = 1, \dots, n$, denote by $Z_i = (X_{i-1} \cap X_i) \setminus W$. The *adhesion* of the vortex decomposition is the maximum of $|Z_i|$, for $i = 1, \dots, n$. The vortex decomposition is *linked* if for $i = 1, \dots, n-1$, the subgraph of G induced on the vertex set $X_i \setminus W$ contains a collection of disjoint paths linking Z_i with Z_{i+1}. Clearly, in that case $|Z_i| = |Z_{i+1}|$, and the paths corresponding to $Z_i \cap Z_{i+1}$ are trivial. Note that (V1) and (V3) imply that there are no edges between nonconsecutive society vertices of the vortex. Let us remark that every vortex (G, W), in which w_i, w_j are non-adjacent for $|i - j| \ge 2$, admits a linked vortex decomposition; just take $X_i = (V(G) \setminus W) \cup \{w_i, w_{i+1}\}$.

The (*linked*) *adhesion of the vortex* is the minimum adhesion taken over all (linked) decompositions of the vortex. Let us observe that in a linked decomposition of adhesion q, there are q disjoint paths linking Z_1 with Z_n in $G - W$. For us it is important to note that a vortex with adhesion less than k does not admit a transaction of order more than k.

Let G be a graph, H an r-wall in G, Σ a surface, and $\alpha \ge 0$ an integer. We say that G can be α-*nearly embedded* in Σ if there is a set of at most α cuffs $C_1, \dots, C_b$ ($b \le \alpha$) in Σ, and there is a set A of at most α vertices of G such that $G - A$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_b$ where $G_0, G_1, \ldots, G_b$ are edge-disjoint subgraphs of G and the following conditions hold:

(N1) G_0 can be embedded in Σ up to 3-separations with G'' being the corresponding reduction of G_0.

(N2) If $1 \leq i < j \leq b$, then $V(G_i) \cap V(G_j) = \emptyset$.

(N3) $W_i = V(G_0) \cap V(G_i) = V(G'') \cap C_i$ for every $i = 1, \ldots, b$.

(N4) For every $i = 1, \ldots, b$, the pair (G_i, W_i) is a vortex of adhesion less than α, where the ordering of W_i is consistent with the (cyclic) order of these vertices on C_i.

The vertices in A are called the *apex vertices* of the *α-near embedding*. The subgraph G_0 of G is said to be the *embedded subgraph* with respect to the α-near embedding and the decomposition $G_0, G_1, \ldots, G_b$. The pairs (G_i, W_i), $i = 1, \ldots, b$, are the *vortices* of the α-near embedding. The vortex (G_i, W_i) is said to *be attached to the cuff C_i* of Σ containing W_i.

If G is α-near-embedded in S, let $G_0, G_1, \ldots, G_b$ be as above and let G'' be the reduction of G_0 that is embedded in S. If H is an r-wall in G, we say that H is *captured in the embedded subgraph G_0* of the α-near-embedding if H is preserved in the reduction G'' and for every separation $G = K \cup L$ of order less than r, where $G_0 \subseteq K$, at least $\frac{2}{3}$ of the degree-3 vertices of H lie in K.

We shall use the following theorem which is a simplified version of one of the cornerstones of Robertson and Seymour's theory of graph minors, the Excluded Minor Theorem, as stated in [8]. For a detailed explanation of how the version in this paper can be derived from the version in [8], see the appendix of [1].

Theorem 3.2 (Excluded Minor Theorem)**.** For every graph R, there is a constant α such that for every positive integer w, there exists a positive integer $r = r(R, \alpha, w)$, which tends to infinity with w for any fixed R and α, such that every graph G that does not contain an R-minor either has tree-width at most w or contains an r-wall H such that G has an α-near embedding in some surface Σ in which R cannot be embedded, and H is captured in the embedded subgraph of the near-embedding.

We can add the following assumptions about the r-wall in Theorem 3.2.

Theorem 3.3. It may be assumed that the r-wall H in Theorem 3.2 has the following properties:

(a) H is contained in the reduction G'' of the embedded subgraph G_0.

(b) H is planarly embedded in Σ, i.e., every cycle in H is contractible in Σ and the outer cycle of H bounds a disk in Σ that contains H.

(c) We may prespecify any constant ρ and ask that the face-width of G'' be at least ρ.

(d) G'' is 3-connected.

Proof. The starting point is Theorem 3.2. By making additional elementary reductions if necessary, we can achieve (d). The property (c) is attained as follows. If the face-width is too small, then there is a set of less than ρ vertices whose removal reduces the genus of the embedding of G''. We can add these vertices in the apex set and repeat the procedure as long as the face-width is still smaller than ρ. The only subtlety here is that the constant α in Theorem 3.2 now depends not only on R but also on ρ. See also [4].

After removing the apex set A, we are left with an $(r-\alpha)$-wall in $G - A$. By applying Lemma 3.1, we may assume that H is contained in the reduced graph $G'' \cup G_1 \cup \cdots \cup G_b$. The wall H contains a large cylindrical wall Q. Since the vortices have bounded adhesion, they do not have large transactions. Since the wall is captured in G'', edges of many meridians of Q lie in G''. Therefore, we can apply Theorem 2.3 for the near embedding of the reduced graph together with the vortices. This shows that a large cylindrical subwall of Q is planarly embedded in the surface. The size r' of this smaller wall still satisfies the condition that $r' = r'(R, \alpha, w) \to \infty$ as w increases. □

It is worth mentioning that there are other ways to show that a graph with large enough tree-width that does not contain a fixed graph R as a minor contains a subgraph that is α-near-embedded in some surface Σ in which R cannot be embedded, and moreover, there is an r-wall planarly embedded in Σ (after reductions taking care of at most 3-separations). Let us describe two of them:

(A) *Large face-width argument*: One can use property (c) in Theorem 3.3 that the face-width ρ can be made as large as we want if $\alpha = \alpha(R, w, \rho)$ is large enough. Once we have that, it follows from [6] that there is a planarly embedded r-wall, where $r = r(R, \rho) \to \infty$ as $\rho \to \infty$. While this easy argument is sufficient for most applications, it appears to be slightly weaker than Theorem 3.3 since the quantifiers change. The difference is that the number of apex vertices is no longer bounded as a function of $\alpha = \alpha(R)$ but rather as a function depending on R and r, where the upper bound has linear dependence on r, i.e. it is of the form $\beta(R)r$. However, other parameters of the near-embedding keep being only dependent on R.

(B) *Irrelevant vertex*: The third way of establishing the same result is to go through the proof of Robertson and Seymour that there is an *irrelevant vertex*, i.e. a vertex v such that G has an R-minor if and only if $G - v$ has. (Compared to the later, more abstract parts of the graph minors series of papers, this part is very clean and well understood; it could (and should) be explained in a(ny) serious graduate course on graph minors.) In that proof, one starts with an arbitrary wall W that is large enough. A large wall exists since the tree-width is large. Then one compares the W-bridges attached to W. They may give rise to $\leq$ 3-separations, to *jumps* (paths in bridges whose addition to W yields a nonplanar graph), *crosses* (pairs of disjoint paths attached to the same planar face of W whose addition to W yields a nonplanar graph). If there are many disjoint jumps or crosses on distinct faces of W, one can find an R-minor. If there are just a few, there is a large planar wall. If there are many of them on the same face, we get a structure of a vortex with bounded transactions (or else an R-minor can be discovered). The proof then discusses ways for many jumps and crosses but no large subset of them being disjoint. One way is to have a small set of vertices whose removal destroys most of these jumps and crosses. This gives rise to the apex vertices. The final conclusion is that the jumps and crosses can affect only a bounded part of the wall, so after the removal of the apex vertices and after elementary reductions which eliminate $\leq$ 3-separations, there is a large subwall W_0 such that no jumps or crosses are involved in it. The "middle" vertex in W_0 is then shown to be irrelevant.

For our reference, only this *planar* wall is needed. By being planar, we mean that the rest of the graph is attached only to the outer face of this wall. Then we define the tangle corresponding to this wall and the proof of the EMT preserves this tangle while making the modifications yielding to an α-near-embedding.

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Maximum independent sets of the 120-cell and other regular polytopes

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Abstract

A d-code in a graph is a set of vertices such that all pairwise distances are at least d. As part of a study of d-codes of three-and four-dimensional regular polytopes, the maximum independent set order of the 120-cell is calculated. A linear program based on counting arguments leads to an upper bound of 221. An independent set of order 110 in the antipodal collapse of the 120-cell (also known as the hemi-120-cell) gives a lower bound of 220 for the 120-cell itself. The gap is closed by the computation described here, with the result that the maximum independent set order of the 120-cell is 220. All maximum d-code orders of the icosahedron, dodecahedron, 24-cell, 600-cell and 120-cell are reported.

Keywords: Graphs, independent sets, polyhedra, polytopes.

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1 Introduction

An *independent set* in a graph is a set of pairwise non-adjacent vertices such that all pairs are at distance two or more. A *clique* (a subset of the vertices that are pairwise adjacent) in a graph corresponds to an independent set in the complement of the graph. Hence algorithms for maximum clique can be applied to find maximum independent sets.

The problem of finding a maximum independent set in a graph is NP-Hard [12]. The DIMACS Clique Challenge arose from the need to find practical algorithms for the maximum clique problem, and the proceedings volume is an excellent place to start looking for information about practical algorithms for clique finding [14]. The challenge also included a database of difficult clique problems. Bomze, Budinich, Pardalos, and Pelillo [1] provide a comprehensive survey on the maximum clique problem. Östergård focusses on solving maximum clique problems on graphs arising from various combinatorial problems. Both surveys cite the problem of finding a maximum clique of the Keller graph of dimension 7 as a open problem. This problem has subsequently been solved [5].

In general, a d*-code* is a set of vertices such that pairwise distances are all at least d. The concept of distance between vertices a and b may be cast in terms of graphs (number of edges in the shortest path between the vertices), coding theory (the Hamming distance between $(0, 1)$ sequences representing coordinates of vertices), or sets (the cardinality of the symmetric difference between the two subsets that represent the vertices). A d-code in a graph G corresponds to an independent set in the graph H which has the same vertex set as G and the property that two vertices u and v are adjacent in H if and only if the distance between them in G is at most $d-1$. The interest for applications is usually to find *maximum* d-codes, and one standard problem in the theory of error-correcting codes [16, 2] is to find the largest d-code in the n-cube. Here we consider the problem of finding largest d-codes in the graphs corresponding to other regular polytopes.

For polycyclic and polyhedral graphs in two and three dimensions, the construction of d-codes has applications to chemistry [15, 4, 7, 8]. For example, sets of codes with increasing d may be seen as templates for addition to an underlying molecular framework by addends of increasing steric demand. Codes have been presented for chemically relevant regular and semi-regular polyhedra [15] and arguments based on d-codes, coupled with spectral information, give useful rationalisations of the extent and symmetry of addition in fullerene chemistry, for example [9, 10, 11]. Although not invoked in chemistry so far, d-anticodes, defined by the requirement that pairwise distances should not exceed d, would model the opposite regime of attachment to a framework where the added groups cluster under strong inter-addend attraction. Extension of the existing lists [15] to d-codes in the graphs of all regular polytopes is a finite task, as there are only the following convex regular polytopes [3]: in dimension n the n-simplex (α_n), the n-cross-polytope (β_n), the n-cube (γ_n), and additionally in dimension 2 the regular polygons, and in dimensions 3 and 4, five sporadic polytopes. In dimension 3, α_3 is the tetrahedron, β_3 the octahedron and γ_3 the cube, and there is a dual pair of sporadic polyhedra: the icosahedron and the dodecahedron. In dimension 4, the analogues of the icosahedron and dodecahedron are the 600-cell (all of its independent sets have been enumerated previously [6]) and the 120-cell (again a dual pair), and there is also the self-dual 24-cell, without analogue in higher or lower dimensions [3].

Codes for the polytopes common to all dimensions (α_n, β_n and γ_n) are well studied. The 1-skeleton of α_n is the complete graph K_n, of diameter 1, and the 1-skeleton of β_n is the complete multipartite graph (the Cocktail-Party graph) $K_{n(2)} \equiv C_p(n) \equiv K_{2,2,\ldots,2}$, of diameter 2, so coding problems are trivial for both. Codes for γ_n are the subject of classical coding theory [16]. As a bipartite graph with equal partite sets, γ_n has independence number 2^{n-1}. The coding problem is also trivial in two dimensions, where the order of the d-code is $\lfloor n/d \rfloor$ for the cycle of length n. It only remains to study the five exceptional regular polytopes in dimensions 3 and 4. Most of these problems are easy (see the summary in Table 3, §3). The difficult case is that of the independence number of the 600-vertex 120-cell, which does not appear to be computable in a reasonable amount of time by use of standard algorithms. The solution to this problem is described in the following.

2 Maximum Independent Sets of the 120-cell

The 120-cell is the largest regular polytope in 4 dimensions. Its properties are described in Coxeter's book on Regular Polytopes [3] and Stillwell's survey paper [20], for example. The 1-skeleton of the 120-cell is a 4-regular graph with 600 vertices, 720 pentagonal faces and 120 dodecahedral cells. Models exhibiting three-dimensional projections of the complete object have been constructed; photographs of Donchian's models are shown in [3]. A partial model of the 120-cell is given as an example of a construction using Zome Models [13, Ch. 21]; this 330-vertex subgraph has 45 of the 120 dodecahedra, arranged in concentric shells of 1, 12 and 32 face-sharing cells.

In the following subsections, the steps leading to the solution of the problem of finding the maximum independent set order of the 120-cell are described. First, a description is given of how the vertices of the graph are numbered and how its automorphism group is computed (§2.1). Then (§2.2) a lower bound of 220 for the maximum independent set order is derived from an independent set of order 110 in the antipodal collapse of the 120-cell. An upper bound of 221 is established by use of a linear program (§2.3), and the information from the solution to the integer program is then exploited (§2.4) to infer structural information about a putative independent set of order 221. This information is subsequently used in the computational search described in the remaining subsections, which establishes that the maximum independent set order of the 120-cell is 220.

2.1 Numbering the Graph and its Automorphism Group

A special breadth-first search was used for numbering the 120-cell and finding a permutation representation of its automorphism group on the 600 vertices. The search in question is performed as follows:

Clockwise BFS Labelling Algorithm:

Input: an adjacency list for the 120-cell or its collapse.
Output: a canonical labelling for the graph and its automorphisms expressed in terms of permutations of the vertices of the canonically labelled graph.

To obtain the initial canonical labelling, select one vertex to be labelled as vertex 0 and then choose one way to label its four neighbours as 1, 2, 3 and 4. The remaining vertices are labelled using a breadth-first search starting at vertex 0, and visiting its neighbours 1, 2, 3, and 4 in order. In order to make the breath-first search labelling deterministic, the neighbours of a vertex v are visited in an order which is decided as follows. Each unlabelled neighbour u of vertex v is in one pentagon with vertex v and the breadth-first search parent p of vertex v. Let the other two vertices of the pentagon be x and y so that the vertices of

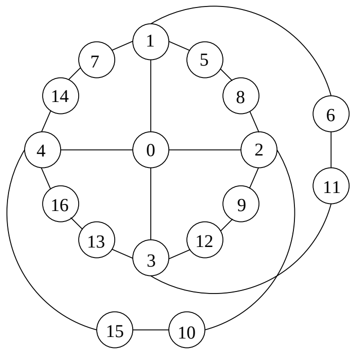

Figure 1: A subgraph of the 120-cell, as labelled by the Clockwise BFS Labelling Algorithm.

this pentagon in cyclic order are u, v, p, x, y. Since x is a neighbour of p (and p is the breadth-first search parent of v), x has already been labelled. The order of the neighbours of vertex v is selected so that the labels of the vertices indicated by x in their pentagons are sorted in increasing order. Figure 1 shows a portion of the 120-cell labelled this way.

The 120-cell has an automorphism group of order 14,400. Obtaining the permutations of the automorphism group is easy using the clockwise BFS as described above, as they correspond to choosing a start vertex for the BFS in each of the 600 possible ways, and for each start, considering each of the 4! permutations of its neighbours (14,400 = 600 $\times$ 4!).

2.2 A Lower Bound from the Antipodal Collapse

Given a vertex v of a graph, its *antipodal* vertices are those at maximum distance from v. The 120-cell has the property that each of its vertices has a unique antipodal vertex. The *antipodal collapse* of the 120-cell is obtained by identifying each vertex of the 120-cell with its antipodal vertex. If $\{u, u'\}$ and $\{v, v'\}$ are two sets of antipodal pairs of vertices of a graph G, then in the collapse, there is one edge between $\{u, u'\}$ and $\{v, v'\}$ corresponding to each edge of the form (u, v), (u, v'), (u', v), or (u', v') of the original graph G. Since multiple edges are inconsequential for the independent-set problem, each multiple edge is replaced by a single edge. The result is a 4-regular graph on 300 vertices which has the same local structure as the 120-cell. This graph is the 1-skeleton of the hemi-120-cell, one of the projective regular polytopes of rank 4 in projective 3-space [17, Section 6C].

The automorphism group order of the collapse is 300 $\times$ 4! = 7200, and the Clockwise BFS Labelling Algorithm from Section 2.1 is first used to label the vertices and find the automorphisms. An independent set of order k in the antipodal collapse can be lifted to one of order $2k$ in the 120-cell (if a vertex is in the independent set in the collapse, then include the two corresponding vertices of the 120-cell).

A non-exhaustive computer search indicated that the antipodal collapse has at least 60 independent sets of order 110 up to isomorphism. This shows that the 120-cell has an independent set of order 220. The most symmetrical of the sets that we found in the antipodal collapse has stabiliser group of order 8. This set is lifted (Table 1) to give an independent set of order 220 in the 120-cell, with 16 automorphisms.

2.3 A Linear Programming Upper Bound

An upper bound of 221 is not difficult to prove by solving a linear programming problem which sets up necessary constraints for a maximum independent set of the 120-cell. The nine variables for this linear program are as follows:

$R =$ the number of red (independent set) vertices

B_i for $i = 0, 1, 2, 3, 4 =$ the number of blue (not in the independent set) vertices having i red neighbours.

P_i for $i = 0, 1, 2 =$ the number of pentagons with i red vertices. Each of the nine variables is constrained to be non-negative. The LP has six further constraints (five equalities and one inequality). These are introduced after proving some theorems required to justify the sixth constraint. The other constraints are all trivial conditions.

A *blue pentagon* is a pentagon whose vertices are all blue (i.e., none of them are in the independent set). An *isolated blue pentagon* is defined to be a blue pentagon such that all of its ten incident vertices (i.e., the ten vertices that are adjacent to a vertex of the pentagon but are not themselves in the pentagon) are red. A blue pentagon with at least one incident blue vertex is called a *non-isolated blue pentagon*. A blue vertex with one red and three blue neighbours is called a *key*. A blue vertex with four red neighbours is called an *isolated blue vertex*.

Remark 2.1. The independent set of order 220 listed in Table 1 has no isolated blue vertices and no keys.

Theorem 2.2. *For any maximum independent set, the number of non-isolated blue pentagons is at most B_1 (the number of keys).*

Proof. Note that for a maximum independent set, it is not possible to have a blue vertex v with four blue neighbours since otherwise the independent set order could be increased by colouring v red. Therefore, if there is a blue pentagon which is a non-isolated blue

0	5	6	9	10	13	14	21	22	23	24	29	32	37	39
42	46	47	55	58	60	61	64	68	69	71	74	76	78	81
83	85	89	90	91	93	95	98	100	102	105	108	109	113	114
116	119	122	129	132	133	136	138	142	148	150	154	155	162	167
171	172	173	178	182	185	186	190	193	194	195	196	197	199	202
210	211	216	217	220	222	227	228	229	230	232	236	242	243	248
249	253	259	260	263	265	267	274	277	280	281	282	283	284	286
289	290	292	293	297	300	304	309	311	312	316	317	318	319	322
324	326	328	329	333	334	335	336	343	346	348	350	357	366	370
373	374	375	379	380	381	385	390	391	398	400	406	410	411	414
417	419	421	423	426	427	428	431	433	435	436	437	440	441	442
443	455	458	462	465	466	470	475	476	480	482	489	493	495	497
500	505	507	508	509	510	511	514	517	518	520	524	525	528	529
533	535	537	540	542	544	545	551	556	557	560	563	566	570	571
574	579	581	584	586	588	589	592	594	599					

Table 1: An independent set of order 220 in the 120-cell generated from a set of order 110 in the antipodal collapse.

pentagon, there must be at least one blue vertex on that pentagon which has one red and three blue neighbours (a key). The number of vertices like this is B_1. This does not complete the proof however because a key can be on 0, 1, 2, or 3 blue pentagons.

To finish the proof, start by assigning a weight to each vertex v of the graph which is a key: assign a weight of one if v is contained in at least one non-isolated blue pentagon and zero otherwise. The sum of the weights of the keys is at most B_1.

Next, assign fractional weights to the non-isolated blue pentagons. If a key v is on r blue pentagons, this key contributes a weight of $1/r$ to each of its blue pentagons. The sum of the weights of the non-isolated blue pentagons is equal to the sum of the weights of the keys.

To finish the proof, we argue that for each of the non-isolated blue pentagons, the sum of the contributions from its keys is at least one, meaning that the number of non-isolated blue pentagons is at most B_1. This argument is broken down into three cases according to the types of keys on each non-isolated blue pentagon P.

Case 1: Pentagon P contains a key v which is only in one non-isolated blue pentagon. In this case, the weight that v contributes to P is one and so P has weight at least one.

Case 2: Pentagon P contains a key v which is in two non-isolated blue pentagons. Let A and B the the two non-isolated blue pentagons containing v and let (v, x) be the edge common to A and B. Vertex x is also a key. If it is a key which is in exactly two non-isolated blue pentagons then the weight of P is at least one, since each of v and x contributes $1/2$ to the weight of P. If x is in three blue pentagons, then consider Case 3 instead of Case 2.

Case 3: Pentagon P contains a key v which is in three non-isolated blue pentagons. Let the three blue neighbours of v be x, y and z where x and y are the vertices which are on P. Since v is on three non-isolated blue pentagons, x and y are either on two or three non-isolated blue pentagons and hence they contribute at least $1/3$ to each pentagon they are on. Since P has contributions of at least $1/3$ from v, x, and y, the sum of its contributions is at least one, as required. □

Corollary 2.3. *For any maximum independent set of the 120-cell, the number of isolated blue pentagons is at least* $P_0 - B_1$.

Theorem 2.4. *For a maximum independent set of the 120-cell, if* I *is the number of isolated blue pentagons, then* $2I \leq P_1$.

Proof. In a dodecahedron, an isolated blue pentagon P appears as a blue pentagon with five incident red vertices. This means that the only possibility for another blue pentagon in the dodecahedron is the pentagon Q antipodal to P (all other pentagons contain at least one of the five reds). But the vertices in the dodecahedron incident to Q cannot be red (they have neighbours which are red) and therefore, a dodecahedron contains at most one isolated blue pentagon.

Any edge (u, v) of the pentagon Q antipodal to P with both endpoints blue is in a pentagon P' with exactly one red vertex in the dodecahedron which contains P and Q. Since the pentagon Q has at least one edge with both endpoints blue, there is at least one pentagon P' with exactly one red vertex in the dodecahedron with P and Q.

Each pentagon of the 120-cell falls into exactly two dodecahedra. To finish the proof, we argue that a pentagon P' with exactly one red vertex occurs as one which must be present as described above because of an isolated blue pentagon in at most one of its two dodecahedra. Suppose that P' corresponds to isolated blue pentagons in both of its two dodecahedra. Then the picture must be as in Figure 2 where the isolated blue pentagons are A and B, and P' is the pentagon with the bold edges. Vertex x is incident to A and vertex y is incident to B so both x and y must be red. This is a contradiction since x and y are adjacent to each other in the 120-cell.

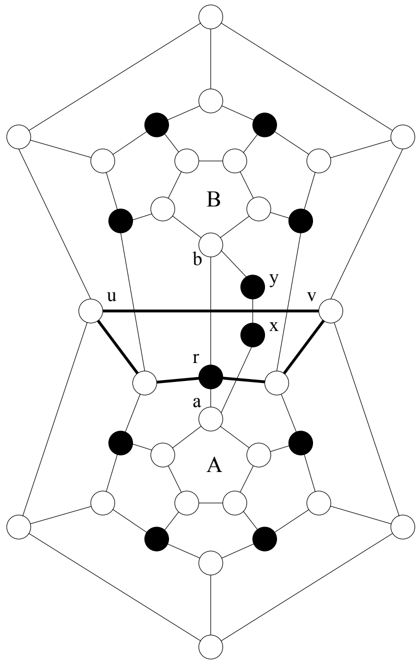

Figure 2: Two isolated blue pentagons A and B sharing a pentagon P' that has one red vertex (P' outlined in bold).

The conclusion is that each isolated blue pentagon maps to at least one pentagon with exactly one red in each of its two dodecahedra. Further, such pentagons with one red correspond to at most one isolated blue pentagon of the graph. This implies that $2I \leq P_1$. □

We now have the necessary theory to justify an integer programming problem which provides necessary constraints on a maximum independent set of the 120-cell. The conditions for the integer programming problem are:

1. $B_1 + B_2 + B_3 + B_4 + R = 600$
2. $P_0 + P_1 + P_2 = 720$
3. $4R = 1B_1 + 2B_2 + 3B_3 + 4B_4$
4. $6R = 0P_0 + 1P_1 + 2P_2$
5. $5P_0 + 2P_1 = 3B_1 + 1B_2$
6. $P_1 \geq 2(P_0 - B_1)$

The justifications for these constraints are:

1. The 120-cell has 600 vertices and for a maximum independent set $B_0 = 0$ as noted earlier.

2. The 120-cell has 720 pentagons.

3. Each red vertex is incident to four blue vertices. Hence, four times the number of red vertices is equal to the number of times a blue vertex is adjacent to a red one.

4. Each vertex is in six pentagons. Hence, six times the number of red vertices is equal to the number of times a red vertex occurs in a pentagon.

5. A *blue 2-path* is a path on 3 vertices (and hence two edges) whose vertices are all blue. Since each 2-path of the graph is in a unique pentagon, the number of blue 2-paths is $5P_0 + 2P_1$. The number of blue 2-paths is also equal to $3B_1 + 1B_2$ since a blue vertex with three blue neighbours is the centre of three blue 2-paths, a blue vertex with two blue neighbours is the centre of one blue 2-path, and a blue vertex with zero or one blue neighbours is not the central vertex of any blue 2-path.

6. This constraint comes from combining Corollary 2.3 with Theorem 2.4.

To get an upper bound for the maximum independent set order, the objective function is to maximize the value of R. Solving the linear programming relaxation gives an upper bound of $2880/13 = 221.538\ldots$ which gives an upper bound of 221 on the integer programming problem. (The optimum solution is attained for the vector $P_0 = 360/13$, $P_1 = 720/13$, $P_2 = 8280/13$, $B_1 = 0$, $B_2 = 3240/13$, $B_3 = 1680/13$, $B_4 = 0$, and the polytope thus defined is three-dimensional and has nine-vertices.) Applying the same tactic to the antipodal collapse gives an upper bound of 110 for the collapse, implying that the independent set of order 110 found in §2.2 is a maximum independent set of the antipodal collapse.

2.4 Exploiting the Integer Program Information

The example in §2.2 gives a lower bound of 220 for the order of a maximum independent set of the 120-cell. On the other hand, §2.3 proves an upper bound of 221. This implies that if the independent set from §2.2 is not optimal, then there is an independent set of the 120-cell of order 221. The next step is to examine the solutions of the integer programming problem from §2.3 which have the number R of red vertices equal to 221 to gain structural information as to what a solution of order 221 must look like.

Table 2 shows all the integer solutions that could result in an independent set of order 221. Correctness of the LP code is not an issue since it is not hard to check the final solutions by hand.

Scanning the table of solutions, we observe that $P_0 - B_1$ is always at least 25. From Corollary 2.3, the implication is that any independent set of order 221 has at least 25 isolated blue pentagons. Observe also that all cases satisfy the constraint that the number B_1 of keys plus two times the number B_4 of isolated blues is at most seven.

The existence of an independent set of order 221 requires that there is some way to add at least 25 isolated blue pentagons to the 120-cell without creating too many keys or isolated blue vertices ($B_1 + 2B_4 \leq 7$). The next two sections explain how we first tried planting a smaller number of isolated blue pentagons in part of the graph in all ways up to isomorphism and give an account of how the search for the 25 isolated blue pentagons was completed.

P_0	P_1	P_2	B_1	B_2	B_3	B_4
25	64	631	0	253	126	0
26	62	632	0	254	124	1
27	60	633	0	255	122	2
28	58	634	0	256	120	3
26	62	632	1	251	127	0
27	60	633	1	252	125	1
28	58	634	1	253	123	2
29	56	635	1	254	121	3
27	60	633	2	249	128	0
28	58	634	2	250	126	1
29	56	635	2	251	124	2
28	58	634	3	247	129	0
29	56	635	3	248	127	1
30	54	636	3	249	125	2
29	56	635	4	245	130	0
30	54	636	4	246	128	1
30	54	636	5	243	131	0
31	52	637	5	244	129	1
31	52	637	6	241	132	0
32	50	638	7	239	133	0

Table 2: Solutions to the Linear Program that would correspond to an independent set of order 221.

2.5 Planting Blue Pentagons

A typical approach to trying to plant 25 isolated blue pentagons into the 120-cell that covers all possibilities up to isomorphism is to choose some smaller number of pentagons (for example, seven) that are placed in all ways up to isomorphism and then add the rest without concern for duplication since at some point, isomorphism screening is too costly for the amount of duplication prevented. This approach was taken first and it resulted in too many cases for a practical solution. The next strategy applied was to consider only a subgraph of the 120-cell for which it is possible to prove that at least some number k of isolated pentagons must be present in order to reach an independent set of order 221, and then to place the k pentagons in this region in all ways up to isomorphism.

It is assumed that vertices of the 120-cell are labelled by the Clockwise BFS Labelling Algorithm from Section 2.1 The *restricted region* for consideration is defined by first sorting the pentagons. Before sorting, a list of five integers is created for each pentagon which contains the labels of its vertices in reverse sorted order (which is not necessarily the same as a cyclic ordering). Then these lists are compared lexicographically while sorting the pentagons. This gives a sorted order of pentagons $P_0, P_1, \ldots, P_{719}$. The first six pentagons are the ones pictured in Figure 1. The sequences used to sort them are:

$P_0 : 8, 5, 2, 1, 0$
$P_1 : 11, 6, 3, 1, 0$
$P_2 : 12, 9, 3, 2, 0$
$P_3 : 14, 7, 4, 1, 0$

$P_4 : 15, 10, 4, 2, 0$ and
$P_5 : 16, 13, 4, 3, 0$.
The last two pentagons (illustrating how lexicographic order is used to break ties) are:
$P_{718} : 599, 598, 596, 592, 591$ and
$P_{719} : 599, 598, 597, 594, 593$.
This (slightly unnatural) ordering was selected so that the maximum vertex number occurring in the pentagons numbered $P_0, P_1, \ldots, P_k$ is minimized given a chosen value of k. Intuitively, this helps to compress the first k pentagons into a small subgraph of the 120-cell.

After some experimentation, it was decided that planting seven pentagons in all ways up to isomorphism was the best compromise between the number of cases created and the difficulty for finishing the cases. The restricted region for planting these pentagons is shown to consist of the first 173 pentagons ($P_0, P_1, \ldots, P_{172}$) in the following lemma.

Lemma 2.5. *If the 120-cell has an independent set of order 221 then it is possible to find an independent set of order 221 such that there are at least seven isolated blue pentagons in the first 173 pentagons ($P_0, P_1, \ldots, P_{172}$).*

Proof. We already know from the results in §2.3 that the entire graph contains at least 25 isolated blue pentagons if there is an independent set of order 221. The idea for this proof is to count the number of isolated blue pentagons in the graph by considering the sets of pentagons numbered $P_0, P_1, \ldots, P_{172}$ for each of the automorphisms of the graph. If the average count over each of these sets $P_0, P_1, \ldots, P_{172}$ is greater than six, then we can conclude that there is at least one automorphism of the graph such that the count for $P_0, P_1, \ldots, P_{172}$ is at least seven.

Owing to the structure of the automorphism group of the graph, taking into consideration the sets of pentagons labelled $P_0, P_1, \ldots, P_{172}$ over all automorphisms accounts for each pentagon the same number of times; each is included $14,400 \times 173/720$ times. If the graph has 25 or more isolated blue pentagons, then the sum of the number of isolated blue pentagons over each choice for $P_0, P_1, \ldots, P_{172}$ is equal to at least $25 \times 14400 \times 173/720$. Hence, the average term is equal to at least $25 \times 173/720$. But $25 \times 173/720 > 6$ and therefore, since the average is greater than six, at least one case must be greater than six. □

The total number of ways to plant seven isolated blue pentagons in the set $P_0, \ldots, P_{172}$ is equal to 8,211,380. It is a little more difficult than usual to define a canonical form for these, because some of the automorphisms of the graph can map a choice of seven pentagons selected from $P_0, P_1, \ldots, P_{172}$ to another choice of seven pentagons which is lexicographically smaller, but is no longer a selection from the pentagons $P_0, P_1, \ldots, P_{172}$ because the new set contains a pentagon numbered 173 or higher. To accommodate this difficulty, the canonical form is selected so that it is the lexicographically minimum set of seven pentagons with the additional property that the pentagon with the largest number in the set corresponds to some P_k for $k \leq 172$.

The algorithms used for this phase were very simple. A nested set of seven loops was used to plant all possible choices for seven isolated blue pentagons. For each isolated blue pentagon selected, the ten incident vertices are coloured red. Vertices adjacent to a red vertex are coloured blue. To determine if an additional choice for an isolated blue pentagon is compatible with a previously chosen set, it suffices to check if its ten incident vertices can all legally be coloured red (that is, they are either uncoloured or red already, but cannot be blue).

Then the 8,211,380 ways to place the isolated blue pentagons were run through a screen which kept only the canonical cases. For this step, the automorphism group of the graph was precomputed as described in §2.1. As a check on the computation, for each of the 1,379,646 cases retained, we determined the number of valid images it had (that is, the number of ways to renumber it with an automorphism such that the largest label on a pentagon is 172). The sum of these was equal to 8,211,380 (the number of cases possible without removing duplicates), a necessary condition for correctness. As an additional check of correctness, the number of cases to consider up to isomorphism matches that from a computation done earlier with a different approach.

2.6 Finishing the Search by a Backtrack

For each of the 1,379,646 non-isomorphic ways of planting seven blue pentagons in the pentagons $P_0, P_1, ...P_{172}$ (described in §2.5), the next step is to determine if it is possible to extend the configuration so that it contains at least 25 isolated blue pentagons. The possibilities for an independent set of order 221 outlined in §2.4 indicate that a solution of order 221 does not have many keys or isolated blues, more specifically, that $B_1 + 2B_4 \leq 7$.

A backtracking routine was used to try to extend each of the cases with the seven isolated blue pentagons to 25 isolated blue pentagons without creating too many isolated blues or keys in the process. Some tricks were used to make this computation finish in a reasonable amount of time.

The backtracking algorithm at level k considers two cases: one where the pentagon P_k is not included as an isolated blue pentagon, and if feasible, a second case where the pentagon P_k is included as an isolated blue pentagon (which means that its ten incident vertices are coloured red).

The colour of a vertex is recorded as an integer which is 0 if a vertex is not coloured. The colour is decremented each time a vertex is coloured red, or incremented each time a vertex is coloured blue. This permits the algorithm to colour vertices then backtrack by uncolouring the vertices without using a data structure such as a stack to indicate vertices with a status change. Only blue or white vertices can legally be coloured blue. Only red or white vertices can legally be coloured red. If a vertex is coloured red, then its neighbours are immediately coloured blue. When the colour of a vertex returns to zero, it returns to the uncoloured status.

As the algorithm progresses, certain vertices can safely be coloured red. These are characterized in the next theorem.

Theorem 2.6. *Suppose a 120-cell has an independent set of vertices coloured red, the neighbours of these are coloured blue, and the remaining vertices are uncoloured. If there is an uncoloured vertex v with three blue neighbours and one uncoloured neighbour w, then if there is a maximum independent set of the 120-cell which is consistent with the entire colouring, there is a maximum independent set with v coloured red.*

Proof. If v is red in the maximum independent set then there is nothing to prove. If v is not red, then w is red because if w is blue instead, v is a blue vertex with four blue neighbours, contradicting the maximality of the independent set (colouring v red increases the independent set order). An independent set of the same order can then be found by changing the colouring so that v is red and w is blue. □

The algorithm first inserts the initial seven isolated blue pentagons, waiting until they are all included before applying Theorem 2.6. The delay is needed because applying the theorem earlier can result in a colouring inconsistent with the initial pentagons. If there are two uncoloured vertices u and v which are adjacent to each other and also each is adjacent to three blues, there are two choices for how to apply Theorem 2.6, and it is possible that only one of these is consistent with the initial selection of the seven isolated blue pentagons.

During the course of the backtrack, each time an isolated blue pentagon is added to the current configuration, a queue is used to record vertices which evolve to being white with three blue neighbours. As the goal is to try to add 25 isolated blue pentagons, 25 queues suffice.

After addition of the ten incident reds of the isolated blue pentagon, the algorithm traverses the queue, and each vertex in the queue which is not blue is coloured red (as noted in the last paragraph, applying Theorem 2.6 at a vertex may prevent its subsequent use at another vertex). This process can trigger the addition of further vertices to the queue. When the isolated blue pentagon is removed (when the computation backtracks), the queue is first traversed in the reverse order to undo these changes.

New isolated blue vertices are recorded at the point when the fourth neighbour of the isolated blue is initially coloured red. The number is decremented when this fourth neighbour becomes uncoloured again. Keys arise either when an uncoloured vertex with three blue neighbours is coloured blue or when a third neighbour of a blue vertex is initially coloured blue. To facilitate the detection of isolated blue vertices and keys, respectively, the number of red neighbours and the number of blue neighbours of each vertex are maintained.

The algorithm takes exponential time to run, which is not surprising as the problem is hard. A careful selection of the data structures results in an approach such that the work it does to maintain the data structures is at most a constant times the number of times a vertex is coloured red.

The algorithm also used some precomputed upper bounds. We determined the maximum number of isolated blue pentagons possible if the pentagons are chosen from P_k, $P_{k+1}, \ldots, P_{719}$ such that the *penalty* (equal to $B_1 + 2B_4$) is at most seven (as required for an independent set of order 221). There is no point in continuing this computation past the point where 18 isolated blue pentagons are possible: since we start with seven, only 18 more are required. Theorem 2.6 was not used for computing these upper bounds, owing to its interference with what we were trying to compute. At a given level of the backtrack for placing the 25 blue pentagons, if the number of isolated blue pentagons included so far plus the bound for the level is less than 25, the algorithm backs up, since it is necessary to have at least 25 isolated blue pentagons for an independent set of order 221.

It is possible in the course of the algorithm that an isolated blue pentagon which has yet not been considered ends up with all ten of its incident vertices red. This however does not preclude the algorithm from adding it: the incident vertices just get coloured red more than once.

The algorithm for this last backtrack was coded independently twice to ensure correctness. The 1,379,646 cases were split into 64 slices, and run in parallel on a 64-processor array, with the run of the C program for a typical slice taking $16 - 18$ hours. Both programs concluded that it is not possible to include 25 blue pentagons in the 120-cell with a penalty of seven or less after applications of Theorem 2.6 are taken into account. Because this must be possible for an independent set of order 221 to exist, the maximum independent set order of the 120-cell is 220.

Polytope	n	m	f	r	g	D	$\|C_d\|$
Icosahedron	12	30	20	5	3	3	3, 2
Dodecahedron	20	30	12	3	5	5	8, 4, 2, 2
24-cell	24	96	96	8	3	3	8, 2
600-cell	120	720	1200	12	3	5	24, 8, 3, 2
120-cell	600	1200	720	4	5	15	220, 120, 48, 28, 24, 10, 8, 5, 5, 3, 2, 2, 2, 2

Table 3: Exceptional polytopes in dimensions three and four. The columns n, m, and f give the numbers of vertices, edges and two-dimensional faces of the polytope; r, g and D are the vertex degree, girth and diameter of the graph. The entries $|C_d|$ are the maximum orders of d-codes for $d = 2, 3, ...D - 1$.

3 Other Results

Table 3 lists the orders of the maximum d-codes for all five exceptional polytopes. Apart from the 2-code of the 120-cell, the only case requiring special tactics is the 4-code of the 120-cell, which was solved as described in [18]. All five polytopes have antipodal pairs as their d-codes for $d = D$, the diameter of the graph. When the codes are considered in terms of their 'contact polytopes' [15], some interesting 'Russian Doll'-like interconnections are seen. In the sense used in previous work [15], the contact polytope of a d-code has the same vertices as the independent set, and two vertices of the contact polytope are joined by an edge if the independent-set vertices are at distance d in the parent graph. Simplices of dimensions two, three and four appear: the triangle (α_2) is the contact polygon of the 3-code of the icosahedron, the 4-code of the 600-cell and the 11-code of the 120-cell; the tetrahedron (α_3) is the contact polyhedron of the 3-code of the dodecahedron; the four-dimensional simplex (α_4) is the contact polytope of the 9-code of the 120-cell. The cube appears (γ_3) appears as the contact polytope of the 2-code of the dodecahedron. The hyperoctahedron (β_4) appears as the contact polytope of 2-code of the 24-cell, 3-code of the 600-cell and the 8-code of the 120-cell. The 24-cell itself is the contact polytope of the 2-code of the 600-cell. The 3-code of the 120-cell is a *perfect* code [10] in the sense that each vertex of the code is at the centre of a ball of radius 1 containing one vertex of the 120-cell and its four nearest neighbours; the 120-cell is then partitioned exactly into 120 such balls, with centres on the vertex set of a 600-cell whose edges are paths of length 3 in the 120-cell. These observations are closely related to the fact, pointed out by Coxeter [3], that the vertices of the 120-cell embedded as an equilateral object in four-dimensional space include the vertices of all fifteen of the other regular polytopes in four dimensions, a property that has no analogy in three dimensions, where the dodecahedron contains the vertices of the cube and tetrahedron, but not those of the octahedron or icosahedron.

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Wiener index of iterated line graphs of trees homeomorphic to the claw $K_{1,3}$

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Abstract

Let G be a graph. Denote by $L^i(G)$ its i-iterated line graph and denote by $W(G)$ its Wiener index. Dobrynin, Entringer and Gutman stated the following problem: Does there exist a non-trivial tree T and $i \geq 3$ such that $W(L^i(T)) = W(T)$? In a series of five papers we solve this problem. In a previous paper we proved that $W(L^i(T)) > W(T)$ for every tree T that is not homeomorphic to a path, claw $K_{1,3}$ and to the graph of "letter H", where $i \geq 3$. Here we prove that $W(L^i(T)) > W(T)$ for every tree T homeomorphic to the claw, $T \neq K_{1,3}$ and $i \geq 4$.

Keywords: Wiener index, iterated line graph, tree, claw.

Math. Subj. Class.: 05C12

1 Introduction

Let G be a graph. For any two of its vertices, say u and v, denote by $d_G(u, v)$ (or by $d(u, v)$ if no confusion is likely) the distance from u to v in G. The *Wiener index* of G, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of G. Wiener index was introduced by Wiener in [12]. It is related to boiling point, heat of evaporation, heat of

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formation, chromatographic retention times, surface tension, vapour pressure, partition coefficients, total electron energy of polymers, ultrasonic sound velocity, internal energy, etc., see [8]. For this reason Wiener index is widely studied by chemists. The interest of mathematicians was attracted in 1970's. It was reintroduced as the distance and transmission, see [5] and [11], respectively. Recently, there are whole special issues of journals devoted to (mathematical properties) of Wiener index, see [6] and [7], as well as several surveys, see e.g. [3] and [4].

By definition, if G has a unique vertex, then $W(G) = 0$. In this case, we say that the graph G is *trivial*. We set $W(G) = 0$ also when the set of vertices (and hence also the set of edges) of G is empty.

The line graph of G, $L(G)$, has vertex set identical with the set of edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1] we have the following statement.

Theorem 1.1. Let T be a tree on n vertices. Then $W(L(T)) = W(T) - \binom{n}{2}$.

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees T satisfying $W(L^2(T)) = W(T)$, see e.g. [2]. In [3], the following problem was posed:

Problem 1.2. Is there any tree T satisfying the equality $W(L^i(T)) = W(T)$ for some $i \geq 3$?

As observed above, if T is a trivial tree then $W(L^i(T)) = W(T)$ for every $i \geq 1$, although here the graph $L^i(T)$ is empty.

Denote by H the tree on six vertices out of which two have degree 3 and four have degree 1. Since H can be drawn to resemble the letter H, it is often called the H-graph. Graphs G_1 and G_2 are homeomorphic if and only if the graphs obtained from G_1 and G_2, respectively, by substituting the vertices of degree two together with the two incident edges with a single edge, are isomorphic. In [10] we proved the following:

Theorem 1.3. Let T be a tree, not homeomorphic to a path, claw $K_{1,3}$ and the graph H. Then $W(L^i(T)) > W(T)$ for all $i \geq 3$.

Since the case when T is a path is trivial, it remains to consider graphs homeomorphic to the claw $K_{1,3}$ and those homeomorphic to H. In this paper we concentrate on graphs homeomorphic to the claw $K_{1,3}$. The remaining two cases, namely the trees homeomorphic to H for $i \geq 3$ and trees homeomorphic to $K_{1,3}$ for $i = 3$, are dealt with in a forthcoming paper.

First, consider the case of the claw $K_{1,3}$ itself. Then $L^i(K_{1,3})$ is a cycle of length 3 for every $i \geq 1$. Since $W(K_{1,3}) = 9$ and the Wiener index of the cycle of length 3 is 3, we have $W(L^i(K_{1,3})) < W(K_{1,3})$ for every $i \geq 1$. For other trees homeomorphic to $K_{1,3}$, we prove the opposite inequality, provided that $i \geq 4$:

Theorem 1.4. Let T be a tree homeomorphic to $K_{1,3}$, such that $T \neq K_{1,3}$. Then it holds that $W(L^i(T)) > W(T)$ for every $i \geq 4$.

In [9] we proved the following statement:

Theorem 1.5. Let G be a connected graph. Then $f_G(i) = W(L^i(G))$ is a convex function in variable i.

Hence, to prove Theorem 1.4 it suffices to prove:

Theorem 1.6. Let T be a tree homeomorphic to $K_{1,3}$, such that $T \neq K_{1,3}$. Then it holds $W(L^4(T)) > W(T)$.

2 Proofs

Let $a, b, c \geq 1$. Denote by $C_{a,b,c}$ a tree that has three paths of lengths a, b and c, starting at a common vertex of degree 3. Obviously, $C_{a,b,c}$ is homeomorphic to $K_{1,3}$ and $C_{1,1,1} = K_{1,3}$. By symmetry, we may assume $a \geq b \geq c$, see Figure 1 for $C_{5,4,3}$.

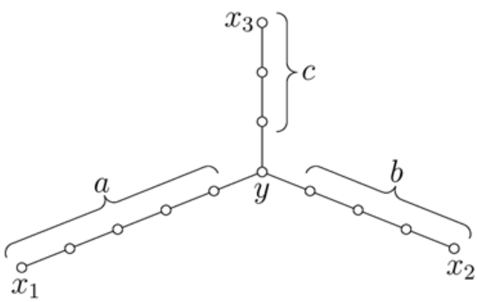

Figure 1: The graph $C_{5,4,3}$.

Denote

$$\Delta C_{a,b,c} = W(L^4(C_{a,b,c})) - W(C_{a,b,c}).$$

Our aim is to prove $\Delta C_{a,b,c} > 0$ if $a \geq 2$. We start with the case $a \leq 3$. This case will serve as the base of induction in the proof of Theorem 1.6.

Lemma 2.1. Let $3 \geq a \geq b \geq c \geq 1$ and $a \neq 1$. Then $\Delta C_{a,b,c} > 0$.

Proof. Since $3 \geq a \geq b \geq c \geq 1$ and $a \neq 1$, there are 9 cases to consider. In Table 1 we present $\Delta C_{a,b,c}$ for each of these cases. The results were found by a computer, though it is rather easy to find $W(C_{a,b,c})$ by hand, and $W(L^4(C_{a,b,c}))$ can be found by applying Proposition 2.3 to $L^2(C_{a,b,c})$. □

(a,b,c)	$W(C_{a,b,c})$	$W(L^4(C_{a,b,c}))$	$\Delta C_{a,b,c}$
$(3,3,3)$	138	642	504
$(3,3,2)$	102	533	431
$(3,3,1)$	75	257	182
$(3,2,2)$	72	435	363
$(3,2,1)$	50	192	142
$(3,1,1)$	32	65	33
$(2,2,2)$	48	348	300
$(2,2,1)$	31	138	107
$(2,1,1)$	18	38	20

Table 1: $\Delta C_{a,b,c}$ for $a \leq 3$.

In what follows we assume that $a \geq 4$. Denote

$$\begin{aligned}\delta_0(a,b,c) &= W(C_{a,b,c}) - W(C_{a-1,b,c})\\ \delta_4(a,b,c) &= W(L^4(C_{a,b,c})) - W(L^4(C_{a-1,b,c})).\end{aligned}$$

Then

$$\Delta C_{a,b,c} - \Delta C_{a-1,b,c} = \delta_4(a,b,c) - \delta_0(a,b,c), \tag{2.1}$$

so if we prove $\delta_4(a,b,c) - \delta_0(a,b,c) \geq 0$, we obtain $\Delta C_{a,b,c} \geq \Delta C_{a-1,b,c}$.

We distinguish 4 vertices in $C_{a,b,c}$. Denote by y the vertex of degree 3, and denote by x_1, x_2 and x_3 the pendant vertices so that $d(x_1,y) = a$, $d(x_2,y) = b$ and $d(x_3,y) = c$, see Figure 1. As is the custom, by $V(G)$ we denote the vertex set of G.

Lemma 2.2. Let $a,b,c \geq 1$. Then

$$\delta_0(a,b,c) = \binom{a+b+1}{2} + \binom{a+c+1}{2} - \binom{a+1}{2}.$$

Proof. Since $C_{a-1,b,c}$ is a subgraph of $C_{a,b,c}$ with $V(C_{a,b,c}) - V(C_{a-1,b,c}) = \{x_1\}$, we have

$$\delta_0(a,b,c) = W(C_{a,b,c}) - W(C_{a-1,b,c}) = \sum_u d(u,x_1),$$

where the sum goes through all $u \in V(C_{a,b,c}) \setminus \{x_1\}$. For vertices u of the $x_1 - x_2$ path, the sum of all $d(u,x_1)$ is $1+2+\cdots+(a+b) = \binom{a+b+1}{2}$. For vertices of the $x_1 - x_3$ path which do not lay on $x_1 - x_2$ path, the sum of $d(u,x_1)$ is $(a+1)+(a+2)+\cdots+(a+c) = \binom{a+c+1}{2} - \binom{a+1}{2}$, see Figure 1. Since the paths $x_1 - x_2$ and $x_1 - x_3$ contain all vertices of $C_{a,b,c}$, we have $\delta_0(a,b,c) = \binom{a+b+1}{2} + \binom{a+c+1}{2} - \binom{a+1}{2}$. □

For two subgraphs S_1 and S_2 of G, by $d(S_1,S_2)$ we denote the shortest distance in G between a vertex of S_1 and a vertex of S_2. If S_1 and S_2 share an edge then we set $d(S_1,S_2) = -1$.

Analogously as a vertex of $L(G)$ corresponds to an edge of G, a vertex of $L^2(G)$ corresponds to a path of length two in G. For $x \in V(L^2(G))$ we denote by $B_2(x)$ the corresponding path in G. Let x and y be two distinct vertices of $L^2(G)$. It was proved in [9] that

$$d_{L^2(G)}(x,y) = d_G(B_2(x),B_2(y)) + 2.$$

Let $u, v \in V(G)$, $u \neq v$. Denote by $\beta_i(u,v)$ the number of pairs $x, y \in V(L^2(G))$, with u being the center of $B_2(x)$ and v being the center of $B_2(y)$, such that $d(B_2(x), B_2(y)) = d(u,v) - 2 + i$. Since $d(u,v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u,v)$, we have $\beta_i(u,v) = 0$ for all $i \notin \{0,1,2\}$. Denote by $\deg(w)$ the degree of w in G. In [9] we have the following statement:

Proposition 2.3. *Let G be a connected graph. Then*

$$\begin{aligned} W(L^2(G)) &= \sum_{u \neq v} \left[\binom{\deg(u)}{2} \binom{\deg(v)}{2} d(u,v) + \beta_1(u,v) + 2\beta_2(u,v) \right] \\ &+ \sum_{u} \left[3\binom{\deg(u)}{3} + 6\binom{\deg(u)}{4} \right], \end{aligned} \tag{2.2}$$

where the first sum goes through unordered pairs $u, v \in V(G)$ and the second one goes through $u \in V(G)$.

We apply Proposition 2.3 to $L^2(C_{a,b,c})$ and $L^2(C_{a-1,b,c})$. This enables us to calculate $\delta_4(a,b,c)$ using degrees and distances of the second iterated line graph.

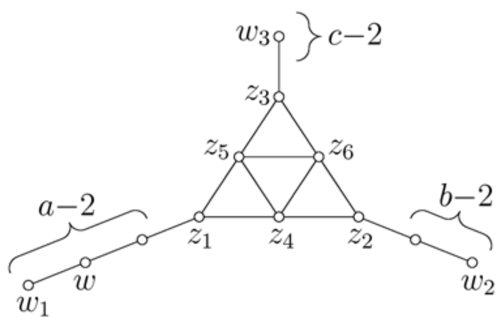

Figure 2: The graph $L^2(C_{5,4,3})$.

Denote by w_1 the pendant vertex of $L^2(C_{a,b,c})$ corresponding to the path of length 2 terminating at x_1. Since $a \geq 4$, the unique neighbour of w_1 has degree 2. Denote by w this neighbour, see Figure 2. For every vertex $u \in V(L^2(C_{a,b,c})) \setminus \{w, w_1\}$, denote by $n(u)$ the number of neighbours of u, whose distance to w is at least $d(u,w)$. We have:

Lemma 2.4. Let $a \geq 4$ and $b, c \geq 1$. Then

$$\delta_4(a,b,c) = \sum_u \left[\binom{\deg(u)}{2} d(u,w) + \binom{n(u)}{2} \right],$$

where the sum goes through all vertices of $V(L^2(C_{a,b,c})) \setminus \{w, w_1\}$.

PROOF. Observe that $L^2(C_{a-1,b,c})$ is a subgraph of $L^2(C_{a,b,c})$ and $V(L^2(C_{a,b,c})) \setminus V(L^2(C_{a-1,b,c})) = \{w_1\}$. Since $\deg(w_1) = 1$, the vertex w_1 cannot be the center of a path of length 2, implying that $\beta_i(u, w_1) = 0$ for every u and i. Since $\binom{\deg(w_1)}{2} =$

0, all summands of (2.2) containing w_1 contribute 0 to $W(L^4(C_{a,b,c}))$. The vertices of $L^2(C_{a-1,b,c})$, except w, have the same degree in $L^2(C_{a,b,c})$ as in $L^2(C_{a-1,b,c})$. The degree of w is 1 in $L^2(C_{a-1,b,c})$, and it is 2 in $L^2(C_{a,b,c})$. Therefore $\sum_u [3\binom{\deg(u)}{3} + 6\binom{\deg(u)}{4}]$ has the same value in $L^2(C_{a,b,c})$ as in $L^2(C_{a-1,b,c})$, so these sums will cancel out. Thus, we have

$$\begin{aligned}\delta_4(a,b,c) &= W(L^2(L^2(C_{a,b,c}))) - W(L^2(L^2(C_{a-1,b,c}))) \\ &= \sum_u \left[\binom{\deg(u)}{2}\binom{2}{2} d(u,w) + \beta_1(u,w) + 2\beta_2(u,w)\right],\end{aligned}$$

where the sum goes through $u \in V(L^2(C_{a-1,b,c})) \setminus \{w\}$.

Let $u \in V(L^2(C_{a-1,b,c})) \setminus \{w\}$. Since $\deg(w_1) = 1$ and $\deg(w) = 2$ in $L^2(C_{a,b,c})$, the unique path of length 2 centered at w contains an endvertex closer to u than w. Hence, $\beta_2(u,w) = 0$. Consequently, $\beta_1(u,w)$ equals the number of paths of length 2 centered at u, both endvertices of which have distance to w at least $d(u,w)$. Hence, $\beta_1(u,w) = \binom{n(u)}{2}$, which completes the proof. □

Using Lemma 2.4 we prove the induction step.

Lemma 2.5. Let $a \geq b \geq c \geq 1$ and $a \geq 4$. Then $\delta_4(a,b,c) \geq \delta_0(a,b,c)$.

Proof. We distinguish 8 more vertices in $L^2(C_{a,b,c})$. Denote by w_2 and w_3 pendant vertices corresponding to the paths of length 2 containing x_2 and x_3, respectively, see Figure 1 and 2. Denote by z_1, z_2 and z_3 the vertices corresponding to the paths of length 2, whose endvertex is y; and denote by z_4, z_5 and z_6 the vertices corresponding to the paths of length 2 centered at y. Of course, if $b \leq 2$ or $c \leq 2$, then some of these vertices are not defined.

For $u \in V(L^2(C_{a-1,b,c})) \setminus \{w\}$, denote

$$h(u) = \binom{\deg(u)}{2} d(u,w) + \binom{n(u)}{2}.$$

By Lemma 2.4, we have $\delta_4(a,b,c) = \sum_u h(u)$, where the sum goes through all vertices of $V(L^2(C_{a,b,c})) \setminus \{w, w_1\}$. If $u \in \{w_2, w_3\}$ then $\deg(u) = 1$ and $n(u) = 0$, so $h(u) = 0$. Thus, vertices of degree 1 contribute 0 to $\sum_u h(u)$. Denote

$$S_i = \sum_u h(u),$$

where the sum is taken over all interior vertices u of the $w_i - z_i$ path, $u \neq w$ and $1 \leq i \leq 3$. Then $\delta_4(a,b,c) = \sum_{i=1}^3 S_i + \sum_{i=1}^6 h(z_i)$.

Regarding the values of a, b and c, we distinguish 4 cases:

Case 1. $a \geq 4$ *and* $b, c \geq 3$.

If u is a vertex of degree 2, then $n(u) = 1$ and $\binom{\deg(u)}{2} = 1$. Hence $h(u) = d(u,w)$. Thus,

$$\begin{aligned}S_1 &= 1 + 2 + \cdots + (a-4) = \binom{a-3}{2} \\ S_2 &= a + (a+1) + \cdots + (a+b-4) = \binom{a+b-3}{2} - \binom{a}{2} \\ S_3 &= a + (a+1) + \cdots + (a+c-4) = \binom{a+c-3}{2} - \binom{a}{2}.\end{aligned}$$

If $u \in \{z_1, z_2, z_3\}$, then $\deg(u) = 3$ and $n(u) = 2$. Thus $h(u) = 3d(u,w) + 1$. If $u \in \{z_4, z_5\}$, then $\deg(u) = 4$ and $n(u) = 3$, so $h(u) = 6d(u,w) + 3$. Finally, if $u = z_6$, then $\deg(u) = 4$ and $n(u) = 2$, so $h(u) = 6d(u,w) + 1$. This gives

$$h(z_1) = 3(a-3) + 1 \qquad h(z_4) = h(z_5) = 6(a-2) + 3$$
$$h(z_2) = h(z_3) = 3(a-1) + 1 \qquad h(z_6) = 6(a-1) + 1.$$

Since $\delta_4(a,b,c) = \sum_{i=1}^{3} S_i + \sum_{i=1}^{6} h(z_i)$, we have

$$\begin{aligned} \delta_4(a,b,c) &= \binom{a-3}{2} + \binom{a+b-3}{2} + \binom{a+c-3}{2} - 2\binom{a}{2} \\ &\quad +(3a-8) + 2(3a-2) + 2(6a-9) + (6a-5). \end{aligned}$$

Denote $P = \delta_4(a,b,c) - \delta_0(a,b,c)$. By Lemma 2.2 we have $\delta_0(a,b,c) = \binom{a+b+1}{2} + \binom{a+c+1}{2} - \binom{a+1}{2}$. Expanding the terms we get

$$P = 17a - 4b - 4c - 17.$$

Since $a \geq b$ and $a \geq c$, we have $P \geq 9a - 17$. Finally, since $a \geq 4$, we have $P = \delta_4(a,b,c) - \delta_0(a,b,c) \geq 0$.

Case 2. *$a \geq 4$, $b \geq 3$ and $c \leq 2$.*

We calculate first $\delta_4(a,b,1)$. In $L^2(C_{a,b,1})$ we have $S_3 = 0$; note that z_3 is not defined here and that $\deg(z_5) = \deg(z_6) = 3$ (see Figure 2). Analogously as in Case 1 we get:

$$S_1 = \binom{a-3}{2} \qquad h(z_2) = 3(a-1) + 1$$
$$S_2 = \binom{a+b-3}{2} - \binom{a}{2} \qquad h(z_4) = 6(a-2) + 3$$
$$S_3 = 0 \qquad h(z_5) = 3(a-2) + 1$$
$$h(z_1) = 3(a-3) + 1 \qquad h(z_6) = 3(a-1)$$

since $n(z_5) = 2$ and $n(z_6) = 1$. Thus,

$$\begin{aligned} \delta_4(a,b,1) &= \binom{a-3}{2} + \binom{a+b-3}{2} - \binom{a}{2} + (3a-8) \\ &\quad +(3a-2) + (6a-9) + (3a-5) + (3a-3). \end{aligned}$$

Denote $P = \delta_4(a,b,1) - \delta_0(a,b,2)$. By Lemma 2.2 we have $\delta_0(a,b,2) = \binom{a+b+1}{2} + \binom{a+3}{2} - \binom{a+1}{2}$. Expanding the terms we get

$$P = 9a - 4b - 18.$$

Since $a \geq b$, we have $P \geq 5a - 18$, and as $a \geq 4$, we have $P \geq 0$. Since $\delta_4(a,b,2) \geq \delta_4(a,b,1)$ and $\delta_0(a,b,2) \geq \delta_0(a,b,1)$, we conclude $\delta_4(a,b,i) - \delta_0(a,b,i) \geq P \geq 0$ for $i \in \{1,2\}$.

Case 3. *$a \geq 4$, $b = 2$ and $c \leq 2$.*

We find $\delta_4(a,2,1)$. In $L^2(C_{a,2,1})$ we have $S_2 = S_3 = 0$. Again, the vertex z_3 is not defined here, $\deg(z_2) = 2$ and $\deg(z_5) = \deg(z_6) = 3$ (see Figure 2). Analogously as in the previous cases we get:

$$S_1 = \binom{a-3}{2} \qquad h(z_4) = 6(a-2) + 3$$
$$h(z_1) = 3(a-3) + 1 \qquad h(z_5) = 3(a-2) + 1$$
$$h(z_2) = (a-1) \qquad h(z_6) = 3(a-1)$$

since $n(z_2) = 1$, $n(z_5) = 2$ and $n(z_6) = 1$. Thus,

$$\delta_4(a,2,1) = \binom{a-3}{2} + (3a-8) + (a-1) + (6a-9) + (3a-5) + (3a-3).$$

Denote $P = \delta_4(a,2,1) - \delta_0(a,2,2)$. By Lemma 2.2 we have $\delta_0(a,2,2) = 2\binom{a+3}{2} - \binom{a+1}{2}$. Expanding the terms we get

$$P = 8a - 26.$$

Since $a \geq 4$, we have $P \geq 0$. Since $\delta_4(a,2,2) \geq \delta_4(a,2,1)$ and $\delta_0(a,2,2) \geq \delta_0(a,2,1)$, we conclude $\delta_4(a,2,i) - \delta_0(a,2,i) \geq P \geq 0$ for $i \in \{1,2\}$.

Case 4. $a \geq 4$ *and* $b = c = 1$.

In $L^2(C_{a,1,1})$ we have $S_2 = S_3 = 0$. Note that the vertices z_2 and z_3 are not defined, while $\deg(z_4) = \deg(z_5) = 3$ and $\deg(z_6) = 2$ (see Figure 2). Analogously as in the previous cases we get:

$$S_1 = \binom{a-3}{2} \qquad h(z_4) = h(z_5) = 3(a-2) + 1$$
$$h(z_1) = 3(a-3) + 1 \qquad h(z_6) = (a-1)$$

since $n(z_4) = n(z_5) = 2$ and $n(z_6) = 0$. Thus,

$$\delta_4(a,1,1) = \binom{a-3}{2} + (3a-8) + 2(3a-5) + (a-1).$$

Denote $P = \delta_4(a,1,1) - \delta_0(a,1,1)$. By Lemma 2.2 we have $\delta_0(a,1,1) = 2\binom{a+2}{2} - \binom{a+1}{2}$. Expanding the terms we get

$$P = 4a - 15.$$

Since $a \geq 4$, we have $P \geq 0$, and hence $\delta_4(a,1,1) - \delta_0(a,1,1) \geq P \geq 0$. □

Proof of Theorem 1.6. Let T be the tree $C_{a,b,c}$ with $a \geq b \geq c \geq 1$, such that $a \neq 1$. If $a \leq 3$, then $\Delta C_{a,b,c} = W(L^4(C_{a,b,c})) - W(C_{a,b,c}) > 0$, by Lemma 2.1.

Now suppose that $a \geq 4$. Consider lexicographical ordering of triples (a', b', c') with $a' \geq b' \geq c' \geq 1$ and $a' \geq 2$. Further, assume that $\Delta C_{a',b',c'} > 0$ for every triple (a', b', c'), such that $a' \geq b' \geq c' \geq 1$ and $a' \geq 2$, which is in the lexicographical ordering smaller than (a, b, c).

Let (a^*, b^*, c^*) be ordering of triple $(a-1, b, c)$, such that the multisets $\{a^*, b^*, c^*\}$ and $\{a-1, b, c\}$ are the same and $a^* \geq b^* \geq c^*$. Then $C_{a-1,b,c}$ and C_{a^*,b^*,c^*} are isomorphic graphs. Moreover, since $a \geq 4$, we have $a^* \geq 3$. By (2.1) and Lemma 2.5 we see that

$$\begin{aligned} \Delta C_{a,b,c} - \Delta C_{a^*,b^*,c^*} &= \Delta C_{a,b,c} - \Delta C_{a-1,b,c} \\ &= \delta_4(a,b,c) - \delta_0(a,b,c) \\ &\geq 0. \end{aligned}$$

Since (a^*, b^*, c^*) is in the lexicographical ordering smaller than (a, b, c) and $a^* \geq 2$, by the induction hypothesis we have $\Delta C_{a^*,b^*,c^*} > 0$. Hence, $\Delta C_{a,b,c} = W(L^4(C_{a,b,c})) - W(C_{a,b,c}) > 0$. □

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Orienting and separating distance-transitive graphs

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Abstract

It is shown that exactly 7 distance-transitive cubic graphs among the existing 12 possess a particular ultrahomogeneous property with respect to oriented cycles realizing the girth that allows the construction of a related Cayley digraph with similar ultrahomogeneous properties in which those oriented cycles appear minimally "pulled apart", or "separated" and whose description is truly beautiful and insightful. This work is proposed as the initiation of a study of similar ultrahomogeneous properties for distance-transitive graphs in general with the aim of generalizing to constructions of similar related "separator" Cayley digraphs.

Keywords: Distance-transitive graph, ultrahomogeneous graph, Cayley graph.

Math. Subj. Class.: 05C62, 05B30, 05C20, 05C38

1 Introduction

A graph is said to be distance-transitive if its automorphism group acts transitively on ordered pairs of vertices at distance i, for each $i \geq 0$ [3, 10, 15]. In this paper we deal mainly with finite cubic distance-transitive graphs. While these graphs are classified and very well-understood since there are only twelve examples, for this very restricted class of graphs we investigate a property called ultrahomogeneity that plays a very important role in logic, see for example [7, 18]. For ultrahomogeneous graphs (resp. digraphs), we refer the reader to [5, 9, 11, 17, 19] (resp. [6, 8, 16]). Distance-transitive graphs and ultrahomogeneous graphs are very important and worthwhile to investigate. However, to start with, the following question is answered in the affirmative for 7 of the 12 existing cubic distance-transitive graphs G and negatively for the remaining 5:

Question 1.1. If k is the largest ℓ such that G is ℓ-arc-transitive, is it possible to orient all shortest cycles of G so that each two oppositely oriented $(k-1)$-arcs of G are just in two corresponding oriented shortest cycles?

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The answer (below) to Question 1 leads to 7 connected digraphs $\mathcal{S}(G)$ in which all oriented shortest cycles of G are minimally "pulled apart" or "separated". Specifically, it is shown that all cubic distance-transitive graphs are $\{C_g\}_{P_k}$-ultrahomogeneous, where $g =$ girth, but only the 7 cited G are $\{\vec{C}_g\}_{\vec{P}_k}$-ultrahomogeneous digraphs and in each of these 7 digraphs G, the corresponding "separator" digraph $\mathcal{S}(G)$ is: **(a)** vertex-transitive digraph of indegree = outdegree = 2, underlying cubic graph and automorphism group as that of G; **(b)** $\{\vec{C}_g, \vec{C}_2\}$-ultrahomogeneous digraph, where $\vec{C}_g =$ induced oriented g-cycle, with each vertex taken as the intersection of exactly one such $\vec{C}_g$ and one $\vec{C}_2$; **(c)** a Cayley digraph. The structure and surface-embedding topology [2, 12, 20] of these $\mathcal{S}(G)$ are studied as well. We remark that the description of these $\mathcal{S}(G)$ is truly beautiful and insightful.

It remains to see how Question 1 can be generalized and treated for distance-transitive graphs of degree larger than 3 and what separator Cayley graphs could appear via such a generalization.

2 Preliminaries

We may consider a graph G as a digraph by taking each edge e of G as a pair of oppositely oriented (or O-O) arcs $\vec{e}$ and $(\vec{e})^{-1}$ inducing an oriented 2-cycle $\vec{C}_2$. Then, *fastening* $\vec{e}$ and $(\vec{e})^{-1}$ allows to obtain precisely the edge e in the graph G. Is it possible to orient all shortest cycles in a distance-transitive graph G so that each two O-O $(k-1)$-arcs of G are in just two oriented shortest cycles, where $k =$ largest ℓ such that G is ℓ-arc transitive? It is shown below that this is so just for 7 of the 12 cubic distance-transitive graphs G, leading to 7 corresponding minimum connected digraphs $\mathcal{S}(G)$ in which all oriented shortest cycles of G are "pulled apart" by means of a graph-theoretical operation explained in Section 4 below.

Given a collection $\mathcal{C}$ of (di)graphs closed under isomorphisms, a (di)graph G is said to be $\mathcal{C}$*-ultrahomogeneous* (or $\mathcal{C}$-UH) if every isomorphism between two induced members of $\mathcal{C}$ in G extends to an automorphism of G. If $\mathcal{C}$ is the isomorphism class of a (di)graph H, we say that such a G is $\{H\}$-UH or H-UH. In [14], $\mathcal{C}$-UH graphs are defined and studied when $\mathcal{C}$ is the collection of either the complete graphs, or the disjoint unions of complete graphs, or the complements of those unions.

Let M be an induced subgraph of a graph H and let G be both an M-UH and an H-UH graph. We say that G is an $\{H\}_M$*-UH graph* if, for each induced copy H_0 of H in G and for each induced copy M_0 of M in H_0, there exists exactly one induced copy $H_1 \neq H_0$ of H in G with $V(H_0) \cap V(H_1) = V(M_0)$ and $E(H_0) \cap E(H_1) = E(M_0)$. The vertex and edge conditions above can be condensed as $H_0 \cap H_1 = M_0$. We say that such a G is *tightly fastened*. This is generalized by saying that an $\{H\}_M$-UH graph G is an ℓ*-fastened* $\{H\}_M$-UH graph if given an induced copy H_0 of H in G and an induced copy M_0 of M in H_0, then there exist exactly ℓ induced copies $H_i \neq H_0$ of H in G such that $H_i \cap H_0 \supseteq M_0$, for each $i = 1, 2, \ldots, \ell$, with at least $H_1 \cap H_0 = M_0$.

Let $\vec{M}$ be an induced subdigraph of a digraph $\vec{H}$ and let the graph G be both an $\vec{M}$-UH and an $\vec{H}$-UH digraph. We say that G is an $\{\vec{H}\}_{\vec{M}}$*-UH digraph* if for each induced copy $\vec{H}_0$ of $\vec{H}$ in $\vec{G}$ and for each induced copy $\vec{M}_0$ of $\vec{M}$ in $\vec{H}_0$ there exists exactly one induced copy

$\vec{H}_1 \neq \vec{H}_0$ of $\vec{H}$ in G with $V(\vec{H}_0) \cap V(\vec{H}_1) = V(\vec{M}_0)$ and $A(\vec{H}_0) \cap \bar{A}(\vec{H}_1) = A(\vec{M}_0)$, where $\bar{A}(\vec{H}_1)$ is formed by those arcs $(\vec{e})^{-1}$ whose orientations are reversed with respect to the orientations of the arcs $\vec{e}$ of $A(\vec{H}_1)$. Again, we say that such a G is *tightly fastened*. This case is used in the constructions of Section 4.

Given a finite graph H and a subgraph M of H with $|V(H)| > 3$, we say that a graph G is (*strongly fastened*) *SF* $\{H\}_M$-UH if there is a descending sequence of connected subgraphs $M = M_1, M_2 \dots, M_t \equiv K_2$ such that: **(a)** M_{i+1} is obtained from M_i by the deletion of a vertex, for $i = 1, \dots, t-1$ and **(b)** G is a $(2^i - 1)$-fastened $\{H\}_{M_i}$-UH graph, for $i = 1, \dots, t$.

This paper deals with the above defined $\mathcal{C}$-UH concepts applied to cubic distance-transitive (CDT) graphs [3]. A list of them and their main parameters follows:

CDT graph G	n	d	g	k	η	a	b	h	κ
Tetrahedral graph K_4	4	1	3	2	4	24	0	1	1
Thomsen graph $K_{3,3}$	6	2	4	3	9	72	1	1	2
3-cube graph Q_3	8	3	4	2	6	48	1	1	1
Petersen graph	10	2	5	3	12	120	0	0	0
Heawood graph	14	3	6	4	28	336	1	1	0
Pappus graph	18	4	6	3	18	216	1	1	0
Dodecahedral graph	20	5	5	2	12	120	0	1	1
Desargues graph	20	5	6	3	20	240	1	1	3
Coxeter graph	28	4	7	3	24	336	0	0	3
Tutte 8-cage	30	4	8	5	90	1440	1	1	2
Foster graph	90	8	10	5	216	4320	1	1	0
Biggs-Smith graph	102	7	9	4	136	2448	0	1	0

where $n =$ order; $d =$ diameter; $g =$ girth; $k =$ AT or arc-transitivity (= largest ℓ such that G is ℓ-arc transitive); $\eta =$ number of g-cycles; $a =$ number of automorphisms; b (resp. h) $= 1$ if G is bipartite (resp. hamiltonian) and $= 0$ otherwise; and κ is defined as follows: let P_k and $\vec{P}_k$ be respectively a $(k-1)$-path and a directed $(k-1)$-path (of length $k-1$); let C_g and $\vec{C}_g$ be respectively a cycle and a directed cycle of length g; then (see Theorem 3 below): $\kappa = 0$, if G is not $(\vec{C}_g; \vec{P}_k)$-UH; $\kappa = 1$, if G is planar; $\kappa = 2$, if G is $\{\vec{C}_g\}_{\vec{P}_k}$-UH with $g = 2(k-1)$; $\kappa = 3$, if G is $\{\vec{C}_g\}_{\vec{P}_k}$-UH with $g > 2(k-1)$.

In Section 3 below, Theorem 2 proves that every CDT graph is an SF $\{C_g\}_{P_k}$-UH graph, while Theorem 3 establishes exactly which CDT graphs are not $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs; in fact 5 of them. Section 4 shows that each of the remaining 7 CDT graphs G yields a digraph $\mathcal{S}(G)$ whose vertices are the $(k-1)$-arcs of G, an arc in $\mathcal{S}(G)$ between each two vertices representing corresponding $(k-1)$-arcs in a common oriented g-cycle of G and sharing just one $(k-2)$-arc; additional arcs of $\mathcal{S}(G)$ appearing in O-O pairs associated with the reversals of $(k-1)$-arcs of G. Moreover, Theorem 4 asserts that each $\mathcal{S}(G)$ is as claimed and itemized at end of the Introduction above.

3 (C_g, P_k)-UH properties of CDT graphs

Theorem 3.1. Let G be a CDT graph of girth $= g$, AT $= k$ and order $= n$. Then, G is an SF $\{C_g\}_{P_k}$-UH graph. In particular, G has exactly $2^{k-2}3ng^{-1}$ g-cycles.

Proof. We have to see that each CDT graph G with girth $= g$ and AT $= k$ is a $(2^{i+1}-1)$-fastened $\{C_g\}_{P_{k-i}}$-UH graph, for $i = 0, 1, \dots, k-2$. In fact, each $(k-i-1)$-path $P = P_{k-i}$ of any such G is shared by exactly 2^{i+1} g-cycles of G, for $i = 0, 1, \dots, k-2$. For example if $k = 4$, then any edge (resp. 2-path, resp. 3-path) of G is shared by 8 (resp. 4, resp. 2) g-cycles of G. This means that a g-cycle C_g of G shares a P_2 (resp. P_3, resp. P_4) with exactly other 7 (resp. 3, resp. 1) g-cycles. Thus G is an SF $\{C_g\}_{P_{i+2}}$-UH graph, for $i = 0, 1, \dots, k-2$. The rest of the proof depends on the particular cases analyzed in the proof of Theorem 3 below and on some simple counting arguments for the pertaining numbers of g-cycles. □

Given a CDT graph G, there are just two g-cycles shared by each $(k-1)$-path. If in addition G is a $\{\vec{C}_g\}_{\vec{P}_k}$-UH graph, then there exists an assignment of an orientation for each g-cycle of G, so that the two g-cycles shared by each $(k-1)$-path receive opposite orientations. We say that such an assignment is a $\{\vec{C}_g\}_{\vec{P}_k}$-*O-O assignment* (or $\{\vec{C}_g\}_{\vec{P}_k}$-OOA). The collection of η oriented g-cycles corresponding to the η g-cycles of G, for a particular $\{\vec{C}_g\}_{\vec{P}_k}$-OOA will be called an $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC. Each such g-cycle will be expressed with its successive composing vertices expressed between parentheses but without separating commas, (as is the case for arcs uv and 2-arcs uvw), where as usual the vertex that succeeds the last vertex of the cycle is its first vertex.

Theorem 3.2. The CDT graphs G of girth $= g$ and AT $= k$ that are not $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs are the graphs of Petersen, Heawood, Pappus, Foster and Biggs-Smith. The remaining 7 CDT graphs are $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs.

Proof. Let us consider the case of each CDT graph sequentially. The graph K_4 on vertex set $\{1,2,3,0\}$ admits the $\{4\,\vec{C}_3\}_{\vec{P}_2}$-OOC $\{(123), (210), (301), (032)\}$. The graph $K_{3,3}$ obtained from K_6 (with vertex set $\{1,2,3,4,5,0\}$) by deleting the edges of the triangles $(1,3,5)$ and $(2,4,0)$ admits the $\{9\,\vec{C}_4\}_{\vec{P}_3}$-OOC $\{(1234), (3210), (4325), (1430), (2145), (0125), (5230), (0345), (5410)\}$. The graph Q_3 with vertex set $\{0,\dots,7\}$ and edge set $\{01, 23, 45, 67, 02, 13, 46, 57, 04, 15, 26, 37\}$ admits the $\{6\,\vec{C}_4\}_{\vec{P}_2}$-OOC $\{(0132), (1045), (3157), (2376), (0264), (4675)\}$.

The Petersen graph Pet is obtained from the disjoint union of the 5-cycles $\mu^\infty = (u_0u_1u_2u_3u_4)$ and $\nu^\infty = (v_0v_2v_4v_1v_3)$ by the addition of the edges (u_x, v_x), for $x \in \mathbf{Z}_5$. Apart from the two 5-cycles given above, the other 10 5-cycles of Pet can be denoted by $\mu^x = (u_{x-1}u_xu_{x+1}v_{x+1}v_{x-1})$ and $\nu^x = (v_{x-2}v_xv_{x+2}u_{x+2}u_{x-2})$, for each $x \in \mathbf{Z}_5$. Then, the following sequence of alternating 5-cycles and 2-arcs starts and ends up with opposite orientations:

$$\mu^2_- \quad u_3u_2u_1 \quad \mu^\infty_+ \quad u_0u_1u_2 \quad \mu^1_- \quad u_2v_2v_0 \quad \nu^0_- \quad v_3u_3u_2 \quad \mu^2_+,$$

where the subindexes $\pm$ indicate either a forward or backward selection of orientation and each 2-path is presented with the orientation of the previously cited 5-cycle but must be present in the next 5-cycle with its orientation reversed. Thus Pet cannot be a $\{\vec{C}_5\}_{\vec{P}_3}$-UH digraph.

Another way to see this is via the auxiliary table indicated below, that presents the form in which the 5-cycles above share the vertex sets of 2-arcs, either O-O or not. The table details, for each one of the 5-cycles $\xi = \mu^\infty, \nu^\infty, \mu^0, \nu^0$, (expressed as $\xi = (\xi_0, \dots, \xi_4)$ in the shown vertex notation), each 5-cycle η in $\{\mu^i, \nu^i; i = \infty, 0, \dots, 4\}\setminus\{\xi\}$ that intersects ξ in the succeeding 2-paths $\xi_i\xi_{i+1}\xi_{i+2}$, for $i = 0, \dots, 4$, with additions involving i taken mod 5. Each such η in the auxiliary table has either a preceding minus sign, if the corresponding 2-arcs in ξ and η are O-O, or a plus sign, otherwise. Each $-\eta_j$ (resp. η_j) shown in the table has the subindex j indicating the equality of initial vertices $\eta_j = \xi_{i+2}$ (resp. $\eta_j = \xi_i$) of those 2-arcs, for $i = 0, \dots, 4$:

$$\mu^\infty:(+\mu_0^1,+\mu_0^2,+\mu_0^3,+\mu_0^4,+\mu_0^0),\quad \nu^\infty:(+\nu_0^2,+\nu_0^4,+\nu_0^1,+\nu_0^3,+\nu_0^0),$$
$$\mu^0:(+\mu_4^\infty,+\nu_3^3,-\nu_1^4,-\nu_4^1,+\nu_2^2),\quad \nu^0:(+\nu_4^\infty,-\mu_2^1,+\mu_4^3,+\mu_1^2,-\mu_3^4).$$

This partial auxiliary table is extended to the whole auxiliary table by adding $x \in \mathbf{Z}_4$ uniformly mod 5 to all superindexes $\neq \infty$, reconfirming that Pet is not $\{\vec{C}_5\}_{\vec{P}_3}$-UH.

For each positive integer n, let I_n stand for the n-vertex cycle $(0, 1, \dots, n-1)$. The Heawood graph Hea is obtained from I_{14} by adding the edges $(2x, 5 + 2x)$, where $x \in \{1, \dots, 7\}$ and operations are in $\mathbf{Z}_{14}$. The 28 6-cycles of Hea include the following 7 6-cycles:

$$\gamma^x=(2x, 2x+1, 2x+2, 2x+3, 2x+4, 2x+5),\quad \delta^x=(2x, 2x+5, 2x+6, 2x+7, 2x+8, 2x+13),$$
$$\epsilon^x=(2x, 2x+5, 2x+4, 2x+9, 2x+8, 2x+13),\quad \zeta^x=(2x+12, 2x+3, 2x+4, 2x+5, 2x, 2x+13),$$

where $x \in \mathbf{Z}_7$. Now, the following sequence of alternating 6-cycles and 3-arcs starts and ends with opposite orientations for γ_0:

$$\gamma_+^0\ 2345\ \gamma_-^1\ 7654\ \gamma_+^2\ 6789\ \gamma_-^3\ ba98\ \gamma_+^4\ abcd\ \gamma_-^5\ 10dc\ \gamma_+^6\ 0123\ \gamma_-^0,$$

(where tridecimal notation is used, up to $d = 13$). Thus Hea cannot be a $\{\vec{C}_7\}_{\vec{P}_4}$-UH digraph. Another way to see this is via an auxiliary table for Hea obtained in a fashion similar to that of the one for Pet above from:

$$\gamma^0:(+\gamma_2^6,+\delta_1^5,+\gamma_0^1,+\zeta_1^6,-\epsilon_1^5,-\zeta_4^0);\quad \epsilon^0:(+\epsilon_2^5,-\gamma_4^2,+\epsilon_0^2,+\zeta_5^4,+\delta_4^0,-\zeta_2^6);$$
$$\delta^0:(+\zeta_0^0,+\gamma_1^2,-\zeta_3^3,+\delta_5^4,+\epsilon_4^0,+\delta_3^3);\quad \zeta^0:(+\delta_0^0,+\gamma_3^1,-\epsilon_5^1,-\delta_2^4,-\gamma_5^0,+\epsilon_3^3).$$

This reaffirms that Hea is not $\{\vec{C}_6\}_{\vec{P}_4}$-UH.

The Pappus graph Pap is obtained from I_{18} by adding to it the edges $(1+6x, 6+6x)$, $(2+6x, 9+6x)$, $(4+6x, 11+6x)$, for $x \in \{0, 1, 2\}$, with sums and products taken mod 18. The 6-cycles of Pap are expressible as: $A_0 = (123456)$, $B_0 = (3210de)$, $C_0 = (34bcde)$, $D_0 = (165gh0)$, $E_0 = (329ab4)$ (where octodecimal notation is used, up to $h = 17$), the 6-cycles A_x, B_x, C_x, D_x, E_x obtained by uniformly adding $6x$ mod 18 to the vertices of A_0, B_0, C_0, D_0, E_0, for $x \in \mathbf{Z}_3 \setminus \{0\}$, and $F_0 = (3298fe)$, $F_1 = (hg54ba)$, $F_2 = (167cd0)$. No orientation assignment makes these cycles into an $\{18\,\vec{C}_6\}_{\vec{P}_3}$-OOC, for the following sequence of alternating 6-cycles and 2-arcs (with orientation reversed between each preceding 6-cycle to corresponding succeeding 6-cycle) reverses the orientation of its initial 6-cycle in its terminal one:

$$D_1^{-1}654\,A_0123\,B_0210\,C_1h01\,D_0^{-1}g56\,C_2^{-1}876\,B_1^{-1}789\,A_1^{-1}cba\,D_2abc\,A_1^{-1}987\,B_1^{-1}678\,C_2^{-1}765D_1$$
$$=(654bc7)\,654\,(123456)\,123\,(3210de)\,210\,(0129ah)\,h01\,(10hg56)\,g56\,(5gf876)\,876$$
$$(216789)\,789(cba987)\,cba(d0habc)\,abc(cba987)\,987\,(216789)\,678\,(5gf876)\,765(cb4567).$$

Another way to see this is via an auxiliary table for Pap obtained in a fashion similar to those above for Pet and Hea, where $x = 0, 1, 2$ (mod 3):

$$-A_x:(B_x,E_x,E_{x+2},D_{x+1},D_x,B_{x+1});\quad -F_0:(E_0,B_1,E_1,B_2,E_2,B_0);$$
$$-B_x:(A_x,C_{x+1},F_2,A_{x+2},C_x,F_0);\quad -F_1:(D_0,E_2,D_1,E_0,D_2,E_1);$$
$$-C_x:(E_x,D_{x+1},D_{x+2},B_{x+2},B_x,E_{x+2});\quad -F_2:(B_1,D_1,B_2,D_2,B_0,D_0);$$
$$-D_x:(A_x,C_{x+2},F_1,A_{x+2},C_{x+1},F_2);$$
$$-E_x:(F_0,C_{x+1},A_{x+1},F_1,C_x,A_x).$$

This reaffirms that Pap is not a $\{\vec{C}_6\}_{\vec{P}_3}$-UH digraph. In fact, observe that any two 6-cycles here that share a 2-path possess the same orientation, in total contrast to what happens in the 7 cases that are being shown to be $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs, in the course of this proof.

The Desargues graph Des is obtained from the 20-cycle I_{20}, with vertices $4x, 4x+1, 4x+2, 4x+3$ redenoted alternatively x_0, x_1, x_2, x_3, respectively, for $x \in \mathbf{Z}_5$, by adding the edges $(x_3, (x+2)_0)$ and $(x_1, (x+2)_2)$, where operations are mod 5. Then, Des admits a $\{20\,\vec{C}_6\}_{\vec{P}_3}$-OOC formed by the oriented 6-cycles A^x, B^x, C^x, D^x, for $x \in \{0, \ldots, 4\}$, where

$$A^x = (x_0 x_1 x_2 x_3 (x+1)_0 (x+4)_3), \qquad B^x = (x_1 x_0 (x+4)_3 (x+4)_2 (x+2)_1 (x+2)_2),$$
$$C^x = (x_2 x_1 x_0 (x+3)_3 (x+3)_2 (x+3)_1), \qquad D^x = (x_0 (x+4)_3 (x+1)_0 (x+1)_1 (x+3)_2 (x+3)_3).$$

The successive copies of $\vec{P}_3$ here, when reversed in each case, must belong to the following remaining oriented 6-cycles:

$$A^x{:}(C^x, C^{x+2}, B^{x+1}, D^{x+1}, D^x, B^x); \qquad B^x{:}(A^x, A^{x+4}, D^{x+1}, C^{x+4}, C^{x+2}, D^{x+4});$$
$$C^x{:}(A^x, D^{x+4}, D^x, A^{x+3}, B^{x+1}, B^{x+3}); \qquad D^x{:}(A^x, C^{x+1}, B^{x+1}, B^{x+4}, C^x, A^{x+4});$$

showing that they constitute effectively an $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC.

The dodecahedral graph Δ is a 2-covering graph of the Petersen graph H, where each vertex u_x, (resp., v_x), of H is covered by two vertices a_x, c_x, (resp. b_x, d_x). A $\{12\,\vec{C}_5\}_{\vec{P}_2}$-OOC of Δ is given by the oriented 5-cycles $(a_0a_1a_2a_3a_4)$, $(c_4c_3c_2c_1c_0)$ and, for each $x \in \mathbf{Z}_5$, also by $(a_xd_xb_{x-2}d_{x+1}a_{x+1})$ and $(d_xb_{x+2}c_{x+2}c_{x-2}b_{x-2})$.

