BLED WORKSHOPS
IN PHYSICS
VOL. 8, NO. 1
p. 43

Proceedings of the Mini-Workshop
Hadron Structure and Lattice QCD
Bled, Slovenia, July 9-16, 2007

Topology of non-commutative U(1) gauge theory on the lattice*

R. Achleitner[a], W. Frisch[a], H. Grosse[b], H. Markum**[a], F. Teitschinger[a]

[a] Atominstitut, Vienna University of Technology, Austria
[b] Department for Theoretical Physics, University of Vienna, Austria

Abstract. Theories with non-commutative space-time coordinates represent alternative candidates of grand unified theories. We discuss U(1) gauge theory in 2 dimensions on a lattice with N sites. The mapping to a U(N) one-plaquette model in the sense of Eguchi and Kawai can be used for computer simulations. We are discussing the formulation and evaluation of topological objects. We performed quantum Monte Carlo simulations and calculated the topological charge for different matrix sizes and several values of the coupling constant. We constructed classical gauge field configurations with large topological charge and used them to initialize quantum simulations. It turned out that the value of the topological charge is decreasing during a Monte Carlo history. Our results show that the topological charge is in general suppressed. The situation is similar to lattice QCD where quantum gauge field configurations are topologically trivial and one needs to apply some cooling procedure on the gauge fields to unhide the integer number of the instantons. A few recent analyses are added to this paper.

1 Motivation

In non-commutative geometry, where the coordinate operators $\hat{x}_\mu$ satisfy the commutation relation $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$, a mixing between ultraviolet and infrared degrees of freedom takes place [1]. Lattice simulations are a promising tool to get deeper insight into non-commutative quantum field theories. In this work we have studied non-commutative U(1) gauge theory on a two-dimensional torus. The advantage of this theory is that there exists an equivalent matrix model which makes numerical calculations feasible [2].

The main topic of the underlying contribution is to study the topological charge in two-dimensional non-commutative U(1) gauge theory. The instanton configurations carry a topological charge q which can be non-integer in this case [3]. We performed Monte Carlo simulations with different values of the coupling constant β and looked at the topological charge q in the equilibrium [4].

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* Talk delivered by H. Markum

** Thanks to the organizers of the Mini-Workshop 2007 on Hadron Structure and Lattice QCD in Bled.

2 Topology and instantons in QCD

The Lagrangian of pure gluodynamics (the Yang-Mills theory with no matter fields) in Euclidean spacetime can be written as

$$\mathcal{L} = \frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} \tag{1}$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abs} A^b_\mu A^c_\nu \tag{2}$$

and f^{abs} are structure constants of the gauge group considered. The classical action of the Yang-Mills fields can be identically rewritten as

$$S = \frac{1}{8g^2} \int dx^4 (G^a_{\mu\nu} \pm \tilde{G}^a_{\mu\nu})^2 \mp \frac{8\pi^2}{g^2} Q \tag{3}$$

where Q denotes the topological charge

$$Q = \frac{1}{32\pi^2} \int dx^4 G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} \tag{4}$$

with

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^a_{\alpha\beta} \,. \tag{5}$$

3 Topological charge in two dimensions

3.1 Lattice regularization of non-commutative gauge theory

The lattice regularized version of the theory can be defined by an analog of Wilson's plaquette action

$$S = -\beta \sum_x \sum_{\mu<\nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^\dagger \star U_\nu(x)^\dagger + \text{c.c.} \tag{6}$$

where the symbol $\hat{\mu}$ represents a unit vector in the μ-direction and we have introduced the lattice spacing a. The link variables $U_\mu(x)$ ($\mu = 1, 2$) are complex fields on the lattice satisfying the star-unitarity condition. The star-product [1] on the lattice can be obtained by rewriting its definition within non-commutative derivatives in terms of Fourier modes and restricting the momenta to the Brillouin zone.

Let us define the topological charge for a gauge field configuration on the discretized two-dimensional torus. In the language of fields, we define the topological charge as

$$q = \frac{1}{4\pi i} \sum_x \sum_{\mu\nu} \epsilon_{\mu\nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^\dagger \star U_\nu(x)^\dagger \tag{7}$$

which reduces to the usual definition of the topological charge in 2d gauge theory

$$q = \frac{1}{4\pi} \int d^2x \, \epsilon_{\mu\nu} G_{\mu\nu} \tag{8}$$

in the continuum limit.

