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Cayley graphs on nilpotent groups with cyclic commutator subgroup are hamiltonian

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Abstract

We show that if G is any nilpotent, finite group, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.

Keywords: Cayley graph, hamiltonian cycle, nilpotent group, commutator subgroup.

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1 Introduction

It has been conjectured that every connected Cayley graph has a hamiltonian cycle. See [4, 10, 11, 14, 15, 17] for references to some of the numerous results on this problem that have been proved in the past forty years, including the following theorem that is the culmination of papers by Marušič [12], Durnberger [5, 6], and Keating-Witte [9]:

Theorem 1.1 (D. Marušič, E. Durnberger, K. Keating, and D. Witte, 1985)**.** Let G be a nontrivial, finite group. If the commutator subgroup $[G, G]$ of G is cyclic of prime-power order, then every connected Cayley graph on G has a hamiltonian cycle.

It is natural to try to prove a generalization that only assumes the commutator subgroup is cyclic, without making any restriction on its order, but that seems to be an extremely difficult problem: at present, it is not even known whether all connected Cayley graphs on dihedral groups have hamiltonian cycles. (See [1, 2] and [15, Cor. 5.2] for the main results that have been proved for dihedral groups.) In this paper, we replace the assumption on the order of $[G, G]$ with the rather strong assumption that G is nilpotent:

Theorem 1.2. Let G be a nontrivial, finite group. If G is nilpotent, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.

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The proof of this theorem is based on a variant of the method of D. Marušič [12] that established theorem 1.1 (cf. [9, Lem. 3.1]).

Remark 1.3. Here are some previous results on the hamiltonicity of the Cayley graph $\mathrm{Cay}(G;S)$ when G is nilpotent:

1. Assume G is nilpotent, the commutator subgroup of G is cyclic, and S has only two elements. Then a hamiltonian cycle in $\mathrm{Cay}(G;S)$ was found in [9, §6] (see proposition 3.16). The present paper generalizes this by eliminating the restriction on the cardinality of the generating set S.
2. For Cayley graphs on nilpotent groups (without any assumption on the commutator subgroup) it was recently shown that if the valence is at most 4, then there is a hamiltonian *path* (see [13]).
3. Every nilpotent group is a direct product of p-groups. For p-groups, it is known that every Cayley graph has a hamiltonian cycle ([16], see theorem 3.13). Unfortunately, we do not know how to extend this to direct products.
4. Every abelian group is nilpotent. It is well known (and easy to prove) that Cayley graphs on abelian groups always have hamiltonian cycles. In fact, there is usually a hamiltonian path from any vertex to any other vertex (see [3]).

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2 Assumptions and notation

We begin with some standard notation:

Notation 2.1. Let G be a group, and let S be a subset of G.

- $\mathrm{Cay}(G;S)$ denotes the *Cayley graph* of G with respect to S. Its vertices are the elements of G, and there is an edge joining g to gs for every $g \in G$ and $s \in S$.
- $G' = [G,G]$ denotes the commutator subgroup of G.
- $S^r = \{\, s^r \mid s \in S \,\}$ for any $r \in \mathbb{Z}$. (Similarly, $G^r = \{\, g^r \mid g \in G \,\}$.)
- $S^{\pm 1} = S \cup S^{-1}$.
- $\#S$ is the cardinality of S.

Note that if S happens to be a cyclic subgroup of G, then S^r is a subgroup of S.

We now fix notation designed specifically for our proof of theorem 1.2:

Notation 2.2.

- G is a nilpotent, finite group,
- N is a cyclic, normal subgroup of G that contains G',
- $g \mapsto \overline{g}$ is the natural homomorphism from G to $G/N = \overline{G}$,
- $S = \{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$ is a subset of G, such that

- $\overline{S}$ is a minimal generating set for $\overline{G}$, and
- $\ell = \#S = \#\overline{S} \geq 2$,

The minimality of $\overline{S}$ implies that $e \notin S$, and that if $s \in S$ and $|s| \geq 3$, then $s^{-1} \notin S$.

- $S_k = \{ \sigma_i \mid i \leq k \}$ for $1 \leq k \leq \ell$,
- $G_k = \langle S_k \rangle N$, and
- $m_k = |\overline{G_k} : \overline{G_{k-1}}|$.

Definition 2.3.

- If $(s_i)_{i=1}^n$ is a sequence of elements of $S^{\pm 1}$ and $\overline{g} \in \overline{G}$, we use $\overline{g}(s_i)_{i=1}^n$ to denote the walk in $\mathrm{Cay}(\overline{G}; \overline{S})$ that visits (in order) the vertices

$$\overline{g}, \overline{gs_1}, \overline{gs_1s_2}, \ldots, , \overline{gs_1s_2 \cdots s_n}.$$

- If $C = \overline{g}(s_i)_{i=1}^n$ is any oriented cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$, its *voltage* is $\prod_{i=1}^n s_i$. This is an element of N, and it may be denoted ΠC.
- For $S_0 \subset S$, we say the walk $\overline{g}(s_i)_{i=1}^n$ *covers* $S_0^{\pm 1}$ if it contains an oriented edge labeled s and a (different) oriented edge labeled s^{-1}, for every $s \in S_0$. (That is, there exist i, j with $i \neq j$, such that $s_i = s$ and $s_j = s^{-1}$. When $|s| = 2$, this means $s_i = s_j = s$.)
- $\mathcal{V}_k$ is the set of voltages of oriented hamiltonian cycles in the graph $\mathrm{Cay}(\overline{G_k}; \overline{S_k})$ that cover $S_k^{\pm 1}$.

Notation 2.4. For $k \in \mathbb{Z}^+$, we use $(s_1, \ldots, s_n)^k$ to denote the concatenation of k copies of the sequence $(s_1, \ldots, s_n)$. Abusing notation, we often write s^k and s^{-k} for

$$(s)^k = (s, s, \ldots, s) \text{ and } (s^{-1})^k = (s^{-1}, s^{-1}, \ldots, s^{-1}),$$

respectively. Furthermore, we often write $\big((s_1, \ldots, s_m), (t_1, \ldots, t_n)\big)$ to denote the concatenation $(s_1, \ldots, s_m, t_1, \ldots, t_n)$. For example, we have

$$\big((a^3, b)^2, c^{-2}\big)^2 = (a, a, a, b, a, a, a, b, c^{-1}, c^{-1}, a, a, a, b, a, a, a, b, c^{-1}, c^{-1}).$$

The following well-known, elementary observation is the foundation of our proof:

Lemma 2.5 ("Factor Group Lemma" [17, §2.2])**.** Suppose

- N is a cyclic, normal subgroup of G,
- $C = \overline{g}(s_i)_{i=1}^n$ is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$, and
- the voltage ΠC generates N.

Then $(s_1, \ldots, s_n)^{|N|}$ is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

With this in mind, we let $N = G'$, and we would like to find a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage generates N. In almost all cases, we will do this by induction on

$\ell = \#S$, after substantially strengthening the induction hypothesis. Namely, we consider the following assertion (α_k^ϵ) for $2 \le k \le \ell$ and $\epsilon \in \{1, 2\}$:

$$\begin{gathered}\text{there exists } h_k \in N, \text{ such that, for every } x \in N,\\ \big(\mathcal{V}_k \cap h_k(G'_k)^\epsilon\big)x \text{ contains a generator of}\\ \text{a subgroup of } N \text{ that contains } (G'_k)^\epsilon.\end{gathered} \tag{α_k^ϵ}$$

For $\epsilon = 2$, we also consider the following slightly stronger condition, which we call α_k^{2+}:

$$\alpha_k^2 \text{ holds, and } \langle h_k, (G'_k)^2\rangle \text{ contains } G'_k. \tag{α_k^{2+}}$$

Lemma 2.6. Let $N = G'$. If either α_ℓ^1 or α_ℓ^{2+} holds, then there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage generates N.