The Tutte 8-cage Tut is obtained from I_{30}, with vertices $6x, 6x+1, 6x+2, 6x+3, 6x+4, 6x+5$ denoted alternatively $x_0, x_1, x_2, x_3, x_4, x_5$, respectively, for $x \in \mathbf{Z}_5$, by adding the edges $(x_5, (x+2)_0)$, $(x_1, (x+1)_4)$ and $(x_2, (x+2)_3)$. Then, Tut admits the $\{90\,\vec{C}_8\}_{\vec{P}_5}$-OOC formed by the oriented 8-cycles:

$$A^0=(4_50_00_10_20_30_40_51_0), \quad B^0=(4_24_34_44_51_01_11_21_3), \quad C^0=(0_20_30_44_14_02_52_42_3),$$
$$D^0=(3_33_23_14_44_34_21_31_2), \quad E^0=(4_51_00_50_44_14_03_50_0), \quad F^0=(4_50_03_54_02_52_41_11_0),$$
$$G^0=(1_01_12_42_30_20_10_04_5), \quad H^0=(2_32_41_11_00_50_40_30_2), \quad I^0=(0_10_20_30_44_14_21_31_4),$$
$$J^0=(1_00_50_40_33_23_14_44_5), \quad K^0=(3_13_20_30_20_10_04_54_4), \quad L^0=(2_32_42_53_03_13_20_30_2),$$
$$M^0=(3_54_04_10_40_30_20_10_0), \quad N^0=(0_00_11_41_52_02_13_43_5), \quad O^0=(4_24_32_22_13_43_31_21_3),$$
$$P^0=(4_54_44_34_24_10_40_51_0), \quad Q^0=(4_04_14_21_31_41_53_02_5), \quad R^0=(0_10_20_33_23_13_01_51_4),$$

together with those obtained by adding $y \in \mathbf{Z}_5$ uniformly mod 5 to all numbers x of vertices x_i in $A^0, \ldots, R^0$, for each $y = 1, 2, 3, 4$, yielding in each case oriented 8-cycles $A^y, \ldots, R^y$.

The Coxeter graph Cox is obtained from three 7-cycles $(u_1u_2u_3u_4u_5u_6u_0)$, $(v_4v_6v_1v_3v_5v_0v_2)$, $(t_3t_6t_2t_5t_1t_4t_0)$ by adding a copy of $K_{1,3}$ with degree-1 vertices u_x, v_x, t_x and a central degree-3 vertex z_x, for each $x \in \mathbf{Z}_7$. Cox admits the $\{24\,\vec{C}_7\}_{\vec{P}_3}$-OOC:

$$\{0^1=(u_1u_2u_3u_4u_5u_6u_0), \quad 0^2=(v_1v_3v_5v_0v_2v_4v_6), \quad 0^3=(t_1t_5t_2\,t_6\,t_3\,t_0\,t_4),$$
$$1^1=(u_1z_1v_1\,v_3\,z_3u_3u_2), \quad 1^2=(z_4v_4v_2v_0z_0\,t_0t_4), \quad 1^3=(t_6t_2t_5z_5u_5u_6z_6),$$
$$2^1=(v_5z_5u_5u_4u_3\,z_3\,v_3), \quad 2^2=(t_6z_6v_6v_4v_2\,z_2t_2), \quad 2^3=(u_1z_1t_1t_4t_0z_0u_0),$$
$$3^1=(v_5v_0z_0u_0u_6\,u_5\,z_5), \quad 3^2=(z_4t_4t_1z_1v_1\,v_6v_4), \quad 3^3=(t_6t_2z_2u_2u_3z_3t_3),$$
$$4^1=(u_1u_0z_0v_0v_2\,z_2\,u_2), \quad 4^2=(t_6t_3z_3v_3v_1v_6\,z_6), \quad 4^3=(z_4u_4u_5z_5t_5t_1t_4),$$
$$5^1=(z_4u_4u_3u_2z_2\,v_2\,v_4), \quad 5^2=(v_5v_3v_1z_1t_1t_5\,z_5), \quad 5^3=(t_6z_6u_6u_0z_0t_0t_3),$$
$$6^1=(z_4v_4v_6z_6u_6\,u_5\,u_4), \quad 6^2=(v_5v_3z_3t_3t_0\,z_0v_0), \quad 6^3=(u_1u_2z_2t_2t_5t_1z_1),$$
$$7^1=(u_1u_0u_6z_6v_6\,v_1\,z_1), \quad 7^2=(v_5z_5t_5t_2z_2\,v_2v_0), \quad 7^3=(z_4t_4t_0t_3z_3u_3u_4)\}.$$

The Foster graph Fos is obtained from I_{90}, with vertices $6x, 6x+1, 6x+2, 6x+3, 6x+4, 6x+5$ denoted alternatively x_0, x_1, x_2, x_3, x_4, x_5, respectively, for $x \in \mathbf{Z}_{15}$, by adding the edges $(x_4, (x+2)_1)$, $(x_0, (x+2)_5)$ and $(x_2, (x+6)_3)$. The 216 10-cycles of Fos include the following 15 10-cycles, where $x \in \mathbf{Z}_{15}$:

$$\phi^x = (x_4 x_5 (x+1)_0 (x+1)_1 (x+1)_2 (x+1)_3 (x+1)_4 (x+1)_5 (x+2)_0 (x+2)_1).$$

Then, the following sequence of alternating 10-cycles and 4-arcs:

$$\phi^0_+[1_4]\phi^1_-[3_1]\phi^2_+[3_4]\phi^3_-[5_1]\phi^4_+[5_4]\phi^5_-[7_1]\phi^6_+[7_4]\phi^7_-[9_1]\phi^8_+[9_4]\phi^9_-[b_1]\phi^a_+[b_4]\phi^b_-[d_1]\phi^c_+[d_4]\phi^d_-[0_1]\phi^e_+[0_4]$$

may be continued with ϕ^0_-, with opposite orientation to that of the initial ϕ^0_+, where $[x_j]$ stands for a 3-arc starting at the vertex x_j in the previously cited (to the left) oriented 10-cycle. Thus Fos cannot be a $\{\vec{C}_{10}\}_{\vec{P}_5}$-UH digraph. Another way to see this is via the following table of 10-cycles of Fos, where the 10-cycle ϕ_0 intervenes as 10-cycle 0^0:

$$\begin{array}{ll}
0^0=(0_4 0_5 1_0 1_1 1_2 1_3 1_4 1_5 2_0 2_1) & 8^0=(0_2 0_3 9_2 9_1 7_4 7_5 8_0 8_1 6_4 6_3) \\
1^0=(0_0 0_1 0_2 0_3 0_4 2_1 2_2 2_3 2_4 2_5) & 9^0=(0_4 0_5 d_0 d_1 d_2 4_3 4_4 4_5 2_0 2_1) \\
2^0=(0_3 0_4 0_5 1_0 1_1 1_2 7_3 7_4 9_1 9_2) & a^0=(0_4 0_5 d_0 d_1 b_4 b_3 b_2 2_3 2_2 2_1) \\
3^0=(0_3 0_4 0_5 d_0 c_5 a_0 9_5 9_4 9_3 9_2) & b^0=(0_4 0_5 1_0 3_5 3_4 5_1 5_0 4_5 2_0 2_1) \\
4^0=(0_0 0_1 0_2 6_3 6_2 6_1 6_0 5_5 3_0 2_5) & c^0=(0_2 0_3 0_4 2_1 2_2 8_3 8_2 8_1 6_4 6_3) \\
5^0=(0_4 0_5 1_0 3_5 4_0 4_1 2_4 2_3 2_2 2_1) & d^0=(0_3 0_4 2_1 2_0 4_5 5_0 7_5 7_4 9_1 9_2) \\
6^0=(0_3 0_4 0_5 1_0 3_5 3_4 3_3 3_2 9_3 9_2) & e^0=(0_5 1_0 3_5 4_0 6_5 7_0 9_5 a_0 c_5 d_0) \\
7^0=(0_0 0_1 R_2 0_3 9_2 9_3 3_2 3_1 3_0 2_5) & f^0=(0_2 0_3 9_2 9_3 3_2 3_3 c_2 c_3 6_2 6_3)
\end{array}$$

where **(a)** hexadecimal notation of integers is used; **(b)** the first 14 10-cycles x^0, ($x = 0, \ldots, 13 = d$), yield corresponding 10-cycles x^j , ($j \in \mathbf{Z}_{15}$). via translation modulo 15 of all indexes; and **(c)** the last two cycles, e^0 and f^0, yield merely additional 10-cycles e^1, e^2, f^1 and f^2 by the same index translation. A corresponding auxiliary table as in the discussions for Pet, Hea and Pap above, in which the $\pm$ assignments are missing and left as an exercise for the reader is as follows:

$$\begin{array}{ll}
0^0:(2^0,4^a,1^1,1^e,3^7,2^1,8^9,b^c,b^0,8^8) & 8^0:(c^7,d^0,5^7,0^7,0^6,b^4,d^0,c^0,6^6,7^6) \\
1^0:(0^e,5^d,c^0,c^9,5^0,0^1,6^e,9^d,9^2,7^0) & 9^0:(1^d,4^d,2^c,2^4,3^4,1^2,a^4,b^0,b^c,a^0) \\
2^0:(0^e,0^0,7^d,9^3,d^7,c^1,c^7,d^0,9^b,6^0) & a^0:(9^b,d^b,5^9,c^b,6^8,7^2,c^9,5^0,d^8,9^0) \\
3^0:(6^c,d^8,e^0,5^9,6^9,0^8,6^6,c^9,6^0,9^b) & b^0:(6^0,d^b,9^3,0^3,5^0,7^2,d^0,9^0,0^0,5^0) \\
4^0:(7^c,c^d,7^6,0^5,7^3,5^2,e^e,d^d,7^0,9^2) & c^0:(1^0,2^8,8^8,4^2,a^6,1^6,2^e,8^0,3^6,a^4) \\
5^0:(3^6,4^d,b^e,8^b,a^6,1^2,1^0,a^0,8^8,a^0) & d^0:(a^4,b^0,4^2,3^7,b^4,a^7,8^0,2^0,2^8,8^9) \\
6^0:(3^6,b^0,3^3,1^1,3^9,a^7,f^3,8^9,3^0,2^0) & e^0:(3^6,4^2,3^9,4^5,3^c,4^8,3^0,4^b,3^3,4^e) \\
7^0:(4^9,a^d,f^3,8^9,4^3,2^2,4^c,b^d,4^0,1^0) & f^0:(7^0,6^0,7^3,6^3,7^6,6^6,7^9,6^9,7^c,6^c)
\end{array}$$

Let $A = (A_0, A_1, \ldots, A_g)$, $D = (D_0, D_2, \ldots, D_f)$, $C = (C_0, C_4, \ldots, C_d)$, $F = (F_0, F_8, \ldots, F_9)$ be 4 disjoint 17-cycles. Each $y = A, D, C, F$ has vertices y_i with i expressed as an heptadecimal index up to $g = 16$. We assume that i is advancing in 1,2,4,8 units mod 17, stepwise from left to right, respectively for $y = A, D, C, F$. Then the Biggs-Smith graph B-S is obtained by adding to the disjoint union $A \cup D \cup C \cup F$, for each $i \in \mathbf{Z}_{17}$, a 6-vertex tree T_i formed by the edge-disjoint union of paths $A_iB_iC_i$, $D_iE_iF_i$, and B_iE_i, where the vertices A_i, D_i, C_i, F_i are already present in the cycles A, D, C, F, respectively, and where the vertices B_i and E_i are new and introduced with the purpose of defining the tree T_i, for $0 \le i \le g = 16$. Now, $\mathcal{S}$ has the collection $\mathcal{C}_9$ of 9-cycles formed by:

$$\begin{array}{l|l}
S^0=(A_0 A_1 B_1 C_1 C_5 C_9 C_d C_0 B_0), & W^0=(A_0 A_1 B_1 E_1 F_1 F_9 F_0 E_0 B_0), \\
T^0=(C_0 C_4 B_4 A_4 A_3 A_2 A_1 A_0 B_0), & X^0=(C_0 C_4 B_4 E_4 D_4 D_2 D_0 E_0 B_0), \\
U^0=(E_0 F_0\ F_9 F_1 F_a F_2 E_2 D_2 D_0,) & Y^0=(E_0 B_0 A_0 A_1 A_2 B_2 E_2 D_2 D_0), \\
V^0=(E_0 D_0 D_2 D_4 D_6 D_8 E_8 F_8 F_0), & Z^0=(F_0\ F_8\ E_8 B_8 C_8 C_4 C_0 B_0 E_0),
\end{array}$$

and those 9-cycles obtained from these, as $S^x, \dots, Z^x$, by uniformly adding $x \in \mathbf{Z}_{17}$ mod 17 to all subindexes i of vertices y_i, so that $|\mathcal{C}_9| = 136$.

An auxiliary table presenting the form in which the 9-cycles above share the vertex sets of 3-arcs, either O-O or not, is shown below, in a fashion similar to those above for Pet, Hea, Pap and Fos, where minus signs are set but plus signs are tacit now:

S^0:($-T_1^e$, T_7^1, $-Z_4^1$, S_4^d, S_3^4, $-Z_3^9$, T_0^d, $-T_6^0$, U_8^0),	W^0:($-U_4^0$, W_2^8, W_1^9, $-U_3^1$, X_7^2, $-X_4^b$, Y_6^0, $-X_8^0$, X_5^9),
T^0:(S_6^4, $-S_0^3$, $-Y_2^2$, T_4^1, T_3^g, $-Y_1^0$, $-S_7^0$, S_1^g, V_8^0),	X^0:($-V_4^0$, X_2^f, X_1^2, $-V_3^4$, $-W_5^6$, W_8^8, $-Z_8^0$, W_4^f, $-W_7^0$),
U^0:(Y_3^g, $-U_6^1$, Z_7^1, $-W_3^g$, $-W_0^0$, Z_0^9, $-U_1^g$, Y_0^0, S_8^0),	Y^0:(U_7^0, $-T_5^0$, $-T_2^f$, U_0^1, $-Y_8^2$, V_2^f, W_6^0, V_5^0, $-Y_4^f$),
V^0:($-Z_2^d$, $-V_6^4$, Y_5^2, $-X_3^d$, $-X_0^0$, Y_7^0, $-V_1^d$, $-Z_5^0$, T_8^0),	Z^0:(U_5^8, $-Z_6^8$, $-V_0^4$, $-S_5^8$, $-S_2^g$, $-V_7^0$, $-Z_1^9$, U_2^g, $-X_6^0$),

This table is extended by adding $x \in \mathbf{Z}_{17}$ uniformly mod 17 to all superindexes, confirming that B-S is not $\{\vec{C}_9\}_{\vec{P}_4}$-UH.. $\square$

4 Separator digraphs of 7 CDT graphs

For each of the 7 CDT graphs G that are $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs according to Theorem 3, the following construction yields a corresponding digraph $\mathcal{S}(G)$ of outdegree and indegree two and having underlying cubic graph structure and the same automorphism group of G. The vertices of $\mathcal{S}(G)$ are defined as the $(k-1)$-arcs of G. We set an arc in $\mathcal{S}(G)$ from each vertex $a_1a_2\dots a_{k-1}$ into another vertex $a_2\dots a_{k-1}a_k$ whenever there is an oriented g-cycle $(a_1a_2\dots a_{k-1}a_k\dots)$ in the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC provided by Theorem 3 to G. Thus each oriented g-cycle in the mentioned $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC yields an oriented g-cycle of $\mathcal{S}(G)$. In addition we set an edge e in $\mathcal{S}(G)$ for each transposition of a $(k-1)$-arc of G, say $h = a_1a_2\dots a_{k-1}$, taking it into $h^{-1} = a_{k-1}a_{k-2}\dots a_1$. Thus the ends of e are h and h^{-1}. As usual, the edge e is considered composed by two O-O arcs.

The polyhedral graphs G here are the tetrahedral graph $G = K_4$, the 3-cube graph $G = Q_3$ and the dodecahedral graph $G = \Delta$. The corresponding graphs $\mathcal{S}(G)$ have their underlying graphs respectively being the truncated-polyhedral graphs of the corresponding dual-polyhedral graphs that we can refer as the truncated tetrahedron, the truncated octahedron and the truncated icosahedron. In fact:

(A) $\mathcal{S}(K_4)$ has vertices $01, 02, 03, 12, 13, 23, 10, 20, 30, 21, 31, 32$; the cycles (123), (210), (301), (032) of the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of K_4 give place to the oriented 3-cycles $(12, 23, 31)$, $(21, 10, 02)$, $(30, 01, 13)$, $(03, 32, 20)$ of $\mathcal{S}(K_4)$; the additional edges of $\mathcal{S}(K_4)$ are $(01, 10)$, $(02, 20)$, $(03, 30)$, $(12, 21)$, $(13, 31)$, $(23, 32)$.

(B) The of oriented cycles of $\mathcal{S}(Q_3)$ corresponding to the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of Q_3 are $(01, 13, 32, 20)$, $(10, 04, 45, 51)$, $(31, 15, 57, 73)$,$(23, 37, 76, 62)$, $(02, 26, 64, 40)$, $(46, 67, 75, 54)$; the additional edges of $\mathcal{S}(Q_3)$ are $(01, 10)$, $(23, 32)$, $(45, 54)$, $(67, 76)$, $(02, 20)$, $(13, 31)$, $(46, 64)$, $(57, 75)$, $(04, 40)$, $(15, 51)$, $(26, 62)$, $(37, 73)$.

(C) The oriented cycles of $\mathcal{S}(\Delta)$ corresponding to the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of Δ are $(a_0a_1, a_1a_2, a_2a_3, a_3a_4, a_4\ a_0)$, $(c_4c_3, c_3c_2, c_2c_1, c_1c_0, c_0c_4)$, and both $(a_xd_x, d_xb_{x-2}, b_{x-2}d_{x+1}, d_{x+1}\ a_{x+1}, a_{x+1}a_x)$ and $(d_xb_{x+2}, b_{x+2}c_{x+2}, c_{x+2}c_{x-2},\ c_{x-2}b_{x-2}, b_{x-2}d_x)$, for each $x \in \mathbf{Z}_5$;

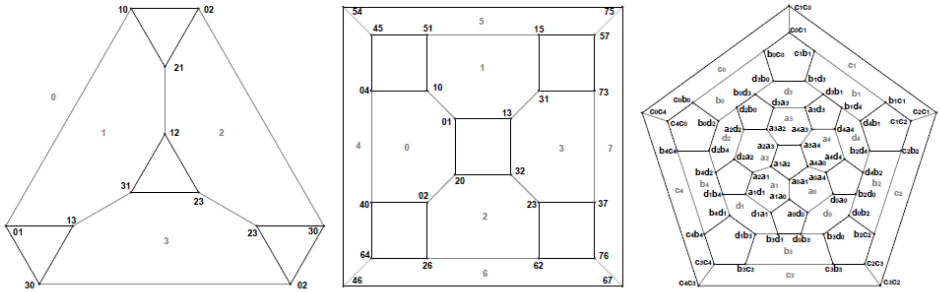

Figure 1: $\mathcal{S}(K_4)$, $\mathcal{S}(Q_3)$ and $\mathcal{S}(\Delta)$

the additional edges are $(a_xa_{x+1}, a_{x+1}a_x)$, $(c_xc_{x+1}, c_{x+1}c_x)$, $(a_xd_x, d_x\ a_x)$, $(d_xb_{x+2}, b_{x+2}d_x)$, $(b_xd_{x+2}, d_{x+2}b_x)$, and (b_xc_x, c_xb_x), for each $x \in \mathbf{Z}_5$.

Among the 7 CDT graphs G that are $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs, the polyhedral graphs treated above are exactly those having arc-transitivity $k = 2$. Figure 1 contains representations of these graphs $\mathcal{S}(G)$, namely $\mathcal{S}(K_4)$, $\mathcal{S}(Q_3)$ and $\mathcal{S}(\Delta)$, with the respective 3-cycles, 4-cycles and 5-cycles in black trace to be considered clockwise oriented, but for the external cycles in the cases $\mathcal{S}(Q_3)$ and $\mathcal{S}(\Delta)$, to be considered counterclockwise oriented. The remaining edges (to be referred as transposition edges) are gray colored and considered bidirectional. The cycles having alternate black and gray edges here, arising respectively from arcs from the oriented cycles and from the transposition edges, are 6-cycles. Each such 6-cycle has its vertices sharing the notation, indicated in gray, of a unique vertex of the corresponding G. Each vertex of G is used as such gray 6-cycle indication.

The truncated tetrahedron, truncated octahedron and truncated icosahedron, oriented as indicated for Figure 1, are the Cayley digraphs of the groups A_4, S_4 and A_5, with respective generating sets $\{(123), (12)(34)\}$, $\{(1234), (12)\}$ and $\{(12345), (23)(45)\}$. Thus

$$\mathcal{S}(K_4)\equiv Cay(A_4,\{(123),(12)(34)\}),\mathcal{S}(Q_3)\equiv Cay(S_4,\{(1234),(12)\}),\mathcal{S}(\Delta)\equiv Cay(A_5,\{(12345),(23)(45)\}).$$

(D) The oriented cycles of $\mathcal{S}(K_{3,3})$ corresponding to the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of $K_{3,3}$ are:

$$\begin{array}{lll}(123,234,341,412), & (321,210,103,032), & (432,325,254,543),\\ (143,430,301,014), & (214,145,452,521), & (012,125,250,501),\\ (523,230,305,052), & (034,345,450,503), & (541,410,105,054);\end{array}$$

and additional edges of $\mathcal{S}(K_{3,3})$ are:

$$\begin{array}{l}(123,321),(234,432),(341,143),(412,214),(210,012),(103,301),\\ (032,230),(325,523),(254,452),(543,345),(430,034),(014,410),\\ (145,541),(521,125),(250,052),(501,105),(305,503),(450,054).\end{array}$$

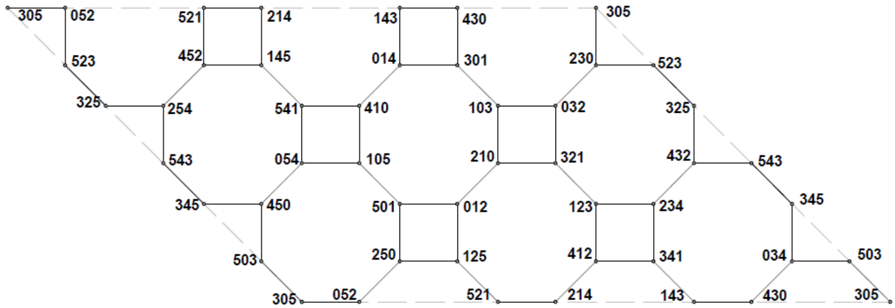

Figure 2: $\mathcal{S}(K_{3,3})$

A cut-out toroidal representation of $\mathcal{S}(K_{3,3})$ is in Figure 2, where black-traced 4-cycles are considered oriented clockwise, corresponding to the oriented 4-cycles in the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of $K_{3,3}$, and where gray edges represent transposition edges of $\mathcal{S}(K_{3,3})$, which gives place to the alternate 8-cycles, constituted each by an alternation of 4 transposition edges and 4 arcs of the oriented 4-cycles. In $\mathcal{S}(K_{3,3})$, there are 9 4-cycles and 9 8-cycles. Now, the group $<(0,5,4,1)(2,3),(0,2)(1,5)>$ has order 36 and acts regularly on the vertices of $\mathcal{S}(K_{3,3})$. For example, $(0,2)(1,5)$ stabilizes the edge $(145,541)$, and the permutation $(0,5,4,1)(2,3)$ permutes (clockwise) the black oriented 4-cycle $(541,410,105,054)$. Thus $\mathcal{S}(K_{3,3})$ is a Cayley digraph. Also, observe the oriented 9-cycles in $\mathcal{S}(K_{3,3})$ obtained by traversing alternatively 2-arcs in the oriented 4-cycles and transposition edges; there are 6 such oriented 9-cycles.

(E) The collection of oriented cycles of $\mathcal{S}(Des)$ corresponding to the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of Des is formed by the following oriented 6-cycles, where $x \in \mathbf{Z}_5$:

$$\begin{array}{l}
(x_0x_1x_2,\ x_1x_2x_3,\ x_2x_3x_0^1,\ x_3x_0^1x_3^4,\ x_0^1x_3^4x_0,\ x_3^4x_0x_1),\\
(x_1x_0x_3^4,\ x_0x_3^4x_2^4,\ x_3^4x_2^4x_1^2,\ x_2^4x_1^2x_2^2,\ x_1^2x_2^2x_1,\ x_2^2x_1x_0),\\
(x_2x_1x_0,\ x_1x_0x_3^3,\ x_0x_3^3x_2^3,\ x_3^3x_2^3x_1^3,\ x_2^3x_1^3x_2,\ x_1^3x_2x_1),\\
(x_0x_3^4x_0^1,\ x_3^4x_0^1x_1^1,\ x_0^1x_1^1x_2^3,\ x_1^1x_2^3x_3^3,\ x_2^3x_3^3x_0,\ x_3^3x_0x_3^4),
\end{array}$$

respectively for the 6-cycles A^x, B^x, C^x, D^x, where x_i^j stands for $(x+j)_i$; each of the participant vertices here is an end of a transposition edge. Figure 3 represents a subgraph $\mathcal{S}(M_3)$ of $\mathcal{S}(Des)$ associated with the matching M_3 of Des indicated in its representation "inside" the left-upper "eye" of the figure, where vigesimal integer notation is used (up to $j = 19$); in the figure, additional intermittent edges were added that form 12 square pyramids, 4 such edges departing from a corresponding extra vertex; so, 12 extra vertices appear that can be seen as the vertices of a cuboctahedron whose edges are 3-paths with inner edge in $\mathcal{S}(M_3)$ and intermittent outer edges. There is a total of 5 matchings, like M_3, that we denote M_x, where $x = 3, 7, b, f, j$. In fact, $\mathcal{S}(Des)$ is obtained as the union $\bigcup\{\mathcal{S}(M_x); x = 3, 7, b, f, j\}$. Observe that the components of the subgraph induced by the matching M_x in Des are at mutual distance 2 and that M_x can be divided into three pairs of edges with the ends of each pair at minimum distance 4, facts that can be used to establish the properties of $\mathcal{S}(Des)$.

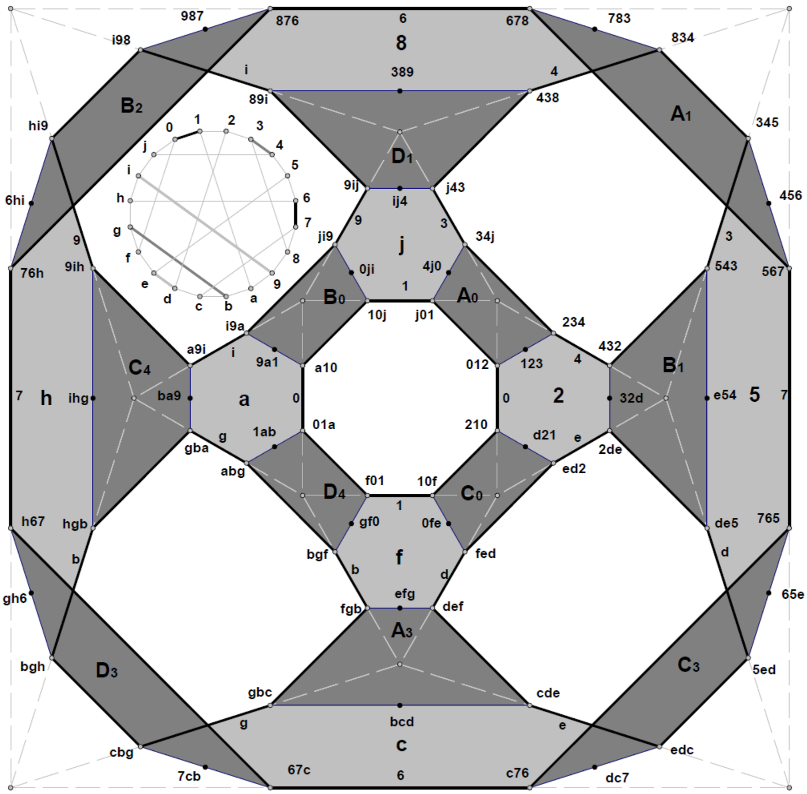

Figure 3: M_3 and the subgraph of $\mathcal{S}(Des)$ associated to it

In Figure 3 there are: 12 oriented 6-cycles (dark-gray interiors); 6 alternate 8-cycles (thick-black edges); and 8 9-cycles with alternate 2-arcs and transposition edges (light-gray interiors). The 6-cycles are denoted by means of the associated oriented 6-cycles of Des. Each 9-cycle has its vertices sharing the notation of a vertex of $V(Des)$ and this is used to denote it. Each edge e in M_3 has associated a closed walk in Des containing every 3-path with central edge e; this walk can be used to determine a unique alternate 8-cycle in $\mathcal{S}(M_3)$, and viceversa. Each 6-cycle has two opposite (black) vertices of degree two in $\mathcal{S}(M_3)$. In all, $\mathcal{S}(Des)$ contains 120 vertices; 360 arcs amounting to 120 arcs in oriented 6-cycles and 120 transposition edges; 20 dark-gray 6-cycles; 30 alternate 8-cycles; and 20 light-gray 9-cycles. By filling the 6-cycles and 8-cycles here with 2-dimensional faces, then the 120 vertices, 180 edges (of the underlying cubic graph) and resulting $20 + 30 = 50$ faces yield a surface of Euler characteristic $120 - 180 + 50 = -10$, so this surface genus is 6. The automorphism group of Des is $G = S_5 \times \mathbf{Z}_2$. Now, G contains three subgroups of index 2: two isomorphic to to S_5 and one isomorphic to $A_5 \times \mathbf{Z}_2$. One of the two subgroups of G isomorphic to S_5 (the diagonal copy) acts regularly on the vertices of $\mathcal{S}(Des)$ and hence $\mathcal{S}(Des)$ is a Cayley digraph.

(F) The collection of oriented cycles of $\mathcal{S}(Cox)$ corresponding to the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of Cox is formed by oriented 7-cycles, such as:

$$\underline{0^1} = (u_1u_2u_3, u_2u_3u_4, u_3u_4u_5, u_4u_5u_6, u_5u_6u_0, u_6u_0u_1, u_0u_1u_2),$$

and so on for the remaining oriented 7-cycles $\underline{x^y}$ with $x \in \{0, \dots, 7\}$ and $y \in \{1, 2, 3\}$, based on the corresponding table in the proof of Theorem 3. Moreover, each vertex of $\mathcal{S}(Cox)$ is adjacent via a transposition edge to its reversal vertex. Thus $\mathcal{S}(Cox)$ has: underlying cubic graph; indegree = outdegree = 2; 168 vertices; 168 arcs in 24 oriented 7-cycles; 84 transposition edges; and 42 alternate 8-cycles. Its underlying cubic graph has 252 edges. From this information, by filling the 7- and 8-cycles mentioned above with 2-dimensional faces, we obtain a surface with Euler characteristic $168 - 252 + (24 + 42) = -18$, so its genus is 10. On the other hand, $\mathcal{S}(Cox)$ is the Cayley digraph of the automorphism group of the Fano plane, namely $PSL(2, 7) = GL(3, 2)$ [4], of order 168, with a generating set of two elements, of order 2 and 7, representable by the 3×3-matrices $(100, 001, 010)^T$ and $(001, 101, 010)^T$ over the field F_2, where T stands for transpose.

Figure 4 depicts a subgraph of $\mathcal{S}(Cox)$ containing in its center a (twisted) alternate 8-cycle that we denote (in gray) z_1u_1, and, around it, four oriented 7-cycles adjacent to it, (namely $\underline{1^1}$, $\underline{7^1}$, $\underline{2^3}$, $\underline{6^3}$, denoted by their corresponding oriented 7-cycles in Cox, also in gray), plus four additional oriented 7-cycles (namely $\underline{0^0}$, $\underline{3^2}$, $\underline{4^1}$, $\underline{5^2}$), related to four 9-cycles mentioned below. Black edges represent arcs, and the orientation of these 8 7-cycles is taken clockwise, with only gray edges representing transposition edges of $\mathcal{S}(Cox)$. Each edge of Cox determines an alternate 8-cycle of $\mathcal{S}(Cox)$. In fact, Figure 4 contains not only the alternate 8-cycle corresponding to the edge z_1u_1 mentioned above, but also those corresponding to the edges u_1u_2, v_1z_1, t_1z_1 and u_0u_1. These 8-cycles and the 7-cycles in the figure show that alternate 8-cycles C and C' adjacent to a particular alternate 8-cycle C'' in $\mathcal{S}(Cox)$ on opposite edges e and e' of C'' have the same opposite edge e'' both to e and e' in C and C', respectively. There are two instances of this property in Figure 4, where the two edges taking the place of e'' are the large central diagonal gray ones, with C'' corresponding to u_1z_1. As in (E) above, the fact that each edge e of Cox determines

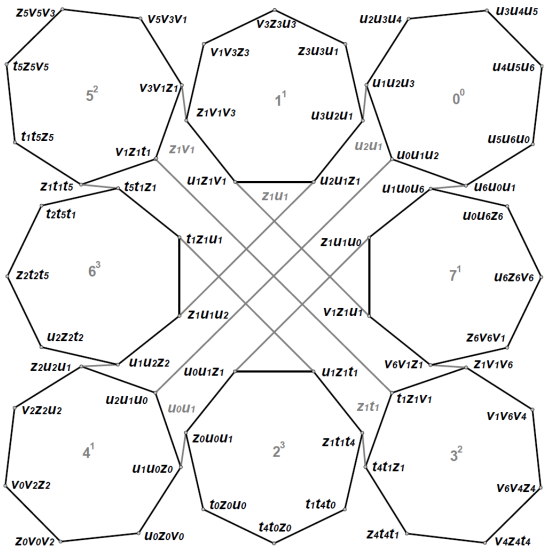

Figure 4: A subdigraph of $\mathcal{S}(Cox)$ associated with an edge of Cox

an alternate 8-cycle of $\mathcal{S}(Cox)$ is related with the closed walk that covers all the 3-paths having e as central edge, and the digraph $\mathcal{S}(Cox)$ contains 9-cycles that alternate 2-arcs in the oriented 7-cycles with transposition edges. In the case of Figure 4, these 9-cycles are, in terms of the orientation of the 7-cycles:

$$(u_2u_1z_1, u_1z_1v_1, z_1v_1v_3, v_3v_1z_1, v_1z_1t_1, z_1t_1t_5, t_5t_1z_1, t_1z_1u_1, z_1u_1u_2),$$
$$(v_1z_1u_1, z_1u_1u_0, u_1u_0u_6, u_6u_0u_1, u_0u_1u_2, u_1u_2u_3, u_3u_2u_1, u_2u_1z_1, u_1z_1v_1),$$
$$(u_0u_1z_1, u_1z_1t_1, z_1t_1t_4, t_4t_1z_1, t_1z_1v_1, z_1v_1v_6, v_6v_1z_1, v_1z_1u_1, z_1u_1u_0),$$
$$(t_1z_1u_1, z_1u_1u_2, u_1u_2z_2, z_2u_2u_1, u_2u_1u_0, u_1u_0z_0, z_0u_0u_1, u_0u_1z_1, u_1z_1t_1).$$

A convenient description of alternate 8-cycles, as those denoted in gray in Figure 4 by the edges $z_1u_1, z_1v_1, u_2u_1, u_0u_1, z_1t_1$ of Cox, is given by indicating the successive passages through arcs of the oriented 7-cycles, with indications by means of successive subindexes in the order of presentation of their composing vertices, which for those 5 alternate 8-cycles looks like:

$$(1^1_{60}, 7^1_{56}, 2^3_{60}, 6^3_{56}), \quad (5^2_{12}, 3^2_{23}, 7^1_{45}, 1^1_{01}), \quad (0^1_{01}, 1^1_{56}, 6^3_{60}, 4^1_{56}), \quad (2^3_{56}, 7^1_{60}, 0^0_{56}, 4^1_{60}), \quad (3^2_{12}, 5^2_{23}, 6^3_{45}, 2^3_{01}).$$

In a similar fashion, the four bi-alternate 9-cycles displayed just above can be presented by means of the shorter expressions:

$$(1^1_{61}, 5^2_{13}, 6^3_{46}), \quad (7^1_{50}, 0^1_{50}, 1^1_{50}), \quad (2^3_{61}, 3^2_{13}, 7^1_{46}), \quad (6^3_{50}, 4^1_{50}, 2^3_{50}).$$

By the same token, there are twenty four tri-alternate 28-cycles, one of which is expressible as:

$$(0^1_{03}, 6^1_{03}, 3^2_{40}, 2^3_{36}, 5^3_{36}, 4^2_{62}, 1^1_{40}).$$

At this point, we observe that $\mathcal{S}(K_4)$, $\mathcal{S}(Q_3)$ and $\mathcal{S}(\Delta)$ have alternate 6-cycles, while $\mathcal{S}(K_{3,3})$, $\mathcal{S}(Des)$ and $\mathcal{S}(Cox)$ have alternate 8-cycles.

(G) The collection of oriented cycles of $\mathcal{S}(Tut)$ corresponding to the $\{\eta\vec{C}_g\}_{\vec{P}_k}$-OOC of Tut is formed by oriented 8-cycles, such as: $\underline{A^0} =$

$$(4_5 0_0 0_1 0_2 0_3, 0_0 0_1 0_2 0_3 0_4, 0_1 0_2 0_3 0_4 0_5, 0_2 0_3 0_4 0_5 1_0, 0_3 0_4 0_5 1_0 4_5, 0_4 0_5 1_0 4_5 0_0, 0_5 1_0 4_5 0_0 0_1, 1_0 4_5 0_0 0_1 0_2)$$

and so on for the remaining oriented 8-cycles $\underline{X^y}$ with $X \in \{A, \dots, R\}$ and $y \in \mathbf{Z}_5$ based on the corresponding table in the proof of Theorem 3. Moreover, each vertex of $\mathcal{S}(Tut)$ is adjacent via a transposition edge to its reversal vertex. Thus $\mathcal{S}(Tut)$ has: underlying cubic graph; indegree = outdegree = 2; 720 vertices; 720 arcs in 90 oriented 8-cycles; 360 transposition edges; and 180 alternate 8-cycles, (36 of which are displayed below); its underlying cubic graph has 1080 edges. From this information, by filling the 900 8-cycles above with 2-dimensional faces, a surface with Euler characteristic $720 - 1080 + 240 = -120$ is obtained, so genus = 61. On the other hand, the automorphism group of Tut is the projective semilinear group $G = P\Gamma L(2,9)$ [13], namely the group of collineations of the projective line $PG(1,9)$. The group G contains exactly three subgroups of index 2 (and so of order 720), one of which (namely M_{10}, the Mathieu group of order 10, acts regularly on the vertices of $\mathcal{S}(Tut)$. Thus $\mathcal{S}(Tut)$ is a Cayley digraph.

A fifth of the 180 alternate 8-cycles of $\mathcal{S}(Tut)$ can be described by presenting in each case the successive pairs of vertices in each oriented 8-cycle $\underline{X^y}$ as follows, each such pair denoted by means of the notation $X^y_{u(u+1)}$, where u stands for the 4-arc in position u in $\underline{X^y}$, with 0 indicating the first position:

$(A^0_{01},$	$M^0_{34},$	$B^4_{34},$	$K^0_{12})$	$(A^0_{12},$	$P^1_{01},$	$I^0_{70},$	$M^0_{23})$	$(A^0_{23},$	$H^0_{34},$	$B^1_{70},$	$P^1_{70})$
$(A^0_{34},$	$J^0_{70},$	$L^3_{70},$	$H^0_{23})$	$(A^0_{45},$	$E^0_{70},$	$P^0_{45},$	$J^0_{67})$	$(A^0_{56},$	$E^1_{45},$	$F^1_{70},$	$E^0_{67})$
$(A^0_{67},$	$G^0_{45},$	$M^1_{67},$	$E^1_{34})$	$(A^0_{70},$	$K^0_{23},$	$L^2_{12},$	$G^0_{34})$	$(B^0_{01},$	$C^2_{34},$	$Q^3_{23},$	$P^0_{67})$
$(B^0_{12},$	$R^3_{34},$	$N^1_{56},$	$C^2_{23})$	$(B^0_{23},$	$M^1_{45},$	$Q^2_{67},$	$R^3_{23})$	$(B^0_{45},$	$D^3_{67},$	$R^1_{70},$	$K^1_{01})$
$(B^0_{56},$	$D^0_{34},$	$O^0_{56},$	$D^3_{56})$	$(B^0_{67},$	$H^4_{45},$	$C^4_{67},$	$D^0_{23})$	$(C^0_{01},$	$G^3_{70},$	$H^3_{70},$	$M^0_{01})$
$(C^0_{12},$	$N^4_{67},$	$F^3_{45},$	$G^3_{67})$	$(C^0_{45},$	$L^0_{67},$	$J^2_{01},$	$Q^1_{12})$	$(C^0_{56},$	$H^0_{56},$	$O^4_{34},$	$L^0_{56})$
$(C^0_{70},$	$M^0_{12},$	$I^0_{01},$	$D^1_{12})$	$(D^0_{01},$	$I^4_{12},$	$O^1_{12},$	$I^3_{56})$	$(D^0_{45},$	$O^2_{67},$	$L^1_{45},$	$O^0_{45})$
$(D^0_{70},$	$I^3_{67},$	$P^4_{12},$	$R^3_{67})$	$(E^0_{01},$	$N^4_{01},$	$K^1_{45},$	$P^0_{34})$	$(E^0_{12},$	$N^2_{34},$	$F^3_{34},$	$N^4_{70})$
$(E^0_{23},$	$M^0_{70},$	$H^3_{01},$	$N^2_{23})$	$(E^0_{56},$	$F^1_{01},$	$Q^2_{45},$	$F^0_{67})$	$(F^0_{12},$	$J^3_{56},$	$P^3_{56},$	$Q^1_{34})$
$(F^0_{23},$	$N^4_{45},$	$R^1_{45},$	$J^3_{45})$	$(F^0_{56},$	$Q^2_{56},$	$M^1_{56},$	$G^0_{56})$	$(G^0_{01},$	$I^1_{45},$	$O^4_{23},$	$H^0_{67})$
$(G^0_{12},$	$Q^1_{01},$	$J^2_{12},$	$I^1_{34})$	$(G^0_{23},$	$L^2_{23},$	$R^2_{12},$	$Q^1_{70})$	$(H^0_{12},$	$L^3_{01},$	$K^1_{34},$	$N^4_{12})$
$(I^0_{23},$	$J^1_{23},$	$K^1_{67},$	$O^2_{01})$	$(J^0_{34},$	$R^3_{56},$	$P^4_{23},$	$K^0_{56})$	$(K^0_{70},$	$R^0_{01},$	$L^0_{34},$	$O^1_{70})$

Again, as in the previously treated cases, we may consider the oriented paths that alternatively traverse two arcs in an oriented 8-cycle and then a transposition edge, repeating this operation until a closed path is formed. It happens that all such bi-alternate cycles are 12-cycles. For example with a notation akin to the one in the last table, we display the first row of the corresponding table of 12-cycles:

$(A^0_{02},$	$P^1_{02},$	$R^0_{60},$	$K^0_{02})$	$(A^0_{13},$	$H^0_{35},$	$C^0_{60},$	$M^0_{13})$	$(A^0_{24},$	$J^0_{71},$	$Q^4_{13},$	$P^1_{60})$
(......,	......,	......,	......)	(......,	......,	......,	......)	(......,	......,	......,	......)

As in the case of the alternate 8-cycles above, which are 180, there are 180 bi-alternate 12-cycles in $\mathcal{S}(Tut)$. On the other hand, an example of a tri-alternate 32-cycle in $\mathcal{S}(Tut)$ is given by:

$$(A^0_{03},\quad H^0_{36},\quad O^4_{36},\quad D^2_{50},\quad I^0_{61},\quad D^1_{14},\quad O^1_{50},\quad K^0_{72}).$$

There is a total of 90 such 32-cycles. Finally, an example of a tetra-alternate 15-cycle in $\mathcal{S}(Tut)$ is given by $(A^0_{04}, J^0_{73}, K^0_{62})$, and there is a total of 240 such 15-cycles. More can be said about the relative structure of all these types of cycles in $\mathcal{S}(Tut)$.

The automorphism groups of the graphs $\mathcal{S}(G)$ in items (A)-(G) above coincide with those of the corresponding graphs G because the construction of $\mathcal{S}(G)$ depends solely on the structure of G as analyzed in Section 3 above. Salient properties of the graphs $\mathcal{S}(G)$ are contained in the following statement.

Theorem 4.1. For each CDT graph that is a $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraph, $\mathcal{S}(G)$ is: **(a)** a vertex-transitive digraph with indegree $=$ outdegree $= 2$, underlying cubic graph and the automorphism group of G; **(b)** a $\{\vec{C}_g, \vec{C}_2\}$-ultrahomogeneous digraph, where $\vec{C}_g$ stands for oriented g-cycle coincident with its induced subdigraph and each vertex is the intersection of exactly one such $\vec{C}_g$ and one $\vec{C}_2$; **(c)** a Cayley digraph. Moreover, the following additional properties hold, where $s(G) =$ subjacent undirected graph of $\mathcal{S}(G)$:

(A) $\mathcal{S}(K_4) \equiv Cay(A_4, \{(123), (12)(34)\})$, $s(K_4) =$ truncated octahedron;

(B) $\mathcal{S}(Q_3) \equiv Cay(S_4, \{(1234), (12)\})$, $s(Q_3) =$ truncated octahedron;

(C) $\mathcal{S}(\Delta) \equiv Cay(A_5, \{(12345), (23)(45)\})$, $s(\Delta) =$ truncated icosahedron;

(D) $\mathcal{S}(K_{3,3})$ is the Cayley digraph of the subgroup of S_6 on the vertex set $\{0,1,2,3,4,5\}$ generated by $(0,5,4,1)(2,3)$ and $(0,2)(1,5)$ and has a toroidal embedding whose faces are delimited by 9 oriented 4-cycles and 9 alternate 8-cycles;

(E) $\mathcal{S}(Des)$ is the Cayley digraph of a diagonal copy of S_5 in the automorphism group $S_5 \times \mathbf{Z}_2$ of Des and has a 6-toroidal embedding whose faces are delimited by 20 oriented 6-cycles and 30 alternate 8-cycles;

(F) $\mathcal{S}(Cox) \equiv Cay(GL(3,2), \{(100, 001, 010)^T, (001, 101, 010)^T\})$, has a 10-toroidal embedding whose faces are delimited by 24 oriented 7-cycles and 42 alternate 8-cycles;

(G) $\mathcal{S}(Tut)$ is the Cayley digraph of a subgroup M_{10} of order 2 in the automorphism group $P\Gamma L(2,9)$ of Tut and has a 61-toroidal embedding whose faces are delimited by 90 oriented 8-cycles and 180 alternate 8-cycles.

Corollary 4.2. The bi-alternate cycles in the graphs $\mathcal{S}(G)$ above are 9-cycles unless either $G = Q_3$ or $G = \Delta$, in which cases they are respectively 12-cycles and 15-cycles.

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A parallel algorithm for computing the critical independence number and related sets

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Abstract

An independent set I_c is a *critical independent set* if $|I_c| - |N(I_c)| \geq |J| - |N(J)|$, for any independent set J. The *critical independence number* of a graph is the cardinality of a maximum critical independent set. This number is a lower bound for the independence number and can be computed in polynomial-time. The existing algorithm runs in $\mathcal{O}(n^{2.5}\sqrt{m/\log n})$ time for a graph G with $n = |V(G)|$ vertices and m edges. It is demonstrated here that there is a parallel algorithm using n processors that runs in $\mathcal{O}(n^{1.5}\sqrt{m/\log n})$ time. The new algorithm returns the union of all maximum critical independent sets. The graph induced on this set is a König-Egerváry graph whose components are either isolated vertices or which have perfect matchings.

Keywords: Critical independent set, critical independence number, independence number, matching number, König-Egerváry graph.

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1 Introduction

A new faster parallel algorithm is given for finding maximum critical independent sets and calculating the critical independence number of an arbitrary graph.

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The following notation is used throughout: the vertex set of a graph G is $V(G)$, the order of G is $n = n(G) = |V(G)|$, the set of neighbors of a vertex v is $N_G(v)$ (or simply $N(v)$ if there is no possibility of ambiguity), the set of neighbors of a set $S \subseteq V(G)$ in G is $N_G(S) = \cup_{u\in S} N(u)$ (or simply $N(S)$ if there is no possibility of ambiguity), the set $N[S] = N(S) \cup S$, and the graph induced on S is $G[S]$. All graphs are assumed to be finite and simple.

A set $I \subseteq V(G)$ of vertices is an *independent set* if no pair of vertices in the set are adjacent. The *independence number* $\alpha = \alpha(G)$ is cardinality of a maximum independent set (MIS) of vertices in G. An independent set of vertices I_c is a *critical independent set* if $|I_c| - |N(I_c)| \geq |J| - |N(J)|$, for any independent set J. A *maximum critical independent set* (MCIS) is a critical independent set of maximum cardinality. The *critical independence number* of a graph G, denoted $\alpha' = \alpha'(G)$, is the cardinality of a maximum critical independent set. If I_c is a maximum critical independent set, and so $\alpha'(G) = |I_c|$, then clearly $\alpha' \leq \alpha$. The *critical difference* d is $max\{|I_c| - |N(I_c)| : I_c \text{ is an independent set}\}$.

Critical independent sets are of interest for both practical and theoretical reasons. By a theorem of Butenko and Trukhanov, any critical independent set can be extended to a maximum independent set [4]. Zhang and Ageev gave polynomial-time algorithms for finding critical independent sets [17, 1]. Thus, finding a critical independent set is a polynomial-time technique for reducing the search for the well-known widely-studied NP-hard problem of finding a maximum independent set in a graph [7]. Maximum critical independent sets are central in the investigation of the structure of maximum independent sets, a connection via the Independence Decomposition Theorem, recounted in the next section.

The existing algorithm for finding a MCIS and calculating α' runs in $\mathcal{O}(n^{2.5}\sqrt{m/\log n})$ time [11]. It is demonstrated here that there is a parallel algorithm using n processors that runs in $\mathcal{O}(n^{1.5}\sqrt{m/\log n})$ time. The new algorithm finds the set H of vertices which are in *some* maximum critical independent set, that is, the union of all MCISs. The graph induced on this set is a König-Egerváry graph whose non-trivial components each have a perfect matching.

2 The set H of vertices in some MCIS

The correctness of the algorithm presented in the next section depends on the following decomposition theorem, a corollary of, and equivalent to, the Independence Decomposition Theorem in [12]. A *matching* in a graph is a set of pairwise non-incident (or independent) edges. The *matching number* μ of a graph is the cardinality of a maximum matching. A König-Egerváry graph is one where $\alpha + \mu = n$.

Theorem 2.1. *(Larson, [12]) For any graph G, there is a unique set $X \subseteq V(G)$ such that*

1. $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
2. *$G[X]$ is a König-Egerváry graph,*
3. *for every non-empty independent set I in $G[X^c]$, $|N(I)| > |I|$, and*
4. *for every maximum critical independent set J_c of G, $X = J_c \cup N(J_c)$.*

According to the theorem there is a unique set $X \subseteq V(G)$ such that, for any maximum critical independent set I_c, $I_c \cup N(I_c) = X$. For any graph G let $X = X(G)$ be the set guaranteed by Theorem 2.1. Call $G[X]$ the *distinguished König-Egerváry subgraph* of G. König-Egerváry graphs were first characterized by Deming [6] and Sterboul [16] in the

1970s. The first author's Graffiti.pc program conjectured (number 329 in [5]) a characterization in terms of the critical independence number: a graph G is a König-Egerváry graph if, and only if, $\alpha(G) = \alpha'(G)$. The conjecture was first proven by Larson in [12]. In [14] Levit & Mandrescu extended the statement of this result as follows.

Theorem 2.2. *(Levit & Mandrescu, [14]) The following are equivalent:*

1. *G is a König-Egerváry graph,*
2. *there is a maximum independent set of G that is a MCIS,*
3. *every maximum independent set of G is a MCIS,*

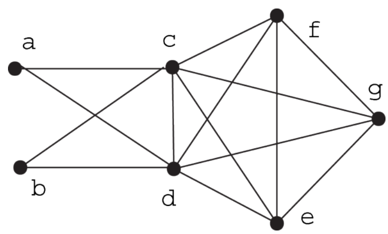

Figure 1: The vertices $I_c = \{a, b\}$ form a (maximum cardinality) critical independent set; this set of vertices can be extended to a maximum independent set of the graph. According to Theorem 2.1 the sets $X = I_c \cup N(I_c) = \{a, b, c, d\}$ and $X^c = V \setminus X = \{e, f, g\}$ induce a decomposition of the graph into a König-Egerváry subgraph $G[X]$ and one, $G[X^c]$, where every non-empty independent set of vertices I has more than $|I|$ neighbors.

It will now be shown that the graph induced on the set H (the union of all MCISs) is König-Egerváry. This fact will be used in the proof of correctness of the parallel algorithm. While the class of König-Egerváry graphs contains all the bipartite graphs (by the König-Egerváry Theorem, [15]) and subgraphs of bipartite graphs are bipartite, it is not true in general that subgraphs of König-Egerváry graphs are König-Egerváry. So it is worth noting that $G[X]$ is König-Egerváry, $G[H]$ is a subgraph of $G[X]$, and $G[H]$ is König-Egerváry.

The following results are required in the proof of Theorem 2.5.

Lemma 2.3. *(The Matching Lemma, [11]) If I is a critical independent set, then there is a matching from $N(I)$ to I.*

Let $\Omega = \Omega(G)$ be the set of maximum independent sets in G. The *core* of a graph G, denoted $core(G)$, is defined to be $\cap\{S : S \in \Omega\}$, namely, the set of vertices which are in every maximum independent set; and $\xi = \xi(G) = |core(G)|$. This notation follows [3].

Theorem 2.4. *(Levit & Mandrescu, [13]) If G is a König-Egerváry graph, then G has a perfect matching if, and only if, $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V(G) - S : S \in \Omega\}|$.*

Theorem 2.5. *If I_c is a maximum critical independent set of a graph G, $X = I_c \cup N(I_c)$, and H is the union of all maximum critical independent sets, then*

1. *$H \cup N(H) = X$,*
2. *$G[H]$ is a König-Egerváry graph,*

3. *I is a maximum independent set of $G[H]$ if, and only if, I is a MCIS of G and $\alpha'(G) = \alpha(G[H])$,*
4. *The non-trivial components of $G[H]$ have a perfect matching,*
5. *If I_0 are the isolated vertices in $G[H]$ then $\alpha(G[H]) = |I_0| + \frac{1}{2}|H \setminus I_0|$.*

Proof. Let I_c be a MCIS of a graph G and $X = I_c \cup N(I_c)$. By Theorem 2.1 it follows that, for any MCIS J_c of G, $J_c \cup N(J_c) = X$. Let Ω_c be the set of MCISs of G. Then $H \cup N(H) = \cup\{J_c : J_c \in \Omega_c\} \cup N(\cup\{J_c : J_c \in \Omega_c\}) = [\cup\{J_c : J_c \in \Omega_c\}] \cup [\cup\{N(J_c) : J_c \in \Omega_c\}] = \cup\{J_c \cup N(J_c) : J_c \in \Omega_c\} = X$, proving 1.

Now, $I_c \subseteq H$. Let $H' = H \setminus I_c$. So $n(G[H]) = |I_c| + |H'|$. Furthermore, $\alpha(G[H]) \geq |I_c|$. By construction $H' \subseteq N(I_c)$. By the Matching Lemma there is a matching from $N(I_c)$ into I_c in G. Thus there is a matching from H' into I_c in $G[H]$ and $\mu(G[H]) \geq |H'|$. So $\alpha(G[H]) + \mu(G[H]) \geq |I_c| + |H'| = n(H)$ and, for any graph, $\alpha + \mu \leq n$, it follows that $\alpha(G[H]) + \mu(G[H]) = n(G[H])$, that is, $G[H]$ is König-Egerváry, proving 2. It now follows easily that $\alpha(G[H]) = |I_c|$ and thus that I_c is a maximum independent set of $G[H]$, proving one direction of 3.

Now let I be a maximum independent set of $G[H]$. So I is an independent set in $G[X]$, $|I| \geq |I_c|$, and $\alpha(G[X]) \geq |I|$. It will now be argued that $\alpha(G[X]) = |I_c|$ and, hence, $|I_c| = |I|$. Theorem 2.1 implies that $G[X]$ is König-Egerváry. Then it is not difficult to see that the Matching Lemma implies that $\mu(G[X]) = |N(I_c)|$. Finally, we have $n(G[X]) = \alpha(G[X]) + \mu(G[X]) \geq |I_c| + |N(I_c)| = |X| = n(G[X])$. The claim them follows. Then, since I and I_c are maximum independent sets of $G[H]$, $I \cup N(I) \subseteq H \cup N(H) = X$. $N(I) \subseteq X \setminus I$ and $N(I_c) \subseteq X \setminus I_c$. It is worth noting here that, $N(I)$ is the set of neighbors of I in graph G (as opposed to graph $G[H]$). No use is made in this proof of neighbors of a set of vertices in graph $G[H]$ and no subscripts are ever required for clarity. To continue, it follows that $|N(I)| \leq |X \setminus I|$ and $|N(I_c)| = |X \setminus I_c|$. Since $|X \setminus I| = |X \setminus I_c|$, it follows that $|N(I)| \leq |N(I_c)|$ and, thus, that $|I| - |N(I)| \geq |I| - |N(I_c)|$. But, if $|I| - |N(I)| > |I_c| - |N(I_c)|$, I_c is not a critical independent set, contradicting our assumption. Thus $|I| - |N(I)| = |I_c| - |N(I_c)|$, and I is a critical independent set of G, proving the other direction of 3.

Let I_0 be the set of isolated vertices in $G[H]$. These are contained in any maximum independent set of $G[H]$. Let $I_c' = I_c \setminus I_0$ and $H' = H \setminus I_0$. It is then claimed that $G[H']$ has a perfect matching. Let $v \in H'$. Suppose $v \in core(G[H'])$, that is, v is in every maximum independent set of $G[H']$. Then v is in every maximum independent set of $G[H]$ and, thus, in every maximum critical independent set of G. But H is the set of vertices in some maximum critical independent set of G. So no vertex in $N(v)$ is in any maximum independent set of $G[H]$, or in any maximum critical independent set of G, which is a contradiction. Thus $|\cap\{S : S \in \Omega(G[H'])\}| = 0$. By similar reasoning it can be shown that $|\cap\{V(G[H']) - S : S \in \Omega(G[H'])\}| = 0$. Thus $|\cap\{S : S \in \Omega(G[H'])\}| = |\cap\{V(G[H']) - S : S \in \Omega(G[H'])\}|$. Theorem 2.4 then implies that $G[H']$ has a perfect matching, proving 4.

It is clear that, since $G[H]$ is König-Egerváry, and $G[H']$ has a perfect matching, $G[H']$ is also König-Egerváry; that is, $\alpha(G[H']) + \mu(G[H']) = n(G[H'])$. Since $n(G[H']) = 2\mu(G[H'])$, it follows that $\alpha(G[H']) = \frac{1}{2}n(G[H']) = \frac{1}{2}|H \setminus I_0|$. Finally $\alpha(G[H]) = |I_0| + \alpha(G[H']) = |I_0| + \frac{1}{2}|H \setminus I_0|$, proving 5.

□

3 A parallel MCIS algorithm

The criterion given for testing whether a vertex belongs to a critical independent set begins by passing to a certain bipartite graph. The computational speed of the following algorithm is due to the fact that the independence number of a bipartite graph can be computed in polynomial time.

Definition 3.1. For a graph G, the *bi-double* graph $B(G)$ has vertex set $V \cup V'$, where V' is a copy of V. If $V = \{v_1, v_2, \ldots, v_n\}$, let $V' = \{v_1', v_2', \ldots, v_n'\}$. Then, $(x, y') \in E(B(G))$ if, and only if, $(x, y) \in E(G)$.

The bi-double graph $B(G)$ of G can also be described as $K_2 \Box G$, the cartesian product of K_2 and G.

Corollary 3.2. *(Larson, [11]) A graph G contains a non-empty critical independent set if, and only if, there is a vertex $v \in V(G)$ such that $\alpha(B(G)) = \alpha(B(G) - \{v, v'\} - N(\{v, v'\}) + 2$.*

In fact, the proof of this corollary actually shows that a vertex v satisfying the specified condition is in *some* critical set. It is also shown in [11] that any critical independent set can be extended to a MCIS. These results are now combined in a form directly useful in the current context.

Theorem 3.3. *(MCIS Criterion) A vertex v in a graph G is in some MCIS if, and only if, $\alpha(B(G)) = \alpha(B(G) - \{v, v'\} - N(\{v, v'\}) + 2$.*

The following algorithm results in the set of all vertices which are in some maximum critical independent set. Step 4 requires n iterations—but, due to the MCIS Criterion, these n tests can be run independently on n processors. This is where the parallelization takes place.

MCIS subgraph algorithm

1. Construct $B(G)$.
2. $a := \alpha(B(G))$.
3. $H := \emptyset$.
4. For each vertex v in $V(G)$,
   (a) $t := \alpha(B(G) - \{v, v'\} - N(\{v, v'\}) + 2$.
   (b) If $t = a$, $H := H \cup \{v\}$.

According to Theorem 3.3 these steps will result in the construction of a set H consisting of all vertices which are in some MCIS. This can be extended in various ways to find the following invariants or sets.

1. Find α'. In order to calculate α', the remaining step is to identify the trivial and non-trivial components of H. Let I_0 be the isolated vertices in H. Then, by Theorem 2.5, $\alpha'(G) = |I_0| + \frac{1}{2}(|H \setminus I_0|)$.

2. Find X. In order to calculate the decomposition guaranteed by the Independence Decomposition Theorem, it remains to find the neighbors of the vertices in H. Again by Theorem 2.5, $H \cup N(H) = X$. Let $X^c = V(G) \setminus X$. Then $G[X^c]$ will have the property that, for every non-empty independent set J, $|N(J)| > |J|$.
3. Find a MCIS I_c of G. In order to find a MCIS, Theorem 2.5 implies that it suffices to find a maximum independent set I_c in H. Then I_c is a MCIS in G.

Since $B(G)$ is a bipartite graph on $2n$ vertices, calculating a maximum matching of $B(G)$ and, hence, calculating $\alpha(B(G))$ requires $\mathcal{O}(n^{1.5}\sqrt{m/\log n})$ operations, using the algorithm of Alt, et al. [2]. This algorithm will be run once and then a second time independently on each of n processors. So the total running time is still $\mathcal{O}(n^{1.5}\sqrt{m/\log n})$.

If M is a matching in a graph G and w is a vertex incident to an edge in M, let w' be the vertex matched to w under M. The new algorithm can now be stated. The parallelism occurs in step 1.

The parallel MCIS algorithm

1. Find H.
2. Find the set I_0 of isolated vertices in $G[H]$. $H_0 := I_0$.
3. If $H \setminus H_0 = \emptyset$, $I := I_0$. STOP.
4. Find a maximum matching M of $G[H]$.
5. Let $w \in H \setminus H_0$.
6. $N_1 := N(I_0 \cup \{w\})$, $M_1 := \{v' : v \in N_1\}$, $I_1 := I_0 \cup M_1$, $H_1 := I_1 \cup N_1$ (=$H_0 \cup N_1 \cup M_1$).
7. $i := 1$.
8. While $H \setminus H_i \neq \emptyset$:
   (a) i. If $H_i \neq H_{i-1}$: $N_{i+1} := N(I_i)$.
       ii. Else if $H_i = H_{i-1}$:
           A. Let $w \in H \setminus H_i$.
           B. $N_{i+1} := N(I_i \cup \{w\})$,
   (b) $M_{i+1} := \{v' : v \in N_{i+1}\}$, $I_{i+1} := I_i \cup M_{i+1}$. $H_{i+1} := I_{i+1} \cup N_{i+1}$, $i := i + 1$.
9. $I := I_i$.

Theorem 3.4. *If G is a graph then the set I produced by the Parallel MCIS algorithm is a maximum critical independent set of G.*

Proof. Let G be a graph, H be the set of vertices in some maximum critical independent set of G, and M be the maximum matching of $G[H]$ produced by the Parallel MCIS algorithm. Theorem 2.5 implies that $\alpha'(G) = \alpha(G[H])$. Thus it is enough to show that the set I produced by this algorithm is a maximum independent set of $G[H]$.

Let I_0 be the set of isolated vertices in $G[H]$ and $H' = H \setminus I_0$. It was shown that $G[H]$ is a König-Egerváry graph whose non-trivial components have perfect matchings. $G[H']$ is the union of the non-trivial components. So M is a perfect matching of $G[H']$.

The algorithm will first be described for the convenience of the reader. The first step is to identify the isolated vertices. These can be extended to a maximum independent set of $G[H]$. Then choose any vertex w among the remaining vertices. By the definition of the set H, there is a MCIS and, by Theorem 2.5, this is a maximum independent set of $G[H]$. So there is maximum independent set I of $G[H]$ which contains w. The neighbors of this vertex cannot be in I but each of these vertices is incident to some edge in the perfect matching M of $G[H]$ and, since one vertex from every edge of M must be in I, it follows that the vertices matched to $N(w)$ under M must be in I. In general, having constructed an independent set J, the neighbors of J cannot be in the maximum independent set but, since one vertex from every edge in M must be in the maximum independent set, the vertices matched to $N(J)$ under M must be in the set. If at some point there are no new vertices in $N(J)$, but the vertices in the graph have not been used up, an arbitrary vertex can be selected from the remaining vertices, added to the independent set, and the process continued.

The set I_0 is an independent set, $H_0 = I_0$, and there is a maximum independent set of $G[H]$ containing I_0. Assume that after the $(k-1)$th iteration of the while loop, I_k is an independent set and there is a maximum independent set of $G[H]$ containing I_k. It will be shown that I_{k+1} is an independent set and there is a maximum independent set of $G[H]$ which contains I_{k+1}.

If $H \setminus H_k = \emptyset$ after the $(k-1)$th iteration of the while loop, then $H = H_k$ and I is a maximum independent set of $G[H]$. Assume then that $H \setminus H_k \neq \emptyset$ after the $(k-1)$th iteration of the while loop. H_k is formed by (possibly) adding vertices to H_{k-1}, namely, $N(I_{k-1}) \setminus H_{k-1}$ together with the vertices matched to these under M. Either $H_k \neq H_{k-1}$ or $H_k = H_{k-1}$. Note that, in either case, by construction, $N_k \subseteq N_{k+1}$, $M_k \subseteq M_{k+1}$, $I_k \subseteq I_{k+1}$, and $H_k \subseteq H_{k+1}$.

In the first case, $H_k \setminus H_{k-1} \neq \emptyset$. H_k is formed by adding the vertices $N_k \setminus N_{k-1}$ and their neighbors $M_k \setminus M_{k-1} = I_k \setminus I_{k-1}$ under the matching M. The vertices $M_k \setminus M_{k-1}$ may or may not have neighbors outside of H_k. $N_{k+1} = N(I_k)$, $M_{k+1} = \{v' : v \in N_{k+1}\}$, $I_{k+1} = I_k \cup M_{k+1}$, and $H_{i+1} = I_{i+1} \cup N_{i+1}$. By assumption I_k is an independent set and there is a maximum independent set of $G[H]$ containing I_k. The vertices in I_{k+1} are the vertices in I_k together with the vertices matched to the neighbors of I_k under M. Let I be a maximum independent set of $G[H]$ containing I_k. It cannot contain any neighbor of I_k. Since any maximum independent set I of $G[H]$ must contain one vertex from each edge of M, I must contain the vertices matched to $N(I_k)$ under M. Thus I_{k+1} is an independent set and it can be extended to a maximum independent set of $G[H]$.

In the case where $H_k = H_{k-1}$, the kth step in the while loop of the algorithm (step 8) works as follows: A vertex w is selected from from $H \setminus H_k$. Since I_k is independent and $w \notin N(w)$, $I_{k+1} = I_k \cup \{w\}$ is an independent set. By assumption there is a MCIS containing w and, following Theorem 2.5, there is a maximum independent set I containing w. Each edge in M must be incident to some vertex in I. Let $I' = (I \setminus H_k) \cup I_k$. It remains to be shown that I' is a maximum independent set of $G[H]$. Since $H_k = I_k \cup N(I_k)$ it follows that I' is an independent set. It is now enough to show that, for every edge xy in M, either x or y is in I'. Either x or y is in I. Assume $x \in I$. If $x \notin I'$ then $x \in H_k$. But then by the construction of H_k, y is matched to x under M and y is also in H_k. But I_k is a maximum independent set in $G[H_k]$. So either x or y must be in I_k and, thus I'. So I_{k+1} is contained in a maximum independent set.

□

The MCIS Subgraph algorithm, which produces H, requires $\mathcal{O}(n^{1.5}\sqrt{m/\log n})$ operations. Finding a maximum matching of $G[H]$ requires the same order or of operations. The remaining steps only require finding the neighbors of sets of vertices. So the total running time of Parallel MCIS Algorithm is $\mathcal{O}(n^{1.5}\sqrt{m/\log n})$.

4 Acknowledgment and future research

We would like to thank an anonymous referee who pointed us to state-of-the-art references on maximum matching algorithms (including [8, 9, 10]). The referee notes that algorithm performance depends on edge density: some algorithms perform better for sparse graphs while others perform better for dense graphs.

The referee also suggests that future investigation of the time complexity of the algorithm presented above would be of interest in the case that there are n^2 or n^3 processors. The presented algorithm requires use of n processors; the problem naturally breaks into n cases following Theorem 3.3. We do not know of a way to break the problem down further.

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A note on homomorphisms of matrix semigroup

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Abstract

Let $\mathbb{F}$ be a field. We classify multiplicative maps from $\mathcal{M}_n(\mathbb{F})$ to $\mathcal{M}_{\binom{n}{k}}(\mathbb{F})$ which annihilate a zero matrix and map rank-k matrix into a rank-one matrix.

Keywords: Matrix semigroup, Homomorphism, Representation.

Math. Subj. Class.: 20M15, 15A33, 20G05

1 Introduction and preliminaries

Let $\mathcal{M}_n(\mathbb{F})$ denote the semigroup of all n–by–n matrices with coefficients in a field $\mathbb{F}$, let E_{ij} be its matrix units, and let $\mathrm{Id} = \mathrm{Id}_n := \sum E_{ii}$ be its identity. In [5], Jodeit and Lam classified nondegenerate semigroup homomorphisms $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$, that is, maps which are (i) multiplicative $\pi(AB) = \pi(A)\pi(B)$ and (ii) their restriction on singular matrices is nonconstant. It was shown that the semigroup of such maps is generated by three simple types: (i) a similarity, (ii) a fixed field homomorphism applied entry-wise on a matrix, and (iii) the map which sends A to a matrix of its cofactors. We refer below for more precise definitions.

The complete classification of degenerate maps on $\mathcal{M}_n(\mathbb{F})$ is more involved. They are all of the type $A \mapsto \pi_1(A) \oplus \mathrm{Id}_{n-m}$ for some integer $m \in \{0, \dots, n\}$ and some degenerate multiplicative $\pi_1 : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ with $\pi_1(0) = 0$ [5]. When $m = 1$, Đoković [2, Theorem 1] proved the following.

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Lemma 1.1. Let $\mathbb{F}$ be a field, and $n \geq 2$. If $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ is multiplicative, then there exists multiplicative $\phi : \mathbb{F} \to \mathbb{F}$ so that $\pi(X) = \phi(\det X)$. □

When $m < n$ and the characteristic of $\mathbb{F}$ differs from 2, Đoković [2, Theorem 2] also showed π_1 factors through determinant so that $\pi_1 = f \circ \det$ for some multiplicative $f : \mathbb{F} \to M_m(\mathbb{F})$. The classification of those seems to be difficult, and as far as we know they are known only in case $\mathbb{F} = \mathbb{C}$ is the filed of complex numbers, by the work of Omladič, Radjavi, and Šemrl [8]. Later, Guralnick, Li, and Rodman [4], extended the result of Đoković to include also the case $n = m$.

Semigroup homomorphisms mapping into higher dimensional algebras are less known. Kokol-Bukovšek [6, 7] classified them in case they are nondegenerate and map 2–by–2 matrices into 3–by–3 or into 4–by–4. Under additional assumption that a degenerate homomorphism is a polynomial in matrix entries, the classification is well-known, see a book by Weyl [9] (see also Fulton and Harris [3] for holomorphic homomorphisms over a field of complex numbers).

It is our aim to show that all homomorphisms from n–by–n matrices to $\binom{n}{k}$–by–$\binom{n}{k}$ matrices which map a rank-k matrix into a rank-one come from exterior product. Both assumptions on the dimension of the target space as well as on the rank of the matrices are essential; otherwise there are many more maps as we show in Remark 1.4 below. We remark that the main idea, that rank-k idempotents are mapped into rank-1 idempotents, is essentially due to Jodeit and Lam [5].

To be self-contained, we briefly repeat the basics about exterior products. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of column vectors in $\mathbb{F}^n$. Given a linear operator X on $\mathbb{F}^n$, denote by $\bigwedge^k(X)$ the k-th exterior product of X, acting on $\bigwedge^k(\mathbb{F}^n)$, i.e., a k-th exterior product of $\mathbb{F}^n$. Recall [3] that, as a vector space, $\bigwedge^k(\mathbb{F}^n)$ has a basis consisting of $\binom{n}{k}$ elements $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k};\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$, where $x \wedge y = -y \wedge x$ and $x \wedge x = 0$ is the alternating tensor. Then by definition, $\bigwedge^k(X) : \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \mapsto (X\mathbf{e}_{i_1}) \wedge \cdots \wedge (X\mathbf{e}_{i_k})$. It follows easily that $\bigwedge^k(AB) = \bigwedge^k(A)\bigwedge^k(B)$. Also, in lexicographic order of a basis $\left(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}\right)_{1 \leq i_1 < \cdots < i_k \leq n}$, the matrix of $\bigwedge^k(X)$ equals the $\binom{n}{k}$–by–$\binom{n}{k}$ matrix of all k–by–k minors of X, where the element at position corresponding to $(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}, \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_k})$ is the minor obtained by taking columns $i_1, \dots, i_k$ and rows $j_1, \dots, j_k$ of a matrix X. In particular, $\bigwedge^n(X) = \det X$ and $\bigwedge^{n-1}(X)$ is similar to a matrix of cofactors under similarity $S = \sum_{i=1}^n (-1)^{i+1} E_{i\,(n-i+1)}$.

Besides the $(n-1)$-st exterior product there are at least two additional multiplicative maps from $\mathcal{M}_n(\mathbb{F})$ to itself. One is an inner automorphism $X \mapsto SXS^{-1}$ where $S \in \mathcal{M}_n(\mathbb{F})$ is fixed, invertible. The other is induced by a field homomorphism $\phi : \mathbb{F} \to \mathbb{F}$ (i.e. an additive multiplicative map) applied entry-wise, that is, it maps a matrix $\sum x_{ij} E_{ij}$ into $\sum \phi(x_{ij}) E_{ij}$. With a slight abuse of notation, we denote this map again by $\phi : X \mapsto \phi(X)$.

Theorem 1.2. Let $\mathbb{F}$ be a field, let $n \geq 2$ be an integer, and let $m = \binom{n}{k}$ for some integer $k = 1, \dots, n$. If $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, $\pi(0) = 0$, is a multiplicative map such that $\operatorname{rk}(\pi(A_0)) = 1$ for some matrix A_0 of rank-k, then

$$\pi(X) = S\phi(\bigwedge^k(X))S^{-1}$$

where $\phi : \mathbb{F} \to \mathbb{F}$ is a multiplicative map and $S \in \mathcal{M}_m(\mathbb{F})$ is invertible.

Moreover, if $k < n$ then ϕ is also additive, hence a field embedding.

Remark 1.3. Without the assumption $\pi(0) = 0$, there are more possibilities. Say, $\pi(A) = 1 \oplus \mathrm{Sym}^2(\bigwedge^{n-1} A) \oplus f(\det A)$, where Sym^2 is the second symmetric power (see [3]) and $f : \mathbb{F} \to \mathcal{M}_{m-1-\binom{n+1}{2}}(\mathbb{F})$ is multiplicative.

However, we remark that to classify multiplicative maps π it suffices to assume $\pi(0) = 0$. In fact, if $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is multiplicative, then $P := \pi(0)$ is an idempotent, and from $\pi(X)P = \pi(X0) = \pi(0) = P = \pi(0X) = P\pi(X)$ we deduce that, relative to decomposition $\mathbb{F}^m = \operatorname{Ker} P \oplus \operatorname{Im} P$ we have

$$\pi(X) = \pi_1(X) \oplus \mathrm{Id}_r,$$

where $r := \operatorname{rk} P$ and $\pi_1 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m-r}(\mathbb{F})$ is multiplicative with $\pi_1(0) = 0$.

Remark 1.4. If $m \neq \binom{n}{k}$ there are more possibilities, say $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n^2}{4}}(\mathbb{F})$, defined by $A \mapsto \bigwedge^4(A \otimes A)$, is multiplicative and maps a rank-2 matrix $E_{11} + E_{22}$ into matrix of rank-one but is not of the form in the Theorem. This is because if $\operatorname{rk} A = r$ then $\operatorname{rk}(\bigwedge^k A) = \binom{r}{k}$, while π maps a rank-3 matrix $E_{11} + E_{22} + E_{33}$ into a matrix whose rank equals 126.

If $\operatorname{rank}(\pi(A_0)) \neq 1$ there are more possibilities as can be seen by the map $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n}{k}}(\mathbb{F})$, defined by $A \mapsto A \oplus 0_{\binom{n}{k}-n}$.

Proof of Theorem 1.2. If $k = n$ then $m = 1$, so $\pi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$. Such multiplicative maps were proven to be in accordance with our results by Lemma 1.1.

Hence, we may assume in the sequel that $k < n$. Clearly, $\pi(\mathrm{Id})$ is an idempotent, and from $\pi(X)\pi(\mathrm{Id}) = \pi(X \cdot \mathrm{Id}) = \pi(X) = \pi(\mathrm{Id})\pi(X)$ we deduce that, relative to decomposition $\mathbb{F}^m = \operatorname{Im}\pi(\mathrm{Id}) \oplus \operatorname{Ker}\pi(\mathrm{Id})$ we have $\pi(X) = \pi_1(X) \oplus 0_{m-r} \in \mathcal{M}_r(\mathbb{F}) \oplus 0_{m-r}$, where $r := \operatorname{rk}\pi(\mathrm{Id})$ and π_1 is multiplicative with $\pi_1(0) = 0$ and $\pi_1(\mathrm{Id}) = \mathrm{Id}_r$. As such, if X is invertible, then $\mathrm{Id}_r = \pi_1(\mathrm{Id}) = \pi_1(XX^{-1}) = \pi_1(X)\pi_1(X^{-1})$, so $\pi_1(X)$ is also invertible and $\pi_1(X)^{-1} = \pi_1(X^{-1})$.

Let X be any matrix of rank-k. Then, there exist invertible $S, T \in \mathcal{M}_n(\mathbb{F})$ with $SXT = \mathrm{Id}_k \oplus 0_{n-k}$, and in particular, there exist invertible S_1, T_1 such that $X = S_1 A_0 T_1$. Consequently,

$$1 = \operatorname{rk}\pi(A_0) = \operatorname{rk}\big(\pi_1(S_1^{-1}XT_1^{-1}) \oplus 0_{m-r}\big) = \operatorname{rk}\big(\pi_1(S_1)^{-1}\pi_1(X)\pi_1(T_1)^{-1} \oplus 0_{m-r}\big),$$

wherefrom $\operatorname{rk}\pi(X) = 1$ for every X of rank-k. Consequently, $\pi(\mathrm{Id}_k \oplus 0_{n-k})$ is an idempotent of rank-1, and by appropriate similarity we may assume it equals E_{11}.

Given $X = \hat{X} \oplus 0_{n-k} \in \mathcal{M}_k(\mathbb{F}) \oplus 0_{n-k}$, we have $X = (\mathrm{Id}_k \oplus 0)X(\mathrm{Id}_k \oplus 0)$, wherefrom $\pi(X) = E_{11}\pi(X)E_{11} \in \mathbb{F}E_{11}$. Hence, π induces a multiplicative map $\hat{\pi} : \mathcal{M}_k(\mathbb{F}) \to \mathbb{F}$ by

$$\hat{\pi}(\hat{X})E_{11} := \pi(\hat{X} \oplus 0_{n-k}).$$

It follows by Lemma 1.1 that there exists a nonzero multiplicative map $\phi_1 : \mathbb{F} \to \mathbb{F}$ such that

$$\hat{\pi}(\hat{X}) = \phi_1(\det \hat{X}).$$

In particular, if the rank of $X = \hat{X} \oplus 0_{n-k}$ is smaller than k, then $\pi(X) = \hat{\pi}(\hat{X})E_{11} = 0$. By multiplicativity, $\pi(X) = 0$ for every $X \in \mathcal{M}_n(\mathbb{F})$ with $\operatorname{rk} X \leq k-1$. Moreover, given any $A \in \mathcal{M}_n(\mathbb{F})$, letting $\hat{A}$ be the compression of A to the upper-left k–by–k block, we have

$$\pi\big((\mathrm{Id}_k \oplus 0)A(\mathrm{Id}_k \oplus 0)\big) = \pi(\hat{A} \oplus 0_{n-k}) = \phi_1(\det \hat{A}).$$

Next, among diagonal idempotent matrices, there exists exactly $m := \binom{n}{k}$ of them that have k ones and $n-k$ zeros on diagonal. We order them lexicographically according to position of ones on diagonal, and denote them

$$P_1 = (\mathrm{Id}_k \oplus 0_{n-k}), \dots, P_m = (0_{n-k} \oplus \mathrm{Id}_k).$$

Given two such diagonal idempotents P_i, P_j, we have $\mathrm{rk}(P_i P_j) \le k$ and the equality holds only if $P_i = P_j$. Hence $\pi(P_1), \dots, \pi(P_m)$ are pairwise orthogonal idempotents of rank-one. It is well-known (say, [1, Lemma 2.2]) that, by applying appropriate similarity, we can achieve $\pi(P_i) = E_{ii}$ for $i = 1, \dots, m$. Combined with $\pi(P_i) = \pi_1(P_i) \oplus 0_{m-r} \in \mathcal{M}_r(\mathbb{F}) \oplus 0_{n-r} \subseteq \mathcal{M}_m(\mathbb{F})$ we see that $r = 0$. Hence, $\pi = \pi_1$ is already unital.

As above for $\pi(P_1 A P_1) = \phi_1(\det \hat{A})$ we see that for each $i = 1, \dots, m$ there exist nonzero multiplicative map $\phi_i : \mathbb{F} \to \mathbb{F}$ so that

$$\pi(P_i A P_i) = \phi_i(\det A^{(ii)}) E_{ii}, \tag{1.1}$$

where, for a matrix $X \in \mathcal{M}_n(\mathbb{F})$, we denote $X^{(ij)}$ the k–by–k submatrix of X which lies on the rows where P_i has nonzero entries and on the columns where P_j has nonzero entries. Observe that a nonzero multiplicative ϕ_i satisfies $\phi_i(1) = 1$.

Consider any $A \in \mathcal{M}_n(\mathbb{F})$. Then,

$$\pi(A) = \mathrm{Id}\, \pi(A)\, \mathrm{Id} = \Big(\sum_{i=1}^{m} E_{ii}\Big)\pi(A)\Big(\sum_{j=1}^{m} E_{jj}\Big) = \sum_{i,j} E_{ii}\pi(A)E_{jj} = \sum_{i,j} \pi(P_i A P_j).$$

Given indices $i \neq j$, there exists $B_{ji} \in \mathcal{M}_n$ of rank-k such that $B_{ji} = P_j B_{ji} P_i$ and $\det (B_{ji})^{(ji)} = 1$; for instance, if $P_i = \sum_{t \in \{t_1,\dots,t_k\}} E_{tt}$ and $P_j = \sum_{s \in \{s_1,\dots,s_k\}} E_{ss}$, with $t_1 < \cdots < t_k$ and $s_1 < \cdots < s_k$, we can take

$$B_{ji} = \sum_{i=1}^{k} E_{s_i t_i} \tag{1.2}$$

and then $(B_{ji})^{(ji)} = \mathrm{Id}_k$. In particular then, $\pi(B_{ji}) = \gamma_{ji} E_{ji} \neq 0$. It follows that

$$\pi(P_i A P_j)\pi(B_{ji}) = \pi(P_i(AP_jP_jB_{ji})P_i) = \phi_i\big(\det (P_i A P_j P_j B_{ji} P_i)^{(ii)}\big) E_{ii}. \tag{1.3}$$

Observe that

$$(P_i A P_j P_j B_{ji} P_i)^{(ii)} = A^{(ij)} B_{ji}^{(ji)}.$$

(This follows easily by writing $P_i = \sum_{t \in \{t_1,\dots,t_k\}} E_{tt}$ and $P_j = \sum_{s \in \{s_1,\dots,s_k\}} E_{ss}$, $t_1 < t_2 < \cdots < t_k$ and $s_1 < s_2 < \cdots < s_k$, and observing that

$$P_i X P_j = \sum_{(t,s) \in \{t_1,\dots,t_k\} \times \{s_1,\dots,s_k\}} x_{ts} E_{ts}$$

and

$$P_j Y P_i = \sum_{(t,s) \in \{t_1,\dots,t_k\} \times \{s_1,\dots,s_k\}} y_{st} E_{st},$$

and hence $P_iXP_j \cdot P_jYP_i = \sum_{t,t' \in \{t_1,\dots,t_k\}} \sum_{s\in\{s_1,\dots,s_k\}} x_{ts}y_{st'}E_{tt'}$.)
Hence,

$$\begin{aligned}\phi_i(\det(P_iAP_jP_jB_{ji}P_i)^{(ii)}) &= \phi_i(\det A^{(ij)})\phi_i(\det(B_{ji})^{(ji)}) = \phi_i(\det A^{(ij)}) \cdot \phi_i(1)\\ &= \phi_i(\det A^{(ij)}).\end{aligned}$$

On the other hand, $\pi(P_iAP_j) = \pi(P_i)\pi(A)\pi(P_j) = E_{ii}\pi(A)E_{jj} = \alpha_{ij}(A)E_{ij}$ wherefrom $\pi(P_iAP_j)\cdot\pi(B_{ji}) = \alpha_{ij}(A)E_{ij}\cdot\gamma_{ji}E_{ji} = \alpha_{ij}(A)\gamma_{ji}E_{ii}$, and hence, by (1.3)

$$\alpha_{ij}(A) = \frac{1}{\gamma_{ji}}\phi_i(\det A^{(ij)}).$$

By similar arguments we also have that

$$\begin{aligned}\gamma_{ji}\,\alpha_{ij}(A)E_{jj} &= \pi(B_{ji})\pi(P_iAP_j) = \pi(B_{ji}P_iAP_j) = \pi(P_jB_{ji}P_iAP_j)\\ &= \phi_j(\det A^{(ij)})E_{jj}\end{aligned}$$

and since A was arbitrary, we see that $\phi_i = \phi_j =: \phi$ is independent of i,j. Hence,

$$\pi(X) = \sum_{i,j}\alpha_{ij}(X)E_{ij} = \sum_{i,j}\frac{1}{\gamma_{ij}}\phi(\det X^{(ij)})E_{ij},$$

where, in accordance with (1.1), we define $\gamma_{ii} = 1$ for $i = 1,\dots,m$. Recall also that $\phi(1) = \phi_i(1) = 1$.

We only need to show that multiplicativity of π forces that ϕ is additive and that $\gamma_{ij}\gamma_{jv} = \gamma_{iv}$. To prove additivity of ϕ, choose a scalar α and consider a rank-$(k+1)$ matrix $A_\alpha := \left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right) \oplus \mathrm{Id}_{k-1} \oplus 0_{n-k-1}$. It is easy to see that in A_α the number of k–by–k submatrices of rank-k equals $(k+2)$, and they are all obtained from the principal $(k+1)$–by–$(k+1)$ block by deleting one of the following (i) the same row and column, or (ii) second row and first column. Under (i) the resulting submatrix equals Id_k, while under (ii) it equals $\alpha \oplus \mathrm{Id}_{k-1}$. Thus, there exist indices $i_1,\dots,i_{k+1}$ and $i,j \in \{i_1,\dots,i_{k+1}\}$, $i \neq j$, so that

$$\pi(A_\alpha) = \sum_{t=1}^{k+1}\phi(1)E_{i_ti_t} + \tfrac{1}{\gamma_{ij}}\phi(\alpha)E_{ij}.$$

(A deeper analysis reveals that, in lexicographic order, $i = \binom{n-2}{k-2}+1$ and $j = \binom{n-1}{k-1}+1$). As $A_\alpha A_\beta = A_{\alpha+\beta}$, the multiplicativity of π together with $\phi(1) = 1$ yields

$$\sum_t E_{i_ti_t} + \tfrac{1}{\gamma_{ij}}\phi(\alpha+\beta)E_{ij} = \pi(A_\alpha A_\beta) = \pi(A_\alpha)\pi(A_\beta) = \sum_t E_{i_ti_t} + \tfrac{\phi(\alpha)+\phi(\beta)}{\gamma_{ij}}E_{ij},$$

wherefrom ϕ is additive.

It remains to prove $\gamma_{ij}\gamma_{jv} = \gamma_{iv}$. Take matrices B_{ij} and B_{jv} defined in (1.2). Then, $\det\big((B_{ij}B_{jv})^{(iv)}\big) = \det((B_{ij})^{(ij)})\det((B_{jv})^{(jv)}) = 1\cdot 1 = 1$. Hence,

$$\frac{1}{\gamma_{iv}}E_{iv} = \frac{1}{\gamma_{iv}}\phi(\det(B_{ij}B_{jv})^{(iv)})E_{iv} = \pi(B_{ij}B_{jv}) = \pi(B_{ij})\pi(B_{jv}) = \frac{1}{\gamma_{ij}\gamma_{jv}}E_{ij}\cdot E_{jv},$$

wherefrom $\gamma_{iv} = \gamma_{ij}\gamma_{jv}$.

Consider now an invertible diagonal matrix $D = \mathrm{diag}(\gamma_{11},\dots,\gamma_{1m})$. Then, $\pi(X) = \sum_{i,j}\frac{1}{\gamma_{ij}}\phi(\det X^{(ij)})E_{ij} = D^{-1}\sum_{i,j}\phi(\det X^{(ij)})E_{ij}D = D^{-1}\phi(\bigwedge^k(X))D$. □

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On geometric trilateral-free (n_3) configurations

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Abstract

This note presents the first known examples of a geometric trilateral-free (23_3) configuration and a geometric trilateral-free (27_3) configuration. The (27_3) configuration is also pentalateral-free.

