ON THE ISOCHRONICITY OF PERIODIC SOLUTIONS AT A CENTRE MANIFOLD

O IZOHRONOSTI PERIODIČNIH REŠITEV NA CENTRALNI MNOGOTEROSTI

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Abstract

The problem of isochronicity is discussed from the historical and dynamical systems point of view. The model of Huygen's cycloidal chops is mathematically explained. We consider two dynamical systems arising from a three-dimensional system with a centre manifold. Based on the period function approach we find necessary and sufficient criteria on the coefficients of the system to distinguish between the cases of isochronous and non-isochronous oscillations.

Povzetek

Problem izohronosti je obravnavan s stališča dinamičnih sistemov ter iz zgodovinskega stališča. Matematično je razložen Huygensov model ure s cikloidnim nihalom. Na osnovi analize funkcije periode obravnavamo dva dinamična podsistema tridimenzionalnega sistema s centralno mnogoterostjo in poiščemo potrebne in zadostne pogoje za izohronost centra na raznoterosti določeni s koeficienti sistema.

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1   INTRODUCTION

A simple pendulum is a bob of mass m on a massless rigid wire of length l (see Fig. 1). Assuming no damping (i.e. the only force acting on the system is the weight of the bob), the differential equation governing a simple pendulum is

$$\frac{d^2\varphi}{dt^2} + \frac{g}{l} \cdot \sin\varphi = 0 \qquad (1.1)$$
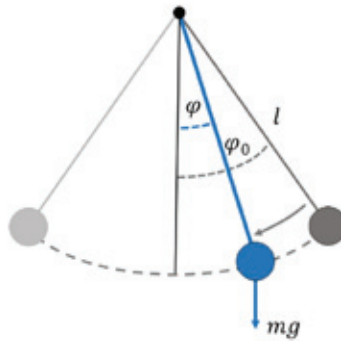
Figure 1: A simple pendulum of length l driven by the weight $F = -mg$ of the bob.

There are several approximations for a period of a simple pendulum (see Fig.1) of length l. One of them (see [1]) is

$$T(l; \varphi_0) = 2\pi \left(\frac{\varphi_0}{\sin\varphi_0}\right)^{\frac{3}{8}} \cdot \sqrt{\frac{l}{g}}, \qquad (1.2)$$

where φ_0 is the initial angle displacement (amplitude) of the bob.

Note that substituting $x = \frac{d\varphi}{dt}$, $y = \varphi$ and $\tau = \sqrt{\frac{g}{l}} \cdot t$ into (1.1) one can easily derive the following system

$$\frac{dx}{d\tau} = -\sin y \approx -y + \frac{y^3}{3!} - \cdots, \quad \frac{dy}{d\tau} = x. \qquad (1.3)$$

The qualitative behaviour of this system is determined by how $x(\tau)$ and $y(\tau)$ behave with the change of τ. It is convenient to indicate how a solution behaves in the phase plane, that is in the x, y −plane. The qualitative behavior is represented by a family of curves, directed with increasing τ. These phase curves are called trajectories or orbits of system (1.1.3). The geometrical representation of the qualitative picture of orbits of system (1.1.3) is called its phase portrait. System (1.1.3) admits a centre at (0,0), that is, all the orbits close to the origin are closed or, in other words, the origin is enclosed with simple closed curves (solutions are periodic). Obviously,

because of (1.1.2) the centre of (1.1.3) is not isochronous, i.e. not all periodic solutions have the same period. Isochronicity, as will be stated in the next section (see Definition 2), was not known at the time of the great Dutch astronomer, physicist and mathematician Christian Huygens. However, he was aware that if the amplitude of the pendulum's swing changed, the time of swing would also change, which means that the pendulum was not isochronous, [2]. In the limit of small amplitudes, the period (1.1.2) is approximated by $T(l) = 2\pi \cdot \sqrt{\frac{l}{g}}$. The following system is obtained from the equation of a mathematical pendulum (1.1) (again using the change of time $\tau = \sqrt{\frac{g}{l}} \cdot t$)

$$\frac{dx}{d\tau} = -y, \quad \frac{dy}{d\tau} = x. \tag{1.4}$$

However, for larger amplitudes, the period T is a function of the amplitude. The approximation formula (1.1.2) is just one of many existing [1]. Huygens' ingenious idea, which he put into practice, was to vary the effective length of the pendulum by allowing its cord to wrap partially around an obstruction as it swings. What should the shape of this obstruction be in order to ensure that the period is strictly independent of the amplitude?