Proof. Note that $G'_\ell = G' = N$. Since $\mathcal{V}_\ell$ consists of voltages of hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{S})$, it suffices to find an element of $\mathcal{V}_\ell$ that generates G'_ℓ.

If we assume α_ℓ^1, then the desired conclusion is immediate, by taking $x = e$ in that assertion.

Similarly, if we assume α_ℓ^{2+}, then taking $x = e$ in α_ℓ^2 tells us that some element γ of $\mathcal{V}_\ell \cap h_\ell(G'_\ell)^2$ generates a subgroup of N that contains $(G'_\ell)^2$. Then, since $\gamma \in (G'_\ell)^2 h_\ell$, and $\langle h_\ell, (G'_\ell)^2\rangle$ contains G'_ℓ, we have

$$N \supset \langle\gamma\rangle = \langle\gamma, (G'_\ell)^2\rangle = \langle h_\ell, (G'_\ell)^2\rangle \supset G'_\ell = N. \qquad \square$$

Remark 2.7.

1. If $|G'_k|$ is odd, then $(G'_k)^2 = G'_k$, so we have $\alpha_k^1 \Leftrightarrow \alpha_k^2 \Leftrightarrow \alpha_k^{2+}$ in this case. Thus, the parameter ϵ is only of interest when $|G'|$ is even.
2. It is not difficult to see that $\alpha_k^1 \Rightarrow \alpha_k^{2+}$, but we do not need this fact.

Our proof of α_ℓ^1 or α_ℓ^{2+} is by induction on k. Here is the outline:

I. We prove a base case of the induction: α_2^2 is usually true (see proposition 4.1).

II. We prove an induction step: under certain conditions, $\alpha_k^1 \Rightarrow \alpha_{k+1}^1$ and $\alpha_k^{2+} \Rightarrow \alpha_{k+1}^{2+}$ (see proposition 5.4).

III. We prove α_ℓ^1 or α_ℓ^{2+} is usually true, by bridging the gap between α_2^2 and either α_3^1 or α_3^{2+}, and then applying the induction step (see corollary 6.1 and proposition 6.2).

3 Preliminaries

3A Remarks on voltage

Remark 3.1. By definition, it is clear that all translates of an oriented cycle C in $\mathrm{Cay}(\overline{G}; \overline{S})$ have the same voltage. That is,

$$\Pi\big(\overline{g}(s_i)_{i=1}^n\big) = \Pi\big((s_i)_{i=1}^n\big).$$

Remark 3.2. If $|N|$ is square-free (which is usually the case in this paper), then N is contained in the center of G (because N is the direct product of normal subgroups of prime order, and it is well known that those are all in the center [8, Thm. 4.3.4]). In this situation,

the voltage of a cycle in $\mathrm{Cay}(\overline{G};\overline{S})$ is independent of the starting point that is chosen for its representation. That is, if $(t_i)_{i=1}^n$ is a cyclic rotation of $(s_i)_{i=1}^n$, so there is some $r \in \{0, 1, 2, \ldots, n\}$ with $t_i = s_{i+r}$ for all i (where subscripts are read modulo n), then

$$\Pi(t_i)_{i=1}^n = s_{r+1}s_{r+2}\cdots s_n\, s_1 s_2 \cdots s_r = (s_1 \cdots s_r)^{-1}\big(\Pi(s_i)_{i=1}^n\big)s_1 \cdots s_r = \Pi(s_i)_{i=1}^n,$$

because $\Pi(s_i)_{i=1}^n \in N \subset Z(G)$.

The following observation is useful for calculating voltages:

Lemma 3.3. If $a, b \in G$, $G' \subset Z(G)$, and $p_1, q_1, \ldots, p_r, q_r \in \mathbb{Z}$, then

$$a^{p_1}b^{q_1}a^{p_2}b^{q_2}\cdots a^{p_r}b^{q_r} = a^{p_1+\cdots+p_r}b^{q_1+\cdots+q_r}[a,b]^{-\Sigma},$$

where $\Sigma = \sum_{i>j} p_i q_j$.

Proof. The desired conclusion is easily proved by induction on r, using the fact that, since $G' \subset Z(G)$, we have $[a^p, b^q] = [a, b]^{pq}$ for $p, q \in \mathbb{Z}$ [7, Lem. 2.2(i)]. □

3B Facts from group theory

Lemma 3.4. If $|G'_k|$ is square-free, then $|G'_k/G'_{k-1}|$ is a divisor of both $|\overline{G_{k-1}}|$ and $|\overline{G_k}/\overline{G_{k-1}}|$.

Proof. We may assume $k = \ell$, so $G = G_k$. Let p be a prime factor of $|G'/G'_{k-1}|$, let P be the Sylow p-subgroup of G, and let $\varphi\colon G \to P$ be the natural projection. Since $|G'|$ is square-free, it suffices to show that $|\overline{G_{k-1}}|$ and $|\overline{G_k}/\overline{G_{k-1}}|$ are divisible by p.

We may assume $|G'| = p$ and $G'_{k-1} = \{e\}$, by modding out the unique subgroup of index p in G'. Therefore $\varphi(G_{k-1})$ is abelian, so it is a proper subgroup of P. Since $G' = P' \subset \Phi(P)$, this implies $\varphi(G_{k-1})G'$ is a proper subgroup of P, so its index is divisible by p. Hence $|\overline{G}/\overline{G_{k-1}}|$ is divisible by p.

There must be some $t \in S_{k-1}$, such that $[\sigma_k, t]$ is nontrivial. Hence $\varphi(t) \notin Z(G) \supset G'$, so p is a divisor of $|\overline{\varphi(t)}|$, which is a divisor of $|\overline{G_{k-1}}|$. □

The following fact is well known and elementary, but we do not know of a reference in the literature. It relies on our assumption that G' is cyclic.

Lemma 3.5. We have $\langle\, [s,t] \mid s, t \in S \,\rangle = G'$ if $N \subset Z(G)$.

Proof. Let $H = \langle\, [s,t] \mid s, t \in S \,\rangle$. Then H is a normal subgroup of G, because every subgroup of a cyclic, normal subgroup is normal. In G/H, every element of S commutes with all of the other elements of S (and with all of N), so G/H is abelian. Hence $G' \subset H$. □

3C Elementary facts about cyclic groups of square-free order

Lemma 3.6. Assume $|N|$ is square-free, and H and K are two subgroups of N. Then:

1. There is a unique subgroup $K^\perp$ of N, such that $N = K \times K^\perp$.
2. $K^\perp$ is a normal subgroup of G.
3. $K \subseteq H$ iff $\underline{H} = \underline{N}$ in $\underline{G} = G/K^\perp$.

Proof. (1) Since N is cyclic, it has a unique subgroup of any order dividing $|N|$; let $K^\perp$ be the subgroup of order $|N/K|$. Since $|N|$ is square-free, we have $\gcd\big(|K|, |K^\perp|\big) = 1$, so $N = K \times K^\perp$.