Keywords: Configurations, trilaterals.

Math. Subj. Class.: 05B30, 51E30, 05C38

1 Introduction

A *(combinatorial)* (n_3) *configuration* is an incidence structure consisting of n distinct points and n distinct lines for which each point lies on exactly three lines, each line is incident with exactly three points, and any two points are incident with at most one common line. If an (n_3) configuration may be depicted in the Euclidean plane using points and (straight) lines, it is said to be *geometric*. As observed in [5] (pg. 17–18), it is evident that every geometric (n_3) configuration is combinatorial, but the converse of this statement does not hold.

Adopting the terminology from [2], we say that a *g-lateral* in a configuration is a cyclically ordered set $\{p_0, l_0, p_1, l_1, \ldots, l_{g-2}, p_{g-1}, l_{g-1}\}$ of pairwise distinct points p_i and pairwise distinct lines such that p_i is incident with l_{i-1} and l_i for each $i \in \mathbb{Z}_g$. Hence a 3-lateral is a trilateral, or triangle, a 4-lateral is a quadrilateral, and a 5-lateral is a pentalateral, according to the previously established nomenclature. A configuration is *g-lateral-free*, for a particular $g \in \{3, 4, 5\}$, if no g-lateral exists within the configuration.

Several recent papers (see [1], [3]) have examined triangle-free (n_3) configurations. The smallest example of a triangle-free configuration is the Cremona-Richmond (15_3) configuration. A theorem mentioned in [5] (Theorem 5.4.3, pg. 333) states the following:

Theorem 1.1. *For every* $n \geq 15$ *except* $n = 16$ *and possibly* $n = 23$ *and* $n = 27$*, there are geometric trilateral-free* (n_3) *configurations.*

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In this note we provide new examples of a geometric, triangle-free (23_3) configuration and a geometric, triangle-free (27_3) configuration, so that this theorem may now be modifed:

Theorem 1.2. *For every $n \geq 15$ except $n = 16$, there are geometric trilateral-free (n_3) configurations.*

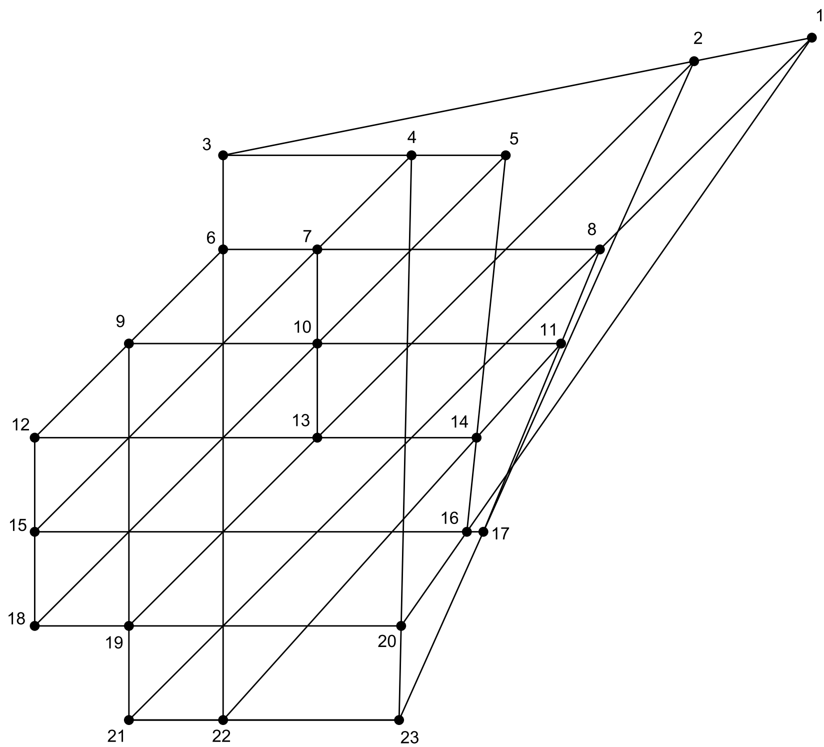
Additionally, the (27_3) configuration is also pentalateral-free. It serves as the smallest known example of a geometric (n_3) configuration that is both 3-lateral-free and 5-lateral-free; the formerly smallest known example of such a configuration is a (51_3) configuration [2].

2 The examples

Configuration tables and diagrams of both of these new configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ are provided below, and rational coordinates for their geometric realizations are given. Verification that the former configuration is trilateral-free, and that the latter configuration is trilateral-free and pentalateral-free, has been conducted using *Mathematica*.

2.1 $\mathcal{C}_1$, a geometric triangle-free (23_3) configuration

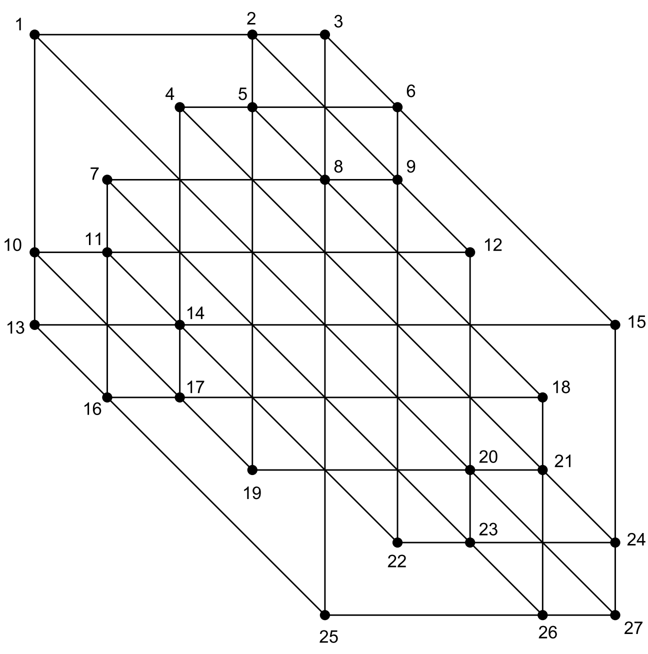
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 8 & 9 & 9 & 11 & 12 & 12 & 15 & 18 & 21 \\ 2 & 8 & 16 & 13 & 17 & 4 & 6 & 7 & 20 & 10 & 14 & 7 & 9 & 10 & 11 & 10 & 19 & 14 & 13 & 15 & 16 & 19 & 22 \\ 3 & 21 & 20 & 19 & 23 & 5 & 22 & 15 & 23 & 18 & 16 & 8 & 12 & 13 & 17 & 11 & 21 & 22 & 14 & 18 & 17 & 20 & 23 \end{pmatrix}$$



Point	*Coordinates*	*Point*	*Coordinates*	*Point*	*Coordinates*
1	$(33/4, 29/4)$	2	$(7, 7)$	3	$(2, 6)$
4	$(4, 6)$	5	$(5, 6)$	6	$(2, 5)$
7	$(3, 5)$	8	$(6, 5)$	9	$(1, 4)$
10	$(3, 4)$	11	$(542/97, 4)$	12	$(0, 3)$
13	$(3, 3)$	14	$(455/97, 3)$	15	$(0, 2)$
16	$(445/97, 2)$	17	$(462/97, 2)$	18	$(0, 1)$
19	$(1, 1)$	20	$(1132/291, 1)$	21	$(1, 0)$
22	$(2, 0)$	23	$(1876/485, 0)$		

2.2 $\mathcal{C}_2$, a geometric triangle-free, pentalateral-free (27_3) configuration

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 6 & 7 & 7 & 7 & 10 & 10 & 11 & 12 & 13 & 13 & 15 & 16 & 18 & 19 & 22 & 25 \\ 2 & 10 & 20 & 5 & 9 & 6 & 8 & 5 & 14 & 21 & 8 & 9 & 8 & 11 & 23 & 11 & 17 & 14 & 20 & 14 & 16 & 24 & 17 & 21 & 20 & 23 & 26 \\ 3 & 13 & 27 & 19 & 12 & 15 & 25 & 6 & 17 & 24 & 18 & 22 & 9 & 16 & 26 & 12 & 19 & 22 & 23 & 15 & 25 & 27 & 18 & 26 & 21 & 24 & 27 \end{pmatrix}$$



Point	*Coordinates*	*Point*	*Coordinates*	*Point*	*Coordinates*
1	$(0, 8)$	2	$(3, 8)$	3	$(4, 8)$
4	$(2, 7)$	5	$(3, 7)$	6	$(5, 7)$
7	$(1, 6)$	8	$(4, 6)$	9	$(5, 6)$
10	$(0, 5)$	11	$(1, 5)$	12	$(6, 5)$
13	$(0, 4)$	14	$(2, 4)$	15	$(8, 4)$
16	$(1, 3)$	17	$(2, 3)$	18	$(7, 3)$
19	$(3, 2)$	20	$(6, 2)$	21	$(7, 2)$
22	$(5, 1)$	23	$(6, 1)$	24	$(8, 1)$
25	$(4, 0)$	26	$(7, 0)$	27	$(8, 0)$

3 Motivation for the results

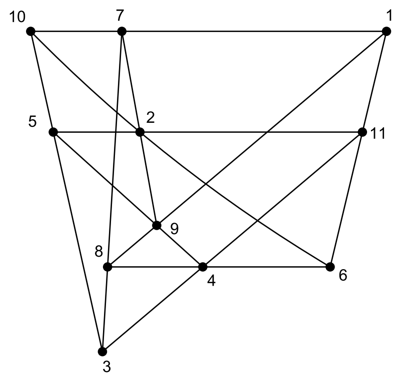
Both $\mathcal{C}_1$ and $\mathcal{C}_2$ have arisen serendipitously in conjunction with the author's study of *magic* (n_3) configurations. An (n_3) configuration is said to be magic if it is possible to assign the integers $\{1, 2, \ldots, n\}$ as labels for its n points, where each integer is used exactly once, in such a manner that the sum of the point labels along each line of the configuration is always the same magic constant, M. Since each point of the configuration is involved in three such sums, we see that

$$\begin{aligned} nM &= 3\sum_{i=1}^{n} i = 3\frac{n(n+1)}{2} \\ M &= \frac{3}{2}(n+1) \end{aligned}$$

Hence n must be odd (and at least 7) for a magic configuration to be possible. The smallest example of a magic configuration turns out to one of the 31 (11_3) configurations. Its combinatorial table is

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & 5 & 7 & 5 & 6 \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & 10 & 8 & 9 & 8 \end{pmatrix}$$

This configuration is $(11_3)_{17}$, according to the (11_3) configuration labeling scheme initiated in [4] and referenced in [6],[7].



Magic (n_3) configurations have not, to the author's knowledge, been previously considered in the literature on configurations, although other magic configurations such as magic stars have been studied [8].

$\mathcal{C}_1$ is dual to the magic (23_3) configuration

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 9 \\ 13 & 14 & 15 & 11 & 12 & 15 & 10 & 15 & 16 & 12 & 13 & 14 & 8 & 9 & 14 & 9 & 10 & 13 & 8 & 10 & 11 & 12 & 11 \\ 22 & 21 & 20 & 23 & 22 & 19 & 23 & 18 & 17 & 20 & 19 & 18 & 23 & 22 & 17 & 21 & 20 & 17 & 21 & 19 & 18 & 16 & 16 \end{pmatrix}$$

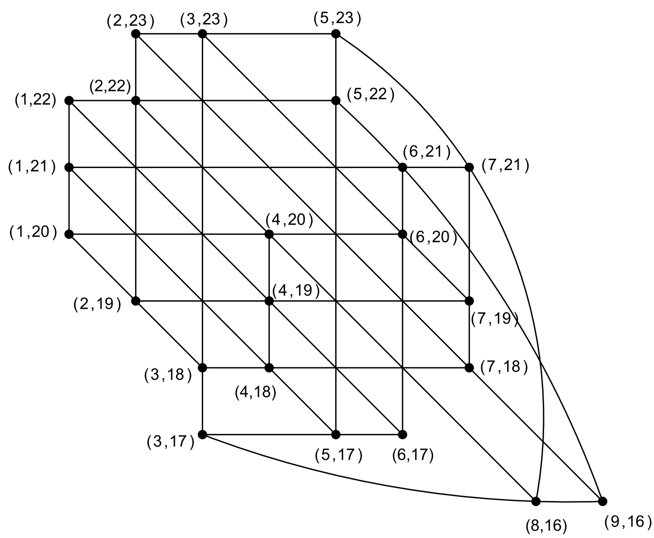
with magic constant $\frac{3}{2}(23+1) = 36$. Also, $\mathcal{C}_2$ is dual to the magic (27_3) configuration

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 \\ 14 & 17 & 18 & 15 & 16 & 18 & 13 & 16 & 17 & 11 & 12 & 17 & 10 & 12 & 18 & 10 & 11 & 16 & 11 & 14 & 15 & 12 & 13 & 15 & 10 & 13 & 14 \\ 27 & 24 & 23 & 25 & 24 & 22 & 26 & 23 & 22 & 27 & 26 & 21 & 27 & 25 & 19 & 26 & 25 & 20 & 24 & 21 & 20 & 22 & 21 & 19 & 23 & 20 & 19 \end{pmatrix}$$

with magic constant $\frac{3}{2}(27+1) = 42$. This means that in each case there exists an isomorphism between $\mathcal{C}_i$ and its dual that interchanges the roles of points and lines while preserving incidence structure. We say that the dual configuration of a magic configuration is *comagic*; hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are comagic. So for both $\mathcal{C}_1$ and $\mathcal{C}_2$ it is possible to label its lines in such a manner that the sum of the labels of the three lines incident to any point of the configuration is always the same magic constant, again $\frac{3}{2}(n+1)$.

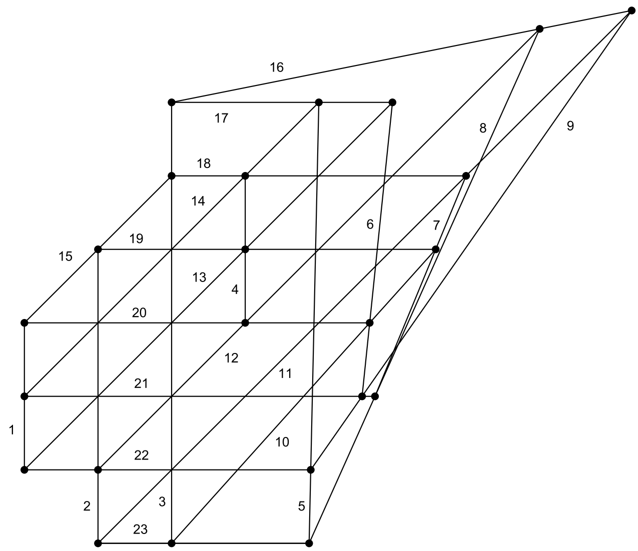
It turns out that a diagram associated with a comagic configuration may be conveniently constructed. Suppose that $(x_1\ x_2\ x_3)^T$ is a line in the configuration table of the original magic configuration, where $x_1 < x_2 < x_3$. It follows that the point $(x_1, x_2, x_3) \in \mathbb{R}^3$ lies in the plane $\{(x, y, z) \in \mathbb{R}^3 \colon x + y + z = \frac{3}{2}(n+1)\}$. After plotting each corresponding point in this plane, for $k = 1, \ldots, n$ we connect three points with an arc (labeled k) if the three points share k as a coordinate. We thereby produce a diagram within the plane $x + y + z = \frac{3}{2}(n+1)$.

Next, we project the diagram onto the xz-plane by simply eliminating the y-coordinate. No information about the configuration is lost when doing this, since for any point we may recapture $x_2 = \frac{3}{2}(n+1) - x_1 - x_3$. Below is a diagram for $\mathcal{C}_1$ achieved in this fashion with each (x_1, x_3) point indicated.



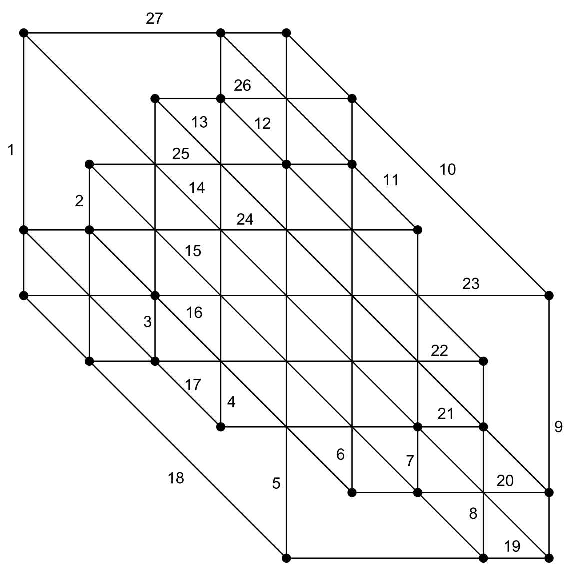
Observe that this diagram has three nonlinear arcs. After some algebraic manipulation involving shifting seven of the 23 points, we find that it is possible to recast the diagram so that all of the arcs indeed are straight lines. After rescaling the points (via the transformation $(x, z) \mapsto (x - 1, 23 - z)$) we arrive at the geometric realization for $\mathcal{C}_1$ provided in Section 2.1.

We again depict the diagram for $\mathcal{C}_1$, this time with its associated magic line labeling.



When undergoing this process for the (27_3) configuration, we discover pleasantly that no shifting of arcs is required. This is a consequence of each line $(x_1 \;\; x_2 \;\; x_3)^T$ satisfying the conditions $1 \leq x_1 \leq 9$, $10 \leq x_2 \leq 18$, and $19 \leq x_3 \leq 27$. After lopping off the x_2-coordinates and rescaling the resulting points (via the transformation $(x, z) \mapsto (x - 1, z - 19)$) we arrive at the geometric realization for $\mathcal{C}_2$ provided in Section 2.2.

We display the diagram of $\mathcal{C}_2$ again with its associated magic line labeling.



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Interlacing–extremal graphs*

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Abstract

A graph G is *singular* if the zero-one adjacency matrix has the eigenvalue zero. The multiplicity of the eigenvalue zero is called the *nullity* of G. For two vertices y and z of G, we call (G, y, z) a *device* with respect to y and z. The nullities of G, $G - y$, $G - z$ and $G - y - z$ classify devices into different *kinds*. We identify two particular classes of graphs that correspond to distinct kinds. In the first, the devices have the minimum allowed value for the nullity of $G - y - z$ relative to that of G for all pairs of distinct vertices y and z of G. In the second, the nullity of $G - y$ reaches the maximum possible for all vertices y in a graph G. We focus on the non–singular devices of the second kind.

Keywords: Adjacency matrix, singular graphs, nut graphs, uniform–core graphs, nuciferous graphs, interlacing.

Math. Subj. Class.: 05C50,05C35, 05C60, 05B20, 92E10, 74E40

1 Introduction

A graph $G = G(\mathcal{V}, \mathcal{E})$ of *order* n has a labelled vertex set $\mathcal{V} = \{1, 2, ..., n\}$. The set $\mathcal{E}$ of m edges consists of unordered pairs of adjacent vertices. We write $\mathcal{V}(G)$ for a graph G when the graph G needs to be specified. A subset of $\mathcal{V}$ is said to be *independent* if no two of its vertices are adjacent, i.e., no two are connected by an edge. For a subset $\mathcal{V}_1$ of $\mathcal{V}$, $G - \mathcal{V}_1$ denotes the subgraph of G induced by $\mathcal{V}\backslash\mathcal{V}_1$. The subgraph of G obtained by deleting a particular vertex y is denoted by $G - y$ and that obtained by deleting two distinct vertices

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y and z is denoted by $G - y - z$. A graph is said to be *bipartite* if its vertex set $\mathcal{V}$ may be partitioned into two independent subsets $\mathcal{V}_1$ and $\mathcal{V}_2$. The *cycle* and the *complete graph* on n vertices are denoted by C_n and K_n, respectively. The complete bipartite graph K_{n_1,n_2} has a vertex partition into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ of independent vertices of sizes n_1 and n_2, respectively, and has edges between each member of $\mathcal{V}_1$ and each member of $\mathcal{V}_2$.

1.1 The adjacency matrix

The graphs we consider are *simple*, that is, without loops or multiple edges. We use $\mathbf{A}(G)$ (or just $\mathbf{A}$ when the context is clear) to denote the 0-1 *adjacency matrix* of a graph G, where the entry a_{ik} of the symmetric matrix $\mathbf{A}$ is 1 if $\{i, k\} \in \mathcal{E}$ and 0 otherwise. We note that the graph G is determined, up to isomorphism, by $\mathbf{A}$. The adjacency matrix $\mathbf{A}^C$ of the *complement* G^C of G is $\mathbf{J} - \mathbf{I} - \mathbf{A}$, where each entry of $\mathbf{J}$ is one and $\mathbf{I}$ is the identity matrix. The *degree* of a vertex i is the number of non–zero entries in the i^{th} row of $\mathbf{A}$. If the adjacency matrix $\mathbf{A}$ of a n-vertex graph G satisfies $\mathbf{Ax} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x}$ then $\mathbf{x}$ is said to be an *eigenvector* belonging to the *eigenvalue* λ. There are n linearly independent eigenvectors. The eigenvalues of $\mathbf{A}$ are said to be *the eigenvalues of* G and to form the *spectrum* of G. They are obtained as the roots of the *characteristic polynomial* $\phi(G, \lambda)$ of the adjacency matrix of G, defined as the polynomial $\det(\lambda\mathbf{I} - \mathbf{A})$ in λ.

Cauchy's inequalities for a Hermitian matrix M (also collectively known as the *Interlacing Theorem*) place restrictions on the multiplicity of the eigenvalues of principal submatrices relative to those of $\mathbf{M}$ (See [6] for instance). When they are applied to graphs we have:

Theorem 1.1. Interlacing Theorem: *Let G be an n-vertex graph and $w \in \mathcal{V}$. If the eigenvalues of G are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and those of $G - w$ are $\xi_1, \xi_2, \ldots, \xi_{n-1}$, both in non-increasing order, then $\lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \ldots \geq \xi_{n-1} \geq \lambda_n$.*

1.2 Cores of singular graphs

For the linear transformation $\mathbf{A}$, the *kernel*, $\ker(\mathbf{A})$, of $\mathbf{A}$ is defined as the subspace of $\mathbb{R}^n$ mapped to zero by $\mathbf{A}$. It is also referred to as the *nullspace* of $\mathbf{A}$. A graph G is said to be *singular* of nullity η_G if the dimension of the nullspace $\ker(\mathbf{A})$ of $\mathbf{A}$ is η_G and $\eta_G > 0$. If there exists a non-zero vector $\mathbf{x}$ in the nullspace of the adjacency matrix $\mathbf{A}$, then $\mathbf{x}$ is said to be a *kernel eigenvector* of the singular graph G and satisfies $\mathbf{Ax} = \mathbf{0}$. It is therefore an eigenvector of $\mathbf{A}$ for the eigenvalue zero whose multiplicity η_G is also the number of roots of $\phi(G, \lambda)$ equal to zero. A vertex corresponding to a non-zero entry of $\mathbf{x}$ is said to be a *core vertex* CV of G. The core vertices corresponding to $\mathbf{x}$ *induce* a subgraph of G termed the *core* of G with respect to $\mathbf{x}$. The core structure of a singular graph will be the basis of our classification of all graphs relative to η_G.

A *core graph* is a singular graph in which every vertex is a core vertex. The empty graph $(K_4)^C$ and the four cycle C_4 are examples of 4–vertex core graphs of nullity four and two, respectively. A core graph of order at least three and nullity one is known as a *nut graph*. It is connected and non–bipartite [12].

For singular graphs, the vertices can be partitioned into core and core-forbidden vertices. The set $\mathcal{CV}$ of *core vertices* consists of those vertices lying on some core of G. A *core-forbidden vertex* (CFV) corresponds to a zero entry in *every* kernel eigenvector. The set $\mathcal{V}\backslash\mathcal{CV}$ is the set of CFVs. It follows that, in a core graph, the set of CFVs is empty.

Let y and z be two distinct vertices of a graph G. By interlacing, when a vertex y or z is deleted from G, the nullity η_{G-y} or η_{G-z}, that is the multiplicity of the eigenvalue zero of $G-y$ or $G-z$, respectively, may take one of three values from $\eta_G - 1$ to $\eta_G + 1$. If the two distinct vertices y and z are deleted, then the nullity η_{G-y-z} of $G-y-z$ may take values in the range from $\eta_G - 2$ to $\eta_G + 2$. Let us call the graph having two particular distinct vertices y and z a *device* (G, y, z). The set of devices can be partitioned into three main *varieties*, namely *variety* 1 when both vertices are CVs, *variety* 2 when one vertex is a CFV and one a CV and *variety* 3 when both vertices are CFVs. A device (G, y, z) is said to be of *kind* $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z})$. Since η_{G-y} and η_{G-z} can take three values each and η_{G-y-z} can take five values, there are potentially 45 *kinds* of graphs relative to η_G. Interlacing further restricts the values of η_{G-y-z}. Moreover, there are *kinds* of graphs that exclude certain combinatorial properties, such as that of being bipartite, as we shall see in Section 5. In Section 2, we express the characteristic polynomial of $\phi(G-y, \lambda)$ as the sum of two terms in λ^{η_G} and $\lambda^{\eta_G} - 1$ with coefficients $f_a(\lambda)$ and $f_b(\lambda)$, respectively, each of which is a polynomial expanded in terms of the entries of the eigenvectors of $\mathbf{A}$ forming an orthonormal basis for $\mathbb{R}^n$. By comparing the diagonal entries of the adjugate of $(\lambda\mathbf{I} - \mathbf{A})$ and of the spectral decomposition of $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ we obtain, in Section 3, an expression for $\phi(G-y-z)$ as the sum of three terms in λ^{η_G}, $\lambda^{\eta_G} - 1$, $\lambda^{\eta_G} - 2$, respectively, with polynomial coefficients. Moreover, the well known *Jacobi's identity* (see, for instance, [4]), relating the entries of the adjugate of $(\lambda\mathbf{I} - \mathbf{A})$ with the characteristic polynomials of a graph G and those of particular subgraphs of G, is used to determine which *kinds* are not realized by any graph G.

In Section 4, the vertices of a graph are partitioned into three subsets of *type lower, middle or upper*, respectively, according to the vanishing or otherwise of $f_a(0)$ and $f_b(0)$. The Interlacing Theorem and Jacobi's identity impose restrictions on the 45 *kinds*, so that not all are possible. In Sections 5 and 6, we show why there exist exactly twelve *kinds* of device (G, y, z) and how they are partitioned into the three main *varieties*. In Section 7, we identify two interesting classes of graphs that in a certain sense have extremal nullities. The first one has the minimum possible nullity η_{G-y-z}, that is $\eta_G - 2$, for all pairs of distinct vertices y and z in a graph G. A graph G in the second class has the maximum possible nullity η_{G-y}, that is $\eta_G + 1$, for all vertices y of G. We show that devices within the second class can reach the maximum allowed $\eta_G + 2$ for the nullity η_{G-y-z} for some but *not* for *all* pairs of distinct vertices y and z in a graph G. A characterization is given of the non–singular devices within the second class having the inverse $\mathbf{A}^{-1}$ of the adjacency matrix $\mathbf{A}$ with zero entries only on the diagonal.

2 Characteristic polynomials

We first need to define some necessary notation.

Associated with the $n \times n$ adjacency matrix $\mathbf{A}$ of a n–vertex graph of nullity η_G, there is an *ordered orthonormal basis* $\mathbf{x}^r$, $1 \leq r \leq n$, for $\mathbb{R}^n$, consisting of eigenvectors of $\mathbf{A}$, with the η_G eigenvectors in the nullspace being labelled first. Let the $n \times 1$ column vector

$\mathbf{x}^r$ be (x^r_y), where for vertex y, $1 \le y \le n$. If

$$\mathbf{P} = \begin{pmatrix} x^1_1 & x^2_1 & \cdots & x^n_1 \\ x^1_2 & x^2_2 & \cdots & x^n_2 \\ \vdots & \vdots & \vdots & \vdots \\ x^1_n & x^2_n & \cdots & x^n_n \end{pmatrix},$$

where the i^{th} column of $\mathbf{P}$ is the eigenvector $\mathbf{x}^i$ belonging to the eigenvalue λ_i in the spectrum of $\mathbf{A}$, diagonalization of $\mathbf{A}$ is given by $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}[\lambda_i]$, where $\mathbf{D}[\lambda_i]$ is the diagonal matrix having λ_i as the i^{th} entry on the main diagonal. Expressing $\mathbf{A}$ in terms of $\mathbf{D}$ and $\mathbf{P}$ leads to the spectral decomposition theorem, which can also be applied to $(\lambda\mathbf{I} - \mathbf{A})^{-1}$. This leads to an expression for the characteristic polynomial of the adjacency matrix $\phi(G - y, \lambda)$ of $G - y$ which is given explicitly in terms of the eigenvector entries $\{x^i_y\}$. Together with Jacobi's identity, it will serve as a basis for the characterization of graphs according to those *kinds* that can exist.

Lemma 2.1.

$$\phi(G - y, \lambda) = \sum_{i=1}^{n} \frac{(x^i_y)^2}{(\lambda - \lambda_i)} \phi(G, \lambda).$$

Proof. The characteristic polynomial of the adjacency matrix $\phi(G - y, \lambda)$ of $G - y$ is the y^{th} diagonal entry $(\text{adj}(\lambda\mathbf{I} - \mathbf{A}))_{yy}$ of the adjugate of $(\lambda\mathbf{I} - \mathbf{A})$. For arbitrary λ, the matrix $(\lambda\mathbf{I} - \mathbf{A})$ is invertible and $\phi(G - y, \lambda) = ((\lambda\mathbf{I} - \mathbf{A})^{-1})_{yy}\phi(G, \lambda)$. Since $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}[\lambda_i]$, it follows that $\dfrac{\text{adj}(\lambda\mathbf{I} - \mathbf{A})}{\phi(G, \lambda)} = (\lambda\mathbf{I} - \mathbf{A})^{-1} = \mathbf{P}\mathbf{D}[\dfrac{1}{\lambda - \lambda_i}]\mathbf{P}^{-1}$.

Taking the y^{th} diagonal entry,

$$\frac{\phi(G - y, \lambda)}{\phi(G, \lambda)} = (x^1_y \; x^2_y \cdots x^n_y)\mathbf{D}[\frac{1}{\lambda - \lambda_i}] \begin{pmatrix} x^1_y \\ x^2_y \\ \vdots \\ x^n_y \end{pmatrix}$$

$$= \sum_{i=1}^{n} \frac{(x^i_y)^2}{(\lambda - \lambda_i)}. \tag{2.1}$$

□

For a graph G with adjacency matrix $\mathbf{A}$ of nullity η_G, let $s(\lambda)$ denote $\phi(G, \lambda)$. If the spectrum of $\mathbf{A}$ is $\lambda_1, \lambda_2, \cdots, \lambda_n$, starting with the zero eigenvalues (if any), we write

$$s(\lambda) = \prod_{\ell=1}^{n} (\lambda - \lambda_\ell) = \lambda^{\eta_G} s_0(\lambda) \quad \text{with} \quad s_0(0) \neq 0. \tag{2.2}$$

Partitioning the range of summation in Equation (2.1),

$$\frac{\phi(G - y, \lambda)}{\phi(G, \lambda)} = \sum_{i=1}^{\eta_G} \frac{(x^i_y)^2}{\lambda} + \sum_{i=\eta_G+1}^{n} \frac{(x^i_y)^2}{\lambda - \lambda_i}$$

Hence

$$\phi(G-y,\lambda) = \sum_{k=1}^{\eta_G}(x_y^k)^2 s_0(\lambda)\lambda^{\eta_G-1} + \sum_{k=\eta_G+1}^{n} \frac{(x_y^k)^2 s_0(\lambda)\lambda^{\eta_G}}{\lambda-\lambda_k} \tag{2.3}$$

which we shall express as

$$\phi(G-y,\lambda) = f_b\lambda^{\eta_G-1} + f_a\lambda^{\eta_G}. \tag{2.4}$$

3 Jacobi's Identity

Relative to (G, y, z), let us denote by $j(\lambda)$, or j, the entry of the adjugate adj$(\lambda\mathbf{I}-\mathbf{A})$ in the yz position, obtained by taking the determinant of the submatrix of $(\lambda\mathbf{I}-\mathbf{A})$ after deleting row y and column z and multiplying it by $(-1)^{y+z}$. We use the convention that $\eta_{G-y} \geq \eta_{G-z}$. Throughout the paper, where the context is clear, we may write s_0 for $s_0(\lambda)$, j for $j(\lambda)$, etc.

Let $s(\lambda), t(\lambda), u(\lambda), v(\lambda)$, often referred to simply as s, t, u and v respectively, be the characteristic polynomials $\phi(G,\lambda)$, $\phi((G-y),\lambda)$, $\phi((G-z),\lambda)$, $\phi((G-y-z),\lambda)$ of the graphs G, $G-y$, $G-z$ and $G-y-z$, respectively, that is, the determinants

$$\begin{aligned} s(\lambda) &= \det(\lambda\mathbf{I}-\mathbf{A}(G)) \\ t(\lambda) &= \det(\lambda\mathbf{I}-\mathbf{A}(G-y)) \\ u(\lambda) &= \det(\lambda\mathbf{I}-\mathbf{A}(G-z)) \\ v(\lambda) &= \det(\lambda\mathbf{I}-\mathbf{A}(G-y-z)). \end{aligned} \tag{3.1}$$

From Lemma 2.1,

$$t(\lambda) = \sum_{k=1}^{n}(x_y^k)^2\prod_{\ell\neq k}(\lambda-\lambda_\ell) \tag{3.2}$$

and

$$u(\lambda) = \sum_{k=1}^{n}(x_z^k)^2\prod_{\ell\neq k}(\lambda-\lambda_\ell). \tag{3.3}$$

We shall see that the characteristic polynomial $v(\lambda)$ of $G-y-z$ can also be expressed in terms of the eigenvector entries $\{x_y^r\}$ and $\{x_z^r\}$ associated with distinct vertices y and z.

Lemma 3.1. *For $y \neq z$, Jacobi's identity expresses the entry j of the adjugate of $\lambda\mathbf{I}-\mathbf{A}$ in the yz position, for a symmetric matrix $\mathbf{A}$, in terms of the characteristic polynomials s, u, t and v:*

$$j^2 = ut - sv$$

Expressing Equations (3.2) and (3.3) as in (2.4),

$$t(\lambda) = \sum_{k=1}^{\eta_G}(x_y^k)^2 s_0(\lambda)\lambda^{\eta_G-1} + \sum_{k=\eta_G+1}^{n} \frac{(x_y^k)^2 s_0(\lambda)\lambda^{\eta_G}}{\lambda-\lambda_k} = t_b\lambda^{\eta_G-1} + t_a\lambda^{\eta_G}, \tag{3.4}$$

and

$$u(\lambda) = \sum_{k=1}^{\eta_G}(x_z^k)^2 s_0(\lambda)\lambda^{\eta_G-1} + \sum_{k=\eta_G+1}^{n} \frac{(x_z^k)^2 s_0(\lambda)\lambda^{\eta_G}}{\lambda-\lambda_k} \tag{3.5}$$

$$= u_b\lambda^{\eta_G-1} + u_a\lambda^{\eta_G}$$

Now we consider pairs of vertices of G.

Since $\dfrac{\mathrm{adj}(\lambda I - A)}{\phi(G,\lambda)} = (\lambda I - A)^{-1} = PD[\dfrac{1}{\lambda-\lambda_i}]P^{-1}$,

$$j(\lambda) = \sum_{k=1}^{n}(x_y^k x_z^k)\prod_{\ell\neq k}(\lambda-\lambda_\ell.) \tag{3.6}$$

We can write

$$j(\lambda) = \sum_{k=1}^{\eta_G} x_y^k x_z^k s_0(\lambda)\lambda^{\eta_G-1} + \sum_{k=\eta_G+1}^{n}\frac{x_y^k x_z^k s_0(\lambda)\lambda^{\eta_G}}{\lambda-\lambda_k} \tag{3.7}$$

$$= j_b\lambda^{\eta_G-1} + j_a\lambda^{\eta_G}$$

The characteristic polynomial $v(\lambda)$ can be written as $v(\lambda) = \dfrac{u(\lambda)t(\lambda) - j^2(\lambda)}{s(\lambda)}$, that is $v(\lambda) = v_a\lambda^{\eta_G} + v_b\lambda^{\eta_G-1} + v_c\lambda^{\eta_G-2}$, where

$$v_c = \frac{1}{s_0}(u_b t_b - j_b^2) = \frac{1}{2}s_0\sum_{i=1}^{\eta_G}\sum_{\ell=1}^{\eta_G}(x_z^i x_y^\ell - x_z^\ell x_y^i)^2$$

$$v_b = \frac{1}{s_0}(u_a t_b + u_b t_a - 2j_a j_b) = s_0\sum_{i=1}^{\eta_G}\sum_{\ell=\eta_G+1}^{n}\frac{(x_z^i x_y^\ell - x_y^i x_z^\ell)^2}{\lambda-\lambda_\ell}$$

$$v_a = \frac{1}{s_0}(u_a t_a - j_a^2) = \frac{1}{2}s_0\sum_{i=\eta_G+1}^{n}\sum_{\ell=\eta_G+1}^{n}\frac{(x_y^i x_z^\ell - x_y^\ell x_z^i)^2}{(\lambda-\lambda_i)(\lambda-\lambda_\ell)} \tag{3.8}$$

4 Three types of vertex

By interlacing, we can identify three types of vertex according to the effect on the nullity on deletion. We call a vertex y *lower*, *middle* or *upper* if the nullity of $G-y$ is $\eta_G - 1$, η_G or $\eta_G + 1$, respectively. We shall distinguish among these three types of vertex according to the values of the functions f_a and f_b in Equation (2.4).

In Table 1 we show the entries of the orthonormal eigenvectors $\{\mathbf{x}^r\}$ in an ordered basis for $\mathbb{R}^n$ as presented in Section 2. We choose a vertex labelling such that the core vertices are labelled first. Note the zero submatrix corresponding to the CFVs.

We consider $\dfrac{\phi(G-y,\lambda)}{s_0\lambda^{\eta_G}}$ from Equation 2.3. It has poles at $\lambda = \mu_i$, $1 \le i \le h$, where, for $1 \le i \le h$, the μ_i are the h distinct non-zero eigenvalues of G. Moreover, the gradient of $\displaystyle\sum_{k=\eta_G+1}^{n}\frac{(x_y^k)^2}{\lambda-\lambda_k}$ is less than 0 for all $\lambda \neq \mu_i$. It follows that $\dfrac{\phi(G-y,\lambda)}{s_0\lambda^{\eta_G}}$ has at most $(h-1)$ roots strictly interlacing the h distinct eigenvalues of $\mathbf{A}$. Note that $\displaystyle\sum_{k=1}^{\eta_G}(x_y^k)^2 \ge 0$ with equality if and only if y is a CFV. Thus at $\lambda = 0$, f_b is non–zero if y is a CV and zero

vertex-entries \ eigenvector	x^1	…	x^{η_G}	x^{η_G+1}	…	x^n
x_1	*	…	*	*	…	*
x_2	*	…	*	*	…	*
⋮	*	…	*	*	…	*
$x_{\lvert CV\rvert}$	*	…	*	*	…	*
$x_{\lvert CV\rvert+1}$	0	…	0	*	…	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_n	0	…	0	*	…	*

Table 1: Ordered orthonormal basis of eigenvectors of **A** with * representing a possibly non–zero entry.

if it is a CFV. For a CFV y, $\sum_{k=\eta_G+1}^{n} \frac{(x_y^k)^2}{\lambda-\lambda_k}$ vanishes at $\lambda = 0$ when y is upper, and does not vanish when y is middle. Note that when $\sum_{k=1}^{\eta_G}(x_y^k)^2 = 0$, one of the $(h-1)$ interlacing roots may be zero. (†)

Different cases occur depending on the vanishing or otherwise of the real constant $\sum_{k=1}^{\eta_G}(x_y^k)^2$ and $\sum_{k=\eta_G+1}^{n} \frac{(x_y^k)^2}{\lambda-\lambda_k}$ at $\lambda = 0$. Equation (2.3) and the analysis in the previous paragraph (marked (†)) lead to the result that $\eta_G - 1 \leq \eta_{G-y} \leq \eta_G + 1$. This can be generalized for the multiplicity of any eigenvalue of G other than zero by replacing the cores and the nullspace of G by the μ_i–cores and μ_i–eigenspace of G (concepts introduced in [10]), thus giving another proof of the Interlacing Theorem.

Proposition 4.1. *The values of f_b and f_a of Expression (2.4) for $\phi(G-y,\lambda)$ at $\lambda = 0$ distinguish the three types of vertex as follows:*

Vertex y	*Status of y*	*The values of f_b and f_a*
Lower	CV	$f_b(0) \neq 0$
Middle	CF	$f_b(0) = 0$ *and* $f_a(0) \neq 0$
Upper	CFV	$f_b(0) = 0$ *and* $f_a(0) = 0$

Proof. Let y be a core vertex of a graph of nullity $\eta_G > 0$. There exists $x_y^k \neq 0$ for some k, $1 \leq k \leq \eta_G$. Then $f_b(0) \neq 0$, which is a necessary and sufficient condition for the multiplicity of the eigenvalue zero to be $\eta_G - 1$ for $G - y$. It follows that a vertex is *lower* if and only if it is a CV.

If y is a CFV, then $f_b(0) = 0$. For $G - y$, the multiplicity of the eigenvalue zero is at least η_G. If one of the roots of $\sum_{k=\eta_G+1}^{n} \frac{(x_y^k)^2}{\lambda-\lambda_k}$ is zero, then λ divides $\sum_{k=\eta_G+1}^{n} \frac{(x_y^k)^2}{\lambda-\lambda_k}$, the multiplicity of the eigenvalue zero is exactly $\eta_G + 1$ for $G - y$ and the vertex y is upper. Otherwise the multiplicity of the eigenvalue zero remains η_G for $G - y$ and the vertex y is middle. □

We consider three *varieties* of devices $\{(G, y, z)\}$ with pairs (y, z) of vertices, namely *variety* 1 with both y and z being CVs, *variety* 2 with z being a CV and y a CFV and *variety* 3 with both y and z being CFVs. Since a CFV can be upper or middle, *varieties* 2 and 3 are subdivided further, as seen in Table 3.

From Proposition 4.1,

for *variety* 1, $u_b \neq 0;\ t_b \neq 0$;

for *variety* 2, $u_b \neq 0;\ j_b = t_b = v_c = 0$;

for *variety* 3: $u_b = j_b = t_b = v_b = v_c = 0$.

Some of these *varieties* can be further subdivided according to the values at $\lambda = 0$ of v_c, v_b and v_a or j_a. From Proposition 4.1, $t_b(0) \neq 0$ if and only if y is a core vertex. Similarly $u_b \neq 0$ if and only if z is a core vertex. If at least one of z or y is core forbidden, then $j_b(0) = 0$. However, there are 'accidental' cases where $j_b(0)$ vanishes when both z and y are CVs, for example in C_4 and $K_{2,3}$ if the vertices y and z are connected by an edge. Indeed this is true for all bipartite core graphs of nullity at least two, since each of u and t has zero as a root. It follows that $E^{2\eta}$ is a factor of $j^2 = ut - sv = (j_b E^{\eta-1} + j_a E^{\eta})^2$ and therefore $j_b(0) = 0$.

5 Restrictions on the nullity of $G - y - z$

It is our aim to classify all graphs according to their *kind* defined by the quadruple

$$(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z}).$$

Not all the 45 *kinds* mentioned in Section 1 exist, as we shall discover. The classification will be given in Table 3 on Page 272. It is best possible since each kind is realized by some graph.

5.1 Restrictions arising from interlacing

In a device (G, y, z) of *kind* $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z})$, interlacing restricts the values that η_{G-y-z} can take. The following result shows an instance when η_{G-y-z} is determined by interlacing alone.

Lemma 5.1. *For* $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z}) = (\eta_G, \eta_G + 1, \eta_G - 1, \eta_{G-y-z})$*, the nullity* η_{G-y-z} *of* $G - y - z$ *is* η_G*.*

Hence, $(\eta_G, \eta_G + 1, \eta_G - 1, \eta_G)$ is the only *kind* where the nullities η_{G-y} and η_{G-z} differ by two. We say that it belongs to *variety* 2a.

In *kinds* where the nullities η_{G-y} and η_{G-z} differ by one, interlacing allows η_{G-y-z} to take either the value η_{G-y} or η_{G-z}. All three possible values of η_{G-y-z} are allowed by interlacing when $\eta_{G-y} = \eta_{G-z}$.

The symmetry about zero of the spectrum of a bipartite graph G (See for instance [8]) requires that the number of zero eigenvalues is $2k$, if G has an even number of vertices and $2k + 1$ if G has an odd number of vertices, for some $k \geq 0$. This implies that on deleting a vertex from a bipartite graph, the nullity changes parity. Therefore if the nullity of a graph G and of its vertex–deleted subgraph $G - y$ are the same, then G is not bipartite. Since on deleting a vertex a bipartite graph remains bipartite, it follows that a graph G of a *kind* where $\eta_G = \eta_{G-y}$ or $\eta_{G-y} = \eta_{G-y-z}$ cannot be bipartite.

Lemma 5.2. *If a vertex of a graph is middle, then the graph is not bipartite.*

Figure 1 shows a device (G, y, z) with a middle vertex z which becomes upper in $G - y$.

Figure 1: A graph with two middle vertices y and z.

5.2 Restrictions arising from Jacobi's Identity

Lemma 3.1 requires that $ut - sv$ which is j^2 has $2k,\ k \geq 0$, zero roots. Let g_f denote the number of zero roots of the real function f. Therefore, for kinds of graph that imply
(i) $g_{ut} = g_{sv} - 1$ and $g_u \neq g_t$
or (ii) $g_{ut} = g_{sv} + 1$ and $g_u = g_t$,
there is a contradiction and those *kinds* of graphs do not exist.

Lemma 5.3. *The following kinds of graphs do not exist:*
(i) $(\eta_G, \eta_G, \eta_G - 1, \eta_G)$*;*
(ii) $(\eta_G, \eta_G + 1, \eta_G + 1, \eta_G + 1)$*;*
(iii) $(\eta_G, \eta_G, \eta_G, \eta_G - 1)$.

Furthermore, if $g_{ut} = g_{sv}$ and g_{ut} is odd, then a graph of that *kind* exists if $ut - sv$ is zero at $\lambda = 0$, otherwise j^2 would have an odd number of zeros. Therefore, if $g_{ut} = g_{sv}$ and g_{ut} is odd, $j_b = 0$ at $\lambda = 0$.

Lemma 5.4. *Graphs with* $g_{ut} = g_{sv}$ *and* g_{ut} *odd exist provided* $j_b = 0$ *at* $\lambda = 0$. *They are non–bipartite and of one of the following kinds:*
(i) $(\eta_G, \eta_G, \eta_G - 1, \eta_G - 1)$*;*
(ii) $(\eta_G, \eta_G + 1, \eta_G, \eta_G + 1)$.

We shall call *kinds* (i) and (ii), in Lemma 5.4 above, *variety* 2b and 3b(i), respectively (See Table 3).

Lemma 5.5. *If* (G, y, z) *is a singular graph with* $g_{ut} < g_{sv}$ *and* g_{sv} *odd, then* (G, y, z) *is non–bipartite and of kind* $(\eta_G, \eta_G - 1, \eta_G - 1, \eta_G - 1)$.

Proof. If y and z are CVs, $g_{ut} < g_{sv}$, then $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z})$ is
(i) $(\eta_G, \eta_G - 1, \eta_G - 1, \eta_G)$ or
(ii) $(\eta_G, \eta_G - 1, \eta_G - 1, \eta_G - 1)$.

Now if furthermore, g_{sv} is given to be odd, then $\eta_{G-y-z} = \eta_G - 1$. It follows that $\eta_{G-y} = \eta_{G-y-z}$ Therefore, G is not bipartite. □

We shall call the graphs in Lemma 5.5 above, *variety* 1(iii) (See Table 3).

6 Kinds of graphs

In this section we determine the properties of a *kind* $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z})$ within each of the three *varieties*.

6.1 Graphs of variety 1

Graphs of *variety* 1, are necessarily singular and therefore have at least one core. There are at least two vertices in a core.

Lemma 6.1. *For a device (G, y, z) of variety 1 and nullity one, $j_b(0) \neq 0$ for core vertices y and z.*

Proof. For $\eta_G = 1$, a non-zero column of the adjugate adj$(\mathbf{A})$ is a kernel eigenvector of G [9]. The non–zero entries occur only at core vertices. Therefore, $j_b(0) \neq 0$. □

There are three types of pairs of vertices (CV,CV) for graphs of *variety* 1, depending on the nullity of $G - y - z$. Since $\eta_G \geq 1$ and $g_u = g_t = \eta_G - 1$, then the nullity g_v of $G - y - z$ can take any of the three values $\eta_G - 2, \eta_G$ and $\eta_G - 1$, corresponding to *variety* 1(i), 1(ii) and 1(iii), respectively.

Theorem 6.2. *For a device (G, y, z) of variety 1(iii), $j(0) \neq 0$ for core vertices y and z.*

Proof. For nullity one the result follows from Lemma 6.1. Now consider a graph with $\eta_G > 1$ of *variety* 1(iii), that is when $g_v = \eta_G - 1$. The number of zeros g_{ut} of ut is $2\eta_G - 2$ and less than that of sv which is odd. If j^2, which is $ut - sv$, is not to have an odd number of zeros, it follows, from $j = j_b\lambda^{\eta_G - 1} + j_a\lambda^{\eta_G}$, that $j_b \neq 0$ at $\lambda = 0$. □

For *variety* 1(i), the vertices y and z are CVs. Moreover, without loss of generality, the vertex z is a CV of the subgraph $G - y$. Only for *variety* 1(i) is $v_c \neq 0$.

Definition 6.3. The connected graphs G in the devices $\{(G, y, z)\}$ with all pairs of vertices $(y, z) \in \mathcal{V} \times \mathcal{V}$ being of *variety* 1(i) are said to form the class of *uniform–core graphs*.

Equivalently, $\eta_{G-y-z} = \eta_G - 2$, that is z is a CV of $G - y$ for all vertex pairs (y, z). It is clear that all vertices of a uniform–core graph are CVs, and that they remain so even in a vertex–deleted subgraph $G - y$ for any vertex y of G. Note that this is not the case in general; if y and z are two distinct core vertices of a graph G, then z need not remain a core vertex of $G - y$. We shall consider uniform–core graphs in more detail in Section 7.

6.2 Graphs of variety 2

In a device (G, y, z) of *variety* 2, (y, z) is a mixed vertex pair, that is exactly one vertex z of the pair (y, z) is a CV.

From Lemmas 5.1 and 5.3, the following result follows immediately.

Proposition 6.4. *In a device (G, y, z) of variety 2,*
(i) there is only one kind when y is upper, namely kind $(\eta_G, \eta_G + 1, \eta_G - 1, \eta_G)$ in variety 2a
and (ii) only one kind when y is middle, namely kind $(\eta_G, \eta_G, \eta_G - 1, \eta_G - 1)$ in variety 2b.

From Lemma 5.2, the graphs of *variety* 2b are non-bipartite.

Theorem 6.5. *In a device (G, y, z) of variety 2b, the term in $\lambda^{2\eta_G - 1}$ of j^2 is identically equal to zero.*

Proof. In *variety* 2b, a graph is of *kind* $(\eta_G, \eta_G, \eta_G - 1, \eta_G - 1)$. The parameter v_c vanishes and $v_b(\lambda) = \dfrac{u_b t_a}{s_0} \neq 0$. The number of zeros of ut is the same as that of sv. Therefore, $j^2 = ut - sv$ has at least $2\eta_G - 1$ zeros. In *variety* 2b, the term in $\lambda^{2\eta_G - 1}$ in its expansion is $u_b t_a - s_0 v_b$. Also v_c vanishes and $v_b(\lambda) = \dfrac{u_b t_a}{s_0} \neq 0$. Hence, $s_0 v_b = u_b t_a$ and the term in $\lambda^{2\eta_G - 1}$ in the expansion of j^2 is identically equal to zero, as expected from the fact that j^2 is a perfect square. □

The parameter v_b distinguishes between a graph in *variety* 2a and one in *variety* 2b.

Theorem 6.6. *For a graph in variety 2a, v_b vanishes at $\lambda = 0$. For a graph in variety 2b, $v_b \neq 0$ at $\lambda = 0$.*

Proof. For both *kinds* in *variety* 2, $u_b \neq 0$. For an upper vertex, $t_a = 0$ at $\lambda = 0$ and for a middle vertex $t_a \neq 0$ at $\lambda = 0$. Since $s_0 \neq 0$, it follows that for a graph in *variety* 2a $v_b = 0$ at $\lambda = 0$ and, for a graph in *variety* 2b, $v_b \neq 0$ at $\lambda = 0$.

□

6.3 Graphs of variety 3

We now consider *variety* 3 for (CFV,CFV) pairs, when t_b, u_b, j_b, v_b and v_c all vanish.

Interlacing provides three types of vertex pairs depending on whether a CFV in the pair (y, z) is upper or middle. When both vertices are upper (*variety* 3a), by Lemma 5.3 only *variety* 3a(i) for $g_v = \eta_G$ and *variety* 3a(ii), when $g_v = \eta_G + 2$ are allowed. The values at $\lambda = 0$ of v_a or j_a suffice to distinguish between graphs of *variety* 3(i) and 3(ii).

Theorem 6.7. *For variety 3a(i), both v_a and j_a are non–zero at $\lambda = 0$. For variety 3a(ii), both v_a and j_a vanish at $\lambda = 0$.*

Proof. For *variety* 3, $v_b = 0$. *Variety* 3a(i) is $(\eta_G, \eta_G + 1, \eta_G + 1, \eta_G)$. Since $v = v_a \lambda^{\eta_G}$ and $\eta_{G-y-z} = \eta_G$, $v_a \neq 0$ at $\lambda = 0$. Also $g_{j^2} = 2\eta_G$ so that $j_a \neq 0$ at $\lambda = 0$. *Variety* 3a(ii) is $(\eta_G, \eta_G +, \eta_G - 1, \eta_G - 1)$. Since $g_v = \eta_G + 2$, λ^2 divides v_a and λ divides all of the functions t_a, u_a and j_a. □

For *variety* 3b, one vertex is upper and one is middle. Interlacing allows only $g_v = \eta_G + 1$ and η_G, corresponding to *variety* 3b(i) and *variety* 3b(ii), respectively. Both v_b and j_b vanish at $\lambda = 0$. The value of j_a at $\lambda = 0$ distinguishes between *variety* 3b(i) and *variety* 3b(ii).

Theorem 6.8. *For variety 3b(i), j_a vanishes at $\lambda = 0$. For variety 3a(ii), j_a is non–zero at $\lambda = 0$.*

Proof. For *variety* 3b(i), λ divides j_a, as otherwise $ut - sv$ is not the perfect square j^2. *variety* 3b(ii) $g_v = \eta_G$ requires $j_a \neq 0$ at $\lambda = 0$. □

For *variety* 3c, both vertices are middle. The values at $\lambda = 0$ of t_a and u_a are non–zero. By Lemma 5.3, $g_v = \eta_G + 1$ or η_G, corresponding to *variety* 3c(i) and *variety* 3c(ii), respectively.

For *variety* 3c(ii), when $g_v = \eta_G$, v_a is non–zero at $\lambda = 0$. Two cases may occur. Either $j_a \neq 0$ at $\lambda = 0$ or the number of zeros of j_a is at least one. The former case is

Vertex y	Vertex z	*variety*
1	7	*variety* 1(i)
1	4	*variety* 1(ii)
1	2	*variety* 1(iii)
1	15	*variety* 2a
1	5	*variety* 2b
17	18	*variety* 3a(i)
15	17	*variety* 3a(ii)
5	15	*variety* 3b(i)
15	16	*variety* 3b(ii)
11	16	*variety* 3c(i)
5	6	*variety* 3c(iiA)
5	17	*variety* 3c(iiB)

Table 2: All *varieties* and *kinds* for the same graph G illustrated in Figure 2.

denoted by *variety* 3c(iiA). The latter case is *variety* 3c(iiB) for which the terms in $\lambda^{2\eta_G-2}$ and in $\lambda^{2\eta_G-1}$ of j^2 vanish.

The remaining case is for *variety* 3c(i) when $g_v = \eta_G + 1$ and λ divides v_a.

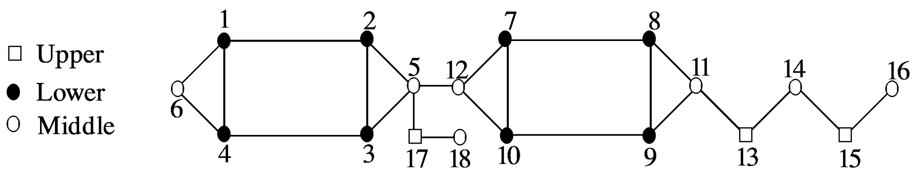

Figure 2: A device (G, y, z) of all possible *kinds* for various (y, z).

The graph in Figure 2 exhibits a device (G, y, z) of all *varieties* and *kinds* for different choices of (y, z).

The classification of devices into *kinds* and *varieties* has an application in chemistry in the identification of molecules (with carbon atoms in particular) that conduct or else bar conduction at the Fermi level. In the chemistry paper [3], conductors and insulators are classified into eleven cases that are essentially the twelve kinds of Table 3, with case 7 in [3] corresponding to the kinds $(\eta_G, \eta_G, \eta_G, \eta_G)$ in variety 3c(iiA) and $(\eta_G, \eta_G, \eta_G, \eta_G)$ in variety 3c(iiB). The latter two varieties are distinguishable by the non–vanishing or otherwise of $j_a(0)$.

7 Graphs with analogous vertex pairs

In general, vertex pairs in a graph may be of different varieties and kinds. We shall explore two interesting classes of graphs with the same extremal nullity (allowed by interlacing) for all vertex–deleted subgraphs. These emerge in the classification of devices $\{(G, y, z)\}$ according to their *kind* $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z})$. A pair of vertices y and z for which $\eta_{G-y} = \eta_{G-z}$ is said to be an *analogous vertex pair*.

Kind	Characterization	*Variety*	G bipartite
Two CVs		1	
$(g_s, g_t, g_u) = (\eta_G, \eta_G - 1, \eta_G - 1)$ $g_v = \eta_G - 2$	$v_c \neq 0$ & $t_b \neq 0$ & $u_b \neq 0$ & $\eta_G \geq 2$	1(i)	Allowed
$g_v = \eta_G$	$v_c = 0$ & $t_b \neq 0$ & $u_b \neq 0$ & $v_b(0) = 0$ & $\eta_G \geq 1$	1(ii)	Allowed
$g_v = \eta_G - 1$	$v_c = 0$ & $t_b \neq 0$ & $u_b \neq 0$ & $v_b(0) \neq 0$ & $\eta_G \geq 1$	1(iii)	Forbidden
CV and CFV		2	
$(g_s, g_t, g_u) = (\eta_G, \eta_G + 1, \eta_G - 1)$ $g_v = \eta_G$	$v_c = 0$ & $t_b = 0$ & $u_b \neq 0$ & $v_b(0) = 0$ & $\eta_G \geq 1$	2a	Allowed
$(g_s, g_t, g_u, g_v) = (\eta_G, \eta_G, \eta_G - 1)$ $g_v = \eta_G - 1$	$v_c = 0$ & $t_b = 0$ & $u_b \neq 0$ & $v_b(0) \neq 0$ & $\eta_G \geq 1$	2b	Forbidden
Two CFVs		3	
$(g_s, g_t, g_u) = (\eta_G, \eta_G + 1, \eta_G + 1)$		3a	
$g_v = \eta_G$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) = 0$ & $u_a(0) = 0$ & $v_a(0) \neq 0$	3a(i)	Allowed
$g_v = \eta_G + 2$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) = 0$ & $u_a(0) = 0$ & $v_a(0) = 0$	3a(ii)	Allowed
$(g_s, g_t, g_u) = (\eta_G, \eta_G + 1, \eta_G)$		3b	
$g_v = \eta_G + 1$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) = 0$ & $u_a(0) \neq 0$ & $v_a(0) = 0$	3b(i)	Forbidden
$g_v = \eta_G$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) = 0$ & $u_a(0) \neq 0$ & $v_a(0) \neq 0$	3b(ii)	Forbidden
$(g_s, g_t, g_u) = (\eta_G, \eta_G, \eta_G)$		3c	
$g_v = \eta_G + 1$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) \neq 0$ & $u_a(0) \neq 0$ & $v_a(0) = 0$	3c(i)	Forbidden
$g_v = \eta_G$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) \neq 0$ & $u_a(0) \neq 0$ & $v_a(0) \neq 0$	3c(ii)	Forbidden
$g_v = \eta_G$ & $j_a(0) \neq 0$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) \neq 0$ & $u_a(0) \neq 0$ & $v_a(0) \neq 0$ & $j_a(0) \neq 0$	3c(iiA)	Forbidden
$g_v = \eta_G$ & $j_a(0) = 0$	$v_c = 0$ & $t_b = 0$ & $u_b = 0$ & $v_b(0) = 0$ & $t_a(0) \neq 0$ & $u_a(0) \neq 0$ & $v_a(0) \neq 0$ & $j_a(0) = 0$	3c(iiB)	Forbidden

Table 3: A characterization of all devices (G, y, z) according to their *variety* and *kind*.

The first of these two classes consists of graphs G with the minimum possible nullity η_{G-y-z} for *all* pairs of distinct vertices y and z, (i.e., $\eta_G - 2$) and therefore also the minimum possible nullities η_{G-y} and η_{G-z} (i.e., $\eta_G - 1$). By Definition 6.3, these graphs form precisely the class of uniform–core graphs. On the other hand, the second of the two classes consists of graphs with the maximum possible nullity η_{G-y-z}, that is $\eta_G + 2$, for *some* pair of distinct vertices y and z, and therefore also the maximum possible nullities η_{G-y} and η_{G-z} (i.e., $\eta_G + 1$).

7.1 Uniform–core graphs

By Definition 6.3, each vertex pair in a uniform–core graph corresponds to a graph of variety (1i). Since the nullity of a graph is non–negative, and $\eta_{G-y-z} = \eta_G - 2$ for all vertex pairs $y,\ z$ of a uniform–core graph G, then the nullity of G is at least two. To understand better the core–structure of uniform–core graphs and be able to characterize them as a subclass of singular graphs, it is necessary to use their core structure with respect to a basis for their nullspace.

Let B be a basis for the η–dimensional nullspace of $\mathbf{A}$ of a singular graph G (with no isolated vertices) of nullity $\eta \geq 1$. As seen in [11], *Hall's Marriage* problem for sets, or the *Rado–Hall* Theorem for matroids, guarantees a vertex–subset S of distinct vertex representatives [1, 11], to represent a system $\mathcal{S}_{Cores}$ of cores corresponding to the vectors of B. This implies that deleting a vertex v representing a core F eliminates the core F from $G - v$, which will now have a new system of $\eta - 1$ cores. Also any $k \geq 1$ cores in a system $\mathcal{S}_{Cores}$ of η_G cores cover at least $k + 1$ vertices.

Theorem 7.1. *A device (G, y, z) is of variety 1(i) if and only if the two vertices y and z do not lie in one core only, i.e. at least two cores are needed to cover the vertices y and z.*

Proof. Consider a basis B for the nullspace of $\mathbf{A}$. The vertices y and z lie on at least one core of G. There are two possibilities. Firstly, B has exactly one vector with non–zero entries at positions associated with y and z. In this case $\eta_{G-y-z} = \eta_{G-y} = \eta_G - 1$, which does not correspond to *variety* 1(i). Secondly, B has at least two vectors with non–zero entries at positions associated with y or z, when $\eta_{G-y-z} = \eta_{G-y} - 1 = \eta_G - 2$, which corresponds to *variety* 1(i). The two core vertices must represent two distinct cores in a system $\mathcal{S}_{Cores}$ of η_G cores corresponding to a basis B for the nullspace [11]. □

A subclass $\mathcal{U}$ of uniform–core graphs can be constructed from nut graphs. A graph $G \in \mathcal{U}$ is obtained from a nut graph H on n vertices and m edges by duplicating each of the n vertices of H. Then G has $2n$ vertices and $4m$ edges. Figure 3 shows the uniform–core graph $G \in \mathcal{U}$ obtained from the smallest nut graph H. The nullity of G is $\dfrac{|\mathcal{V}(G)|}{2} + 1$. Deletion of any $\dfrac{|\mathcal{V}(G)|}{2} + 1$ vertices reduces the graph to a non–singular graph.

Let the vertices of G be labelled $1, 2, ..., n, 1', 2', ...n'$ where $\{1, 2, ...\}$ are the vertices of the nut graph H and $\{1', 2', ...\}$ are the duplicate vertices of $\{1, 2, ...\}$ in that order in G. Note that a vertex labelled r for $1 \leq r \leq n$ is adjacent to the original neighbours in H and also to precisely those primed vertices with the same numeric label. For instance, vertex 1 is adjacent to 2 and 7 in H and to 2, 2', 7 and 7' in G. The following result, expressing the adjacency matrix of $G \in \mathcal{U}$ in terms of the adjacency matrix of H, is immediate.

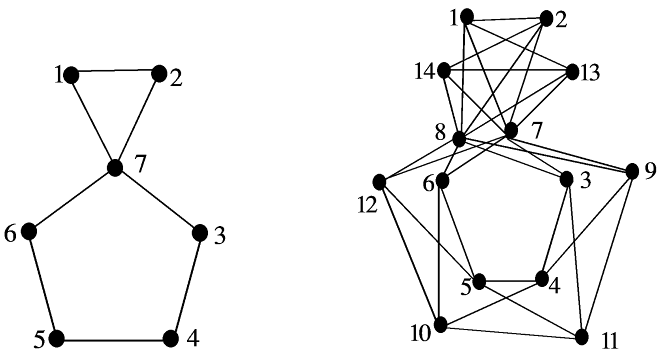

Figure 3: The smallest nut graph H and the uniform–core graph G derived from H.

Theorem 7.2. *If* $\mathbf{H}$ *is the adjacency matrix of the nut graph* H*, then the adjacency matrix of the uniform–core graph* $G \in \mathcal{U}$ *is* $\begin{pmatrix} \mathbf{H} & \mathbf{H} \\ \mathbf{H} & \mathbf{H} \end{pmatrix}$. *The spectrum of* G *consists of* n *eigenvalues equal in value to double the eigenvalues of* H *and an additional* n *zero eigenvalues corresponding to the* n *duplicate vertex pairs. If* $(x_1, x_2, \cdots, x_n)^t$ *is an eigenvector of* H *for an eigenvalue* μ*, then* $(x_1, x_2, \cdots, x_n, x_1, x_2, \cdots, x_n)^t$ *is an eigenvector of* G *for an eigenvalue* 2μ*.*

We shall now characterize uniform–core graphs by requiring that a set of vertex representatives of a system $\mathcal{S}_{Cores}$ of cores be an arbitrary subset of the vertices for all systems of cores.

Theorem 7.3. *A graph of nullity* η_G *is a uniform–core graph if and only if it is a singular graph such that the deletion of any subset of* η_G *vertices produces a non–singular graph.*

Proof. Let us relate the nullspace of $\mathbf{A}$ to the vertices of a uniform–core graph G of nullity η_G. Let S be any subset of η_G vertices of G labelled $\{1, 2, \cdots, \eta_G\}$ and let B be an ordered basis for the nullspace of $\mathbf{A}$. If all pairs of vertices give a graph of *variety* 1(i), then no two vertices lie in only one core of $\mathcal{S}_{Cores}$. Therefore, it is possible to obtain a new ordered basis B' for the nullspace of $\mathbf{A}$, by linear combination of the vectors in B, such that, for $1 \leq i \leq \eta_G$, only the vector i of B' has a non–zero entry at position i [11]. Removal of any vertex in S destroys precisely one eigenvector of B' reducing the nullity by one. Deletion of all the vertices in S destroys all the kernel eigenvectors and leaves a non–singular graph. □

A characterization of the subclass $G \in \mathcal{U}$ of uniform–core graphs uses the operation NEPS (*non–complete extended p-sum*) of a nut graph and K_2. The graph product NEPS is described for instance in [2].

Definition 7.4. Given a collection $\{G_1, G_2, \cdots, G_k, \cdots, G_n\}$ of graphs and a corresponding set $\mathcal{B} \subseteq \{0,1\}^n \backslash \{(0, 0, ..., 0)\}$, called the *basis*, of non–zero binary n-tuples, the NEPS of $G_1, G_2, ..., G_n$ is the graph with vertex set $\mathcal{V}(G_1) \times \mathcal{V}(G_2) \times \cdots \times \mathcal{V}(G_n)$ in which two vertices $\{w_1, w_2, \cdots, w_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ are adjacent if and only if there exists $(\beta_1, \beta_2, \cdots, \beta_n) \in \mathcal{B}$ such that $w_i = y_i$ whenever $\beta_i = 0$ and w_i is adjacent to y_i whenever $\beta_i = 1$.

Lemma 7.5. [2] *If for* $1 \leq i \leq n$, $\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in_i}$ *is the spectrum of* G_i*, of order* n_i *for* $1 \leq i \leq n$*, then the spectrum of the NEPS of* $G_1, G_2, \cdots, G_n$ *with respect to basis* $\mathcal{B}$ *is* $\{\sum_{\beta \in \mathcal{B}} \lambda_{1i_1}^{\beta_1}, \lambda_{2i_2}^{\beta_2}, \cdots, \lambda_{ni_n}^{\beta_n} : i_k = 1, 2, ..., n_k \quad \& \quad k = 1, 2, ..., n\}$.

The following result follows from the construction of a uniform–core graph $G \in \mathcal{U}$.

Theorem 7.6. *A uniform–core graph* $G \in \mathcal{U}$ *is the NEPS of a nut graph* G_1 *and* K_2 *with basis* $\{(1,0),(1,1)\}$.

From Lemma 7.5 and Theorem 7.6, the spectrum of the uniform–core graph $G \in \mathcal{U}$ is $\lambda_i + \lambda_i\lambda_j$ where $\{\lambda_i\}$ is the spectrum of the nut graph H and $\{\lambda_j\} = \{1, -1\}$ is the spectrum of K_2. This agrees with the result in Theorem 7.2.

7.2 Non-singular graphs with a complete weighted inverse

We shall now look into the second class of devices. Such a graph G is a device (G, y, z), of variety 3a(ii), for some pair of distinct vertices y and z. Graphs which are devices (G, y, z), of variety 3a(ii), for a particular pair of vertices y and z exist, as shown in the example of Figure 2 for vertex connections 15 and 17. Can a graph G be a device (G, y, z), of variety 3a(ii), for *all* vertex pairs $\{y, z\}$? The question amounts to determining whether it is possible to have $\eta(G - y - z)$ equal to the maximum allowed nullity relative to $\eta(G)$, that is $\eta(G) + 2$, for all vertex pairs $\{y, z\}$. The answer is in the negative.

Lemma 7.7. *It is impossible that a graph G is a device* (G, y, z) *of variety 3a(ii) for all pairs of distinct vertices* y *and* z.

Proof. Suppose G is a graph which is a device (G, y, z) of *variety* 3a(ii) for all pairs of distinct vertices y and z. This requires that each of the graphs $G - y$ and $G - z$ is singular and therefore has CVs. Deletion of a CV from $G - y$, restores the nullity back to $\eta(G)$. Hence it is impossible to achieve $\eta(G - y - z) = \eta(G) + 2$, for *all* vertex pairs $\{y, z\}$. □

By Lemma 5.3(ii), the kind $(\eta_G, \eta_{G-y}, \eta_{G-z}, \eta_{G-y-z})$=$(\eta_G, \eta_G+1, \eta_G, \eta_G+1)$ is impossible. Hence the only devices (G, y, z) within the second class that have the maximum value of $\eta(G - y)$ relative to η_G, for all vertices y, are of kind $(\eta_G, \eta_G + 1, \eta_G + 1, \eta_G)$. Our focus is on the non–singular graphs of this kind having the inverse $\mathbf{A}^{-1}$ equal to the adjacency matrix of the complete graph with real non–zero weighted edges and no loops.

The smallest candidate is K_2. Indeed $\mathbf{A}(K_2) = \mathbf{A}(K_2))^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 7.8. Let G be a non–singular graph G with the off–diagonal entries of the inverse $\mathbf{A}^{-1}$ of its adjacency matrix $\mathbf{A}$ being non–zero and real, and all the diagonal entries of $\mathbf{A}^{-1}$ being zero. Then G is said to be a *nuciferous graph.*

The motivation for the name nuciferous graph (meaning nut–producing graph) will become clear from Theorem 7.9. To characterize this class of graphs, let us consider the deck $\{G - v : v \in \mathcal{V}\}$ of subgraphs, as in the investigation of the polynomial reconstruction problem [10].

Theorem 7.9. *Let G be a nuciferous graph. Then G is either* K_2 *or each vertex–deleted subgraph* $G - v$ *is a nut graph.*

Proof. Let $\mathbf{Q}$ be the $n-1 \times n$ matrix obtained from $\mathbf{A}^{-1}$ by suppressing the diagonal entry from each column. Therefore each entry of $\mathbf{Q}$ is non–zero.

Let the i^{th} column of $\mathbf{Q}$ be $\mathbf{q}_i := (q_{(1)i}, q_{(2)i}, ..., q_{(i-1)i}, q_{(i+1)i}, q_{(i+2)i}, ..., q_{(n)i})^t$ for $2 \leq i \leq n-1$. The first and last columns are $\mathbf{q}_1 := (q_{(2)1}, q_{(3)1}, ..., q_{(n)1})^t$ and $\mathbf{q}_n := (q_{(1)n}, q_{(2)n}, ..., q_{(n-1)n})^t$, respectively.

Since $\mathbf{A}\mathbf{A}^{-1}$ is the identity matrix $\mathbf{I}$, then $\mathbf{A}(G-i)\mathbf{q}_i = \mathbf{0}$ for all $1 \leq i \leq n$. Therefore $\mathbf{q}_i$ is a kernel eigenvector (with non–zero entries) of $G-i$ for all the vertices i. Hence $G-i$ is a core graph. By interlacing, it has nullity one. It follows that each vertex–deleted subgraph is a nut graph. □

From Lemma 7.7, nuciferous devices (G, y, z) are not of type of *variety* 3a(ii) for all pairs of distinct vertices y and z. Moreover, from Theorem 7.9, for $G \neq K_2$, each vertex–deleted subgraph is a nut graph and therefore has nullity one. On deleting a vertex from a nut graph, the nullity becomes zero. Hence a candidate graph G cannot be of *variety* 3a(ii) for *any* pair of vertices y and z.

Theorem 7.10. *Let G be a nuciferous graph G. If G is not K_2, then*
(i) it has order at least eight;
(ii) the device (G, y, z) is of variety 3a(i) for **all** *pairs of distinct vertices y and z;*
(iii) the graph G is not bipartite.

Proof. (i) Since nut graphs exist for order at least seven [12], it follows, from Theorem 7.9, that a nuciferous graph G, of order at least three, has at least eight vertices.

(ii) From the proof of Lemma 7.7, a nuciferous graph G is of *kind* $(\eta_G, \eta_G + 1, \eta_G + 1, \eta_G)$. Thus G is a device (G, y, z) of *variety* 3a(i) for all pairs of distinct vertices y and z.

(iii) From Theorem 7.9, $G-y$ and $G-z$ are nut graphs and therefore cannot be bipartite [12]. Hence G has odd cycles and cannot be bipartite. □

To date, no graph (except K_2) has been found to satisfy the condition of Theorem 7.9. An exhaustive search on all graphs on up to 10 vertices and all chemical graphs on up to 16 vertices reveals no counter example. We conjecture the following result.

Conjecture 7.11. *There are no graphs for which* every *vertex–deleted subgraph is a nut graph.*

8 Chemical implications

Graph theory has strong connections with the study of physical and chemical properties of all-carbon frameworks such as those in benzenoids, fullerenes and carbon nanotubes. The eigenvalues and eigenvectors of the adjacency matrix of the molecular graph (the graph of the carbon skeleton) are used in qualitative models of the energies and spatial distributions of the mobile π electrons of such systems. Specifically, graphs and their nullities figure in simple theories of ballistic conduction of electrons by conjugated systems. In the simplest formulation [3] of the SSP (Source and Sink Potential) [5] approach to molecular conduction, the variation of electron transmission with energy is qualitatively modelled in terms of the characteristic polynomials of G, $G-y$, $G-z$, $G-y-z$, where G is the molecular graph and vertices y and z are in contact with wires. (This is the motivation for the definition of a *device* in the present paper.) As a consequence, the transmission at the Fermi level (corresponding here to $\lambda = 0$) obeys selection rules couched in terms of the nullities η_G, η_{G-y}, η_{G-z}, and η_{G-y-z} [7], motivating the definition of *kinds here. In terms of the* varieties defined here, the SSP theory predicts conduction at the Fermi level for connection across the vertex pair (y, z) for $1(ii)$, $1(iii)$, $3a(i)$, $3b(ii)$, $3c(i)$ and $3c(iiA)$, and, conversely, insulation at the Fermi level for $1(i)$, $2a$, $2b$, $3a(ii)$, $3b(i)$ and $3c(iiB)$.

The two classes of graphs with analogous vertex pairs and certain extremal conditions on the nullity of their vertex-deleted subgraphs, explored in Section 7 are envisaged to have interesting developments in spectral graph theory. Moreover, the classification of graphs into *varieties* and *kinds* has an application in chemistry in the identification of molecules (with carbon atoms in particular) that conduct or else bar conduction at the Fermi level that has already been investigated in [3]. According to the SSP theory, the first class, the *uniform–core graphs*, corresponds to insulation at the Fermi–level for all two vertex connections and the second class, the nuciferous graphs, to Fermi–level conducting devices (G, y, z) for all pairs of distinct vertices y and z. The latter class has the additional properties that it consists of devices corresponding to non-singular graphs that are Fermi–level insulators when $y = z$. Therefore nuciferous graphs have no non–bonding orbital and are conductors for all distinct vertex connection pairs and insulators for all one vertex connections. We conjecture that the only nuciferous graph is K_2.

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Unordered multiplicity lists of wide double paths

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Abstract

Recently, Kim and Shader analyzed the multiplicities of the eigenvalues of a Φ-binary tree. We carry this discussion forward extending their results to a larger family of trees, namely, the wide double path, a tree consisting of two paths that are joined by another path. Some introductory considerations for dumbbell graphs are mentioned regarding the maximum multiplicity of the eigenvalues. Lastly, three research problems are formulated.

Keywords: Graph, tree, matrix, eigenvalues, multiplicities, inverse eigenvalue problem.

Math. Subj. Class.: 15A18, 05C50

1 Preliminaries

For a given $n \times n$ real symmetric matrix $A = (a_{ij})$, we define the graph of A, and write $G(A)$, as the undirected graph whose vertex set is $\{1, \ldots, n\}$ and edge set is $\{ij \mid i \neq j \text{ and } a_{ij} \neq 0\}$. On the other hand, for a given (weighted) graph G, we may define $A(G) = (a_{ij})$ to be the (real) symmetric matrix whose graph $G(A)$ is G. We focus our attention to the set

$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n} \mid A = A^T \text{ and } G(A) = G\} ,$$

i.e., the set of all symmetric matrices sharing a common graph G on n vertices. Nevertheless, all results can easily be extended to complex Hermitian matrices.

If G is a tree, then the matrix $A(G)$ is called *acyclic*. In particular, if G is a path, we order the vertices of G such that $A(G)$ is a tridiagonal matrix.

We will often omit the mention of the graph of the matrix if it is clear from the context.

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Let us denote the (algebraic) multiplicity of the eigenvalue θ of a symmetric matrix $A = A(G)$ by $m_A(\theta)$. The $(n-1) \times (n-1)$ principal submatrix, formed by the deletion of row and column indexed by i, which is equivalent to removing the vertex i from G, is designated by $A(G\backslash i)$.

Among the linear algebra community, most of the results on multiplicities of eigenvalues are mainly confined to trees motivated by the Parter-Wiener Theorem [16] and to Cauchy's Interlacing Theorem.

For a more detailed account on the subject the reader is referred to [16]. We remark that the Parter-Wiener Theorem was reformulated in the survey work [7], by the second author, motivated by the earlier seminal work of C. Godsil on matchings polynomials [9, 10, 11]. The same approach produced a result for the multiplicities of an eigenvalue of a matrix involving certain paths of the underlying graph, with many interesting applications to general graphs [4].

Theorem 1.1. *[6, 8] Let P be a path that does not contain any edge of any cycle in G. Then*

$$m_{A(G\backslash P)}(\theta) \geqslant m_{A(G)}(\theta) - 1\,. \tag{1.1}$$

Since a tree has no cycles, the inequality (1.1) is true for any path in a tree, which generalizes a result for the standard adjacency acyclic matrices [9].

The inequality (1.1) can provide us an upper bound for the multiplicity of an eigenvalue of a graph. The next result was established by R.A. Beezer in [3, Lemma 2.1] and it gives a lower bound. It was originally stated for standard adjacency matrices, but it can be proved for weighted adjacency matrices.

Lemma 1.2. *Let us suppose that $H_1, \ldots, H_k$ be graphs, and let $v_1, \ldots, v_t$ be additional vertices. Construct a graph H by adding new edges that have one endpoint in the set $\{v_1, \ldots, v_t\}$ and the other endpoint in a vertex of some H_i. If*

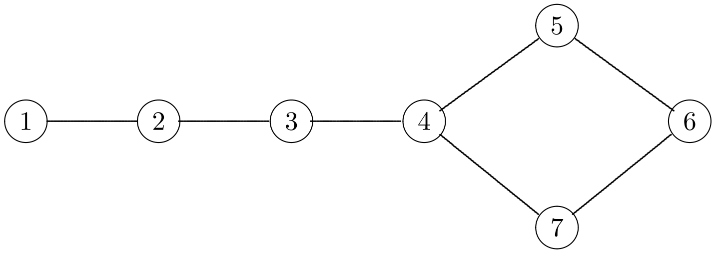
$$A = \left(\begin{array}{ccc|c} A_1 & & 0 & \\ & \ddots & & C^T \\ 0 & & A_k & \\ \hline & C & & D \end{array}\right) \in \mathcal{S}(H)\,,$$

where $A_i \in \mathcal{S}(H_i)$, for $i = 1 \ldots k$, D is a real diagonal block, and C has t rows, then

$$m_A(\lambda) \geqslant \sum_{i=1}^{k} m_{A_i}(\lambda) - t\,. \tag{1.2}$$

We remark that (1.2) is a special case of Cauchy-type interlacing theorems for block Hermitian matrices. In fact, if λ is an eigenvalue of the upper block decomposition $A_1 \oplus \cdots \oplus A_k$, a block vector calculation shows that the dimension the eigenspace of λ of A is at least as great as the dimension of the intersection of the eigenspace of λ of B and the null space of C (for more details the reader is referred to [15]).

Interestingly, Lemma 1.2 provides us an algorithm construct matrices of certain graphs where the maximum multiplicity is attained. For example, let us consider the $(4, 3)$-tadpole graph T



Considering the path joining vertices 1 and 4 in (1.1) we see that, for any eigenvalue λ of $A \in \mathcal{S}(T)$, $m_A(\lambda) \leqslant 2$ (see also [2, 8]). On the other hand, from (1.2), setting A_1 for the Jacobi matrix whose graph is the path joining vertices 1 and 2 (an edge) and A_2 for the matrix whose graph is the cycle containing vertices 4, 5, 6, and 7, we have

$$m_A(\lambda) \geqslant m_{A_1}(\lambda) + m_{A_2}(\lambda) - 1 \geqslant 1 + 2 - 1 = 2\,.$$

Therefore, if we want to construct a matrix in $\mathcal{S}(T)$ with an eigenvalue, say $\sqrt{2}$ of maximal multiplicity, we may set

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

constructed as in [5]. Then any matrix of the form

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & * & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

has $\sqrt{2}$ (and, in this case, $-\sqrt{2}$ too) as an eigenvalue of maximal multiplicity 2.

In general, for a given (m, n)-tadpole $G_{m,n}$, if λ is an eigenvalue of $A \in \mathcal{S}(G_{m,n})$ but not eigenvalue of $A(C_m)$, then λ is simple, from (1.1).

In this note, we show how to use inequality (1.1) to generalize a result on the maximum multiplicity of an eigenvalue of a Φ-binary tree due to Kim and Shader [18] to a more general family of trees: a *wide double path*. A wide double path is consists of two paths that are joined by another path. Unordered multiplicity lists for two classes of wide double paths are established generalizing previous results. At the end three new research problems are pointed out.

First, we show how combining both bounds (1.1) and (1.2) in order to give necessary and sufficient conditions for an eigenvalue has maximal multiplicity in the case of a dumb-bell graph.

2 Dumbbell graphs

A *dumbbell graph* is obtained by joining two cycles by a path. We will assume that the length of this path is greater than 2. Nevertheless, all results are true for lower lengths, with slight modifications.

Dumbbells graphs are a special class of bicyclic graphs, i.e., connected graphs in which the number of edges equals the number of vertices plus one. These graphs are well considered in graph theory, combinatorics, and optimization literature [12, 13, 14, 19, 20, 21]. Much attention has recently been paid to the spectral properties of these non-acyclic graphs [22, 23].

We start this section establishing an upper bound for the multiplicity of an eigenvalue of a dumbbell graph.

Proposition 2.1. *The maximum multiplicity of an eigenvalue of a dumbbell graph is* 3*.*

Proof. Let P be the path intersecting both pendant cycles of the dumbbell graph D. Observe that the (disconnect) subgraph $D\backslash P$ is the union of two paths. Then, from (1.1),

$$m_{A(D)}(\lambda) \leqslant m_{A(D\backslash P)}(\lambda) + 1 \leqslant 2 + 1 = 3\,,$$

for any eigenvalue λ of D. □

Next we characterize the matrices where an eigenvalue attains the maximum multiplicity.

Proposition 2.2. *Let D be a dumbbell graph and let P be the path intersecting both pendant cycles C_1 and C_2 at the vertices v_1 and v_ℓ, respectively. If λ is an eigenvalue of $A \in \mathcal{S}(D)$ of multiplicity* 3*, then λ is an eigenvalue of $A(C_1 \setminus v_1)$ and of $A(C_2 \setminus v_\ell)$.*

Proof. Again, by Theorem 1.1, if $m_{A(D)}(\lambda) = 3$, then

$$m_{A(D\backslash P)}(\lambda) \geqslant 3 - 1 = 2\,.$$

Since $D\backslash P = (C_1 \setminus v_1) \cup (C_2 \setminus v_\ell)$ and both $C_1 \setminus v_1$ and $C_2 \setminus v_\ell$ are paths, λ must be an eigenvalue of each $A(C_1 \setminus v_1)$ and of $A(C_2 \setminus v_\ell)$. □

Corollary 2.3. *Let D be a dumbbell graph and let $P = v_1 v_2 \cdots v_\ell$ be the path intersecting both pendant cycles C_1 and C_2 at the vertices v_1 and v_ℓ, respectively. If λ is an eigenvalue of $A \in \mathcal{S}(D)$ of multiplicity* 3*, then λ is an eigenvalue of both $A(C_1)$ and $A(C_2)$.*

Proof. Considering the path $P' = v_2 \cdots v_\ell$ and (1.1), we have

$$m_{A(D\backslash P')}(\lambda) \geqslant 2\,.$$

Now we only have to observe that $D \setminus P' = C_1 \cup (C_2 \setminus v_\ell)$ and $m_{A(C_2 \setminus v_\ell)}(\lambda) = 1$. □

Note that we can conclude a result more general than Corollary 2.3. In fact, λ should be an eigenvalue of any tadpole graph obtaining by joining, for example, the path $v_i \cdots v_\ell$, for any $i = 2, \ldots, \ell - 1$, to the cycle C_2 at v_ℓ.

The next result is a straightforward consequence of (1.1).

Proposition 2.4. *Let D be a dumbbell graph and let P be the path intersecting both pendant cycles C_1 and C_2 at the vertices v_1 and v_ℓ, respectively. If λ is an eigenvalue of $A \in \mathcal{S}(D)$ but neither an eigenvalue of $A(C_1 \setminus v_1)$ nor of $A(C_2 \setminus v_\ell)$ or neither an eigenvalue of $A(C_1)$ nor of $A(C_2)$, then $m_A(\lambda) = 1$, i.e., λ is a simple eigenvalue of A.*

As we mentioned before, Lemma 1.2 provides an interesting algorithm producing matrices of certain graphs with eigenvalues of maximum multiplicity. For, let us consider the real number λ. For a given path of order k_1, let A_1 be a tridiagonal matrix of order k_1, with eigenvalue λ, using (according Jean Favard) the *simple et élégant* Wendroff's algorithm [24], appeared a long time ago but somehow has not received so far the appropriate consideration by the linear algebra community. Now, using the general algorithm established in [5], it is possible to construct periodic Jacobi matrices, say A_2 and A_3, whose cycles are of orders k_2 and k_3, respectively, with λ being an eigenvalue of both matrices. Let us set

$$A = \left(\begin{array}{ccc|cc} A_1 & & & & \\ & A_2 & & x^T & y^T \\ & & A_3 & & \\ \hline & x & & 0 & 0 \\ & y & & 0 & 0 \end{array}\right) \in \mathcal{S}(H)\,,$$

where x is the $0,1$ vector with 1's in the position 1 and $k_1 + 1$ and 0 elsewhere, and, analogously, y is the $0,1$ vector with 1's in the position k_1 and $k_1 + k_2 + 1$ and 0 elsewhere. Then λ is an eigenvalue of A of multiplicity 3. In fact, from (1.2),

$$m_A(\lambda) \geqslant m_{A_1}(\lambda) + m_{A_2}(\lambda) + m_{A_3}(\lambda) - 2 = 1 + 2 + 2 - 2 = 3\,.$$

3 Maximum multiplicities

We now turn back our attention to a family of binary trees. Recall that a *binary tree* is a tree such that the degree of each vertex is no more than three. In this section we will consider the family constituted by the trees of the following form: take five paths $P_1, \ldots, P_5$ and two vertices u and v; join any terminal vertex of P_1, P_2, and P_5 to u; the other terminal vertex of P_5 and any terminal vertex of P_3 and of P_4 are added to v. These trees can also be seen as consisting of two paths that are joined by another path. Therefore, we will call them *wide double paths*. The paths $P_1, \ldots, P_4$ are the *legs* (or *branches*) of such tree.