In the middle of the 17th century, the accuracy of clocks was not measured in minutes; even the best of them gained or lost several minutes per day because they were driven by falling weights or springs. Galileo Galilei was probably the first who wanted to design a pendulum driven clock in order to achieve better accuracy. He believed that he had mathematically proven that a (simple) pendulum is isochronous. As already stated, for the case of the pendulum this simply means that the time it takes to complete one full swing is independent of the size of the swing. Nowadays, it is well-known that this is not the case, and that Galileo was mistaken (see approximation formula (1.1.2)). The first scientist who produced a pendulum-driven clock to keep time errors to within one minute per day was Christian Huygens in 1656; within two years the clocks with accuracy of about 10 seconds per day were produced (by Huygens and others).
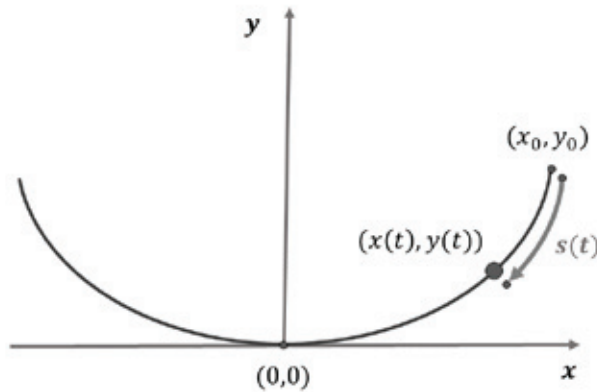

Figure 2: *A bead on a wire of shape* $(x(t), y(t))$.

In order to solve the problem of an "isochronous pendulum", a generalization of the simple pendulum in the sense depicted in Fig. 2 was considered. Thus, if we allow the shape connecting (x_0, y_0) with $(0,0)$ to be different from a semicircle (i.e. a general shape $(x(t), y(t))$), then Huygens's question (of isochronous swing) becomes for what shape will the bead descend to the origin in the same time regardless of its starting point (x_0, y_0)?

Denoting by s(t) the distance along the string that the bead has traveled at time t and by $v(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ the velocity of the bead (at time t) which is (in contrast) equal to $v(t) = -\sqrt{2g(y_0 - y(t))}$, where g is the acceleration of a freely falling body due to gravity. Furthermore, assume that the functorial dependence of s(t) in terms of y(t) is s = f(y). Then the time taken by a bead to descend from (x_0, y_0) to $(x(t), y(t))$ becomes:

$$T(y; y_0) = \int_{y_0}^{0} \frac{dt}{ds}\frac{ds}{dy}\,dy = \int_{y_0}^{0} \frac{-1}{\sqrt{2g(y_0 - y)}} f'(y)\,dy. \tag{1.5}$$

Substituting a dimensionless variable $z = \frac{y}{y_0}$ into (1.1.5) and rearranging we obtain

$$T(z) = \frac{1}{\sqrt{2g}} \int_0^1 \frac{f'(zy_0)\sqrt{y_0}}{\sqrt{1 - z}}\,dz, \tag{1.6}$$

which is independent of y_0 (i.e. a constant function of y_0) for $0 < z < 1$, if $f'(zy_0)\sqrt{y_0}$ is a function only of z (and not of y_0). This is equivalent to $\frac{\partial}{\partial y}\left(f'(zy)\sqrt{y}\right) = 0$ or explicitly

$$2f''(u)u + f'(u) = 0; \quad 0 < u < y_0, \tag{1.7}$$

where u = zy.