(2) It is well known that every subgroup of a cyclic, normal subgroup is normal (because no other subgroup of N has the same order).

(3) We prove only the nontrivial direction. Since $\underline{H} = \underline{N}$, we know that $|K| = |\underline{N}|$ is a divisor of $|H|$. So $|H|$ has a subgroup whose order is $|K|$. Since K is the only subgroup of N with this order, we must have $K \subseteq H$. □

Remark 3.7. When we want to show that some subgroup H of N contains some other subgroup K, Lemma 3.6 often allows us to assume $K = N$ (by modding out $K^\perp$), which means we wish to prove $H = N$.

Lemma 3.8. Suppose

- γ is a generator of N,
- $x \in N$, and
- $a \geq \max\big(|N|, 5\big)$.

Then, for some i with $1 \leq i \leq \lfloor (a-1)/2 \rfloor$, we have $N^2 \subseteq \langle \gamma^{-2i} x \rangle$.

Proof. Write $x = \gamma^h$, where $1 \leq h \leq |N|$, choose $r \in \{1, 2\}$ such that $h - r$ is even, and let

$$i = \begin{cases} r & \text{if } h \in \{1, 2\}, \\ (h-r)/2 & \text{if } h > 2. \end{cases}$$

Then $h - 2i \in \{\pm r\} \subset \{\pm 1, \pm 2\}$, so $N^2 \subseteq \langle \gamma^{h-2i} \rangle = \langle \gamma^{-2i} x \rangle$. □

Lemma 3.9. If

- N is a cyclic group of square-free order,
- $m \geq |N|$,
- $k \geq 2$,
- $T = \{\gamma_1, \ldots, \gamma_k\}$ generates N,
- $h \in N$, and
- $\mathrm{Cay}(N; T)$ is not bipartite,

then we may choose a sequence $(j_i)_{i=1}^{m-1}$ of elements of $\{1, 2, \ldots, k\}$, and $\gamma_i^* \in \{\gamma_{j_i}^{\pm 1}\}$ for each i, such that $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$, and

$$\langle h \gamma_1^* \gamma_2^* \cdots \gamma_{m-1}^* \rangle = N. \tag{3.10}$$

Proof. Let us assume $|N| > 3$. (The smaller cases are very easy to address individually.)

We begin by finding $\gamma_1^*, \gamma_2^*, \ldots, \gamma_{m-1}^* \in T^{\pm 1}$, such that $\langle h \gamma_1^* \gamma_2^* \cdots \gamma_{m-1}^* \rangle$ contains N^2 (or N, if appropriate), but without worrying about the requirement that $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$.

Let γ be a generator of N, and assume $h^{-1}\gamma \neq e$ (by replacing γ with its inverse, if necessary). Since $\mathrm{Cay}(N; T)$ is not bipartite, there is a walk $(\gamma_i^*)_{i=1}^r$ from e to $h^{-1}\gamma$, such that $r \equiv m - 1 \pmod 2$.

We now show the walk can be chosen to satisfy the additional constraint that $r < |N|$ (so $r \leq m-1$). We know that $\mathrm{Cay}(N;T)$ has a hamiltonian cycle C (since N is abelian). Since $\mathrm{Cay}(N;T)$ is not bipartite, C must have a chord L of even length. We may assume one endpoint of L is e, since $\mathrm{Cay}(N;T)$ is vertex transitive. Let z be the other endpoint of L. Being a hamiltonian cycle, C can be written as the union of two edge-disjoint paths from e to $h^{-1}\gamma$. Let P be the one of these paths that contains a subpath of even length from e to z, and let $\widehat{P}$ be obtained from P by replacing this subpath with the edge L. Then P and $\widehat{P}$ are two paths from e to $h^{-1}\gamma$. Both have length less than $|N|$, and their lengths are of opposite parity.

Now

$$h\gamma_1^*\gamma_2^*\cdots\gamma_r^*(\gamma_1\gamma_1^{-1})^{(m-1-r)/2} = \gamma \text{ generates } N,$$

as desired.

To complete the proof of the lemma, we modify the above sequence $\gamma_1^*, \gamma_2^*, \ldots, \gamma_{m-1}^*$ to satisfy the condition that $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$. First of all, since N is commutative, we may collect like terms, and thereby write

$$\gamma_1^*\gamma_2^*\cdots\gamma_{m-1}^* = \gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_k^{m_k}\gamma_1^{-n_1}\gamma_2^{-n_2}\cdots\gamma_k^{-n_k}$$

where $m_1+\cdots+m_k+n_1+\cdots+n_k = m-1$. Notice that if m_k and n_1 are both nonzero, then no occurrence of γ_i is immediately followed by γ_i^{-1}; so we have $\gamma_{i+1}^* = \gamma_i^*$ whenever $j_{i+1} = j_i$, as desired. Therefore, by permuting $\gamma_1, \ldots, \gamma_k$, we may assume $m_i = n_i = 0$ for all $i > 1$. Also, we may assume m_1 and n_1 are both nonzero, for otherwise we have $\gamma_i^* = \gamma_j^*$ for all i and j. Then, since $\gamma_1\gamma_1^{-1} = \gamma_2\gamma_2^{-1}$, we have

$$\gamma_1^*\gamma_2^*\cdots\gamma_{m-1}^* = \gamma_1^{m_1}\gamma_1^{-n_1} = \gamma_1^{m_1-1}\gamma_2\gamma_1^{-(n_1-1)}\gamma_2^{-1}.$$

We can assume $n_1 \geq m_1$ (by replacing γ_1 with its inverse, if necessary). Then

$$n_1 \geq \left\lceil \frac{m-1}{2} \right\rceil \geq \left\lceil \frac{|N|-1}{2} \right\rceil \geq 2,$$

so this new representation of the same product satisfies the condition that γ_i is never immediately followed by γ_i^{-1}. This completes the proof. □

Corollary 3.11. Assume $|N|$ is square-free and $k \geq 2$ (and $\epsilon \in \{1,2\}$, as usual). For convenience, let $m = m_{k+1}$ and $a = \sigma_{k+1}$. If $h \in N$, then there exists a sequence $(s_i)_{i=1}^{m-1}$ of elements of S_k, and $s_i^* \in \{s_i^{\pm 1}\}$ for each i, such that $s_{i+1}^* = s_i^*$ whenever $s_{i+1} = s_i$, and

$$\left\langle h\prod_{i=1}^{m-1}[a,s_i^*],\ (G_k')^\epsilon \right\rangle \text{ contains } \begin{cases} G_{k+1}' & \text{if there exist } s,s' \in S_k, \text{ such that} \\ & |[a,s]| \text{ is even and } |[a,s']| \text{ is odd,} \\ (G_{k+1}')^\epsilon & \text{if } |G_{k+1}'/G_k'| \text{ is odd,} \\ (G_{k+1}')^2 & \text{otherwise.} \end{cases} \tag{3.12}$$

Proof. For each $s \in S_k$, let $\gamma_s = [a,s]$. Also, let

$$T = \{\, \gamma_s \mid s \in S_k \,\} \subseteq G_{k+1}' \subseteq N.$$

Let

$$\widehat{\epsilon} = \begin{cases} 1 & \text{if there exist } t, t' \in T, \text{ such that } |t| \text{ is even and } |t'| \text{ is odd,} \\ \epsilon & \text{if } |G'_{k+1}/G'_k| \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 3.6 allows us to assume $(G'_{k+1})^{\widehat{\epsilon}} = N$, by modding out $\big((G'_{k+1})^{\widehat{\epsilon}}\big)^{\perp}$. We can also assume $(G'_k)^{\epsilon}$ is trivial, by modding it out.