In [18] Kim and Shader studied several spectral properties of the so-called Φ-binary trees. It is a particular case of the trees under discussion now: P_5 has size 1 (i.e., a single vertex) and the longest legs among the four legs are connected to different terminal vertices.

Theorem 3.1. *Let T be a wide double path and $A \in \mathcal{S}(T)$. Then the maximum multiplicity of an eigenvalue of A is 3.*

Proof. We only have to apply (1.1), for example, to the path $P_1 - u - P_5 - v - P_3$. □

Theorem 3.2. *For a given wide double path T, λ is an eigenvalue of $A \in \mathcal{S}(T)$ of maximum multiplicity if and only if λ is an eigenvalue of each path $P_1, \ldots, P_5$.*

Proof. Set $A_i = A(P_i)$, for $i = 1, \ldots, 5$. Let us assume first that $m_A(\lambda) = 3$. Considering the path $P_1 - u - P_5 - v - P_3$ in T, from (1.1), it follows

$$m_{A_2}(\lambda) + m_{A_4}(\lambda) \geqslant 3 - 1 = 2\,.$$

Thus, $m_{A_2}(\lambda) = m_{A_4}(\lambda) = 1$. Analogously, we prove $m_{A_1}(\lambda) = m_{A_3}(\lambda) = 1$. It remains to prove that $m_{A_5}(\lambda) = 1$. In fact, since P_5 can be obtained from T deleting the paths $P_1 - u - P_2$ and $P_3 - u - P_4$, we have again, from (1.1),

$$1 \geqslant m_{A_5}(\lambda) \geqslant m_{\tilde{A}}(\lambda) - 1 \geqslant m_A(\lambda) - 2 = 1\,,$$

where $\tilde{A} = A(H)$, with H being the generalized star with center u and legs P_1, P_2, and P_5.

Conversely, if $m_{A_i}(\lambda) = 1$, for $i = 1, \ldots, 5$, then

$$3 \geqslant m_A(\lambda) \geqslant \sum_{i=1}^{5} m_{A_i}(\lambda) - 2 = 3\,.$$

from Theorem 3.1 and (1.2). □

Let us set ℓ_i for the order of the path P_i, with $i = 1, \ldots, 5$, in a wide double path W.

Corollary 3.3. *The number n_3 of eigenvalues with multiplicity 3 of a wide double path is at most $\min\{\ell_1, \ldots, \ell_5\}$.*

Corollary 3.4. *[18, Theorem 2(a)] Let T be a Φ-binary tree and $A \in \mathcal{S}(T)$. Then there are no eigenvalues of multiplicity 4 or more, and the number n_3 of eigenvalues with multiplicity 3 is at most one. Furthermore, if $\lambda \in \sigma(A)$ with $m_A(\lambda) = 3$, then the diagonal entry of A corresponding to the axis vertex of T is λ.*

We now investigate the eigenvalues of multiplicity 2. The first result is a consequence of Lemma 1.2 and Theorem 3.2.

Lemma 3.5. *Let T be a wide double path and let $A \in \mathcal{S}(T)$. If $\lambda \in \sigma(A)$ is an eigenvalue of exactly four of the paths $P_1, \ldots, P_5$, then $m_A(\lambda) = 2$.*

As before, n_2 denotes the number of eigenvalues of multiplicity 2 of a given matrix.

Theorem 3.6. *Let W be a wide double path and $r = \min\{\ell_i + \ell_j \mid i = 1, 2,\ j = 3, 4\}$. Then*

$$n_2 \leqslant r - 2n_3\,. \tag{3.1}$$

Proof. By Theorem 3.2, if $\lambda_1, \ldots, \lambda_{n_3}$ are the distinct eigenvalues of A of multiplicity 3, then they must be (simple) eigenvalues of both $A(P_2)$ and $A(P_4)$. Taking into account Lemma 3.5, the inequality (3.1) follows. □

We remark that Theorem 3.2 is in fact much more general. An analogous result can be proved for any generalized caterpillar, i.e., a tree for which removing the legs produces a path, and the maximum multiplicity of any eigenvalue is equal to the number of legs minus one. In particular we have the following result.

Lemma 3.7. *Let S be a generalized star and $A \in \mathcal{S}(S)$. Then $m_A(\lambda) = 2$ if and only if λ is an eigenvalue of each leg.*

Theorem 3.8. *Let T be a wide double path and $A \in \mathcal{S}(T)$. Then $m_A(\lambda) = 2$ if and only if λ is a simple eigenvalue of the paths P_1, P_2, and of the generalized star with center v and legs P_3, P_4, P_5, or is a simple eigenvalue of the paths P_3, P_4, and of the generalized star with center u and legs P_1, P_2, P_5.*

Proof. Let us assume that $m_A(\lambda) = 2$. From (1.1), for any path P in T, $m_{A(T\setminus P)}(\lambda) \geqslant 1$. Therefore, if λ is not an eigenvalue of P_1 (P_2), then it must be an eigenvalue of both P_3 and P_4. Moreover, if S denotes the generalized star with center u and legs P_1, P_2, P_5, then $m_{A(S)}(\lambda) \geqslant 1$. The other assertion is set in a similar fashion.

The converse follows from (1.1) and (1.2). □

We now address the question on the number n_1 of simple eigenvalues of $A \in \mathcal{S}(W)$. Since

$$n = n_1 + 2n_2 + 3n_3 = \ell_1 + \cdots + \ell_5 + 2\,,$$

on the one hand, we have

$$n_1 \leqslant n\,. \tag{3.2}$$

In fact, the equality is attained when we construct $A(L_1), \ldots, A(L_5)$, with distinct eigenvalues. On the other hand,

$$\begin{aligned} n_1 &= \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + 2 - 2n_2 - 3n_3 \\ &\geqslant \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + 2 - 2r + n_3 \\ &\geqslant |\ell_1 - \ell_2| + |\ell_3 - \ell_4| + \ell_5 + 2\,. \end{aligned} \tag{3.3}$$

Interestingly, we observe that, from the lower bound (3.3), any matrix in $\mathcal{S}(W)$ must have at least $\ell_5 + 2$ simple eigenvalues. This generalizes [18, Corollary 3].

4 Unordered multiplicity lists

Recall that if $m_1 \geqslant \cdots \geqslant m_k$, with $m_1 + \cdots + m_k = n$, are the multiplicities of the distinct eigenvalues of an n-by-n symmetric matrix A, then $(m_1, \ldots, m_k)$ is the *unordered multiplicity list* (or *list*) of the eigenvalues of A. By *unordered multiplicity list of a graph* G we mean the set of unordered multiplicity lists for all matrices in $\mathcal{S}(G)$.

Without loss of generality, we will assume that $\ell_1 \geqslant \ell_2$ and $\ell_3 \geqslant \ell_4$. Moreover, we convention that for a finite sequence of real numbers $a_1, \ldots, a_i$, with $i \leqslant 0$, is empty.

We are able now to find the unordered multiplicity lists of the wide double path under discussion.

Theorem 4.1. *Let W be a wide double path of order n, with $\ell_1 \geqslant \ell_2$ and $\ell_3 \geqslant \ell_4$. Then the set of unordered multiplicity lists of W consists of the positive integer lists of the form*

$$(\underbrace{3, \ldots, 3}_{n_3}, \underbrace{2, \ldots, 2}_{n_2}, \underbrace{1, \ldots, 1}_{n_1})\,, \tag{4.1}$$

with $0 \leqslant n_3 \leqslant \min\{\ell_2, \ell_4, \ell_5\}$, $0 \leqslant n_2 \leqslant \ell_2 + \ell_4 - 2n_3$, and $n_1 = n - 2n_2 - 3n_3$.

Proof. From our discussion in the previous section, it is clear that any unordered multiplicity list of W is of the form (4.1).

Conversely, let S_1 (resp., S_2) be the generalized star with center vertex u (resp., v) and legs L_1, L_2, L_5 (resp., L_3, L_4, L_5). Now, for $k = 0, \ldots, \min\{\ell_2, \ell_4, \ell_5\}$, $p = 0, \ldots, \ell_2 - k$, and $q = 0, \ldots, \ell_4 - k$, let us consider the $\ell_1 + \ell_5 - k + p + 1$ distinct real numbers

$$\beta_1, \ldots, \beta_{\ell_4 - k - q}, \theta_1, \ldots, \theta_{\ell_1 - \ell_4 + \ell_5 + p + q + 1}$$

strictly interlacing with the $\ell_1 + \ell_5 - k + p$ (distinct) real numbers

$$\lambda_1, \ldots, \lambda_{\ell_5}, \alpha_1, \ldots, \alpha_{\ell_1-k}, \tilde{\alpha}_{\ell_2-k-p+1}, \ldots, \tilde{\alpha}_{\ell_2-k}\,,$$

and the $\ell_3 + \ell_5 - k + q + 1$ distinct real numbers

$$\alpha_1, \ldots, \alpha_{\ell_2-k-p}, \mu_1, \ldots, \mu_{\ell_3-\ell_2+\ell_5+q+p+1}$$

strictly interlacing with the $\ell_3 + \ell_5 - k + q$ (distinct) real numbers

$$\lambda_1, \ldots, \lambda_{\ell_5}, \beta_1, \ldots, \beta_{\ell_3-k}, \tilde{\beta}_{\ell_4-k-q+1}, \ldots, \tilde{\beta}_{\ell_4-k}\,.$$

Now we consider $A \in \mathcal{S}(W)$ such that

$$\begin{aligned}
\sigma(A(L_5)) &= \{\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{\ell_5}\}\\
\sigma(A(L_1)) &= \{\lambda_1, \ldots, \lambda_k, \alpha_1, \ldots, \alpha_{\ell_2-k-p}, \alpha_{\ell_2-k-p+1}, \ldots, \alpha_{\ell_1-k}\}\\
\sigma(A(L_2)) &= \{\lambda_1, \ldots, \lambda_k, \alpha_1, \ldots, \alpha_{\ell_2-k-p}, \tilde{\alpha}_{\ell_2-k-p+1}, \ldots, \tilde{\alpha}_{\ell_2-k}\}\\
\sigma(A(S_1)) &= \{\lambda_1, \lambda_1, \ldots, \lambda_k, \lambda_k, \alpha_1, \ldots, \alpha_{\ell_2-k-p}, \beta_1, \ldots, \beta_{\ell_4-k-q}, \theta_1, \ldots,\\
&\qquad \theta_{\ell_1-\ell_4+\ell_5+p+q+1}\}\\
\sigma(A(L_3)) &= \{\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_{\ell_4-k-q}, \beta_{\ell_4-k-q+1}, \ldots, \beta_{\ell_3-k}\}\\
\sigma(A(L_4)) &= \{\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_{\ell_4-k-q}, \tilde{\beta}_{\ell_4-k-q+1}, \ldots, \tilde{\beta}_{\ell_4-k}\}\\
\sigma(A(S_2)) &= \{\lambda_1, \lambda_1, \ldots, \lambda_k, \lambda_k, \beta_1, \ldots, \beta_{\ell_4-k-q}, \alpha_1, \ldots, \alpha_{\ell_2-k-p}, \mu_1, \ldots,\\
&\qquad \mu_{\ell_3-\ell_2+\ell_5+q+p+1}\},
\end{aligned}$$

The existence of the Jacobi matrices is granted by [24] and the two generalized stars by [17, Theorem 11].

It is clear that the unordered multiplicity list of A is

$$(\underbrace{3, \ldots, 3}_{t_3}, \underbrace{2, \ldots, 2}_{t_2}, \underbrace{1, \ldots, 1}_{t_1})\,, \tag{4.2}$$

with $t_3 = k$, $t_2 = \ell_2 + \ell_4 - 2k - p - q$, and $t_1 = (\ell_1 - \ell_2) + (\ell_3 - \ell_4) + \ell_5 + 2 + k + 2(p+q)$. Note that $0 \leqslant k + 2(p+q) \leqslant 2(\ell_2 + \ell_4)$. □

Observe that with $1 = \ell_5 \leqslant \ell_2, \ell_4, \leqslant \ell_1, \ell_3$, we are able to recover the results in [18] for Φ-binary trees. Moreover, Theorem 4.1 can also be applied for $\ell_5 = 0$ [1, 17].

Finally, a tree is *minimal* provided there is a matrix such that number of distinct eigenvalues is equal to the diameter (counting edges) plus one. From Theorem 4.1, we conclude that the wide double paths are minimal, for $\ell_2, \ell_4 \leqslant \ell_1, \ell_3 \leqslant \ell_5$. In fact, with $t_3 = 0$, $t_2 = \ell_2 + \ell_4$, and $t_1 = (\ell_1 - \ell_2) + (\ell_3 - \ell_4) + \ell_5 + 2$, since the number of distinct eigenvalues is $\ell_1 + \ell_3 + \ell_5 + 2$ and the diameter is $\ell_1 + \ell_3 + \ell_5 + 1$.

5 Open problems

In this paper, we provided the solution for the inverse eigenvalue problem of a wide double path, generalizing the results for the very particular case of the family of the so-called Φ-binary trees, recently established. The approach adopted here is different, offering a general result with a more concise proof.

A natural generalization a wide double path is when we have more than 2 legs adjacent to the “central” vertices u and v. Let us suitably call such trees as *wide double generalized stars*.

Problem 1. *Characterize the unordered multiplicity lists of a wide double generalized star.*

It seems this is not a difficult problem to handle and an elegant characterization similar to Theorem 4.1 may be achieved.

A more hard problem is related an analogous characterization for binary trees.

Problem 2. *Characterize the unordered multiplicity lists of a binary tree.*

Our results may also be seen as the starting point for a more meaningful study:

Problem 3. *What are the unordered multiplicity lists of trees having maximum multiplicity* 3*?*

Of course, from our approach, the previous question can be extended to general graphs. Probably new techniques will need to be developed for this attractive and vast area of research. Some computational experiments allow us to assert that there will be some surprising multiplicity lists.

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GCD-Graphs and NEPS of Complete Graphs

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Abstract

A gcd-graph is a Cayley graph over a finite abelian group defined by greatest common divisors. Such graphs are known to have integral spectrum. A non-complete extended p-sum, or NEPS in short, is well-known general graph product. We show that the class of gcd-graphs and the class of NEPS of complete graphs coincide. Thus, a relation between the algebraically defined Cayley graphs and the combinatorially defined NEPS of complete graphs is established. We use this link to show that gcd-graphs have a particularly simple eigenspace structure, to be precise, that every eigenspace of the adjacency matrix of a gcd-graph has a basis with entries $-1, 0, 1$ only.

Keywords: Integral graphs, Cayley graphs, graph products.

Math. Subj. Class.: 05C25, 05C50

1 Introduction

Given a set $B \subseteq \{0,1\}^n$ and graphs $G_1, \ldots, G_n$, the NEPS (non-complete extended p-sum) of these graphs with respect to *basis* B, $G = \mathrm{NEPS}(G_1, \ldots, G_n; B)$, has as its vertex set the Cartesian product of the vertex sets of the individual graphs, $V(G) = V(G_1) \times \cdots \times V(G_n)$. Distinct vertices $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in V(G)$ are adjacent in G, if and only if there exists some n-tuple $(\beta_1, \ldots, \beta_n) \in B$ such that $x_i = y_i$, whenever $\beta_i = 0$, and $x_i,\ y_i$ are distinct and adjacent in G_i, whenever $\beta_i = 1$. In particular, $\mathrm{NEPS}(G_1; \{(1)\}) = G_1$ and $\mathrm{NEPS}(G_1; \emptyset) = \mathrm{NEPS}(G_1; \{(0)\})$ is the graph without edges on the vertices of G_1.

The NEPS operation generalizes a number of known graph products, all of which have in common that the vertex set of the resulting graph is the Cartesian product of the input vertex sets. For example, $\mathrm{NEPS}(G_1, \ldots, G_n; \{(1, 1, \ldots, 1)\}) = G_1 \otimes \ldots \otimes G_n$ is the

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product of $G_1, \ldots, G_n$ (cf. [10], "direct product" in [15]). As can be seen, unfortunately, the naming of graph products is not standardized at all. The "Cartesian product" of graphs in [15] is even known as the "sum" of graphs in [10]. With respect to this seemingly arbitrary mixing of sum and product terminology, let us point out that here the term "sum" (and also the "p-sum" contained in the NEPS acronym) indicates that the adjacency matrix of the constructed product graph arises from a certain sum of matrices (involving the adjacency matrices of the input graphs). Refer to [10] or [11] for the history of the notion of NEPS. We remark that the NEPS operation can be generalized even further, see e.g. [12] and [21].

Next, we consider the important class of Cayley graphs [13]. These graphs have been and still are studied intensively because of their symmetry properties and their connections to communication networks, quantum physics and other areas [8], [13]. Let Γ be a finite, additive group. A subset $S \subseteq \Gamma$ is called a *symbol* (also: connection set, shift set) of Γ if $-S = \{-s : s \in S\} = S$, $0 \notin S$. The undirected *Cayley graph over* Γ *with symbol* S, denoted by $\mathrm{Cay}(\Gamma, S)$, has vertex set Γ; two vertices $a,\ b \in \Gamma$ are adjacent if and only if $a - b \in S$.

Let us now construct the class of gcd-graphs. The greatest common divisor of nonnegative integers a and b is denoted by $\gcd(a, b)$, $\gcd(0, b) = \gcd(b, 0) = b$. If $x = (x_1, \ldots, x_r)$ and $m = (m_1, ..., m_r)$ are tuples of nonnegative integers, then we set

$$\gcd(x, m) = (d_1, \ldots, d_r) = d, \;\; d_i = \gcd(x_i, m_i) \text{ for } i = 1, \ldots, r.$$

For an integer $n \geq 1$ we denote by Z_n the additive group of integers modulo n, the ring of integers modulo n, or simply the set $\{0, 1, \ldots, n-1\}$. The particular choice will be clear from the context. Let Γ be an (additive) finite abelian group represented as a direct sum of cyclic groups,

$$\Gamma = Z_{m_1} \oplus \ldots \oplus Z_{m_r}\ ,\ m_i \geq 1 \text{ for } i = 1, \ldots, r.$$

Suppose that d_i is a divisor of m_i, $1 \leq d_i \leq m_i$, for $i = 1, \ldots, r$. For the divisor tuple $d = (d_1, \ldots, d_r)$ of $m = (m_1, \ldots, m_r)$ we define

$$S_\Gamma(d) \;=\; \{x = (x_1, \ldots, x_r) \in \Gamma : \gcd(x, m) = d\}.$$

Let $D = \{d^{(1)}, \ldots, d^{(k)}\}$ be a set of distinct divisor tuples of m and define

$$S_\Gamma(D) \;=\; \bigcup_{j=1}^{k} S_\Gamma(d^{(j)}).$$

Observe that the union is actually disjoint. The sets $S_\Gamma(D)$ shall be called *gcd-sets* of Γ. We define the class of *gcd-graphs* as the Cayley graphs $\mathrm{Cay}(\Gamma, S)$ over a finite abelian group Γ with symbol S a gcd-set of Γ. The most prominent members of this class are perhaps the unitary Cayley graphs $X_n = \mathrm{Cay}(Z_n, U_n)$, where $U_n = S_{Z_n}(1)$ is the multiplicative group of units of Z_n (cf. [16], [17], [22]).

The main goal of this paper is to show in Section 2 that every gcd-graph is isomorphic to a NEPS of complete graphs. Conversely, every NEPS of complete graphs is isomorphic to a gcd-graph over some abelian group. This relation is remarkable since it allows us to define gcd-graphs either algebraically (via Cayley graphs) or purely combinatorially (via NEPS). The characterization of gcd-graphs as NEPS of complete graphs reveals some new access to structural properties of gcd-graphs. As a first application, we show in Section

3 that every gcd-graph has simply structured eigenspace bases for all of its eigenvalues. This means that for every eigenspace a basis can be found whose vectors only have entries from the set $\{0, 1, -1\}$. It is known that other graph classes exhibit a similar eigenspace structure, although not necessarily for all of their eigenspaces [9], [20], [25]. Finally, we present some open problems in Section 4.

2 Isomorphisms between NEPS of complete graphs and gcd-graphs

We are going to show in several steps that gcd-graphs and NEPS of complete graphs are the same.

Lemma 2.1. *Let* $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$ *and* $d = (d_1, \ldots, d_r)$ *a tuple of positive divisors of* $m = (m_1, \ldots, m_r)$. *Define* $b = (b_i) \in \{0, 1\}^r$ *by*

$$b_i = \begin{cases} 1 & \text{if } d_i < m_i, \\ 0 & \text{if } d_i = m_i. \end{cases}$$

Then we have

$$\mathrm{Cay}(\Gamma, S_\Gamma(d)) = \mathrm{NEPS}(\mathrm{Cay}(Z_{m_1}, S_{Z_{m_1}}(d_1)), \ldots, \mathrm{Cay}(Z_{m_r}, S_{Z_{m_r}}(d_r)); \{b\}).$$

Proof. Both $\mathrm{Cay}(\Gamma, S_\Gamma(d))$ and the above NEPS have the same vertex set Γ. It remains to show that they have the same edge set.

Let $x,\ y \in \Gamma$ with $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r)$ and suppose that $x \neq y$. Now x and y are adjacent in $\mathrm{Cay}(\Gamma, S_\Gamma(d))$ if and only if $\gcd(x_i - y_i, m_i) = d_i$ for $i = 1, \ldots, r$. The latter condition means that in case $d_i < m_i$ the vertices x_i and y_i are adjacent in $G_i = \mathrm{Cay}(Z_{m_i}, S_{Z_{m_i}}(d_i))$, and in case $d_i = m_i$ we have $x_i = y_i$. But this is exactly the condition for adjacency of x and y in $\mathrm{NEPS}(G_1, \ldots, G_r; \{b\})$. □

The following lemma allows us to break down the Cayley graphs that form the factors of the NEPS mentioned in Lemma 2.1. Each factor can be transformed into a gcd-graph over a product of cyclic groups of prime power order. Using Lemma 2.1 once again, we obtain a representation of the original graph as a NEPS of NEPS of gcd-graphs over cyclic groups of prime power order.

Lemma 2.2. *Let the integer* $m \geq 2$ *and a proper divisor* $d \geq 1$ *of* m *be given as products of powers of distinct primes,*

$$m = \prod_{i=1}^{r} m_i, \quad m_i = p_i^{\alpha_i},\ \alpha_i > 0 \text{ for } i = 1, \ldots, r,$$
$$d = \prod_{i=1}^{r} d_i, \quad d_i = p_i^{\beta_i}\ ,\ 0 \leq \beta_i \leq \alpha_i \text{ for } i = 1, \ldots, r.$$

If we set $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$ *and* $\tilde{d} = (d_1, \ldots, d_r)$*, then there exists an isomorphism*

$$\mathrm{Cay}(Z_m, S_{Z_m}(d)) \ \simeq\ \mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d})).$$

Proof. By the Chinese remainder theorem [23] we know that every $z \in Z_m$ is uniquely determined by the congruences

$$z \equiv z_i \mod m_i, \ z_i \in Z_{m_i} \text{ for } i = 1, \dots, r.$$

This gives rise to a bijection $Z_m \to \Gamma$ by virtue of $z \mapsto (z_1, \dots, z_r) =: \tilde{z}$. We show that this bijection induces an isomorphism between $\mathrm{Cay}(Z_m, S_{Z_m}(d))$ and $\mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d}))$.

Let $x, y \in Z_m$, $x \neq y$. Note that $\tilde{x}$ and $\tilde{y}$ are vertices of $\mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d}))$. The vertices x and y are adjacent in $\mathrm{Cay}(Z_m, S_{Z_m}(d))$ if and only if $\gcd(x-y, m) = d$. This is equivalent to $\gcd(x_i - y_i, m_i) = d_i$ for every $i = 1, \dots, r$. Now this means $\gcd(\tilde{x} - \tilde{y}, \tilde{m}) = \tilde{d}$, with $\tilde{m} = (m_1, \dots, m_r)$, which is the condition for adjacency of $\tilde{x}$ and $\tilde{y}$ in $\mathrm{Cay}(\Gamma, S_\Gamma(\tilde{d}))$. □

Next we shall prove a lemma that helps us consolidate the nesting of NEPS operations into a single NEPS operation. As a result, we then know that every single-divisor tuple gcd-graph is isomorphic to a NEPS of gcd-graphs over cyclic groups of prime power order.

Lemma 2.3. *Let*

$$H = \mathrm{NEPS}(H^{(1)}, \dots, H^{(t)}; B) \tag{2.1}$$

be a NEPS of graphs $H^{(j)}$ with respect to basis B such that each graph $H^{(j)}$ is itself a NEPS of graphs $G_i^{(j)}$ with respect to basis $B^{(j)}$,

$$H^{(j)} = \mathrm{NEPS}(G_1^{(j)}, \dots, G_{r_j}^{(j)}; B^{(j)}) \text{ for } j = 1, \dots, t. \tag{2.2}$$

Then there exists a set $B' \subseteq \{0,1\}^r$, $r = r_1 + \ldots + r_t$, such that

$$H \simeq \mathrm{NEPS}(G_1^{(1)}, \dots, G_{r_1}^{(1)}, \dots, G_1^{(t)}, \dots, G_{r_t}^{(t)}; B'). \tag{2.3}$$

Proof. We show that in (2.1) the graph $H^{(1)}$ can be replaced by $G_1^{(1)}, \dots, G_{r_1}^{(1)}$. More precisely, we construct a set $\tilde{B}$ such that

$$H \simeq \mathrm{NEPS}(G_1^{(1)}, \dots, G_{r_1}^{(1)}, H^{(2)}, \dots, H^{(t)}; \tilde{B}), \ \ \tilde{B} \subseteq \{0,1\}^{r_1+t-1}. \tag{2.4}$$

An analogous procedure can be repeated for $H^{(2)}, \dots, H^{(t)}$ until we end up with the representation (2.3) of H.

In the original representation (2.1) every vertex x of the vertex set $V(H)$ has the form

$$x = (x^{(1)}, \dots, x^{(t)}), \ x^{(j)} \in V(H^{(j)}) \text{ for } j = 1, \dots, t. \tag{2.5}$$

By (2.2) each coordinate $x^{(j)}$ is itself an r_j-tuple, in particular

$$x^{(1)} = (x_1^{(1)}, \dots, x_{r_1}^{(1)}), \ x_i^{(1)} \in V(G_i^{(1)}) \text{ for } i = 1, \dots, r_1 .$$

Expansion of $x^{(1)}$ in (2.5) yields

$$\begin{aligned} &\tilde{x} = (x_1^{(1)}, \dots, x_{r_1}^{(1)}, x^{(2)}, \dots, x^{(t)}), \\ &x_i^{(1)} \in V(G_i^{(1)}) \text{ for } i = 1, \dots, r_1, \ \ x^{(j)} \in V(H^{(j)}) \text{ for } j = 2, \dots, t. \end{aligned} \tag{2.6}$$

This is the representation of vertices for (2.4).

Now we adapt the basis set B to the new representation of vertices of H such that adjacencies remain unchanged. Let the distinct vertices x and y of H be given in their original representation according to (2.5) and in their new representation $\tilde{x}$, $\tilde{y}$ according to (2.6).

$$x = (x^{(1)}, \ldots, x^{(t)}), \ \ y = (y^{(1)}, \ldots, y^{(t)}),$$
$$\tilde{x} = (x_1^{(1)}, \ldots, x_{r_1}^{(1)}, x^{(2)}, \ldots, x^{(t)}), \ \ \tilde{y} = (y_1^{(1)}, \ldots, y_{r_1}^{(1)}, y^{(2)}, \ldots, y^{(t)}).$$

For each $b = (b_1, \ldots, b_t) \in B$ we define a set $\tilde{B}(b) \subseteq \{0,1\}^{r_1+t-1}$ such that

$$x,\ y \text{ adjacent with respect to } b \ \Leftrightarrow\ \tilde{x},\ \tilde{y} \text{ adjacent with respect to } \tilde{B}(b). \tag{2.7}$$

Case 1: $b_1 = 0$.
For x and y to be adjacent with respect to b we must have $x^{(1)} = y^{(1)}$. If this is satisfied, then x and y are adjacent, if and only if $(x^{(2)}, \ldots, x^{(t)})$ and $(y^{(2)}, \ldots, y^{(t)})$ are adjacent with respect to $(b_2, \ldots, b_t)$. We achieve (2.7) by setting $\tilde{b} = (0, \ldots, 0, b_2, \ldots, b_t)$ (first r_1 entries equal to zero) and $\tilde{B}(b) = \{\tilde{b}\}$.
Case 2: $b_1 = 1$.
Now x and y are adjacent with respect to b, if and only if $x^{(1)}$ and $y^{(1)}$ are adjacent in $H^{(1)}$ and $x^{(2)}, \ldots, x^{(t)}$ and $y^{(2)}, \ldots, y^{(t)}$ are equal or adjacent with respect to $b_2, \ldots, b_t$, respectively. By (2.2) vertices $x^{(1)}$ and $y^{(1)}$ of $H^{(1)}$ are adjacent, if and only if they are adjacent with respect to some $b^{(1)} = (b_1^{(1)}, \ldots, b_{r_1}^{(1)}) \in B^{(1)}$. In this case we satisfy (2.7) by setting

$$\tilde{B}(b) = \{(b_1^{(1)}, \ldots, b_{r_1}^{(1)}, b^{(2)}, \ldots, b^{(t)}) :\ b^{(1)} \in B^{(1)}\}.$$

Finally, we collect the new basis tuples in $\tilde{B} = \cup\{\tilde{B}(b) \ :\ \ b \in B\}$ and thus achieve (2.4). □

The next step towards our goal is to show that a single-divisor gcd-graph over a cyclic group of prime power order is actually isomorphic to a NEPS of complete graphs.

We denote the complete graph on n vertices by K_n. For our purposes, we assume that the vertex set of K_n is $Z_n = \{0, 1, \ldots, n-1\}$.

Lemma 2.4. *Let $m = p^\alpha$ be a prime power, $d = p^\beta$ a divisor of m, $0 \le \beta \le \alpha$. Then the gcd-graph over Z_m with respect to d is isomorphic to a NEPS of α copies of the complete graph K_p, i.e.*

$$\mathrm{Cay}(Z_m, S_{Z_m}(d)) \simeq \mathrm{NEPS}(K_p, \ldots, K_p; B) \textit{ for some } B \subseteq \{0,1\}^\alpha.$$

Proof. In case $\beta = \alpha$ we have $\mathrm{Cay}(Z_m, S_{Z_m}(m)) \simeq \mathrm{NEPS}(K_p, \ldots, K_p; \{(0, \ldots, 0)\})$. So we may now assume $\beta < \alpha$.

Let us denote $G = \mathrm{Cay}(Z_m, S_{Z_m}(d))$ and $H = \mathrm{NEPS}(K_p, \ldots, K_p; B)$ (where the basis B is not yet fixed). For every $x \in Z_m$ let $(x_0, \ldots, x_{\alpha-1})$ be defined by the p-adic representation of x,

$$x = \sum_{i=0}^{\alpha-1} x_i p^i,\ 0 \le x_i < p \text{ for } i = 0, \ldots, \alpha - 1.$$

We shall assume that the vertex set of K_p is Z_p. Define the bijection $\varphi:\ Z_m \to Z_p \oplus \cdots \oplus Z_p = Z_p^\alpha$ by $\varphi(x) = (x_0, \ldots, x_{\alpha-1})$. We now construct a basis set $B \subseteq \{0,1\}^\alpha$ such that φ induces an isomorphism between G and H. Observe that for every $z \in Z_m$,

$$\gcd(z,m) = d \Leftrightarrow z_i = 0 \text{ for every } i < \beta \text{ and } z_\beta \neq 0.$$

This leads to the definition of B as follows:

$$B = \{(b_0, \ldots, b_{\alpha-1}) \in \{0,1\}^\alpha :\ b_i = 0 \text{ for every } i < \beta,\ b_\beta = 1\}.$$

Let $x, y \in Z_m$, $x \neq y$, $\varphi(x) = (x_0, \ldots, x_{\alpha-1})$, $\varphi(y) = (y_0, \ldots, y_{\alpha-1})$. Now x and y are adjacent in G if and only if $\gcd(x-y, m) = d$, which means $x_i - y_i = 0$ for every $i < \beta$ and $x_\beta - y_\beta \neq 0$. Thanks to our choice of B, this is exactly the condition for $\varphi(x)$ and $\varphi(y)$ being adjacent in H. □

Theorem 2.5. *Let G be an arbitrary gcd-graph, $G = \mathrm{Cay}(\Gamma, S_\Gamma(D))$, $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$, $D = \{d^{(1)}, \ldots, d^{(k)}\}$ a set of divisor tuples of $m = (m_1, \ldots, m_r)$. If $n = p_1 \cdots p_t$ is the prime factorization of $n = m_1 \cdots m_r$, then*

$$G \simeq \mathrm{NEPS}(K_{p_1}, \ldots, K_{p_t}; B) = H \textit{ for some } B \subseteq \{0,1\}^t.$$

Proof. Each divisor tuple in D gives rise to a graph $G^{(j)} = \mathrm{Cay}(\Gamma, S_\Gamma(d^{(j)}))$, $j = 1, \ldots, k$. By application of the preceding lemmas of this section we know that

$$G^{(j)} \simeq \mathrm{NEPS}(K_{p_1}, \ldots, K_{p_t}; B^{(j)}) = H^{(j)} \text{ for some } B^{(j)} \subseteq \{0,1\}^t.$$

The graphs $G^{(j)}$ constitute an edge disjoint decomposition of G. Now, for every divisor tuple $d^{(1)}, \ldots, d^{(k)} \in D$, we perform the decomposition process outlined by the lemmas in exactly the same way, in the sense that the vertex numberings of the resulting graphs $H^{(j)}$ are coherent. Then the graphs $H^{(j)}$ also constitute an edge disjoint decomposition of G:

$$E(G) = \bigcup_{j=1}^{k} E(G^{(j)}),\ \ E(H) = \bigcup_{j=1}^{k} E(H^{(j)})$$

The binary sets $B^{(j)}$, $1 \leq j \leq k$, are also pairwise disjoint. The disjoint union of the edge sets $E(H^{(j)})$, $1 \leq j \leq k$, is generated in the NEPS of $K_{p_1}, \ldots, K_{p_t}$ by

$$B = \bigcup_{j=1}^{k} B^{(j)}.$$

With this choice of B the isomorphisms between the subgraphs $G^{(j)}$ and $H^{(j)}$, $1 \leq j \leq k$, extend to an isomorphism between G and H. □

Theorem 2.6. *Let G be a NEPS of complete graphs, $G = \mathrm{NEPS}(K_{m_1}, \ldots, K_{m_r}; B)$. Then G is isomorphic to a gcd-graph over $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$.*

Proof. The vertex set of G can be represented by $\Gamma = Z_{m_1} \oplus \cdots \oplus Z_{m_r}$. Edges of G are generated by the binary r-tuples $b = (b_i)$ of the basis set B. Vertices $x = (x_1, \ldots, x_r) \neq y = (y_1, \ldots, y_r)$ are adjacent in G with respect to b, if $x_i = y_i$, whenever $b_i = 0$, and

$x_i \neq y_i$, whenever $b_i = 1$. Let the set $D(b)$ consist of all positive divisor tuples $d = (d_1, \ldots, d_r)$ of $m = (m_1, \ldots, m_r)$ such that $d_i = m_i$, whenever $b_i = 0$, and d_i a proper divisor of m_i, whenever $b_i = 1$. Then x and y are adjacent with respect to b, if and only if $\gcd(x - y, m) \in D(b)$. If we define $D = \cup\{D(b) : b \in B\}$, then the gcd-graph $\mathrm{Cay}(\Gamma, S_\Gamma(D))$ is isomorphic to G. □

Theorems 2.5 and 2.6 imply the following corollary.

Corollary 2.7. *Let $n = p_1 \cdots p_t$ be the prime factorization of the integer $n \geq 2$. Every gcd-graph with n vertices is isomorphic to a gcd-graph over $\Gamma = Z_{p_1} \oplus \cdots \oplus Z_{p_t}$.*

We conclude this section with some examples.

Example 2.8. We generalize the definition of a Hamming graph given in [15]. The Hamming graph $G = \mathrm{Ham}(m_1, \ldots, m_r; D)$ has vertex set $V(G) = Z_{m_1} \oplus \ldots \oplus Z_{m_r}$. Distinct vertices are adjacent in G, if their Hamming distance is in D. It can be easily shown that G is a NEPS of the complete graphs $K_{m_1}, \ldots, K_{m_r}$.

Example 2.9. Sudoku graphs arise from the popular game of Sudoku. The Sudoku graph $\mathrm{Sud}(n)$ models the number restrictions imposed when filling out an $n^2 \otimes n^2$ Sudoku puzzle. Each vertex represents a cell of the Sudoku puzzle. Two vertices are adjacent if the two corresponding cells are required to contain different numbers (which is the case when they lie in the same row, column or block of the puzzle). It has been shown that Sudoku graphs are NEPS of complete graphs [25].

Example 2.10. This is an example that demonstrates the application of Theorem 2.5. Let $\Gamma = Z_4 \oplus Z_{18}$ and $D = \{(1,6), (4,2), (2,9)\}$. We want to represent the graph $\mathrm{Cay}(\Gamma, S_\Gamma(D))$ as a NEPS of complete graphs. To start with, let us consider the graph $\mathrm{Cay}(\Gamma, S_\Gamma((1,6)))$. Application of Lemma 2.1, Lemma 2.2, once again Lemma 2.1, then Lemma 2.3, Lemma 2.4, and finally once again Lemma 2.3 gives us:

$$\begin{aligned}
&\mathrm{Cay}(Z_4 \oplus Z_{18}, S((1,6))) \\
&\quad \simeq \mathrm{NEPS}(\mathrm{Cay}(Z_4, S(1)), \mathrm{Cay}(Z_{18}, S(6)); \{(1,1)\}) \\
&\quad \simeq \mathrm{NEPS}(\mathrm{Cay}(Z_4, S(1)), \mathrm{Cay}(Z_2 \oplus Z_9, S((2,3))); \{(1,1)\}) \\
&\quad \simeq \mathrm{NEPS}(\mathrm{Cay}(Z_4, S(1)), \mathrm{NEPS}(\mathrm{Cay}(Z_2, S(2)), \mathrm{Cay}(Z_9, S(3)); \{(0,1)\}); \{(1,1)\}) \\
&\quad \simeq \mathrm{NEPS}(\mathrm{Cay}(Z_4, S(1)), \mathrm{Cay}(Z_2, S(2)), \mathrm{Cay}(Z_9, S(3)); \{(1,0,1)\}) \\
&\quad \simeq \mathrm{NEPS}(\mathrm{NEPS}(K_2, K_2; \{(1,0), (1,1)\}), \mathrm{NEPS}(K_2; \{(0)\}), \\
&\qquad\quad \mathrm{NEPS}(K_3, K_3; \{(0,1)\}); \{(1,0,1)\}) \\
&\quad \simeq \mathrm{NEPS}(K_2, K_2, K_2, K_3, K_3; \{(1,0,0,0,1), (1,1,0,0,1)\}).
\end{aligned}$$

Note that for the sake of simplicity we have dropped the subscripts of the symbol sets since the respective groups are clear from the context. Regarding the application of Lemma 2.3 note that, trivially, $G \simeq \mathrm{NEPS}(G; \{(1)\})$.

$$\begin{aligned}
\mathrm{Cay}(Z_4 \oplus Z_{18}, S((4,2))) &\simeq \mathrm{NEPS}(K_2, K_2, K_2, K_3, K_3; \\
&\qquad \{(0,0,0,1,0), (0,0,0,1,1)\}), \\
\mathrm{Cay}(Z_4 \oplus Z_{18}, S((2,9))) &\simeq \mathrm{NEPS}(K_2, K_2, K_2, K_3, K_3; \{(0,1,1,0,0)\}).
\end{aligned}$$

The graph $\mathrm{Cay}(\Gamma, S_\Gamma(D))$ is the disjoint union of the graphs $\mathrm{Cay}(\Gamma, S_\Gamma(d))$ with $d \in D$ which we have considered above, so we arrive at:

$$\begin{aligned}\mathrm{Cay}(\Gamma, S_\Gamma(D)) \quad \simeq \quad & \mathrm{NEPS}(K_2, K_2, K_2, K_3, K_3; \\ & \{(1,0,0,0,1),(1,1,0,0,1),(0,0,0,1,0), \\ & (0,0,0,1,1),(0,1,1,0,0)\}).\end{aligned}$$

3 Eigenspace bases of gcd-graphs

The *eigenvalues* and *eigenspaces* of an undirected graph G are the eigenvalues and eigenspaces, respectively, of any adjacency matrix of G. The multiset of all eigenvalues of a graph is called its *spectrum*. According to HARARY and SCHWENK [14], a graph G is defined to be *integral* if all of its eigenvalues are integers. Integral graphs have been a focus of research for some time; see [4] for a survey.

In particular, many notable results on integrality of Cayley graphs have been obtained. Integral cubic and quartic Cayley graphs on abelian groups have been characterized in [1] and [2], respectively. Circulant graphs are the Cayley graphs over $Z_n,\ n \geq 1$. SO [26] showed that the integral circulant graphs with n vertices are exactly the gcd-graphs over Z_n. This result was extended in [18] to groups of the form $Z_2 \oplus \ldots \oplus Z_2 \oplus Z_n$, $n \geq 2$. A complete characterization of integral Cayley graphs over abelian groups has recently been achieved by ALPERIN and PETERSON [3].

The eigenvalues of $G = \mathrm{NEPS}(G_1, \ldots, G_n; B)$ are certain sums of products of the eigenvalues of the G_i, cf. [10]:

Theorem 3.1. *Let $G_1, \ldots, G_n$ be graphs with $n_1, \ldots, n_r$ vertices, respectively. Further, for $i = 1, \ldots, r$ let $\lambda_{i1}, \ldots, \lambda_{in_i}$ be the eigenvalues of G_i. Then, the spectrum of the graph $G = \mathrm{NEPS}(G_1, \ldots, G_n; B)$ with respect to basis B consist of all possible values*

$$\mu_{i_1,\ldots,i_n} = \sum_{(\beta_1,\ldots,\beta_n) \in B} \lambda_{1i_1}^{\beta_1} \cdot \ldots \cdot \lambda_{ni_n}^{\beta_n}$$

with $1 \leq i_k \leq n_k$ for $1 \leq k \leq n$.

A first consequence is that every NEPS of integral graphs is integral. It is easily checked that the complete graph K_n on $n \geq 2$ vertices has the simple eigenvalue $n-1$ and the eigenvalue -1 with multiplicity $n-1$. Hence NEPS of complete graphs are integral. Using Theorem 2.5, we now readily confirm the following result of [18]:

Proposition 3.2. *Every gcd-graph is integral.*

An interesting property of a graph is the ability to choose an eigenspace basis such that its vectors have entries from a very small set only. This may be possible only for certain or for all of its eigenvalues. For example, in [9] a construction is given for a basis of the eigenspace of eigenvalue -2 of a generalized line graph whose vectors contain only entries from $\{0, \pm 1, \pm 2\}$.

Imposing an even greater restriction on the admissible entries, we call an eigenspace basis *simply structured* if it consists of vectors containing only entries from $\{0, 1, -1\}$. Accordingly, an eigenspace is considered as simply structured if it has a simply structured basis. Observe that the eigenvalue belonging to a simply structured eigenspace is necessarily integral.

For a trivial example of a simply structured eigenspace basis, consider a connected r-regular graph. Here the all ones vector constitutes a basis of the eigenspace corresponding to the eigenvalue r. Moreover, for several graph classes, the eigenspaces corresponding to the eigenvalues 0 or -1 are simply structured, cf. [5],[20],[24].

It is somewhat remarkable if *all* of the eigenspaces of a graph are simply structured. In [25] is has been shown that Sudoku graphs are NEPS of complete graphs (recall Example 2.9) and admit simply structured eigenspace bases for all eigenvalues. As we shall see, this is true for any NEPS of complete graphs. For this we require the following theorem [11]:

Theorem 3.3. *If X and Y are graphs of orders n and m with linearly independent eigenvectors $x^{(1)}, \ldots, x^{(n)}$ and $y^{(1)}, \ldots, y^{(m)}$, respectively, then the nm tensor products*

$$x^{(i)} \otimes y^{(j)} \qquad (i = 1, \ldots, n; j = 1, \ldots, m)$$

form a set of linearly independent eigenvectors of any NEPS of X and Y. This fact readily extends to NEPS with more factors.

Corollary 3.4. *Any NEPS of graphs for which all eigenspaces are simply structured inherits that very property.*

Proof. Using the notation of the previous theorem, it is obvious that $x^{(i)} \otimes y^{(j)}$ has only entries from $\{0, 1, -1\}$ if the same holds for $x^{(i)}$ and $y^{(j)}$. This remains true for an arbitrary number of factors. □

We can now prove the following result:

Proposition 3.5. *All eigenspaces of a gcd-graph are simply structured.*

Proof. Consider the complete graph $K_n,\ n \geq 2$. The all-ones vector $(1, 1, \ldots, 1)$ forms a basis of the eigenspace of eigenvalue $n - 1$. A basis of the eigenspace of eigenvalue -1 is formed by the vectors

$$\begin{aligned} x^{(1)} &= (-1, 1, 0, 0, \ldots, 0, 0), \\ x^{(2)} &= (-1, 0, 1, 0, \ldots, 0, 0), \\ \vdots & \qquad\qquad \vdots \\ x^{(n-1)} &= (-1, 0, 0, 0, \ldots, 0, 1). \end{aligned}$$

Thus the result follows from Corollary 3.4 and Theorem 2.5. □

4 Open problems

Let us conclude with a number of open problems we think are worth investigating in the future:

1. Does every integral Cayley graph over a finite abelian group have a simply structured eigenspace basis for every eigenvalue?
2. Find a small class of integral graphs such that every integral Cayley graph over an abelian group is a NEPS of some graphs of this class.
3. It has been shown by So [26] that integral Cayley graphs over Z_{p^α}, p prime, are uniquely determined by their spectrum. Find more groups Γ such that cospectral integral Cayley graphs $\mathrm{Cay}(\Gamma, S_1)$, $\mathrm{Cay}(\Gamma, S_2)$ are necessarily isomorphic.

4. Try to determine or estimate the number $\varrho(n)$ of nonisomorphic gcd-graphs on n vertices. In [18] we showed that for a prime $p \geq 5$ we have $\varrho(p^2) = 6$. Observe that $\varrho(2^\alpha)$ is the number of nonisomorphic cubelike graphs on 2^α vertices, cf. [19].
5. Determine graph invariants for gcd-graphs such as connectivity, clique number, and chromatic number, cf. [6], [7].

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The bipartite graphs of abelian dessins d'enfants

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Abstract

Let S be a closed Riemann surface and let $\beta : S \to \widehat{\mathbb{C}}$ be a regular branched holomorphic covering, with an abelian group as deck group, whose branch values are contained in the set $\{\infty, 0, 1\}$. Three dessins d'enfants are provided by $\beta^{-1}([0,1])$, $\beta^{-1}([1,\infty])$ and $\beta^{-1}([0,\infty])$. In this paper we provide a description of the bipartite graphs associated to these dessins d'enfants using simple arguments.

Keywords: Dessins d'enfants, Belyi curves, Algebraic curves, Riemann Surfaces.

Math. Subj. Class.: 11G32, 14H37, 30F10

1 Introduction

As a consequence of the Riemann-Roch theorem, there is a bijective correspondence between isomorphism classes of closed Riemann surfaces and isomorphism classes of complex algebraic curves. A closed Riemann surface S is called a Belyi curve if there is a non-constant meromorphic function $\beta : S \to \widehat{\mathbb{C}}$ whose branch values are contained in $\{\infty, 0, 1\}$. The function β is called a Belyi function for S and (S, β) is called a Belyi pair. If the branch orders of β at 0, 1 and ∞ are p, q and r, respectively, then we say that the Belyi pair (S, β) is of type (p, q, r).

If (S, β) is a Bely pair, then the pre-image $D_1 = \beta^{-1}([0,1])$ ($D_2 = \beta^{-1}([1,\infty])$ and $D_3 = \beta^{-1}([0,\infty])$, respectively) defines a dessin d'enfant on S (see [7]), that is, a bipartite map on S (the pre-image of 0 are the white vertices and the pre-image of 1 are the black vertices of D_1). Conversely, by the Uniformization Theorem, each dessin d'enfant D on a closed orientable surface induces a unique (up to isomorphisms) Riemann surface structure S on it and a Belyi map $\beta : S \to \widehat{\mathbb{C}}$ so that D and $\beta^{-1}([0,1])$ are equivalent bipartite maps on S. A famous result due to Belyi [1, 2] states that a closed Riemann surface S is a Belyi curve if and only if S can be defined by an algebraic curve over $\overline{\mathbb{Q}}$. This relationship was

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observed by Grothendieck in his famous *Esquisse d'un programme* [3] to propose a study of the structure of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by its action on the dessins d'enfants. As a consequence, a natural link between Galois theory, Belyi pairs and dessins d'enfants appears and, moreover, Galois invariants should be expressed in a purely combinatorial form. Unfortunately, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins d'enfants is not well understood.

A particular class of dessins d'enfants are those produced by Belyi pairs (S,β) for which β is a regular branched holomorphic cover; in which case we say that S is a quasiplatonic curve. Wolfart [8] noticed that quasiplatonic curves (and also the corresponding regular dessins d'enfants) are definable over their field of moduli. In the particular case when the deck group of β is an abelian group, we say that S is an abelian quasiplatonic curve, that (S,β) is an abelian Belyi pair (the corresponding dessins d'enfants are called abelian dessins d'enfants). In this case, if (p,q,r) is the type of (S,β), then we say that signature $(0;p,q,r)$ is an abelian triangular signature. In [4] it was noticed that every abelian Belyi pair (and the corresponding abelian dessins d'enfants) can be defined over $\mathbb{Q}$, that is, they are fixed points for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In this paper we describe, using simple arguments, the underlying bipartite graphs of the abelian dessins d'enfants (see Theorem 1). Next, we proceed to describe a couple of classical examples.

If n,m,d are positive integers, then we denote by $K^d_{n,m}$ the bipartite graph obtained from the complete bipartite graph $K_{n,m}$ by replacing each edge by d edges. In this way, $K^1_{n,m} = K_{n,m}$.

(1) If (S,β) is an abelian Belyi pair of type (k,k,k), where $k \geq 2$ is an integer, and whose deck group of β is $\mathbb{Z}_k^2$, then (S,β) is isomorphic to (F_k,β_k), where F_k is the classical Fermat curve $\{x^k + y^k + z^k = 0\} \subset \mathbb{P}^2$, $\beta_k([x:y:z]) = -(y/x)^k$ and the deck group of β_k is the abelian group generated by $a([x:y:z]) = [\omega_k x : y : z]$ and $b([x:y:z]) = [x:\omega_k y:z]$, where $\omega_k = e^{2\pi i/k}$. The fixed points of a (respectively, b and ab) are given by the k points in $\beta_k^{-1}(\infty)$ (respectively, $\beta_k^{-1}(0)$ and $\beta_k^{-1}(1)$). The abelian dessins d'enfants $D_1 = \beta_k^{-1}([0,1])$, $D_2 = \beta_k^{-1}([1,\infty])$ and $D_3 = \beta_k^{-1}([0,\infty])$ have as bipartite graph the complete bipartite graph $K^1_{k,k}$ [5, 6].

(2) If $(\widehat{\mathbb{C}},\beta)$ is an abelian Belyi pair, with deck group H, then either $H \cong \mathbb{Z}_n$ or $H \cong \mathbb{Z}_2^2$. The associated bipartite graphs of the corresponding abelian dessins d'enfants are in the first case equal to $K^1_{1,n}$ and $K^n_{1,1}$ and in the second case equal to $K^1_{2,2}$.

Theorem 1 below generalizes the above to the case of abelian dessins d'enfants of any type.

Theorem 1. Let (S,β) be an abelian Belyi pair of type (p,q,r) and let d be the degree of β. Then the bipartite graphs associated to the three abelian dessins d'enfants are given by

$$\mathcal{G}_1 = K^{pq/d}_{d/p,d/q}, \quad \mathcal{G}_2 = K^{qr/d}_{d/q,d/r}, \quad \mathcal{G}_3 = K^{pr/d}_{d/p,d/r}.$$

Remark 2. Particular classes of abelian dessins d'enfants are those provided by the maximal ones with respect to its type (as the case provided by classical Fermat curves). An abelian dessin d'enfant associated to an abelian Belyi pair (S,β), with abelian group H as deck group of β, is called an homology dessin d'enfant (and (S,β) is called an homology Belyi pair) if there is no an abelian Belyi pair (R,η), with abelian group G as deck group of η, so that $S = R/L$ for some non-trivial subgroup $L < G$ acting freely on R with $H = G/L$. If (S,β) is an homology Belyi pair, then equations over $\mathbb{Q}$ for S and β were found in [4].

Clearly every abelian dessin d'enfant is covered by an homology dessin d'enfant of the same type. If the genus of S is at least two, then a homology Belyi pair (S,β) of type (p,q,r) can be uniformized as follows. Let Γ be a Fuchsian group of signature $(0;p,q,r)$ and let Γ' be its derivative subgroup. Then (S,β) is equivalent to $(\mathbb{H}^2/\Gamma',\beta_\Gamma)$, where β_Γ is the natural quotient map $\mathbb{H}^2/\Gamma' \to \mathbb{H}^2/\Gamma$. In this way, not only the bipartite graphs may be described, but also the corresponding dessins d'enfants.

2 Proof of Theorem 1

Let (S,β) be an abelian Belyi pair of type (p,q,r) and let H be the abelian group being the deck group of β (so $d = |H|$, the order of H).

Let D_1 be the dessin d'enfant whose edges are the pre-images under β of the arc $[0,1]$, the black vertices are the pre-images of 0 and the white vertices are the pre-images of 1. The number of black vertices is equal to $|H|/p$, the number of white vertices is equal to $|H|/q$ and the number of faces is $|H|/r$. The degree of a black vertex is p, the degree of a white vertex is q and the degree of a face is r.

Let $x_1 \in S$ (respectively, $y_1 \in S$) be such that $\beta(x_1) = 0$ (respectively, $\beta(y_1) = 1$). Let $\mathbb{Z}_p \cong \langle a\rangle < H$ (respectively, $\mathbb{Z}_q \cong \langle b\rangle < H$) be the H-stabilizer of x_1 (respectively, the H-stabilizer of y_1). As H acts transitively on $\beta^{-1}(0)$ (respectively, $\beta^{-1}(1)$) and H is abelian, we may see that:

1. the H-stabilizer of every point in $\beta^{-1}(0)$ (respectively, $\beta^{-1}(1)$) is $\langle a\rangle$ (respectively, $\langle b\rangle$);
2. $H = \langle a,b\rangle$;
3. $\langle b\rangle$ (respectively, $\langle a\rangle$) acts transitively on $\beta^{-1}(0)$ (respectively, $\beta^{-1}(1)$).

As there is a black vertex and a white vertex connected with an edge, condition (3) above ensures that every black vertex and every white vertex is connected by an edge.

Again from (3), the $\langle b\rangle$-stabilizer of x_1 (respectively, the $\langle a\rangle$-stabilizer of y_1) is its cyclic subgroup of $\langle b\rangle$ (respectively, $\langle a\rangle$) of order $pq/|H|$. It follows that every pair of black and white vertices are connected with $pq/|H|$ edges.

All the above information permits to obtain that the graph associated to D_1 is the bipartite graph

$$\mathcal{G}_1 = K^{pq/|H|}_{|H|/p,|H|/q}.$$

Similarly, let D_2 (respectively, D_3) be the dessin d'enfant obtained as the pre-image of the arc $[1,\infty]$ (respectively, $[\infty,0]$) and the corresponding graph $\mathcal{G}_2$ (respectively, $\mathcal{G}_3$). Then, working in the same way as for D_1 one obtains that

$$\mathcal{G}_2 = K^{qr/|H|}_{|H|/q,|H|/r}, \quad \mathcal{G}_3 = K^{pr/|H|}_{|H|/p,|H|/r}.$$

Remark 3. It is well know that a signature $(0;p,q,r)$ is an abelian triangular signature if and only if $\mathrm{lcm}(p,q,r) = \mathrm{lcm}(p,q) = \mathrm{lcm}(p,r) = \mathrm{lcm}(q,r)$, where lcm stands for the least common multiple. In that case we may write

$$p = AA_{12}A_{13}, \; q = AA_{12}A_{23}, \; r = AA_{13}A_{23},$$

where $\gcd(A_{12},A_{13}) = \gcd(A_{12},A_{23}) = \gcd(A_{13},A_{23}) = 1$ and $A = \gcd(p,q,r)$, where gcd stands for the greatest common divisor.

In [4] we proved that if (S, β) is an homology Belyi pair, then $H \cong \mathbb{Z}_A \times \mathbb{Z}_\mu$, where $\mu = \mathrm{lcm}(p, q, r) = AA_{12}A_{13}A_{23}$. In particular, the bipartite graphs in Theorem 1 are given by

$$\mathcal{G}_1 = K^{A_{12}}_{AA_{23},AA_{13}}, \quad \mathcal{G}_2 = K^{A_{23}}_{AA_{13},AA_{12}}, \quad \mathcal{G}_3 = K^{A_{13}}_{AA_{23},AA_{12}}.$$

In the general situation, that is, for any abelian Bely pair of type (p, q, r) with β of degree d, the bipartite graphs of the three dessins d'enfants are given by

$$\mathcal{G}_1 = K^{lA_{12}}_{AA_{23}/l,AA_{13}/l}, \quad \mathcal{G}_2 = K^{lA_{23}}_{AA_{13}/l,AA_{12}/l}, \quad \mathcal{G}_3 = K^{lA_{13}}_{AA_{23}/l,AA_{12}/l},$$

where $l = A^2A_{12}A_{13}A_{23}/d$.

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On stratifications for planar tensegrities with a small number of vertices

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Abstract

In this paper we discuss several results about the structure of the configuration space of two-dimensional tensegrities with a small number of points. We briefly describe the technique of surgeries that is used to find geometric conditions for tensegrities. Further we introduce a new surgery for three-dimensional tensegrities. Within this paper we formulate additional open problems related to the stratified space of tensegrities.

Keywords: Tensegrities, equilibrium, surgeries.

Math. Subj. Class.: 52C30, 05C10

1 Introduction

In this paper we study the stratified spaces of tensegrities with a small number of points. We work mostly with planar tensegrities. In the case of 4 and 5 point configurations we give an explicit description of all the strata and present a visualization of the entire stratified space. Further we give a geometric description of the strata for 6 and 7 points and use the technique of surgeries to find new geometric conditions adding to the list of already known ones. In particular, we introduce a new surgery for tensegrities in $\mathbb{R}^3$.

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1.1 Configuration space of tensegrities

The first steps in the study of rigidity and flexibility of tensegrities were made by B. Roth and W. Whiteley in [9] and further developed by R. Connelly and W. Whiteley in [3], see also the survey about rigidity in [13]. N. L. White and W. Whiteley in [12] started the investigation of geometric conditions for a tensegrity with prescribed bars and cables. In the preprint [7] M. de Guzmán describes several other examples of geometric conditions for tensegrities.

Let us recall standard definitions of tensegrities (as in [2], [4], etc.). See also [10] for a collection of open problems and a good bibliography.

Definition 1.1. Fix a positive integer d. Let $G = (V, E)$ be an arbitrary graph without loops and multiple edges. Let it have n vertices $v_1, \ldots, v_n$.

- A *configuration* is a finite collection P of n labeled points $(p_1, p_2, \ldots, p_n)$, where each point p_i (also called a *vertex*) is in a fixed Euclidean space $\mathbb{R}^d$.
- The embedding of G with straight edges, induced by mapping v_j to p_j is called a *tensegrity framework* and it is denoted as $G(P)$.
- We say that a *load* or *force* F acting on a framework $G(P)$ in $\mathbb{R}^d$ is an assignment of a vector f_i in $\mathbb{R}^d$ to each vertex i of G.
- We say that a *stress* w for a framework $G(P)$ in $\mathbb{R}^d$ is an assignment of a real number $w_{i,j} = w_{j,i}$ (we call it an *edge-stress*) to each edge p_ip_j of G. An edge-stress is regarded as a tension or a compression in the edge p_ip_j. For simplicity reasons we put $w_{i,j} = 0$ if there is no edge between the corresponding vertices. We say that w *resolves* a load F if the following vector equation holds for each vertex i of G:

  $$f_i + \sum_{\{j|j\neq i\}} w_{i,j}(p_j - p_i) = 0.$$

  By $p_j - p_i$ we denote the vector from the point p_i to the point p_j.
- A stress w is called a *self-stress* if, the following equilibrium condition is fulfilled at every vertex p_i:

  $$\sum_{\{j|j\neq i\}} w_{i,j}(p_j - p_i) = 0.$$
- A couple $(G(P), w)$ is called a *tensegrity* if w is a self-stress for the framework $G(P)$.
- If $w_{i,j} < 0$ then we call the edge p_ip_j a *cable*, if $w_{i,j} > 0$ we call it a *strut*.

Let $W(n)$ denote the linear space of dimension n^2 of all edge-stresses $w_{i,j}$. Consider a framework $G(P)$ and denote by $W(G, P)$ the subset of $W(n)$ of all possible self-stresses for $G(P)$. By definition the set $W(G, P)$ is a linear subspace of $W(n)$.

Definition 1.2. The *configuration space of tensegrities* corresponding to the graph G is the set

$$\Omega_d(G) := \big\{(G(P), w) \mid P \in (\mathbb{R}^d)^n, w \in W(G, P)\big\}.$$

The set $\{G(P) \mid P \in (\mathbb{R}^d)^n\}$ is said to be the *base of the configuration space*, we denote it by $B_d(G)$.

1.2 Stratification of the base of a configuration space of tensegrities

Suppose we have some framework $G(P)$ and we want to find a cable-strut construction on it. Then *which edges can be replaced by cables, and which by struts? What is the geometric position of points for which given edges may be replaced by cables and the others by struts?* These questions lead to the following definition.

Definition 1.3. A set $W(G, P_1)$ is said to be *equivalent* to a set $W(G, P_2)$ if there exists a homeomorphism ξ between $W(G, P_1)$ and $W(G, P_2)$, such that for any self-stress w in $W(G, P_1)$ the self-stress $\xi(w)$ satisfies

$$\operatorname{sgn}\big(\xi(w)\big) = \operatorname{sgn}\big(w\big).$$

Henceforth we call a set $W(G, P)$ a *linear fiber*. The described equivalence relation on linear fibers gives us a stratification of the base $B_d(G) = (\mathbb{R}^d)^n$. A *stratum* is by definition a maximal connected set of points with equivalent linear fibers. In the paper [4] we prove that all strata are semialgebraic sets (which implies for instance that they are path connected).

The idea of this paper is to make the first steps in the study of particular configuration spaces of tensegrities. We present the techniques to find geometric conditions and open problems for further study that already arise in very simple situations of 9 point configurations.

Let us, first, make the following three *general remarks*.

GR1. The majority of the strata of codimension k can be defined by algebraic equations and inequalities that define the strata of codimension 1. The exceptions here are mostly in high codimension (the simplest one is as follows: for two points connected by an edge there is no codimension 1 stratum, but there is one codimension 2 stratum corresponding to coinciding points; actually it is interesting to find the complete list of such exceptions). So the most important case to study is the codimension 1 case.

GR2. A stratification of a subgraph is a substratification of the original graph (i.e., each stratum for a subgraph is the union of certain strata for the original graph), hence below we skip the description of $B_2(G)$ for graphs with 5 vertices other than K_5.

GR3. For any stratum there exists a certain subgraph that *locally identifies* the stratum (i.e., for any point x of the stratum there exists a neighborhood $B(x)$ such that any configuration in $B(x)$ has a nonzero self-stress for the subgraph if and only if this point is on the stratum).

According to general remarks GR1 and GR2 the most interesting case is to study the strata of codimension 1 for the complete graph on n vertices (denoted further by K_n). It is possible to find some of the strata of K_n directly. For the other strata one, first, should find an appropriate subgraph that locally identifies the stratum, and then find appropriate surgeries (explained in Section 3) to reduce the complexity of the subgraph to find geometric conditions.

This paper is organized as follows. In Section 2 we study the stratification of configuration spaces of tensegrities in the plane with a small number of vertices. In Subsections 2.1 and 2.2 we briefly describe the trivial cases of two and three point configurations. Further in Subsections 2.3 and 2.4 we study the four and the five point cases. In each of the cases we describe the geometry and the number of strata. In addition we introduce the adjacency diagram of full dimension and codimension 1 strata. In Subsections 2.5 and 2.6 we describe geometric conditions for the codimension 1 strata of 6, 7, and 8 point tensegrities. In Section 3 we present the technique of surgeries to find geometric descriptions for the strata. In Subsection 3.1 we describe surgeries that do not change graphs, and in Subsection 3.2 we show a couple of surgeries in the two-dimensional case. We introduce a new three-dimensional surgery in Subsection 3.3. In conclusion, we formulate several open questions in Subsection 3.4.

2 Stratification of the space $B_2(K_n)$ for small n

In this section we study the geometry of tensegrity stratifications for graphs with a small number of vertices. The cases of $n = 2, 3, 4, 5$ are studied in full detail. Starting from $n = 6$ there are some gaps in the understanding of tensegrities. Still for $n = 6, 7, 8$ the complete description of the geometric conditions for the strata is known, we briefly describe several results on them here (see [4] for more information).

2.1 Case of two points

Consider, first, the case of two points ($n = 2$). There are only two graphs on two points: a complete one K_2 and a graph without edges (denote it by $G_{0,2}$).

All the fibers of the base $B_2(G_{0,2}) = \mathbb{R}^4$ are of dimension 0, and, therefore, they are equivalent. Hence the stratification is trivial.

The complete graph K_2 here has only one edge. If two points of the graph do not coincide then the stress at this edge should be zero. When two points coincide then the stress at the edge can be arbitrary, and we have a one-dimensional set of solutions (i.e., a fiber). So the base $B_2(K_2) = \mathbb{R}^4$ has a codimension 2 stratum (a 2-dimensional plane). The complement to this stratum is a stratum of codimension 0.

2.2 Three point configurations

There are four different types of graphs here: let $G_{i,3}$ be the graph with i edges for $i = 0, 1, 2, 3$.

In cases $G_{0,3}$ and $G_{1,3}$ the base stratifications are the following direct products:

$$B_2(G_{0,3}) = B_2(G_{0,2}) \times \mathbb{R}^2 \quad \text{and} \quad B_2(G_{1,3}) = B_2(K_2) \times \mathbb{R}^2.$$

So $B_2(G_{0,3})$ is trivial and $B_2(G_{1,3})$ has a 4-dimensional subspace and its complement as strata.

The base $B_2(G_{2,3})$ contains five strata. One of them corresponds to the configuration where three points coincide: the fiber here is 2-dimensional, this stratum is isometric to $\mathbb{R}^2$. There are three strata where one of the edges of the graph vanishes: they are isometric to $\mathbb{R}^4 \setminus \mathbb{R}^2$. Finally, the complement to the union of these strata is the only stratum of maximal dimension. There are no nonzero tensegrities for a configuration in this stratum.

For the complete graph on three vertices we have, for the first time, codimension 1 strata. There are three codimension 1 strata, all of them correspond to the following configuration: three points are in one line. Different strata correspond to having a different point between the two others.

Let us briefly describe one of such strata. Let $P_i = (x_i, y_i)$ be the points of the graph ($i = 1, 2, 3$). Then the condition that the three points are in a line is defined by a quadratic equation:

$$(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = 0$$

This quadric divides the space into two connected components: corresponding to positively and negatively oriented triangles.

To sum up we present for $B_2(K_3)$ the following table.

Dimension of a stratum	0	1	2	3	4	5	6
Number of such strata	0	0	1	0	3	3	2

2.3 Stratification of $B_2(K_4)$

In this subsection we restrict ourselves to the complete graph K_4 (for its subgraphs we apply the reasoning of GR2 above). A plane configuration of four points in general position admits a unique tensegrity (up to a multiplicative constant), which is called an *atom*. In [8] it was proved that any self-stress for K_n is a sum of self-stressed atoms in K_n (i.e., a sum of certain $K_4 \subset K_n$ with scalars). For K_4 there are exactly 14 strata of general position.

The strata of codimension 1 correspond to three of four points of the graph lying in a line. Actually in this case there is no jump of dimension of the fiber: there is also a unique (up to scalar) solution corresponding to the three points in a line. But the stresses on the edges from the fourth point are all zero, and hence a fiber of this stratum is not equivalent to general fibers. The number of such strata is 24.

In codimension 2 we have two different types of strata corresponding to

- four points in a line: the dimension of a fiber is 2 (twelve strata);
- two points coincide: the dimension of a fiber is 1 (twelve strata).

In codimension 3 there is one type of strata with configurations of four points in a line, two of which coincide. Six of them with the double point in the middle and twelve of them with the double point not in the middle.

In codimension 4, there are two types of strata:

- three points coincide (4 strata);
- two pairs of points coincide (3 strata).

And, finally, there is a codimension 6 stratum when all four points coincide. We remark that for none of the strata the fiber is 3-dimensional.

The cardinalities of strata are shown in the following table.

Dimension of a stratum	0	1	2	3	4	5	6	7	8
Number of strata	0	0	1	0	7	18	24	24	14

2.3.1 The space of formal configurations

Let us draw schematically the adjacency of the strata of maximal dimension via strata of codimension 1. The dimension of the stratified space is 8, let us reduce it to two via factoring by proper affine transformations. We will use the following simple proposition.

Proposition 2.1. *Invertible affine transformations of the plane do not change the equivalence class of a fiber $W(G, P)$. In other words if P is a configuration and T an invertible affine transformation of the plane then*

$$W(G, P) \simeq W(G, T(P)).$$

□

So instead of studying the stratification itself we restrict to the set of formal configurations with respect to proper affine transformations of the plane.

Definition 2.2. We say that a four point configuration v_1, v_2, v_3, v_4 is *formal* in one of the following cases:

i) nondegenerate case: *a configuration* $P_{x,y,+}$ *with vertices* $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (x,y)$, $v_4 = (x, y{+}1)$ *for arbitrary* (x,y).

ii) nondegenerate case: *a configuration* $P_{x,y,-}$ *with* $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (x,y)$, $v_4 = (x, y{-}1)$ *for arbitrary* (x,y).

iii) degenerate case: *a configuration* $P_{\Delta,+}$ *with* $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (0,1)$, $v_4 = (\Delta, 1)$ *for an arbitrary* Δ.

iv) degenerate case: *a configuration* $P_{\Delta,-}$ *with* $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (0,-1)$, $v_4 = (\Delta, -1)$ *for an arbitrary* Δ.

v) closure: we add two formal configurations $P_{\pm\infty}$ with vertices $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,0)$, $v_4 = (1, \pm\infty)$.

We denote the set of all formal configurations by Λ_4.

In some sense the space Λ_4 is the space of all codimension 0 and codimension 1 configurations factored by the group of proper affine transformations.

Proposition 2.3. *For any codimension 0 and codimension 1 configuration there exists a unique formal configuration to which the first configuration can be affinely deformed.* □

The space Λ_4 is endowed with a natural topology of a quotient space.

Proposition 2.4. *There is a natural topology of a sphere* S^2 *for the set* Λ_4.

Proof. Let us introduce a topology of the unit sphere S^2 for Λ_4. Consider the configurations of case *i*) on the plane $z = 1$: we identify the point $P_{x,y,+}$ with the point $(x, y, 1)$. Consider the projection of this plane to the upper unit hemisphere S^2 from the origin. So we have a one to one correspondence between the configurations of case *i*) and the upper hemisphere.

Similarly we take the plane $z = -1$ for the case *ii*) identifying the point $(-x, -y, -1)$ with the configuration $P_{x,y,-}$ and projecting it to the lower hemisphere.

For the equator of the unit sphere we use all the other cases as asymptotic directions. First, we associate the configuration $P_{\Delta,+}$ with the point

$$(\cos(\pi - \operatorname{arccotan} \Delta), \sin(\pi - \operatorname{arccotan} \Delta), 0).$$

Let us explain the topology at one of such points of the equator. Suppose we start with $P_{x,y,+}$. The transformation sending the first three points to $(0,0)$, $(1,0)$, and $(0,1)$ is linear with matrix

$$\begin{pmatrix} 1 & -x/y \\ 0 & 1/y \end{pmatrix}.$$

Then the image of the fourth point of $P_{x,y,+}$ is $(-x/y, 1{+}1/y)$. While x tends to infinity and x/y tends to Δ the last point tends to $(-\Delta, 1)$, and hence the configuration $P_{x,y,+}$ tends to $P_{-\Delta,+}$, as in the above formula.

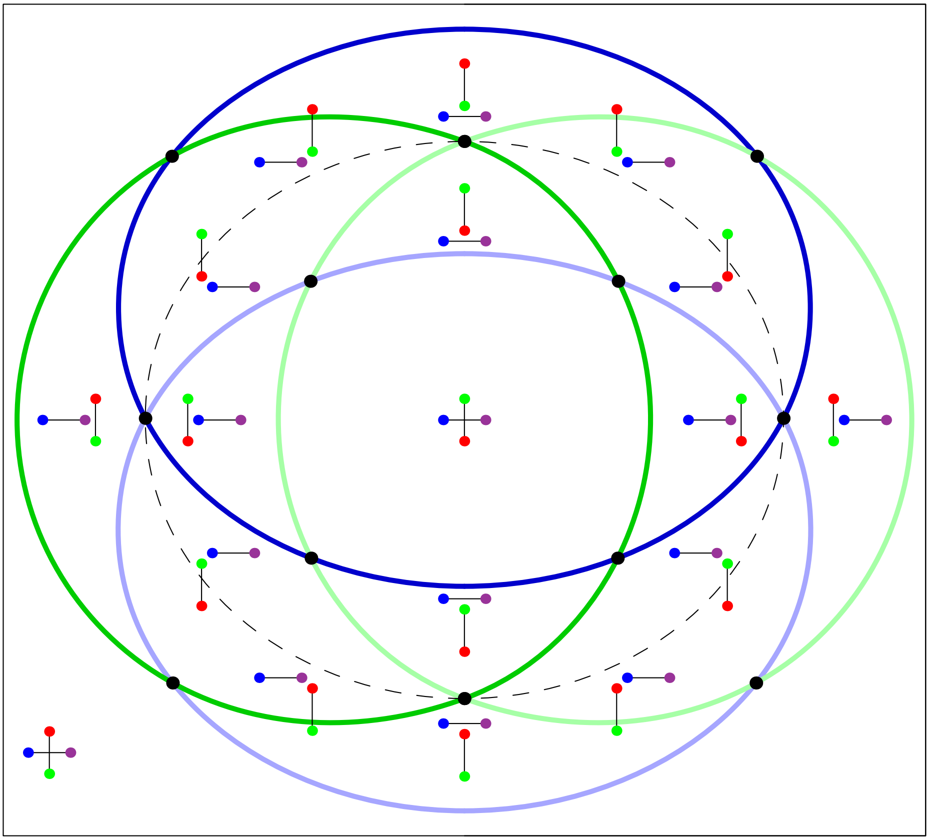


Figure 1: Stratification of $B_2(K_4)$.

Secondly, we associate $P_{\Delta,-}$ with the point

$$(\cos(-\operatorname{arccotan}\Delta), \sin(-\operatorname{arccotan}\Delta), 0)$$

in a similar way.

Finally, we glue $P_{+\infty}$ and $P_{-\infty}$ to the points $(1,0,0)$ and $(-1,0,0)$ respectively. □

So, the codimension 0 and 1 stratification of $B_2(K_4)$ can be derived from the stratification of the sphere. We show the stereographic projection of Λ_4 from the point $(0,0,-1)$ to the plane $z = 1$ on Figure 1. There are four types of strata of codimension 1, they correspond to the fact that certain three points are in a line. They separate the plane into 14 connected components. In each of the connected components we draw a typical type of configuration: (v_1, v_2, v_3, v_4). Here v_1 is blue, v_2 is purple, v_3 is red and v_4 is green.

Remark 2.5. Different geometric conditions are represented by different colors in the picture, the correspondence is as follows.

- Light blue strata (6 strata forming a circle) correspond to configurations with v_1, v_2, and v_3 in a line.

- Dark blue strata (6 strata) contain configurations with v_1, v_2, and v_4 in a line.
- Light green strata (6 strata) contain configurations with v_1, v_3, and v_4 in a line.
- Dark green strata (6 strata) correspond to configurations with v_2, v_3, and v_4 in a line. We have 24 strata of codimension 1 in total.
- The dashed black line is the projection of the equator. It corresponds to the degenerate case of parallel segments. The dashed line is not a stratum, it has the same fiber as all the points in its neighborhood. While one passes the dashed line the red-green segment "rotates" around the blue-purple segment.

Remark 2.6. The 14 connected components of the plane are in one-to-one correspondence with the 14 faces of a cuboctahedron (accordingly, the 12 points on these circles correspond to its vertices). Thus, the four circles are those circumscribed around the equatorial regular hexagons of the cuboctahedron. The vertices of this polytope lie on a sphere, hence, through stereographic projection the four circumcircles in question project in fact to circles in the image plane.

2.4 Stratification of $B_2(K_5)$

2.4.1 General description of the strata

We have 264 strata of general position.

As in the two previous cases the strata of codimension 1 correspond to three points of the graph lying in a line. The number of such strata is 600.

The following strata are of codimension 2:

- twice three points in a line: 270 strata;
- four points in a line: 120 strata;
- two points coincide: 420 strata.

In codimension 3 we have the following cases:

- three points in a line and one double point: 60 strata;
- four points in a line two of which coincide: 180 strata;
- five points in a line: 60 strata.

For codimension 4 we have the following list:

- one triple point: 20 strata;
- five points in a line two of which coincide: 120 strata;
- two double points: 30 strata.

In codimension 5 we get:

- five points in a line three of which coincide: 30 strata;
- five points in a line with two pairs of points coinciding: 45 strata.

In codimension 6 there are the following strata:

- a triple point and a double point: 10 strata;
- one point and one point of multiplicity four: 5 strata.

And, finally, there is a codimension 8 stratum when all five points coincide.

The cardinalities of the strata are shown in the following table.

Dimension of a stratum	0	1	2	3	4	5	6	7	8	9	10
Number of strata	0	0	1	0	15	75	170	300	810	600	264

2.4.2 Visualization of $B_2(K_5)$

Let us now describe the structure of the stratification $B_2(K_5)$. Like in case of $B_2(K_4)$ we introduce a set Λ_5 which represents the adjacency of strata of full dimension and of codimension 1. By definition we put

$$\Lambda_5 = \Lambda_4 \times \mathbb{R}^2,$$

i.e., we consider all the four point configurations of Λ_4, and to each configuration we add the fifth point. We take the product topology for Λ_5.

So at each point of Λ_4 we attach an $\mathbb{R}^2$-fiber. It will soon become clear that for any full dimension stratum of Λ_4 the corresponding fibration is trivial, but the adjacency is not.

On Figures 2 and 3 we show Λ_5 in the following way. We draw the stratification of Λ_4 and inside each connected component we show the typical fiber of the component. The first four points are represented by purple, blue, green, and red points. The lines passing through any pair of them divide the fiber into 18 connected components, that correspond to strata of full dimension. At each such component we write a letter of the Latin alphabet (we consider capital and small letters as distinct).

- Two regions denoted by the same letter and lying in neighboring connected components of Λ_4 separated by light red, dark red, and black strata are in the same stratum.
- Two regions denoted by the same letter and lying in neighboring connected components of Λ_4 separated by light blue, dark blue, light green, and dark green strata are in distinct strata which are adjacent to the same codimension 1 stratum.
- Two regions denoted by a distinct letter and lying in neighboring connected components of Λ_4 are not in one stratum and are not adjacent to the same codimension 1 stratum.

The light blue, dark blue, light green, and dark green strata represent the same geometric conditions as in Remark 2.5 above. For the remaining strata we have:

- The dark red stratum symbolizes that the line through the red and blue points is parallel to the line through the green and purple points.
- The light red stratum symbolizes that the line through the red and purple points is parallel to the line through the green and blue points.
- The black stratum symbolizes that the line through the red and green points is parallel to the line through the purple and blue points.

Remark 2.7. The configuration space $B_2(K_5)$ has several obvious symmetries. First, there is the group of permutations S_5 that acts on the points of $B_2(K_5)$; these symmetries are hardly seen from Figures 2 and 3 since the representation is not S_5-symmetric. Secondly, there is a symmetry about the origin that sends configurations from $B_2(K_5)$ to themselves, on Figures 2 and 3 we used capital and small letters to indicate this symmetry (for instance, the strata of "a" contain centrally symmetric configurations to the configurations of the strata "A").

As in the case of 4 point configurations we skip the subgraphs of K_5, see the second general remark above (GR2).

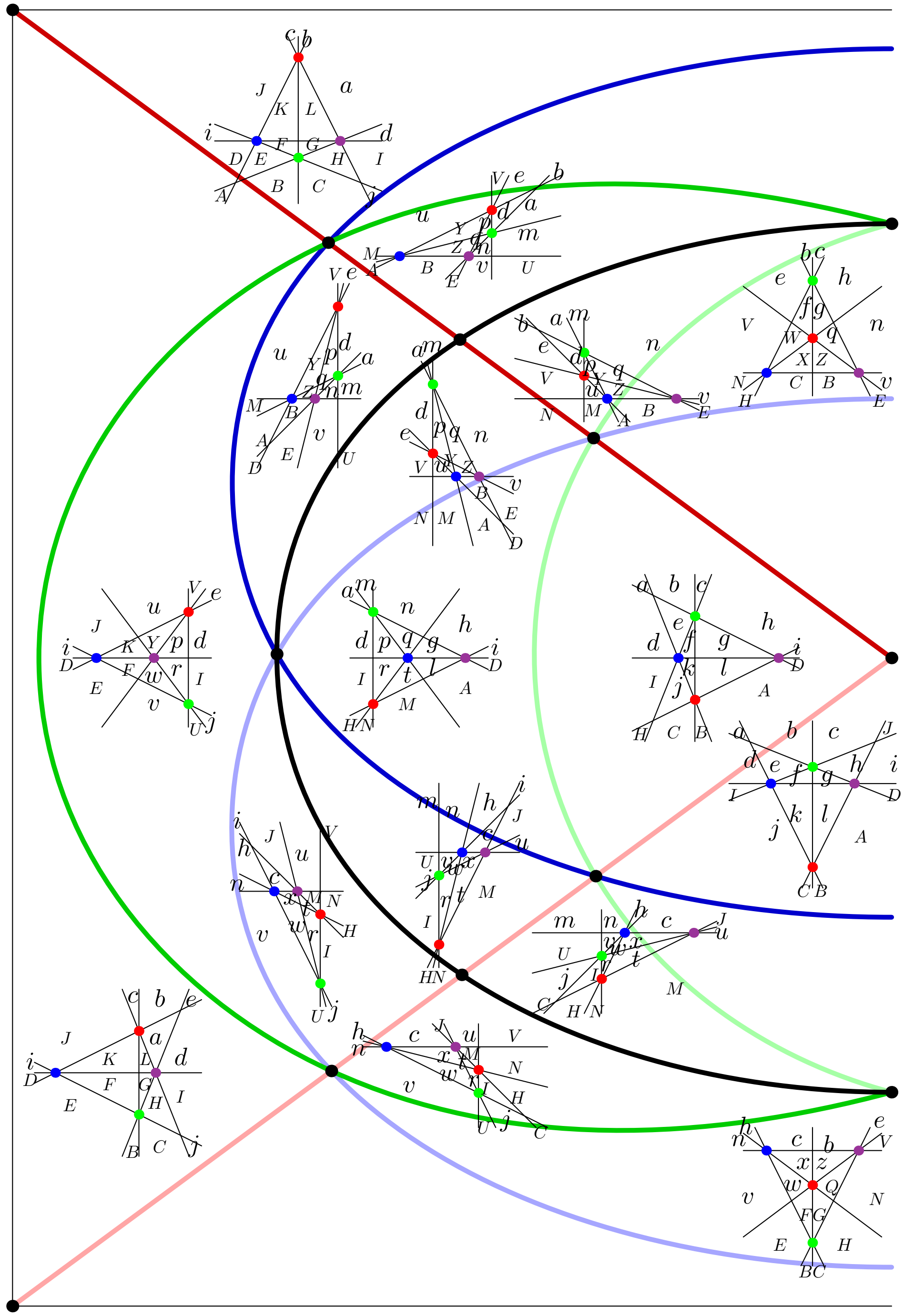


Figure 2: Stratification of $B_2(K_5)$ (Left part).

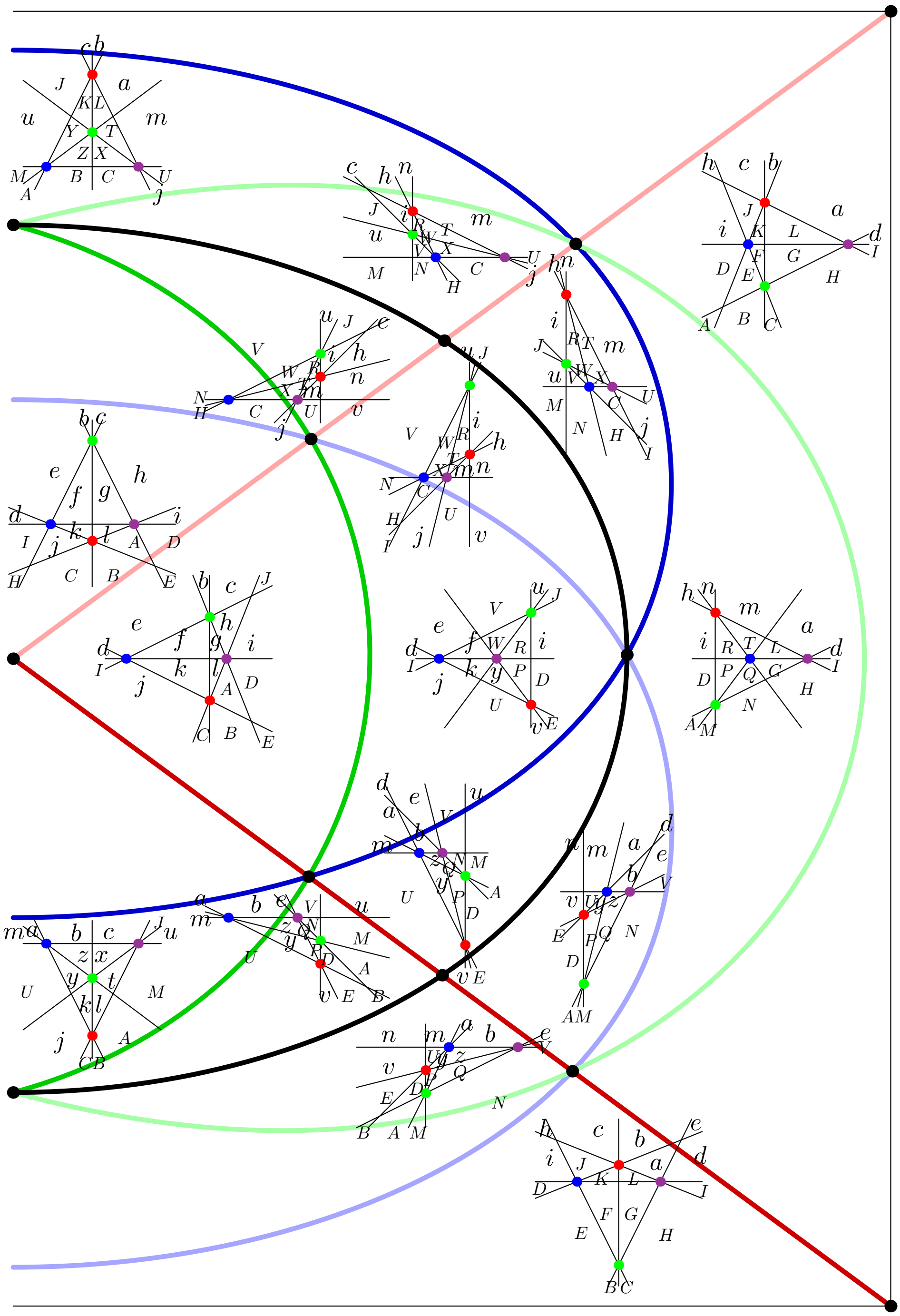


Figure 3: Stratification of $B_2(K_5)$ (Right part).

2.5 Essentially new strata in $B_2(K_6)$

The stratification of $B_2(K_6)$ is much more complicated, at this moment we do not even know how many strata of distinct dimension are present in the stratification.

According to GR1 the first step in studying the stratification of $B_2(K_6)$ is to study all possible distinct types of strata of codimension 1. In the examples of K_n for $n < 6$ we only have strata corresponding to the following geometric condition: three points are in a line. For the case of 6 points we get two additional types of strata: six points on a conic, and three lines passing through three pairs of points have a unique point of intersection.

So the following are three codimension 1 strata (appeared in [12] by N. L. White and W. Whiteley):

- three points in a line;
- the lines v_1v_2, v_3v_4, and v_5v_6 meet in one point (or all parallel);
- all the six points are on a conic.

We conclude this subsection with the following problems.

Problem 2.8. Find a description of $B_2(K_6), B_3(K_4)$ and $B_3(K_5)$ similar to the ones for $B_2(K_4)$ and $B_2(K_5)$ shown in the previous subsections.

2.6 A few words about the case $n > 6$

In [4] we have studied strata of the 7 and 8 point configurations. There are 4 distinct types of codimension 1 strata for 7 points and 17 types for 8 points.

The 4 types of codimension 1 strata for 7 points are defined by the following geometric conditions:

- three points in a line;
- the lines v_1v_2, v_3v_4, and v_5v_6 meet in one point (or all parallel);
- the lines v_1v_2, v_3v_4, and v_5p (where p is the intersection of the lines v_2v_6 and v_3v_7) have a common nonempty intersection;
- the six points v_1, v_2, v_3, v_4, v_5, and p (where p is the intersection of the lines v_1v_6 and v_3v_7) are on a conic.

For the list of strata of 8 point configurations we refer to [4].

It turns out that the geometric conditions of any codimension 1 stratum can be obtained by the following procedure. Consider the points of configuration P; for each two pairs of points (v_i, v_j) and (v_k, v_l) of this configuration consider the point of intersection of the lines v_iv_j and v_kv_l. This leads to a bigger configuration of points including P and the above intersections, we denote it by $U(P)$. This operation can be iteratively applied infinitely many times, which results in a *universal set*

$$U^\infty(P) = \bigcup_{m=0}^{\infty} U^m(P).$$

Any condition for a codimension 1 stratum is always as follows: *three certain points of $U^\infty(P)$ are in a line* (for the details, see for instance [9] and [4]).