Obviously, the solution to (1.1.7) expressed in terms of y as independent variable is $f(y) = C_1 + C_2\sqrt{y}$. This implies

$$f'(y) = \sqrt{\frac{C}{y}}, \quad C > 0. \tag{1.8}$$

Recall that s = f(y), yielding $\frac{ds}{dy} = f'(y)$. Dividing $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ by $\left(\frac{dy}{dt}\right)^2$ yields $\left(\frac{ds}{dy}\right)^2 = (f'(y))^2 = \left(\frac{dx}{dy}\right)^2 + 1$ and finally from (1.1.8) we obtain

$$x = \int_{y_0}^{y} \sqrt{\frac{C - y}{y}}\,dy. \tag{1.9}$$

The solution to (1.1.9) containing $(x, y) = (0,0)$ can be parametrized (using $y = C \sin^2 \frac{\varphi}{2}$, for $0 < \varphi < \pi$, and the "half angle identities") by

$$x(\varphi) = \frac{C}{2}(\varphi + \sin \varphi), \qquad y(\varphi) = \frac{C}{2}(1 - \cos \varphi), \tag{1.10}$$

which passes through (x_0, y_0) for $\varphi = \varphi_0 \leq \tan^{-1} \frac{2}{\pi}$ (satisfying $\frac{y_0}{x_0} = \frac{1-\cos \varphi_0}{\varphi_0 + \sin \varphi_0}$) and $C = \frac{2y_0}{1 - \cos \varphi}$.
The shape of the curve $(x(t), y(t))$ from (1.10) is called an (inverted) cycloid (see Fig. 3).

Definition 1. The cycloid (studied and named by Galileo in 1599) is the locus of a point on the rim of a circle of radius a rolling along a straight line. If the point considered on a circumference of the rolling circle is initially in the origin $(0,0)$, the cycloid has a cusp at the origin and its humps are oriented upward. Its parametric equation is

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t) \tag{1.11}$$
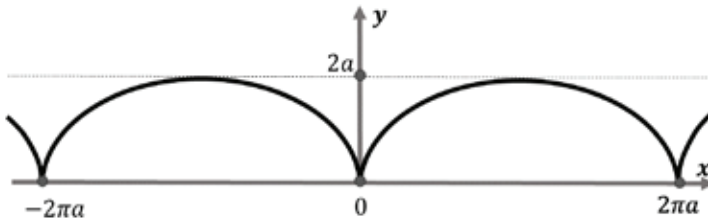
Figure 3: *Cycloid: The path followed by a point on the circumference of a circle as that circle rolls along a straight line.*

An involute is a curve obtained from another given curve by attaching an imaginary taut string to the given curve and tracing its free end as it is (un)wound onto that given curve. *The evolute* of a curve is the locus of all its centres of curvature. That is to say that when the centre of curvature of each point on a curve is drawn, the resultant shape will be the evolute of that curve. For instance: the evolute of a circle is a single point at its centre, the evolute of a quadratic parabola $y = x^2$ is a semicubical parabola $y = \frac{1}{2} + \frac{3}{4} \sqrt[3]{4x^2}$. The only curve for which the evolute (and involute) is "of the same size" (neglecting translation) as the original curve is the cycloid (see Fig. 3 and Fig. 4). The evolute of an involute is the original curve (fewer portions of zero or undefined curvature). Alternatively, another way to construct the involute of a curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute (Fig. 3).

Figure 4: *Inverted cycloid (black) with its involute (red) which is a (inverted) cycloid with the same parameter a, just translated for a vector $(\pi a, -2a)$.*

The above results led Huygens to invent a pendulum with cycloidal chops (Fig. 5).
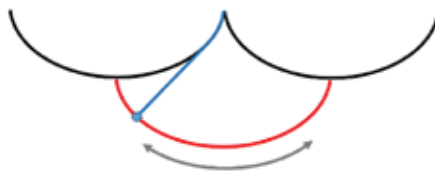
Figure 5: *The cycloidal chops pendulum invented by C. Huygens.*

However, it is not possible to determine the centre of oscillation for pendula suspended between cycloids, as seen in Fig. 4, since the motion of a pendulum with cycloidal chops is not planar (which was known by Huygens; such a pendulum is a multiple system, i.e. a system having two degrees of freedom). Therefore, the next step of improvements of the pendulum clocks was the so-called escapement mechanism (which was driven primarily by the gravity force and later by the force of a spring).