We claim that $\langle T \rangle = N$. We have

$$\begin{aligned} \big\langle T, G'_k \big\rangle &\supseteq \big\langle \{\, [\sigma_{k+1}, s] \mid s \in S_k \,\}, \{\, [s,t] \mid s,t \in S_k \,\} \big\rangle \\ &= \big\langle \{\, [s,t] \mid s,t \in S_{k+1} \,\} \big\rangle \\ &= G'_{k+1} \qquad \text{(see Lemma 3.5)} \\ &= N. \end{aligned}$$

Since $(G'_k)^{\epsilon}$ is trivial, this implies $|N : \langle T \rangle| \leq \epsilon$. Thus, if the claim is not true, must have $|N : \langle T \rangle| = \epsilon = 2$. In particular, $|N|$ is even. Since $(G'_{k+1})^{\widehat{\epsilon}} = N$, we conclude that $\widehat{\epsilon} = 1$, so the definition of $\widehat{\epsilon}$ (together with the fact that $\epsilon = 2$) implies that T contains an element of even order. So $|\langle T \rangle|$ is even, which contradicts the fact that $|N : \langle T \rangle| = 2$ is even (and $|N|$ is square-free). This completes the proof of the claim.

We also claim that $\mathrm{Cay}(N; T)$ is not bipartite. We may assume $|N|$ is even (for otherwise the claim is obvious). Since $(G'_{k+1})^{\widehat{\epsilon}} = N$, this implies $\widehat{\epsilon} = 1$. However, the claim is also obviously true if there exist $t, t' \in T$, such that $|t|$ is even and $|t'|$ is odd. Hence, we may assume $|G'_{k+1}/G'_k|$ is odd, and $\epsilon = \widehat{\epsilon} = 1$. Since $(G'_k)^{\epsilon}$ is trivial, this implies $|G'_k|$ is odd, which contradicts the fact that $|N|$ is even. This completes the proof of the claim.

The desired conclusion is now immediate from Lemma 3.9 (since $\gamma_s = [a, s]$ and $(\gamma_s)^{-1} = [a, s^{-1}]$). □

3D Results from [9] and [16]

The following result from [16] allows us to assume G is not a 3-group. (Since we always assume that G' is cyclic, a short proof of the special case we need can be found in [15, Thm. 6.1].)

Theorem 3.13 (Witte [16])**.** If $|G|$ is a power of some prime p, then every connected Cayley graph on G has a hamiltonian cycle.

The following simple observation usually allows us to assume $|N|$ is square-free.

Lemma 3.14 ([9, Lem. 3.2])**.** Let $\underline{G} = G/\Phi(N)$, where $\Phi(N)$ is the Frattini subgroup of N [8, §10.4]. Then:

1. $|\underline{N}|$ is square-free, and
2. if there is a hamiltonian cycle in $\mathrm{Cay}(\underline{G}/\underline{N}; S)$ whose voltage generates $\underline{N}$, then there is a hamiltonian cycle in $\mathrm{Cay}(G/N; S)$ whose voltage generates N.

Lemma 3.15 (Keating-Witte [9, Case 6.1])**.** If $|\overline{G_2}|$ is even, then $\mathrm{Cay}\big(\overline{G_2}; \overline{S_2}\big)$ has a hamiltonian cycle whose voltage is a generator of G'_2.

Proof. For the reader's convenience, we provide a proof. We may assume $|\overline{\sigma_1}|$ is even (by interchanging σ_1 and σ_2 if necessary). For convenience, let $n = |\overline{\sigma_1}|$ and $m = m_2$. Then

$$\big(\sigma_2^{m-1}, (\sigma_1, \sigma_2^{-(m-2)}, \sigma_1, \sigma_2^{m-2})^{(n-2)/2}, \sigma_1, \sigma_2^{-(m-1)}, \sigma_1^{-(n-1)}\big)$$

is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_2}; \overline{S_2})$.

Lemma 3.14 allows us to assume $|G_2'|$ is square-free, which implies G_2' is in the center of G_2 (see remark 3.2). Also, from Lemma 3.4, we know that $|[\sigma_1, \sigma_2]|$ is a divisor of both m and n. Therefore

$$\begin{aligned}(\sigma_1\sigma_2^{-(m-2)}\sigma_1\sigma_2^{m-2})^{(n-2)/2} &= \big(\sigma_1^2\,[\sigma_1, \sigma_2^{m-2}]\big)^{(n-2)/2}\\ &= \sigma_1^{n-2}\,[\sigma_1, \sigma_2]^{(m-2)(n-2)/2} = \sigma_1^{n-2}\,[\sigma_1, \sigma_2]^2\end{aligned}$$

and

$$[\sigma_2^{-(m-1)}, \sigma_1^{-(n-1)}] = [\sigma_2, \sigma_1]^{(m-1)(n-1)} = [\sigma_2, \sigma_1] = [\sigma_1, \sigma_2]^{-1},$$

so the voltage of this cycle is

$$\begin{aligned}&\sigma_2^{m-1}\big(\sigma_1\sigma_2^{-(m-2)}\sigma_1\sigma_2^{m-2}\big)^{(n-2)/2}\sigma_1\sigma_2^{-(m-1)}\sigma_1^{-(n-1)}\\ &\quad= \sigma_2^{m-1}\big(\sigma_1^{n-2}\,[\sigma_1, \sigma_2]^2\big)\sigma_1\sigma_2^{-(m-1)}\sigma_1^{-(n-1)}\\ &\quad= \sigma_2^{m-1}\sigma_1^{n-1}\sigma_2^{-(m-1)}\sigma_1^{-(n-1)}\,[\sigma_1, \sigma_2]^2\\ &\quad= [\sigma_2^{-(m-1)}, \sigma_1^{-(n-1)}]\,[\sigma_1, \sigma_2]^2\\ &\quad= [\sigma_1, \sigma_2],\end{aligned}$$

which generates G_2' (see Lemma 3.5). □

The following result allows us to assume $\ell \geq 3$.

Proposition 3.16 (Keating-Witte [9, §6])**.** If $\ell = 2$ and $N = G'$, then $\mathrm{Cay}\big(G; S\big)$ has a hamiltonian cycle.

Proof. For the reader's convenience, we provide a proof (using the main result of Section 4 below). We may assume $|G/G'|$ is odd, for otherwise a hamiltonian cycle is obtained by combining Lemma 3.15 with the Factor Group Lemma (2.5). We may also assume that $|G|$ is not a power of 3, for otherwise theorem 3.13 applies. This implies it is not the case that $|\overline{s}| = 3$ for every $s \in S$.

If $|G'|$ is square-free, then proposition 4.1 tells us that α_2^2 is true. Also, since $|G/G'|$ is odd, we know $|G'|$ is odd (cf. Lemma 3.4), so α_2^2 implies that α_2^1 is true (see remark 2.7(1)). Therefore, the Factor Group Lemma (2.5) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$ (see Lemma 2.6). Then Lemma 3.14 tells us there is a hamiltonian cycle even without the assumption that $|G'|$ is square-free. □

4 Base case of the inductive construction

Recall that the condition α_k^ϵ is defined in Section 2.