Example 2.9. The condition *the lines v_1v_2, v_3v_4, and v_5v_6 meet in one point* in terms of points of $U^1(P) = U(P)$ is as follows. *The points v_1, v_2, and $p = v_3v_4 \cap v_5v_6$ are in a line.*

Remark 2.10. For simplicity reasons we omit discussions of cases where certain lines v_iv_j and v_kv_l are parallel, due to the fact that this situation is never generic for codimension 1 strata. In general one may think that if the lines v_iv_j and v_kv_l are parallel, then their intersection point is in the line at infinity in the projectivization of $\mathbb{R}^2$.

Remark 2.11. At first glance, the condition *six points are on a conic* is of different nature. Nevertheless, it is a relation on the points of the configuration in $U^1(P)$ described by Pascal's theorem: *The intersections of the extended opposite sides of a hexagon inscribed in a conic lie on the Pascal line.* See also Example 2.15 below.

Problem 2.12. Describe all the possible different types of strata for 9 points.

Problem 2.13. How to calculate the number of different types of strata for n points with arbitrary n?

It is also interesting to have an answer for the following question: *how many iterations does one need to perform (i.e., find the minimal m for $U^m(P)$) to describe all conditions for the codimension 1 strata of n-point configurations P?*

Problem 2.14. Which configurations of $U^m(P)$ define the same geometric condition?

This problem is a kind of question of finding generators and relations for the set of all conditions. Let us show one type of such "relations" in the following example.

Example 2.15. Consider the condition: six points $v_1, v_2, \ldots, v_6$ are on a conic. This condition is described by configurations contained in $U^1(P)$ via Pascal's theorem:

$$\text{The points } p, q, r \text{ are in a line for} \quad \begin{cases} p = v_{\sigma(1)}v_{\sigma(2)} \cap v_{\sigma(4)}v_{\sigma(5)} \\ q = v_{\sigma(2)}v_{\sigma(3)} \cap v_{\sigma(5)}v_{\sigma(6)} \\ r = v_{\sigma(3)}v_{\sigma(4)} \cap v_{\sigma(6)}v_{\sigma(1)} \end{cases},$$

where σ is an arbitrary permutation of the set of six elements. So, there are 60 different configurations of $U^1(P)$ defining the same geometric condition.

3 Further study of strata: surgeries

We now look into subgraphs contained in a particular stratum and ask the basic question on the dimension of the fiber.

Even graphs of very low connectivity admit non-zero tensegrities, for disconnected or one-connected graphs we may simply examine the connected or 2-connected components. Also 2-connected graphs may be decomposed via the 2-sum, see [11]: Consider graphs G_1 and G_2, their configurations P_1 and P_2 admitting tensegrities with p_1q_1 a cable in $G_1(P_1)$ and p_2q_2 a strut in $G_2(P_2)$. We form the *2-sum* $G_1 \bigoplus G_2$ by identifying p_1 with p_2 and q_1 with q_2 and removing the identified edge. We can inherit a configuration P from P_1 and P_2 by fixing P_1 and properly dilating, rotating and translating P_2. It is clear that

$$\dim W(G_1 \bigoplus G_2, P) = \dim W(G_1, P_1) + \dim W(G_2, P_2) - 1.$$

Since 2-sum decomposition is canonical, we can describe geometric conditions for 2-connected graphs by geometric conditions on their 3-blocks. For example the geometric condition for G in Figure 4 is that the lines v_1v_2, v_3v_4, and v_5v_6 meet in one point.

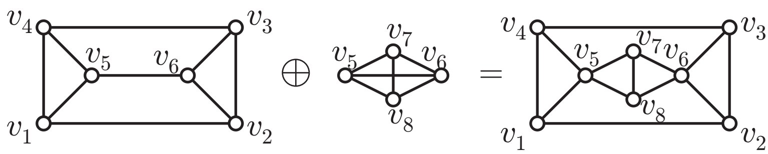


Figure 4: The 2-sum of a triangular prism with K_4

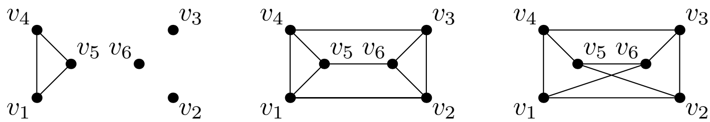


Figure 5: Examples of subgraphs of K_6 admitting tensegrities at codimension 1 strata of $B_2(K_6)$.

3.1 Subgraphs related to codimension 1 strata

As we have already mentioned in GR3, for any codimension 1 stratum there exists at least one subgraph of K_n that generically does not admit tensegrities but at this stratum admits a one-dimensional family of tensegrities. Let us show such subgraphs for the codimension one strata of $B_2(K_6)$ and $B_2(K_7)$.

Example 3.1. In the case of K_6 we have three strata of different geometrical nature. The first triangular subgraph (Figure 5, left) is related to the strata with three points in a line. The second (Figure 5, middle) corresponds to the strata whose three pairs of points generate lines passing through one point. The last one (Figure 5, right) corresponds to the configurations of six points on a conic.

Example 3.2. In the case of K_7 there are the following new examples of subgraphs, corresponding to the main 4 different types of strata.

From the left to the right we have the following geometric conditions

- v_1, v_2, and v_3 are in a line;
- the lines v_1v_2, v_3v_4, and v_5v_6 meet in one point;
- the lines v_1v_2, v_3v_4, and v_5p (where $p = v_2v_6 \cap v_3v_7$) have a common point;
- the six points v_1, v_2, v_3, v_4, v_5, and p (where $p = v_1v_6 \cap v_3v_7$) are on a conic.

Note that the example for three points in a line is actually the 2-sum of a triangle with two atoms, so the only way for a non-zero self-stress on the edges is to have v_1, v_2, and v_3,

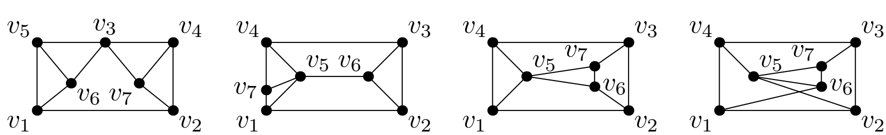


Figure 6: Examples of subgraphs of K_7 admitting tensegrities at codimension 1 strata of $B_2(K_7)$.

the vertices of the triangle, in a line.

Remark 3.3. Geometric conditions for the graphs with 8 and fewer vertices are given in [4]. Several of those geometric conditions were described before in terms of bracket polynomials in [12] by N. L. White and W. Whiteley. We also refer to the paper [1] by E. D. Bolker and H. Crapo for the relation of bipartite graphs with rectangular bar constructions.

3.2 Surgeries on subgraphs that change geometric conditions in a predictable way

In this subsection we present several surgeries that allow to guess the geometric conditions for new strata (characterized by certain subgraphs) via other strata (characterized by these graphs modified in a certain way). We call such modifications of graphs *surgeries*.

3.2.1 Surgeries that do not change geometric conditions

Let G be a graph, denote by G_e the graph with an edge e removed.

Proposition 3.4. (Edge exchange) *Consider a graph G and a subgraph H, and let e_1 and e_2 be two edges of H. Let P be a configuration for which $\dim W(H, P) = 1$. Suppose also that the self-stresses of H do not vanish at the edges e_1 and e_2. Then we have*

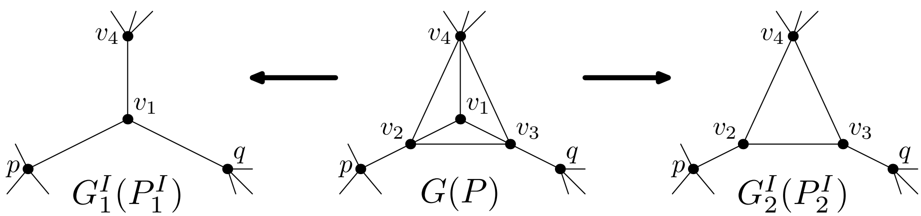
$$\dim W(G_{e_1}, P) = \dim W(G_{e_2}, P). \qquad \square$$

In the situation of Proposition 3.4 the strata of $G_{e_1}(P)$ and $G_{e_2}(P)$ are defined by the same geometrical conditions.

3.2.2 Two two-dimensional surgeries that change geometric conditions

The first surgery is described in the following proposition.

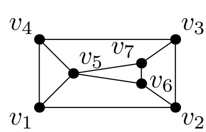
Proposition 3.5. *Consider the frameworks $G(P)$, $G_1^I(P_1^I)$, and $G_2^I(P_2^I)$ as on the figure:*



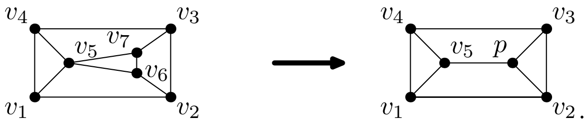
If none of the triples of points (p, v_2, v_3), (q, v_2, v_3), (p, v_2, v_4), (q, v_3, v_4) and (v_2, v_3, v_4) are on a line then we have

$$\dim W(G_1^I, P_1^I) = \dim W(G_2^I, P_2^I).$$

Example 3.6. Let us consider a simple example of how to get a geometric condition for the graph



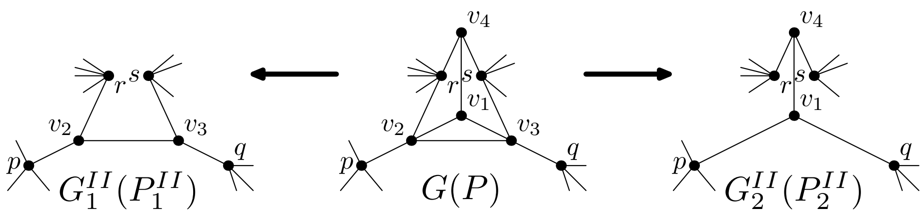
to admit a tensegrity knowing all geometric conditions for 6-point graphs. Let us apply Surgery I to the points v_5, v_6, v_7. We have:



The geometric condition to admit a tensegrity for the graph on the right is:
the lines v_1v_2, v_3v_4 and v_5p intersect in a point.
Hence the geometric condition for the original graph is:
the lines v_1v_2, v_3v_4 and v_5p intersect in a point, where $p = v_2v_6 \cap v_3v_7$.

Now let us show the second surgery.

Proposition 3.7. *Consider the frameworks $G(P)$, $G_1^{II}(P_1^{II})$, and $G_2^{II}(P_2^{II})$ as on the following figure:*



If none of the triples of points (p, q, v_1), (p, v_1, v_4), (r, v_1, v_4), (q, v_1, v_4), (s, v_1, v_4), or (r, s, v_4) lie on a line then we have

$$\dim W(G_1^{II}, P_1^{II}) = \dim W(G_2^{II}, P_2^{II}). \qquad \square$$

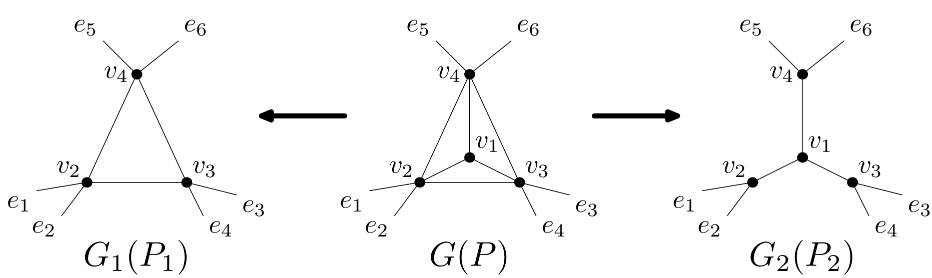
Remark 3.8. Both surgeries were shown in [4]. There is a certain analogy of the first surgery to ΔY exchange in matroid theory (see for instance [13] and [5] for the connections between matroids and rigidity theory), but it is not exactly the same.

Remark 3.9. Actually these surgeries are valid in the multidimensional case as well under the condition that certain points are in one plane.

3.3 A new tensegrity surgery in $\mathbb{R}^3$

We conclude this paper with a single surgery for tensegrities in $\mathbb{R}^3$.

Proposition 3.10. *Consider a graph G and frameworks $G(P)$, $G_1(P_1)$, and $G_2(P_2)$ as follows:*



Denote the plane $v_2v_3v_4$ by π_1. Suppose that the couples of edges e_1 and e_2, e_3 and e_4, e_5 and e_6 define planes π_2, π_3, and π_4, different from π_1. Assume that $\pi_2 \cap \pi_3 \cap \pi_4$ is a one point intersection.

If $G_1(P_1)$ and $G_2(P_2)$ have nonzero stress on the edges connecting v_1, v_2, v_3, and v_4 then

$$\pi_1 \cap \pi_2 \cap \pi_3 \cap \pi_4 = v_1.$$

In this case we additionally have

$$\dim W(G_1, P_1) = \dim W(G_2, P_2).$$

Proof. The first statement follows since v_1 only has valency 3 in $G_2(P_2)$, so v_1, v_2, v_3, and v_4 need to be coplanar to have a nonzero edge-stress. Now we explain how to map $W(G_1, P_1)$ to $W(G_2, P_2)$. The inverse map is simply given by the reverse construction. By the conditions v_1 is the intersection point of the planes π_1, π_2, and π_3. We add the uniquely defined plane atom on v_1, v_2, v_3, v_4 to $G_1(P_1)$ that cancels the edge-stress on v_2v_3. Since the plane π_1 does not coincide with the plane π_2 spanned by the forces on e_1 and e_2, the edge-stress on v_2v_4 is also canceled. By the same reasons the edge-stress on v_3v_4 is canceled as well. This uniquely defines a self-stress on $G_2(P_2)$. □

3.4 Some related open problems

The next goal in this approach is to continue to study the geometry of the strata. Ideally *one would like to find techniques that will give geometric conditions for a graph via its combinatorics*. This question seems to be a very hard open problem. The study of surgeries is the first step to solve it at least in codimension 1.

For a start we propose the following open question.

Problem 3.11. Find all geometric conditions for the strata of 9 point tensegrities.

The surgeries introduced in this section were extremely useful for the study of 8 point configurations (see in [4]). We think that it is not enough to know only these surgeries to find all the geometric conditions. This gives rise to another question.

Problem 3.12. Find other surgeries on graphs that predictably change the geometric conditions.

As far as we know there is no systematic study of strata for tensegrities in $\mathbb{R}^3$ or higher dimensions: these cases are much more complicated than the planar case. At least the stratification of $B_3(K_5)$ should have a description similar to that of $B_2(K_4)$, since 5 points in general position in $\mathbb{R}^3$ admit a unique non-zero self-stress.

Additionally one should examine the rigidity properties of subgraphs in a stratum. For K_4 we have 14 strata of full dimension. For 8 of them the convex hull is a triangle, in 5 of the strata the points are in convex position. A tensegrity for the convex position has 4 struts (cables) and two cables (struts), while in the non-convex case there are three cables and three struts. All of these tensegrities are (infinitesimally) rigid and struts and cables may be exchanged without destroying rigidity. However, when viewed as graphs embedded in $\mathbb{R}^3$ only half of them are rigid. For the convex case, there must be cables on the convex hull and two struts. In the non-convex case there must be a triangle of struts on the convex hull and three cables in the interior, termed a spider web by R. Connelly. None of these are proper forms in the sense of B. Grünbaum. They are minimally rigid, but in the convex case they have members intersecting in a vertex other than a vertex of the graph, in the non-convex case there is a vertex without a strut. B. Grünbaum in his lectures on lost mathematics [6] asks about the number of proper forms given n struts. On 3 struts, there is only one tensegrity which is minimally rigid with edges only intersecting at vertices and such that every vertex is endpoint of at least one strut. For 4 struts there are at least 20 forms, but it is not known how many there are. The number of forms on n struts is bounded by the number of strata on $B_3(K_n)$. For the hierarchies of the various kinds of rigidity see [3].

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Relations between graphs

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Abstract

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, we ask under which conditions there is a relation $R \subseteq V_G \times V_H$ that generates the edges of H given the structure of the graph G. This construction can be seen as a form of multihomomorphism. It generalizes surjective homomorphisms of graphs and naturally leads to notions of R-retractions, R-cores, and R-cocores of graphs. Both R-cores and R-cocores of graphs are unique up to isomorphism and can be computed in polynomial time.

Keywords: Generalized surjective graph homomorphism, R-reduced graph, R-retraction, binary relation, multihomomorphism, R-core, cocore.

Math. Subj. Class.: 05C60, 05C76

1 Introduction

1.1 Motivation

Graphs are frequently employed to model natural or artificial systems [3, 11]. In many applications separate graph models have been constructed for distinct, but at least conceptually related systems. One might think, e.g., of traffic networks for different means of transportation (air, ship, road, railroad, bus). In the life sciences, elaborate network models are considered for gene expression and the metabolic pathways regulated by these genes, or for the co-occurrence of protein domains within proteins and the physical interactions of proteins among each other.

Let us consider an example. Most proteins contain several functional domains, that is, parts with well-characterized sequence and structure features that can be understood as functional units. Protein domains for instance mediate the catalytic activity of an enzyme and they are responsible for specific binding to small molecules, nucleic acids, or other proteins. Databases such as `SuperFamily` compile the domain composition of a large number of proteins. We can think of these data as a relation $R \subset D \times P$ between the set D of domains and the set of P proteins which contain them. Protein-protein interaction networks (PPIs) have been empirically determined for several species and are among the best-studied biological networks [16]. From this graph, which has P as its vertex set, and the relation R we can obtain a new graph whose vertex set are the protein domains D, with edges connecting domains that are found in physically interacting proteins. This "domain interaction graph" conveys information e.g. on the functional versatility of protein complexes. On the other hand, we can use R to construct the domain-cooccurrence networks (DCNs) [14] as simple relational composition $R \circ R^+$. In examples like these, the detailed connections between the various graphs have remained unexplored. In fact, there may not be a meaningful connection between some of them, e.g. between PPIs and DCNs, while in other cases there is a close connection: the domain interaction graph, for example, is determined by the PPI and R.

A second setting in which graph structures are clearly related to each other is coarse-graining. Here, sets of vertices are connected to a single coarse-grained vertex, with coarse-grained edges inherited from the original graph. In the simplest case, we deal with quotient graphs [15], although other, less stringent constructions are conceivable. Similarly, we would expect that networks that are related by some evolutionary process retain some sort

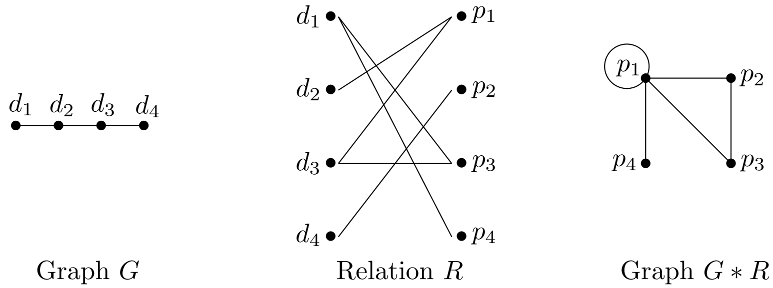


Figure 1: The graph $G * R$ is determined by the graph G and the relation R.

of structural relationship.

1.2 Main definitions

A well-defined mathematical problem is hidden in this setting: Given two networks, can we identify whether they are related in meaningful ways? The usual mathematical approach to this question, namely to ask for the existence of structure-preserving *maps*, appears to be much too restrictive. Instead, we set out here to ask if there is a *relation* between the two networks that preserves structures in a less restrained sense.

The idea is to transfer edges from a graph G to a graph H with the help of a relation R between the vertex set V of G and the vertex set B of H. In this context, R is simply a set of pairs (v, b), with $v \in V, b \in B$. Since graphs can be regarded as representations of binary relations, we can also consider G as a relation on its vertex set, with $(x, y) \in G$ if and only if x and y are connected by an edge of G. We then have the composition $G \circ R$ given by all pairs (x, b) for which there exists a vertex $y \in V$ connected by an edge of G to x and $(y, b) \in R$. This, however, like R is a relation between elements of different sets. In order to equip the target set B with a graph structure, we simply connect elements u and v in B if they stand in relation to connected elements of G. In the following, we give a formal definition, and we shall then relate it to the composition of relations just described.

A *directed graph* G is a pair $G = (V_G, E_G)$ such that E_G is a subset of $V_G \times V_G$. We denote by V_G the *set of vertices of* G and by E_G *the set of edges of* G. We consider only finite graphs and allow loops on vertices.

An *undirected graph* (or simply a *graph*) G is any directed graph such that $(u, v) \in E_G$ if and only if $(v, u) \in E_G$. We thus consider undirected graphs to be special case of directed graphs and we still allow loops on vertices. A *simple graph* is an undirected graph without loops.

Definition 1.1. Let $G = (V_G, E_G)$ be a graph, B a finite set, and $R \subseteq V \times B$ a binary relation, where for every element $b \in B$, we can find an element $v \in V_G$ such that $(v, b) \in R$. Then the graph $G * R$ has vertex set B and edge set

$$E_{G*R} = \{(u, v) \in B \times B |\text{ there is } (x, y) \in E_G \text{ and } (x, u), (y, v) \in R\} . \tag{1.1}$$

An example of the $*$ operation is depicted in Fig. 1.

Graphs with loops are not always a natural model, however, so that it may appear more appealing to consider the slightly modified definition.

Definition 1.2. Let $G = (V_G, E_G)$ be a simple graph, B a finite set, R a binary relation, where for every element $b \in B$, we can find an element $v \in V_G$ such that $(v, b) \in R$. Then the (simple) graph $G \star R$ has vertex set B and edge set

$$E_{G\star R} = \{(u, v) \in B \times B | u \neq v \text{ and there is } (x, y) \in E_G \text{ and } (x, u), (y, v) \in R\} . \tag{1.2}$$

We shall remark that these definitions remain meaningful for directed graphs, weighted graphs (where the weight of edge is a sum of weights of its pre-images) as well as relational structures. For simplicity, we restrict ourselves to undirected graphs (with loops). Most of the results can be directly generalized.

Graphs can be regarded as representations of symmetric binary relations. Using the same symbol for the graph and the relation it represents, we may re-interpret definition 1.1 as a conjugation of relations. R^+ is the *transpose* of R, i.e., $(u, x) \in R^+$ if and only if $(x, u) \in R$. The double composition $R^+ \circ G \circ R$ contains the pair (u, v) in $B \times B$ if and only if there are x and y such that $(u, x) \in R^+$, $(y, v) \in R$, and $(x, y) \in E_G$. Thus

$$G * R = R^+ \circ G \circ R. \tag{1.3}$$

Simple graphs, analogously, correspond to the irreflexive symmetric relations. For any relation R, let R^ι denote its irreflexive part, also known as the *reflexive reduction* of R. Since definition 1.2 explicitly excludes the diagonals, it can be written in the form

$$G \star R = (R^+ \circ G \circ R)^\iota. \tag{1.4}$$

We have $G \star R = (G * R)^\iota$, and hence $E_{G\star R} \subseteq E_{G*R}$. The composition $G * R$ is of particular interest when G is also a simple graph, i.e., $G = G^\iota$.

The main part of this contribution will be concerned with the solutions of the equation $G * R = H$. The weak version, $G \star R = H$, will turn out to have much less convenient properties, and will be discussed only briefly in section 7.

Throughout this paper we use the following standard notations and terms.

For relation $R \subseteq X \times Y$ we define by $R(x) = \{p \in Y | (x, p) \in R\}$ the *image of* x *under* R and $R^{-1}(p) = \{x \in X | (x, p) \in R\}$ the *pre-image of* p *under* R.

The *domain* of R is defined by $\operatorname{dom} R = \{x \in X | \exists p \in Y \text{ s.t. } (x, p) \in R\}$, and the *image* of R is defined by $\operatorname{img} R = \{p \in Y | \exists x \in X \text{ s.t. } (x, p) \in R\}$. We say that the *domain of* R *is full* if for any $x \in X$ we have $R(x) \neq \emptyset$. Analogously, the *image is full* if for any $p \in Y$ we have $R^{-1}(p) \neq \emptyset$.

Let $R \subseteq X \times Y$ is a binary relation, then R is *injective*, if for all x and z in X and y in Y it holds that if $(x, y) \in R$ and $(z, y) \in R$ then $x = z$. R is *functional*, if for all x in X, and y and z in Y it holds that if $(x, y) \in R$ and $(x, z) \in R$ then $y = z$.

We denote by I_G the *identity map on* G, i.e., $\{(x, x) | x \in V_G\}$.

Let $G = (V_G, E_G)$ be a graph and let $W \subseteq V_G$. The *induced subgraph* $G[W]$ has vertex set W and (x, y) is an edge of $G[W]$ if $x, y \in W$ and $(x, y) \in E_G$.

A graph P_k is a path of length k. Similarly, C_k is an (elementary) cycle of length k with vertex set $\{0, 1, \ldots, k - 1\}$. Finally, K_k is the complete (loopless) graph with k vertices.

An *isolated vertex* is a vertex with degree 0. Note that the vertex with a loop is not isolated in this sense.

1.3 Matrix multiplication

The operation $*$ can also be formulated in terms of matrix multiplication. To see this, consider the following variant of the operation on weighted graphs.

Definition 1.3. If G is a weighted graph, we use $w(x, y)$ to denote the weight between x and y. Given a finite set B and a binary relation $R \subseteq V_G \times B$, $G \circledast R$ is defined as a weighted graph H with vertex set B, for any $u, v \in B$, $w(u, v) = \sum_{(x,u)\in R,(y,v)\in R} w(x, y)$.

Ignoring the weights, operations $*$ and $\circledast$ are equivalent.

Using the language of matrices, $G \circledast R = H$ can be interpreted as matrix multiplication:

$$\mathbf{W}_{G\circledast R} = \mathbf{R}^{+}\mathbf{W}_G\mathbf{R} \tag{1.5}$$

where $\mathbf{R}$ is the matrix representation of the relation R, i.e., $\mathbf{R}_{xu} = 1$ if and only if $(x, u) \in R$, otherwise $\mathbf{R}_{xu} = 0$, $\mathbf{R}^{+}$ denotes the transpose of $\mathbf{R}$, and $\mathbf{W}_G$ is the matrix of edge weights of G.

1.4 Graph homomorphisms and multihomomorphisms

The notion of relations between graphs is in many ways similar (but not equivalent) to the well studied notion of graph homomorphisms. The majority of our results focus on similarities and differences between those two concepts. We give here only the basic definitions of graph homomorphisms. For more details see [7].

A *homomorphism* from a graph G to a graph H is a mapping $f : V_G \to V_H$ such that for every edge (x, y) of G, $(f(x), f(y))$ is an edge of H. Note that homomorphisms require loops in H whenever $(x, y) \in E_G$ and $f(x) = f(y)$. In contrast, f is a *weak homomorphism* if $(x, y) \in E_G$ implies that either $f(x) = f(y)$ or $(f(x), f(y)) \in E_H$. Every homomorphism from G to H induces also a weak homomorphism, but not conversely [9].

Since every homomorphism preserves adjacency, it naturally defines a mapping $f^1 : E_G \to E_H$ by setting $f^1((x, y)) = (f(x), f(y))$ for all $(x, y) \in E_G$. If f is surjective, we call f a *vertex surjective homomorphism*, and if f^1 is surjective, we call f an *edge surjective homomorphism*. f is *surjective homomorphism* if it is both vertex- and edge-surjective [7].

A map $f : V_G \to V_H$ is, of course, a special case of a relation. This is seen by setting $F = \{(x, f(x))|x \in V_G\}$. Hence, there is a surjective homomorphism from G to H if and only if there is a functional relation F such that $G * F = H$. Another important connection to the graph homomorphisms is the following simple lemma.

Lemma 1.4. If $G * R = H$, and the domain of R is full, then there is a homomorphism f from G to H contained in R.

Proof. If $G * R = H$, then take any functional relation $f \subseteq R$, we have $G * f \subseteq H$, where f is a homomorphism from G to H. □

Analogously, there is a *weak surjective homomorphism* from G to H if and only if there is a functional relation F such that $G \star F = H$, and there is a weak homomorphism from G to H if there is a functional relation $F \subseteq V_G \times V_H$ such that $G \star F$ is a subgraph of H. The existence of relations between graphs thus can be seen as a proper generalization of graph homomorphisms or weak graph homomorphisms, respectively.

Finally, a *full homomorphism* from a graph G to a graph H is a vertex mapping f such that for distinct vertices u and v of G, we have (u, v) an edge of G if and only if $(f(u), f(v))$ is an edge of H, see [4].

Relation between graphs can be regarded also as a variant of multihomomorphisms. Multihomomorphisms are building blocks of Hom-complexes, introduced by Lovász, and are related to recent exciting developments in topological combinatorics [10], in particular to deep results involved in proof of the Lovász hypothesis [1].

A *multihomomorphism* $G \to H$ is a mapping $f : V_G \to 2^{V_H} \setminus \{\emptyset\}$ (i.e., associating a nonempty subset of vertices of H with every vertex of G) such that whenever $\{u_1, u_2\}$ is an edge of G, we have $(v_1, v_2) \in E_H$ for every $v_1 \in f(u_1)$ and every $v_2 \in f(u_2)$.

The functions from vertices to sets can be seen as representation of relations. A relation with full domain thus can be regarded as *surjective multihomomorphism*, a multihomomorphism such that pre-image of every vertex in H is non-empty and for every edge (u, v) in H we can find an edge (x, y) in G satisfying $u \in f(x)$, $v \in f(y)$.

1.5 Examples

Similarly to graph homomorphisms, the equation $G * R = H$ (or $G \star R = H$ respectively) may have multiple solutions for some pairs of graphs (G, H), while there may be no solution at all for other pairs.

As an example, consider K_2 (two vertices x, y connected by an edge) and C_3 (a cycle of three vertices u, v, w). Denote $R_1 = \{(u, x), (v, y)\}$, $R_2 = \{(v, x), (w, y)\}$, $R_3 = \{(w, x), (u, y)\}$, then it is easily seen that $C_3 * R_i = K_2$ for each $1 \leq i \leq 3$, i.e. the equation $C_3 * R = K_2$ has more than one solution.

On the other hand, there is no relation R such that $K_2 * R = C_3$. Otherwise, each vertex of C_3 is related to at most one vertex of K_2, since C_3 is loop free; hence there exists a vertex in K_2 which has no relation to at least two vertices in C_3, w.l.o.g., one can assume $(x, u), (x, v) \notin R$; then the definition of $*$ implies that there is no edge between u and v, which causes a contradiction.

Because relations do not need to have full domain (unlike graph homomorphisms), there is always an relation from a graph G to its induced subgraph $G[W]$.

Relations with full domain are not restricted to surjective homomorphisms. As a simple example, consider paths P_1 with vertex set $\{x, y\}$ and P_2 with vertex set $\{u, v, w\}$, respectively, and set $R = \{(x, u), (x, w), (y, v)\}$. One can easily verify $P_1 * R = P_2$ by direct computation. Here, R is not functional since x has two images.

1.6 Outline and main results

This paper is organized as follows.

In section 2 the basic properties of the strong relations between graphs are compiled. It is shown that relations compose and every relation can be decomposed in a standard way into a surjective and an injective relation (Corollary 2.3). We discuss some structural properties of graph preserved by the relations.

Equivalence on the class of graphs induced by the existence of relations between graphs is the topic of section 3. We consider two forms: the strong relational equivalence, where relations are required to be reversible, and weak relational equivalence. Equivalence classes of strong relational equivalence are characterized in Theorem 3.8. To describe equivalence classes of the weak relational equivalence we introduce the notion of an R-core of a graph and show that it is in many ways similar to the more familiar construction of the graph core (Theorem 3.17). We explore in particular the differences between core and R-core and an effective algorithm to compute the R-core of given graph is provided.

Section 4 is concerned with the partial order induced on relations between two fixed graphs G and H. Focusing on the special case $G = H$ the minimal elements of this partial order are described. In Theorem 4.7 we give a, perhaps surprisingly simple, characterization of those graphs G for which all relations of G to itself are automorphisms.

R-retraction is defined in section 5 in analogy to retractions. It naturally gives rise to a notion of R-reduced graphs that we show to coincide with the concept of graph cores. By reversing the direction of relations, however, we obtain the concept of a cocore of a graph, which does not have a non-trivial counterpart in the world of ordinary graph homomorphisms, and explore its properties.

The computational complexity of testing for the existence of a relation between two graphs is briefly discussed in section 6. In Theorem 6.1 we describe the reduction of this problem to the surjective homomorphism problem.

Finally, in section 7 we briefly summarize the most important similarities and differences between weak and strong relational composition.

2 Basic properties

2.1 Composition

Recall that the composition of binary relations is associative, i.e., suppose $R \subseteq W \times X$, $S \subseteq X \times Y$, and $T \subseteq Y \times Z$. Then $R \circ (S \circ T) = (R \circ S) \circ T$. Furthermore, the transposition of relations satisfies $(R \circ S)^+ = S^+ \circ R^+$. Interpreting the graph G as a relation on its vertex set, we easily derive the following identities:

Lemma 2.1 (Composition law)**.** $(G * R) * S = G * (R \circ S)$.

Proof. $(G * R) * S = S^+ \circ (R^+ \circ G \circ R) \circ S = (S^+ \circ R^+) \circ G \circ (R \circ S)$
$= (R \circ S)^+ \circ G \circ (R \circ S) = G * (R \circ S)$. □

Now we show that every relation R can be decomposed, in a standard way, to a relation R_D duplicating vertices and a relation R_C contracting vertices.

Lemma 2.2. Let $R \subseteq X \times Y$ be a relation. Then there exists a subset A of X, a set B, an injective relation with full domain $R_D \subseteq A \times B$ and a functional relation $R_C \subseteq B \times Y$, such that $R = I_A \circ R_D \circ R_C$, where I_A is the identity on X restricted to A.

Proof. Put $A = \operatorname{dom} R$. Then the relation I_A removes vertices in $X \setminus \operatorname{dom} R$. It remains to show, therefore, that any relation $R \subseteq X \times Y$ with full domain can be decomposed into an injective relation $R_D \subseteq X \times B$ and a functional relation $R_C \subseteq B \times Y$. To see this, set $B = R$ and declare $(x, \alpha) \in R_D$ if and only if $\alpha = (x, p) \in R$ for some $p \in Y$, and $(\beta, q) \in R_C$ if and only if $\beta = (y, q) \in R$ for some $y \in X$. By construction R_D is injective and R_C is functional. Furthermore, $(x_0, p_0) \in R_D \circ R_C$ if and only if there is $\alpha \in R$ that is simultaneously of the form (x_0, p) and (x, p_0), i.e., $x = x_0$ and $p = p_0$. Hence $(x_0, p_0) \in R$. □

Note that this decomposition is not unique. For instance, we could construct B from multiple copies of R. More precisely, let $B = R \times \{1, 2, \cdots, k\}$, then we would set $\big(x, (\alpha, i)\big) \in R_D$ ($1 \le i \le k$) if and only if $\alpha = (x, p) \in R$ for some $p \in Y$, etc.

The set B as constructed in the proof of Lemma 2.2 has minimal size. To see this, it suffices to show that, given B there is a mapping from B onto R. Since R_D is injective and R_C is functional we may set

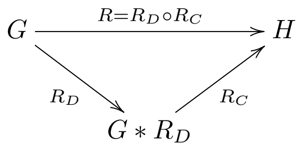
$$\alpha \in B \mapsto (R_D^{-1}(\alpha), R_C(\alpha)).$$

Since $R = I_A \circ R_D \circ R_C$ we conclude that the mapping is surjective, and hence $|B| \geq |R|$.

According to Lemma 2.1, the decomposition of R in Lemma 2.2 can be restated as follows:

Corollary 2.3. *Suppose $G * R = H$. Then there is a set B, an injective relation $R_D \subseteq \operatorname{dom} R \times B$ with full domain, and a surjective relation $R_C \subseteq B \times \operatorname{img} R$ such that $G[\operatorname{dom} R] * R_D * R_C = H$.*

In diagram form, this is expressed as



(2.1)

We shall remark that from the fact the relations compose it follows that the existence of a relation implies a quasi-order on graphs that is related to the homomorphism order. This order is studied more deeply in [8].

2.2 Structural properties preserved by relations

In this subsection we investigate structural properties of H that can be derived from knowledge about certain properties of G and the fact that there is some relation R such that $G * R = H$.

2.2.1 Connected components

Proposition 2.4. Let $G * R = H$ and denote by $H_1, \cdots, H_k$ the connected components of H. Then there are relations $R_i \subseteq V_G \times V_{H_i}$ for each $1 \leq i \leq k$ such that $G * R_i = H_i$ and $R = \bigcup_{i=1}^k R_i$. Furthermore, set $G_i = G[R^{-1}(V_{H_i})]$. Then there are no edges between G_i and G_j for arbitrary $i \neq j$.

Proof. Define the restriction of R to the connected components of H as $R_i = \{(x, y) \in R | y \in V_{H_i}\}$. Clearly, R is the disjoint union of the R_i and $G * R_i \subseteq H_i$. The definition of $*$ implies $H = G * R = (\bigcup_i R_i)^+ \circ G \circ \left(\bigcup_j R_j\right) = \bigcup_i \bigcup_j R_i^+ \circ G \circ R_j$. Since R_i and R_j relate vertices of G to different connected components of H, we have $R_i^+ \circ G \circ R_j = \emptyset$. It follows that $H = \bigcup_i \bigcup_j R_i^+ \circ G \circ R_j = \bigcup_i R_i^+ \circ G \circ R_i = \bigcup_i G * R_i$. Hence $G * R_i = H_i$.

Any edge between G_i and G_j would generate edges between H_i and H_j, thus causing a contradiction to our assumptions. □

Denote by $b_0(G)$ the number of connected components of G, then from Proposition 2.4 we arrive at:

Corollary 2.5. *Suppose both G and H do not have isolated vertices. If $G * R = H$ and R has full domain, then $b_0(G) \geq b_0(H)$.*

Proof. Our notations is the same as in Proposition 2.4. We claim for arbitrary connected component C of graph G, there exists a unique i, such that C is a connected component of G_i. Otherwise one can find two vertices $x, y \in C$, x and y adjacent, such that $x \in V_{G_i}$ and $y \in V_{G_j}$, since G has no isolated vertices, which contradicts $E(G_i, G_j) = \emptyset$. Thus $b_0(G) \geq b_0(H)$ is easily followed. □

From corollary 2.5, we know that H is connected whenever G is connected. The connectedness of G, however, cannot be deduced from the connectedness of H. For example, consider $G = P_1 \cup P_1$ with vertex set $\{x_1, x_2, x_3, x_4\}$ and edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$, and $H = P_2$ with vertex set $\{v_1, v_2, v_3\}$. Set $R = \{(x_1, v_1), (x_2, v_2), (x_3, v_2), (x_4, v_3)\}$. One can easily verify that $G * R = H$. On the other hand, H is connected but G has 2 connected components. The point here is, of course, that R is not injective.

2.2.2 Colorings

Graph homomorphisms of simple graphs can be seen as generalizations of colorings: A *(vertex) k-coloring* of G is a mapping $c : V_G \to \{1, 2, \ldots, k\}$ such that adjacent vertices have distinct colors, i.e., $c(u) \neq c(v)$ whenever $(u, v) \in E_G$. Every k-coloring c can be also seen as a homomorphism $c : G \to K_k$.

The *chromatic number* χ is defined as the minimal of colors needed for a coloring, see e.g. [7]. Thus, if R is a functional relation describing a vertex coloring, then $G * R \subseteq K_k$. Conversely, $G * R \subseteq K_k$, where R has full domain, then from Lemma 1.4, there exists a homomorphism from G to K_k, which is a coloring of G.

Lemma 2.6. If G is a simple graph and R has full domain, then $\chi(G) \leq \chi(G * R)$.

Proof. Suppose $G * R = H$ and the domain of R is full, from Lemma 1.4 we know $G \to H$, so $\chi(G) \leq \chi(G * R)$. □

2.2.3 Distances

Observation 2.7. If $P_k * R = G$, G is a simple graph and the domain of R is full, P_k with the vertex set $0, 1, \cdots, k$, then there is a walk $[v_0, v_1, \ldots, v_k]$ in G, where $(i, v_i) \in R$ for $0 \leq i \leq k$.

Observation 2.8. If $C_k * R = G$, G is a simple graph and the domain of R is full, then there is a closed walk $[v_0, v_1, \ldots, v_{k-1}]$ in G, where $(i, v_i) \in R$ for $0 \leq i \leq k - 1$.

Let $d_G(x, y)$ denote the *canonical distance* on graph G, i.e., $d_G(x, y)$ is the minimal length of a path in graph G that connects vertices x and y; if there is no path connects vertices x and y, then the distance is infinite.

Lemma 2.9. Suppose there exists a relation R with full domain s.t. $G * R = H$, $x, y \in V_G$, $u, v \in V_H$ and $(x, u) \in R$, $(y, v) \in R$. If $x \neq y$, then $d_H(u, v) \leq d_G(x, y)$; If $x = y$ and x is not an isolated vertex, then $d_H(u, v) \leq 2$.

Proof. If $x = y$ and x is not isolated, pick a vertex z of graph G which is adjacent to vertex x, and find a vertex $w \in H$ satisfying $(z, w) \in R$. Then $(w, u) \in E_H$ and similarly $(w, v) \in E_H$. So $d_H(u, v) \leq 2$.

If $x \neq y$, choose the shortest path $P = x, x_1, x_2, \cdots, x_k, y$ between x and y, and find corresponding vertices $u_1, u_2, \cdots, u_k \in H$ such that$(x_i, u_i) \in R$ for any $1 \leq i \leq k-1$ it is easily seen that $(u, u_1) \in E_H$, $(u_i, u_{i+1}) \in E_H$ and $(u_k, v) \in E_H$, then $d(u, v) \leq d(x, y)$. □

The *eccentricity* ϵ of a vertex v is the greatest distance between v and any other vertex. The *radius* of a graph G, denoted by $rad(G)$, is the minimum eccentricity of any vertex. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum eccentricity of any vertex in the graph, i.e., the largest distance between any pair of vertices.

Corollary 2.10. *Suppose $G * R = H$, G and H are connected graphs, and R has full domain, then $rad(H) \leq \max\{rad(G), 2\}$.*

An analogous result holds for the diameters. In particular, if G is not a complete graph, then $diam(G) \geq diam(G * R)$.

Corollary 2.11. *There is a relation from the path of length k, P_k, to the path of length l, P_l, if and only if either $k \geq l$ or $k = 1, l = 2$.*

Proof. For $k \geq l$ there is a surjective homomorphism f from P_k to P_l and hence by Lemma 1.4 there is also a relation from P_k to P_l. In Section 1.5 we already showed a relation from P_1 to P_2.

To show that $P_1 * R = P_2$ is the only case with $k < l$ we first observe that Lemma 2.9 excludes the existence of relation from P_k to P_l for $1 < k < l$. Now suppose R satisfies $P_1 * R = P_k$ for $k > 2$. Since P_k has at least 4 vertices, either one of the vertices of P_1 has at least 3 images so that $P_1 * R$ has a vertex with degree at least 3, or both of the vertices in P_1 have at least 2 images, in which case all vertices of $P_1 * R$ have degree at least 2. In both cases $P_1 * R$ cannot be a path. □

In particular, $\{P_1, P_2\}$ is the only pair of paths such that there is a relation between them in both directions.

2.2.4 Complete graphs

The *complement graph* $\overline{H}$ of a simple graph H has the same vertex set as H, and two vertices are connected in $\overline{H}$ if and only if they are not connected in H.

Note that in this subsection we do not require that the domain of R is full.

Proposition 2.12. Let H be a simple graph. Then there exists a relation R such that $K_k * R = H$ if and only if $\overline{H}$ is the disjoint union of at most k complete graphs.

Proof. Denote the connected components of $\overline{H}$ by $H_1, \ldots, H_m$. If $m \leq k$ and every connected component of $\overline{H}$ is a complete graph, let $R = \{(i, u) | i = 1, \cdots, m, u \in V_{H_i}\}$ and by the definition of complement graph, for any $i = 1, \cdots, m$, all the vertices in H_i are independent in H, and u is adjacent to v whenever $u \in V_{H_i}$ and $v \in V_{H_j}$ for distinct i, j. Hence it is easily seen that $K_k * R = H$.

Conversely, if $K_k * R = H$, denote the vertices in K_k by $1, \cdots, k$, s.t. $\operatorname{dom} R = \{1, \cdots, m\}$. We claim that R is injective, otherwise H would have loops. Thus V_H is the disjoint union of $R(1), \cdots, R(m)$. For any two distinct vertices u, v in $R(i)$, u and v are independent in H and for distinct i and j every vertex in $R(i)$ are adjacent with every vertex in $R(j)$ whenever $R(i) \neq \emptyset$. Therefore for any i, $R(i)$ is the vertex set of a connect component of $\overline{H}$, which is a complete graph. □

2.2.5 Subgraphs

Relations between graphs intuitively imply relations between local subgraphs. In this section we make this concept more precise. Denote by

$$N_G[x] := \{z \in V_G | z = x \vee (x, z) \in E_G\} \tag{2.2}$$

the *closed neighborhood* of x in G. Furthermore, we let $\overline{N_G[x]} := V_G \setminus N_G[x]$ be the set of vertices that are not adjacent (or identical) to x in G and denote by $\overline{G_x} := G[\overline{N_G[x]}]$ the induced subgraph of G that is obtained by removing the closed neighborhood of a vertex x.

Analogously, for a subset $S \subseteq V_G$ we define

$$\overline{S} = G\left[V_G \setminus \bigcup_{x \in S} N_G[x]\right] \tag{2.3}$$

as the induced subgraph obtained by removing all vertices in S and their neighbors.

Then we have the following result about relations between local subgraphs.

Proposition 2.13. Suppose $G{*}R = H$ and S and D are subsets of V_G and V_H, respectively, such that $G[S] * R|_{(S \times D)} = H[D]$, $R|_{(S \times D)}$ has full domain on S, and there is no isolated vertex in $\overline{D}$. Then $\overline{S} * \tilde{R} = \overline{D}$, where $\tilde{R} = R|_{(\overline{S} \times \overline{D})}$ is the corresponding restriction of R.

Proof. Obviously, $\overline{S} * \tilde{R}$ is an induced subgraph of $\overline{D}$. We have to show the reverse inclusion. Given $u \in V_{\overline{D}}$ and $x \in R^{-1}(u)$, we first show that there are two possibilities:

1. x is a vertex of $\overline{S}$.
2. x is an isolated vertex of S.

Assume that is not the case, i.e., that $x \notin V_{\overline{S}}$ and that x is either a non-isolated vertex of S or x is in the neighborhood of some vertex of S. In either case there is $y \in S$ connected by an edge to x. Consequently there is also $v \in D$, such that $v \in R(y)$, connected by an edge to u. It follows $u \notin V_{\overline{D}}$, a contradiction.

Now consider an arbitrary edge $(u, v) \in E_{\overline{D}}$. We have $(x, y) \in E_G$ such that $u \in R(x)$ and $v \in R(y)$. It follows that x and y are not isolated and thus x, y are vertices of $\overline{S}$. Consequently $\overline{S} * \tilde{R}$ has precisely the same edges as $\overline{D}$. Because $\overline{D}$ has no isolated vertices and thus every vertex is an endpoint of some edge, we know that the vertex set of $\overline{S} * \tilde{R}$ is same as the vertex set of $\overline{D}$. □

This result is of particular practical use in the special case where S and D consist of a single vertex. When looking for a relation R such that $G * R = H$ one can remove a vertex including its neighborhood from G as well as the prospective image including the neighborhood from H and solve the problem on the subgraphs.

3 Relational equivalence

Graphs G and H are *homomorphism equivalent* (or *hom-equivalent*) if there exists homomorphisms $G \to H$ and $H \to G$. It is well known that every equivalence class of the homomorphism order contains a minimal representative that is unique up to isomorphism: the *graph core* [7].

We define similar equivalences implied by the existence of (special) relations between graphs. In this section, we require all relations to have full domain unless explicitly stated otherwise. With this condition we will show that these equivalences produce a rich structure closely related to but distinct from the structure of homomorphism equivalences.

This may come as a surprise: the equivalence implied by the existence of surjective homomorphisms is not interesting. Consider two graphs G and H and suppose there are surjective homomorphisms $f : G \to H$ and $g : H \to G$. Since every vertex in V_G has at most one image under f, we have $|V_G| \geq |V_H|$. Analogously $|V_H| \geq |V_G|$, and hence $|V_G| = |V_H|$. Thus f and g are both bijective, and G is isomorphic to H.

3.1 Reversible relations

Definition 3.1. A relation R is *reversible with respect to graph* G if $(G * R) * R^+ = G$.

We write $N_G(x) := \{z \in V_G | (x, z) \in E_G\}$ for the *open neighborhood* of vertex x in graph G.

Proposition 3.2. Suppose $R = R_D \circ R_C$, where R_D and R_C are constructed as in the proof of Proposition 2.2. Then R is reversible with respect G if and only if for every α and β satisfying $R_C(\alpha) = R_C(\beta)$ we have $N_{G*R_D}(\alpha) = N_{G*R_D}(\beta)$.

Proof. We set $G_1 = G * R_D$, then from Lemma 2.1 we have $G_1 * R_C = H$. If $R_C(\alpha) = R_C(\beta)$ implies $N_{G_1}(\alpha) = N_{G_1}(\beta)$, then $H * R_C^+ = G_1$. Since $G_1 * R_D^+ = H$, we have $H * R_C^+ * R_D^+ = H * R^+ = G$, i.e., R is reversible.

Conversely, since R is reversible, i.e., $H * R^+ = G$, setting $G_2 = H * R_C^+$ gives $G_2 * R_D^+ = G$. Hence $G_1 * R_C * R_C^+ = G_2$ and $G_2 * R_D^+ * R_D = G_1$. From $I_{G_1} \subseteq R_C * R_C^+$ we conclude $G_1 \subseteq G_2$, and similarly $I_{G_2} \subseteq R_D^+ * R_D$ yields $G_1 \supseteq G_2$. Hence $G_1 = G_2$. R_C^+ is injective, hence $\alpha, \beta \in V_{G_2} = V_{G_1}$ has the same open neighborhood whenever the pre-image of α and β under R_C^+ coincide, i.e. $R_C(\alpha) = R_C(\beta)$. □

R_D is an injective relation, hence one can easily get $N_{G*R_D}(\alpha) = R_D(N_G(x))$ provided that $(x, \alpha) \in R_D$. On the other hand, if we define R to be the image of R_D as in the proof of Proposition 2.2, then $R_C(\alpha) = R_C(\beta)$ implies there are two distinct vertices $x, y \in V_G$, s.t. $(x, u), (y, u) \in R$, where $u = R_C(\alpha) = R_C(\beta)$, and verse visa. Using Proposition 3.2 we thus obtain

Proposition 3.3. A relation R is reversible with respect to G if and only if for every two vertices x and y such that $R(x) \cap R(y) \neq \emptyset$ we have $N_G(x) = N_G(y)$.

3.2 Strong relational equivalence

Definition 3.4. Two graphs G and H are *(strongly) relationally equivalent*, $G \backsim H$, if there is a relation R such that $G * R = H$ and $H * R^+ = G$.

Lemma 3.5. Relational equivalence is an equivalence relation on graphs.

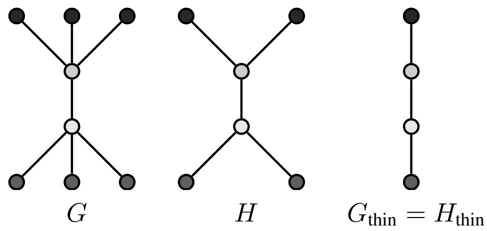


Figure 2: Non-isomorphic graphs G and H with isomorphic thin graphs.

Proof. The relation $\backsim$ is reflexive since $G * I_G = G$. Symmetry also follows directly from the definition. Suppose $G * R = H$ and $H * R^+ = G$ and $H * Q = K$ and $K * Q^+ = H$, i.e., $(G * R) * Q = K$ and $(K * Q^+) * R^+ = G$, i.e., $G * (R \circ Q) = K$ and $K * (Q^+ \circ R^+) = K * (R \circ Q)^+ = G$, i.e., $\backsim$ is also transitive. □

Definition 3.6. The *thinness relation* S of G is the equivalence relation on V_G defined by $(x, y) \in S$ if and only if $N_G(x) = N_G(y)$. A graph G is called *thin* if every vertex forms its own class in S.

Thin graphs are also known as "point determining graphs" [13].

We denote by $\mathcal{S}$ the corresponding partition of V_G, and write $R_S \subseteq V_G \times \mathcal{S}$ for the relation that associates each vertex with its S-equivalence class, i.e., $(x, \beta) \in R_S$ if and only if $x \in \beta$.

Definition 3.7. The *thin graph of* G, denoted by G_{thin}, is the quotient graph G/S, i.e., G_{thin} has vertex set $\mathcal{S}$ and two equivalence classes σ and τ of S are adjacent in G_{thin} if and only if (x, y) is an edge of G with $x \in \sigma$ and $y \in \tau$.

As noted e.g. in [6, p.81], G_{thin} is itself a thin graph. Furthermore, R_S is a full homomorphism of G to G_{thin}, see [4].

Thinness and the quotients w.r.t. the thinness relation play an important role in particular in the context of product graphs, see [9]. In this context it is well known that G can be reconstructed from G_{thin} and the knowledge of the S-equivalence classes. In fact, we have

$$G_{\text{thin}} * {R_S}^+ = G. \tag{3.1}$$

Theorem 3.8. *G and H are in the same equivalence class w.r.t. $\backsim$ if and only if their thin graphs are isomorphic.*

Proof. Assume $G \backsim H$. From Equation(3.1) we know that $G \backsim G_{\text{thin}}$, $H \backsim H_{\text{thin}}$, so $G_{\text{thin}} \backsim H_{\text{thin}}$. Now we claim that G_{thin} and H_{thin} are isomorphic. Suppose $G_{\text{thin}} * R = H_{\text{thin}}$, then the pre-image of R is unique. Otherwise, there exist distinct vertices $x, y \in V_{G_{\text{thin}}}$ such that $R(x) = R(y)$, then $N_{G_{\text{thin}}}(x) = N_{G_{\text{thin}}}(y)$, contradicting thinness. Likewise, the pre-image of R^{-1} is unique, i.e., the image of R is unique. Hence R is one-to-one. □

The example in Fig. 2 shows that thin graphs can be isomorphic while G and H themselves are not isomorphic. Relational equivalence thus is coarser than graph isomorphism (surjective homomorphic equivalence) but stronger than homomorphic equivalence.

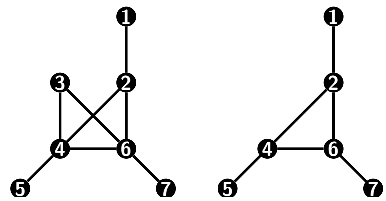


Figure 3: G and H are weakly relationally equivalent but have non-isomorphic thin graphs.

3.3 Weak relational equivalence

Definition 3.9. Two graphs G and H are *weak relationally equivalent*, $G \backsim_w H$, if there are relations R and S such that $G * R = H$ and $H * S = G$.

Lemma 3.10. Weak relational equivalence is an equivalence relation on graphs.

Proof. By definition $\backsim_w$ is symmetric. Because $G * I_G = G$, relation $\backsim_w$ is reflexive. Suppose $G \backsim_w G'$ and $G' \backsim_w G''$. Thus there are relations R, S, R', and S', such that $G' = G * R$, $G'' = G' * R'$, $G = G' * S$, and $G' = G'' * S'$. By the composition law (Lemma 2.1) it follows that $G'' = G * (R \circ R')$ and $G = G'' * (S' \circ S)$, i.e, $G \backsim_w G''$. Hence $\backsim_w$ is transitive. □

Strong relational equivalence implies weak relational equivalence. To see this, simply observe that the definition of the weak form is obtained from the strong one by setting $S = R^+$.

The converse is not true, as shown by the graphs G and H in Fig. 3: It is easy to see that their thin graphs are different and thus G and H are not strongly relationally equivalent. However, are relationally equivalent. To get relation from G to H contract vertices 2 and 3 and keep other vertices on place, i.e.,

$$R = \{(1,1),(2,2),(3,2),(4,4),(5,5),(6,6),(7,7)\}.$$

To get relation from H to G, duplicate 5 and 7 and contract them together to 3,

$$S = \{(1,1),(2,2),(4,4),(5,5),(6,6),(7,7),(5,3),(7,3)\}.$$

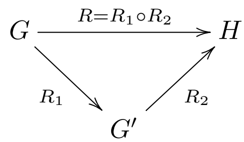
Consequently, weak relational equivalence is coarser than strong relational equivalence.

3.4 R-cores

A graph is an *R-core*, if it is the smallest graph (in the number of vertices) in its equivalence of $\backsim_w$.

This notion is analogous to the definition of graph cores. In this section we show properties of R-cores that are similar to the properties of graph cores. To this end we first need to develop a simple characterization of R-cores.

Again we start from a decomposition of relations. Consider a relation R such that $G * R = H$. We seek for pair of relations R_1 and R_2 such that $R = R_1 \circ R_2$. In contrast to Lemma 2.2, however, we now look for a decomposition so that the graph $G' = G * R_1$ is smaller (in the number of vertices) than G.



(3.2)

The existence of such a decomposition follows from a translation of the well-known Hall Marriage Theorem [12] to the language of relations. We say that the relation $R \subseteq A \times B$ satisfies the *Hall condition*, if for every $S \subseteq A$ we have $|S| \leq |R(S)|$.

Theorem 3.11 (Hall's theorem)**.** *If $G * R = H$ and R satisfies the Hall condition, then R contains a monomorphism $f : G \to H$.*

Proof. The Hall Marriage Theorem is usually described on set systems. For set systems satisfying the Hall condition, the theorem guarantees the existence of a system of distinct representatives, see i.e. [12]. Relations can be seen as set systems (defined by the images of individual vertices). Furthermore, in our setting the system of distinct representatives directly corresponds to a monomorphism contained in the relation R. □

Lemma 3.12. If $G * R = H$ and relation R does not satisfy the Hall condition, then there are relations R_1 and R_2 such that $R = R_1 \circ R_2$, and the number of vertices of graph $G' = G * R_1$ is strictly smaller than the number of vertices of G.

Proof. Without loss of generality assume that $V_G \cap V_H = \emptyset$. If R does not satisfy the Hall condition, then there exists a vertex set $S \subset V_G$ such that $|S| > |R(S)|$. Now we define relations R_1 and R_2 as follows:

$$R_1(x) = \begin{cases} R(x) \text{ for } x \in S, \\ x \text{ otherwise}, \end{cases} \qquad R_2(x) = \begin{cases} x \text{ for } x \in R(S), \\ R(x) \text{ otherwise}. \end{cases} \tag{3.3}$$

Obviously $R_1 \circ R_2 = R$ and $|V_{G'}| = |V_G| - (|S| - |R(S)|) < |V_G|$. □

This immediately gives a necessary, but in general not sufficient, condition for a graph to be an R-core.

Corollary 3.13. *If G is an R-core, then every relation R such that $G * R = G$ satisfies the Hall condition and thus contains a monomorphism.*

Proof. Assume that there is a relation R that does not satisfy the Hall condition. Then there is a graph G', $|V_{G'}| < |V_G|$, and relations R_1 and R_2 such that $G * R_1 = G'$ and $G' * R_2 = G$. Consequently G' is a smaller representative of the equivalence class of $\backsim_w$, a contradiction with G being R-core. □

To see that the condition of Corollary 3.13 is not sufficient consider a graph consisting of two independent vertices.

Next we show that R-cores are, up to isomorphism, unique representatives of the equivalence classes of $\backsim_w$.

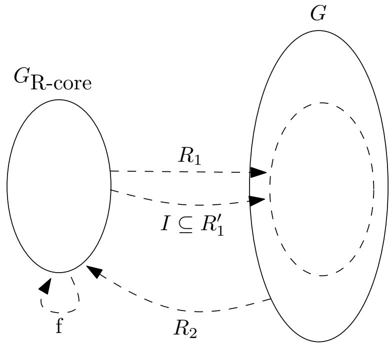


Figure 4: Construction of an embedding from $G_{\text{R-core}}$ to G.

Proposition 3.14. If both G and H are R-cores in the same equivalence class of $\backsim_w$, then G and H are isomorphic.

Proof. Because both G and H are R-cores, we know that $|V_G| = |V_H|$.

Consider relations R_1 and R_2 such that $G*R_1 = H$ and $H*R_2 = G$. Applying Lemma 3.12 we know that R_1 satisfies the Hall condition. Otherwise there would be a graph G' with $|V_{G'}| < |V_G|$ so that G' is relationally equivalent to both G and H contradicting the fact that G and H are R-cores. Similarly, we can show that R_2 also satisfies the Hall condition.

From Theorem 3.11 we know that there is a monomorphism f from G to H, and monomorphism g from H to G. It follows that number of edges of G is not larger than the number of edges of H and vice versa. Because G and H have the same number of edges and same number of vertices, G and H must be isomorphisms. □

It thus makes sense to define a construction analogous to the core of a graph.

Definition 3.15. H is an *R-core of graph* G if H is an R-core and $H \backsim_w G$.

All R-cores of graph G are isomorphic as an immediate consequence of Prop. 3.14. We denote the (up to isomorphism) unique R-core of graph G by $G_{\text{R-core}}$.

Lemma 3.16. $G_{\text{R-core}}$ is isomorphic to a (not necessarily induced) subgraph of G.

Proof. Take any relation R such that $G_{\text{R-core}}*R = G$. By the same argument as in Corollary 3.13, there is a monomorphism $f : G_{\text{R-core}} \to G$ contained in R. Consider the image of f on G. □

Theorem 3.17. *$G_{R\text{-}core}$ is isomorphic to an induced subgraph of G.*

Proof. Fix R_1 and R_2 such that $G_{\text{R-core}} * R_1 = G$ and $G * R_2 = G_{\text{R-core}}$.

$R = R_1 \circ R_2$ is a relation such that $G_{\text{R-core}} * R = G_{\text{R-core}}$. By Corollary 3.13, R contains a monomorphism $f : G_{\text{R-core}} \to G_{\text{R-core}}$. Because such a monomorphism is a permutation, there exists n such that f^n, the n-fold composition of f with itself, is the identity.

Put $R'_1 = R^{n-1} \circ R_1$. Because R^n contains the identity and $R^n = R'_1 \circ R_2$, it follows that for every $x \in V_{G_{\text{R-core}}}$, there is a vertex $I(x) \in V_G$ such that $I(x) \in R'_1(x)$ and $x \in R_2(I(x))$.

We show that for two vertices $x \neq y$, we have $I(x) \neq I(y)$ and thus both I and I^{-1} are monomorphisms. Assume, that is not the case, i.e., that there are two vertices $x \neq y$ such that $I(x) = I(y)$. Consider an arbitrary vertex z in the neighborhood of x. It follows that $I(z)$ must be in the neighborhood of $I(x)$ and consequently z is in the neighborhood of y. Thus the neighborhoods of x and y are the same. By Theorem 3.8, however, we know that the R-core is a thin graph (because weak relational equivalence is coarser than strong relational equivalence), a contradiction.

Finally observe that I is an embedding from $G_{\text{R-core}}$ to G. For every edge $(x, y) \in E_{G_{\text{R-core}}}$ we also have edge $(I(x), I(y)) \in E_G$ because I is contained in relation R'_1. Similarly because I^{-1} is contained in relation R_2, every edge $(I(x), I(y)) \in E_G$ corresponds to an edge $(x, y) \in E_{G_{\text{R-core}}}$. □

We close the section with an algorithm computing the R-core of a graph. In contrast to graph cores, where the computation is known to be NP-complete, there is a simple polynomial algorithm for R-cores.

Observe that the R-core of a graph containing isolated vertices is isomorphic to the disjoint union of the R-core of the same graph with the isolated vertices removed and a single isolated vertex. The R-core of a graph without isolated vertices can be computed by Algorithm 1.

Algorithm 1 The R-core of a graph

Input:
Graph G with loops allowed and without isolated vertices, vertex set denoted by V, neighborhoods $N_G(i)$, $i \in V$.

for $i \in V$ **do**
  $W(i) = \emptyset$
  found = FALSE
  for $j \in V \setminus \{i\}$ **do**
    if $N(j) \subseteq N(i)$ **then**
      $W(i) := W(i) \cup N(j)$
    end if
    if $N(i) \subseteq N(j)$ **then**
      found = TRUE
    end if
  end for
  if $W(i) = N(i) \wedge$ found **then**
    delete i from V
    $N(i) = \emptyset$
  end if
end for
return The R-core $G[V]$ of G.

The algorithm removes all vertices $v \in G$ such that (1) the neighborhood of v is union of neighborhood of some other vertices $v_1, v_2, \ldots, v_n$ and (2) there is vertex u such that

$N_G(v) \subseteq N_G(u)$.

It is easy to see that the resulting graph H is relationally equivalent to G. Condition (1) ensures the existence of a relation R_1 such that $H * R_1 = G$, while the condition (2) ensures the existence of a relation R_2 such that $G * R_2 = H$.

We need to show that H is isomorphic to $G_{\text{R-core}}$. By Theorem 3.17 we can assume that $G_{\text{R-core}}$ is an induced subgraph of H that is constructed as an induced subgraph of G.

We also know that there are relations R_1 and R_2 such that $G_{\text{R-core}} * R_1 = H$ and $G * R_2 = G_{\text{R-core}}$. By the same argument as in the proof of Theorem 3.17 we can assume both R_1 and R_2 to contain an (restriction of) identity.

Now assume that there is a vertex $v \in V_H \setminus V_{G_{\text{R-core}}}$. We can put $u = R_2(v)$ and because R_2 contains an identity we have $N_G(v) \subseteq N_G(u)$. We can also put $\{v_1, v_2 \dots v_n\}$ to be set of all vertices such that $v \in R_1(v_i)$. It follows that the neighborhood of v is the union of neighborhoods of $v_1, v_2, \dots, v_n$ and consequently we have $v \notin V_H$, a contradiction.

4 The partial order $\mathrm{Rel}(G, H)$

4.1 Basic properties

For fixed graphs G and H we consider partial order $\mathrm{Rel}(G, H)$. The vertices of this partial order are all relations R such that $G * R = H$. We put $R_1 \leq R_2$ if and only if $R_1 \subseteq R_2$.

This definition is motivated by Hom-complexes, see [10]. In this section we show the basic properties of this partial order and concentrate on minimal elements in the special case of $\mathrm{Rel}(G, G)$.

Proposition 4.1. Suppose $G * R' = H$, $G * R'' = H$ and $R' \subseteq R''$, then any relation R with $R' \subseteq R \subseteq R''$ also satisfies $G * R = H$.

Proof. From $R' \subseteq R \subseteq R''$ we conclude $G*R' \subseteq G*R \subseteq G*R''$. Hence $G*R' = G*R''$ implies $G * R = H$. □

Hence it is possible to describe the partial order $\mathrm{Rel}(G, H)$ by listing minimal and maximal solutions R of $G * R = H$ w.r.t. set inclusion.

For example, if G is P_3 with vertices v_0, v_1, v_2, v_3 and H is P_1 with vertices x_0, x_1, it is easily seen that $R'' = \{(v_0, x_0), (v_2, x_0), (v_1, x_1), (v_3, x_1)\}$ is a maximal solution of $G * R = H$ and $R' = \{(v_0, x_0), (v_1, x_1)\}$ is a minimal solution, because $R' \subset R''$, then all the relations R with $R' \subseteq R \subseteq R''$ satisfy $G * R = H$. We note that minimal and maximal solutions need not be unique.

4.2 Solutions of $G * R = G$

For simplicity, we say that a relation R is an *automorphism* of G if it is of the form $R = \{(x, f(x)) | x \in V_G\}$ and $f : V_G \to V_G$ is an automorphism of G.

We shall see that conditions related to thinness again play a major role in this context. Recall that G is thin if no two vertices have the same neighborhood, i.e., $N_G(x) = N_G(y)$ implies $x = y$. Here we need an even stronger condition:

Definition 4.2. A graph G satisfies *condition N* if $N_G(x) \subseteq N_G(y)$ implies $x = y$.

In particular, graph satisfying condition N is thin.

Proposition 4.3. For a given graph G, the set $\mathrm{Rel}(G,G)$ of all relations satisfying $G * R = G$ forms a monoid.

Proof. Firstly, because G is a finite graph, the set $\mathrm{Rel}(G,G)$ is also finite. Furthermore, $R, S \in \mathrm{Rel}(G,G)$ implies $G * R = G$ and $G * S = G$ and thus $G * (R \circ S) = G$, so that $R \circ S \in \mathrm{Rel}(G,G)$. Finally, the identity relation I_G is a left and right identity for relational composition: $I_G \circ R = R \circ I_G = R$. □

A relation $R \subset V_G \times V_G$ can be interpreted as a directed graph $\vec{R}$ with vertex set V_G and a directed edge $u \to v$ if and only if $(u,v) \in R$. Note that $\vec{R}$ may have loops. We say that $v \in V_G$ is *recurrent* if and only if there exists a walk (of length at least 1) from v to itself. Let S_G be the set of all the recurrent vertices. Furthermore, we define an equivalence relation ξ on S_G by setting $(u,v) \in \xi$ if there is a walk in $\vec{R}$ from u to v and vice versa. The equivalence classes w.r.t. ξ are denoted by $\vec{R}/\xi = \{D_1, D_2, \cdots, D_m\}$. We furthermore define a binary relation $\geq$ over $\vec{R}/\xi$ as follows: if there is a walk from a vertex u in D_i to a vertex v in D_j, then we say $u \geq v$. It is easily seen that $\geq$ is reflexive, antisymmetric, and transitive, hence $(\vec{R}/\xi, \geq)$ is a partially ordered set. W.l.o.g. we can assume $\{D_1, D_2, \ldots, D_s\}$ are the maximal elements w.r.t. $\geq$. Now let $G_r = G[D_1 \cup \cdots \cup D_s]$ be the subgraph of G induced by these maximal elements.

In the following we write R^l for the l-fold composition of R with itself.

Lemma 4.4. For arbitrary $x \in V_G$, there exists $l \in \mathbb{N}$ and a recurrent vertex v such that $(v,x) \in R^l$.

Proof. Set $x_0 = x$ and choose $x_i \in R^{-1}(x_{i-1})$ for all $i \geq 1$. Since $|V_G| < \infty$, there are indices $j,k \in \mathbb{N}$, $j < k$, $x_j = x_k$. Then x_j is recurrent vertex. The lemma follows by setting $l = j$ and $v = x_i$. □

Lemma 4.5. For every $v \in V_{G_r}$, $R^{-1}(v) \subseteq V_{G_r}$.

Proof. Suppose $x \in R^{-1}(v)$ is not recurrent. Lemma 4.4 implies that there is $l \in \mathbb{N}$ and a recurrent vertex w such that $(w,x) \in R^l$. Hence the definitions of E and $\geq$ imply $[w] \geq [v]$, where $[v]$ denotes the equivalent class (w.r.t. E) containing the vertex v. Since $[v]$ is maximal w.r.t. $\geq$, we have $[v] = [w]$. Consequently, there exists an index $k \in \mathbb{N}$ such that $(v,w) \in R^k$. On the other hand, we have $(x,x) = (x,v) \circ (v,w) \circ (w,x) \in R^{k+l+1}$. Thus, x is recurrent, a contradiction.

Therefore, every vertex $x \in R^{-1}(v)$ is recurrent. Hence $[x] \geq [v]$ together with the maximality of $[v]$ gives $[x] = [v]$, and thus $x \in V_{G_r}$. □

Lemma 4.6. For every $x \in V_G$, there is $l \in \mathbb{N}$ such that, for arbitrary $i \geq l$, there exists $u \in V_{G_r}$ satisfying $(u,x) \in R^i$.

Proof. From Lemma 4.4 and Lemma 4.5 we conclude that it is sufficient to show that for an arbitrary recurrent vertex v there is a $k \in \mathbb{N}$ and $w \in V_{G_r}$ such that $(w,v) \in R^k$. The lemma now follows easily from the finiteness of V_G. □

From these three lemmata we can deduce

Theorem 4.7. *All solutions of $G * R = G$ are automorphisms if and only if G has property N.*

Proof. Suppose there are distinct vertices $x, y \in V_G$ such that $N_G(x) \subseteq N_G(y)$. Then $R = I_G \cup (x, y)$, which is not functional, satisfies $G * R = G$. Thus $G * R = G$ is also solved by relations that are not automorphisms of G. This proves the "only if" part.

Conversely, suppose G has property N. **Claim:** There is a $k \in \mathbb{N}$ such that $R^k \cap (V_{G_r} \times V_{G_r}) = I_{G_r}$.

For each $v_i \in V_{G_r}$ there is a walk of length $s_i \geq 1$ from v_i to itself. Hence $(v_i, v_i) \in R^{s_i}$. Let s be the least common multiple of the s_i. Then $(v_i, v_i) \in R^s$ for all $v_i \in V_{G_r}$. Define $Q := R^s \cap (V_{G_r} \times V_{G_r})$. Thus $I_{G_r} \subseteq Q$ and moreover $Q^j \subseteq Q^{j+1}$ for all $j \in \mathbb{N}$. Since V_{G_r} is finite there is an $n \in \mathbb{N}$ such that $Q^{n+1} = Q^n$, and hence $Q^{2n} = Q^n$. Let us write $R^{-i}(v) := \{u \in V_G : (u, v) \in R^i\}$. For $v \in V_{G_r}$ we have $R^{-i}(v) \in V_{G_r}$ (from Lemma 4.5) and hence $Q^{-n}(v) = R^{-sn}(v)$ for all $v \in V_{G_r}$. If $Q^n \neq I_{G_r}$, then there are two distinct vertices $u, v \in V_{G_r}$, such that $(u, v) \in Q^n$. $N_G(u) \nsubseteq N_G(v)$ and $G = G * R^{sn}$ allows us to conclude that $R^{-sn}(u) \nsubseteq R^{-sn}(v)$ and $R^{-sn}(v) \nsubseteq R^{-sn}(u)$. Hence, there is a vertex w, such that $(w, u) \in Q^n$ and $(w, v) \notin Q^n$. From $(u, v) \in Q^n$ and $(w, u) \in Q^n$ we conclude $(w, v) \in Q^n \circ Q^n = Q^{2n}$, contradicting to $Q^{2n} = Q^n$. Therefore $Q^n = I_{G_r}$. Setting $k = sn$ now implies the claim.

Finally, we show $V_{G_r} = V_G$. For any $v \in V_G \setminus V_{G_r}$, Lemma 4.6 implies the existence of $w \in V_{G_r}$ and $m \in \mathbb{N}$ such that $(w, v) \in R^{mk}$. However, we have claimed $R^{-k}(w) = \{w\}$, hence $R^{-mk}(w) = \{w\}$. This, however, implies $N_G(w) \subseteq N_G(v)$ and thus contradicts property N. Therefore, $V_G = V_{G_r}$ and moreover $R^k = I_G$. This R is an automorphism. □

5 R-retraction

A particularly important special case of ordinary graph homomorphisms are homomorphisms to subgraphs, and in particular so-called retractions: Let H be a subgraph of G, a *retraction* of G to H is a homomorphism $r : V_G \to V_H$ such that $r(x) = x$ for all $x \in V_H$.

We introduced the graph cores in section 3 as minimal representatives of the homomorphism equivalence classes. The classical and equivalent definition is the following: A *(graph) core* is a graph that does not retract to a proper subgraph. Every graph G has a unique core H (up to isomorphism), hence one can speak of H as *the core of* G, see [7].

Here, we introduce a similar concept based on relations between graphs. Again to obtain a structure related to graph homomorphisms, in this section we require all relations to have full domain unless explicitly stated otherwise.

Definition 5.1. Let H be a subgraph of G. An *R-retraction* of G to H is a relation R such that $G * R = H$ and $(x, x) \in R$ for all $x \in V_H$. If there is an R-retraction of G to H we say that H is a *retract* of G.

Lemma 5.2. If H is an R-retract of G and K is an R-retract of H, then K is an R-retract of G.

Proof. Suppose T is an R-retraction of H to K and S is an R-retraction of G to H. Then $(G * S) * T = G * (S \circ T) = K$. Furthermore $(x, x) \in T$ for all $x \in V_K \subseteq V_H$, and $(u, u) \in S$ for all $u \in V_H$, hence $(x, x) \in S \circ T$ for all $x \in V_k$. Therefore $S \circ T$ is an R-retraction from G to K.

□

Hence, the following definition is meaningful.

Definition 5.3. A graph is *R-reduced* if there is no R-retraction to a proper subgraph.

Thus, we can also speak about "the R-reduced graph of a graph G" as the smallest subgraph on which it can be retracted. We shall see below that the R-reduced graph of a graph is always unique up to isomorphism.

We shall remark that R-reduced graphs differs from R-cores introduced in section 3, thus we choose an alternative name used also in homomorphism setting (cores are also called reduced graphs).

Lemma 5.4. Let G be a graph with loops and o a vertex of G with a loop on it. Then the R-reduced graph of G is the subgraph induced by $\{o\}$.

Proof. Let O be the graph induced by $\{o\}$, and $R = \{(x, o)|x \in V_G\}$, then it is easily seen R is a R-retraction of G to O. Moreover, since O has only one vertex, thus there is no R-retraction to its subgraphs. So O is a R-reduced graph of G.

Conversely, let H be a R-reduced graph of G and denote by R the R-retraction from G to H. Then a loop of G must generate a loop of H via R, denote it by O. Similarly to above, we see O is a R-retract of H, hence it is also a R-retract of G (by Lemma 5.2). Therefore the definition of R-reduced graph implies $H = O$. □

In the remainder of this section, therefore, we will only consider graphs without loops.

Lemma 5.5. If G is R-reduced, then G has property N.

Proof. Suppose there are two distinct vertices $x, y \in V_G$ with $N_G(x) \subseteq N_G(y)$ and consider the induced graph $G/x := G[V_G \setminus \{x\}]$ obtained from G by deleting the vertex x and all edges incident with x. The relation $R = \{(z, z)|z \in V_G \setminus \{x\}\} \cup \{(x, y)\}$ satisfies $G{*}R = G/x$: the first part is the identity on G/x and already generates all necessary edges in G/x. The second part transforms edges of the form $(x, z) \in E_G$ to edges (y, z). Since R has full domain and contains the identity relation restricted to G/x, it is an R-retraction of graph G, and hence G is not R-reduced.

□

Proposition 5.6. A graph G is R-reduced if and only if it has no relation to a proper subgraph.

Proof. The "if" part is trivial. Now we suppose that H is a proper induced subgraph of graph G with the minimal number of vertices such that there is a relation R satisfying $G * R = H$. Then H does not have a relation to a proper subgraph of itself. We claim that H has property N; otherwise, one can find a vertex $u \in V_H$ and construct a retraction from H to H/u as in Lemma 5.5, which causes a contradiction. Denote $\tilde{R} = R \cap (V_H \times V_H)$, then $K = H * \tilde{R}$ is a subgraph of H. From our assumptions on H we obtain $K = H$. By virtue of Theorem 4.7, $\tilde{R}$ is induced by an automorphism of H. Hence $R \circ \tilde{R}^+$ is again a relation of G to H that contains the identity on H, i.e., it is an R-retraction. □

Since graph cores are induced subgraphs and retractions are surjective they also imply relations. Proposition 5.6 is also a consequence of this fact. We refer to [7] for a formal proof.

We call R a *minimal R-retraction* if there is no R-retraction R' such that $R \supset R' \supset I_H$.

Lemma 5.7. Let H be an R-retract of G. Then any minimal R-retraction of G to H is functional.

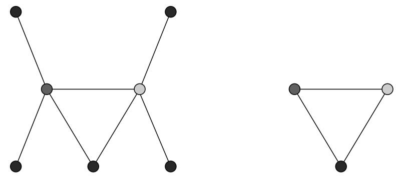


Figure 5: A graph G and its core.

Proof. Suppose R is a minimal R-retraction of G to H. If R is not functional, then there exist distinct $x, y \in V_H$ such that $(u, x), (u, y) \in R$. Hence we could always pick a vertex from $\{x, y\}$ which is different of u, w.l.o.g. suppose it is x. Then $R/(u, x)$ is an R-retraction, which contradicts minimality. To see this, set $R' = R/(u, x)$, then $R \supset R' \supset I_H$ and moreover $H = G * I_H \subseteq G * R' \subseteq G * R = H$, and thus $G * R' = H$. □

Proposition 5.8. A graph is R-reduced if and only if it is a graph core.

Proof. If H is R-reduced from G there is an R-retraction from G to H which can be chosen minimal and hence by Lemma 5.7 is functional and hence is a homomorphism retraction. Conversely, a homomorphism retraction is also an R-retraction. Hence the R-reduced graphs coincide with the graph cores. □

Proposition 5.9. Suppose H is the core of graph G. If $H * R = K$ then there is a relation R' such that $G * R' = K$. If $K * S = G$, then there is a relation S' such that $K * S' = H$.

Proof. Since H is the core of graph G, there is a relation R_1 such that $G * R_1 = H$. If $H * R = K$ we have $G * R_1 * R = K$ and $R' = R_1 \circ R$ satisfies $G * R' = K$. If $K * S = G$ we have $K * S * R_1 = H$ and $S' = S \circ R_1$ satisfies $K \circ S' = G$. □

5.1 Cocores

In the classical setting of maps between graphs, one can only consider retractions from a graph to its subgraphs, since graph homomorphisms of an induced subgraph to the original graph are just the identity maps. In the setting of relations between graphs, however, it appears natural to consider relations with identity restriction between a graph and an induced subgraph. This gives rise to notions of R-coretraction and R-cocore in analogy with R-retractions and R-reduced graphs.

Definition 5.10. Let H be a subgraph of graph G. An *R-coretraction* of H to G is a relation R such that $H * R = G$ and $(x, x) \in R$ for all $x \in V_H$. We say that H is an *R-coretract* of G.

Lemma 5.11. If H is an R-coretract of graph G and K is an R-coretract of H, then K is an R-coretract of G.

Proof. Suppose T is an R-coretraction of K to H and S is an R-coretraction of H to G. Then $(K * T) * S = K * (T \circ S) = G$. Furthermore $(x, x) \in T$ for all $x \in V_K \subseteq V_H$, and $(v, v) \in S$ for all $v \in V_H$, hence $(x, x) \in T \circ S$ for all $x \in V_K$. Therefore $T \circ S$ is an R-coretraction from K to G. □

Hence, the following definition is meaningful.

Definition 5.12. An R-coretract H of a graph G is an *R-cocore of* G if H does not have a proper subgraph that is an R-coretract of H (and hence of G).

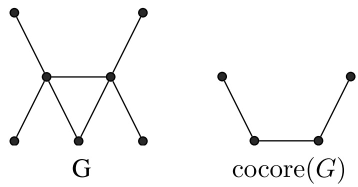


Figure 6: A graph and its cocore

Clearly, the reference to G is irrelevant: A graph G is an *R-cocore* if there is no proper subgraph of G that is an R-coretract of G. Similarly, we call R to be a *minimal R-coretraction* of H to G if there exists no R-coretraction R', such that $R' \subset R$.

Lemma 5.13. Let H be an R-coretract of graph G, and let R be a minimal R-coretraction of H to G. Then the restriction of R to H equals I_H.

Proof. Suppose $R \cap (V_H \times V_H) \neq I_H$ and define $R_1 = R \setminus \{(x, y) \in R : x, y \in V_H, x \neq y\}$. Then $H * R_1 \subseteq H * R = G$. We claim that $H * R_1 = H * R$ and thus R_1 is an R-coretraction of H to R, contradicting the minimality of R.

To prove this claim, it is sufficient to show that any edge $e \in E_G$ is contained in $H * R_1$. If e is not incident with any vertex in V_H or $e \in E_H$, the conclusion is trivial. So we only need to consider $e = (z, u)$ with $z \in E_H$ and $u \in V_G \setminus V_H$. Since $G = H * R$, one can find $x_1, x_2 \in V_H$ such that $(x_1, z), (x_2, u) \in R$ and $(x_1, x_2) \in E_H$. Because $H \subseteq H * \big(I_H \cup (x_1, z)\big) \subset H * \big(R \cap (V_H \times V_H)\big) = H$, we get $N_H(x_1) \subseteq N_H(z)$. It follows that $(z, x_2) \in E_H$ and hence $e = (z, u) \in G * R_1$. $\square$

Like R-reduced graphs, R-cocores satisfy a stringent condition on their neighborhood structure.

Definition 5.14. A graph G satisfies property N* if, for every vertex $x \in V_G$, there is no subset $U_x \subseteq V_G \setminus \{x\}$ such that

$$N_G(x) = \bigcup_{y \in U_x} N_G(y) \tag{5.1}$$

In other words, no neighborhood can be represented as the union of neighborhoods of other vertices of graph G.

Proposition 5.15. G is an R-cocore if and only if G has property N*.

Proof. Consider a vertex set U_x as in Definition 5.14 and suppose that there is a vertex $x \in V_G$ such that $N_G(x) = \bigcup_{y \in U_x} N_G(y)$. Then the relation $R := I \setminus (x, x) \cup \{(y, x) : y \in U_x\}$ is an R-coretraction from G/x to G. Thus G is not a R-cocore.

Conversely, suppose that G is not an R-cocore, let H be a coretract of G, and denote by R a minimal R-coretraction of H to G. Then, by Lemma 5.13, $R \cap (V_H \times V_H) = I_H$. Consider a vertex $v \in V_G \setminus V_H$ and set $R^{-1}(v) = \{x_1, \cdots, x_i\}$. Then $N(v) = \bigcup_i N(x_i)$, contradicting property N*. $\square$

Proposition 5.16. The R-cocore of G is unique up to isomorphism.

Proof. We denote by $\mathcal{N}$ the collection of all open neighborhoods of vertices in G, i.e., $\mathcal{N} = \{N_G(x_1), N_G(x_2), \cdots, N_G(x_k)\}$, where $V_G = \{x_1, x_2, \cdots, x_k\}$. From the definition of the R-cocore we know that the subcollection $\mathcal{M}$ of $\mathcal{N}$ consisting of all the open neighborhoods of vertices in R-cocore is a basis of $\mathcal{N}$, i.e., any set in $\mathcal{N}$ can be expressed by the union of some sets in $\mathcal{M}$. W.l.o.g., we denote the vertex set in a R-cocore C of G is $\{x_1, x_2, \cdots, x_m\}$ where $m \leq k$, then $\mathcal{M} = \{N_G(x_1), N_G(x_2), \cdots, N_G(x_m)\}$. We claim that any element in $\{N_G(x_1), N_G(x_2), \cdots, N_G(x_m)\}$ cannot be expressed as the union of other elements, i.e., $\mathcal{M}$ is a minimal basis. Otherwise, w.l.o.g., suppose $N_G(x_1) = \cup_{x_k} N_G(x_k), x_k \in \{x_2, \ldots, x_m\}$. For any $1 \leq k \leq m$, $N_G(x_k) = N_C(x_k)$ or $N_G(x_k) = N_C(x_k) \cup \{x_i | (x_i, x_k) \in E_G, m+1 \leq i \leq n\}$, so either $N_C(x_1) = \cup_{x_k} N_C(x_k), x_k \in \{x_2, \ldots, x_m\}$ or $N_C(x_1) = \cup_{x_k} N_C(x_k) \cup \{x_i | (x_i, x_k) \in E_G, m+1 \leq i \leq n, x_k \in \{x_2, \ldots, x_m\}\}$, the former contradicts to Proposition 5.15, which implies any element in $\{N_C(x_1), N_C(x_2), \cdots, N_C(x_m)\}$ cannot be expressed as the union of other elements, the latter is impossible because $\{x_i | (x_i, x_k) \in E_G, m+1 \leq i \leq n, x_k \in \{x_2, \ldots, x_m\}\} \nsubseteq C$.

Now we prove that this minimal basis is unique. Note that in $\mathcal{N}$ we view any vertex with the same neighborhood as the same, since any vertex in R-cocore has different neighborhoods. Let us consider two minimal sub-collections $\mathcal{A}, \mathcal{B}$. Neither contains the other by their minimality. Since everything is finite, let $A \in \mathcal{A}/\mathcal{B}$ be an element of minimal size. Now A can be expressed as a union of elements of $\mathcal{B}$, which all need to be of smaller cardinality than A (or same but $A \notin \mathcal{B}$), but $\mathcal{A}$ then contains all of them, letting A be expressed by a union of elements of $\mathcal{A}$ contradicting the minimality of $\mathcal{A}$. □

These results allow us to construct an algorithm that computes the cocore of given graph G in polynomial time. First observe that the cocore of a graph G that contains isolated vertices is the disjoint union of cocore of the graph G' obtained from G by removing isolated vertices and the graph consisting of a single isolated vertex. It is thus sufficient to compute cocores for graphs without isolated vertices in Algorithm 2.

Proposition 5.17. Suppose H is a cocore of G. If $K * R = H$, then there is a relation R' such that $K * R' = G$. If $G * S = K$, then there is a relation S' such that $H * S' = K$.

Proof. Since H is a cocore of G, there exists an R-coretraction R_1 such that $H * R_1 = G$. If $K * R = H$, then letting $R' = R \circ R_1$ implies $K * R' = G$. If $G * S = K$, we have $H * R_1 * S = K$. Let $S' = R_1 \circ S$, then $H * S' = K$. □

6 Computational complexity

In this section we briefly consider the complexity of computational problems related to graph homomorphisms. The *homomorphism problem* HOM(H) takes as input some finite G and asks whether there is a homomorphism from G to H. The computational complexity of the homomorphism problem is fully characterized. It is known that HOM(H) is NP-complete if and only if H has no loops and contains odd cycles. All the other cases are polynomial, see [7].

The analogous problem for relations between graphs can be phrased as follows: The *full relation problem* FUL-REL(H) takes as input some finite G and asks whether there is a relation with full domain from G and asks whether there is a relation from G to H. We

Algorithm 2 The cocore of a graph

Input:

Graph G with loops and without isolated vertices specified by its vertex set V and the neighborhoods $N_G(i)$, $i \in V$.

for $i \in V$ **do**
  $W(i) = \emptyset$
  for $j \in V \setminus \{i\}$ **do**
    if $N(j) \subseteq N(i)$ **then**
      $W(i) := W(i) \cup N(j)$
    end if
  end for
  if $W(i) = N(i)$ **then**
    delete i from V
    $N(i) = \emptyset$
  end if
end for
return $G[V]$, the cocore of G.

show that this problem can be easily converted to a related problem on surjective homomorphisms. The *surjective homomorphism problem* $\text{SUR-HOM}(H)$ takes as input some finite G and asks whether there is a surjective homomorphism from G to H.

Let $\leq_{\text{P}}^{\text{Tur}}$ indicate polynomial time Turing reduction.