2   ISOCHRONICITY OF PLANAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

We discuss differential systems with analytical right sides

$$\frac{dx}{dt} = -y + \sum_{i+j=2}^{\infty} p_{ij} x^i y^j, \frac{dy}{dt} = x + \sum_{i+j=2}^{\infty} q_{ij} x^i y^j, \tag{2.1}$$

where p_{ij} and q_{ij} are real constants. There has been a longstanding problem, [3], called the Poincaré centre-focus problem; for the system (2.1) to find explicit conditions under which (2.1) has a *centre at the origin*. The problem is equivalent to an analogue for a corresponding periodic equation

$$\frac{dr}{d\varphi} = \frac{\sum_{i=1}^{\infty} A_i(\varphi) r^{i+1}}{1 + \sum_{i=1}^{\infty} B_i(\varphi) r^i} = \sum_{i=2}^{\infty} R_i(\varphi) r^i \tag{2.2}$$

to have periodic solutions. To see this let us note that the phase curves of (2.1) near the origin in polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$ are determined by (2.2), where $A_i(\varphi)$ and $B_i(\varphi)$ are polynomials in $\cos \varphi$ and $\sin \varphi$ and $R_i(\varphi)$ are 2π −periodic functions of φ and the series on the right-hand side of (2.2) is convergent for all φ and all sufficiently small r, [4].

Since the closed orbits of (2.1) correspond to periodic solutions of (2.2), the planar analytic vector field (2.1) has a centre at (0,0) if and only if (2.2) has a centre at $r = 0$; that is, all the solutions nearby are periodic.

All methods for determining the nature of the critical point of (2.1) theoretically require determining infinitely many coefficients of some function or as we will see later the corresponding ideal of the infinitely many polynomials, [4,5,6,7]. The most widely used method to resolve the Poincare centre-focus problem are the so-called Lyapunov method (combined with the Bautin method) and Mironenko's method, [8]. Solving the Poincaré centre-focus problem for a given system (2.1) may be a tedious job. However, once we know the origin of (2.1) is a centre, the problem of isochronicity becomes sensible, and the definition of the period function makes sense.

Definition 2 ([4]). Suppose $r^* > 0$ is so small that it lies in the so-called period annulus (the maximal neighbourhood of the origin that is enclosed with simple closed curves). Let us consider the line segment $\Sigma = \{(x,y) \colon 0 < x < r^*, y = 0\}$. For any $x = (r, 0) \in \Sigma$ satisfying $0 < r < r^*$ let $T(r)$ denote the least period of the corresponding trajectory (through $(x, y) = (r, 0) \in \Sigma$). The function $T(r)$ is the *period function* of the centre. If the function $T(r)$ is constant, then the centre $(0,0)$ of (2.1) is said to be *isochronous.*

In (2.2), the corresponding time-dependent equations are of the form

$$\frac{dr}{dt} = \sum_{i=1}^{\infty} A_i(\varphi) r^{i+1} \tag{2.3}$$

and

$$\frac{d\varphi}{dt} = 1 + \sum_{i=1}^{\infty} B_i(\varphi) r^i. \tag{2.4}$$

For a sufficiently small (fixed) r_0 a solution to (2.3) satisfying the initial condition $(r, \varphi) = (r_0, 0)$ has a unique solution $r(\varphi) = f(\varphi; r_0)$. Inserting $f(\varphi; r_0)$ into (2.4) yields

$$\frac{dt}{d\varphi} = \frac{1}{1 + \sum_{i=1}^{\infty} M_i(\varphi) r_0^i} = 1 + \sum_{i=1}^{\infty} F_i(\varphi) r_0^i.$$