Proposition 4.1 (cf. [9, Case 6.2])**.** Assume $|N|$ is square-free (and $\ell \geq 2$). Then α_2^2 is true unless $|G_2'| = m_2 = |\overline{\sigma_1}| = |\overline{\sigma_2}| = 3$.

Proof. For convenience, let

$$a = \sigma_1,\ b = \sigma_2,\ \text{and}\ \ m = m_2,$$

and define r by

$$\overline{b}^m = \overline{a}^r \ \text{ and } \ 0 < r \leq |\overline{a}|.$$

We may assume:

- $\ell = 2$, so $S = S_2 = \{a, b\}$ and $G = G_2$.
- $(G')^2$ is nontrivial. (Otherwise, the condition about generating $(G')^2$ is automatically true, so it suffices to show $\mathcal{V}_2 \neq \emptyset$, which is easy.)
- Either $|\overline{a}|$ is even, or m is odd (by interchanging σ_1 and σ_2 if necessary).
- $|\overline{a}| \neq 3$ (by interchanging σ_1 and σ_2 if necessary: if $|\overline{\sigma_1}| = |\overline{\sigma_2}| = 3$, then $m = 3$ and, from Lemma 3.4, we also have $|G'| = 3$, which means we are in a case in which the statement of the proposition does not make any claim).
- $r \geq |\overline{a}|/2$ (by replacing a with its inverse if necessary).

Note that $|G'|$ is a divisor of both $|\overline{a}|$ and m (see Lemma 3.4). Since $(G')^2$ is nontrivial, this implies that $|\overline{a}|$ and m both have at least one odd prime divisor.

Case 1. *Assume* $m = 3$. Since $|G'|$ is a divisor of m, we must have $|G'| = 3$, so $|\overline{a}|$ must be divisible by 3. Then, since $|\overline{a}| \neq 3$, we must have $|\overline{a}| \geq 6$. Furthermore, by applying Lemma 3.4 with a and b interchanged, we see that $|\overline{G}/\langle\overline{b}\rangle|$ is also divisible by $|G'| = 3$, which means that r is divisible by 3.

We claim that it suffices to find two elements $\gamma_1, \gamma_2 \in \mathcal{V}_2$, such that $\gamma_1 \neq \gamma_2$ and $\gamma_1 \in \gamma_2 G'$. To see this, note that, for any $x \in N$, there is some $i \in \{1, 2\}$, such that $\langle\gamma_i x\rangle$ has nontrivial projection to G' (with respect to the unique direct-product decomposition $N = G' \times (G')^{\perp}$). Since $|G'|$ is prime, this implies that the projection is all of G', so Lemma 3.6 tells us that $\langle\gamma_i x\rangle$ contains G'. This establishes α_2^1, which is equivalent to α_2^2 (see remark 2.7(1)). This completes the proof of the claim.

Assume, for the moment, that $r = 3$. Then, since $r \geq |\overline{a}|/2$ and $|\overline{a}| \geq 6$, we must have $|\overline{a}| = 6$. Here are two hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$ that cover $S^{\pm 1}$:

$$(b^{-1}, a^{-2}, b^{-4}, a^{-2}, b^{-1}, a^3, b^2, a, b^{-2})$$

and

$$(b^{-1}, a^{-2}, b^{-1}, a, b^{-1}, a^{-1}, b^{-2}, a^{-1}, b^{-1}, a^2, b^2, a, b^{-2})$$

(see Figure 1). By using Lemma 3.3, we see that their voltages are

$$b^{-1}a^{-2}b^{-4}a^{-2}b^{-1}a^3b^2ab^{-2} = b^{-6}[a,b]^{-(-10)} = b^{-6}[a,b]$$

and

$$b^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b^{-2}a^{-1}b^{-1}a^2b^2ab^{-2} = b^{-6}[a,b]^{-(-8)} = b^{-6}[a,b]^2,$$

respectively. So we may let $\gamma_1 = b^{-6}[a,b]$ and $\gamma_2 = b^{-6}[a,b]^2$.

We may now assume $r \geq 6$ (since r is divisible by 3). Let

$$I = \begin{cases} \{0, 1\} & \text{if } r \neq |\overline{a}|, \\ \{1, 2\} & \text{if } r = |\overline{a}|. \end{cases}$$

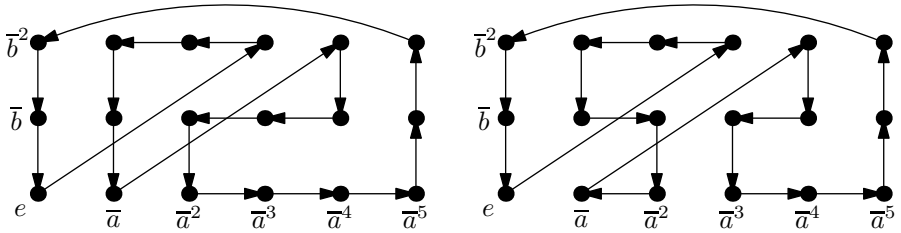

Figure 1: Two hamiltonian cycles in $\mathrm{Cay}\big(\overline{G}; \{\overline{a}, \overline{b}\}\big)$ when $m = r = 3$.

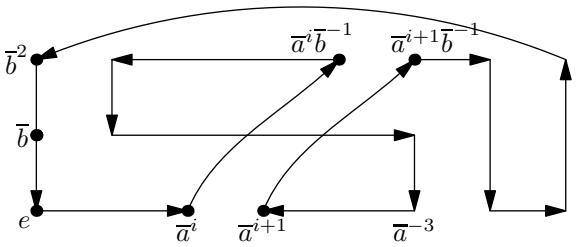

Figure 2: A hamiltonian cycle in $\mathrm{Cay}\big(\overline{G}; \{\overline{a}, \overline{b}\}\big)$ when $m = 3$ and $r \geq 6$.

Then, for $i \in I$, we have $0 \leq i \leq |\overline{a}| - 4$, and $4 \leq r - i \leq |\overline{a}| - 1$, so the walk

$$C_i = \big(a^i, b^{-1}, a^{-(|\overline{a}|-r+i-1)}, b^{-1}, a^{|\overline{a}|-4}, b^{-1}, a^{-(|\overline{a}|-i-4)}, b^{-1}, a^{r-i-3}, b^{-2}, a, b^2, a, b^{-2}\big)$$

is as pictured in Figure 2. It is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$ that covers $S^{\pm 1}$. Furthermore, since

$$a^i b^{-1} a^{-i} = b^{-1}(b a^i b^{-1} a^{-i}) = b^{-1}[b^{-1}, a^{-i}] = b^{-1}[b, a]^i = b^{-1}[a, b]^{-i},$$

its voltage is of the form $h_2\,[a, b]^{-2i}$, where h_2 is independent of i. Thus, we may let

$$\{\gamma_1, \gamma_2\} = \{\, h_2\,[a, b]^{-2i} \mid i \in I\,\}.$$

Case 2. *Assume $m \neq 3$.* (Cf. [9, Case 4.3].) Since m and $|\overline{a}|$ both have at least one odd prime divisor, we must have $m \geq 5$ and $|\overline{a}| \geq 5$. Let

$$X = \begin{cases} \big(b^{-(m-2)}, a, b^{m-3}, a^{|\overline{a}|-3}, b^{-1}, (a^{-(|\overline{a}|-4)}, b^{-1}, a^{|\overline{a}|-4}, b^{-1})^{(m-3)/2}\big) & \text{if } |\overline{a}| \text{ is odd,} \\ \big(b^{-1}, (b^{-(m-3)}, a, b^{m-3}, a)^{(|\overline{a}|/2)-1}, b^{-(m-2)}\big) & \text{if } |\overline{a}| \text{ is even.} \end{cases}$$