Theorem 6.1. *For finite H our relation problem sits in the following relationship.*

$$\text{HOM}(H) \leq_P^{Tur} \text{FUL-REL}(H) \leq_P^{Tur} \text{SUR-HOM}(H). \tag{6.1}$$

Proof. First we show that $\text{HOM}(H)$ is polynomially reducible to $\text{FUL-REL}(H)$. If there is a homomorphism from G to H, then there is also a surjective homomorphism from $G + H$ to H. On the other hand, suppose G has no homomorphism to H. From Lemma 1.4 we conclude that $G + H$ has no relation to H since G has no relation to H.

The relation problem $\text{FUL-REL}(H)$ is polynomially reducible to $\text{SUR-HOM}(H)$. From Corollary 2.3 we know $G * R = H$ if and only if there is a graph $G' = G * R_D$ which has a full homomorphism to G and has a surjective homomorphism to H.

We construct G'', by duplicating all the vertices of G precisely $|V_H|$ times. It is easy to see that if G' exists, we can also put $G' = G''$ because the surjective homomorphism can easily undo the redundant duplications.

It remains to check whether there is surjective homomorphism from G'' to H. This gives the polynomial reduction from $\text{FUL-REL}(H)$ to $\text{SUR-HOM}(H)$. □

To our knowledge, $\text{SUR-HOM}(H)$ is not fully classified. A recent survey of the closely related complexity problem concerning the existence of vertex surjective homomorphisms [2] provides some arguments why the characterization of complexity is likely to be hard, see also [5]. We observe that the existence of a homomorphism from G to H is equivalent to the existence of a surjective homomorphism from $G + H$ to H. Thus $\text{SUR-HOM}(H)$ is clearly hard for all graphs for which $\text{HOM}(H)$ is hard, i.e., for all loop-less graphs with odd cycles.

Testing the existence of a homomorphism from a fixed G to H is polynomial (there is only a polynomial number $|V_H|^{|V_G|}$ of possible functions from G to H). Similarly the existence of a relation from a fixed G to H is also polynomial. In fact, an effective algorithm exists. For fixed G there are finitely many thin graphs T which G has relation to. The algorithm thus first constructs the thin graph of H and then, using a decision tree recognizes all isomorphic copies of all thin graphs G has relation to.

7 Weak relational composition

In this section we will briefly discuss the "loop-free" version, i.e., equations of the form $G \star R = H$.

Most importantly, there is no simple composition law analogous to Lemma 2.1. The expression

$$(G \star R) \star S = (S^+ \circ (R^+ \circ G \circ R)^\iota \circ S)^\iota \tag{7.1}$$

does not reduce to relational composition in general. For example, let $G = K_3$ with vertex set $V = \{x, y, z\}$ and consider the relations $R = \{(x, 1), (z, 1), (y, 2)\} \subseteq \{x, y, z\} \times \{1, 2\}$ and $S = \{(1, x')(1, z')(2, y')\} \subseteq \{1, 2\} \times \{x', y', z'\}$. One can easily verify

$$(G \star R) \star S = P_2 \neq G \star (R \circ S) = K_3 \tag{7.2}$$

The most important consequence of the lack of a composition law is that R-retractions cannot be meaningfully defined for the weak composition. Similarly, the results related to R-equivalence heavily rely on the composition law.

Nevertheless, many of the results, in particular basis properties derived in section 2, remain valid for the weak composition operation. As the proofs are in many cases analogous, we focus here mostly on those results where strong and weak composition differ, or where we need different proofs. In particular, Lemma 2.2 also holds for the weak composition. Thus, we still have a result similar to corollary 2.3, but the proof is slightly different.

Corollary 7.1. *Suppose $G \star R = H$. Then there is a set C, an injective relation $R_D \subseteq \operatorname{dom} R \times C$, and a surjective relation $R_C \subseteq C \times \operatorname{img} R$ such that $G[\operatorname{dom} R] \star R_D \star R_C = H[\operatorname{img} R]$.*

Proof. From Proposition 2.2 we know $R = I' \circ R_D \circ R_C \circ I''$. And we know $G[\operatorname{dom} R] \star R_D = G[\operatorname{dom} R] * R_D$. From the properties of $\star$, we have

$$\begin{aligned}
G[\operatorname{dom} R] \star R &= (R^+ \circ G[\operatorname{dom} R] \circ R)^l \\
&= ((R_D \circ R_C)^+ \circ G[\operatorname{dom} R] \circ R_D \circ R_C)^l \\
&= (R_C^+ \circ R_D^+ \circ G[\operatorname{dom} R] \circ R_D \circ R_C)^l \\
&= (R_C^+ \circ (R_D^+ \circ G[\operatorname{dom} R] \circ R_D) \circ R_C)^l \\
&= (R_C^+ \circ G[\operatorname{dom} R] * R_D \circ R_C)^l \\
&= (R_C^+ \circ G[\operatorname{dom} R] \star R_D \circ R_C)^l \\
&= G[\operatorname{dom} R] \star R_D \star R_C \\
&= H[\operatorname{img} R]
\end{aligned}$$

□

Assume $G \star R = H$ and let $H_1, \cdots, H_k$ the connected components of H. From the definition of $\star$ and $*$, if we denote $\tilde{H} = G * R$, then $\tilde{H}$ could be decomposed into the union of connected components $\tilde{H}_i (1 \leq i \leq k)$, such that $(\tilde{H}_i)^\iota = H_i$. Hence the conclusion of the proposition 2.4 also holds true for weak relations.

Lemma 2.6 does not hold for weak relations. For example, there is a weak relation of K_5 to K_3, but $\chi(K_5) = 5 > \chi(K_3) = 3$.

Lemma 2.7 and Lemma 2.8 do not hold for weak relations. For example, if G is a graph consisting of a single isolated vertex, then $P_3 \star R = G$ and $C_3 \star R = G$, but there are no walk in G.

With respect to complete graphs, weak relational composition also behaves different from strong composition. If $K_k \star R = H$ then $R(i)$ can contain more that one vertex in V_H. Compared to Proposition 2.12, we also obtain a different result:

Theorem 7.2. *There is a relation R such that $K_k \star R = H$ if and only if every connected component of $\overline{H}$ is a complete graph, and the number of connected components of $\overline{H}$ containing at least 2 vertices is at most k.*

Proof. If every connected component of $\overline{H}$ is a complete graph, denoted the vertex sets of the connected components containing at least 2 vertices by $H_1, \ldots, H_m$, $m \leq k$ and the vertices of K_k by $1, \cdots, k$. Let $R = \{(i, u) | i = 1, \cdots, k, u \in V_{H_i}\} \cup \{(j, v) : 1 \leq j \leq k, v \in V_H \setminus \bigcup_{i=1}^m V_{H_i}\}$. One easily checks that $K_k \star R = H$.

Conversely, let R be a relation satisfying $K_k \star R = H$. Consider the set $U_i = \{u \in V_H | R^{-1}(u) = \{i\}\}$. Then u and v are not adjacent for arbitrary $u, v \in U_i$, while u is adjacent to w for every $w \in V_H \setminus U_i$. Hence $\overline{H}(U_i)$ is a connected component of $\overline{H}$, which is also a complete graph. Given $w \in V_H \setminus \bigcup_{i=1}^m U_i$, $R^{-1}(w)$ must have at least 2 vertices in K_k, hence w is adjacent to every vertex in H except itself; in other words, w is an isolated vertex in $\overline{H}$. Therefore the number of connected components of $\overline{H}$ containing at least 2 vertices is no more than k. □

The results in subsection 3.1 also remain true for weak relations.

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CI-groups with respect to ternary relational structures: new examples

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Abstract

We find a sufficient condition to establish that certain abelian groups are not CI-groups with respect to ternary relational structures, and then show that the groups $\mathbb{Z}_3 \times \mathbb{Z}_2^2$, $\mathbb{Z}_7 \times \mathbb{Z}_2^3$, and $\mathbb{Z}_5 \times \mathbb{Z}_2^4$ satisfy this condition. Then we completely determine which groups $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, p a prime, are CI-groups with respect to color binary and ternary relational structures. Finally, we show that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures.

Keywords: CI-group, ternary relation.

Math. Subj. Class.: 05C15, 05C10

1 Introduction

In recent years, there has been considerable interest in which groups G have the property that any two Cayley graphs of G are isomorphic if and only if they are isomorphic by a group automorphism of G. Such a group is a called a *CI-group with respect to graphs*, and this problem is often referred to as the Cayley isomorphism problem. The interested

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reader is referred to [11] for a survey on CI-groups with respect to graphs. Of course, the Cayley isomorphism problem can and has been considered for other types of objects (see, for example, [9, 14, 16] for work on this problem on codes and on designs). Before proceeding we give the relevant definitions. (There are several equivalent definitions of combinatorial object [1, 15], here we follow [13].)

Definition 1.1. A *k-ary relational structure* is an ordered pair $X = (V, E)$, with V a set and E a subset of V^k. Furthermore, a *color k-ary relational structure* is an ordered pair $X = (V, (E_1, \ldots, E_c))$, with V a set and $E_1, \ldots, E_c$ pairwise disjoint subsets of V^k. If $k = 2, 3$, or 4, we simply say that X is a (color) binary, ternary, or quaternary relational structure. A *combinatorial object* is a pair $X = (V, E)$, with V a set and E a subset of $\bigcup_{i=1}^{\infty} V^i$.

The following two definitions are due to Babai [1].

Definition 1.2. For a group G, define $g_L : G \to G$ by $g_L(h) = gh$, and let $G_L = \{g_L : g \in G\}$. Then G_L is a permutation group on G, called the *left regular representation of G*. We will say that a (color) k-ary relational structure X is a *Cayley (color) k-ary relational structure of G* if $G_L \le \mathrm{Aut}(X)$ (note that this implies $V = G$). In general, a combinatorial object X will be called a *Cayley object of G* if $G_L \le \mathrm{Aut}(X)$.

Definition 1.3. For a class $\mathcal{C}$ of Cayley objects of G, we say that G is a *CI-group with respect to $\mathcal{C}$* if whenever $X, Y \in \mathcal{C}$, then X and Y are isomorphic if and only if they are isomorphic by a group automorphism of G.

It is clear that if G is a CI-group with respect to *color* k-ary relational structures, then G is a CI-group with respect to k-ary relational structures.

Perhaps the most significant result in this area is a well-known theorem of Pálfy [15] which states that a group G of order n is a CI-group with respect to every class of combinatorial objects if and only if $n = 4$ or $\gcd(n, \varphi(n)) = 1$, where φ is the Euler phi function. In fact, in proving this result, Pálfy showed that if a group G is not a CI-group with respect to some class of combinatorial objects, then G is not a CI-group with respect to quaternary relational structures. As much work has been done on the case of binary relational structures (i.e., digraphs), until recently there was a "gap" in our knowledge of the Cayley isomorphism problem for k-ary relational structures with $k = 3$. As additional motivation to study this problem, we remark that a group G that is a CI-group with respect to ternary relational structures is necessarily a CI-group with respect to binary relational structures, see [5, page 227].

Although Babai [1] showed in 1977 that the dihedral group of order $2p$ is a CI-group with respect to ternary relational structures, no additional work was done on this problem until the first author considered the problem in 2003 [5]. Indeed, in [5] a relatively short list of groups is given and it is proved that every CI-group with respect to ternary relational structures lies in this list (although not every group in this list is necessarily a CI-group with respect to ternary relational structures). Additionally, several groups in the list were shown to be CI-groups with respect to ternary relational structures. Recently, the second author [17] has shown that two groups given in [5] are not CI-groups with respect to ternary relational structures, namely $\mathbb{Z}_3 \ltimes Q_8$ and $\mathbb{Z}_3 \times Q_8$. In this paper, we give a sufficient condition to ensure that certain abelian groups are not CI-groups with respect to ternary relational structures (Theorem 2.1), and then show that $\mathbb{Z}_2^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_7$, and $\mathbb{Z}_2^4 \times \mathbb{Z}_5$

satisfy this condition in Corollary 2.4 (and so are not CI-groups with respect to ternary relational structures). We then show that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to ternary relational structures. As the first author has shown [6] that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to ternary relational structures provided that $p \geq 11$, we then have a complete determination of which groups $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, p a prime, are CI-groups with respect to ternary relational structures.

Theorem A. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color ternary relational structures if and only if $p \notin \{3, 7\}$.

We will show that both $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups with respect to color binary relational structures. As it is already known that $\mathbb{Z}_2^4$ is a CI-group with respect to binary relational structures [11], we have the following result.

Corollary A. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color binary relational structures for all primes p.

We are then left in the situation of knowing whether or not any subgroup of $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color binary or ternary relational structures, with the exception of $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ with respect to color ternary relational structures (as $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color binary relational structures [10]). We show that $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color ternary relational structures (which generalizes a special case of the main result of [10]) and we prove the following.

Corollary B. The group $\mathbb{Z}_2^2 \times \mathbb{Z}_p$ is a CI-group with respect to color ternary relational structures if and only if $p \neq 3$.

Finally, using Magma [2] and GAP [8], we show that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures. We conclude this introductory section by recalling the following.

Definition 1.4. For g, h in G, we denote the commutator $g^{-1}h^{-1}gh$ of g and h by $[g, h]$.

2 The main ingredient and Theorem A

We start by proving the main ingredient for our proof of Theorem A.

Theorem 2.1. *Let G be an abelian group and p an odd prime. Assume that there exists an automorphism α of G of order p fixing only the zero element of G. Then $\mathbb{Z}_p \times G$ is not a* CI-*group with respect to color ternary relational structures. Moreover, if there exists a ternary relational structure Z on G with $\mathrm{Aut}(Z) = \langle G_L, \alpha\rangle$, then $\mathbb{Z}_p \times G$ is not a* CI-*group with respect to ternary relational structures.*

Proof. Since α fixes only the zero element of G, we have $|G| \equiv 1 \pmod p$ and so $\gcd(p, |G|) = 1$.

For each $g \in G$, define $\hat{g} : \mathbb{Z}_p \times G \to \mathbb{Z}_p \times G$ by $\hat{g}(i, j) = (i, j + g)$. Additionally, define $\tau, \gamma, \bar{\alpha} : \mathbb{Z}_p \times G \to \mathbb{Z}_p \times G$ by $\tau(i, j) = (i + 1, j)$, $\gamma(i, j) = (i, \alpha^i(j))$, and $\bar{\alpha}(i, j) = (i, \alpha(j))$. Then $(\mathbb{Z}_p \times G)_L = \langle \tau, \hat{g} : g \in G\rangle$.

Clearly, $\langle G_L, \alpha\rangle = G_L \rtimes \langle\alpha\rangle$ is a subgroup of $\mathrm{Sym}(G)$ (where G_L acts on G by left multiplication and α acts as an automorphism). Note that the stabilizer of 0 in $\langle G_L, \alpha\rangle$ is $\langle\alpha\rangle$. As α fixes only 0, we conclude that for every $g \in G$ with $g \neq 0$, the point-wise stabilizer of 0 and g in $\langle G_L, \alpha\rangle$ is 1. Therefore, by [18, Theorem 5.12], there exists a color Cayley ternary relational structure Z of G such that $\mathrm{Aut}(Z) = \langle G_L, \alpha\rangle$. If there exists also a ternary relational structure with automorphism group $\langle G_L, \alpha\rangle$, then we let Z be one such ternary relational structure.

Let

$$\begin{aligned} U &= \{((0_{\mathbb{Z}_p}, g), (0_{\mathbb{Z}_p}, h)) : (0_G, g, h) \in E(Z)\}, \text{ and} \\ S &= \{([\hat{g}, \gamma](1, 0_G), [\hat{g}, \gamma](2, 0_G)) : g \in G\} \cup U \end{aligned}$$

and define a (color) ternary relational structure X by

$$V(X) = \mathbb{Z}_p \times G \quad \text{and} \quad E(X) = \{k(0_{\mathbb{Z}_p\times G}, s_1, s_2) : (s_1, s_2) \in S, k \in (\mathbb{Z}_p \times G)_L\}.$$

If Z is a color ternary relational structure, then we assign to the edge $k(0_{\mathbb{Z}_p\times G}, s_1, s_2)$ the color of the edge $(0_G, g, h)$ in Z if $(s_1, s_2) \in U$ and $(s_1, s_2) = ((0_{\mathbb{Z}_p}, g), (0_{\mathbb{Z}_p}, h))$, and otherwise we assign a fixed color distinct from those used in Z. By definition of X we have $(\mathbb{Z}_p \times G)_L \le \mathrm{Aut}(X)$ and so X is a (color) Cayley ternary relational structure of $\mathbb{Z}_p \times G$.

We claim that $\bar{\alpha} \in \mathrm{Aut}(X)$. As $\bar{\alpha}$ is an automorphism of $\mathbb{Z}_p \times G$, we see that $\bar{\alpha} \in \mathrm{Aut}(X)$ if and only if $\bar{\alpha}(S) = S$ and $\bar{\alpha}$ preserves colors (if X is a color ternary relational structure). By definition of Z and U, we have $\bar{\alpha}(U) = U$ and $\bar{\alpha}$ preserves colors (if X is a color ternary relational structure). So, it suffices to consider the case $s \in S - U$, i.e., $s = ([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0))$ for some $g \in G$. Note that now we need not consider colors as all the edges in $S - U$ are of the same color. Then $\bar{\alpha}\hat{g}(i, j) = (i, \alpha(j) + \alpha(g)) = \widehat{\alpha(g)}\bar{\alpha}(i, j)$. Thus $\bar{\alpha}\hat{g} = \widehat{\alpha(g)}\bar{\alpha}$. Similarly, $\bar{\alpha}\hat{g}^{-1} = \widehat{\alpha(g)}^{-1}\bar{\alpha}$. Clearly $\bar{\alpha}$ commutes with γ, and so $\bar{\alpha}[\hat{g}, \gamma] = [\widehat{\alpha(g)}, \gamma]\bar{\alpha}$. As $\bar{\alpha}$ fixes $(1, 0)$ and $(2, 0)$, we see that

$$\begin{aligned} \bar{\alpha}(s) = \bar{\alpha}([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0)) &= (\bar{\alpha}[\hat{g}, \gamma](1, 0), \bar{\alpha}[\hat{g}, \gamma](2, 0)) \\ &= ([\widehat{\alpha(g)}, \gamma]\bar{\alpha}(1, 0), [\widehat{\alpha(g)}, \gamma]\bar{\alpha}(2, 0)) \\ &= ([\widehat{\alpha(g)}, \gamma](1, 0), [\widehat{\alpha(g)}, \gamma](2, 0)) \in (S - U). \end{aligned}$$

Thus $\bar{\alpha}(S) = S$, $\bar{\alpha}$ preserves colors (if X is a color ternary relational structure) and $\bar{\alpha} \in \mathrm{Aut}(X)$.

We claim that $\gamma^{-1}(\mathbb{Z}_p \times G)_L\gamma$ is a subgroup of $\mathrm{Aut}(X)$. We set $\tau' = \gamma^{-1}\tau\gamma$ and $g' = \gamma^{-1}\hat{g}\gamma$, for $g \in G$. Note that $\tau' = \tau\bar{\alpha}^{-1}$. As $\bar{\alpha} \in \mathrm{Aut}(X)$, we have $\tau' \in \mathrm{Aut}(X)$. Therefore it remains to prove that $\langle g' : g \in G\rangle$ is a subgroup of $\mathrm{Aut}(X)$. Let $e \in E(X)$ and $g \in G$. Then $e = k((0, 0), s)$, where $s \in S$ and $k = \tau^a\widehat{l}$, for some $a \in \mathbb{Z}_p$, $l \in G$. We have to prove that $g'(e) \in E(X)$ and has the same color as e (if X is a color ternary relational structure).

Assume that $s \in U$. As $g'(i, j) = (i, j + \alpha^{-i}(g))$, by definition of U, we have $g'[k((0, 0), s)] \in E(X)$ and has the same color of e (if X is a color ternary relational structure). So, it remains to consider the case $s \in S - U$, i.e., $s = ([\widehat{x}, \gamma](1, 0), [\widehat{x}, \gamma](2, 0))$ for some $x \in G$. As before, we need not concern ourselves with colors because all the edges in $S - U$ are of the same color.

Set $m = k\widehat{\alpha^{-a}(g)}$. Since $\bar{\alpha}\widehat{g} = \widehat{\alpha(g)}\bar{\alpha}$ and $\bar{\alpha}, \gamma$ commute, we get $\bar{\alpha}g' = (\alpha(g))'\bar{\alpha}$. Also observe that as G is abelian, g' commutes with $\widehat{h}$ for every $g, h \in G$. Hence

$$\begin{aligned} g'k &= \gamma^{-1}\widehat{g}\gamma\tau^a\widehat{l} = \gamma^{-1}\widehat{g}\tau^a\gamma\bar{\alpha}^a\widehat{l} = \gamma^{-1}\tau^a\widehat{g}\bar{\alpha}^a\gamma\widehat{l} \\ &= \tau^a\gamma^{-1}\bar{\alpha}^{-a}\widehat{g}\bar{\alpha}^a\gamma\widehat{l} = \tau^a(\alpha^{-a}(g))'\widehat{l} = \tau^a\widehat{l}(\alpha^{-a}(g))' \\ &= k\widehat{\alpha^{-a}(g)}\widehat{\alpha^{-a}(g)}^{-1}\gamma^{-1}\widehat{\alpha^{-a}(g)}\gamma = m[\widehat{\alpha^{-a}(g)},\gamma] \end{aligned}$$

and

$$\begin{aligned} g'[k((0,0),s)] &= g'k((0,0),[\widehat{x},\gamma](1,0),[\widehat{x},\gamma](2,0)) \\ &= m[\widehat{\alpha^{-a}(g)},\gamma]((0,0),[\widehat{x},\gamma](1,0),[\widehat{x},\gamma](2,0)) \\ &= m((0,0),[\widehat{\alpha^{-a}(g)},\gamma][\widehat{x},\gamma](1,0),[\widehat{\alpha^{-a}(g)},\gamma][\widehat{x},\gamma](2,0)) \\ &= m((0,0),[\widehat{\alpha^{-a}(g)x},\gamma](1,0),[\widehat{\alpha^{-a}(g)x},\gamma](2,0)) \in E(X). \end{aligned}$$

This proves that $g' \in \mathrm{Aut}(X)$. Since g is an arbitrary element of G, we have $\gamma^{-1}G_L\gamma \subseteq \mathrm{Aut}(X)$. As claimed, $\gamma^{-1}(\mathbb{Z}_p \times G)_L\gamma$ is a regular subgroup of $\mathrm{Aut}(X)$ conjugate in $\mathrm{Sym}(\mathbb{Z}_p \times G)$ to $(\mathbb{Z}_p \times G)_L$.

We now see that $Y = \gamma(X)$ is a Cayley (color) ternary relational structure of $\mathbb{Z}_p \times G$ as $\mathrm{Aut}(Y) = \gamma\mathrm{Aut}(X)\gamma^{-1}$. We will next show that $Y \neq X$. Assume by way of contradiction that $Y = X$. As $\gamma(0,g) = (0,g)$ for every $g \in G$, the permutation γ must map edges of U to themselves, so that $\gamma(S-U) = S-U$. We will show that $\gamma(S-U) \neq S-U$. Note that we need not concern ourselves with colors as all the edges derived from $S-U$ via translations of $(\mathbb{Z}_p \times G)_L$ have the same color. Observing that

$$\begin{aligned} ([\hat{g},\gamma](1,0),[\hat{g},\gamma](2,0)) &= (\hat{g}^{-1}\gamma^{-1}\hat{g}\gamma(1,0),\hat{g}^{-1}\gamma^{-1}\hat{g}\gamma(2,0)) \\ &= (\hat{g}^{-1}\gamma^{-1}\hat{g}(1,0),\hat{g}^{-1}\gamma^{-1}\hat{g}(2,0)) \\ &= (\hat{g}^{-1}\gamma^{-1}(1,g),\hat{g}^{-1}\gamma^{-1}(2,g)) \\ &= (\hat{g}^{-1}(1,\alpha^{-1}(g),\hat{g}^{-1}(2,\alpha^{-2}(g)) \\ &= ((1,\alpha^{-1}(g)-g),(2,\alpha^{-2}(g)-g)), \end{aligned}$$

we see that $\gamma(S-U) = \{((1,g-\alpha(g)),(2,g-\alpha^2(g))) : g \in G\}$. Moreover, as $S-U = \{(1,\alpha^{-1}(g)-g),(2,\alpha^{-2}(g)-g) : g \in G\}$, we conclude that for each $g \in G$, there exists $h_g \in G$ such that

$$g-\alpha(g) = \alpha^{-1}(h_g)-h_g \quad \text{and} \quad g-\alpha^2(g) = \alpha^{-2}(h_g)-h_g.$$

Setting $\iota : G \to G$ to be the identity permutation, we may rewrite the above equations as

$$(\iota-\alpha)(g) = (\alpha^{-1}-\iota)(h_g) \quad \text{and} \quad (\iota-\alpha^2)(g) = (\alpha^{-2}-\iota)(h_g).$$

Computing in the endomorphism ring of the abelian group G, we see that $(\alpha^{-2}-\iota) = (\alpha^{-1}+\iota)(\alpha^{-1}-\iota)$. Applying the endomorphism $(\alpha^{-1}+\iota)$ to the first equation above, we then have

$$(\alpha^{-1}+\iota)(\iota-\alpha)(g) = (\alpha^{-1}+\iota)(\alpha^{-1}-\iota)(h_g) = (\alpha^{-2}-\iota)(h_g) = (\iota-\alpha^2)(g).$$

Hence $(\alpha^{-1}+\iota)(\iota-\alpha)=\iota-\alpha^2$, and so

$$0=(\alpha^{-1}+\iota)(\iota-\alpha)-(\iota-\alpha^2)=((\alpha^{-1}+\iota)-(\iota+\alpha))(\iota-\alpha)=(\alpha^{-1}-\alpha)(\iota-\alpha),$$

(here 0 is the endomorphism of G that maps each element of G to 0). As α fixes only 0, the endomorphism $\iota-\alpha$ is invertible, and so we see that $\alpha^{-1}-\alpha=0$, and $\alpha=\alpha^{-1}$. However, this implies that $p=|\alpha|=2$, a contradiction. Thus $\gamma(S-U)\neq S-U$ and so $Y\neq X$.

We set $T=\gamma(S)$, so that $((0,0),t)\in E(Y)$ for every $t\in T$, where if X is a color ternary relational structure we assume that γ preserves colors. Now suppose that there exists $\beta\in \mathrm{Aut}(\mathbb{Z}_p\times G)$ such that $\beta(X)=Y$. Since $\gcd(p,|G|)=1$, we obtain that $\mathbb{Z}_p\times 1_G$ and $1_{\mathbb{Z}_p}\times G$ are characteristic subgroups of $\mathbb{Z}_p\times G$. Therefore $\beta(i,j)=(\beta_1(i),\beta_2(j))$, where $\beta_1\in\mathrm{Aut}(\mathbb{Z}_p)$ and $\beta_2\in\mathrm{Aut}(G)$.

As β fixes $(0,0)$, we must have $\beta(S)=T$. Observe that every element of S and of T is of the form $((0,g),(0,h))$ or $((1,g),(2,h))$, for some $g,h\in G$. In particular, we must have $\beta_1(1)=1$ and hence $\beta_1=1$. As $\bar{\alpha}\in\mathrm{Aut}(X)$ and $X\neq Y$, we have $\beta_2\notin\langle\alpha\rangle$. Now observe that $\beta(U)=U$. Thus $\beta_2\in\mathrm{Aut}(Z)=\langle G_L,\alpha\rangle$. We conclude that $\beta_2\in\langle\alpha\rangle$, a contradiction. Thus X,Y are not isomorphic by a group automorphism of $\mathbb{Z}_p\times G$, and the result follows. □

The following two lemmas, which in our opinion are of independent interest, will be used (together with Theorem 2.1) in the proof of Corollary 2.4.

Lemma 2.2. *Let G be a transitive permutation group on Ω. If $x\in\Omega$ and $\mathrm{Stab}_G(x)$ in its action on $\Omega-\{x\}$ is the automorphism group of a k-ary relational structure with vertex set $\Omega-\{x\}$, then G is the automorphism group of a $(k+1)$-ary relational structure.*

Proof. Let Y be a k-ary relational structure with vertex set $\Omega-\{x\}$ and automorphism group $\mathrm{Stab}_G(x)$ in its action on $\Omega-\{x\}$. Let $W=\{(x,v_1,\ldots,v_k):(v_1,\ldots,v_k)\in E(Y)\}$, and define a $(k+1)$-ary relational structure X by $V(X)=\Omega$ and $E(X)=\{g(w):w\in W \text{ and } g\in G\}$. We claim that $\mathrm{Aut}(X)=G$. First, observe that $\mathrm{Stab}_G(x)$ maps W to W. Also, if $e\in E(X)$ and $e=(x,v_1,\ldots,v_k)$ for some $v_1,\ldots,v_k\in\Omega$, then there exists $(x,u_1,\ldots,u_k)\in W$ and $g\in G$ with $g(x,u_1,\ldots,u_k)=(x,v_1,\ldots,v_k)$. We conclude that $g(x)=x$ and $g(u_1,\ldots,u_k)=(v_1,\ldots,v_k)$. Hence $g\in\mathrm{Stab}_G(x)$ and $(v_1,\ldots,v_k)\in E(Y)$. Then W is the set of all edges of X with first coordinate x.

By construction, $G\leq\mathrm{Aut}(X)$. For the reverse inclusion, let $h\in\mathrm{Aut}(X)$. As G is transitive, there exists $g\in G$ such that $g^{-1}h\in\mathrm{Stab}_{\mathrm{Aut}(X)}(x)$. Note that as $g\in G$, the element $g^{-1}h\in G$ if and only if $h\in G$. We may thus assume without loss of generality that $h(x)=x$. Then h stabilizes set-wise the set of all edges of X with first coordinate x, and so $h(W)=W$ and h induces an automorphism of Y. As $\mathrm{Aut}(Y)=\mathrm{Stab}_G(x)\leq G$, the result follows. □

Lemma 2.3. *Let $m\geq 2$ be an integer and $\rho\in\mathrm{Sym}(\mathbb{Z}_{ms})$ be a semiregular element of order m with s orbits. Then there exists a digraph Γ with vertex set $\mathbb{Z}_{ms}$ and with $\mathrm{Aut}(\Gamma)=\langle\rho\rangle$.*

Proof. For each $i\in\mathbb{Z}_s$, set

$$\rho_i=(0,1,\ldots,m-1)\cdots(im,im+1,\ldots,im+m-1)\quad\text{and}\quad V_i=\{im+j:j\in\mathbb{Z}_m\}.$$

We inductively define a sequence of graphs $\Gamma_0, \ldots, \Gamma_{s-1} = \Gamma$ such that the subgraph of Γ induced by $\mathbb{Z}_{(i+1)m}$ is Γ_i, the indegree in Γ of a vertex in V_i is $i+1$, and $\mathrm{Aut}(\Gamma_i) = \langle \rho_i \rangle$, for each $i \in \mathbb{Z}_s$.

We set Γ_0 to be the directed cycle of length m with edges $\{(j, j+1) : j \in \mathbb{Z}_m\}$ and with automorphism group $\langle \rho_0 \rangle$. Inductively assume that Γ_{s-2}, with the above properties, has been constructed. We construct Γ_{s-1} as follows. First, the subgraph of Γ_{s-1} induced by $\mathbb{Z}_{(s-1)m}$ is Γ_{s-2}. Then we place the directed m cycle $\{((s-1)m+j, (s-1)m+j+1) : j \in \mathbb{Z}_m\}$ whose automorphism group is $\langle ((s-1)m, (s-1)m+1, \ldots, (s-1)m+m-1) \rangle$ on the vertices in V_{s-1}. Additionally, we declare the vertex $(s-1)m$ to be outadjacent to $(s-2)m$ and to every vertex that $(s-2)m$ is outadjacent to that is not contained in V_{s-2}. Finally, we add to Γ_{s-1} every image of one of these edges under an element of $\langle \rho_{s-1} \rangle$.

By construction, ρ_{s-1} is an automorphism of Γ_{s-1} and the subgraph of Γ_{s-1} induced by $\mathbb{Z}_{(s-1)m}$ is Γ_{s-2}. Then each vertex in $\Gamma_{s-1} \cap V_i$ has indegree $i+1$ for $0 \le i \le s-2$, while it is easy to see that each vertex of V_{s-1} has indegree s. Finally, if $\delta \in \mathrm{Aut}(\Gamma_{s-1})$, then δ maps vertices of indegree $i+1$ to vertices of indegree $i+1$, and so δ fixes set-wise V_i, for every $i \in \mathbb{Z}_s$. Additionally, the action induced by $\langle \delta \rangle$ on V_{s-1} is necessarily $\langle ((s-1)m, (s-1)m+1, \ldots, (s-1)m+m-1) \rangle$ as this is the automorphism group of the subgraph of Γ_{s-1} induced by V_{s-1}. Moreover, arguing by induction, we may assume that the action induced by δ on $V(\Gamma_{s-1}) - V_{s-1}$ is given by an element of $\langle \rho_{s-2} \rangle$. If $\delta \notin \langle \rho_{s-1} \rangle$, then $\mathrm{Aut}(\Gamma_{s-1})$ has order at least m^2, and there is some element of $\mathrm{Aut}(\Gamma_{s-1})$ that is the identity on $V(\Gamma_{s-2})$ but not on V_{s-1} and vice versa. This however is not possible as each vertex of V_{s-2} is inadjacent to exactly one vertex of V_{s-1}. Then $\mathrm{Aut}(\Gamma_{s-1}) = \langle \rho_{s-1} \rangle$ and the result follows. □

Corollary 2.4. *None of the groups $\mathbb{Z}_3 \times \mathbb{Z}_2^2$, $\mathbb{Z}_7 \times \mathbb{Z}_2^3$, or $\mathbb{Z}_5 \times \mathbb{Z}_2^4$ are CI-groups with respect to ternary relational structures.*

Proof. Observe that $\mathbb{Z}_2^2$ has an automorphism α_3 of order 3 that fixes 0 and acts regularly on the remaining 3 elements, and similarly, $\mathbb{Z}_2^3$ has an automorphism α_7 of order 7 that fixes 0 and acts regularly on the remaining 7 elements. As a regular cyclic group is the automorphism group of a directed cycle, we see that $\langle (\mathbb{Z}_3 \times \mathbb{Z}_2^2)_L, \alpha_3 \rangle$ and $\langle (\mathbb{Z}_7 \times \mathbb{Z}_2^3)_L, \alpha_7 \rangle$ are the automorphism groups of ternary relational structures by Lemma 2.2. The result then follows by Theorem 2.1.

Now $\mathbb{Z}_2^4$ has an automorphism α_5 of order 5 that fixes 0 and acts semiregularly on the remaining 15 points. Then $\langle \alpha_5 \rangle$ in its action on $\mathbb{Z}_2^4 - \{0\}$ is the automorphism group of a binary relational structure by Lemma 2.3. By Lemma 2.2, there exists a ternary relational structure with automorphism group $\langle (\mathbb{Z}_5 \times \mathbb{Z}_2^4)_L, \alpha_5 \rangle$. The result then follows by Theorem 2.1. □

Before proceeding, we will need terms and notation concerning complete block systems.

Let $G \le \mathrm{Sym}(n)$ be a transitive permutation group (acting on $\mathbb{Z}_n$, say). A subset $B \subseteq \mathbb{Z}_n$ is a *block for* G if $g(B) = B$ or $g(B) \cap B = \emptyset$ for every $g \in G$. Clearly $\mathbb{Z}_n$ and its singleton subsets are always blocks for G, and are called *trivial blocks*. If B is a block, then $g(B)$ is a block for every $g \in G$, and the set $\mathcal{B} = \{g(B) : g \in G\}$ is called a *complete block system for* G, and we say that G *admits* $\mathcal{B}$. A complete block system is *nontrivial* if its blocks are nontrivial. Observe that a complete block system is a partition of $\mathbb{Z}_n$, and any two blocks have the same size. If G admits $\mathcal{B}$ as a complete block system, then each $g \in G$

induces a permutation of $\mathcal{B}$, which we denote by $g/\mathcal{B}$. We set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$. The kernel of the action of G on $\mathcal{B}$, denoted by $\mathrm{fix}_G(\mathcal{B})$, is then the subgroup of G which fixes each block of $\mathcal{B}$ set-wise. That is, $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$. For fixed $B \in \mathcal{B}$, we denote the set-wise stabilizer of B in G by $\mathrm{Stab}_G(B)$. That is $\mathrm{Stab}_G(B) = \{g \in G : g(B) = B\}$. Note that $\mathrm{fix}_G(\mathcal{B}) = \cap_{B\in\mathcal{B}}\mathrm{Stab}_G(B)$. Finally, for $g \in \mathrm{Stab}_G(B)$, we denote by $g|_B$ the action induced by g on $B \in \mathcal{B}$.

Note that Corollary 2.4, together with the fact that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, $p \geq 11$, is a CI-group with respect to color ternary relational structures [6], settles the question of which groups $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ are CI-groups with respect to color ternary relational structures except for $p = 5$. Our next goal is to show that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to color ternary relational structures. From a computational point of view, the number of points is too large to enable a computer to determine the answer without some additional information. Lemma 6.1 in [6] is the only result that uses the hypothesis $p \geq 11$. For convenience, we report [6, Lemma 6.1].

Lemma 2.5. *Let $p \geq 11$ be a prime and write $H = \mathbb{Z}_2^3 \times \mathbb{Z}_p$. For every $\phi \in \mathrm{Sym}(H)$, there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system consisting of 8 blocks of size p.*

In particular, to prove that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to color ternary relational structures, it suffices to prove that Lemma 2.5 holds true also for the prime $p = 5$. We begin with some intermediate results which accidentally will also help us to prove that $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-group with respect to color binary relational structures. (Here we denote by $\mathrm{Alt}(X)$ the alternating group on the set X and by $\mathrm{Alt}(n)$ the alternating group on $\{1, \ldots, n\}$.)

Lemma 2.6. *Let p be an arbitrary divisor of n with $p \neq 1$ and let P_1 and P_2 be partitions of $\mathbb{Z}_n$ where each block in P_1 and P_2 has size p. Then there exists $\phi \in \mathrm{Alt}(\mathbb{Z}_n)$ such that $\phi(P_1) = P_2$.*

Proof. Let $P_1 = \{\Delta_1, \ldots, \Delta_{n/p}\}$ and $P_2 = \{\Omega_1, \ldots, \Omega_{n/p}\}$. As $\mathrm{Alt}(n)$ is $(n-2)$-transitive, there exists $\phi \in \mathrm{Alt}(n)$ such that $\phi(\Delta_i) = \Omega_i$, for $i \in \{1, \ldots, n/p - 1\}$. As both P_1 and P_2 are partitions, we see that $\phi(\Delta_{n/p}) = \Omega_{n/p}$ as well. □

Lemma 2.7. *Let p be a prime, let $G = \mathbb{Z}_2^3 \times \mathbb{Z}_p$ and let $\delta \in \mathrm{Sym}(G)$. Suppose that $\langle G_L, \delta^{-1}G_L\delta\rangle$ admits a complete block system $\mathcal{C}$ with p blocks of size 8 such that $\mathrm{Alt}(C) \leq \mathrm{Stab}_{\langle G_L,\delta^{-1}G_L\delta\rangle}(C)|_C$, where $C \in \mathcal{C}$. Then there exists $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ admits a complete block system $\mathcal{B}$ with $4p$ blocks of size 2.*

Proof. Write $H = \langle G_L, \delta^{-1}G_L\delta\rangle$, $N = \mathrm{fix}_H(\mathcal{C})$ and $M = [N, N]$. Clearly both G_L and $\delta^{-1}G_L\delta$ are regular, and so both $\mathrm{fix}_{G_L}(\mathcal{C})$ and $\mathrm{fix}_{\delta^{-1}G_L\delta}(\mathcal{C})$ are semiregular of order 8. Moreover, as $\mathrm{fix}_{G_L}(\mathcal{C})|_C$ and $\mathrm{fix}_{\delta^{-1}G_L\delta}(\mathcal{C})|_C$ have exponent 2, we see that they are both consist of even permutations and hence they are contained in $\mathrm{Alt}(C)$, for each $C \in \mathcal{C}$.

From the previous paragraph, as $\mathrm{Alt}(8)$ is simple and $1 \neq N|_C \triangleleft \mathrm{Stab}_{\langle G,\delta^{-1}G\delta\rangle}(\mathcal{C})|_C$, we have $\mathrm{Alt}(C) = M|_C$, for every $C \in \mathcal{C}$. In particular, M is isomorphic to a subgroup of $\mathrm{Alt}(8)^p$.

Denote by $M_{(C)}$ the pointwise stabilizer of $C \in \mathcal{C}$. Define an equivalence relation $\equiv$ on $\mathcal{C}$ by $C \equiv C'$ if and only if $M_{(C)} = M_{(C')}$. Clearly, $\equiv$ is an H-invariant equivalence relation because $M \triangleleft H$. As $|\mathcal{C}| = p$, we see that $\equiv$ is either the identity or the universal relation. From this, we infer that either $M \cong \mathrm{Alt}(8)$ (when $\equiv$ is the identity relation) or $M \cong \mathrm{Alt}(8)^p$ (when $\equiv$ is the universal relation). Observe further that, when $M \cong \mathrm{Alt}(8)$, since $\mathrm{Alt}(8)$ has only one permutation representation of degree 8 [3, Theorem 5.3], the group M induces equivalent actions on C and on C', for every C and C' in $\mathcal{C}$. In particular, in both cases, given a subgroup I of G_L and J of $\delta^{-1}G_L\delta$ both of order 2, there exists $\gamma \in M$ with $I = \gamma^{-1}J\gamma$.

Write $K = \langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$. Clearly, I is centralized by G_L and by $\gamma^{-1}\delta^{-1}G_L\delta\gamma$ because $I \leq G_L$ and $I \leq \gamma^{-1}\delta^{-1}G_L\delta\gamma$. So I is centralized by K. As $I \lhd K$, the orbits of I form a complete block system for K with $4p$ blocks of size 2. □

The proof of the following result is similar to the proof of [6, Lemma 6.1], and generalizes it.

Lemma 2.8. *Let H be an abelian group of order ℓp, where $\ell < p$ and p is prime. Let $\phi \in \mathrm{Sym}(H)$. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system with blocks of size p.*

Proof. Let $\rho \in H$ be of order p. Then H_L admits a complete block system $\mathcal{B}$ of ℓ blocks of size p formed by the orbits of $\langle\rho\rangle$. Note that as $\ell < p$, a Sylow p-subgroup of $\mathrm{Sym}(H)$ has order p^ℓ. In particular, $\langle \rho|_B : B \in \mathcal{B}\rangle$ is a Sylow p-subgroup of $\mathrm{Sym}(H)$ isomorphic to $\mathbb{Z}_p^\ell$, an elementary abelian p-group of order p^ℓ. Let P and P_1 be Sylow p-subgroups of $\langle H_L, \phi^{-1}H_L\phi\rangle$ containing ρ and $\phi^{-1}\rho\phi$, respectively. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\delta^{-1}P_1\delta = P$. Now, every element of H_L normalizes $\langle\rho\rangle$, and so normalizes $\langle \rho|_B : B \in \mathcal{B}\rangle$. This then implies that H_L normalizes P because $P = \langle \rho|_B : B \in \mathcal{B}\rangle \cap \langle H_L, \phi^{-1}H_L\phi\rangle$.

Let $\mathcal{B}'$ be the complete block system of $\delta^{-1}\phi^{-1}H_L\phi\delta$ formed by the orbits of the cyclic group $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta$. Arguing as above, we see that $\delta^{-1}\phi^{-1}H_L\phi\delta$ normalizes $M = \langle(\delta^{-1}\phi^{-1}\rho\phi\delta)|_{B'} : B' \in \mathcal{B}'\rangle \cap \langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$. However, M is the Sylow p-subgroup of $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ containing $\delta^{-1}\phi^{-1}\langle\rho\rangle\delta\phi$, which is P. Thus we have $P \lhd \langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$, and the orbits of P form the required complete block system. □

Lemma 2.9. *Let $p \geq 5$, $H = \mathbb{Z}_2^3 \times \mathbb{Z}_p$, and $\phi \in \mathrm{Sym}(H)$. Then either there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system with blocks of size p or $\langle H_L, \phi^{-1}H_L\phi\rangle$ admits a complete block system $\mathcal{B}$ with blocks of size 8 and $\mathrm{fix}_{\langle H_L,\phi^{-1}H_L\phi\rangle}(\mathcal{B})|_B$ is isomorphic to a primitive subgroup of $\mathrm{AGL}(3,2)$, for $B \in \mathcal{B}$.*

Proof. Set $K = \langle H_L, \phi^{-1}H_L\phi\rangle$. As H has a cyclic Sylow p-subgroup, we have by [4, Theorem 3.5A] that K is doubly-transitive or imprimitive. If K is doubly-transitive, then by [12, Theorem 1.1] we have $\mathrm{Alt}(H) \leq K$. Now Lemma 2.6 reduces this case to the imprimitive case. Thus we may assume that K is imprimitive with a complete block system $\mathcal{C}$.

Suppose that the blocks of $\mathcal{C}$ have size ℓp, where $\ell = 2$ or 4. Notice that $p > \ell$. As H is abelian, $\mathrm{fix}_{H_L}(\mathcal{C})$ is a semiregular group of order ℓp and $\mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})$ is also a semiregular group of order ℓp. Then, for $C \in \mathcal{C}$, both $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ and $\mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})|_C$ are regular groups of order ℓp. Let $C \in \mathcal{C}$. By Lemma 2.8, there exists $\delta \in \langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})\rangle$ such that $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle|_C$ admits a complete block system $\mathcal{B}_C$ consisting of blocks of size p. Let $C' \in \mathcal{C}$ with $C' \neq C$. Arguing as above, there exists $\delta' \in \langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle$ such that $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta'^{-1}\delta^{-1}\phi^{-1}H_L\phi\delta\delta'}(\mathcal{C})\rangle|_{C'}$ admits a complete block system $\mathcal{B}_{C'}$ consisting of blocks of size p. Note that the restriction $\delta'|_C$ is in $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle|_C$ and so $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta'^{-1}\delta^{-1}\phi^{-1}H_L\phi\delta\delta'}(\mathcal{C})\rangle|_C$ admits $\mathcal{B}_C$ as a complete block system. Repeating this argument for every block in $\mathcal{C}$, we find that there exists $\delta \in \langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})\rangle$ such that $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle|_C$ admits a complete block system $\mathcal{B}_C$ consisting of blocks of size p. Let $\mathcal{B} = \cup_C \mathcal{B}_C$. We claim that $\mathcal{B}$ is a complete block system for $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$, which will complete the argument in this case.

Let $\rho \in H_L$ be of order p. By construction, $\rho \in \mathrm{fix}_{H_L}(\mathcal{B})$. As H is abelian, $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ is abelian, for every $C \in \mathcal{C}$. Then $\mathcal{B}_C$ is formed by the orbits of some subgroup of $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ of order p, and as $\langle\rho\rangle|_C$ is the unique subgroup of $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ of order p, we obtain that $\mathcal{B}_C$ is formed by the orbits of $\langle\rho\rangle|_C$. Then $\mathcal{B}$ is formed by the orbits of $\langle\rho\rangle \lhd H_L$ and $\mathcal{B}$ is a complete block system for H_L. An analogous argument for $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta$ gives that $\mathcal{B}$ is a complete block system for $\delta^{-1}\phi^{-1}H_L\phi\delta$. Then $\mathcal{B}$ is a complete block system for $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ with blocks of size p, as required.

Suppose that the blocks of $\mathcal{C}$ have size 8. Now $H_L/\mathcal{C}$ and $\phi^{-1}H_L\phi/\mathcal{C}$ are cyclic of order p, and as $\mathbb{Z}_p$ is a CI-group [1, Theorem 2.3], replacing $\phi^{-1}H_L\phi$ by a suitable conjugate, we may assume that $\langle H_L, \phi^{-1}H_L\phi\rangle/\mathcal{C} = H_L/\mathcal{C}$. Then $K/\mathcal{C}$ is regular and $\mathrm{Stab}_K(C) = \mathrm{fix}_K(\mathcal{C})$, for every $C \in \mathcal{C}$.

Suppose that $\mathrm{Stab}_K(\mathcal{C})|_C$ is imprimitive, for $C \in \mathcal{C}$. By [4, Exercise 1.5.10], the group K admits a complete block system $\mathcal{D}$ with blocks of size 2 or 4. Then $K/\mathcal{D}$ has degree $2p$ or $4p$ and, by Lemma 2.8, there exists $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle/\mathcal{D}$ admits a complete block system $\mathcal{B}'$ with blocks of size p. In particular, $\mathcal{B}'$ induces a complete block system $\mathcal{B}''$ for $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ with blocks of size $2p$ or $4p$, and we conclude by the case previously considered applied with $\mathcal{C} = \mathcal{B}''$. Suppose that $\mathrm{Stab}_K(\mathcal{C})|_C$ is primitive, for $C \in \mathcal{C}$. If $\mathrm{Stab}_K(\mathcal{C})|_C \geq \mathrm{Alt}(C)$, then the result follows by Lemma 2.7, and so we may assume this is not the case. By [12, Theorem 1.1], we see that $\mathrm{Stab}_K(\mathcal{C})|_C \leq \mathrm{AGL}(3,2)$. The result now follows with $\mathcal{B} = \mathcal{C}$. □

Corollary 2.10. *Let $H = \mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\phi \in \mathrm{Sym}(H)$. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system with blocks of size 5.*

Proof. Set $K = \langle H_L, \phi^{-1}H_L\phi\rangle$. By Lemma 2.9, we may assume that K admits a complete block system $\mathcal{B}$ with blocks of size 8 and with $\mathrm{Stab}_K(\mathcal{B})|_B \leq \mathrm{AGL}(3,2)$, for $B \in \mathcal{B}$. As $|\mathrm{AGL}(3,2)| = 8 \cdot 7 \cdot 6 \cdot 4$, we see that a Sylow 5-subgroup of K has order 5. Let $\langle\rho\rangle$ be the subgroup of H_L of order 5. So $\langle\rho\rangle$ is a Sylow 5-subgroup of K. Then $\phi^{-1}\langle\rho\rangle\phi$ is also a Sylow 5-subgroup of K, and by a Sylow theorem there exists $\delta \in K$ such that $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta = \langle\rho\rangle$. We then see that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ has a unique Sylow 5-subgroup, whose orbits form the required complete block system $\mathcal{B}$. □

We are finally ready to prove Theorem A.

Proof of Theorem A. If p is odd, then the paragraph following the proof of Corollary 2.4 shows that it suffices to prove that Lemma 2.5 holds for the prime $p = 5$. This is done in Corollary 2.10. If $p = 2$, then the result can be verified using GAP or Magma. □

3 Proof of Corollaries A and B

Before proceeding to our next result we will need the following definitions.

Definition 3.1. Let G be a permutation group on Ω and $k \geq 1$. A permutation $\sigma \in \mathrm{Sym}(\Omega)$ lies in the k-closure $G^{(k)}$ of G if for every k-tuple $t \in \Omega^k$ there exists $g_t \in G$ (depending on t) such that $\sigma(t) = g_t(t)$. We say that G is k-closed if the permutations lying in the

k-closure of G are the elements of G, that is, $G^{(k)} = G$. The group G is k-closed if and only if there exists a color k-ary relational structure X on Ω with $G = \mathrm{Aut}(X)$, see [18].

Definition 3.2. For color digraphs Γ_1 and Γ_2, we define the *wreath product of* Γ_1 *and* Γ_2, denoted $\Gamma_1 \wr \Gamma_2$, to be the color digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set $E_1 \cup E_2$, where $E_1 = \{((x_1, y_1), (x_1, y_2)) : x_1 \in V(\Gamma_1), (y_1, y_2) \in E(\Gamma_2)\}$ and $E_2 = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(\Gamma_1), y_1, y_2 \in V(\Gamma_2)\}$.

The edge $((x_1, y_1), (x_1, y_2)) \in E_1$ is colored with the same color as (y_1, y_2) in Γ_2 and the edge $((x_1, y_1), (x_2, y_2)) \in E_2$ is colored with the same color as (x_1, x_2) in Γ_1.

Definition 3.3. Let $G \leq \mathrm{Sym}(X)$ and let $H \leq \mathrm{Sym}(Y)$. We define the *wreath product of* G *and* H, denoted by $G \wr H$, to be the semidirect product $G \ltimes H^X$, where H^X is the direct product of $|X|$ copies of H (labeled by the elements of X) and where G acts on H^X as a group of automorphisms by permuting the coordinates according to its action on X. The group $G \wr H$ has a natural faithful action on $X \times Y$, where for $(x, y) \in X \times Y$ the element $g \in G$ acts via $(x, y) \mapsto (g(x), y)$ and the element $(h_z)_{z \in X} \in H^X$ acts via $(x, y) \mapsto (x, h_x(y))$. We refer the reader to [4, page 46] for more details on this construction.

The following very useful result (see [1, Lemma 3.1]) characterizes CI-groups with respect to a class of combinatorial objects.

Lemma 3.4. *Let* H *be a group and let* $\mathcal{K}$ *be a class of combinatorial objects. The following are equivalent.*

1. H *is a CI-group with respect to* $\mathcal{K}$,
2. *whenever* X *is a Cayley object of* H *in* $\mathcal{K}$ *and* $\phi \in \mathrm{Sym}(H)$ *such that* $\phi^{-1} H_L \phi \leq \mathrm{Aut}(X)$, *then* H_L *and* $\phi^{-1} H_L \phi$ *are conjugate in* $\mathrm{Aut}(X)$.

Proof of Corollary A. From Theorem A, it suffices to show that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups with respect to color binary relational structures. As the transitive permutation groups of degree 24 are readily available in GAP or Magma, it can be shown using a computer that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ is a CI-group with respect to color binary relational structures. It remains to consider $H = \mathbb{Z}_2^3 \times \mathbb{Z}_7$.

Fix $\phi \in \mathrm{Sym}(H)$ and set $K = \langle H_L, \phi^{-1} H_L \phi \rangle$. Assume that there exists $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle$ admits a complete block system with blocks of size 7. Now, it follows by [6] (see the two paragraphs following the proof of Corollary 2.4) that H_L and $\delta^{-1}\phi^{-1} H_L \phi\delta$ are conjugate in $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle^{(3)}$. Since $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle^{(3)} \leq \langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle^{(2)}$, the corollary follows from Lemma 3.4 (and from Definition 3.1).

Assume that there exists no $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle$ admits a complete block system with blocks of size 7. By Lemma 2.9, the group K admits a complete block system $\mathcal{C}$ with blocks of size 8 and $\mathrm{fix}_K(\mathcal{C})|_C$ is isomorphic to a primitive subgroup of $\mathrm{AGL}(3, 2)$, for $C \in \mathcal{C}$. Suppose that 7 and $|\mathrm{fix}_K(\mathcal{C})|$ are relatively prime. So, a Sylow 7-subgroup of K has order 7. We are now in the position to apply the argument in the proof of Corollary 2.10. Let $\langle \rho \rangle$ be the subgroup of H_L of order 7. Then $\phi^{-1}\langle \rho \rangle \phi$ is a Sylow 7-subgroup of K, and by a Sylow theorem there exists $\delta \in K$ such that $\delta^{-1}\phi^{-1}\langle \rho \rangle \phi\delta = \langle \rho \rangle$. We then see that $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle$ has a unique Sylow 7-subgroup, whose orbits form a complete block system with blocks of size 7, contradicting our hypothesis on K. We thus assume that 7 divides $|\mathrm{fix}_K(\mathcal{C})|$ and so $\mathrm{fix}_K(\mathcal{C})$ acts doubly-transitively on C, for $C \in \mathcal{C}$.

Fix $C \in \mathcal{C}$ and let L be the point-wise stabilizer of C in $\mathrm{fix}_K(\mathcal{C})$. Assume that $L \neq 1$. Now, we compute $K^{(2)}$ and we deduce that H_L and $\phi^{-1}H_L\phi$ are conjugate in $K^{(2)}$, from which the corollary will follow from Lemma 3.4. As $L \triangleleft \mathrm{fix}_K(\mathcal{C})$, we have $L|_{C'} \triangleleft \mathrm{fix}_K(\mathcal{C})|_{C'}$, for every $C' \in \mathcal{C}$. As a nontrivial normal subgroup of a primitive group is transitive [19, Theorem 8.8], either $L|_{C'}$ is transitive or $L|_{C'} = 1$. Let Γ be a Cayley color digraph on H with $K^{(2)} = \mathrm{Aut}(\Gamma)$. Let $\mathcal{C} = \{C_i : i \in \mathbb{Z}_7\}$ where $C_i = \{(x_1, x_2, x_3, i) : x_1, x_2, x_3 \in \mathbb{Z}_2\}$, and assume without loss of generality that $C = C_0$. Suppose that there is an edge of color κ from some vertex of C_i to some vertex of C_j, where $i \neq j$. Then there is an edge of color κ from some vertex of C_0 to some vertex of C_{j-i}. Additionally, $j - i$ generates $\mathbb{Z}_7$, so there is a smallest integer s such that $L|_{C_{s(j-i)}} = 1$ while $L|_{C_{(s+1)(j-i)}}$ is transitive. As there is an edge of color κ from some vertex of $C_{s(j-i)}$ to some vertex of $C_{(s+1)(j-i)}$, we conclude that there is an edge of color κ from every vertex of $C_{s(j-i)}$ to every vertex of $C_{(s+1)(j-i)}$. This implies that there is an edge of color κ from every vertex of C_i to every vertex of C_j, and then Γ is the wreath product of a Cayley color digraph Γ_1 on $\mathbb{Z}_7$ and a Cayley color digraph Γ_2 on $\mathbb{Z}_2^3$. Since $\mathrm{fix}_K(\mathcal{C})$ is doubly-transitive on C, we have $\mathrm{Aut}(\Gamma_2) \cong \mathrm{Sym}(8)$. Therefore $K^{(2)} = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \cong \mathrm{Aut}(\Gamma_1) \wr \mathrm{Sym}(8)$. By [7, Corollary 6.8] and Lemma 3.4 H_L and $\phi^{-1}H_L\phi$ are conjugate in $K^{(2)}$. We henceforth assume that $L = 1$, that is, $\mathrm{fix}_K(\mathcal{C})$ acts faithfully on C, for each $C \in \mathcal{C}$.

Define an equivalence relation on H by $h \equiv k$ if and only if it holds $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h) = \mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(k)$. The equivalence classes of $\equiv$ form a complete block system $\mathcal{D}$ for K. As $\mathrm{fix}_K(\mathcal{C})|_C$ is primitive and not regular, each equivalence class of $\equiv$ contains at most one element from each block of $\mathcal{C}$. We conclude that $\mathcal{D}$ either consists of 8 blocks of size 7 or each block is a singleton. Since we are assuming that K has no block system with blocks of size 7, we see that each block of $\mathcal{D}$ is a singleton.

Fix C and D in $\mathcal{C}$ with $C \neq D$ and $h \in C$. Now, $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h)$ is isomorphic to a subgroup of $\mathrm{GL}(3,2)$ and acts with no fixed points on D. From [4, Appendix B]), we see that $\mathrm{AGL}(3,2)$ is the only doubly-transitive permutation group of degree 8 whose point stabilizer admits a fixed-point-free action of degree 8. Therefore $\mathrm{fix}_K(\mathcal{C}) \cong \mathrm{AGL}(3,2)$. Additionally, $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h)|_D$ is transitive on D.

Suppose that Γ is a color digraph with $K^{(2)} = \mathrm{Aut}(\Gamma)$ and suppose that there is an edge of color κ from h to $\ell \in E$, with $E \in \mathcal{C}$ and $E \neq D$. Then $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h)|_E$ is transitive, and so there is an edge of color κ from h to every vertex of E. As $\mathrm{fix}_K(\mathcal{C})$ is transitive on both C and E, we see that there is an edge of color κ from every vertex of C to every vertex of D. We conclude that Γ is a wreath product of two color digraphs Γ_1 and Γ_2, where Γ_1 is a Cayley color digraph on $\mathbb{Z}_7$ and Γ_2 is either complete or the complement of a complete graph, and $K^{(2)} = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Sym}(8)$. The result then follows by the same arguments as above. □

Proof of Corollary B. From Corollary 2.4 and Theorem A, it suffices to show that $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color ternary relational structures. As the transitive permutation groups of degree 28 are readily available in GAP or Magma, it can be shown using a computer that $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color ternary relational structures. (We note that a detailed analysis similar to the proof of Corollary A for the group $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ also gives a proof of this theorem.) □

4 Concluding remarks

In the rest of this paper, we discuss the relevance of Theorem A to the study of CI-groups with respect to ternary relational structures. Using the software packages [2] and [8], we have determined that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures. Here we report an example witnessing this fact: the group G has order 2048, V and W are two *nonconjugate* elementary abelian regular subgroups of G, and $X = (\{1, \dots, 32\}, E)$ is a ternary relational structure with $G = \mathrm{Aut}(X)$. The group V is generated by

(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32),

(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13, 15)(14,16)(17,19)(18,20)(21,23)(22,24)(25,27)(26,28)(29,31)(30,32),

(1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16)(17,21)(18,22)(19,23)(20,24)(25,29)(26,30)(27,31)(28,32),

(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)(17,25)(18,26)(19,27)(20,28)(21,29)(22,30)(23,31)(24,32),

(1,17)(2,18)(3,19)(4,20)(5,21)(6,22)(7,23)(8,24)(9,25)(10,26)(11,27)(12,28)(13,29)(14,30)(15,31)(16,32),

the group W is generated by

(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32),

(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13, 15)(14,16)(17,20)(18,19)(21,24)(22,23)(25,28)(26,27)(29,32)(30,31),

(1,5)(2,6)(3,7)(4,8)(9,14)(10,13)(11,16)(12,15)(17,22)(18,21)(19,24)(20, 23)(25,29)(26,30)(27,31)(28,32),

(1,9)(2,10)(3,11)(4,12)(5,14)(6,13)(7,16)(8,15)(17,27)(18,28)(19,25)(20,26)(21,32)(22,31)(23,30)(24,29),

(1,17)(2,18)(3,20)(4,19)(5,22)(6,21)(7,23)(8,24)(9,27)(10,28)(11,26)(12,25)(13,32)(14,31)(15,29)(16,30),

the group G is generated by

$V, W,$ (25,26)(27,28)(29,30)(31,32),(1,11)(2,12)(3,9)(4,10)(5,13)(6, 14)(7,15)(8,16)(17,19)(18,20)(25,27)(26,28),

the set E is defined by

$$\{g((1,3,9)), g((1,5,25)) : g \in G\}.$$

Definition 4.1. For a cyclic group $M = \langle g \rangle$ of order m and a cyclic group $\langle z \rangle$ of order 2^d, $d \geq 1$, we denote by $D(m, 2^d)$ the group $\langle z \rangle \ltimes M$ with $g^z = g^{-1}$.

Combining Theorem A with [5, Theorem 9], [5, Lemma 6], the construction given in [17] and the previous paragraph, we have the following result which lists every group that can be a CI-group with respect to ternary relational structures (although not every group on the list needs to be a CI-group with respect to ternary relational structures).

Theorem 4.2. *If G is a CI-group with respect to ternary relational structures, then all Sylow subgroups of G are of prime order or isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2^d$, $1 \leq d \leq 4$, or Q_8. Moreover, $G = U \times V$, where $\gcd(|U|, |V|) = 1$, U is cyclic of order n, with $\gcd(n, \varphi(n)) = 1$, and V is one of the following:*

1. *$\mathbb{Z}_2^d$, $1 \leq d \leq 4$, $D(m, 2)$, or $D(m, 4)$, where m is odd and $\gcd(nm, \varphi(nm)) = 1$,*
2. *$\mathbb{Z}_4$, Q_8.*

Furthermore,

(a) if $V = \mathbb{Z}_4$*,* Q_8*, or* $D(m, 4)$ *and* $p \mid n$ *is prime, then* $4 \not| (p-1)$*,*

(b) if $V = \mathbb{Z}_2^d$*,* $d \geq 2$*, or* Q_8*, then* $3 \not| n$*,*

(c) if $V = \mathbb{Z}_2^d$*,* $d \geq 3$*, then* $7 \not| n$*,*

(d) if $V = \mathbb{Z}_2^4$*, then* $5 \not| n$*.*

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On the rank two geometries of the groups $\mathrm{PSL}(2, q)$: part II*

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Abstract

We determine all firm and residually connected rank 2 geometries on which $\mathrm{PSL}(2, q)$ acts flag-transitively, residually weakly primitively and locally two-transitively, in which one of the maximal parabolic subgroups is isomorphic to A_4, S_4, A_5, $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$, where q' divides q, for some prime-power q.

Keywords: Projective special linear groups, coset geometries, locally s-arc-transitive graphs.

Math. Subj. Class.: 51E24, 05C25

1 Introduction

In [5], we started the classification of the residually weakly primitive and locally two-transitive coset geometries of rank two for the groups $\mathrm{PSL}(2, q)$. The aim of this paper is to finish this classification. It remains to focus on the cases in which one of the maximal parabolic subgroups is isomorphic to A_4, S_4, A_5, $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$ where q' divides q. For motivation, basic definitions, notations and context of the work we refer to [5].

In Section 3, we sketch the proof of our main result:

Theorem 1.1. *Let $G \cong \mathrm{PSL}(2, q)$ and $\Gamma(G; \{G_0, G_1, G_0 \cap G_1\})$ be a locally two-transitive RWPRI coset geometry of rank two. If G_0 is isomorphic to one of A_4, S_4, A_5, $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$, where q' divides q, then Γ is isomorphic to one of the geometries appearing in Table 1, Table 2, Table 3, Table 4, Table 5, and Table 6.*

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		$G_0 \cong A_5$			$q = 4^r$ with r prime	
	$G_0 \cap G_1$	G_1	♯ Geom. up to conj.	♯ Geom. up to isom.	Extra conditions on q	loc. (G,s)-arc-trans. g.
Γ_1	D_{10}	D_{30}	1	1	$\frac{q\pm1}{15}$ odd	$s=3$
Γ_2	A_4	$E_{16}:3$	1	1	$q=16$	$s=3$
Γ_3	A_4	$E_{16}:3$	5	2	$q=64$	$s=3$
Γ_4	A_4	$E_{16}:3$	$\frac{4^{r-1}-1}{3}$	$\frac{2(4^{r-2}-1)+3.2^{r-2}}{3r}$	$r>3$, r odd prime	$s=3$
		$G_0 \cong A_5$			$q=p=\pm1(5)$ with p odd prime	
	$G_0 \cap G_1$	G_1	♯ Geom. up to conj.	♯ Geom. up to isom.	Extra conditions on q	loc. (G,s)-arc-trans. g.
Γ_5	D_{10}	D_{20}	2	1	$q=\pm1(20)$	$s=3$
Γ_6	D_{10}	D_{30}	2	1	$q=\pm1(30)$	$s=3$
Γ_7	D_{10}	A_5	2	1	$\frac{q\pm1}{10}$ even	$s=2$
Γ_8	D_{10}	A_5	1	1	$\frac{q\pm1}{10}$ odd	$s=2$
Γ_9	A_4	S_4	2	1	$q=\pm1(40)$ or $q=\pm9(40)$	$s=3$
Γ_{10}	A_4	A_5	2	1	$q=\pm1(40)$ or $q=\pm9(40)$	$s=2$
Γ_{11}	A_4	A_5	1	1	$q=\pm11(40)$ or $q=\pm19(40)$	$s=2$
		$G_0 \cong A_5$			$q=p^2=-1(5)$ with p odd prime	
	$G_0 \cap G_1$	G_1	♯ Geom. up to conj.	♯ Geom. up to isom.	Extra conditions on q	loc. (G,s)-arc-trans. g.
Γ_{12}	D_{10}	D_{20}	2	1	$q=-1(20)$	$s=3$
Γ_{13}	D_{10}	D_{30}	2	1	$q=-1(30)$	$s=3$
Γ_{14}	D_{10}	A_5	2	1	$\frac{q+1}{10}$ even	$s=2$
Γ_{15}	D_{10}	A_5	1	1	$\frac{q+1}{10}$ odd	$s=2$
Γ_{16}	A_4	S_4	2	1	$q=-1(40)$ or $q=9(40)$	$s=3$
Γ_{17}	A_4	A_5	2	1	$q=-1(40)$ or $q=9(40)$	$s=2$
Γ_{18}	A_4	A_5	1	1	$q=-11(40)$ or $q=19(40)$	$s=2$

Table 1: The RWPRI and $(2T)_1$ geometries with $G_0 \cong A_5$.

		$G_0 \cong A_4$			$q=p>3$ and $q=3,13,27,37(40)$ or $q=5$	
	$G_0 \cap G_1$	G_1	♯ Geom. up to conj.	♯ Geom. up to isom.	Extra conditions on q	locally(G,s)-arc-transitive graphs
Γ_1	3	Z_6	1	1	$q=13,37,83,107(120)$	$s=3$
Γ_2	3	D_6	$\frac{q+1}{6}$	$\frac{\frac{q+1}{6}+1}{2}$	$\frac{q+1}{6}$ odd	$s=3$
Γ_3	3	D_6	$\frac{q-1}{6}$	$\frac{\frac{q-1}{6}+1}{2}$	$\frac{q-1}{6}$ odd	$s=3$
Γ_4	3	D_6	$\frac{q+1}{6}$	$\frac{q+1}{12}$	$\frac{q+1}{6}$ even	$s=3$
Γ_5	3	D_6	$\frac{q-1}{6}$	$\frac{q-1}{12}$	$\frac{q-1}{6}$ even	$s=3$
Γ_6	3	A_4	$\frac{q+1}{3}-1$	$\frac{q+1}{6}$	$3 \mid q+1$	$s=2$
Γ_7	3	A_4	$\frac{q-1}{3}-1$	$\frac{q-1}{6}$	$3 \mid q-1$	$s=2$

Table 2: The RWPRI and $(2T)_1$ geometries with $G_0 \cong A_4$

		$G_0 \cong S_4$			$q = p > 2$ and $q = \pm 1(8)$	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	locally(G,s)-arc-transitive graphs
Γ_1	D_6	D_{12}	2	1	$q = \pm 1(24)$	$s = 3$
Γ_2	D_6	D_{18}	2	1	$q = \pm 1(72)$ or $q = \pm 17(72)$	$s = 3$
Γ_3	D_6	S_4	2	1	$\frac{q\pm 1}{6}$ even	$s = 2$
Γ_4	D_6	S_4	1	1	$\frac{q\pm 1}{6}$ odd	$s = 2$
Γ_5	D_8	D_{16}	2	1	$q = \pm 1(16)$	$s = 7$
Γ_6	D_8	D_{24}	2	1	$q = \pm 1(24)$	$s = 3$
Γ_7	D_8	S_4	2	1	$\frac{q\pm 1}{8}$ even	$s = 4$
Γ_8	D_8	S_4	1	1	$\frac{q\pm 1}{8}$ odd	$s = 4$
Γ_9	A_4	A_5	2	1	$q = \pm 1(40)$ or $q = \pm 9(40)$	$s = 3$

Table 3: The RWPRI and $(2T)_1$ geometries with $G_0 \cong S_4$.

		$G_0 \cong \mathrm{PSL}(2,2^n)$			$q = 2^{nm}$, with m prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc.(G,s)-arc-trans. graphs
Γ_1	$E_{2^n} : (2^n - 1)$	$E_{2^{mn}} : (2^n - 1)$	1	1	$m = 2, n \neq 1$	$s = 3$
Γ_2	2	D_6	1	1	$q = 4; n = 1, m = 2$	$s = 2$
Γ_3	2	2^2	1	1	$q = 4; n = 1, m = 2$	$s = 3$
Γ_4	3	A_4	1	1	$q = 4; n = 1, m = 2$	$s = 3$
Γ_5	D_{10}	D_{30}	1	1	$q = 4^m; n = 2; \frac{q\pm 1}{15} odd$	$s = 3$

Table 4: The RWPRI and $(2T)_1$ geometries with $G_0 \cong \mathrm{PSL}(2,2^n)$.

		$G_0 \cong \mathrm{PSL}(2,p^n)$			$q = p^{nm}$, p and m odd primes	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	locally(G,s)-arc-transitive graphs
Γ_1	3	A_4	$3^{m-1} - 1$	$\frac{3^{m-1}-1}{2m}$	$q = 3^m; n = 1, m \neq 3$	$s = 2$
Γ_2	3	A_4	8	2	$q = 27; n = 1, m = 3$	$s = 2$

Table 5: The RWPRI and $(2T)_1$ geometries with $G_0 \cong \mathrm{PSL}(2,q')$, q' odd.

		$G_0 \cong \mathrm{PGL}(2,p^n)$			$q = p^{2n}$, with p odd prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc. (G,s)-arc-transitive graphs
Γ_1	$E_{p^n} : (p^n - 1)$	$E_{p^{2n}} : (p^n - 1)$	2	1	none	$s = 3$
Γ_2	$\mathrm{PSL}(2,p^n)$	A_5	2	1	$q = 9$	$s = 3$
Γ_3	D_8	$\mathrm{PGL}(2,3)$	1	1	$q = 9$	$s = 4$

Table 6: The RWPRI and $(2T)_1$ geometries with $G_0 \cong \mathrm{PGL}(2,q')$.

Observe that, geometry Γ_5 in Table 4 is exactly geometry Γ_1 in Table 1.

In Section 4, we recall the subgroup lattice of $\mathrm{PSL}(2,q)$, and we give the two-transitive representations of the maximal subgroups. In Section 5, we prove Theorem 1.1, which is based on the proof of Propositions 5.5, 5.6, 5.10, 5.12, 5.16 and 5.21. For that purpose, we determine the rank two RWPRI and $(2T)_1$ geometries of $\mathrm{PSL}(2,q)$ and their number, up to isomorphism and up to conjugacy. The existence of such geometries is equivalent to the existence of a locally 2-arc transitive bipartite graph for which the action of G is primitive on one of the bipartite halves (see [8]). Our result is also a part of the program initiated in [8].

These graphs are interesting in their own right because of the numerous connections they have with other fields of mathematics (see [8] for more details). We also refer to the classification of these graphs for almost simple groups with socle a Ree simple group $\mathrm{Ree}(q)$ (see [7]). In terms of locally 2-arc-transitive graphs, we obtain here the classification of these graphs with one vertex-stabilizer maximal in $\mathrm{PSL}(2,q)$ and isomorphic to A_4, S_4, A_5, $\mathrm{PSL}(2,q')$ or $\mathrm{PGL}(2,q')$. The last column of Table 1, Table 2, Table 3, Table 4, Table 5 and Table 6 gives, for each geometry Γ, the value of s such that Γ is a locally s-arc-transitive but not a locally $(s+1)$-arc-transitive graph. In section 6, we determine the exact value of s in all cases that are not current by the method of Leemans.

In Tables 1, 2, 3, 4, 5, 6 and 9 most values are $s = 2$ or $s = 3$, but there are some spectacular examples with larger values of s. Indeed we obtain a locally 4-arc transitive graph and a locally 7-arc transitive graph. As one of the referees pointed out, the $(G,2)$-arc transitive graphs with $L_2(q) \leq G \leq P\Gamma L_2(q)$ were classified by Hassani, Nochefranca and Praeger in [9]. Therefore, they already classified the geometries of Theorem 1.1 in which $G_0 \cap G_1$ is of index two in one of G_0 or G_1. Our proof of Theorem 1.1 uses a completely different approach. In cases where our work overlaps with [9], the results are the same.

Also, in Table 3, geometry Γ_5 is due to Wong in [22] and geometries Γ_7 and Γ_8 are the Biggs-Hoare graphs in [1] (see also [14], Table 1).

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2 Definitions and notation

For basic notions on coset geometries and locally s-arc-transitive graphs needed to understand this paper we shall freely use the definitions from Section 2 in [5].

Let us nevertheless recall concepts related to isomorphism. Let G be a group and $Aut(G)$ be its automorphism group. The coset geometries $\Gamma(G;\{G_0,G_1\})$ and $\Gamma(G;\{G'_0,G'_1\})$ are *conjugate* (resp. *isomorphic*) provided there exists an element $g \in G$ (resp. $g \in Aut(G)$) such that $\{G_0^g, G_1^g\} = \{G'_0, G'_1\}$ (resp. $\{g(G_0), g(G_1)\} = \{G'_0, G'_1\}$). We classify geometries up to conjugacy and up to isomorphism. That is, for each triple $\{G_0, G_1, G_0 \cap G_1\}$, we give the number of corresponding classes of geometries with respect to conjugacy and isomorphism.

3 Sketch of the proof of Theorem 1.1

Let $G \cong \mathrm{PSL}(2,q)$. Let G_0 and G_1 be subgroups of G and let $G_{01} = G_0 \cap G_1$. The RWPRI condition in rank two requires that either G_0 or G_1 is a maximal subgroup of G and that G_{01} is a maximal subgroup of G_0 and G_1. The $(2T)_1$ condition requires that both G_0 and G_1 act two-transitively on the respective cosets of G_{01}.

We break down the task by classifying those geometries with a fixed subgroup G_0. Since we may assume without loss of generality that G_0 is maximal in G, we follow Tables 7 and 8 that give all maximal subgroups of $\mathrm{PSL}(2,q)$. The number of RWPRI and $(2T)_1$ geometries of rank 2 depends on the value of $q = p^n$. More precisely, it usually depends on whether $p = 2$ or $p \neq 2$. Knowing that $q = p^n$ with p a prime, the two cases are $q = 2^n$ or q odd.

The way we work to determine the RWPRI and $(2T)_1$ geometries of rank two always follows the same path. To achieve our goal we first choose a subgroup G_0, which is a maximal subgroup of $G \cong \mathrm{PSL}(2,q)$. Then, using the results obtained in Proposition 4.6, we determine the possibilities for $G_{01} := G_0 \cap G_1$. They are the two-transitive pairs (G_0, G_{01}). At last, in Section 5 we determine the possible subgroups G_1 of $\mathrm{PSL}(2,q)$ such that (G_1, G_{01}) is a two-transitive pair. Finally, we determine for each triple (G_0, G_1, G_{01}) the number of geometries it gives rise to, up to conjugacy and up to isomorphism.

4 Structure of subgroups of $\mathrm{PSL}(2,q)$

To follow the approach described above, we first recall the list of subgroups of the projective special linear groups $\mathrm{PSL}(2,q)$. We then give the list of maximal subgroups of $\mathrm{PSL}(2,q)$. Finally we determine the two-transitive representations of the maximal subgroups of $\mathrm{PSL}(2,q)$ in order to be able to check the $(2T)_1$ property easily.

4.1 The subgroups of $\mathrm{PSL}(2,q)$

We recall the complete subgroup structure of $\mathrm{PSL}(2,q)$ for which we refer to Dickson [6], Moore [15], Huppert [10] and Suzuki [16]. In the statement of Lemma 4.1, we make use of the phrasing due to O. H. King [11].

Lemma 4.1. *[Dickson-Moore] The group* $\mathrm{PSL}(2,q)$ *of order* $\frac{q(q^2-1)}{(2,q-1)}$*, where* $q = p^n$ *(*p *prime), contains exactly the following subgroups:*

1. *The identity subgroup.*
2. *A single class of* $q+1$ *conjugate elementary abelian subgroups of order* q*, denoted by* E_q*.*
3. *A single class of* $\frac{q(q+1)}{2}$ *conjugate cyclic subgroups of order* d*, denoted by either* Z_d *or* d*; for every divisor* d *of* $q-1$ *for* q *even and* $\frac{q-1}{2}$ *for* q *odd, with* $d > 1$.
4. *A single class of* $\frac{q(q-1)}{2}$ *conjugate cyclic subgroups of order* d*,denoted by either* Z_d *or* d*; for every divisor* d *of* $q+1$ *for* q *even and* $\frac{q+1}{2}$ *for* q *odd, with* $d > 1$.
5. • *For* q *odd, a single class of* $\frac{q(q^2-1)}{4d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $\frac{q-1}{2}$ *with* $\frac{q-1}{2d}$ *odd, with* $d > 1$*;*

- *For q odd, two classes each of* $\frac{q(q^2-1)}{8d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* $d > 2$ *of* $\frac{q-1}{2}$ *with* $\frac{q-1}{2d}$ *even;*
- *For q even, a single class of* $\frac{q(q^2-1)}{2d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $q-1$*, with* $d > 1$*;*
- *For q odd, a single class of* $\frac{q(q^2-1)}{4d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $\frac{q+1}{2}$ *with* $\frac{q+1}{2d}$ *odd, with* $d > 1$*;*
- *For q odd, two classes each of* $\frac{q(q^2-1)}{8d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* $d > 2$ *of* $\frac{q+1}{2}$ *with* $\frac{q+1}{2d}$ *even;*
- *For q even, a single class of* $\frac{q(q^2-1)}{2d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $q+1$*, with* $d > 1$*.*

6. - *A single class of* $\frac{q(q^2-1)}{24}$ *conjugate dihedral groups of order* 4 *denoted by* 2^2 *when* $q = \pm 3(8)$*;*
   - *Two classes each of* $\frac{q(q^2-1)}{48}$ *conjugate dihedral groups of order* 4 *denoted by* 2^2 *when* $q = \pm 1(8)$*;*
   - *When q is even, the groups* 2^2 *are in the case 7.*

7. *A number of classes of* $\frac{q^2-1}{(2,1,1)(p^k-1)}$ *conjugate elementary abelian subgroups of order* p^m*, denoted by* E_{p^m} *for every natural number* m*, such that* $1 \leq m \leq n-1$*, where* k *is a common divisor of* n *and* m *and* $(2,1,1)$ *is equal to 2 (resp. 1, 1) if* $p > 2$ *and* $\frac{n}{k}$ *is even (resp.* $p > 2$ *and* $\frac{n}{k}$ *is odd,* $p = 2$*).*

8. *A number of classes of* $\frac{(q^2-1)p^{n-m}}{(2,1,1)(p^k-1)}$ *conjugate subgroups* $E_{p^m}:d$ *which are semidirect products of an elementary abelian group* E_{p^m} *and a cyclic group of order* d*,* $d > 1$*, for every natural number* m *such that* $1 \leq m \leq n$ *and every natural number* d *dividing* $\frac{p^k-1}{(1,2,1)}$*, where* k *is a common divisor of* n *and* m *and* $(1,2,1)$ *is one of*

   - *1 for* $p > 2$ *and* $\frac{n}{k}$ *is even*
   - *2 for* $p > 2$ *and* $\frac{n}{k}$ *is odd*
   - *1 for* $p = 2$

   These subgroups are Frobenius groups.

9. - *Two classes each of* $\frac{q(q^2-1)}{48}$ *conjugates of* A_4 *when* $q = \pm 1(8)$*;*
   - *A single class of* $\frac{q(q^2-1)}{24}$ *conjugates of* A_4 *when* $q = \pm 3(8)$*;*
   - *A single class of* $\frac{q(q^2-1)}{12}$ *conjugates of* A_4 *when q is an even power of* 2*.*

10. *Two classes each of* $\frac{q(q^2-1)}{48}$ *conjugates of* S_4 *when* $q = \pm 1(8)$*.*

11. *Two classes each of* $\frac{q(q^2-1)}{120}$ *conjugate alternating groups* A_5 *when* $q = \pm 1(10)$*.*

12. - *Two classes each of* $\frac{q(q^2-1)}{2q'(q'^2-1)}$ *groups* $PSL(2,q')$*, where q is an even power of* q'*, for q odd;*
    - *A single class of* $\frac{q(q^2-1)}{q'(q'^2-1)}$ *groups* $PSL(2,q')$*, where q is an odd power of* q'*, for q odd;*

- *A single class of* $\frac{q(q^2-1)}{q'(q'^2-1)}$ *groups* $PSL(2,q')$*, where* q *is a power of* q'*, for* q *even.*

13. *Two classes each of* $\frac{q(q^2-1)}{2q'(q'^2-1)}$ *groups* $PGL(2,q')$*, where* q *is an even power of* q'*, for* q *odd.*
14. $\mathrm{PSL}(2,q)$ *itself.*

Remark 4.2. Subgroups A_5 are given either by case 11 (when $q = \pm 1(5)$) or by case 12 (when $q = 0(5)$ and $q = 4^m$) of Lemma 4.1. Also, if q is even, the $PGL(2,q')$ are given by case 12, since $PGL(2,q') \cong PSL(2,q')$ provided q is even.

Remark 4.3. Let us mention that in the cases 7 and 8 of Lemma 4.1, the number of conjugacy classes is not given. The number of conjugacy classes of the elementary abelian subgroups E_{p^m} given by Dickson (see [6], §260) is incorrect. For an example we refer to [5] Remark 7.

Notice that Dickson does not give the number of conjugacy classes of the subgroups $E_{p^m} : d$, except in the particular case where $m = n$ and $d = \frac{p^n-1}{(2,q-1)}$. There are $q+1$ subgroups $E_q : \frac{q-1}{(2,q-1)}$, all conjugate.

4.2 Maximal subgroups of $\mathbf{PSL(2,q)}$

In this section, we list the maximal subgroups of $\mathrm{PSL}(2,q)$. As the classification of geometries usually depends on whether q is even or odd, we give in Table 7 and Table 8 the maximal subgroups of $\mathrm{PSL}(2,q)$ in these two cases. We borrowed this result from Suzuki [16], page 417. Notice that the subgroups A_5 appear both as A_5 and $\mathrm{PSL}(2,q')$ for $q' = 5$.

Let us mention that a little error in Suzuki [16] was detected and corrected by Patricia Vanden Cruyce [19] in her thesis: *Indeed the subgroup* A_5 *is maximal if* r *is an odd prime. Because if* $r = 2$ *we have that* $A_5 < \mathrm{PGL}(2,5) < \mathrm{PSL}(2,25)$. However there remains a missing case in Suzuki [16] because, A_4 is maximal if $q = 5$. We include it in Table 8.

4.3 Two-transitive representations of the maximal subgroups of $\mathbf{PSL(2,q)}$

The first lemma is obvious but used often in the next section as a necessary condition to have a two-transitive action.

Lemma 4.4. *Let* G *be a group and let* H *be a subgroup of* G*. If* G *acts two-transitively on the cosets of* H *in* G*, then* $|G|$ *must be divisible by* $[G:H]([G:H]-1)$*.*

A group G is said to act *regularly* on a set Ω if G is transitive on Ω and the stabilizer in G of a point $x \in \Omega$ is the identity.

Lemma 4.5. *[21] Let* (G,Ω) *be a permutation group which is transitive over* Ω *and let* G *be abelian. Then* G *is regular. Moreover, if* G *is two-transitive then* $|\Omega| = 2$*.*

In order to simplify notation used throughout this section and the following one, we need another basic definition (borrowed from [2]). In a group G, an ordered pair of subgroups (A,B) is called *two-transitive* provided that B is a maximal subgroup of A and that the action of A on the left cosets of B is two-transitive.

Structure	Order	Index
$E_q : (q-1)$	$q(q-1)$	$q+1$
$D_{2(q+1)}$ $q \neq 2$	$2(q+1)$	$\frac{q(q-1)}{2}$
$D_{2(q-1)}$	$2(q-1)$	$\frac{q(q+1)}{2}$
A_5 $q = 4^r$ r *is prime*	60	$\frac{q(q^2-1)}{60}$
$\mathrm{PSL}(2, q') \cong \mathrm{PGL}(2, q')$ $q' > 4$, $q = q'^m$, m is *prime* or $q' = 2$, $q = q'^2$	$q'(q'^2-1)$	$\frac{q(q^2-1)}{q'(q'^2-1)}$

Table 7: The maximal subgroups of $\mathrm{PSL}(2, q)$, for q even

Structure	Order	Index
$E_q : \frac{q-1}{2}$	$\frac{q(q-1)}{2}$	$q+1$
$D_{(q+1)}$ $q \neq 7, 9$	$q+1$	$\frac{q(q-1)}{2}$
$D_{(q-1)}$ $q \neq 3, 5, 7, 9, 11$	$q-1$	$\frac{q(q+1)}{2}$
A_4 *if* $q = p > 3$ *and* $q = 3, 13, 27, 37(40)$ or $q = 5$	12	$\frac{q(q^2-1)}{12\times 2}$
S_4 *if* $q = p > 2$ *and* $q = \pm 1(8)$	24	$\frac{q(q^2-1)}{24\times 2}$
A_5 if $\begin{cases} q = 5^r & r \text{ odd prime} & \text{or} \\ p = q = \pm 1(5) & p \text{ prime} & \text{or} \\ q = p^2 = -1(5) & p \text{ prime} & \text{or} \end{cases}$	60	$\frac{q(q^2-1)}{60\times 2}$
$\mathrm{PSL}(2, q')$ $q' \neq 5$, $q = q'^m$ m *odd prime*	$\frac{q'(q'^2-1)}{2}$	$\frac{q(q^2-1)}{q'(q'^2-1)}$
$\mathrm{PGL}(2, q')$ $q = q'^2$	$q'(q'^2-1)$	$\frac{q(q^2-1)}{q'(q'^2-1)}$

Table 8: The maximal subgroups of $\mathrm{PSL}(2, q)$, for q odd

We now provide the classification (existence and uniqueness) of all two-transitive representations of every maximal subgroup of $\mathrm{PSL}(2,q)$, a result borrowed from [2].

For the time being, let U be a group acting 2-transitively on a set Ω. Let $Ker\ U$ be the kernel of the representation, namely, the set of all $u \in U$ such that $u(x) = x$ for every $x \in \Omega$. Let U_0 be the stabilizer in U of some element 0 in Ω.