After integration, we obtain

$$t - \varphi = \sum_{i=1}^{\infty} \Theta_i(\varphi) r_0^i, \tag{2.5}$$

where $\Theta_i(\varphi) = \int_0^{\varphi} F_i(\varphi) d\varphi$ and the series on the right-hand side of (2.5) is convergent (for small enough $0 < r_0 < r^*$) and $0 \le \varphi \le 2\pi$. For any $(r_0, 0) \in \Sigma$ for $\varphi = 2\pi$ we write $t = T$ and obtain

$$T(r_0) = 2\pi \left(1 + \sum_{i=1}^{\infty} T_i r_0^i \right), \tag{2.6}$$

where $T_i = \frac{\Theta_i(2\pi)}{2\pi}$. No, we can finally state the condition for isochronicity of the centre.

Suppose system (2.1) has a centre at the origin, then $(0,0)$ is isochronous if and only if $T_i = 0$ for all $i \in \mathbb{N}$. Thus, the set of all systems (2.1) with isochronous centre on the centre manifold is the set of common zeros of polynomials $T_i(p_{ij}, q_{ij})$. Thus, we have to consider the ideal $I = \langle T_1(p_{ij}, q_{ij}), T_2(p_{ij}, q_{ij}), T_3(p_{ij}, q_{ij}), \dots \rangle$. According to the Hilbert Basis Theorem (see [9]), every ideal in the polynomial ring over a field is finitely generated, and then the direct consequence of this theorem is that the ascending chain of ideals $I_1 = \langle T_1(p_{ij}, q_{ij}) \rangle$, $I_2 = \langle T_1(p_{ij}, q_{ij}), T_2(p_{ij}, q_{ij}), \dots \rangle$, stabilizes. That is, there exists $m \ge 1$ such that ideal I with infinitely many polynomials $T_i(p_{ij}, q_{ij})$ is the same as I_m.

3   CENTRE MANIFOLDS

The theory of centre manifolds, which originates from the works of V. A. Pliss, [10,11], and was then further developed by many others (see, e.g. [12,13] and references therein) is an extremely effective tool for studying the behaviour of trajectories of high-dimensional systems of ordinary differential equations. Note that one of the important applications of centre manifolds (also called the Pliss reduction principle, [11]) is that it allows the reduction of the system and consequently the study of the stability in lower dimensional phase space (thus, instead of analysing the original high-dimensional phase space we can consider the lower-dimensional subsystem).
Consider an **m + n**-dimensional system of ordinary differential equations of the form

$$\dot{x} = Ax + u(x, y)$$
$$\dot{y} = By + v(x, y), \tag{3.1}$$

where $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, $\text{Re}(\sigma(A)) = 0$, $\text{Re}(\sigma(B)) \ne 0$, $\sigma(A)$ and $\sigma(A)$ are spectrums of A and B, respectively, and u, v are C^k- functions, $k \ge 1$ which vanish together with their first derivatives

at the origin. By definitio, a C^k- manifold $W^c \equiv W^\wedge c(0, U)$ in a neighbourhood U of 0 is said to be a centre manifold of (3.1) if W^c is invariant under the flow as long as the solution remains in U nd W^c is the graph of a C^k- function $y = h(x)$ which is tangent at $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ to the x-space.

The following fundamental result shows that for the system (3.1), there is always a centre manifold in a neighbourhood of the origin.

Theorem 1 ([12]). There exists a neighbourhood U of $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that there exists a local centre manifold W^c of (3.1) which is the graph of *a C^k-* function $y = h(x)$.

The simplest examples of the centre manifolds can be seen in low dimensions. For instance, consider two-dimensional system

$$\dot{x} = -x^3, \quad \dot{y} = -y.$$

$$(3.2)$$

For any constants c_1, c_2 the curve $h(x; c_1, c_2)$, where

$$y = h(x; , c_1, c_2) = \begin{cases} -c_1 e^{-\frac{1}{2x^2}}, & x < 0 \\ 0, & x = 0 \\ c_2 e^{-\frac{1}{2x^2}}, & x > 0, \end{cases}$$

consists of orbits of (3.2) and is therefore invariant. Furthermore, this curve is tangent to the x-axis at the origin, which implies that it is a centre manifold. From this example, we can also see that the centre manifold is not necessaily unique. Another simple two-dimensional system with a centre manifold is

$$\dot{x} = -x^3, \quad \dot{y} = -y + x^2.$$

Similar as in the previous case, we see that although the system is analytic, it is not difficult to verify that the centre manifold is not analytic.