For each i with $1 \leq i \leq \lfloor(|\overline{a}| - 1)/2\rfloor$, we have $1 \leq i \leq \min\big(r - 1, |\overline{a}| - 3\big)$ (since $r \geq |\overline{a}|/2$ and $|\overline{a}| \geq 5$), so we may let

$$C_i = \big(a^i, b^{-1}, a^{-(|\overline{a}|+i-r-1)}, X, a^{-(|\overline{a}|-i-2)}, b^{-1}, a^{r-i-1}, b^{-(m-1)}\big)$$

(see Figures 3 and 4). Then C_i is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$.

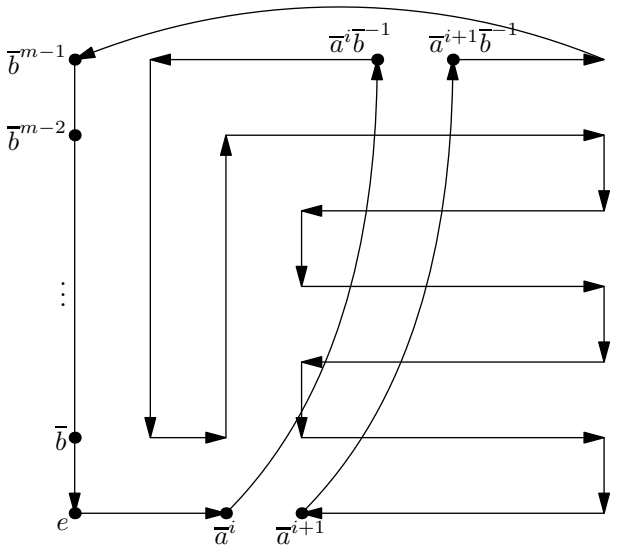

Figure 3: A hamiltonian cycle C_i in $\mathrm{Cay}\big(\overline{G};\{\overline{a},\overline{b}\}\big)$ when $m = |\overline{G}/\langle\overline{a}\rangle|$ is odd.

Note that both possibilities for X contain oriented edges labelled a, b, and b^{-1}. Furthermore, since $|\overline{a}| - i - 2 \geq 1$, we see that C_i also contains at least one oriented edge labelled a^{-1}. Therefore C_i covers $\{a, b, a^{-1}, b^{-1}\} = S^{\pm 1}$.

As in Case 1, the voltage ΠC_i of C_i is of the form $h_2\,[a,b]^{-2i}$, where h_2 is independent of i. Since $|\overline{a}| \geq |G'|$ (see Lemma 3.4) and $\langle [a,b]\rangle = G'$ (see Lemma 3.5), Lemma 3.8 (combined with Lemma 3.6) tells us that for any $x \in N$, we may choose i so that $\langle (\Pi C_i)x\rangle$ contains $(G')^2$. □

5 The main induction step

The induction step of our proof uses the following well-known gluing technique that is illustrated in Figure 5.

Definition 5.1. Let

- C_1 and C_2 be two vertex-disjoint oriented cycles in $\mathrm{Cay}(\overline{G};\overline{S})$,
- $g \in G$, and
- $a, s \in S$.

If

- C_1 contains the oriented edge $\overline{g}(s)$, and
- C_2 contains the oriented edge $\overline{gsa}(s^{-1})$,

then we use $C_1 \#_s^a C_2$ to denote the oriented cycle obtained from $C_1 \cup C_2$ as in Figure 5, by

- removing the oriented edges $\overline{g}(s)$ and $\overline{gsa}(s^{-1})$, and

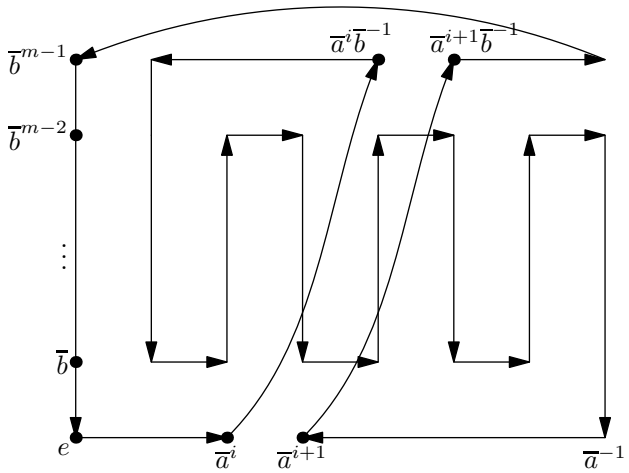

Figure 4: A hamiltonian cycle C_i in $\mathrm{Cay}\big(\overline{G};\{\overline{a},\overline{b}\}\big)$ when $|\overline{a}|$ is even.

- inserting the oriented edges $\overline{g}(a)$ and $\overline{gs\overline{a}}(a^{-1})$.

This is called the *connected sum* of C_1 and C_2.

Lemma 5.2. If C_1, C_2, g, s, and a are as in definition 5.1, and $N \subset Z(G)$, then

$$\Pi(C_1 \#_s^a C_2) = (\Pi C_1)(\Pi C_2)[a,s].$$

Proof. Write $C_1 = \overline{gs}(s_i)_{i=1}^m$ and $C_2 = \overline{g\overline{a}}(t_j)_{j=1}^n$. Then

$$C_1 \#_s^a C_2 = \overline{gs\overline{a}}\big(a^{-1},(s_i)_{i=1}^{m-1},a,(t_j)_{j=1}^{n-1}\big),$$

so

$$\begin{aligned}
\Pi(C_1 \#_s^a C_2) &= a^{-1}\left(\prod_{i=1}^{m-1} s_i\right) a \left(\prod_{j=1}^{n-1} t_j\right)\\
&= a^{-1}\left(\prod_{i=1}^{m} s_i\right) s_m^{-1}\, a \left(\prod_{j=1}^{n} t_j\right) t_n^{-1}\\
&= a^{-1}\,(\Pi C_1)\,s^{-1}\,a\,(\Pi C_2)\,s\\
&= (\Pi C_1)\,(\Pi C_2)\;a^{-1}s^{-1}as \qquad\qquad (\Pi C_i \in N \subset Z(G))\\
&= (\Pi C_1)\,(\Pi C_2)\;[a,s].
\end{aligned}$$

□

Corollary 5.3. Assume

- $2 \le k < \ell$, and (to eliminate some subscripts) $m = m_{k+1}$ and $a = \sigma_{k+1}$,
- $\pi_1, \pi_2, \ldots, \pi_m$ are elements of $\mathcal{V}_k$,

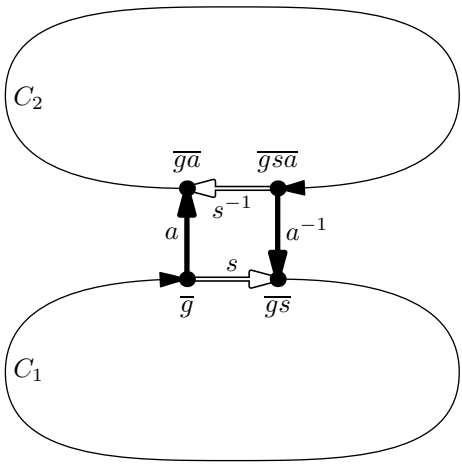

Figure 5: C_1 and C_2 are merged into a single cycle by replacing the two white edges labelled s and s^{-1} with the two black edges labelled a and a^{-1}.