Proposition 4.6. *[2] Let $G \cong \mathrm{PSL}(2,q)$ for some power q of a prime p. Let (U, U_0) be a 2-transitive pair of subgroups of G with U maximal in G. Then one of the following holds:*

1. *$U \cong E_q : \frac{q-1}{2}$, $q = 1(4)$, $Ker\ U$ is the unique subgroup of index 2 of U, $|\Omega| = 2$, $U_0 = Ker\ U$ (unique up to conjugacy);*
2. *$U \cong E_q : (q-1)$, q even, $|\Omega| = q$, $Ker\ U = 1$, U_0 is a cyclic subgroup of order $(q-1)$ (unique up to conjugacy);*
3. *$U \cong \mathrm{PSL}(2,2) \cong S_3$, $|\Omega| = 2$, $Ker\ U = Z_3 = U_0$ (unique up to conjugacy);*
4. *$U \cong \mathrm{PSL}(2,2) \cong S_3$, $|\Omega| = 3$, $Ker\ U = 1$, $U_0 \cong Z_2$ (unique up to conjugacy);*
5. *$U \cong \mathrm{PSL}(2,3) \cong A_4$, $|\Omega| = 4$, $Ker\ U = 1$, $U_0 \cong Z_3$ (unique up to conjugacy);*
6. *$U \cong A_5 \cong \mathrm{PSL}(2,5) \cong \mathrm{PSL}(2,4)$, $p \neq 2$, $p \neq 5$, either $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$. Here $|\Omega| = 5$, $Ker\ U = 1$, $U_0 \cong A_4$; (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,q)$); or $|\Omega| = 6$, $Ker\ U = 1$, $U_0 \cong D_{10}$.*
7. *$U \cong \mathrm{PSL}(2,11)$, $|\Omega| = 11$, $Ker\ U = 1$, $U_0 \cong A_5$ (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,11) = Aut(U)$);*
8. *$U \cong \mathrm{PSL}(2,9) \cong A_6$, $|\Omega| = 6$, $Ker\ U = 1$, $U_0 \cong A_5$ (two such representations, up to conjugacy; they are fused in $PGL(2,9)$);*
9. *$U \cong \mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, $|\Omega| = 7$, $Ker\ U = 1$, $U_0 \cong S_4$ (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,7)$);*
10. *$U \cong \mathrm{PSL}(2,r)$ for every $r = p^s$, $s \geq 1$, $r > 3$ with $q = r^m$ and m prime. Moreover, for $p > 2$ we also require $m > 2$. Here $|\Omega| = r+1$, $Ker\ U = 1$, $U_0 \cong E_r : \frac{r-1}{(2,r-1)}$ (unique up to conjugacy for given r);*
11. *$U \cong \mathrm{PGL}(2,r)$, r odd, $r = p^s$, $q = r^2$, $|\Omega| = 2$, $Ker\ U = U_0 \cong \mathrm{PSL}(2,r)$ (unique up to conjugacy);*
12. *$U \cong \mathrm{PGL}(2,r)$, r odd, $r = p^s$, $s \geq 1$ with $q = r^2$. Here $|\Omega| = r+1$, $Ker\ U = 1$, $U_0 \cong E_r : (r-1)$ (unique up to conjugacy);*
13. *$U \cong \mathrm{PGL}(2,3) \cong S_4$, $q = \pm 1(8)$, $|\Omega| = 3$, $Ker\ U = E_4$, $U/Ker\ U \cong S_3$, $U_0 \cong D_8$ (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,q)$);*
14. *U is dihedral of order $2(q-1)$ or $2(q+1)$, q even. $|\Omega| = 2$, $Ker\ U = U^+ = U_0$ where U^+ is the cyclic subgroup of index 2 of U, (unique up to conjugacy for each of the two possible values of $|U|$);*
15. *U is dihedral of order $(q-1)$ or $(q+1)$, q odd. $|\Omega| = 2$, $Ker\ U = U^+ = U_0$ where U^+ is the cyclic subgroup of index 2 of U, (unique up to conjugacy for each of the two possible values of $|U|$). In the particular case where $q = 3$, the case of $(q+1)$ provides $U = E_4$, $|\Omega| = 2$. Then U_0 is one of the three subgroups of order 2 in U (unique up to conjugacy);*

16. *U is dihedral of order either $2(q-1)$ or $2(q+1)$, q even, and $3 \mid |U|$; $|\Omega| = 3$, $Ker\ U$ is the unique cyclic subgroup of index 6 in U. Then U_0 is one of the three dihedral subgroups of index 3 in U, $U/Ker\ U \cong S_3$ (unique up to conjugacy);*
17. *U is dihedral of order either $(q-1)$ or $(q+1)$, q odd, and $3 \mid |U|$. Here $|\Omega| = 3$, $Ker\ U$ is the unique cyclic subgroup of index 6 in U. Then U_0 is one of the three dihedral subgroups of index 3 in U, $U/Ker\ U \cong S_3$ (unique up to conjugacy);*
18. *U is dihedral of order either $(q-1)$ or $(q+1)$, q odd, $q > 5$, and $4 \mid |U|$. Here $|\Omega| = 2$, $Ker\ U = U_0$ is one of the two dihedral subgroups of index 2 in U (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,q)$); $Ker\ U$ is dihedral, U_0 is dihedral of index 2;*
19. *U is dihedral of order 4, q is one of 3,5; $|\Omega| = 2$, $Ker\ U = U_0$ is one of the three dihedral subgroups of index 2 in U (unique up to conjugacy);*
20. *$U \cong \mathrm{PGL}(2,5) \cong S_5$, $|\Omega| = 5$, $Ker\ U = 1$, $U_0 \cong S_4$ (unique up to conjugacy).*

4.4 Some other useful results

An observation used in our proofs is that $\mathrm{PGL}(2,q)$ can be viewed as a subgroup of $\mathrm{PSL}(2,q^2)$ and also that $\mathrm{PGL}(2,q)$ has a unique subgroup isomorphic to $\mathrm{PSL}(2,q)$. This lets us extract the list of subgroups of $\mathrm{PGL}(2,q)$ from the list of subgroups of $\mathrm{PSL}(2,q^2)$. Therefore we require the properties of the subgroup lattice of $\mathrm{PGL}(2,q)$ for which we refer to [4] (see also [13] and [15]). The next lemma is often used to count the geometries up to isomorphism.

Lemma 4.7.

- *Assume that $\frac{q\pm1}{d(2,q-1)}$ is even. In this case both conjugacy classes of D_{2d} for every $d > 2$ dividing $\frac{q\pm1}{(2,q-1)}$ fuse in $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$.*
- *Assume that $q = \pm1(8)$. In this case both conjugacy classes of S_4 and A_4 fuse in $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$.*
- *Assume that $q = \pm1(5)$. In this case both conjugacy classes of A_5 fuse in $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$.*
- *Assume that $q = p^{2n}$ is odd. In this case both conjugacy classes of $\mathrm{PGL}(2,p^n)$ fuse in $\mathrm{PGL}(2,p^{2n})$ and thus also in $P\Gamma L(2,q)$.*

5 Proof of Theorem 1.1

In this section, we prove the Classification Theorem 1.1 by a case analysis. We determine the rank 2 RWPRI and $(2T)_1$ geometries of $\mathrm{PSL}(2,q)$.

In order to structure this work we introduce a subsection for each type of G_0. There are 5 such subsections left to consider, which are the different types of maximal subgroups of $G \cong \mathrm{PSL}(2,q)$, listed in section 4.2. The cases $E_q : \frac{(q-1)}{(2,q-1)}$, $D_{2\frac{(q-1)}{(2,q-1)}}$ and $D_{2\frac{(q+1)}{(2,q-1)}}$ have been treated in [5].

The various cases for the two-transitive pairs (G_0, G_{01}) with G_0 maximal in G are provided by Proposition 4.6. Those situations are analysed in order to detect the admissible G_1 in a series of Lemmas. During this analysis, candidates for G_1 are represented by the symbol H. They become G_1 only when they resist the analysis.

5.1 The case where $G_0 = A_5$

Recall that following Table 7 and Table 8, the subgroup A_5 is maximal in $\mathrm{PSL}(2,q)$ if

$$\begin{cases} q = 5^r & r \text{ odd prime} & \text{or} \\ q = 4^r & r \text{ prime} & \text{or} \\ q = p = \pm 1(5) & p \text{ odd prime} & \text{or} \\ q = p^2 = -1(5) & p \text{ odd prime} & . \end{cases}$$

In this section we assume these conditions on q. Observe that if $q = 0(5)$ the group A_5 is isomorphic to $\mathrm{PSL}(2,5)$ which is a particular case of the family $\mathrm{PSL}(2,5^n)$ with $q = 5^{nm}$ for m an odd prime. In this section we treat this particular situation. The general situation is treated in Proposition 5.16. If $q = 0(4)$ the group A_5 is isomorphic to $\mathrm{PSL}(2,4)$ which is a particular case of the family $\mathrm{PSL}(2,4^n)$ with $q = 4^{nm}$ for m prime. In this section we analyse this particular situation. The general situation is treated in Proposition 5.12.

In view of (6) in Proposition 4.6 there are two cases for G_{01}, namely the case of D_{10} and A_4. For each of these G_{01} we look for the various possible groups H in one of the two following Lemmas. Remember that H is any subgroup of G such that (H, G_{01}) is a two-transitive pair. In order to determine all H candidates we scan the list of maximal subgroups. For each maximal subgroup we analyse its subgroup lattice.

Lemma 5.1. *Let $G \cong \mathrm{PSL}(2,q)$ with q as required in this section. If H is a subgroup of G such that (H, D_{10}) is a two-transitive pair then one of the three following statements holds:*

- $H \cong D_{20}$ *provided* $10 \mid \frac{q\pm1}{(2,q-1)}$;
- $H \cong D_{30}$ *provided* $15 \mid \frac{q\pm1}{(2,q-1)}$;
- $H \cong \mathrm{PSL}(2,5) \cong A_5$.

Proof. Left to the reader. See Appendix pg 1. (The Appendix contains details for this and several other results to follow.) □

Lemma 5.2. *Let $G \cong \mathrm{PSL}(2,q)$ with q as required in this section. If H is a subgroup of G such that (H, A_4) is a two-transitive pair then one of the five following statements holds:*

- $H \cong E_{16}{:}3$ *provided* $q = 4^r$;
- $H \cong \mathrm{PSL}(2,4) \cong A_5$ *provided* $q = 4^r$;
- $H \cong \mathrm{PSL}(2,5)$ *provided* $q = 5^r$;
- $H \cong S_4$ *provided* $q = \pm1(5)$ *and* $q = \pm1(8)$;
- $H \cong A_5$.

Proof. Left to the reader. See Appendix pg 2. □

In Remark 4.3 of section 4.1. we mention that the number of conjugacy classes of cases 7 and 8 are not given in Lemma 4.1. To prove the following Proposition we need the number of conjugacy classes of a particular situation, treated in the next two Lemmas.

Lemma 5.3. *The number of conjugacy classes of* $E_{16}:3$ *in* $\mathrm{PSL}(2,4^r)$*, for an odd prime* r*, is equal to* $\frac{4^{r-1}-1}{15}$.

Proof. Step 1: We must count the number of conjugacy classes of subgroups $E_{16}:3$ in $\mathrm{PSL}(2,4^r)$. Therefore we first count the total number of subgroups $E_{16}:3$ in $\mathrm{PSL}(2,4^r)$ and divide this number by the length of the conjugacy classes. We shall indeed see that this number is constant.

Step 2: We consider $G \cong \mathrm{PSL}(2,4^r)$ as a permutation group acting on the projective line $PG(1,4^r)$. This group is sharply 3-transitive on 4^r+1 points. Given a point ∞, its stabilizer is $E_{4^r}:4^r-1 \cong AGL(1,4^r)$. The latter contains our $E_{16}:3$. Let H be any subgroup $E_4:3 \cong A_4 \cong AGL(1,4)$. It is contained in a subgroup $K := \mathrm{PGL}(2,4) \cong A_5$ which has an orbit of length five namely $PG(1,4)$.

Step 3: Let us see $AG(1,4^r) = PG(1,4^r)\backslash\{\infty\}$ as an affine space V of dimension r over the field $GF(4)$. The subgroup H stabilizes a line l of V namely $AG(1,4)$. Hence, l contains the points 0 and 1. The space V endowed with the point 0 is a vector space of dimension r on $GF(4)$.

Observe that H fixes a unique point namely ∞. In A_5 there are four conjugate subgroups $E_4:3$ say X_1, X_2, X_3, X_4 other than H, each fixing a unique point which belongs to l. Moreover, H stabilizes no other line l' in V since otherwise $l' \cup \{\infty\}$ is an orbit of length five of A_5 and so each of X_1, X_2, X_3, X_4 fixes a point on l' while this point is on l implying $l = l'$. Therefore, H stabilizes a unique line of V which is l.

Step 4: Observe that $AG(1,4^r)$ is transitive on the lines of V. There are $\frac{4^r(4^r-1)}{12}$ lines in V and, taking the point ∞ into account, we see that the conjugacy class of H in G consists of $\frac{4^r(4^r-1)}{12}$ subgroups $E_{16}:3$.

Step 5: Coming back to the beginning of Step 3, the multiplicative group of $GF(4)$ is a cyclic group Z_3 which is a subgroup of H and so also a subgroup of A_5 namely $E_{16}:3$.

Step 6: The group Z_3 stabilizes the point 0 and every line on 0 in the space V. Therefore, it also stabilizes every plane on 0 in this space, in particular every plane containing l. There are $\frac{4^r-4}{16-4} = \frac{4^{r-1}-1}{3}$ such planes on l.

Step 7: Let π be a plane of V containing l. It is invariant under 16 translations and Z_3. Thus π is invariant under a subgroup $E_{16}:3$ containing H. Conversely, every $E_{16}:3$, say L, containing H also contains Z_3 which fixes the point 0. The orbit of 0 under L is its orbit under E_{16}. And Z_3 acts on this orbit, hence this orbit is a plane. In conclusion, the subgroups $E_{16}:3$ containing H and the planes containing l are in one-to-one correspondence.

Step 8: Combining Steps 3, 6 and 7 we see that the number of conjugacy classes of subgroups $E_{16}:3$ containing H and fixing ∞ is $\frac{4^{r-1}-1}{3}.\frac{1}{5}$ as required. □

For the particular situation of Lemma 5.3, we count the number of geometries up to conjugacy and up to isomorphism in the following Lemma.

Lemma 5.4. *Let* r *be an odd prime. Let* $\alpha_C(r)$ *(resp.* $\alpha_I(r)$*) be the number of geometries of type* $\Gamma(\mathrm{PSL}(2,4^r), A_5, A_4, E_{16}:3)$ *up to conjugacy (resp. isomorphism). Then the following hold:*

1. $\alpha_C(3) = 5$;
2. $\alpha_I(3) = 2$;
3. *if* $r > 3$*, then* $\alpha_C(r) = \frac{4^{r-1}-1}{3}$;

4. if $r > 3$*, then* $\alpha_I(r) = \frac{2(4^{r-2}-1)+3.2^{r-2}}{3r}$.

Proof. Step 1: Lemma 5.3 gives the number of conjugacy classes of $E_{16} : 3$ for a given $\mathrm{PSL}(2,4^r)$. Every $E_{16} : 3$ has five conjugacy classes of subgroups $E_4 : 3$. Moreover, each $E_4 : 3$ is contained in a unique A_5. Therefore, we get the number of triples consisting of a representative G_1 of every conjugacy class of $E_{16} : 3$, a representative G_{01} of every conjugacy class of $E_4 : 3$ in G_1 and the unique subgroup $G_0 \cong A_5$ containing G_{01}. Hence $\alpha_C(r) = \frac{4^{r-1}-1}{15} \cdot 5 \cdot 1$. In particular $\alpha_C(3) = 5$. This is proving respectively (3) and (1).

Step 2: Let ∞, V, H and l be defined as in the proof of Lemma 5.3, Steps 2 and 3. Recall that l contains 0 and 1. To get $\alpha_I(r)$, we still have to figure out how $N_{P\Gamma L(2,4^r)}(H)$ acts on the subgroups $E_{16} : 3$ containing H. In other words, how does $N_{P\Gamma L(2,4^r)}(H)$ act on the planes of V containing l ?

Step 3: To answer the question of Step 2 we shall show that $N_{P\Gamma L(2,4^r)}(H) = H : K$, where K is the group of field automorphisms of $GF(4^r)$. Recall the fact that the group $P\Gamma L(2,4^r)$ is $PSL(2,4^r) : K$. Recall also that K is a cyclic group of order $2r$. The group K leaves every subfield of $GF(4^r)$ invariant. Hence K leaves $GF(4)$ invariant, thus also the line l, and it normalizes H. Applying Lemma 4.1 we see that $N_{PSL(2,4^r)}(H) = H$ in view of the fact that $H \cong A_4$ and of the restrictions on the values taken by q. We get that $N_{P\Gamma L(2,4^r)}(H)$ is a group of order $H.K.\epsilon$ and we want now to show that $\epsilon = 1$. Let N_1 (resp. N_2) be the number of conjugate subgroups of H in G (resp. $P\Gamma L(2,4^r)$). Then $N_1 \leq N_2$, $N_1 = \frac{|G|}{|H|}$, $N_2 = \frac{|P\Gamma L(2,4^r)|}{|H|.|K|.\epsilon} = \frac{|G|}{|H|.\epsilon}$ and so $\epsilon = 1$. Therefore $N_{P\Gamma L(2,4^r)}(H) = H : K$.

Step 4: In our count of triples, we may assume that G_0 and G_{01} are fixed because, up to isomorphism, the chain of subgroups $PSL(2,4^r) - A_5 - A_4$ is unique. Moreover, without loss of generality, we suppose that G_{01} is H.

Step 5: We consider $G \cong PSL(2,4^r)$ and H in it. We recall the $\frac{4^{r-1}-1}{15}$ conjugacy classes of subgroups $E_{16} : 3$ containing H as found in Lemma 5.3. Let Ω be the set of these $\frac{4^{r-1}-1}{15}$ conjugacy classes. Recall that $N_{PSL(2,4^r)}(H) = H$ and so the action of H on Ω is the identity. Next we consider the action of K on Ω which is also the action of $H : K$. The number of orbits of this K-action on Ω is the number $\alpha_I(r)$ we have to determine.

Step 6: As in the proof of Lemma 5.3, Step 2 we consider $G \cong \mathrm{PSL}(2,4^r)$ as a triply transitive permutation group acting on the projective line $PG(1,4^r)$. For every t dividing $2r$ there is a subfield $GF(2^t)$ of $GF(4^r)$. It fixes $2^t + 1$ points on $PG(1,4^r)$. This set of points is called a circle as well as all of its transforms under G. Every triple of distinct points on $PG(1,4^r)$ is contained in one and only circle of $2^t + 1$ points.

Step 7: Given three points ∞, 0 and 1, there is a unique circle C_5 of five points, namely $PG(1,4) = \{\infty\} \cup l$ and there is a unique circle of $2^r + 1$ points C_{2^r+1}, namely $PG(1,2^r)$. The involution $\beta \in K$ fixes all the points of C_{2^r+1}. On C_5, it fixes ∞, 0 and 1, and it permutes the remaining 2 points that we call a and $\beta(a)$. The group induced on C_5 by the stabilizer of C_5 in $PSL(2,4^r)$ is $A_5 \cong \mathrm{PSL}(2,4)$ extended by β, that is, S_5.

The unique subgroup K^+ of K, of order r fixes all points of C_5 and splits the remaining points of $PG(1,4^r)$ in orbits of length r. Therefore, $(2^{2r} + 1) - 5$ must be divisible by r. Indeed, $(2^{2r} + 1) - 5 = 4^r - 4 = 4(4^{r-1} - 1)$, the latter being divisible by r thanks to Fermat's little theorem.

The subgroup $H \cong E_4 : 3$ fixes ∞. Every cyclic subgroup of order 3 of H fixes two points of C_5. This gives ten conjugate subgroups of order 3 in A_5.

The group K^+ fixes C_5 point-wise. Suppose K^+ stabilizes a plane π of V containing l. Then it must decompose the $16 - 4 = 12$ points of $\pi\backslash l$ in orbits of length r.

If $r = 3$, this may occur and K^+ indeed stabilizes two of the five planes containing l, hence it normalizes two of the $E_{16}:3$ containing H. Moreover, it fuses the other three. The two $E_{16}:3$ normalized by K^+ are swapped by β, giving $\alpha_I(3) = 2$. This is proving (2).

If $r > 3$, no plane of V that contains l can be stabilized by K^+. Hence K^+ fuses the $\frac{4^{r-1}-1}{3}$ subgroups $E_{16}:3$ in $\frac{4^{r-1}-1}{3r}$ orbits of length r.

Step 8: It remains to look at the action of β on these orbits. In $GF(4^r)$, there are three proper subfields, namely $GF(2)$, $GF(4)$ and $GF(2^r)$. The involution β fixes all the elements of $GF(2^r)$. Let us show that β stabilizes $\frac{2^r-2}{2}$ planes containing l. Given an element $x \in GF(2^r)$, the plane π containing 0, 1 and x is stabilized since 0, 1 and x are fixed by β. Moreover, π contains the point $x + 1 \in GF(2^r)$. Hence, there are at least four points fixed in π by β. If there are more, there must be at least 8 points fixed and the whole plane π is fixed point-wise, a contradiction with the fact that $a \in \pi$ and a is not fixed by β. Therefore, the elements of $GF(2^r)$ give $\frac{2^r-2}{2}$ distinct planes that are stabilized by β.

Step 9: We claim that the remaining planes of V that contain l are fused in pairs by β. Indeed, suppose that there exists a plane π containing l and no other element of $GF(2^r)$ in V, and such that $\beta(\pi) = \pi$. In π, the only fixed points are thus 0 and 1. For every $x \in \pi\backslash C_5$, the line $x\beta(x)$ is stabilized by β. It is either secant or parallel to l. Suppose first that it is secant. Then, it intersects l in either 0 or 1 and the fourth point of $x\beta(x)$ must be fixed, a contradiction. Suppose then that it is parallel. The other two points of $x\beta(x)$ may be written as y and $\beta(y)$. Let us recall that we denote the points of l as 0, 1, a and $\beta(a)$. The lines ax and $\beta(a)\beta(x)$ are swapped and parallel. One of the lines $1y$ or $1\beta(y)$ must also be parallel to ax. Its image by β is not parallel to ax. This is a contradiction. Therefore, no other plane of V containing l can be stabilized by β.

Step 10: In conclusion, we get $\frac{2^{r-1}-1}{r}$ sets of r isomorphic geometries and $\frac{1}{2r}(\frac{4^{r-1}-1}{3} - (2^{r-1} - 1))$ sets of $2r$ isomorphic geometries. Finally, we obtain $\alpha_I(r) = \frac{2^{r-1}-1}{r} + \frac{1}{2r}(\frac{4^{r-1}-1}{3} - (2^{r-1} - 1))$ and the formula given in the Lemma is obtained by a straightforward simplification. This is proving (4). □

Proposition 5.5. *Let $G \cong \mathrm{PSL}(2, q)$ with q as required in this section. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong A_5$ is isomorphic to one of the geometries appearing in Table 1.*

Proof. Let $G_0 \cong A_5$.
We subdivide our discussion in two cases, namely the two G_{01}-candidates in view of (6) in Proposition 4.6 which are: D_{10} and A_4. In each of these two cases we review all possibilities for G_1 given in the previous Lemmas 5.1 and 5.2, as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism).

Subcase 1: $G_{01} = G_0 \cap G_1 \cong D_{10}$.
This is dealt with in the appendix, (pg 2-6).

Subcase 2: $G_{01} = G_0 \cap G_1 \cong A_4$.
By Lemma 5.2 the possibilities for G_1 are $E_{16} : 3$ if $q = 4^r$, $\mathrm{PSL}(2, 4) \cong A_5$ if $q = 4^r$, $\mathrm{PSL}(2, 5) \cong A_5$ if $q = 5^r$, S_4 if $q = \pm1(5)$ as well as $q = \pm1(8)$ and A_5.

2.1 We consider the case where $G_1 \cong E_{16}\!:\!3$.
The condition on q is $q = 4^r$ with r prime. In this situation there is only one conjugacy class of A_5 and one of A_4 in $\mathrm{PSL}(2, q)$. Notice that there are 5 conjugacy classes of A_4 in $E_{16}\!:\!3$. Since $\mathrm{PSL}(2, 16)$ is simple and A_5 maximal, A_5 is self-normalized. Moreover, A_4 is self-normalized in $\mathrm{PSL}(2, 4^r)$. The normalizer of $E_{16}:3$ depends on whether $r = 2$ or not. We distinguish three cases namely: $r = 2$, $r = 3$ and $r > 3$. In the latter two, notice that since $r \neq 2$, $E_{16}\!:\!3$ is self-normalized in $\mathrm{PSL}(2, 4^r)$.

• Let us first consider the particular case where $r = 2$. In this situation there exists only one conjugacy class of $E_{16} : 3$ in $\mathrm{PSL}(2, 16)$. We also have that $N_{\mathrm{PSL}(2,16)}(E_{16} : 3) = E_{16} : 15$. Therefore the number of subgroups $E_{16} : 3$ containing a given subgroup A_4 in $\mathrm{PSL}(2, 16)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,16) \mid}{\mid E_{16}\!:\!15 \mid} \cdot \frac{\mid E_{16}\!:\!3 \mid}{\mid A_4 \mid} \cdot 5 \cdot \frac{\mid A_4 \mid}{\mid \mathrm{PSL}(2,16) \mid} = 1.$$

Thus the RWPRI and $(2T)_1$ geometry $\Gamma_2 = \Gamma\,(\mathrm{PSL}(2, 16); A_5, E_{16}\!:\!3, A_4)$ exists and is unique up to conjugacy and also up to isomorphism.

• In view of Lemma 5.3 and Lemma 5.4 we know that if $r = 3$ there exist up to conjugacy exactly five RWPRI and $(2T)_1$ geometries $\Gamma_3 := \Gamma(\mathrm{PSL}(2, 64), A_5, A_4, E_{16}\!:\!3)$ and exactly two up to isomorphism.

• In view of Lemma 5.3 and Lemma 5.4 we know that if $r > 3$ there exist up to conjugacy exactly $\frac{4^{r-1}-1}{3}$ RWPRI and $(2T)_1$ geometries $\Gamma_4 := \Gamma(\mathrm{PSL}(2, q), A_5, A_4, E_{16}: 3)$ and exactly $\frac{2(4^{r-2}-1)+3.2^{r-2}}{3r}$ up to isomorphism.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 16, 64$. For $q = 16$, it is also confirmed by [20].

2.2 We consider the case where $G_1 \cong S_4$.
The conditions given on q are $q = \pm 1(5)$ and $q = \pm 1(8)$. They imply that there are two conjugacy classes of S_4, two of A_5 and also two of A_4 in $\mathrm{PSL}(2, q)$. Therefore we consider two situations: either $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$, with p an odd prime. We distinguish these two cases in the discussion below.

• Assume $q = p = \pm 1(5)$ with p prime. All conditions given on q imply that either $q = \pm 1(40)$ or $q = \pm 9(40)$. In both situations we know that S_4 is a maximal subgroup of $\mathrm{PSL}(2, q)$. Therefore $N_{\mathrm{PSL}(2,q)}(A_4) = S_4 = N_{S_4}(A_4)$ and $N_{A_5}(A_4) = A_4$. Now all A_4 in an S_4 are conjugate and this is also the case for all A_4 in an A_5. The number of subgroups A_5 containing a given subgroup A_4 in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid A_5 \mid} \cdot \frac{\mid A_5 \mid}{\mid A_4 \mid} \cdot \frac{\mid S_4 \mid}{\mid \mathrm{PSL}(2,q) \mid} = 2.$$

To count the geometries up to conjugacy we need to know whether the S_4 normalizes each of the A_5. This is not the case because $|N_{\mathrm{PSL}(2,q)}(A_4) \cap N_{\mathrm{PSL}(2,q)}(S_4)| = |S_4| = 2|A_4|$. Hence, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_9 = \Gamma(\mathrm{PSL}(2, q); A_5, S_4, A_4)$ up to conjugacy, provided $q = \pm 1(40)$ or $q = \pm 9(40)$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 4.7 the two classes of S_4, A_4 and A_5 are fused under the action of $\mathrm{PGL}(2, q)$ and thus also under the action of $P\Gamma L(2, q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_9 = \Gamma\,(\mathrm{PSL}(2, q); A_5, S_4, A_4)$ up to isomorphism provided $q = \pm 1(40)$ or $q = \pm 9(40)$.

• Assume $q = p^2 = -1(5)$ with p prime. All conditions given on q imply that either $q = -1(40)$ or $q = 9(40)$. All A_4 in an S_4 are conjugate and $N_{\mathrm{PSL}(2,q)}(A_4) = S_4 = N_{S_4}(A_4)$ and $N_{A_5}(A_4) = A_4$. We also know that $N_{\mathrm{PSL}(2,q)}(S_4) = S_4$. Therefore the number of S_4 containing a given A_4 is one.

To count the geometries up to conjugacy we need to know whether the S_4 normalizes each of the A_5. This is not the case because $|N_{\mathrm{PSL}(2,q)}(A_4) \cap N_{\mathrm{PSL}(2,q)}(S_4)| = |S_4| = 2|A_4|$. Therefore, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{16} = \Gamma\left(\mathrm{PSL}(2,q); A_5, S_4, A_4\right)$ provided either $q = -1(40)$ or $q = 9(40)$, with $q = p^2$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 4.7 the two classes of A_4, S_4 and A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{16} = \Gamma\left(\mathrm{PSL}(2,q); A_5, S_4, A_4\right)$ up to isomorphism, provided either $q = -1(40)$ or $q = 9(40)$, with $q = p^2$.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 31, 41, 49$. For $q = 9$, it is also confirmed by [3].

2.3 Consider the case where $G_0 \cong G_1 \cong A_5$.
With the given conditions on q there are three cases to consider:

• If $q = 4^r$ with r prime, there is only one conjugacy class of A_5 and also one of A_4. Since every A_4 is contained in only one A_5, there is no such geometry.

• Assume $q = 5^r$ with r an odd prime. The number of conjugacy classes of A_4 in $\mathrm{PSL}(2,q)$ depends on whether $q = \pm1(8)$ or $q = \pm3(8)$. If $q = \pm1(8)$ there is a contradiction with r odd in $q = 5^r$. Now $q = \pm3(8)$ implies that there is one conjugacy class of A_4 and also one of A_5. Since every A_4 is contained in only one A_5, there exists no such geometry.

• Assume $q = p = \pm1(5)$ or $q = p^2 = -1(5)$ with p an odd prime.
There are two conjugacy classes of A_5 in $\mathrm{PSL}(2,q)$. The number of conjugacy classes of A_4 in $\mathrm{PSL}(2,q)$ depends on whether $q = \pm1(8)$ or $q = \pm3(8)$. We distinguish these two cases.

If $q = \pm1(8)$ there are two classes of A_4, all A_4 in an A_5 are conjugate, and the normalizer of A_4 in $\mathrm{PSL}(2,q)$ is S_4. All conditions on q imply that if $q = p = \pm1(5)$ either $q = \pm1(40)$ or $q = \pm9(40)$; and if $q = p^2 = -1(5)$ either $q = -1(40)$ or $q = +9(40)$.
The number of subgroups A_5 containing a given subgroup A_4 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ A_5\ |} \cdot \frac{|\ A_5\ |}{|\ A_4\ |} \cdot \frac{|\ S_4\ |}{|\ \mathrm{PSL}(2,q)\ |} = 2.$$

Therefore, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{10} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, A_4)$ up to conjugacy, provided either $q = \pm1(40)$ or $q = \pm9(40)$, with q prime, one for each class of A_5. Also, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{17} = \Gamma\left(\mathrm{PSL}(2,q); A_5, A_5, A_4\right)$ up to conjugacy, provided either $q = -1(40)$ or $q = +9(40)$, with $q = p^2$, one for each class of A_5.

Let us deal with the fusion of non-conjugate classes. Following Lemma 4.7 the two classes of A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{10} = \Gamma\left(\mathrm{PSL}(2,q); A_5, A_5, A_4\right)$ up to isomorphism provided either $q = \pm1(40)$ or $q = \pm9(40)$.

Also, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{17} = \Gamma\,(\mathrm{PSL}(2, q); A_5, A_5, A_4)$ up to isomorphism provided either $q = -1(40)$ or $q = +9(40)$.

If $q = \pm 3(8)$, there is one conjugacy class of A_4 in $\mathrm{PSL}(2, q)$. All conditions on q imply that if $q = p = \pm 1(5)$ either $q = \pm 11(40)$ or $q = \pm 19(40)$; and if $q = p^2 = -1(5)$ either $q = -11(40)$ or $q = +19(40)$. Every A_4 is contained in exactly one A_5, and there are two conjugacy classes of A_5 in $\mathrm{PSL}(2, q)$.

Hence, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{11} = \Gamma(\mathrm{PSL}(2, q); A_5, A_5, A_4)$ up to conjugacy and thus also exactly one up to isomorphism provided either $q = \pm 11(40)$ or $q = \pm 19(40)$, with q prime.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 11, 19, 29, 31, 41, 61$. For $q = 11, 19$, it is also confirmed by [20].

Also, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{18} = \Gamma(\mathrm{PSL}(2, q); A_5, A_5, A_4)$ up to conjugacy and thus also exactly one up to isomorphism provided either $q = -11(40)$ or $q = +19(40)$, with $q = p^2$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 49$. □

5.2 The case where $G_0 = A_4$

Recall that following Table 8, the subgroup A_4 is maximal in $\mathrm{PSL}(2, q)$ provided q is prime, $q > 3$ and either $q = 3, 13, 27, 37(40)$ or $q = 5$. Therefore $q = \pm 3(8)$ and there exists only one conjugacy class of subgroups isomorphic to A_4. In view of (5) in Proposition 4.6 there is only one case for G_{01}, namely the cyclic subgroup of order 3.

The proof of all following propositions are very similar to that of Proposition 5.5. Therefore we do not give the details and we refer to the Appendix. The proof of proposition 5.6 may be found in the Appendix (pg. 6-9).

Proposition 5.6. *Let* $G \cong \mathrm{PSL}(2, q)$ *with q prime,* $q > 3$ *and either* $q = 3, 13, 27, 37(40)$ *or* $q = 5$*. Every RWPRI and* $(2T)_1$ *geometry of rank two* $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ *in which* $G_0 \cong A_4$ *is isomorphic to one of the geometries appearing in Table 2.*

5.3 The case where $G_0 = S_4$

Recall that following Table 7 and Table 8, the subgroup S_4 is maximal in $\mathrm{PSL}(2, q)$ if q is an odd prime and $q = \pm 1(8)$. In this section we assume these conditions on q. Moreover, there are two conjugacy classes of subgroups isomorphic to S_4 in G.

In view of (11), (12) and (13) in Proposition 4.6 there are three cases for G_{01}, namely the case of $D_6 \cong E_3 : 2$, the case of D_8 and the case of A_4. For each of these G_{01} we look for the various possible groups H in one of the three following Lemmas, whose proofs are left to the reader. The proof of proposition 5.10 may be found in the Appendix (pg. 8-12).

Lemma 5.7. *Let* $G \cong \mathrm{PSL}(2, q)$ *with q an odd prime and* $q = \pm 1(8)$ *as required in this section. If H is a subgroup of G such that* (H, D_6) *is a two-transitive pair then one of the three following statements holds:* $H \cong D_{12}$ *provided* $6 \mid \frac{q\pm 1}{2}$*;* $H \cong D_{18}$ *provided* $9 \mid \frac{q\pm 1}{2}$*; or* $H \cong S_4$*.*

Lemma 5.8. *Let* $G \cong \mathrm{PSL}(2, q)$ *with q an odd prime and* $q = \pm 1(8)$ *as required in this section. Then the following statement holds: If H is a subgroup of G such that* (H, D_8) *is a two-transitive pair then* $H \cong D_{16}$ *provided* $8 \mid \frac{q\pm 1}{2}$*,* $H \cong D_{24}$ *provided* $12 \mid \frac{q\pm 1}{2}$*; or*$H \cong S_4$*.*

Lemma 5.9. *Let $G \cong \mathrm{PSL}(2,q)$ with q an odd prime and $q = \pm 1(8)$ as required in this section. Then the following statement holds: If H is a subgroup of G such that (H, A_4) is a two-transitive pair then $H \cong S_4$; or $H \cong A_5$ provided $q = \pm 1(5)$.*

Proposition 5.10. *Let $G \cong \mathrm{PSL}(2,q)$ with q an odd prime and $q = \pm 1(8)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong S_4$ is isomorphic to one of the geometries appearing in Table 3.*

5.4 The case where $G_0 = \mathrm{PSL}(2, q')$

In this section we make a distinction between the cases q odd and q even with $q = p^{nm}$. The subgroups $\mathrm{PSL}(2,q')$ and $\mathrm{PGL}(2,q')$ with $q' = p^n$ are isomorphic provided q is even and they are distinct provided q is odd.

5.4.1 The case q even

Since q is even, $\mathrm{PSL}(2,q') \cong \mathrm{PGL}(2,q')$. Recall that following Table 7, the subgroup $\mathrm{PSL}(2,q') \cong \mathrm{PGL}(2,q')$ is maximal in $\mathrm{PSL}(2,q)$ provided $q' = 2^n$ and $q = q'^m = 2^{n.m}$ for m prime; moreover for $n = 1$ we need $m = 2$. In this section we assume these conditions on q.

In view of (3), (4), (6) and (10) in Proposition 4.6 there are three cases for G_{01}, namely: the case of the cyclic subgroup of order 3 provided $q' = 2$, the case of D_{10} provided $q' = 4$ and the case of $E_{2^n} : (2^n - 1)$.

For each of these G_{01} we look for the various possible groups H; the case of $E_{2^n} : (2^n - 1)$ is treated in the following Lemma, whose proof is left to the reader. The proof of proposition 5.12 may be found in the Appendix (pg. 13-14).

Lemma 5.11. *Assume $q = 2^{nm}$ with m prime and $n \neq 1$ and let $G \cong \mathrm{PSL}(2,q)$. If H is a subgroup of G such that $(H, E_{2^n} : 2^n - 1)$ is a two-transitive pair then one of the two following statements holds: $H \cong E_{2^{2n}} : 2^n - 1$ provided $m = 2$ or $H \cong \mathrm{PSL}(2, 2^n)$.*

Notice that if $n = 2$, $\mathrm{PSL}(2, 2^n) \cong A_5$.

Proposition 5.12. *Assume $q' = 2^n$ and $q = q'^m = 2^{n.m}$ for m prime ; moreover for $n = 1$ we need $m = 2$. Let $G \cong \mathrm{PSL}(2, 2^{n.m})$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong \mathrm{PSL}(2,q')$ is isomorphic to one of the geometries appearing in Table 4.*

5.4.2 The case q odd

Since q is odd we need to consider two distinct maximal subgroups which are $\mathrm{PSL}(2,p^n)$ provided $q = p^{mn}$ where m and p are odd primes and $\mathrm{PGL}(2,p^n)$ provided $q = p^{2n}$ where p is an odd prime. The latter will be treated in section 5.5.

Recall that following Table 8, the subgroup $\mathrm{PSL}(2,p^n)$ is maximal in $\mathrm{PSL}(2,q)$ provided $q = p^{mn}$ with m and p odd primes. In this section we assume these conditions on q.

In view of (5)-(10) in Proposition 4.6 there are four possibilities for G_{01}, namely: A_4 provided $q' = 5$, S_4 provided $q' = 7$, A_5 provided $q' = 9, 11$ and $E_{q'} : \frac{q'-1}{2}$. For each of these G_{01} we look for the various possible groups H in the three following Lemmas, whose proofs are left to the reader. The proof of proposition 5.16 may be found in the Appendix (pg. 14-17). The case of A_4 provided $q' = 5$, will be treated directly in the proof of the Proposition.

Lemma 5.13. *Assume q odd, $q = p^{nm}$ with m prime and let $G \cong \mathrm{PSL}(2,q)$; then the following statement holds: If H is a subgroup of G such that $\left(H, E_{p^n} : \frac{p^n-1}{2}\right)$ is a two-transitive pair then $H \cong \mathrm{PSL}(2,p^n)$.*

Notice that if $p^n = 3$, $\mathrm{PSL}(2,p^n) \cong A_4$ and if $p^n = 5$, $\mathrm{PSL}(2,p^n) \cong A_5$. They are particular cases of $\mathrm{PSL}(2,p^n)$.

Lemma 5.14. *Assume q is either 11^m or 9^m, with m an odd prime and let $G \cong \mathrm{PSL}(2,q)$. Then the following statement holds: If H is a subgroup of G such that (H, A_5) is a two-transitive pair then $H \cong \mathrm{PSL}(2,q')$ provided $q' = 9$ or 11.*

Lemma 5.15. *Assume $q = 7^m$, with m odd prime and let $G \cong \mathrm{PSL}(2,7^m)$. Then the following statement holds: If H is a subgroup of G such that (H, S_4) is a two-transitive pair then $H \cong \mathrm{PSL}(2,7)$.*

Proposition 5.16. *Assume $q = p^{nm}$ with p and m odd primes and let $G \cong \mathrm{PSL}(2,q)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong \mathrm{PSL}(2,p^n)$ is isomorphic to one of the geometries appearing in Table 5.*

5.5 The case where $G_0 = \mathrm{PGL}(2,q')$

If q is even, $\mathrm{PGL}(2,q') \cong \mathrm{PSL}(2,q')$ and this situation has been treated in Section 5.4. Therefore, we assume in this section that q is odd. Recall that following Table 8, the subgroup $\mathrm{PGL}(2,q')$ is maximal in $\mathrm{PSL}(2,q)$ provided $q' = p^n$ and $q = q'^2 = p^{2n}$ with p an odd prime.

In view of (11), (12), (13) and (20) in Proposition 4.6 there are four cases for G_{01}, namely the case of $E_{p^n} : (p^n-1)$, the case of $\mathrm{PSL}(2,q')$, the case of D_8 provided $q = 3^2$ and the case of S_4 provided $q = 5^2$.

For each of these four G_{01} we look for the various possible groups H in one of the four following Lemmas, whose proofs are left to the reader. The proof of proposition 5.21 may be found in the Appendix (pg. 17-19).

Lemma 5.17. *Let $G \cong \mathrm{PSL}(2,3^2)$. Then the following statement holds:*
If H is a subgroup of G such that (H, D_8) is a two-transitive pair then $H \cong \mathrm{PGL}(2,3)$.

Lemma 5.18. *Assume q is odd and let $G \cong \mathrm{PSL}(2,p^{2n})$. Then the following statement holds:*
If H is a subgroup of G such that $(H, E_{p^n} : (p^n-1))$ is a two-transitive pair then $H \cong E_{p^{2n}} : (p^n-1)$ or $H \cong \mathrm{PGL}(2,p^n)$.

Lemma 5.19. *Assume q is odd and let $G \cong \mathrm{PSL}(2,p^{2n})$. Then the following statement holds:*
If H is a subgroup of G such that $(H, \mathrm{PSL}(2,p^n))$ is a two-transitive pair then $H \cong A_5$ provided $p^n = 3$; or $H \cong \mathrm{PGL}(2,p^n)$.

Notice that if $p^n = 3$, $\mathrm{PGL}(2,p^n) \cong S_4$.

Lemma 5.20. *Let $G \cong \mathrm{PSL}(2,5^2)$. Then the following statement holds:*
If H is a subgroup of G such that (H, S_4) is a two-transitive pair then $H \cong \mathrm{PGL}(2,5)$.

Proposition 5.21. *Assume $q' = p^n$ and $q = q'^2 = p^{2n}$ with p an odd prime. Let $G \cong \mathrm{PSL}(2,q)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong \mathrm{PGL}(2,q')$ is isomorphic to one of the geometries appearing in Table 6.*

The proof of Theorem 1.1 readily follows from Propositions 5.6, 5.10, 5.5, 5.12, 5.16 and 5.21.

The main Theorem of [5] and Theorem 1.1 complete the classification of rank two residually weakly primitive and locally two-transitive coset geometries for the groups $\mathrm{PSL}(2,q)$. We also give the number of classes of all such geometries with respect to conjugacy and isomorphism.

This classification includes infinite classes of geometries up to conjugacy and up to isomorphism. This number is dependent on the prime power $q = p^n$; it is a function of n and p.

6 Locally s-arc-transitive graphs

The construction of the $(G,2)$-arc-transitive graphs, using Tits' Theorem, is studied in full detail in Leemans [12]. This construction shows that the rank two incidence structures are also locally-2-arc-transitive graphs in the sense of [8].

All the RWPRI and $(2T)_1$ geometries we have obtained are bipartite graphs and also locally 2-arc-transitive graphs. Now we want the value of s such that the incidence graph of Γ is a locally s-arc-transitive but not a locally $(s+1)$-arc-transitive graph. We mainly use the method of D. Leemans [12] (Lemma 5.1). This provides the value of s in all cases given in Tables 1, 2, 3, 4, 5 and 6 (in the introduction) except those listed in Table 9. We don't give the details in the cases for which the Leemans' method works.

We now discuss the nine cases left over in Table 9. In every case if p is a vertex of the graph, we write $p^\perp$ for the set of neighbours of p which is also the residue of p.
We give the details for four of them, the other five are dealt with in the Appendix (pg. 19-20).

Case of Table 1, geometry Γ_1, case of Table 1, geometries Γ_6 and Γ_{13} and case of Table 4, geometry Γ_5.

We know that $s \geq 2$. Consider a path (a,b,c) such that a is of type 0, b is of type 1, c is of type 0. Here, $G_{abc} = Z_5$. This acts on the five 1-elements $d_1, ..., d_5$ other than b in $c^\perp$. The action is transitive since otherwise Z_5 would be in the kernel of the action of G_c on $c^\perp$ contradicting the simplicity of $G_0 = A_5 = G_c$. This provides $s \geq 3$ for paths starting at a $0-element$.

Next consider a path (h,i,j) such that h is of type 0, i is of type 1, j is of type 0. Here, $G_{hij} = Z_2$. This acts on the two 0-elements k_1, k_2 other than i in $j^\perp$. The action is transitive since otherwise Z_2 would be in the kernel of the action of G_j on $j^\perp$. This kernel for the action of D_{30} on the cosets of D_{10} is a group Z_5, a contradiction. Hence $s \geq 3$.

Applying Leemans' method we get $s = 2$ or 3. Thus $s = 3$.

Case of Table 3, geometry Γ_6.

We know that $s \geq 2$. Consider a path (a,b,c) as in the preceding case. Here, $G_{abc} = Z_4$. This acts on the two 1-elements d_1, d_2 other than b in $c^\perp$. The action is transitive since otherwise Z_4 would be in the kernel of the action of G_c on $c^\perp$. This kernel for the action

	$G_0 \cong A_5$	
G_{01}	G_1	
D_{10}	D_{30}	Table 1, Γ_1
D_{10}	D_{30}	Table 1, Γ_6 and Γ_{13}
	$G_0 \cong S_4$	
G_{01}	G_1	
D_6	D_{18}	Table 3, Γ_2
D_8	D_{16}	Table 3, Γ_5
D_8	D_{24}	Table 3, Γ_6
D_8	S_4	Table 3, Γ_7 and Γ_8
	$G_0 \cong \mathrm{PSL}(2, 2^n)$	
G_{01}	G_1	
$E_{2^n} : (2^n - 1)$	$E_{2^{mn}} : (2^n - 1)$	Table 4, Γ_1
D_{10}	D_{30}	Table 4, Γ_5
	$G_0 \cong \mathrm{PGL}(2, p^n)$	
G_{01}	G_1	
$E_{p^n} : (p^n - 1)$	$E_{p^{2n}} : (p^n - 1)$	Table 6, Γ_1

Table 9: Cases in which s cannot be decided by Leemans' method.

of S_4 on the cosets of D_8 is 2^2, a contradiction. This provides $s \geq 3$ for paths starting at a $0 - element$.

Next consider a path (h, i, j) as in the preceding case. Here, $G_{hij} = 2^2$. This acts on the two 0-elements k_1, k_2 other than i in $j^{\perp}$. The action is transitive since otherwise 2^2 would be in the kernel of the action of G_j on $j^{\perp}$. This kernel for the action D_{18} on the cosets of D_8 is a group Z_4, a contradiction. Hence $s \geq 3$. Applying Leemans' method we get that s equals 3 or 4.

We now prove that s cannot be equal to 4 thanks to the following argument due to an unknown referee: Given the path (a, b, c) starting at a 0-element we have shown that $G_{abc} = Z_4$ and that this is transitive on the two elements adjacent to c other than b. Thus $G_{abcd} = Z_2 = \langle x \rangle$, where x is the square of an element of order 4 in $G_{abc} < G_a \cong S_4$. Thus x lies in the normal subgroup of G_a of order 4 and so acts trivially on the set of neighbours of a. Thus G_{dcba} is not transitive on the set of 4-arcs starting with (d, c, b, a) and so the graph is not locally 4-arc transitive. Hence $s = 3$.

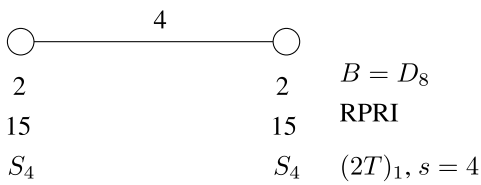
Let us make some observations on the results: In Tables 1, 2, 3, 4, 5, 6 and 9 most values are $s = 2$ or $s = 3$. There are some spectacular examples with larger values of s. Indeed we obtain a locally 4-arc transitive graph and a locally 7-arc transitive graph which are respectively

$$\Gamma\left(\mathrm{PSL}(2, q); S_4, S_4, D_8\right) \text{ due to Biggs-Hoare [1]}$$

$$\text{and } \Gamma\left(\mathrm{PSL}(2, q); D_{16}, S_4, D_8\right) \text{ due to Wong [22]}$$

These examples also appear in Li [14].

However, let us pay more attention to the case $q = 9$. Here we are dealing with a geometry whose Buekenhout diagram is given by



This is the smallest thick generalised quadrangle. Its origin is the symplectic group $Sp_4(2)$; in that context it is known at least from [17]. It is also famous as Tutte's 8-cage [18]. Its incidence graph admits an automorphism group four times as big as group $\mathrm{PSL}(2, 9)$ which is $P\Gamma L(2, 9)$. Under the action of this group we check that the graph is actually 5-arc-transitive and this is also provided by Tutte.

Moreover, for the cases in which $q = 17, 23, 31, 41, 47, 71, 73, 79, 89$ the full automorphism group of the incidence graph is the group $\mathrm{PGL}(2, q)$. This group has a unique conjugacy class of subgroups S_4, according to E.H. Moore as we see in [4]. Thus $\mathrm{PGL}(2, q)$ fuses the two classes of S_4 in $\mathrm{PSL}(2, q)$ and so it cannot provide 5-arc-transitivity. Finally, for the case $\Gamma\left(\mathrm{PSL}(2, q); D_{16}, S_4, D_8\right)$ for $q = 17, 31, 79, 97$, there are two classes of S_4 in $\mathrm{PSL}(2, q)$ that are fused in $\mathrm{PGL}(2, q)$. There are two such geometries for each value of q and so the full automorphism group of Γ is $\mathrm{PSL}(2, q)$. (see Proposition 5.10).

7 Appendix

The Appendix contains details for several results of this paper, except the proofs of Lemmas 5.7, 5.8, 5.9, 5.11, 5.13, 5.14, 5.15, 5.17, 5.18, 5.19, 5.20 which are left to the reader. Appendix is available on-line at: `http://amc-journal.eu/index.php/amc/issue/view/17`.

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A note on a conjecture on consistent cycles

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Abstract

Let Γ denote a finite digraph and let G be a subgroup of its automorphism group. A directed cycle $\vec{C}$ of Γ is called *G-consistent* whenever there is an element of G whose restriction to $\vec{C}$ is the 1-step rotation of $\vec{C}$. In this short note we prove a conjecture on G-consistent directed cycles stated by Steve Wilson.

Keywords: Digraphs, consistent directed cycles.

Math. Subj. Class.: 05C20, 05C38, 05E18

1 Introduction

Let Γ denote a finite digraph (without loops and multiple arcs). By a *directed cycle* in Γ we mean a cyclically ordered set $\vec{C} = \{v_0, v_1, v_2, \ldots, v_{r-1}\}$, $r \geq 3$, of pairwise distinct vertices of Γ such that (v_i, v_{i+1}) is an arc of Γ for every $i \in \mathbb{Z}_r$ (the addition being mod r). Let G be a subgroup of the automorphism group of Γ. Directed cycle $\vec{C}$ is called *G-consistent*, if there exists $g \in G$ such that $v_i^g = v_{i+1}$ for each $i \in \mathbb{Z}_r$. In this case g is called a *shunt* for $\vec{C}$. Clearly, G acts on the set of G-consistent directed cycles: for $h \in G$, $\vec{C}^h = \{v_0^h, v_1^h, v_2^h, \ldots, v_{r-1}^h\}$ is G-consistent with a shunt $h^{-1}gh$.

Consistent cycles in finite arc-transitive graphs were introduced by J. H. Conway in one of his public lectures [3]. Since then a number of papers on consistent cycles and their applications appeared, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11].

Observe that if (u, v) is an arc of Γ and $g \in G$ is such that $u^g = v$, then the orbit of u under g induces a G-consistent directed cycle $\{u, v = u^g, u^{g^2}, \ldots\}$ (provided that $u^{g^2} \neq u$). Steve Wilson [12] stated the following conjecture on consistent cycles.

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Conjecture 1.1. *Let Γ denote a finite digraph (without loops and multiple arcs) and let G be an arc-transitive subgroup of its automorphism group. Pick vertices u, v of Γ, such that (u, v) is an arc of Γ. For a G-orbit $\mathcal{A}$ of G-consistent directed cycles, let $B_{\mathcal{A}}$ denote the set of all automorphisms $g \in G$, such that $u^g = v$, and the orbit of u under g is in $\mathcal{A}$. Let $G_{(u,v)}$ denote the G-stabilizer of the arc (u, v). Then the number of elements in $B_{\mathcal{A}}$ is independent of $\mathcal{A}$, and is equal to the order of $G_{(u,v)}$.*

In this short note we prove the above conjecture.

2 Proof of the conjecture

In this section we prove Conjecture 1.1. We prove Conjecture 1.1 in two steps. In Proposition 2.1 we prove that $|G_{(u,v)}| \leq |B_{\mathcal{A}}|$, and in Proposition 2.2 we prove that $|B_{\mathcal{A}}| \leq |G_{(u,v)}|$.

Proposition 2.1. *With the notation of Conjecture 1.1, we have $|G_{(u,v)}| \leq |B_{\mathcal{A}}|$.*

Proof. Since G is arc-transitive, there exists a G-consistent directed cycle $\vec{C}$ in $\mathcal{A}$, which contains the arc (u, v). Let g denote a shunt for $\vec{C}$. Let $G_{\vec{C}}$ denote the pointwise stabiliser of $\vec{C}$ and let k be the index of $G_{\vec{C}}$ in $G_{(u,v)}$. Let $g_1, \ldots, g_k$ be representatives of cosets of $G_{\vec{C}}$ in $G_{(u,v)}$.

Observe that for each $1 \leq i \leq k$ and each $h \in G_{\vec{C}}$, the automorphism $g_i^{-1} g h g_i$ sends u to v. Furthermore, the orbit of u under $g_i^{-1} g h g_i$ is the directed cycle $\vec{C}^{g_i}$. Namely, since g is a shunt for $\vec{C}$ and $h \in G_{\vec{C}}$, the image of $v^{g^j g_i}$ under $g_i^{-1} g h g_i$ is $v^{g^{j+1} g_i}$. Moreover, $\vec{C}^{g_i}$ is clearly in $\mathcal{A}$. Therefore, $g_i^{-1} g h g_i \in B_{\mathcal{A}}$.

We claim that if either $i \neq j$ or $h_1 \neq h_2$ $(h_1, h_2 \in G_{\vec{C}})$, then $\alpha = g_i^{-1} g h_1 g_i$ and $\beta = g_j^{-1} g h_2 g_j$ are distinct. Indeed, assume first that $i \neq j$. Note that $\vec{C}^{g_i} \neq \vec{C}^{g_j}$ since g_i and g_j are from different cosets of $G_{\vec{C}}$ in $G_{(u,v)}$. Moreover, α is a shunt for $\vec{C}^{g_i}$ and β is a shunt for $\vec{C}^{g_j}$. Since $\vec{C}^{g_i} \neq \vec{C}^{g_j}$ (and since $\vec{C}^{g_i}$ and $\vec{C}^{g_j}$ have at least the arc (u, v) in common), it follows that also $\alpha \neq \beta$. On the other hand, if $i = j$ and $\alpha = \beta$, then $h_1 = h_2$. Therefore, if $h_1 \neq h_2$ and $i = j$, then $\alpha \neq \beta$. This proves the claim.

It follows that $|B_{\mathcal{A}}| \geq k|G_{\vec{C}}| = |G_{(u,v)}|$. □

Proposition 2.2. *With the notation of Conjecture 1.1, we have $|B_{\mathcal{A}}| \leq |G_{(u,v)}|$.*

Proof. Let X denote the set of all G-consistent directed cycles in $\mathcal{A}$, containing the arc (u, v). Clearly, $B_{\mathcal{A}}$ is exactly the set of all shunts of directed cycles from X. Since all directed cycles from X have the arc (u, v) in common, every element of $B_{\mathcal{A}}$ is a shunt for exactly one directed cycle from X. Note also that X is nonempty as G is arc-transitive. We now define a mapping Ψ from $B_{\mathcal{A}}$ to $G_{(u,v)}$ as follows.

Fix $\vec{C} \in X$ and a shunt $g_{\vec{C}}$ of $\vec{C}$. For each $\vec{D} \in X$ there exists an element of G which sends $\vec{D}$ to $\vec{C}$. Pick such an element and denote it by $h(\vec{D})$. Composing $h(\vec{D})$ with an appropriate power of $g_{\vec{C}}$, we could assume that $h(\vec{D}) \in G_{(u,v)}$. For each $g \in B_{\mathcal{A}}$, let $\vec{D}(g)$ denote the unique directed cycle in X, for which g is a shunt (see Figure 1). For $g \in B_{\mathcal{A}}$ define $\Psi(g) = g h(\vec{D}(g)) g_{\vec{C}}^{-1}$ and note that $\Psi(g) \in G_{(u,v)}$.

We now show that Ψ is an injection. Pick $g_1, g_2 \in B_{\mathcal{A}}$ and assume that $\Psi(g_1) = \Psi(g_2)$. Let $\vec{D}(g_1) = \{u, v, v_1, v_2, \ldots, v_{n-1}\}$ and $\vec{D}(g_2) = \{u, v, w_1, w_2, \ldots, w_{n-1}\}$. We first

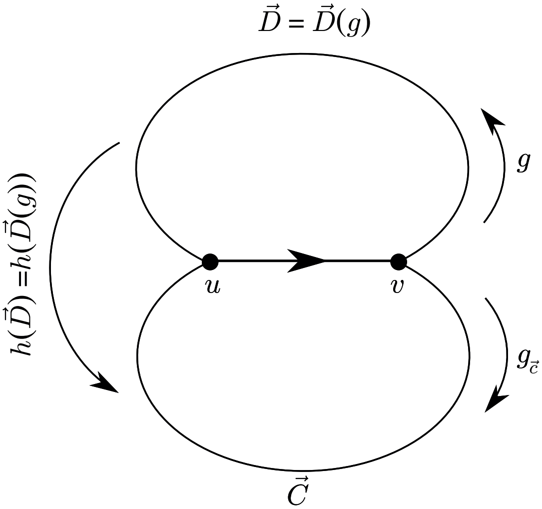

Figure 1: Directed consistent cycles $\vec{C}$ and $\vec{D}$.

show that $\vec{D}(g_1) = \vec{D}(g_2)$. Since $\Psi(g_1) = g_1 h(\vec{D}(g_1)) g_{\vec{C}}^{-1} = g_2 h(\vec{D}(g_2)) g_{\vec{C}}^{-1} = \Psi(g_2)$, we have $g_2^{-1} g_1 = h(\vec{D}(g_2)) h(\vec{D}(g_1))^{-1}$. This implies that $g_2^{-1} g_1$ is in $G_{(u,v)}$. We claim that $v_{n-i} = w_{n-i}$ for $i = 0, 1, \ldots n-1$, where $v_n = w_n = u$. We prove our claim using induction on i. Note that our claim is true for $i = 0$. Assume that our claim is true for $i = 0, 1, \ldots, t$, where $0 \le t \le n-2$. Note that $h(\vec{D}(g_2)) h(\vec{D}(g_1))^{-1}$ fixes the arc $(v_{n-t}, v_{n-t+1}, \ldots v_{n-1}, u, v)$, and therefore also $g_2^{-1} g_1$ fixes this arc. But since

$$v_{n-t-1}^{g_1} = v_{n-t} = v_{n-t}^{g_2^{-1} g_1} = w_{n-t-1}^{g_1},$$

we have $v_{n-t-1} = w_{n-t-1}$, verifying the claim. It follows that $\vec{D}(g_1) = \vec{D}(g_2)$. But since $\vec{D}(g_1) = \vec{D}(g_2)$, also $h(\vec{D}(g_1)) = h(\vec{D}(g_2))$. As $g_1 h(\vec{D}(g_1)) g_{\vec{C}}^{-1} = g_2 h(\vec{D}(g_2)) g_{\vec{C}}^{-1}$, it follows that $g_1 = g_2$. Therefore Ψ is an injection and so $|B_{\mathcal{A}}| \le |G_{(u,v)}|$. □

Corollary 2.3. *With the notation of Conjecture 1.1, we have* $|B_{\mathcal{A}}| = |G_{(u,v)}|$.

Proof. Immediately from Propositions 2.1 and 2.2. □

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Sectional split extensions arising from lifts of groups

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Abstract

Covering techniques have recently emerged as an effective tool used for classification of several infinite families of connected symmetric graphs. One commonly encountered technique is based on the concept of lifting groups of automorphisms along regular covering projections $\wp\colon \tilde{X} \to X$. Efficient computational methods are known for regular covers with cyclic or elementary abelian group of covering transformations $\mathrm{CT}(\wp)$.

In this paper we consider the lifting problem with an additional condition on how a group should lift: given a connected graph X and a group G of its automorphisms, find all connected regular covering projections $\wp\colon \tilde{X} \to X$ along which G lifts as a sectional split extension. By this we mean that there exists a complement $\overline{G}$ of $\mathrm{CT}(\wp)$ within the lifted group $\tilde{G}$ such that $\overline{G}$ has an orbit intersecting each fibre in at most one vertex. As an application, all connected elementary abelian regular coverings of the complete graph K_4 along which a cyclic group of order 4 lifts as a sectional split extension are constructed.

Keywords: Covering projection, graph, group extension, lifting automorphisms, voltage assignment.

Math. Subj. Class.: 05C50, 05E18, 20B40, 20B25, 20K35, 57M10

1 Introduction

Graph covers play a significant role when symmetry properties of graphs are investigated. One of the commonly used techniques is based on the concept of lifting automorphisms along regular covering projections. Applications of this technique have been used to classify families of graphs with given structural properties (see for instance [2, 11, 12, 19, 20]).

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In its most general form the problem of lifting automorphisms is well understood. Much attention has been devoted to finding the necessary and sufficient lifting conditions in combinatorial terms, see [15, 16, 26, 27]. Nevertheless, these general results are rather hopeless to apply when concrete examples and more detailed questions related to symmetry properties of graphs are considered.

In a more specific setting of regular covers in which the group of covering transformations is either cyclic or elementary abelian, the situation changes. For such covers, efficient computational methods are known. For example, in the case of elementary abelian regular covers, the idea behind the approach developed in [19] is to reduce the general lifting problem to that of finding invariant subspaces of matrix groups over prime fields, linearly representing the action of automorphisms on the first homology group of the graph. Applying this method to a number of symmetric graphs – including the complete graphs K_4 [20] and K_5 [13], the Möbius-Kantor graph [18], the complete bipartite graph $K_{3,3}$ [20], the Petersen graph [21], the Pappus graph [25], the octahedron graph [14], and the Heawood graph [19] – has resulted in the classification of connected elementary abelian regular covers admitting various types of subgroups of automorphisms. A similar approach, also based on linear criteria for lifting automorphisms, was proposed in [3], and has been used in order to find connected regular coverings with cyclic or elementary abelian group of covering transformation for the complete graph K_4 [6], the 3-dimensional cube graph Q_3 [7], the complete bipartite graph $K_{3,3}$ [4], and the Petersen graph [5].

Assuming that a group G of automorphisms of X lifts along a regular covering projection $\wp\colon \tilde{X} \to X$, the lifted group $\tilde{G}$ is an extension of the group of covering transformations $\mathrm{CT}(\wp)$ by G. Specific types of extensions have usually a strong impact on structural properties of the covering graph $\tilde{X}$. In this context, the following two cases deserve special attention: (i) $\tilde{G}$ is a split extension of $\mathrm{CT}(\wp)$ by G, and in particular, (ii) $\tilde{G}$ is a direct split extension of $\mathrm{CT}(\wp)$ by G. For short we say that G lifts as a *split extension* or as a *direct split extension*, respectively. In the former case there exists, by definition, a complement $\overline{G}$ of $\mathrm{CT}(\wp)$ within $\tilde{G}$, and a normal complement $\overline{G}$ of $\mathrm{CT}(\wp)$ in the latter. This allows us to compare actions of two isomorphic groups, G on X and $\overline{G}$ on $\tilde{X}$, where $\overline{G}$ projects isomorphically onto G along $\wp$. However, it can happen that the complement is not unique, and what is more, different complements can exhibit different actions on $\tilde{X}$. Therefore, the analysis can be quite complicated. Certain algorithmic aspects related to the question of how difficult is to test conditions (i) and (ii) are considered in [22].

According to particular kinds of actions that can arise from complements, two extremal cases seem to stand out: (iii) there exists a complement $\overline{G}$ that acts transitively on the covering graph $\tilde{X}$, and (iv), there exists a complement $\overline{G}$ that is *sectional*. By this we mean that there is a *section* of $\tilde{X}$ – a set of vertices containing at most one vertex from each fibre – invariant under the action of $\overline{G}$. For short we say that G lifts as a *transitive split extension* or as a *sectional split extension*, respectively. Clearly, one might further restrict conditions (iii) and (iv) to normal complements. Certain particular questions along these lines have been addressed in [1, 8, 16, 17].

Motivated by the above discussion, the following problem is of interest. Given a connected graph X and a group G of its automorphisms, find all connected regular covering projections $\wp\colon \tilde{X} \to X$ along which G lifts in a prescribed way. In this paper we restrict to case (iv) – we introduce a method for finding regular coverings along which G lifts as a sectional split extension.

The basic idea behind our approach is the following. First, we take the cone $\widehat{X}$ over the graph X obtained by adding a new vertex $*$ joined to every vertex of X, together with the group of automorphisms $\widehat{G}$ of $\widehat{X}$ that fixes $*$ and acts on X as the group G. Next, the condition for lifting G as a sectional split extension is reduced to the general lifting problem of finding regular coverings of $\widehat{X}$ admitting the lift of $\widehat{G}$. Consequently, the original problem can be solved as soon as the general lifting problem can be solved. Our approach is illustrated on a concrete example: we construct all connected elementary abelian regular coverings of the complete graph K_4 along which a cyclic group of order 4 lifts as a sectional split extension.

The rest of the paper is organized as follows. In Section 2 we review some preliminary concepts about regular graph covers and lifting automorphisms. In Section 3 we devise a method for constructing connected regular covering projections along which G lifts as a sectional split extension. A detailed example is provided in Section 4.

2 Preliminaries

A *graph* is an ordered quadruple $X = (D, V; \text{beg}, ^{-1})$, where $D_X = D$ and $V_X = V$ are disjoint sets of *darts* and *vertices*, respectively, beg is a mapping that assigns to each dart x its *initial vertex* $\text{beg}(x)$, and $^{-1}$ is an involution interchanging every dart x and its inverse dart x^{-1}. For a dart x, its *terminal vertex* is the vertex $\text{end}(x) = \text{beg}(x^{-1})$. The orbits of $^{-1}$ are called *edges*. An edge $e = \{x, x^{-1}\}$ is called a *link* whenever $\text{beg}(x) \neq \text{end}(x)$. If $\text{beg}(x) = \text{end}(x)$, then the respective edge is either a *loop* or a *semi-edge*, depending on whether $x \neq x^{-1}$ or $x = x^{-1}$, respectively. All graphs in this paper are assumed to be *finite*, meaning that the sets of vertices and darts are finite.

A *graph homomorphism* $f\colon Y \to X$ is an adjacency preserving mapping taking darts to darts and vertices to vertices, or more precisely, $f(\text{beg}(x)) = \text{beg}(f(x))$ and $f(x^{-1}) = f(x)^{-1}$. An *isomorphism* is a bijective homomorphism. An isomorphism of a graph onto itself is an *automorphism*. All automorphisms of a graph X together with composition of automorphisms constitute the *automorphism group* $\text{Aut}(X)$.

A surjective homomorphism $\wp : \tilde{X} \to X$ is called a *regular covering projection* if there exists a semi-regular subgroup $S_\wp$ of $\text{Aut}(\tilde{X})$ such that its vertex orbits and dart orbits coincide with *vertex fibres* $\wp^{-1}(v)$, $v \in V_X$, and *dart fibres* $\wp^{-1}(x)$, $x \in D_X$, respectively. Two regular covering projections $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are *isomorphic* if there exist an automorphism g of X and an isomorphism $\tilde{g}\colon \tilde{X} \to \tilde{X}'$ such that the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X}' \\ \wp\downarrow & & \downarrow\wp' \\ X & \xrightarrow[g]{} & X \end{array}$$

commutes. In particular, if $g = \text{id}$ then $\wp$ and $\wp'$ are *equivalent*. If, in the above setting, $\tilde{X} = \tilde{X}'$ and $\wp = \wp'$, then we say that g *lifts along* $\wp$ or that $\tilde{g}$ is a *lift* of g along $\wp$. A group $G \leq \text{Aut}(X)$ lifts if all $g \in G$ lift. The collection of all lifts of all elements in G forms a subgroup $\tilde{G} \leq \text{Aut}(\tilde{X})$, the *lift* of G. In particular, the lift of the trivial group is known as the *group of covering transformations* and denoted by $\text{CT}(\wp)$. Observe that $\tilde{G}$ is an extension of $\text{CT}(\wp)$ by G. Furthermore, if G lifts along a given projection $\wp$, then it lifts along any covering projection equivalent to $\wp$. This allows us to study lifts of automorphisms combinatorially in terms of voltage assignments, a concept that we are going to describe now.

Let X be a graph and let N be an (abstract) group, called the *voltage group*. Assign to each dart x of X a *voltage* $\zeta_x \in N$ in such a way that $\zeta_{x^{-1}} = \zeta_x^{-1}$. Such a function $\zeta\colon D_X \to N$ is called a *voltage assignment* on X. Further, construct the *derived graph* $\mathrm{Cov}(\zeta)$ with vertex set $V_X \times N$ and dart set $D_X \times N$, where $\mathrm{beg}(x,n) = (\mathrm{beg}(x), n)$ and $(x,n)^{-1} = (x^{-1}, n\,\zeta_x)$. The projection onto the first coordinate $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ is then the *derived regular covering projection*, where the required semi-regular subgroup $S_{\wp_\zeta}$ of $\mathrm{Aut}(\mathrm{Cov}(\zeta))$ arises from the action of N on the second coordinate by left multiplication on itself. Conversely, any regular covering projection $\wp : \tilde{X} \to X$ can be reconstructed by a voltage assignment ζ on X such that the projection $\wp_\zeta$ derived from ζ is equivalent to $\wp$. Moreover, one can assume that the voltage assignment ζ is $\mathcal{T}$-reduced for some arbitrarily chosen spanning tree $\mathcal{T}$ of X, meaning that $\zeta_x = 1$ for all darts x in $\mathcal{T}$, see [9] for more details.

Consider now a regular covering projection $\wp$ of connected graphs. Then we say that $\wp$ is *connected*. Further, the semi-regular group $S_\wp$ is equal to $\mathrm{CT}(\wp)$, and the voltage assignment ζ that reconstructs the projection $\wp$ is valued in the voltage group $N \cong \mathrm{CT}(\wp)$ (viewed as an abstract group). Such a voltage assignment ζ is also called *connected*. It is well known that ζ is connected if and only if each element of N appears as the voltage of some closed walk. Furthermore, by the *basic lifting lemma* [15, 16], an automorphism g of X lifts along $\wp_\zeta$ if and only if each closed walk with trivial voltage is mapped by g to a walk with trivial voltage.

Two assignments ζ and ζ' on X are *equivalent* whenever the respective derived regular covering projections $\wp_\zeta$ and $\wp_{\zeta'}$ are equivalent. Assuming that both assignments are connected and valued in N, then they are equivalent if and only if there exists an automorphism of N mapping ζ_W to ζ'_W for each closed walk W at u_0 [27].

For a given connected graph X and subgroup $G \le \mathrm{Aut}(X)$, the problem of finding regular covering projections $\wp$ along which G lifts is very difficult in general. However, in the case of *elementary abelian* regular coverings $\wp$ – that is, when $\mathrm{CT}(\wp)$ is isomorphic to an elementary abelian group – the necessary and sufficient lifting condition can be stated combinatorially by means of voltages as follows, see [19].

Let p be a prime. The first homology group $H_1(X;\mathbb{Z}_p)$ is generated by the (directed) cycles of X and is isomorphic to the elementary abelian group $\mathbb{Z}_p^r$, where r is the Betti number of the graph X. The group $H_1(X;\mathbb{Z}_p)$ is usually viewed as a vector space over $\mathbb{Z}_p$ of dimension r. Since each automorphism $\alpha \in \mathrm{Aut}(X)$ maps a cycle in X to a cycle in X, there is a natural action of α on $H_1(X;\mathbb{Z}_p)$ which induces a linear transformation $\alpha^\#$ of $H_1(X;\mathbb{Z}_p)$. Choose a spanning tree $\mathcal{T}$ of X and exactly one dart from each edge $\{x, x^{-1}\}$ that is not contained in $\mathcal{T}$. Then the sequence $x_1, x_2, \ldots, x_r \in D_X \backslash D_\mathcal{T}$ of all such darts naturally defines an (ordered) basis $\mathcal{B}_\mathcal{T} = \{C_1, C_2, \ldots, C_r\}$ of $H_1(X;\mathbb{Z}_p)$, where C_i is the cycle arising from the spanning tree $\mathcal{T}$ and the dart x_i. Next, denote the matrix representation of $\alpha^\#$ with respect to the basis $\mathcal{B}_\mathcal{T}$ by $M_\alpha \in \mathbb{Z}_p^{r,r}$. Thus, a subgroup $G \le \mathrm{Aut}(X)$ induces a subgroup $M_G = \{M_g \mid g \in G\} \le \mathrm{GL}(r, \mathbb{Z}_p)$. By M_G^t we denote the dual group consisting of all transposes of matrices in M_G.

Theorem 2.1. *([19, Proposition 6.3, Corollary 6.5]) With the notation above, let* $\zeta\colon D_X \to \mathbb{Z}_p^{d,1}$ *be a* $\mathcal{T}$*-reduced voltage assignment on* X*, and let* $Z \in \mathbb{Z}_p^{d,r}$ *be the matrix with columns*

$$\zeta_{x_1}, \zeta_{x_2}, \ldots, \zeta_{x_r}.$$

If Z *has rank* d*, then the derived graph* $\mathrm{Cov}(\zeta)$ *is connected and the following hold:*

(i) A group $G \leq \mathrm{Aut}(X)$ lifts along $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ if and only if the columns of Z^t form a basis of a M_G^t-invariant d-dimensional subspace $\mathcal{S}(\zeta)$ of $\mathbb{Z}_p^{r,1} \cong H_1(X;\mathbb{Z}_p)$.

(ii) If $\zeta'\colon D_X \to \mathbb{Z}_p^{d,1}$ is another voltage assignment on X satisfying the above conditions, then $\wp_{\zeta'}$ is equivalent to $\wp_\zeta$ if and only if $\mathcal{S}(\zeta') = \mathcal{S}(\zeta)$. Moreover, $\wp_{\zeta'}$ is isomorphic to $\wp_\zeta$ if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(X)$ such that the matrix M_α^t maps $\mathcal{S}(\zeta')$ onto $\mathcal{S}(\zeta)$.

By Theorem 2.1, we can find all pairwise nonequivalent connected elementary abelian regular coverings of X along which G lifts – in terms of voltages – as follows. First find a basis $\{u_1, u_2, \ldots, u_d\}$ for each M_G^t-invariant subspace U of $\mathbb{Z}_p^{r,1}$. Next, for each basis $\{u_1, u_2, \ldots, u_d\}$ consider a matrix Z with rows $u_1^t, u_2^t, \ldots, u_d^t$, and then define the voltage assignment $\zeta^U\colon D_X \to \mathbb{Z}_p^{d,1}$, mapping dart x_i to the i-th column of Z, $i = 1, 2, \ldots, r$, and mapping all darts of $\mathcal{T}$ to the trivial voltage. Observe that the choice of a spanning tree together with a sequence $x_1, x_2, \ldots, x_r$ as well as choosing a basis for an invariant subspace is irrelevant as long as we consider regular coverings up to equivalence. Thus, the problem of finding connected elementary abelian regular coverings along which a given group of automorphisms lifts translates to a purely algebraic question of finding invariant subspaces of finite linear groups.

In this context, let $A \in \mathbb{Z}_p^{n,n}$ be an $n \times n$ matrix over a field $\mathbb{Z}_p$, acting as a linear transformation on the column vector space $\mathbb{Z}_p^{n,1}$. Next, let $\kappa_A(x) = f_1(x)^{n_1} f_2(x)^{n_2} \cdots f_k(x)^{n_k}$ be the characteristic polynomial and $m_A(x) = f_1(x)^{s_1} f_2(x)^{s_2} \cdots f_k(x)^{s_k}$ the minimal polynomial of A where polynomials f_i are pairwise distinct and irreducible over $\mathbb{Z}_p$. Then $\mathbb{Z}_p^{n,1}$ can be written as a direct sum of the A-invariant subspaces

$$\mathbb{F}^{n,1} = \mathrm{Ker} f_1(A)^{s_1} \oplus \mathrm{Ker} f_2(A)^{s_2} \oplus \cdots \oplus \mathrm{Ker} f_k(A)^{s_k}.$$

Moreover, all A-invariant subspaces appear as direct sums of some A-invariant subspaces of $\mathrm{Ker} f_i(A)^{s_i}$.

As for finding common invariant subspaces of a finite linear group, we can often exploit Maschke's theorem which states that if the characteristic of the field does not divide the order of the group, then the representation is completely reducible. In this case one essentially needs to find just the minimal common invariant subspaces. In particular, if the order of the matrix A is not divisible by p, each A-invariant subspace of $\mathbb{Z}_p^{n,1}$ is a direct sum of the minimal ones. For a more detailed description of finding invariant subspaces we refer the reader to [10].

3 Sectional split extensions

We start by giving a more precise definition of a sectional split extension mentioned in the Introduction. Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, and let Ω be a nonempty set of vertices of X. A *section* over Ω is a set of vertices $\bar{\Omega}$ of $\tilde{X}$ containing exactly one vertex from each vertex fibre over Ω. Further, let G be a group of automorphisms of X. Assuming that Ω is invariant under the action of G, we say that G lifts along $\wp$ to $\tilde{G}$ as a *sectional split extension* over Ω if the following two conditions are met: (a) G lifts along $\wp$ and (b) there exist a complement $\overline{G}$ to $\mathrm{CT}(\wp)$ within $\tilde{G}$ and a section $\bar{\Omega}$ over Ω that is invariant under the action of $\overline{G}$. Such a complement is called *sectional* over Ω. The necessary and sufficient conditions for G to lift as a sectional split extension over Ω in terms of voltages were given by Malnič et al. This is summarized in the following theorem.

Theorem 3.1. ([16, Theorem 9.1, Theorem 9.3]) With the notation and assumptions above, a group G lifts along $\wp$ as a sectional split extension over Ω if and only if $\wp$ can be reconstructed by a voltage assignment ζ on X such that the following condition

$$\zeta_W = 1 \Rightarrow \zeta_{gW} = 1 \tag{3.1}$$

holds for each automorphism $g \in G$ and each walk W in X with both its endpoints in Ω .

Firstly, note that this theorem is an extended version of an old result of Biggs [1], retold in a different language. Secondly, Malnič took this result further in [17], and used it to sketch a method for testing whether G lifts along $\wp$ as a sectional split extension over Ω. The approach is based on introducing a new vertex joined to every vertex of Ω, and then converting condition (3.1) to the general lifting problem (but no proof is given). In order to exploit this idea in another direction (see below), we introduce the following notation.

The cone $\widehat{X}(\Omega)$ over the graph X is the graph obtained by adding a new vertex $*$ joined to every vertex of Ω. Assuming that Ω is invariant under the action of G, we denote by $\widehat{G}$ the group of automorphisms of $\widehat{X}(\Omega)$ that fixes $*$ and acts on X as the group G. Also, for any voltage assignment ζ on X, we extend ζ to a voltage assignment $\widehat{\zeta}$ on $\widehat{X}(\Omega)$ by assigning the trivial voltage to the extra darts. More precisely,

$$\widehat{\zeta}_x = \begin{cases} \zeta_x, & x \in D_X; \\ 1, & x \in D_{\widehat{X}(\Omega)} \backslash D_X. \end{cases}$$

Conversely, for a voltage assignment ζ on $\widehat{X}(\Omega)$ being trivial on the set of extra darts we denote by $\bar{\zeta}$ the restriction of ζ to X. Clearly, if ζ is not trivial on the set of extra darts, then we can always find an equivalent assignment that is. For example, we may choose a spanning tree $\mathcal{T}^*$ of $\widehat{X}(\Omega)$ such that all extra darts are included in $\mathcal{T}^*$, and then take an equivalent $\mathcal{T}^*$-reduced voltage assignment. Moreover, the following holds.

Proposition 3.2. Let ζ and ζ' be two equivalent connected voltage assignments on $\widehat{X}(\Omega)$, that are trivial on the set of extra darts $D_{\widehat{X}(\Omega)} \backslash D_X$. Then their restrictions $\bar{\zeta}$ and $\bar{\zeta}'$ to X are also equivalent. Hence they are either both connected or both disconnected.

Proof. By definition of equivalence, there exists an isomorphism $\tilde{g}$ from the derived graph $\mathrm{Cov}(\zeta)$ to the derived graph $\mathrm{Cov}(\zeta')$ such that $\wp_\zeta = \tilde{g}\wp_{\zeta'}$. Clearly, $\tilde{g}$ maps the vertex fibre $\wp_\zeta^{-1}(*)$ to the vertex fibre $\wp_{\zeta'}^{-1}(*)$. Therefore, when restricting to X, the isomorphism $\tilde{g}$ induces an isomorphism from the derived graph $\mathrm{Cov}(\bar{\zeta})$ to $\mathrm{Cov}(\bar{\zeta}')$ that gives rise to an equivalence of $\bar{\zeta}$ and $\bar{\zeta}'$. It is then obvious that isomorphic graphs are either both connected or both disconnected, as required. □

We are now ready to forge a link between connected regular coverings of X along which G lifts as a sectional split extension over Ω, and connected regular coverings of $\widehat{X}(\Omega)$ admitting the lift of $\widehat{G}$. For completeness, we explicitly record the following theorem and provide the proof.

Theorem 3.3. Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, and let G be a group of automorphisms of X. Suppose that a nonempty subset Ω of vertices of X is invariant under the action of G. Then the group G lifts along $\wp$ as a sectional split extension over Ω if and only if $\wp$ can be reconstructed by a voltage assignment ζ on X such that $\widehat{G}$ lifts along the derived regular covering projection $\wp_{\widehat{\zeta}}\colon \mathrm{Cov}(\widehat{\zeta}) \to \widehat{X}(\Omega)$.

Proof. Suppose that G lifts along $\wp$ as a sectional split extension over Ω. By Theorem 3.1, there exists a voltage assignment ζ on X that reconstructs $\wp$ and satisfies condition (3.1). Extend ζ to a voltage assignment $\widehat{\zeta}$. We will show that $\widehat{G}$ lifts along the projection $\wp_{\widehat{\zeta}}$ derived from $\widehat{\zeta}$. Let W^* be a closed walk at $*$ in $\widehat{X}(\Omega)$ with $\widehat{\zeta}_{W^*} = 1$, and let $g^* \in \widehat{G}$. In view of the basic lifting lemma we need to show that $\widehat{\zeta}_{g^*W^*} = 1$. Write W^* as a concatenation $W^* = W_1^* W_2^* \dots W_k^*$ of closed walks at $*$ such that $W_i^* = P_i W_i Q_i^{-1}$, where $W_i \colon u_i \to v_i$ is a walk in X with both its endpoints u_i and v_i in Ω, while $P_i \colon * \to u_i$ and $Q_i \colon * \to v_i$ are walks of length 1, for $i = 1, 2, \dots, k$. Observe that $\zeta_{W_1}\zeta_{W_2}\dots\zeta_{W_k} = 1$. Now choose a vertex $u_0 \in \Omega$. Let $R_i \colon u_0 \to u_i$ and $S_i \colon u_0 \to v_i$ be walks with $\zeta_{R_i} = \zeta_{S_i} = 1$, for $i = 1, 2, \dots, k$ (note that such walks always exist). Then the product of walks $W = \prod_{i=1}^{k} R_i W_i S_i^{-1}$ is a closed walk at u_0 with $\zeta_W = \zeta_{W_1}\zeta_{W_2}\dots\zeta_{W_k} = 1$. By condition (3.1) we have that $\zeta_{gW} = 1$ as well as $\zeta_{gR_j} = \zeta_{gS_j} = 1$, for $i = 1, 2, \dots, k$. Thus $\zeta_{gW_1}\zeta_{gW_2}\dots\zeta_{gW_k} = 1$ implies that $\widehat{\zeta}_{g^*W^*} = 1$, as required.

Conversely, suppose that $\wp$ is reconstructed by a voltage assignment ζ on X such that $\widehat{G}$ lifts along the covering projection $\wp_{\widehat{\zeta}}$. By Theorem 3.1, it is sufficient to prove that ζ satisfies condition (3.1). Consider a walk $W \colon u \to v$ in X with both its endpoints u and v in Ω such that $\zeta_W = 1$. Let $P \colon * \to u$ and $Q \colon * \to v$ be the (unique) walks of length 1 in $\widehat{X}(\Omega)$. Then the closed walk $W^* = PWQ^{-1}$ at $*$ has voltage $\widehat{\zeta}_{W^*} = 1$. By the basic lifting lemma we have $\widehat{\zeta}_{g^*W^*} = 1$ for any automorphism $g^* \in \widehat{G}$. Hence $\zeta_{gW} = 1$, completing the proof. □

Coming back to methods for testing whether G lifts along $\wp$ as a sectional split extension over Ω, one possibility would be to use the latter theorem. However, from computational point of view that would be inefficient, since one has to seek for an appropriate voltage assignment that reconstructs the cover. For a more adequate approach to this problem we refer the reader to [23].

As already mentioned, Theorem 3.3 can be efficiently exploited in another direction: given a connected graph X, a group G of its automorphisms, and a nonempty subset $\Omega \subseteq V_X$ invariant under the action of G, find, up to equivalence, all connected regular coverings $\wp \colon \tilde{X} \to X$ along which G lifts as a sectional split extension over Ω. As a first step towards this aim we need to find, in view of Proposition 3.1 and Theorem 3.3, all pairwise nonequivalent connected regular coverings of $\widehat{X}(\Omega)$ along which the group $\widehat{G}$ lifts – combinatorially reconstructed in terms of voltage assignments ζ being trivial on the set of extra darts. Although each ζ is connected – as it reconstructs a connected cover – its restriction $\bar{\zeta}$ to X, however, might be disconnected. Thus, additional testing whether $\bar{\zeta}$ is connected is required. These remarks are formally gathered in the following theorem.

Theorem 3.4. Let X be a connected graph and Ω a nonempty subset of vertices of X that is invariant under the action of a group of automorphisms $G \leq \mathrm{Aut}(X)$. Further, let ζ be a voltage assignment on $\widehat{X}(\Omega)$ that is trivial on the set of extra darts $D_{\widehat{X}(\Omega)} \backslash D_X$ and gives rise to a connected regular covering projection along which the group $\widehat{G}$ lifts. If the restriction $\bar{\zeta}$ to X is connected, then G lifts along the derived regular covering projection $\wp_{\bar{\zeta}}$ as a sectional split extension over Ω. Moreover, any connected regular covering of X along which G lifts as a sectional split extension over Ω arises in this way.

Remark 3.5. Even if ζ and ζ' are two nonequivalent connected assignments on $\widehat{X}(\Omega)$ such that their restrictions $\bar{\zeta}$ and $\bar{\zeta}'$ to X are connected, it still might happened that $\bar{\zeta}$ and $\bar{\zeta}'$ are equivalent. Thus, additional testing is needed.

Now we can more precisely summarize our approach. First, construct all voltage assignments ζ on $\widehat{X}(\Omega)$ giving rise to pairwise nonequivalent connected regular covering projections along which $\widehat{G}$ lifts. Next, consider their restrictions $\bar{\zeta}$ to X and remove the disconnected ones. Finally, do further reduction to obtain all voltage assignments on X giving rise to pairwise nonequivalent connected regular covering projections along which G lifts as a sectional split extension over Ω.

4 Elementary abelian regular covers of K_4

In light of the discussion in Section 3 we now give an example to illustrate our approach. Let $X = K_4$ be the complete graph on the vertex set $V_X = \{1, 2, 3, 4\}$, and let $\Omega = V_X$. Further, denote by $g = (1234) \in \mathrm{Aut}(X)$ the automorphism of X. We compute all voltage assignments on X giving rise to pairwise nonequivalent connected elementary abelian regular covering projections along which the cyclic group $G = \langle g \rangle$ lifts as a sectional split extension over Ω.

To start with, we need to find all voltage assignments on $\widehat{X}(\Omega)$ giving rise to pairwise nonequivalent connected elementary abelian regular coverings along which the group $\widehat{G} = \langle g^* \rangle$ lifts. Let $\mathcal{T}^*$ be the spanning tree of $\widehat{X}(\Omega)$ consisting of all extra darts, and let

$$x_1 = (1, 2),\ x_2 = (2, 3),\ x_3 = (3, 4),\ x_4 = (4, 1),\ x_5 = (2, 4),\ x_6 = (3, 1)$$

denote the six cotree darts of $\widehat{X}(\Omega)$. Denote by $\mathcal{B}_{\mathcal{T}^*} = \{\vec{x}_i \,|\, 1 \le i \le 6\}$ the ordered basis of the vector space $H_1(\widehat{X}(\Omega); \mathbb{Z}_p)$, where $\vec{x}_i$ is the cycle arising from the spanning tree $\mathcal{T}^*$ and the dart x_i. Next, in view of the remarks given in Preliminaries, let $(g^*)^\#$ be the linear transformation of $H_1(\widehat{X}(\Omega); \mathbb{Z}_p)$ induced by the natural action of g^* on $H_1(\widehat{X}(\Omega); \mathbb{Z}_p)$, and let $M_{g^*} \in \mathbb{Z}_p^{6,6}$ be its matrix representation with respect to the basis $\mathcal{B}_{\mathcal{T}^*}$. By computation we obtain that

$$A = M_{g^*}^t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

By Theorem 2.1, we need to find A-invariant subspaces of $\mathbb{Z}_p^{6,1}$. However, note that every elementary abelian regular $\mathbb{Z}_p^{d,1}$-cover of X is disconnected if the dimension d is higher that the Betti number of X. Since the Betti number of X is three, it is therefore enough to find all A-invariant subspaces of dimension at most three. These subspaces define $\mathcal{T}^*$-reduced voltage assignments

$$\zeta \colon D_{\widehat{X}(\Omega)} \to \mathbb{Z}_p^{d,1}, \quad d = 1, 2, 3$$

on $\widehat{X}(\Omega)$ that give rise to pairwise nonequivalent connected regular coverings of $\widehat{X}(\Omega)$ along which $\widehat{G}$ lifts. In addition, as already explained in the previous section, their restrictions $\bar{\zeta}$ to X might still be disconnected as well as connected but equivalent.