In three dimension, a family of systems with a centre manifold is

$$\begin{aligned} \dot{u} &= -v + P(u, v, w) \\ \dot{v} &= u + Q(u, v, w) \\ w &= -\lambda w + R(u, v, w), \end{aligned}$$

$$(3.3)$$

where λ is a positive real number and P, Q, R are polynomials without constant and linear terms. By Theorem 1, this system has a centre manifold $w = f(u, v)$. There are many systems arising from physics (for instance, the Rikitake system [14], for Earth's magnetic field or the Hide-Acheson Dynamo, [15]) that possess a fixed point at which the linear part has one negative and two purely imaginary eigenvalues and are therefore of the form (3.3). Since system (3.3) has a centre manifold W^c and $\lambda > 0$, the trajectories in a small neighbourhood of the origin tend to the trajectories to the centre manifold as time increases. In systems (3.3), the phase portrait in a neighbourhood of the origin on W^c can be, depending on the added nonlinear terms P and Q, either a centre in which case every trajectory (other than the origin itself) is an oval surrounding the origin, or a focus, in which case every trajectory spirals towards the origin or every trajectory spirals away from the origin as the time increases. The problem of determining the dynamical

behaviour on W^c, that is, distinguishing between a centre and a focus on the centre manifold for a quadratic polynomial system of the form (3.3) was studied in [16].

4   THE STUDY OF TWO EXAMPLES

In [16], the authors studied the dynamics of trajectories at the centre manifolds for the system

$$\dot{u} = -v + au^2 + av^2 + cuw + dvw$$
$$\dot{v} = u + bu^2 + bv^2 + euw + fvw \qquad\qquad (4.1)$$
$$\dot{w} = -w + Su^2 + Sv^2 + Tuw + Uvw,$$

where the coefficients $a, b, c, d, e, f, S, T, U$ are real. They found five conditions for the existence of a centre on the centre manifold:

1. $S = 0$;

2. $a = b = c + f = 8c + T^2 - U^2 = 4(e - d) - T^2U^2 = 2(e + d) + TU$ and $S = 1$;

3. $a = b = c = f = d + e = 0$ and $S = 1$;

4. $d + e = c = f = T - 2a = U - 2b = 0$ and $S = 1$;

5. $c = d = e = f = 0$ and $S = 1$.

The next question that naturally arises is whether the centre at the centre manifold is isochronous, that is, whether all oscillations have the same period. In this paper we study two subfamilies of system (4.1) satisfying conditions 1 and 2 above. For each system, we compute the period function as described in section 2 and use it to find conditions for the centre to be isochronous.

System (4.1) satisfying condition 1 above is written as

$$\dot{u} = -v + au^2 + av^2 + cuw + dvw$$
$$\dot{v} = u + bu^2 + bv^2 + euw + fvw \qquad\qquad (4.2)$$
$$\dot{w} = -w + Tuw + Uvw.$$

It is proven (see [16]) that $w = 0$ isthe centre manifold for system (4.2). The corresponding 2D system

$$\dot{u} = -v + au^2 + av^2 \qquad\qquad (4.3)$$
$$\dot{v} = u + bu^2 + bv^2$$

has a centre at the origin for all $a, b \in \mathbb{R}$ (see [16]).

We now study the isochronicity problem for the above centre. Using the computer algebra software MATHEMATIC, we first turn to the computation of the period function T of the form (2.6). After introducing polar coordinates system, (4.3) becomes

$$\frac{dr}{dt} = r^2(a \cos \varphi + b \sin \varphi)$$

$$\frac{d\varphi}{dt} = 1 + r(b \cos \varphi - a \sin \varphi). \tag{4.4}$$

Following the procedure described in the second section, we find that the first non-zero coefficient of the period function is $T_2 = 2\pi(a^2 + b^2)$. Thus we see that the necessary condition for the isochronicity of system (4.3) is $a = b = 0$, which, obviously is also the sufficient condition.