- $s_1, s_2, \ldots, s_{m-1}$ are elements of S_k, and, for each i, a choice $s_i^* \in \{s_i^{\pm 1}\}$ has been made in such a way that if $s_{i+1} = s_i$, then $s_{i+1}^* = s_i^*$, and
- $N \subset Z(G)$.

Then there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_{k+1}}; \overline{S_{k+1}})$ that covers $S_{k+1}^{\pm 1}$, and whose voltage is

$$\left(\prod_{i=1}^{m} \pi_i\right)\left(\prod_{i=1}^{m-1} [a, s_i^*]\right).$$

Proof. For each i, let C_i be an oriented hamiltonian cycle in $\mathrm{Cay}(\overline{G_k}; \overline{S_k})$ that covers $S_k^{\pm 1}$, and has voltage π_i. We inductively construct sequences $(g_i)_{i=1}^m$ and $(x_i)_{i=1}^m$ of elements of G_k, as follows.

Let $g_1 = e$. Since C_1 covers $S_k^{\pm 1}$, we know there is some $x_1 \in G_k$, such that ag_1C_1 contains the oriented edge $\overline{ax_1}(s_1^*)$.

Now, suppose $g_1, x_1, g_2, x_2, \ldots, g_i, x_i \in G_k$ are given, such that the connected sum

$$ag_1C_1 \,\#_{s_1^*}^a\, a^2g_2C_2 \,\#_{s_2^*}^a \cdots \#_{s_{i-1}^*}^a\, a^ig_iC_i$$

exists, and contains the oriented edge $\overline{a^ix_i}(s_i^*)$. Since C_{i+1} covers $S_k^{\pm 1}$, we know that C_{i+1} contains an oriented edge labelled $(s_i^*)^{-1}$, and a different oriented edge that is labelled s_{i+1}^*. Therefore, there exist $g_{i+1}, x_{i+1} \in G_k$, such that

$a^{i+1}g_{i+1}C_{i+1}$ contains the oriented edges $\overline{a^{i+1}x_is_i^*}\big((s_i^*)^{-1}\big)$ and $\overline{a^{i+1}x_{i+1}}(s_{i+1}^*)$.

The first of these edges is removed when we form the connected sum

$$\big(ag_1C_1 \,\#_{s_1^*}^a\, a^2g_2C_2 \,\#_{s_2^*}^a \cdots \#_{s_{i-1}^*}^a\, a^ig_iC_i\big) \;\#_{s_i^*}^a\; a^{i+1}g_{i+1}C_{i+1},$$

but the second edge remains, and will be used to form the next connected sum (unless $i+1=m$).

Since each C_i is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_k};\overline{S_k})$, the resulting connected sum

$$ag_1C_1 \#^a_{s_1^*} a^2g_2C_2 \#^a_{s_2^*} \cdots \#^a_{s_{m-1}^*} a^m g_m C_m$$

passes through all of the vertices in $\overline{aG_k} \cup \overline{a^2G_k} \cup \cdots \cup \overline{a^mG_k}$. That is, it passes through every element of $\overline{G_{k+1}}$, so it is a hamiltonian cycle in $\mathrm{Cay}(\overline{G_{k+1}};\overline{S_{k+1}})$. Its voltage is calculated by repeated application of Lemma 5.2.

To complete the proof, we verify that the hamiltonian cycle covers $S_{k+1}^{\pm1}$. Since each C_i covers $S_k^{\pm1}$, the disjoint union

$$ag_1C_1 \cup a^2g_2C_2 \cup \cdots \cup a^m g_m C_m$$

contains (at least) m disjoint pairs of edges labelled s and s^{-1}, for each $s \in S_k$. Each invocation of the connected sum removes only one such pair, and the operation is performed only $m-1$ times, so at least one of the m pairs must remain, for each $s \in S_k$. Therefore, the hamiltonian cycle covers $S_k^{\pm1}$. Also, the cycle certainly covers $a^{\pm1}$, since each invocation of the connected sum inserts a pair of edges labelled a and a^{-1}. Hence, the hamiltonian cycle covers $S_k^{\pm1} \cup \{a^{\pm1}\} = S_{k+1}^{\pm1}$. □

We can now prove the main result of this section. (Recall that the condition α_k^ϵ is defined in Section 2.)

Proposition 5.4. Assume $|N|$ is square-free and $|G'_{k+1}/G'_k|$ is odd. Then

1. $\alpha_k^1 \Rightarrow \alpha_{k+1}^1$, and
2. $\alpha_k^{2+} \Rightarrow \alpha_{k+1}^{2+}$ if $|[s,t]|$ is even for all $s,t \in S_{k+1}$ with $s \neq t$.

Proof. For convenience, let $m = m_{k+1}$ and $a = \sigma_{k+1}$. Let $h_k \in N$ be as in (α_k^ϵ), and choose an oriented hamiltonian cycle C in $\mathrm{Cay}(\overline{G_k};\overline{S_k})$ that covers $S_k^{\pm1}$, and has its voltage in $h_k(G'_k)^\epsilon$. There is no harm in assuming that the voltage is precisely h_k. Let

$$h_{k+1} = (h_k)^m [a,\sigma_1]^{m-1}.$$

Given any $x \in N$, corollary 3.11 provides a sequence $(s_i)_{i=1}^{m-1}$ of elements of S_k, and $s_i^* \in \{s_i^{\pm1}\}$ for each i, such that $s_{i+1}^* = s_i^*$ whenever $s_{i+1} = s_i$, and

$$\left\langle x\,(h_k)^m \prod_{i=1}^{m-1} [a, s_i^*],\ (G'_k)^\epsilon \right\rangle \text{ contains } (G'_{k+1})^\epsilon. \tag{5.5}$$

From (α_k^ϵ) we know there exists $\pi \in \mathcal{V}_k \cap h_k\,(G'_k)^\epsilon$, such that, if we let

$$\gamma = \pi\,(h_k)^{m-1} \prod_{i=1}^{m-1} [a, s_i^*],$$

then $\langle x\gamma\rangle$ contains $(G'_k)^\epsilon$. Since $\pi \equiv h_k \pmod{(G'_k)^\epsilon}$, combining this with (5.5) shows that $\langle x\gamma\rangle$ contains $(G'_{k+1})^\epsilon$. Also, since we are assuming $|[a,s_i^*]|$ is even if $\epsilon = 2$, we have $[a,s_i^*] \equiv [a,\sigma_1] \pmod{(G'_{k+1})^\epsilon}$ for all i, so

$$\gamma \in (h_k)^m [a,\sigma_1]^{m-1} (G'_{k+1})^\epsilon = h_{k+1}(G'_{k+1})^\epsilon.$$

Furthermore, corollary 5.3 tells us that there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G};\overline{S})$ whose voltage is γ, and this hamiltonian cycle covers $S^{\pm 1}$. This establishes α^{ϵ}_{k+1}.