In order to test whether the restriction $\bar{\zeta}$ to X stays connected, let $\mathcal{T}$ be the spanning tree of X consisting of the edges $\{1, 2\}$, $\{1, 3\}$ and $\{1, 4\}$. Denote by C_1, C_2 and C_3 cycles arising from the spanning tree $\mathcal{T}$ and darts x_2, x_3 and x_5, respectively. The connectedness condition, relative to the ordered basis $\mathcal{B}_{\mathcal{T}} = \{C_1, C_2, C_3\}$ of $H_1(X; \mathbb{Z}_p)$, translates to the requirement that the voltages

$$\begin{aligned} \bar{\zeta}_{C_1} &= \zeta_{C_1} = \zeta_{x_1} + \zeta_{x_2} + \zeta_{x_6}, \\ \bar{\zeta}_{C_2} &= \zeta_{C_2} = \zeta_{x_3} + \zeta_{x_4} - \zeta_{x_6}, \\ \bar{\zeta}_{C_3} &= \zeta_{C_3} = \zeta_{x_1} + \zeta_{x_4} + \zeta_{x_5} \end{aligned}$$

generate the voltage group $\mathbb{Z}_p^{d,1}$. As for the test of equivalence, let ζ and ζ' be two $\mathcal{T}^*$-reduced voltage assignments on $\widehat{X}(\Omega)$ arising from two different d-dimensional subspaces U and U' of $\mathbb{Z}_p^{6,1}$, respectively. Suppose that their restrictions $\bar{\zeta}$ and $\bar{\zeta}'$ are connected. Then $\bar{\zeta}$ and $\bar{\zeta}'$ are equivalent, in view of [27], if and only if there exists an automorphism of $\mathbb{Z}_p^{d,1}$ mapping

$$\zeta_{C_1} \mapsto \zeta'_{C_1},\ \zeta_{C_2} \mapsto \zeta'_{C_2},\ \text{and } \zeta_{C_3} \mapsto \zeta'_{C_3}.$$

For the purpose of finding A-invariant subspaces, note that $\kappa_A(x) = (x^4 - 1)(x^2 + 1)$ is the characteristic polynomial of A, while

$$m_A(x) = x^4 - 1$$

is its minimal polynomial. Further, observe that the factorization of $m_A(x)$ into irreducible factors over $\mathbb{Z}_p$ depends on the congruence class of p modulo 4, namely

$$m_A(x) = \begin{cases} (x-1)(x+1)(x^2+1), & p \equiv 3 \pmod 4; \\ (x-1)(x+1)(x-i)(x+i), & p \equiv 1 \pmod 4,\ i^2 = -1; \\ (x-1)^4, & p = 2. \end{cases}$$

Therefore the analysis splits into three cases.

Case $p \equiv 3 \pmod 4$.

In this case the representation of the group $\langle A \rangle$ is completely reducible, by Maschke's theorem. The eigenvalues are 1 and -1, both of multiplicity 1. The respective eigenspaces are $\mathcal{L}_A(1) = \langle v_1 \rangle$ and $\mathcal{L}_A(-1) = \langle v_2 \rangle$, where

$$v_1 = (1, 1, 1, 1, 0, 0)^t \text{ and } v_2 = (1, -1, 1, -1, 0, 0)^t.$$

The whole space splits into a direct sum of A-invariant subspaces

$$\mathbb{Z}_p^{6,1} = \mathcal{L}_A(1) \oplus \mathcal{L}_A(-1) \oplus \mathrm{Ker}(A^2 + I).$$

It is obvious that the 1-dimensional A-invariant subspaces are $\mathcal{L}_A(1)$ and $\mathcal{L}_A(-1)$. The respective lists of voltages for the base homology cycles C_1, C_2, C_3 in X are $2, 2, 2$ for the one arising from $\mathcal{L}_A(1)$, and $0, 0, 0$ for the one arising from $\mathcal{L}_A(-1)$. Thus, only $\mathcal{L}_A(1)$ gives rise to a connected cover of X, while $\mathcal{L}_A(-1)$ does not.

Since the 2-dimensional A-invariant subspace arising from the direct sum $\mathcal{L}_A(1) \oplus \mathcal{L}_A(-1)$ does not give a connected cover of X, all others are necessarily contained in $\mathrm{Ker}(A^2 + I)$. These subspaces are of the form $\langle v, Av \rangle$, for $v \in \mathrm{Ker}(A^2 + I)$. There are

p^2+1 distinct subspaces. To check which of these give rise to connected covers of X, choose a basis of $\mathrm{Ker}(A^2+I)$, for instance

$$\begin{aligned} b_1 &= (1,0,-1,0,0,0)^t,\\ b_2 &= (0,1,0,-1,0,0)^t,\\ b_3 &= (0,0,0,0,1,0)^t,\\ b_4 &= (0,0,0,0,0,1)^t. \end{aligned}$$

An arbitrary vector $v \in \mathrm{Ker}(A^2+I)$ is then of the form $v = (a,b,-a,-b,c,d)^t$, for some $a,b,c,d \in \mathbb{Z}_p$, while $Av = (b,-a,-b,a,d,-c)^t$. For convenience we denote

$$W_{a,b,c,d} = \langle (a,b,-a,-b,c,d)^t, (b,-a,-b,a,d,-c)^t \rangle.$$

Checking for connectedness gives that

$$(a,b)^t+(b,-a)^t+(d,-c)^t,\ (-a,-b)^t+(-b,a)^t-(d,-c)^t \text{ and } (a,b)^t+(-b,a)^t+(c,d)^t$$

should generate $\mathbb{Z}_p^{2,1}$. The condition is reduced to requiring that $(a+b+d,-a+b-c)^t$ and $(a-b+c,a+b+d)^t$ are linearly independent in $\mathbb{Z}_p^{2,1}$. Let $x = a+b+d$ and $y = a-b+c$. The vectors $(x,y)^t$ and $(-y,x)^t$ are linearly dependent if and only if $x^2+y^2 \equiv 0 \pmod p$ Since $p \equiv 3 \pmod 4$, we must have $x \equiv 0 \pmod p$ and $y \equiv 0 \pmod p$. Thus a disconnected cover of X is obtained if and only if $c = -a+b$ and $d = -a-b$; in this case $W_{a,b,c,d}$ is generated by

$$v_{a,b} = a(1,0,-1,0,-1,-1)^t + b(0,-1,0,1,-1,1)^t \text{ and } Av_{a,b}.$$

Observe that any $v_{a,b}$ is contained in $\langle v_{1,0}, Av_{1,0}\rangle$. Hence $\langle v_{a,b}, Av_{a,b}\rangle = \langle v_{1,0}, Av_{1,0}\rangle$ for all $a,b \in \mathbb{Z}_p$. This is therefore the only A-invariant 2-dimensional subspace giving rise to a disconnected cover of X. As for the remaining subspaces, these are $W_{a,b,c,d}$ where $(c,d) \neq (-a+b,-a-b)$. Furthermore, these subspaces all give rise to equivalent coverings of X. Indeed. Choose one of these subspaces, say

$$W_{1,1,0,0} = \langle (1,1,-1,-1,0,0)^t, (1,-1,-1,1,0,0)^t \rangle.$$

Let ζ and ζ' be two assignments arising from $W_{a,b,c,d}$ and $W_{1,1,0,0}$, respectively. The base homology cycles C_1, C_2, C_3 in X have the following voltages

$$\begin{aligned} \zeta_{C_1} &= (a+b+d,-a+b-c)^t, & \zeta'_{C_1} &= (2,0)^t,\\ \zeta_{C_2} &= (-a-b-d,a-b+c)^t, & \zeta'_{C_2} &= (-2,0)^t,\\ \zeta_{C_3} &= (a-b+c,a+b+d)^t, & \zeta'_{C_3} &= (0,2)^t. \end{aligned}$$

By computation one can check that there exists a matrix in $\mathrm{GL}(2,\mathbb{Z}_p) \cong \mathrm{Aut}(\mathbb{Z}_p^{2,1})$ taking $\zeta_{C_1}, \zeta_{C_2}, \zeta_{C_3}$ to $\zeta'_{C_1}, \zeta'_{C_2}, \zeta'_{C_3}$, respectively, if and only if $(c,d) \neq (-a+b,-a-b)$, and the claim is proved. As a representative of the above 2-dimensional subspaces we take $W_{1,1,0,0}$.

Any 3-dimensional A-invariant subspace giving rise to a connected cover of X is equivalent to the homological cover of X. So it is enough to find one such a subspace, if it exists. For instance, we may take the subspace $\mathcal{L}_A(1) \oplus W_{1,1,0,0}$, as the reader can easily check.

Case $p \equiv 1 \pmod 4$.

The representation of the group $\langle A \rangle$ is again completely reducible, by Maschke's theorem. The matrix A is diagonalizable, having the diagonal form $\mathrm{diag}_A(1, -1, i, i, -i, -i)$.

Clearly, the 1-dimensional eigenspaces $\mathcal{L}_A(1)$ and $\mathcal{L}_A(-1)$ are the same as before, where only $\mathcal{L}_A(1)$ gives rise to a connected cover of X. As for the eigenvalues i and $-i$ satisfying $i^2 \equiv -1 \pmod p$, the respective eigenspaces $\mathcal{L}_A(i) = \langle u_i, v_i \rangle$ and $\mathcal{L}_A(-i) = \langle u_{-i}, v_{-i} \rangle$ are 2-dimensional, where

$$u_i = (1, i, -1, -i, 1, i)^t, \qquad u_{-i} = (1, -i, -1, i, 1, -i)^t,$$
$$v_i = (1, i, -1, -i, 0, 0)^t, \qquad v_{-i} = (1, -i, -1, i, 0, 0)^t.$$

The 1-dimensional subspaces in $\mathcal{L}_A(i)$ can be conveniently parametrized as

$$W_\infty(i) = \langle u_i \rangle,$$
$$W_s(i) = \langle s u_i + v_i \rangle = \langle (s+1, (s+1)i, -(s+1), -(s+1)i, s, si)^t \rangle, \quad s \in \mathbb{Z}_p,$$

while those in $\mathcal{L}_A(-i)$ can be parametrized as

$$W_\infty(-i) = \langle u_{-i} \rangle,$$
$$W_s(-i) = \langle s u_{-i} + v_{-i} \rangle = \langle (s+1, -(s+1)i, -(s+1), (s+1)i, s, -si)^t \rangle, \quad s \in \mathbb{Z}_p.$$

The conditions for connectedness of covers of X arising from $W_\infty(i)$, $W_s(i)$, $W_\infty(-i)$ and $W_s(-i)$ become $i - 2 \not\equiv 0 \pmod p$, $s(i-2) \not\equiv 1 - i \pmod p$, $-i - 2 \not\equiv 0 \pmod p$ and $s(-i-2) \not\equiv 1 + i \pmod p$, respectively. We need to consider subcases $p \neq 5$ and $p = 5$ separately.

Let $p \neq 5$. Then $i, -i \neq 2$, and there are $(2p+1)$ 1-dimensional subspaces giving rise to connected covers of X, namely the set

$$\mathcal{W}_i = \{W_s(i) \mid s \in (\mathbb{Z}_p \backslash \{(1-i)(i-2)^{-1}\}) \cup \{\infty\}\}$$

of p subspaces in $\mathcal{L}_A(i)$, the set

$$\mathcal{W}_{-i} = \{W_s(-i) \mid s \in (\mathbb{Z}_p \backslash \{(1+i)(-i-2)^{-1}\}) \cup \{\infty\}\}$$

of p subspaces in $\mathcal{L}_A(-i)$, and the subspace $\mathcal{L}_A(1)$. However, all subspaces in $\mathcal{W}_i$ give rise to equivalent coverings of X. To show this, let ζ and ζ' be two assignments arising from $W_s(i)$ and $W_\infty(i)$, respectively. By computation we have

$$\zeta_{C_1} = (s+1)(1+i) + si, \qquad \zeta'_{C_1} = 1 + 2i,$$
$$\zeta_{C_2} = -(s+1)(1+i) - si = -\zeta_{C_1}, \qquad \zeta'_{C_2} = -1 - 2i = -\zeta'_{C_1},$$
$$\zeta_{C_3} = (s+1)(1-i) + s = -i\zeta_{C_1}, \qquad \zeta'_{C_3} = 2 - i = -i\zeta'_{C_1}.$$

Clearly, there exists an automorphism of $\mathbb{Z}_p$ taking $\zeta_{C_1}, \zeta_{C_2}, \zeta_{C_3}$ to $\zeta'_{C_1}, \zeta'_{C_2}, \zeta'_{C_3}$, respectively, if and only if $\zeta_{C_1} \neq 0$. In fact, we do have $\zeta_{C_1} \neq 0$ since $s \neq (1-i)(i-2)^{-1}$. Similarly, all subspaces in $\mathcal{W}_{-i}$ give rise to equivalent coverings of X. As a representative in $\mathcal{W}_i$ we choose $W_0(i)$, while in $\mathcal{W}_{-i}$ we choose $W_0(-i)$. In fact, there are exactly three pairwise nonequivalent connected coverings of X, namely the one arising from $\mathcal{L}_A(1)$, and the two coverings arising from $W_0(i)$ and $W_0(-i)$. The respective lists of voltages for the base homology cycles C_1, C_2, C_3 in X are $2, 2, 2$ for the one arising from $\mathcal{L}_A(1)$, while $1+i, -1-i, 1-i$ and $1-i, -1+i, 1+i$ for the other two covers. The reader may check that there is no automorphism of $\mathbb{Z}_p$ taking any of these triples to any other.

Let $p = 5$. Then for each $s \in \mathbb{Z}_5$ the subspace $W_s(2)$ gives rise to a connected cover of X, while the subspace $W_\infty(2)$ does not. On the other hand, for each $s \neq 3$ we obtain a connected cover of X arising from $W_s(3)$, and one connected cover of X arising from $W_\infty(3)$. Together with the cover of X arising from $\mathcal{L}_A(1)$ we therefore have $2p+1 = 11$ connected covers. If ζ denotes an assignment arising from $W_s(2)$, then the base homology cycles in X have voltages $\zeta_{C_1} = 3(s+1) + 2s = 3$, $\zeta_{C_2} = -3(s+1) - 2s = -\zeta_{C_1}$, $\zeta_{C_3} = -(s+1) + s = -2\zeta_{C_1}$. It is obvious that the subspaces $W_s(2)$, $s \in \mathbb{Z}_5$, give rise to equivalent coverings of X. As a representative we take $W_0(2)$. Let now ζ be an assignment arising from $W_s(3)$, where $s \in \mathbb{Z}_p$ and $s \neq 3$. Further, let ζ' denote an assignment arising from $W_\infty(3)$. Then we have

$$\begin{aligned} \zeta_{C_1} &= 4(s+1) + 3s = 2s - 1, & \zeta'_{C_1} &= 2, \\ \zeta_{C_2} &= -4(s+1) - 3s = -\zeta_{C_1}, & \zeta'_{C_2} &= -2 = -\zeta'_{C_1}, \\ \zeta_{C_3} &= -2(s+1) + s = -3\zeta_{C_1}, & \zeta'_{C_3} &= -1 = -3\zeta'_{C_1}. \end{aligned}$$

Clearly, multiplication by $s+2$ takes $\zeta_{C_1}, \zeta_{C_2}, \zeta_{C_3}$ to $\zeta'_{C_1}, \zeta'_{C_2}, \zeta'_{C_3}$, respectively. As a representative we take $W_0(3)$. The reader may check that $\mathcal{L}_A(1), W_0(2)$ and $W_0(3)$ give rise to pairwise nonequivalent coverings of X.

Let us now consider the 2-dimensional subspaces. We shall need the following lemma.

Lemma 4.1. Let $\mathcal{T}^*$ be a spanning tree of $\widehat{X}(\Omega)$ such that all extra darts are included in $\mathcal{T}^*$, and let the sequence $x_1, x_2, \ldots, x_n$ contain exactly one dart from each edge not contained in $\mathcal{T}^*$. Further, let U, U', W, W' be subspaces of $\mathbb{Z}_p^{n,1}$ such that $U \cap W = \{0\} = U' \cap W'$, and let $\zeta^U, \zeta^{U'}, \zeta^W, \zeta^{W'}, \zeta^{U\oplus W}, \zeta^{U'\oplus W'}$ denote $\mathcal{T}^*$-reduced voltage assignments on $\widehat{X}(\Omega)$, where the voltages of darts x_i arise from U, U', $W, W', U \oplus W, U' \oplus W'$, respectively. Suppose that all their restrictions to X are connected. If the restrictions of ζ^U and $\zeta^{U'}$ are equivalent and the restrictions of ζ^W and $\zeta^{W'}$ are equivalent, then the restrictions of $\zeta^{U\oplus W}$ and $\zeta^{U'\oplus W'}$ are also equivalent.

Proof. Since $U \cap W = \{0\} = U' \cap W'$ we may assume, up to equivalence of regular covering projections, that

$$\zeta_x^{U\oplus W} = \begin{bmatrix} \zeta_x^U \\ \zeta_x^W \end{bmatrix} \text{ and } \zeta_x^{U'\oplus W'} = \begin{bmatrix} \zeta_x^{U'} \\ \zeta_x^{W'} \end{bmatrix},$$

for all darts x in $\widehat{X}(\Omega)$. Let r be the Betti number of X, and let $C_1, C_2, \ldots, C_r$ be an ordered basis of $H_1(X, \mathbb{Z}_p)$. Since the restrictions of ζ^U and $\zeta^{U'}$ are equivalent, there exists an invertible matrix A mapping voltages $\zeta^U_{C_i}$ to voltages $\zeta^{U'}_{C_i}$, $i = 1, 2, \ldots, r$. Similarly, there exists an invertible matrix B mapping voltages $\zeta^W_{C_i}$ to voltages $\zeta^{W'}_{C_i}$, $i = 1, 2, \ldots, r$. Then the matrix

$$\begin{bmatrix} A & \\ & B \end{bmatrix}$$

is invertible and clearly takes voltages $\zeta^{U\oplus W}_{C_i}$ to voltages $\zeta^{U'\oplus W'}_{C_i}$. Hence the restrictions of $\zeta^{U\oplus W}$ and $\zeta^{U'\oplus W'}$ to X are equivalent. □

In order to test which 2-dimensional subspaces give rise to connected and possibly equivalent coverings of X we only need to check, by Lemma 4.1, the subspaces $\mathcal{L}_A(i)$, $\mathcal{L}_A(-i)$, and the following direct sums:

$$\mathcal{L}_A(1) \oplus W_0(i),\ \mathcal{L}_A(1) \oplus W_0(-i),\ W_0(i) \oplus W_0(-i).$$

First note that the two subspaces $\mathcal{L}_A(i)$ and $\mathcal{L}_A(-i)$ give rise to disconnected covers of X since each can be written as a direct sum of two 1-dimensional subspaces, of which one gives rise to a disconnected cover of X. As for the remaining subspaces, they all give rise to connected and pairwise nonequivalent coverings of X. We leave this to the reader.

Finally, any 3-dimensional subspace giving rise to a connected cover of X is equivalent to the homological cover over X. So it is enough to identify one such subspace. The reader can check that $\mathcal{L}_A(1) \oplus W_0(i) \oplus W_0(-i)$ satisfies the connectedness condition. This completes the analysis when p is odd.

Case $p = 2$.

In this case the representation of the group $\langle A\rangle$ is not completely reducible. First we need an appropriate Jordan basis for the matrix A. Observe that the respective Jordan form has two elementary Jordan matrices, one of size 4 and one of size 2. By computation, a Jordan basis is, say,

$$\begin{aligned} v_1 &= (1,1,1,1,0,0)^t, \\ b_1 &= (0,1,0,1,0,0)^t, \\ b_3 &= (0,0,1,1,0,0)^t, \\ b_4 &= (0,0,0,1,0,0)^t, \\ v_2 &= (0,0,0,0,1,1)^t, \\ b_2 &= (0,0,0,0,1,0)^t, \end{aligned}$$

where v_1 and v_2 are the eigenvectors, the 4-dimensional cyclic subspace is spanned by v_1, b_1, b_3, b_4, and the 2-dimensional one by v_2, b_2.

There are exactly three 1-dimensional A-invariant subspaces, all contained in the 2-dimensional eigenspace $\mathcal{L}_A(1)$, namely $W_\infty(1) = \langle v_1\rangle$, $W_0(1) = \langle v_2\rangle$ and $W_1(1) = \langle (1,1,1,1,1,1)^t\rangle$. Only the latter two give rise to connected covers of X. Moreover, both also give rise to equivalent coverings of X. As a representative we choose, say, $W_1(1)$. The resulting cover is the canonical double cover.

As for the 2-dimensional A-invariant subspaces, there are exactly seven of them. One is the eigenspace $\mathcal{L}_A(1) = \langle v_1, v_2\rangle$. The other six subspaces arise from vectors $u \in \mathrm{Ker}(A - I)^2 \setminus \mathcal{L}_A(1)$. Such a 2-dimensional subspace consists of the following vectors: $0, u, Au, u + Au$. Clearly $Au \neq u$ (since u is not an eigenvector), and $Au \in \mathrm{Ker}(A - I)^2 \setminus \mathcal{L}_A(1)$ (as $A^2u = Au$ implies $Au = u$). Therefore the elements in $\mathrm{Ker}(A - I)^2 \setminus \mathcal{L}_A(1)$ in the same 2-dimensional subspace come in pairs. As the set $\mathrm{Ker}(A - I)^2 \setminus \mathcal{L}_A(1)$ contains exactly 12 nontrivial vectors, there are exactly six subspaces of this kind. These can be explicitly represented as $\langle v_1, b_1\rangle$, $\langle v_1, u_1\rangle$, $\langle v_2, b_2\rangle$, $\langle v_2, u_2\rangle$, $\langle v_1 + v_2, u_3\rangle$, $\langle v_1 + v_2, u_4\rangle$, where

$$\begin{aligned} u_1 &= (0,1,0,1,1,1)^t, \\ u_1 &= (1,1,1,1,1,0)^t, \\ u_3 &= (1,0,1,0,0,1)^t, \\ u_4 &= (1,0,1,0,1,0)^t. \end{aligned}$$

The reader can check that the subspaces giving rise to connected covers of X are pairwise equivalent. As a representative we choose, say, $\langle v_2, b_2\rangle$.

Consider now the 3-dimensional A-invariant subspaces. It is enough to find just one (if it exists) giving rise to a connected cover of X (which is then equivalent to the homological cover of X). However, the reader can check that all 3-dimensional subspaces give rise to disconnected covers of X. To this end we only provide a basis for each of them. Note that there are seven 3-dimensional subspaces in all. Indeed, three such subspaces exist in $\mathrm{Ker}(A-I)^2$, namely

$$\langle v_1, v_2, b_1\rangle, \langle v_1, v_2, b_2\rangle, \text{ and } \langle v_1, v_2, u_4\rangle,$$

each containing $\mathcal{L}_A(1)$. The other four arise as cyclic subspaces of the Jordan chains of length 3 (note that there are 16 chains in all, and A acts semi-regularly on the set of vectors in $\mathrm{Ker}(A-I)^3 \setminus \mathrm{Ker}(A-I)^2$ with four orbits of size 4). The respective bases are $\{v_1, b_1, b_3\}$, $\{v_1, b_1, u_5\}$, $\{v_1, u_6, u_7\}$, $\{v_1, u_6, u_8\}$, where

$$\begin{aligned} u_5 &= (0,0,1,1,1,1)^t, \\ u_6 &= (1,0,1,0,1,1)^t, \\ u_7 &= (1,0,0,1,0,1)^t, \\ u_8 &= (1,0,0,1,1,0)^t. \end{aligned}$$

This completes the analysis for $p = 2$.

Remark 4.2. In order to further reduce these coverings up to isomorphism we can follow (ii) of Theorem 2.1. The possibility that the projections in Table 1 are isomorphic is converted to checking the pairs in rows $2, 3, 4$ and those in rows $7, 8, 9$. The reader can check that rows 3 and 4 give rise to isomorphic covers as well as rows 7 and 8.

Remark 4.3. Consider the automorphism $h = (12)$ of X. Clearly, g and h generate the full automorphism group $\mathrm{Aut}(X)$. Let $(h^*)^\#$ be the linear transformation of $H_1(\widehat{X}(\Omega); \mathbb{Z}_p)$ induced by the natural action of h^* on $H_1(\widehat{X}(\Omega); \mathbb{Z}_p)$, and let $M_{h^*} \in \mathbb{Z}_p^{6,6}$ be its matrix representation with respect to the basis $\mathcal{B}_{\mathcal{T}^*}$. By computation we have that

$$M_{h^*}^t = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is now easy to check that among subspaces in Table 1 only $W_1(1)$ is also $M_{h^*}^t$-invariant. Thus, the canonical double cover of X is the only covering along which the full automorphism group $\mathrm{Aut}(X)$ lifts as a sectional split extension over Ω.

Table 1: All voltage assignments on K_4 giving rise to pairwise nonequivalent connected elementary abelian regular covering projections along which the cyclic group $\langle g\rangle$ of automorphisms of K_4 lifts as a sectional split extension. Additionally, coverings in rows 3 and 4 are isomorphic, as well as those in rows 7 and 8.

n	Inv. subsp.	ζ_{x_1}	ζ_{x_2}	ζ_{x_3}	ζ_{x_4}	ζ_{x_5}	ζ_{x_6}	Condition
1	$W_1(1)$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$p=2$
2	$\langle v_1\rangle$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$	$p\neq 2$
3	$\langle v_i\rangle$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}i\end{bmatrix}$	$\begin{bmatrix}-1\end{bmatrix}$	$\begin{bmatrix}-i\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$	$p\equiv 1 \pmod 4$, $i^2=-1$
4	$\langle v_{-i}\rangle$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}-i\end{bmatrix}$	$\begin{bmatrix}-1\end{bmatrix}$	$\begin{bmatrix}i\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$	$p\equiv 1 \pmod 4$, $i^2=-1$
5	$\langle v_2, b_2\rangle$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$p=2$
6	$W_{1,1,0,0}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$p\equiv 3 \pmod 4$
7	$\langle v_1, v_i\rangle$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\i\end{bmatrix}$	$\begin{bmatrix}1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\-i\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$p\equiv 1 \pmod 4$, $i^2=-1$
8	$\langle v_1, v_{-i}\rangle$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\-i\end{bmatrix}$	$\begin{bmatrix}1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\i\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$p\equiv 1 \pmod 4$, $i^2=-1$
9	$\langle v_i, v_{-i}\rangle$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}i\\-i\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}-i\\i\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$p\equiv 1 \pmod 4$, $i^2=-1$
10	$\langle v_1, W_{1,1,0,0}\rangle$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$p\equiv 3 \pmod 4$
11	$\langle v_1, v_i, v_{-i}\rangle$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\i\\-i\end{bmatrix}$	$\begin{bmatrix}1\\-1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\-i\\i\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$p\equiv 1 \pmod 4$, $i^2=-1$

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Augmented down-up algebras and uniform posets

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Abstract

Motivated by the structure of the uniform posets we introduce the notion of an augmented down-up (or ADU) algebra. We discuss how ADU algebras are related to the down-up algebras defined by Benkart and Roby. For each ADU algebra we give two presentations by generators and relations. We also display a $\mathbb{Z}$-grading and a linear basis. In addition we show that the center is isomorphic to a polynomial algebra in two variables. We display seven families of uniform posets and show that each gives an ADU algebra module in a natural way. The main inspiration for the ADU algebra concept comes from the second author's thesis concerning a type of uniform poset constructed using a dual polar graph.

Keywords: Uniform poset, dual polar space, dual polar graph, down-up algebra.

Math. Subj. Class.: 06A07, 05E10, 17B37

1 Introduction

In [10] the first author introduced the notion of a uniform poset, and constructed eleven families of examples from the classical geometries. Among the examples are the polar spaces $\mathrm{Polar}_b(N, \epsilon)$ and the attenuated spaces $\mathrm{A}_b(N, M)$, as well as the posets $\mathrm{Alt}_b(N)$, $\mathrm{Her}_q(N)$, and $\mathrm{Quad}_b(N)$ associated with the alternating, Hermitean, and quadratic forms. Another example is Hemmeter's poset $\mathrm{Hem}_b(N)$. In [12, Proposition 26.4] the second author constructed a new family of uniform posets using the dual polar graphs. We denote these posets by $\mathrm{Polar}_b^{\mathrm{top}}(N, \epsilon)$ and describe them in Section 5 below.

In [2] Benkart and Roby introduced the down-up algebras, and obtained modules for these algebras using $\mathrm{Alt}_b(N)$, $\mathrm{Her}_q(N)$, $\mathrm{Quad}_b(N)$, and $\mathrm{Hem}_b(N)$. A down-up algebra module is obtained from $\mathrm{Polar}_b^{\mathrm{top}}(N, \epsilon)$ in a similar way. However, it appears that the down-up algebra concept is not sufficiently robust to handle $\mathrm{Polar}_b(N, \epsilon)$ or $\mathrm{A}_b(N, M)$. The

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same can be said for the generalized down-up algebras [5]. In the present paper we introduce a family of algebras called augmented down-up algebras, or ADU algebras for short. These algebras seem well suited to handle uniform posets. Indeed, we show that each of the uniform posets $\mathrm{Polar}_b(N,\epsilon)$, $\mathrm{A}_b(N,M)$, $\mathrm{Alt}_b(N)$, $\mathrm{Her}_q(N)$, $\mathrm{Quad}_b(N)$, $\mathrm{Hem}_b(N)$, $\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$ gives an ADU algebra module in a natural way.

The ADU algebras are related to the down-up algebras as follows. Given scalars α, β, γ the corresponding down-up algebra $A(\alpha,\beta,\gamma)$ is defined by generators e, f and relations

$$e^2 f = \alpha efe + \beta fe^2 + \gamma e,$$
$$ef^2 = \alpha fef + \beta f^2 e + \gamma f.$$

See [2, p. 308]. To turn this into an ADU algebra we make three adjustments as follows. Let q denote a nonzero scalar that is not a root of unity. We first require

$$\alpha = q^{-2s} + q^{-2t}, \qquad \beta = -q^{-2s-2t}$$

where s, t are distinct integers. Secondly, we add two generators $k^{\pm 1}$ such that $ke = q^2 ek$ and $kf = q^{-2} fk$. Finally we reinterpret γ as a Laurent polynomial in k for which the coefficients of k^s, k^t are zero.

From the above description the ADU algebras are reminiscent of the quantum univeral enveloping algebra $U_q(\mathfrak{sl}_2)$. To illuminate the difference between these algebras, consider their center. By [6, p. 27] the center of $U_q(\mathfrak{sl}_2)$ is isomorphic to a polynomial algebra in one variable. As we will see, the center of an ADU algebra is isomorphic to a polynomial algebra in two variables.

The results of the present paper are summarized as follows. We define two algebras by generators and relations, and show that they are isomorphic. We call the common resulting algebra an ADU algebra. For each ADU algebra we display a $\mathbb{Z}$-grading and a linear basis. We also show that the center is isomorphic to a polynomial algebra in two variables. We obtain ADU algebra modules from each of the above seven examples of uniform posets.

We have a remark about the place of down-up algebras and ADU algebras in ring theory. A down-up algebra can be viewed as an ambiskew polynomial ring [7, Section 3], which in turn can be viewed as a generalized Weyl algebra [1], [7, Prop. 2.1]. By a comment in [8, p. 48] that cites a preprint version of the present paper, an ADU algebra can also be viewed in this way. Hoping to keep our paper accessible to nonexperts in ring theory, we will avoid this point of view and use only linear algebra.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

2 Augmented down-up algebras

Our conventions for the paper are as follows. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. Let $\mathbb{F}$ denote a field. Let λ denote an indeterminate. Let $\mathbb{F}[\lambda,\lambda^{-1}]$ denote the $\mathbb{F}$-algebra of Laurent polynomials in λ that have all coefficients in $\mathbb{F}$. Pick $\psi \in \mathbb{F}[\lambda,\lambda^{-1}]$ and write $\psi = \sum_{i\in\mathbb{Z}} \alpha_i \lambda^i$. By the *support* of ψ we mean the set $\{i \in \mathbb{Z} | \alpha_i \neq 0\}$. This set is finite.

Fix distinct $s, t \in \mathbb{Z}$. Define

$$\mathbb{F}[\lambda,\lambda^{-1}]_{s,t} = \mathrm{Span}\{\lambda^i | i \in \mathbb{Z},\ i \neq s, i \neq t\}.$$

Note that

$$\mathbb{F}[\lambda,\lambda^{-1}] = \mathbb{F}[\lambda,\lambda^{-1}]_{s,t} + \mathbb{F}\lambda^s + \mathbb{F}\lambda^t \qquad \text{(direct sum)}.$$

For $\psi \in \mathbb{F}[\lambda,\lambda^{-1}]$ the following are equivalent: (i) $\psi \in \mathbb{F}[\lambda,\lambda^{-1}]_{s,t}$; (ii) the integers s, t are not in the support of ψ.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Definition 2.1. For $\varphi \in \mathbb{F}[\lambda,\lambda^{-1}]_{s,t}$ the $\mathbb{F}$-algebra $\mathbb{A} = \mathbb{A}_q(s,t,\varphi)$ has generators e, f, $k^{\pm 1}$ and relations

$$kk^{-1} = 1, \qquad k^{-1}k = 1,$$
$$ke = q^2 ek, \qquad kf = q^{-2} fk,$$
$$e^2 f - (q^{-2s} + q^{-2t}) efe + q^{-2s-2t} fe^2 = e\varphi(k), \tag{2.1}$$
$$ef^2 - (q^{-2s} + q^{-2t}) fef + q^{-2s-2t} f^2 e = \varphi(k) f. \tag{2.2}$$

Remark 2.2. Referring to Definition 2.1, consider the special case in which $\varphi \in \mathbb{F}$. Then the relations (2.1), (2.2) become the defining relations for the down-up algebra $A(q^{-2s} + q^{-2t}, -q^{-2s-2t}, \varphi)$.

Definition 2.3. For $\phi \in \mathbb{F}[\lambda,\lambda^{-1}]_{s,t}$ the $\mathbb{F}$-algebra $\mathbb{B} = \mathbb{B}_q(s,t,\phi)$ has generators C_s, C_t, E, F, $K^{\pm 1}$ and relations

$$C_s,\ C_t \quad \text{are central,}$$
$$KK^{-1} = 1, \qquad K^{-1}K = 1,$$
$$KE = q^2 EK, \qquad KF = q^{-2} FK,$$
$$FE = C_s q^s K^s + C_t q^t K^t + \phi(qK), \tag{2.3}$$
$$EF = C_s q^{-s} K^s + C_t q^{-t} K^t + \phi(q^{-1}K). \tag{2.4}$$

Next we describe how the algebras in Definition 2.1 and Definition 2.3 are related.

Definition 2.4. We define an $\mathbb{F}$-linear map $\mathbb{F}[\lambda,\lambda^{-1}] \to \mathbb{F}[\lambda,\lambda^{-1}]$, $\psi \mapsto \psi_{s,t}$ as follows. For $\psi \in \mathbb{F}[\lambda,\lambda^{-1}]$,

$$\psi_{s,t}(\lambda) = \psi(q^{-1}\lambda) - (q^{-2s} + q^{-2t})\psi(q\lambda) + q^{-2s-2t}\psi(q^3\lambda).$$

Recall the basis $\{\lambda^i\}_{i\in\mathbb{Z}}$ for $\mathbb{F}[\lambda,\lambda^{-1}]$.

Lemma 2.5. *Consider the map $\psi \mapsto \psi_{s,t}$ from Definition 2.4. For $i \in \mathbb{Z}$ the vector λ^i is an eigenvector for the map. The corresponding eigenvalue is $q^{3i}(q^{-2i} - q^{-2s})(q^{-2i} - q^{-2t})$. This eigenvalue is zero if and only if $i \in \{s,t\}$.*

Proof. Use Definition 2.4. □

The following two lemmas are routine consequences of Lemma 2.5.

Lemma 2.6. *For the map $\psi \mapsto \psi_{s,t}$ from Definition 2.4 the image is $\mathbb{F}[\lambda,\lambda^{-1}]_{s,t}$ and the kernel is $\mathbb{F}\lambda^s + \mathbb{F}\lambda^t$.*

Lemma 2.7. *For the map $\psi \mapsto \psi_{s,t}$ from Definition 2.4 the restriction to $\mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ is invertible.*

Let $\varphi, \phi \in \mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ such that $\varphi = \phi_{s,t}$. We are going to show that the algebras $\mathbb{A}_q(s, t, \varphi)$ and $\mathbb{B}_q(s, t, \phi)$ are isomorphic.

Lemma 2.8. *For $\phi \in \mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ the following hold in $\mathbb{B}_q(s, t, \phi)$:*

$$C_s = \frac{q^{-t}FE - q^tEF + q^t\phi(q^{-1}K) - q^{-t}\phi(qK)}{q^{s-t} - q^{t-s}}K^{-s}, \tag{2.5}$$

$$C_t = \frac{q^{-s}FE - q^sEF + q^s\phi(q^{-1}K) - q^{-s}\phi(qK)}{q^{t-s} - q^{s-t}}K^{-t}. \tag{2.6}$$

Moreover the algebra $\mathbb{B}_q(s, t, \phi)$ is generated by $E, F, K^{\pm 1}$.

Proof. We first verify (2.5). In the expression on the right in (2.5), eliminate FE and EF using (2.3) and (2.4). After a routine simplification (2.5) is verified. The equation (2.6) is similarly verified. The last assertion follows from (2.5), (2.6). □

Lemma 2.9. *For $\phi \in \mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ the following hold in $\mathbb{B}_q(s, t, \phi)$:*

$$E^2F - (q^{-2s} + q^{-2t})EFE + q^{-2s-2t}FE^2 = E\varphi(K), \tag{2.7}$$

$$EF^2 - (q^{-2s} + q^{-2t})FEF + q^{-2s-2t}F^2E = \varphi(K)F. \tag{2.8}$$

In the above lines $\varphi = \phi_{s,t}$.

Proof. We first verify (2.7). In the expression on the left in (2.7), view $E^2F = E(EF)$, $EFE = E(FE)$, $FE^2 = (FE)E$ and eliminate each parenthetical expression using (2.3) and (2.4). Simplify the result using $KE = q^2EK$ along with $\varphi = \phi_{s,t}$ and Definition 2.4. The equation (2.7) is now verified. The equation (2.8) is similarly verified. □

The following definition is motivated by Lemma 2.8.

Definition 2.10. For $\varphi \in \mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ let c_s, c_t denote the following elements in $\mathbb{A}_q(s, t, \varphi)$:

$$c_s = \frac{q^{-t}fe - q^tef + q^t\phi(q^{-1}k) - q^{-t}\phi(qk)}{q^{s-t} - q^{t-s}}k^{-s}, \tag{2.9}$$

$$c_t = \frac{q^{-s}fe - q^sef + q^s\phi(q^{-1}k) - q^{-s}\phi(qk)}{q^{t-s} - q^{s-t}}k^{-t}. \tag{2.10}$$

In the above lines ϕ denotes the unique element in $\mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ such that $\varphi = \phi_{s,t}$.

Lemma 2.11. *With the notation and assumptions of Definition 2.10, the elements c_s, c_t are central in $\mathbb{A}_q(s, t, \varphi)$. Moreover*

$$fe = c_sq^sk^s + c_tq^tk^t + \phi(qk), \tag{2.11}$$

$$ef = c_sq^{-s}k^s + c_tq^{-t}k^t + \phi(q^{-1}k). \tag{2.12}$$

Proof. We first show that c_s is central in $\mathbb{A}_q(s,t,\varphi)$. To do this we show $c_s e = ec_s$, $c_s f = fc_s$, $c_s k = kc_s$. To verify these equations, eliminate each occurrence of c_s using (2.9), and simplify the result using the relations in Definition 2.1. We have shown that c_s is central in $\mathbb{A}_q(s,t,\varphi)$. One similarly shows that c_t is central in $\mathbb{A}_q(s,t,\varphi)$. We now verify (2.11). In the expression on the right in (2.11), eliminate c_s, c_t using (2.9), (2.10). After a routine simplification (2.11) is verified. The equation (2.12) is similarly verified. □

Theorem 2.12. *Given $\varphi, \phi \in \mathbb{F}[\lambda, \lambda^{-1}]_{s,t}$ such that $\varphi = \phi_{s,t}$. Then there exists an $\mathbb{F}$-algebra isomorphism $\mathbb{A}_q(s,t,\varphi) \to \mathbb{B}_q(s,t,\phi)$ that sends*

$$e \mapsto E, \qquad f \mapsto F, \qquad k^{\pm 1} \mapsto K^{\pm 1}.$$

The inverse isomorphism sends

$$C_s \mapsto c_s, \qquad C_t \mapsto c_t, \qquad E \mapsto e, \qquad F \mapsto f, \qquad K^{\pm 1} \mapsto k^{\pm 1}$$

where c_s, c_t are from Definition 2.10.

Proof. Combine Lemmas 2.8, 2.9, 2.11. □

Definition 2.13. By an *augmented down-up algebra* we mean an algebra $\mathbb{A}_q(s,t,\varphi)$ from Definition 2.1 or an algebra $\mathbb{B}_q(s,t,\phi)$ from Definition 2.3.

Consider the algebra $\mathbb{B} = \mathbb{B}_q(s,t,\phi)$ from Definition 2.3. In Section 3 we are going to show that the elements C_s, C_t generate the center $Z(\mathbb{B})$, and that $Z(\mathbb{B})$ is isomorphic to a polynomial algebra in two variables. Because of this and following [6, p. 27], it seems appropriate to call C_s, C_t the *Casimir elements* for $\mathbb{B}_q(s,t,\phi)$.

3 A $\mathbb{Z}$-grading and linear basis for $\mathbb{B}_q(s,t,\phi)$

Recall the algebra $\mathbb{B} = \mathbb{B}_q(s,t,\phi)$ from Definition 2.3. In this section we display a $\mathbb{Z}$-grading for $\mathbb{B}$. We also display a basis for the $\mathbb{F}$-vector space $\mathbb{B}$.

Let $\mathcal{A}$ denote an $\mathbb{F}$-algebra. By a *$\mathbb{Z}$-grading of $\mathcal{A}$* we mean a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{Z}}$ consisting of subspaces of $\mathcal{A}$ such that

$$\mathcal{A} = \sum_{n\in\mathbb{Z}} \mathcal{A}_n \qquad \text{(direct sum)},$$

and $\mathcal{A}_m\mathcal{A}_n \subseteq \mathcal{A}_{m+n}$ for all $m,n \in \mathbb{Z}$. Let $\{\mathcal{A}_n\}_{n\in\mathbb{Z}}$ denote a $\mathbb{Z}$-grading of $\mathcal{A}$. For $n \in \mathbb{Z}$ we call $\mathcal{A}_n$ the *n-homogeneous component* of $\mathcal{A}$. We refer to n as the *degree* of $\mathcal{A}_n$. An element of $\mathcal{A}$ is said to be *homogeneous of degree* n whenever it is contained in $\mathcal{A}_n$.

Theorem 3.1. *The algebra $\mathbb{B}$ has a $\mathbb{Z}$-grading $\{\mathbb{B}_n\}_{n\in\mathbb{Z}}$ with the following properties:*

(i) *The $\mathbb{F}$-vector space $\mathbb{B}_0$ has a basis*

$$K^h C_s^i C_t^j \qquad h \in \mathbb{Z}, \quad i,j \in \mathbb{N}. \tag{3.1}$$

(ii) *For $n \geq 1$, the $\mathbb{F}$-vector space $\mathbb{B}_n$ has a basis*

$$F^n K^h C_s^i C_t^j \qquad h \in \mathbb{Z}, \quad i,j \in \mathbb{N}. \tag{3.2}$$

(iii) *For $n \geq 1$, the $\mathbb{F}$-vector space $\mathbb{B}_{-n}$ has a basis*

$$E^n K^h C_s^i C_t^j \qquad h \in \mathbb{Z}, \quad i, j \in \mathbb{N}. \tag{3.3}$$

Moreover the union of (3.1)–(3.3) *is a basis for the $\mathbb{F}$-vector space $\mathbb{B}$.*

Proof. Routinely applying the Bergman diamond lemma [3, Theorem 1.2] one finds that the union of (3.1)–(3.3) is a basis for the $\mathbb{F}$-vector space $\mathbb{B}$. Let $\mathbb{B}_0$ denote the subspace of $\mathbb{B}$ spanned by (3.1). For $n \geq 1$ let $\mathbb{B}_n$ and $\mathbb{B}_{-n}$ denote the subspaces of $\mathbb{B}$ spanned by (3.2) and (3.3), respectively. We show that $\{\mathbb{B}_n\}_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading of $\mathbb{B}$. By construction the sum $\mathbb{B} = \sum_{n\in\mathbb{Z}} \mathbb{B}_n$ is direct. By construction and since C_s, C_t are central we have $C_s\mathbb{B}_n \subseteq \mathbb{B}_n$ and $C_t\mathbb{B}_n \subseteq \mathbb{B}_n$ for $n \in \mathbb{Z}$. Using $KE = q^2EK$ and $KF = q^{-2}FK$ we find $K^{\pm1}\mathbb{B}_n \subseteq \mathbb{B}_n$ for $n \in \mathbb{Z}$. Using (2.3) and (2.4) we find $E\mathbb{B}_n \subseteq \mathbb{B}_{n-1}$ and $F\mathbb{B}_n \subseteq \mathbb{B}_{n+1}$ for $n \in \mathbb{Z}$. By these comments and the construction we see that $\mathbb{B}_m\mathbb{B}_n \subseteq \mathbb{B}_{m+n}$ for all $m, n \in \mathbb{Z}$. Therefore $\{\mathbb{B}_n\}_{\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading of $\mathbb{B}$. The result follows. □

We emphasize a few points from Theorem 3.1.

Corollary 3.2. *With respect to the above $\mathbb{Z}$-grading of $\mathbb{B}$, the generators $C_s, C_t, E, F, K^{\pm1}$ are homogeneous with the following degrees:*

v	C_s	C_t	E	F	$K^{\pm1}$
degree of v	0	0	-1	1	0

Corollary 3.3. *The homogeneous component $\mathbb{B}_0$ is the subalgebra of $\mathbb{B}$ generated by $C_s, C_t, K^{\pm1}$. The algebra $\mathbb{B}_0$ is commutative.*

Let $\{\lambda_i\}_{i=0}^2$ denote mutually commuting indeterminates.

Corollary 3.4. *There exists an $\mathbb{F}$-algebra isomorphism $\mathbb{B}_0 \to \mathbb{F}[\lambda_0^{\pm1}, \lambda_1, \lambda_2]$ that sends*

$$K^{\pm1} \mapsto \lambda_0^{\pm1}, \qquad C_s \mapsto \lambda_1, \qquad C_t \mapsto \lambda_2.$$

The $\mathbb{Z}$-grading $\{\mathbb{B}_n\}_{n\in\mathbb{Z}}$ has the following interpretation.

Lemma 3.5. *Consider the $\mathbb{F}$-linear map $\mathbb{B} \to \mathbb{B}$, $\xi \mapsto K^{-1}\xi K$. For $n \in \mathbb{Z}$ the n-homogeneous component $\mathbb{B}_n$ is an eigenspace of this map. The corresponding eigenvalue is q^{2n}.*

Proof. Use the basis for $\mathbb{B}_n$ given in Theorem 3.1, along with the relations $KE = q^2EK$ and $KF = q^{-2}FK$. □

Corollary 3.6. *The homogeneous component $\mathbb{B}_0$ consists of the elements in $\mathbb{B}$ that commute with K.*

Proof. Immediate from Lemma 3.5. □

4 The center of $\mathbb{B}_q(s,t,\phi)$

Recall the algebra $\mathbb{B} = \mathbb{B}_q(s,t,\phi)$ from Definition 2.3. In this section we describe the center $Z(\mathbb{B})$.

Theorem 4.1. *The following is a basis for the* $\mathbb{F}$*-vector space* $Z(\mathbb{B})$*:*

$$C_s^i C_t^j \qquad i,j \in \mathbb{N}. \tag{4.1}$$

Proof. By Theorem 3.1 the elements (4.1) are linearly independent over $\mathbb{F}$, so they form a basis for a subspace of $\mathbb{B}$ which we denote by Z'. We show $Z' = Z(\mathbb{B})$. The elements C_s, C_t are central in $\mathbb{B}$ so $Z' \subseteq Z(\mathbb{B})$. To obtain the reverse inclusion, pick $\xi \in Z(\mathbb{B})$. The element ξ commutes with K, so $\xi \in \mathbb{B}_0$ by Corollary 3.6. Recall the basis (3.1) for $\mathbb{B}_0$. Writing ξ in this basis, we find $\xi = \sum_{h\in\mathbb{Z}} K^h \xi_h$ where $\xi_h \in Z'$ for $h \in \mathbb{Z}$. Using $KE = q^2 EK$ and $\xi E = E\xi$ we obtain $0 = E\sum_{h\in\mathbb{Z}} K^h \xi_h (q^{2h}-1)$. Combining this with Theorem 3.1 we find $\xi_h = 0$ for all nonzero $h \in \mathbb{Z}$. Therefore $\xi = \xi_0 \in Z'$. We have shown $Z' = Z(\mathbb{B})$ and the result follows. □

Corollary 4.2. *There exists an* $\mathbb{F}$*-algebra isomorphism* $Z(\mathbb{B}) \to \mathbb{F}[\lambda_1, \lambda_2]$ *that sends*

$$C_s \mapsto \lambda_1, \qquad\qquad C_t \mapsto \lambda_2.$$

5 Uniform posets

Recall the algebras $\mathbb{A}_q(s,t,\varphi)$ from Definition 2.1. In this section we discuss how these algebras are related to the uniform posets [10].

Throughout this section we assume that $\mathbb{F}$ is the complex number field $\mathbb{C}$. Let P denote a finite ranked poset with fibers $\{P_i\}_{i=0}^N$ [10, p. 194]. Let $\mathbb{C}P$ denote the vector space over $\mathbb{C}$ with basis P. Let $\mathrm{End}(\mathbb{C}P)$ denote the $\mathbb{C}$-algebra consisting of all $\mathbb{C}$-linear maps from $\mathbb{C}P$ to $\mathbb{C}P$. We now define three elements in $\mathrm{End}(\mathbb{C}P)$ called the lowering, raising, and q-rank operators. For $x \in P$, the lowering operator sends x to the sum of the elements in P that are covered by x. The raising operator sends x to the sum of the elements in P that cover x. The q-rank operator sends x to $q^{N-2i}x$ where $x \in P_i$.

In [10] we introduced a class of finite ranked posets said to be *uniform*. We refer the reader to that article for a detailed description of these posets. See also [2, p. 306] and [9], [11]. In [10, Section 3] we gave eleven examples of uniform posets. We are going to show that six of these examples give an $\mathbb{A}_q(s,t,\varphi)$-module. These six examples are listed in the first six rows of the table below. The remaining row of the table contains an example $\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$ which is defined as follows. Start with the poset $\mathrm{Polar}_b(N,\epsilon)$ which we denote by P. Using P we define an undirected graph Γ as follows. The vertex set of Γ consists of the top fiber P_N of P. Vertices $y,z \in P_N$ are adjacent in Γ whenever they are distinct and cover a common element of P. The graph Γ is often called a *dual polar graph* [4, p. 274], [12, Section 16]. Fix a vertex $x \in P_N$. Using x we define a partial order $\le$ on P_N as follows. For $y,z \in P_N$ let $y \le z$ whenever $\partial(x,y) + \partial(y,z) = \partial(x,z)$, where ∂ denotes path-length distance in Γ. We have turned P_N into a poset. We call this poset $\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$. Using [12, Proposition 26.4] one checks that $\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$ is uniform.

Theorem 5.1. *In each row of the table below we give an example of a uniform poset* P*. For each example we display integers* $s < t$ *and a Laurent polynomial* $\varphi \in \mathbb{F}[\lambda,\lambda^{-1}]_{s,t}$*.*

In each case the vector space $\mathbb{C}P$ becomes an $\mathbb{A}_q(s,t,\varphi)$-module such that the generator e (resp. f) (resp. k) acts on $\mathbb{C}P$ as the lowering (resp. raising) (resp. q-rank) operator for P. For convenience, for each example we display the element $\phi \in \mathbb{F}[\lambda,\lambda^{-1}]_{s,t}$ such that $\varphi = \phi_{s,t}$.

example	s	t	φ	ϕ
$\mathrm{Polar}_b(N,\epsilon)$	0	1	$-(q+q^{-1})(q^{2N+1+2\epsilon}\lambda^2+q^{N-3}\lambda^{-1})$	$-\frac{q^{2N+2\epsilon}\lambda^2+q^{N-1}\lambda^{-1}}{(q-q^{-1})^2}$
$\mathrm{A}_b(N,M)$	-1	0	$-(q+q^{-1})q^{N+2M+1}\lambda$	$-\frac{q^{N+2M-1}}{(q-q^{-1})^2}\lambda$
$\mathrm{Alt}_b(N)$	-2	-1	$-(q+q^{-1})q^{2N+1}$	$-\frac{q^{2N-2}}{(q-q^{-1})^2}$
$\mathrm{Her}_q(N)$	-2	-1	$-(q+q^{-1})q^{2N+2}$	$-\frac{q^{2N-1}}{(q-q^{-1})^2}$
$\mathrm{Quad}_b(N)$	-2	-1	$-(q+q^{-1})q^{2N+3}$	$-\frac{q^{2N}}{(q-q^{-1})^2}$
$\mathrm{Hem}_b(N)$	-2	-1	$-(q+q^{-1})q^{2N+1}$	$-\frac{q^{2N-2}}{(q-q^{-1})^2}$
$\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$	-2	-1	$-(q+q^{-1})q^{2N+3+2\epsilon}$	$-\frac{q^{2N+2\epsilon}}{(q-q^{-1})^2}$

In the above table $b = q^2$.

Proof. For each example except the last, our assertions follow routinely from [10, Theorem 3.2]. For the last example $\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$ our assertions follow from [12, Theorem 1.10]. Note that the parameter denoted ϵ in [12, Theorem 1.10] is one more than the parameter denoted ϵ in [10, p. 201]. □

6 Acknowledgement

The main inspiration for the ADU algebra concept comes from the second author's thesis [12] concerning the uniform poset $\mathrm{Polar}_b^{\mathrm{top}}(N,\epsilon)$. To be more precise, it was his discovery of two central elements that he called C_1, C_2 [12, Section 28] that suggested to us how to define an ADU algebra.

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Simplicial arrangements revisited

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Abstract

In connection with the publication of the catalogue [7] of known simplicial arrangements of lines in the real projective plane, and the note [8] about small simplicial arrangements of pseudolines, several developments of these topics deserve to be mentioned. The present paper puts these results in perspective, and provides appropriate illustrations.

Keywords: Simplicial arrangement.

Math. Subj. Class.: 51M16

1 Simplicial arrangements of pseudolines

Very significant new results on simplicial arrangements of pseudolines are contained in the publications [1] by L. W. Berman and [3] by M. Cuntz. We recall that an arrangement of pseudolines is a family of simple curves in the real projective plane such that each differs from a straight line in a finite part only, and every two have a single point in common at which they cross transversally. Throughout, we model or interpret the real projective plane as the *extended Euclidean plane*, with added points "at infinity" and the line "at infinity" (indicated by ∞ if included in a diagram) consisting of all the points at infinity.

Developing an idea of Eppstein [4], Berman described a method of construction of simplicial arrangements of pseudolines that has a very general applicability; moreover, it is very easily adapted for investigation of linear simplicial arrangements (that is, consisting of straight lines). To explain this approach, we start with the case of linear arrangements. (It needs to be noted that our explanation differs somewhat from Berman's; we shall return to this later on.) Starting with the lines of mirror symmetry of a regular k-gon ($k \geq 2$) centered at the origin, we select one of the $2k$ wedges (angular regions) determined by a pair of adjacent rays formed by these k lines. Considering these rays as mirrors, we shine a (laser) ray (or several such rays) into the wedge, and let it (them) reflect on the two mirrors according to the laws of reflection; this generates a *beam* (or several beams). As is easily seen be elementary considerations, the laser ray will reflect only a finite number of times, and the final fate of each beam will be one of the following:

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(i) The final segment will be perpendicular to one of the reflecting rays; this includes what can be considered a limiting case, where the starting laser ray is aimed at the origin; in particular, it includes the case where the mirrors are part of the arrangement.

(ii) The last part of the beam will be a ray shooting out of the wedge. In this case there are two distinct portions of the beam — the incoming part and the outgoing part. Each of these parts is simple (has no selfintersections) but the two parts may have intersections. Such beams are called *two-ended*.

In case of pseudoline arrangements, the same conditions are assumed, except that:

- The reflections on the mirrors do not follow rules of optics but are simply endpoints of pairs of segments or rays;
- Each segment or ray may be a pseudosegment or pseudoray (the purple line in Figure 1 is an example);
- The orthogonality in (i) is waived, and each of the two parts in (ii) is assumed to be simple. See examples in Figures 1, 2, 3 and 4.

In any case, if the beam(s) satisfy some additional conditions, as detailed in [1], repeated reflection in the $2k$ rays yields a linear or pseudoline simplicial arrangement. We call these *kaleido* arrangements, to distinguish them from more general simplicial arrangements. Examples of the latter kind (non-kaleido) are $A(14, 3)$, $A(16, 7)$, and others, in the notation of [7], as well as the linear arrangement in Figure 7.

In Berman's paper [1], only beams satisfying (i) or its modification for pseudolines are accepted. Detailed discussion of the conditions that lead to linear simplicial arrangements (and of their pseudoline analogs) is presented in [1] for up to three beams other than the mirrors. It may be assumed that analogous investigations may determine conditions under which beams as defined here lead to simplicial arrangements, but I have not determined these conditions.

The main reason for introducing condition (ii) in the definition of kaleido arrangements is that it leads to the following result:

Theorem 1.1. *Each simplicial arrangement, with k-fold dihedral symmetry such that all mirrors are lines of the arrangement, is a kaleido arrangement.*

The theorem is valid equally for linear arrangements and for pseudoline arrangements.

Proof. Let all the beams be marked as far as possible, starting with the incoming rays; the claim is that there are no unmarked segments (of straight or pseudolines) or rays. If any such segment were present, its continuation by reflection in the mirrors would have to close on itself, which is impossible. □

In [3], Cuntz first enumerates simplicial arrangements of at most 27 pseudolines, and then investigates their stretchability, that is, the isomorphism to linear arrangements. The bound 27 is due to limitations of the computing power available, but even with this bound several notable results are obtained and several conjectures of the present writer are resolved.

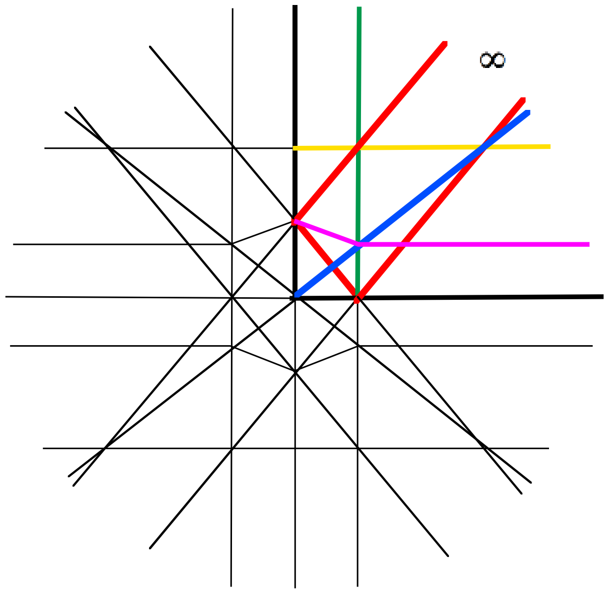

Figure 1: The simplicial pseudoline arrangement $B_1(15)$ (adapted from [7]) is a kaleido arrangement with $k = 2$ and seven beams, one of which (red) is two-ended. The blue beam and the black ones are aimed at the origin, the purple one is a pseudoray, and the green and yellow ones are rays ending at mirrors. The mirrors are heavily drawn black lines.

The enumeration of simplicial arrangements of pseudolines in [3] shows that all simplicial arrangements with at most 14 pseudolines are stretchable, thus confirming a conjecture made in [8]. The computer-assisted enumeration in [3] uses "wiring diagrams" introduced Goodman in [5], and elaborated in Goodman and Pollack [6] and other publications, together with innovative arguments to reduce the computational effort. The results, in particular, disprove another conjecture in [8]: Namely, that there is a single unstretchable simplicial arrangement of 15 pseudolines and four of 16 pseudolines. In the paper [3] Cuntz establishes that there are precisely two such arrangements with 15, and precisely seven with 16 pseudolines. The second 15-pseudoline arrangement is shown in Figures 7 and 8 in two forms. Figure 7 shows a "wiring diagram" of this pseudoline arrangement, modified from Figure 2 of [3]. The presentation in Figure 8 exhibits the 3-fold rotational symmetry of this arrangement in the extended Euclidean model of the real projective plane. The colors of the lines, and the labels, establish the isomorphism between the two diagrams in Figures 7 and 8. As no pseudolines in this example are mapped onto themselves by reflection, this is not a kaleido arrangement.

2 Simplicial arrangements of straight lines

Another result of [3] is the discovery of four new simplicial arrangements of (*straight lines*. A short review of the historical background seems appropriate to explain the significance of Cuntz's results.

The first introduction of the concept of simplicial arrangements of lines occurred in a

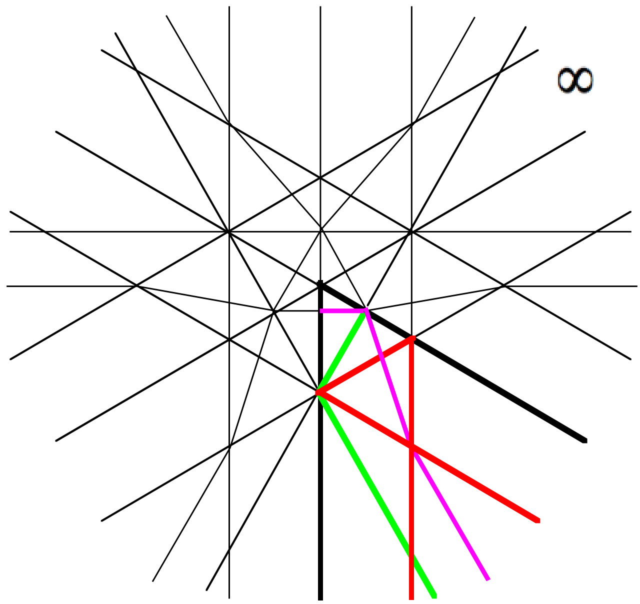

Figure 2: A kaleido simplicial arrangement $B_2(16)$ of 16 pseudolines, with $k = 3$ and with five beams, one of which (red) is two-ended.

paper by Melchior [11] in 1941, but the paper did not seem to have any immediate effect. Close to thirty years later, the fact that Melchior found only few such arrangements piqued my curiosity. Over time, I found that there are three infinite families of simplicial arrangements, and a large number of nonsystematic, "sporadic" ones. Details were published in [9] in 1971; however, the presentation there was very concise, and not "user friendly". More recently, a more detailed version was published [7]. Ninety sporadic arrangements were shown in [9], and this number remained unchanged in [7] although one arrangement was a duplicate and was deleted, and a new one was found. The presentation in [7] seems to have attracted more attention; one of the results was the paper by Cuntz [3].

In this paper Cuntz disproves the present author's longstanding conjecture, first stated in [9] in 1971 and repeated in other publications, notably in [7], that the list of 90 sporadic simplicial arrangements is complete. Cuntz found that the catalog [7] is complete regarding simplicial arrangements with up to 27 lines, *except* for one missing arrangement for each of 22, 23, 24, and 25 lines. These arrangements, missed in [7], form a "family" in the sense that the one with the largest number of lines (25) leads to the other three by omitting 1, 2, or 3 lines. A version of this arrangement, denoted $A(25, 8)$ by Cuntz, is shown in Figure 9. This presentation is geometrically more symmetric than the one in Figure 1 of [3]. The lines that may be omitted are shown heavily drawn, and it is obvious that they play the same role in the arrangement. Therefore only a single additional arrangement arises on omitting 1, 2, or 3 of them.

As a consequence, there are now 94 known sporadic simplicial arrangements of lines. As a further consequence, it is now more open to question whether there exist additional such arrangements with 28 or more lines? An inspection of the twenty known such ar-

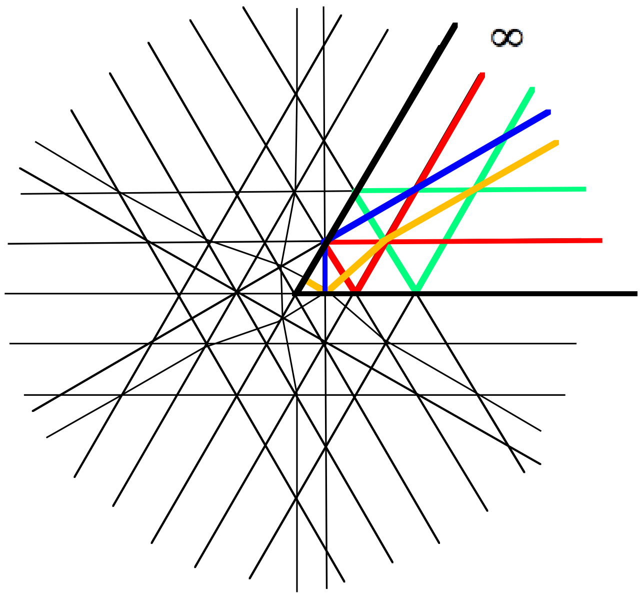

Figure 3: A kaleido simplicial pseudoline arrangement $B(22)$, with $k = 3$ and with six beams, two of which are two-ended. It is the arrangement shown in Figure 22 of [1]

rangements (depicted in [7]) shows clearly that the experimental discovery becomes very complicated with this range of the number of lines. Hence there is a real possibility that some of these arrangements have not been found so far. It would seem very desirable — but challenging — to find ways of ascertaining the completeness of the list in [7] of such arrangements augmented by the four Cuntz arrangements, or the lack of it.

3 Additional remarks on simplicial arrangement of lines and pseudolines

It is not clear how to decide from the combinatorial (or topological) description of a simplicial arrangement of pseudolines what is the minimal number of non-straight ones. Nor is it obvious how that number depends on the order of the automorphism group of the arrangement. Another question is whether it is possible to have different numbers of beams for the same arrangement; this possibility arises since the reflections are not strictly optical ones.

A still different question is what are the restrictions on k, the number of single-ended, and the number of two-ended beams. In particular, for a given number d of two-ended beams, what is the minimal number s of single-ended ones – for linear arrangements, and for pseudoline ones. Figure 3 shows that with $k = 3$, and $d = 2$, as few as $b = 4$ single-ended beams are possible; the new arrangement $A(22, 5)$ shows the same for a linear arrangement. As another example we have in Figure 11 a linear arrangement $A(15, 1)$ with three two-ended beams and two single-ended beams.

As shown by examples, a (linear) kaleido arrangement may have isomorphic realiza-

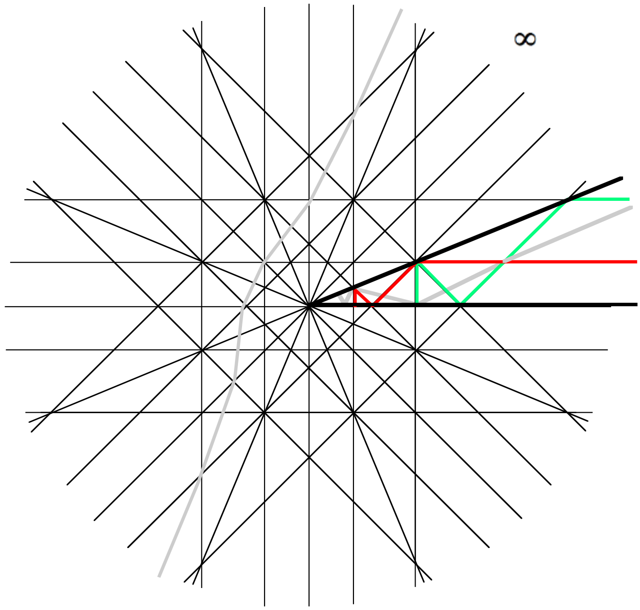

Figure 4: The simplicial linear kaleido arrangement $A(25, 5)$, in the notation of [7], with $k = 8$ and with four beams (two black, one red and one green). It is isomorphic to the second (pseudoline) arrangement in Figure 11 of [1]. Without the line at infinity it is the arrangement $A(24, 2)$ of [6]. With the eight additional pseudolines (only one shown, in gray) generated by the gray beam it is a simplicial kaleido arrangement with 33 pseudolines. It should be noted that the mirrors of a kaleido arrangement need not be parts of lines of the arrangement. Examples of kaleido arrangements with such "virtual" mirrors are shown in Figures 5 and 6.

tions with different geometric symmetry groups. The arrangement $A(6, 1)$ shown in Figure 12 provides an example.

4 Additional remarks on simplicial arrangement of lines and pseudolines

While it is not hard to show that the simplicial pseudoline arrangements shown in the above figures are not stretchable, it is not clear to what *extent* they fail to be stretchable. More precisely, at least how many non-straight pseudolines have to be used in every diagram of these arrangements? In Figure 1 there are two such pseudolines, in Figure 2 there are three, and in Figure 6 there are six. In all these cases this seems to be the minimal number of non-straight pseudolines. The four non-stretchable simplicial arrangements of 16 pseudolines described in Figures 5 and 6 of [8] have at least 2, 3, 3, resp. 1 non-straight pseudolines. According to a private communication by Prof. Cuntz, the four non-stretchable simplicial arrangements of 16 pseudolines described in Figures 5 and 6 of [7] have 2, 1, 1, resp. 1 non-straight pseudolines; also, the three new non-stretchable arrangements of 16 pseudolines

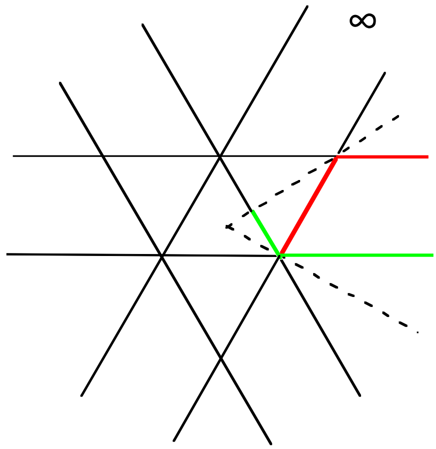

Figure 5: The simplicial linear kaleido arrangement $A(7,1)$, with $k = 3$ and two beams, reflected on two virtual mirrors.

mentioned in [3] but not described there, have each at least 2 non-straight pseudolines.

It is not clear how to decide from the combinatorial (or topological) description of a simplicial arrangement of pseudolines what is the minimal number of non-straight ones. Nor is it obvious how that number depends on the order of the automorphism group of the arrangement.

The pseudoline arrangement $B_2(15)$ of 15 pseudolines is listed in Table 3 of 2] as having 6-fold cyclic symmetry. This seems hard to reconcile with Figure 8 above.

There is a regrettable error in the catalog [7]. The arrangement shown there on page 14 and labeled $A(16,7)$ is, in fact, isomorphic with the arrangement $A(16,5)$ shown just above it. A correct diagram of $A(16,7)$ is shown in Figure 13.

Simplicial arrangements of (straight) lines lead to a number of other problems. Not only is the question of the completeness of the list in [6], as augmented in [3], debatable — but it is conceivable that there are infinitely many arrangements missing. In fact, there seems to be no known family of lines in the plane that could not be imbedded into a simplicial arrangement of lines, or at least of pseudolines.

Even the belief that there are no additional infinite families of simplicial arrangements of lines beyond the three families described in [9] and [7], has no credible supporting evidence. On the other hand, it could be argued that the available facts concerning simplicial pseudoline arrangements make the existence of additional infinite families of straight-line simplicial arrangements more believable.

Here are these facts. Already in [10, p.51] it is mentioned that there are at least seven infinite families of simplicial pseudoline arrangements. But this was rendered insignificant through the work of Berman. In [1] Berman described constructions of many infinite families of simplicial arrangements of pseudolines, based on reflecting kaleidoscopically suitable zigzags in an angle. It may well be that some of these lead to linear arrangements.

The difference between the definition of kaleido arrangements used here, and the one proposed by Berman is not as large as might be thought. In most cases one could replace one two-ended beam by two single-ended ones by accepting that the end-segment does not

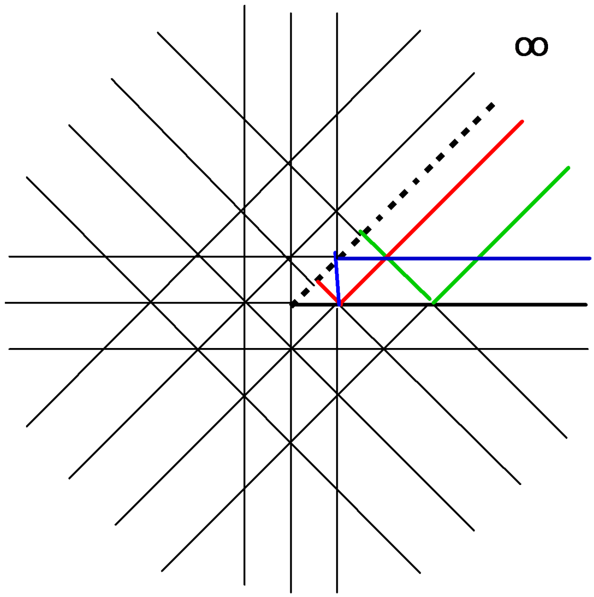

Figure 6: The simplicial linear kaleido arrangement $A(15, 2)$, with $k = 4$ and three beams, one of which is a mirror; one mirror is a virtual mirror.

meet the mirror perpendicularly. On the other hand, our definition of kaleido arrangements could be extended to arrangements that are not simplicial. There seems to be no interesting information available about such more general arrangements, but the concept may well be worth investigating.

Finally, another result of Cuntz and collaborators should be mentioned. They investigated a particular class of linear simplicial arrangements called "crystallographic arrangements"; their definition is too involved to be repeated here and readers are referred to [2] and the references given there. In contrast to the uncertainties discussed above, this class has the notable property that its members have been completely determined and classified.

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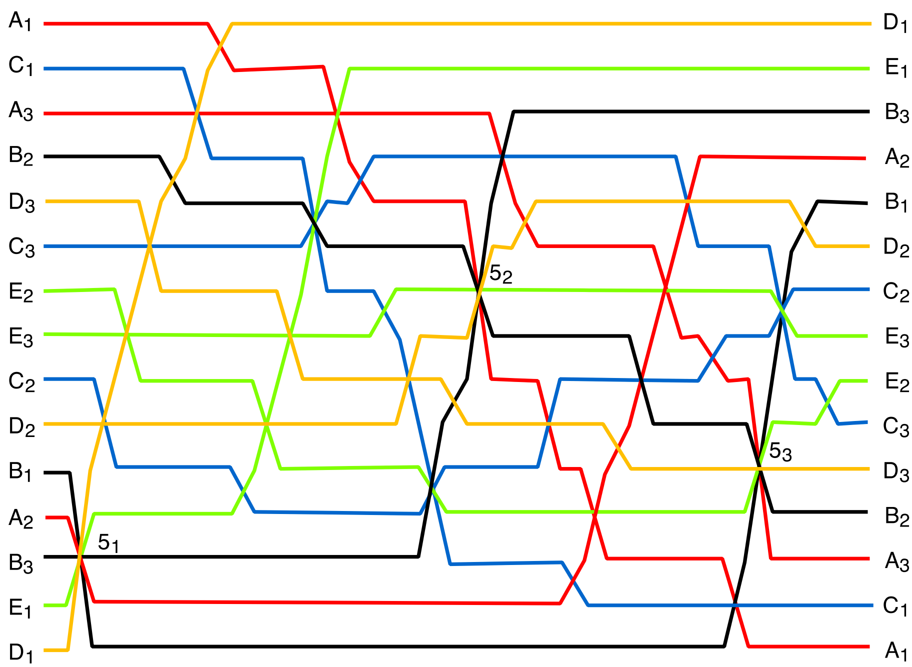

Figure 7: A wiring diagram of the new simplicial arrangement $B_2(15)$ of 15 pseudolines found by Cuntz. Adapted from Figure 2 of [3].

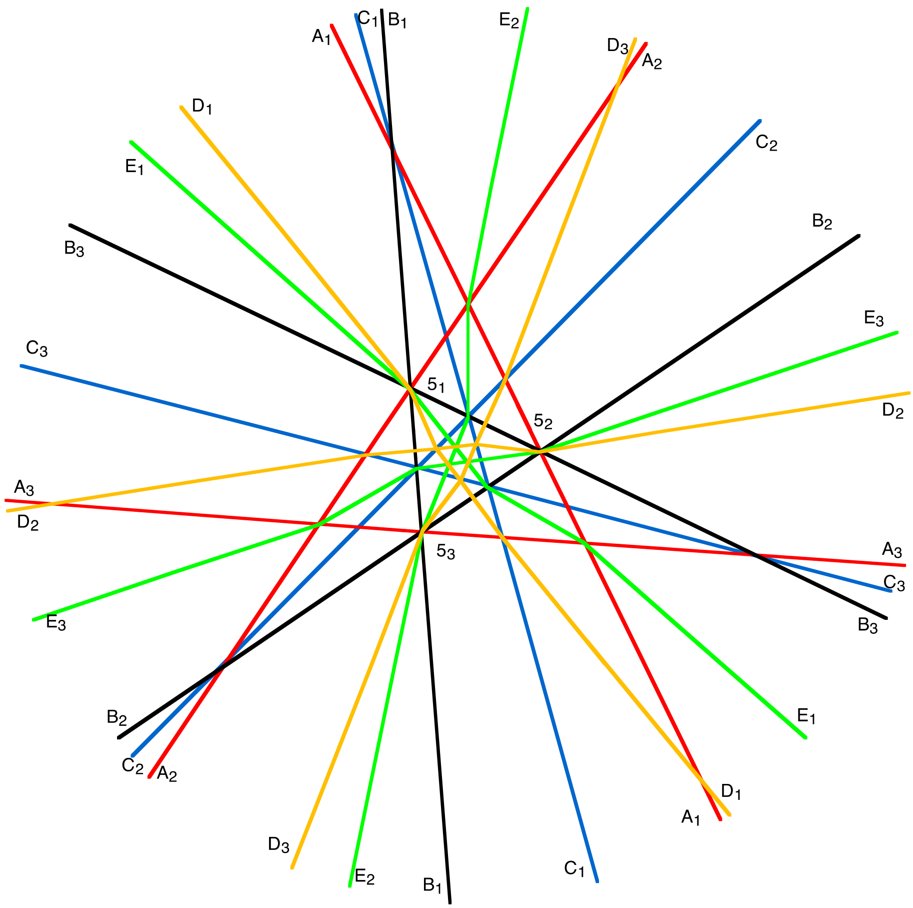

Figure 8: A presentation of the simplicial arrangement $B_2(15)$ of 15 pseudolines in the extended Euclidean model of the real projective plane. The colors and labels of the pseudolines correspond to those in Figure 7.

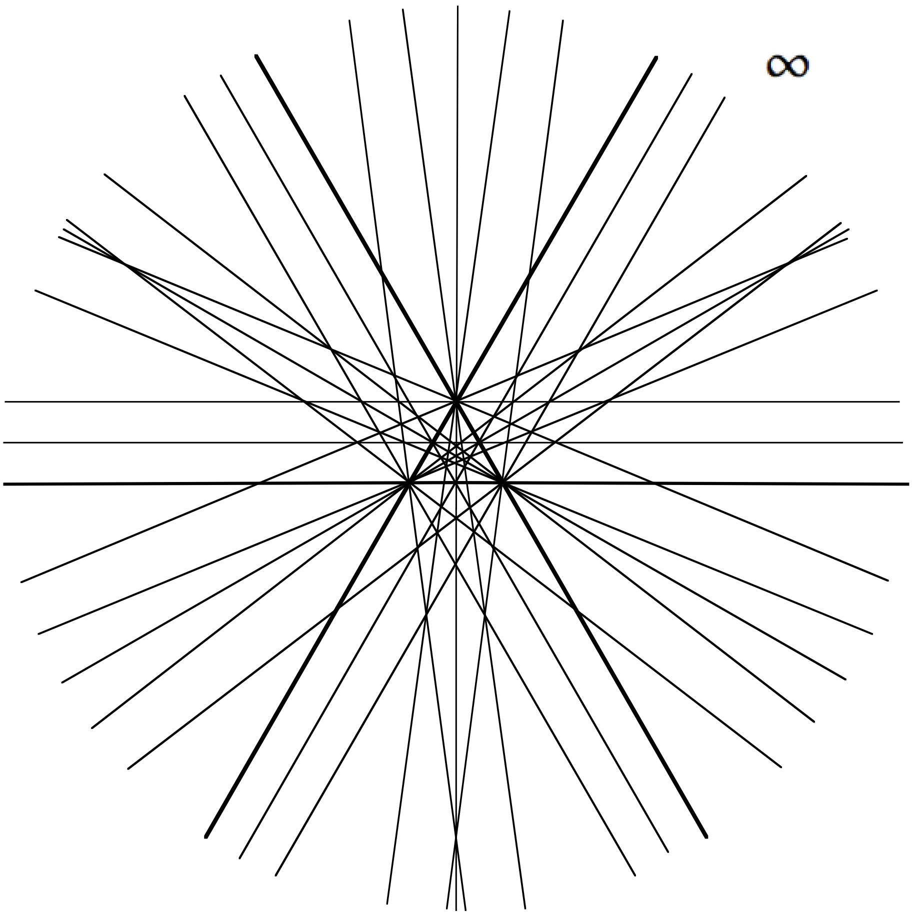

Figure 9: A version of the linear simplicial arrangement denoted $A(25, 8)$ by Cuntz [3]. Any number of the three heavily drawn lines can be deleted, resulting in the simplicial arrangements labeled $A(22, 5)$, $A(23, 2)$, and $A(24, 4)$ in [3].

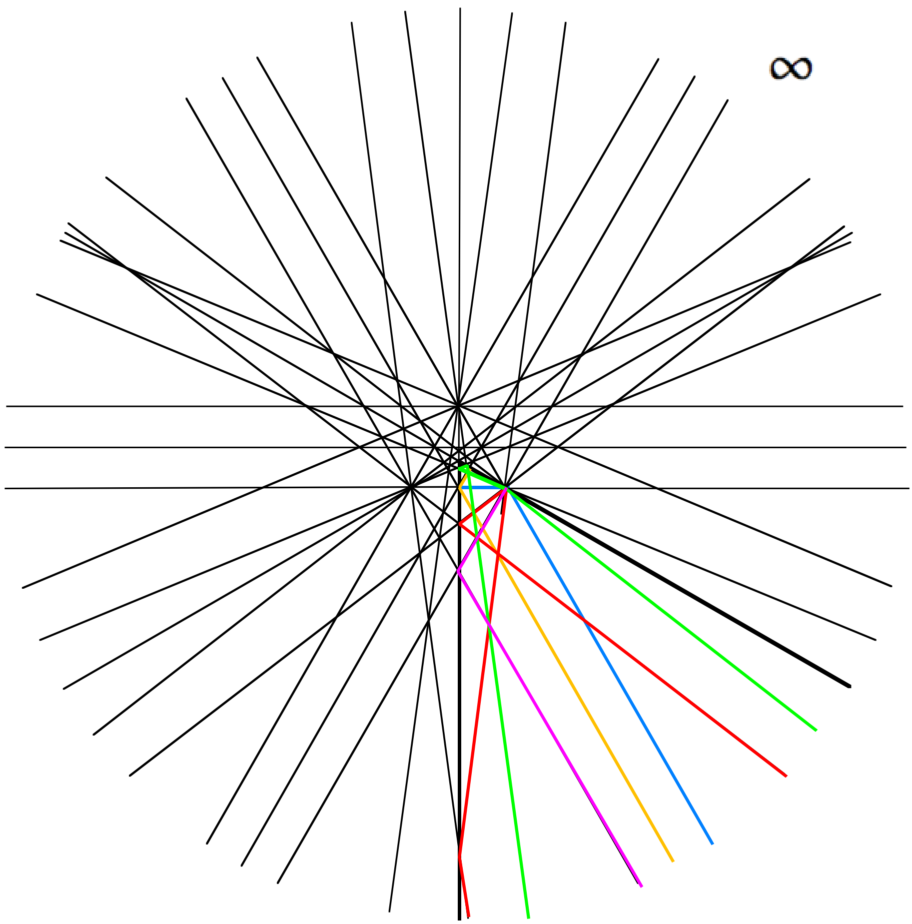

Figure 10: Cuntz's $A(25, 8)$ simplicial arrangement of lines is a kaleido arrangement with $k = 3$; it has two two-ended beams (red and green), and five other beams.

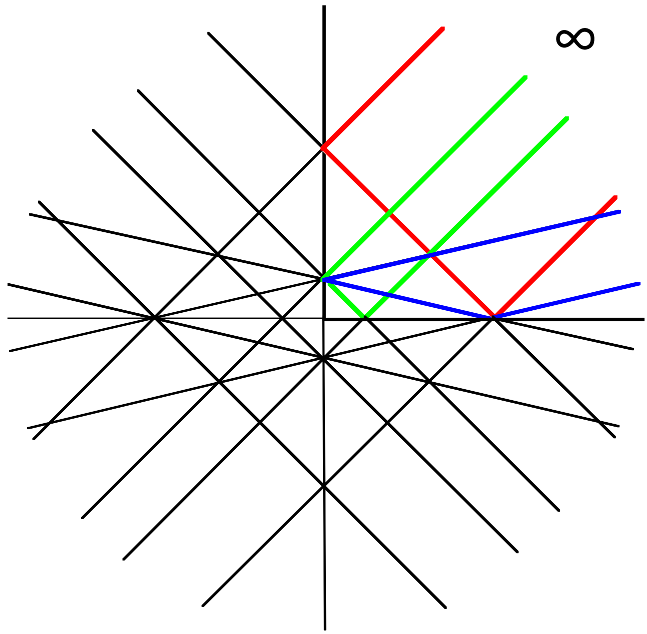

Figure 11: The linear simplicial kaleido arrangement $A(15, 1)$ with $k = 2$ has five beams, three of which are two-ended.

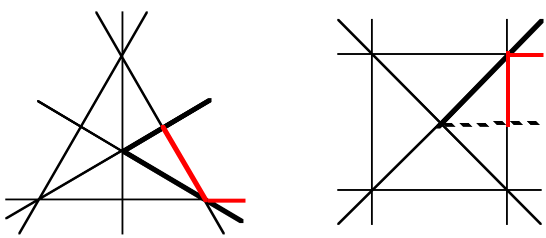

Figure 12: Isomorphic realizations with different symmetries.

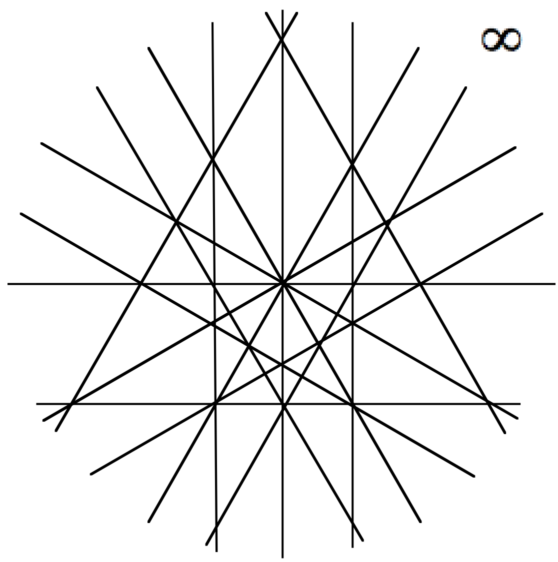

Figure 13: A correct diagram of the simplicial arrangement $A(16, 7)$; the diagram shown in [7] and labeled $A(16, 7)$ is not correct.

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[1] First A. Author, Second B. Author and Third C. Author, Article title, *Journal Title* **121** (1982), 1–100.

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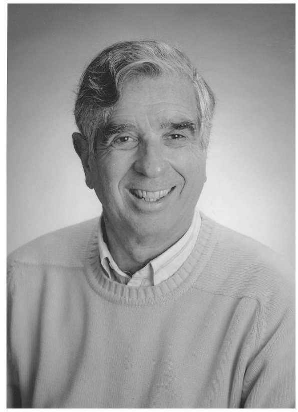
Herbert S. Wilf (June 13, 1931 – January 7, 2012)



Professor Herbert Wilf was an outstanding mathematician, an excellent teacher, and a spirited writer. He started in numerical analysis, went on to complex analysis, and finally to his greatest love, combinatorics. He is known for the Fine-Wilf theorem on periodic functions; the Szekeres-Wilf number of a graph; Wilf's bound on the chromatic number of a graph; the Calkin-Wilf tree; the Wilf-Zeilberger pair; the WZ-method for proving combinatorial identities; and more. Together with Doron Zeilberger, he was awarded the 1998 *Leroy P. Steele Prize of the AMS for a Seminal Contribution to Research.* With Donald Knuth he founded *Journal of Algorithms* in 1980; served as Editor of *The American Mathematical Monthly* 1987–1991; founded *The Electronic Journal of Combinatorics* with Neil Calkin in 1994, and served as its Editor-In-Chief until 2001. Besides over 160 journal papers, he published several well-known books, such as: *Mathematics for the Physical Sciences* in 1962; *Combinatorial Algorithms* with Albert Nijenhuis in 1975; *Algorithms and Complexity* in 1986; *generatingfunctionology* in 1990; and $A = B$ with Doron Zeilberger and Marko Petkovšek in 1996. All of these he made freely available for downloading from his web page. Together with his loving wife Ruth Tumen Wilf, he visited Slovenia twice: in June 1995, he participated at the 3rd Slovenian International Graph Theory Conference at Lake Bled; and in June 2010, at the Symbolic Computation and its Applications Conference in Maribor. His next-but-last paper (with Vittorio Addona and Stan Wagon) appeared in *Ars Mathematica Contemporanea* in 2011.

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