In the case of conditions 2, under the same renaming of parameters c, d, e, f, T, U and using α, β as in [16], the system (4.1) takes the form

$$\dot{u} = -v - \frac{1}{2}\alpha\beta uw - \frac{1}{2}\beta^2 vw$$

$$\dot{v} = u + \frac{1}{2}\alpha^2 uw + \frac{1}{2}\alpha\beta vw \tag{4.5}$$

$$\dot{w} = -w + u^2 + v^2 + (\alpha + \beta)uw + (\beta - \alpha)vw.$$

A search for invariant algebraic surfaces led to the explicit equation of centre manifold W^c given by $w = \frac{u^2+v^2}{1-\alpha u-\beta v}$. Inserting the expression for w into system (4.5), we obtain the system

$$u = -v - \frac{\beta(\alpha u + \beta v)(u^2 + v^2)}{2(1 - \alpha u - \beta v)}$$

$$v = u + \frac{\alpha(\alpha u + \beta v)(u^2 + v^2)}{2(1 - \alpha u - \beta v)}.$$

Using Taylor series expansion up to the order five, we obtain the system

$$\dot{u} = -v + \frac{1}{2}\beta(\alpha u + \beta v)(u^2 + v^2) + \frac{1}{2}\beta(\alpha u + \beta v)^2(u^2 + v^2) + m_1$$

$$v = u - \frac{1}{2}\alpha(\alpha u + \beta v)(u^2 + v^2) - \frac{1}{2}\alpha(\alpha u + \beta v)^2(u^2 + v^2) + m_2, \tag{4.6}$$

where

$$m_1 = \frac{1}{120}(-60\alpha^3\beta u^5 - 180\alpha^2\beta^2 u^4 v + 10u^3 v^2(-6\alpha^3\beta - 18\alpha\beta^3)$$
$$+10u^2 v^3(-18\alpha^2\beta^2 - 6\beta^4) - 180\alpha\beta^3 uv^4 - 60\beta^4 v^5)$$

and

$$m_2 = \frac{1}{120}(60\alpha^4 u^5 + 180\alpha^3\beta u^4 v + 10u^3 v^2(6\alpha^4 + 18\alpha^2\beta^2)$$
$$+10u^2 v^3(18\alpha^3\beta + 6\alpha\beta^3) + 180\alpha^2\beta^2 uv^4 + 60\alpha\beta^3 v^5).$$

Further computation following the computational pattern described in section 2 yields $T_2 = \frac{\pi}{2}(\alpha^2 + \beta^2)$, yielding the following result.

System (4.6) has an isochronous centre if and only if $\alpha = \beta = 0$.

5   CONCLUSIONS

After more than 300 years, the isochronicity problem remains one of the central problems in the theory of dynamical systems. In addition to the period function approach, there are some other approaches. The analysis of the isochronicity problem leads us first to conditions for existence of periodic solutions, i.e. the existence of centre and then the further study yields the conditions on parameters of considered system for the cases of isochronous and non-isochronous oscillations. The approach described in Section 2 requires an integration of trigonometric functions and might be too complex for some systems. Therefore, in some cases, we need to find other ways to solve the isochronicity problem. It turns out that the isochronicity problem for system (2.1) is equivalent to the linearizability problem in which we look for an analytic change of coordinates that reduces (2.1) to the canonical linear centre $\dot{x} = -y, \dot{y} = x$ (see e.g. [4] for more details). The theory of the linearizability of planar systems of ordinary differential equations was applied [17] to system (4.1) corresponding to condition 4 above. In fact, for each of the five conditions above we can find the conditions under which the corresponding system can be reduced to canonical linear centre, and we see that they are the same as conditions obtained using the period function approach.

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