Now, if $\epsilon = 2$, then our assumptions imply that $|h_k|$ and $|[a,\sigma_1]|$ are both even. Since m and $m-1$ are of opposite parity, this implies that $|h_{k+1}|$ is even, so $\langle h_k, (G'_{k+1})^2\rangle$ contains G'_{k+1}. This establishes α^{2+}_{k+1}. □

6 Combining the base case with the induction step

Recall that the condition α^{ϵ}_k is defined in Section 2.

Corollary 6.1. Assume $|N|$ is square-free and $\ell \geq 2$. If $|G'|$ is odd, then α^1_ℓ is true unless $|G'| = |\overline{s}| = 3$ for all $s \in S$.

Proof. Assume it is not the case that $|G'| = |\overline{s}| = 3$ for all $s \in S$. Then we may assume (by permuting the elements of S) that either $|G'_2| \neq 3$ or $|\sigma_1| \neq 3$. Therefore proposition 4.1 tells us that α^2_2 is true. Also, since $|G'|$ is odd, we have $\alpha^2_2 \Leftrightarrow \alpha^1_2$ (see remark 2.7(1)), so α^1_2 is true. Then repeated application of proposition 5.4(1) establishes α^1_ℓ. □

Proposition 6.2. Assume $|N|$ is square-free and $\ell \geq 3$. If $|G'|$ is even, then:

1. α^1_ℓ is true if there exist $s,t \in S$, such that $|[s,t]|$ is odd and $s \neq t$.
2. α^{2+}_ℓ is true if $|[s,t]|$ is even for all $s,t \in S$ with $s \neq t$.

Proof. Since $|G'|$ is even, we may assume (by permuting the elements of S) that $|[\sigma_3,\sigma_1]|$ is even. It suffices to prove α^1_3 or α^{2+}_3 (as appropriate), for then repeated application of proposition 5.4 establishes the desired conclusion. Thus, we may assume $\ell = 3$, so $G_3 = G$. Let $m = m_3$ and $a = \sigma_3 = \sigma_\ell$.

By permuting the elements of S, we may assume that either:

$$\begin{aligned} \text{odd case:}\ & |[\sigma_3,\sigma_2]| \text{ is odd, or} \\ \text{even case:}\ & |[s,t]| \text{ is even for all } s,t \in S \text{ with } s \neq t. \end{aligned}$$

Furthermore, in the even case, we may assume that either:

$$\begin{aligned} \text{even subcase:}\ & \langle \overline{\sigma_1}, \overline{\sigma_2}\rangle \text{ has even index in } \overline{G}, \text{ or} \\ \text{odd subcase:}\ & \langle \overline{s}, \overline{t}\rangle \text{ has odd index in } \overline{G}, \text{ for all } s,t \in S, \text{ such that } s \neq t. \end{aligned}$$

Since $|[\sigma_3,\sigma_1]|$ is even, we know $|\overline{\sigma_1}|$ is even (see Lemma 3.4). In particular, we have $|\overline{\sigma_1}| \neq 3$, so proposition 4.1 tells us that α^2_2 is true.

We now use a slight modification of the proof of proposition 5.4. Let $h_2 \in N$ be as in (α^{ϵ}_k) (with $k = \epsilon = 2$), and choose an oriented hamiltonian cycle C in $\mathrm{Cay}(\overline{G_2};\overline{S_2})$ that covers $S_2^{\pm 1}$, and has its voltage in $h_2(G'_2)^2$. There is no harm in assuming that the voltage is precisely h_2.

Since $|\overline{\sigma_1}|$ is even, we know $|\overline{G_2}|$ is even. Therefore, Lemma 3.15 provides a hamiltonian cycle C' in $\mathrm{Cay}(\overline{G_2};\overline{S_2})$, such that $\Pi C'$ is a generator of G'_2. Let

$$h' = \begin{cases} \Pi C' & \text{in the odd subcase of the even case,} \\ h_2 & \text{in all other cases.} \end{cases}$$

Let $h_3 = (h_2)^{m-1} h' [a,\sigma_1]^{m-1}$.

Given any $x \in N$, corollary 3.11 provides a sequence $(s_i)_{i=1}^{m-1}$ of elements of S_2, and $s_i^* \in \{s_i^{\pm 1}\}$ for each i, such that $s_{i+1}^* = s_i^*$ whenever $s_{i+1} = s_i$, and

$$\left\langle x(h_2)^{m-1}h' \prod_{i=1}^{m-1} [a, s_i^*],\ (G_2')^2 \right\rangle \text{ contains } (G')^2. \tag{6.3}$$

Furthermore, in the odd case, the choices can be made so that (6.3) holds with G' in the place of $(G')^2$.

From α_2^2, we know there exists $\pi \in \mathcal{V}_2 \cap h_2\,(G_2')^2$, such that, if we let

$$\gamma = \pi\,(h_2)^{m-2}h' \prod_{i=1}^{m-1} [a, s_i^*],$$

then

$$\langle x\gamma \rangle \text{ contains } (G')^2. \tag{6.4}$$

It is clear from the definitions that $\gamma \in h_3 G_3'$. Furthermore, we have $\gamma \in h_3 (G_3')^2$ in the even case.

corollary 5.3 tells us that there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage is γ, and this hamiltonian cycle covers $S^{\pm 1}$. We now consider various cases individually.

Case 1. *The odd case.* Recall that, in this case, (6.3) holds with G' in the place of $(G')^2$. Since $\pi \equiv h_2 \pmod{(G_2')^2}$, combining this with (6.4) shows that $\langle x\gamma \rangle$ contains all of G'. This establishes α_3^1.

Case 2. *The even subcase of the even case.* In this subcase, we know m is even, $h' = h_2$, and $|[a, \sigma_1]|$ is even. Since $h_3 = (h_2)^m [a, \sigma_1]^{m-1}$, we see that $|h_3|$ is even, so $\langle h_3, (G_3')^2 \rangle$ contains G_3'. This establishes α_3^{2+}.

Case 3. *The odd subcase of the even case.* In this subcase, we know $m - 1$ is even, and $h' = \Pi C'$ has even order. Therefore $|h_3|$ is even, so $\langle h_3, (G_3')^2 \rangle$ contains G_3'. This establishes α_3^{2+}. □

We can now establish our main theorem:

Proof of theorem 1.2. We may assume:

- $\ell \geq 3$, for otherwise proposition 3.16 applies.
- $|G|$ is not a power of 3, for otherwise theorem 3.13 applies.

Let S be a minimal generating set of G, and let $N = G'$. Note that $\overline{S}$ is a minimal generating set of $\overline{G}$ (because G' is contained in the Frattini subgroup $\Phi(G)$ [8, Thm. 10.4.3]).

We claim there is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ whose voltage generates G'. While proving this, there is no harm in assuming that $|G'|$ is square-free (see Lemma 3.14). Also note that, since $|G|$ is not a power of 3, we cannot have $|G'| = |\overline{s}| = 3$ for all $s \in S$. Then, by applying either corollary 6.1 or proposition 6.2 (depending on the parity of $|G'|$), we obtain either α_ℓ^1 or α_ℓ^{2+}. Each of these yields the desired hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{S})$ (see Lemma 2.6).

Now that the claim has been verified, the Factor Group Lemma (2.5) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

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