3.2 Matrix-model formulation

It is much more convenient for computer simulations to use an equivalent formulation, in which one maps functions on a non-commutative space to operators so that the star-product becomes nothing but the usual operator product, which is non-commutative. The action (6) can then be written as

$$S = -N\beta \sum_{\mu \neq \nu} \mathrm{tr} \left\{ \hat{U}_\mu \, (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) \, (\Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger) \, \hat{U}_\nu^\dagger \right\} + 2\beta N^2 \tag{9}$$

$$= -N\beta \sum_{\mu \neq \nu} \mathcal{Z}_{\nu\mu} \mathrm{tr} \left(V_\mu \, V_\nu \, V_\mu^\dagger \, V_\nu^\dagger \right) + 2\beta N^2 \tag{10}$$

where $V_\mu \equiv \hat{U}_\mu \Gamma_\mu$ is a U(N) matrix with N the linear extent of the original lattice. An explicit representation of Γ_μ in the $d = 2$ case shall be given in Sec. 5. This is the twisted Eguchi-Kawai (TEK) model [5], which appeared in history as a matrix model equivalent to the large N gauge theory [6]. We have added the constant term $2\beta N^2$ to what we would obtain from (6) in order to make the absolute minimum of the action zero.

By using the map between fields and matrices, the topological charge (7) can be represented in terms of matrices as

$$q = \frac{1}{4\pi i} N \sum_{\mu\nu} \epsilon_{\mu\nu} \mathrm{tr} \left\{ \hat{U}_\mu \, (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) \, (\Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger) \, \hat{U}_\nu^\dagger \right\} \tag{11}$$

$$= \frac{1}{4\pi i} N \sum_{\mu\nu} \epsilon_{\mu\nu} \mathcal{Z}_{\nu\mu} \mathrm{tr} \left(V_\mu \, V_\nu \, V_\mu^\dagger \, V_\nu^\dagger \right) . \tag{12}$$

4 Numerical results for the TEK model

We have computed the topological content of gauge field configurations produced by quantum Monte Carlo simulations. In Fig. 1 we display scatter plots of the action S without a factor of β in its definition Eq. (10) and the topological charge q performing a cold start. The size of the matrix is $N = 25$ and the values of the coupling β are chosen to yield a non-commutativity parameter $\theta = 2.55, 1.27, 0.63$, respectively. One observes a decrease of the action with increasing β due to stronger coupling of the matrices in analogy to lower temperature in an Ising model. The importance sampling of the system with smaller action generates smaller values of its topological content. This can also be seen from the distributions of the topological charge in Fig. 2 where the peaks become narrower with increasing β. Similar plots have been obtained for a larger matrix size $N = 35$, for more results see Ref. [4]. To compare the topology-action diagrams on the same scale, we display in Fig. 3 our simulation for $N = 25$ with all β-values considered in a single plot.

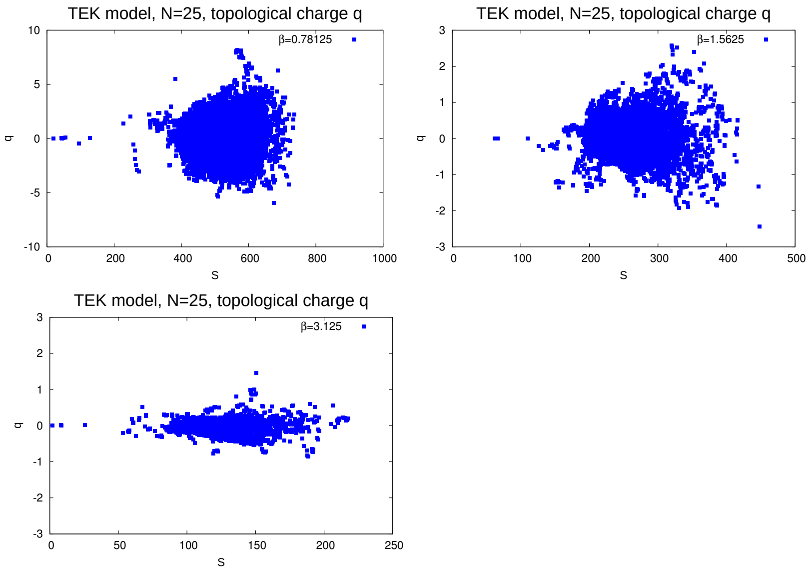

Fig. 1. Scatter plots of the action S divided by β (x-axis) and the topological charge q (y-axis) for a Monte Carlo simulation (cold start) at $N = 25$ and $\beta = 0.78125, 1.5625$ and 3.125.

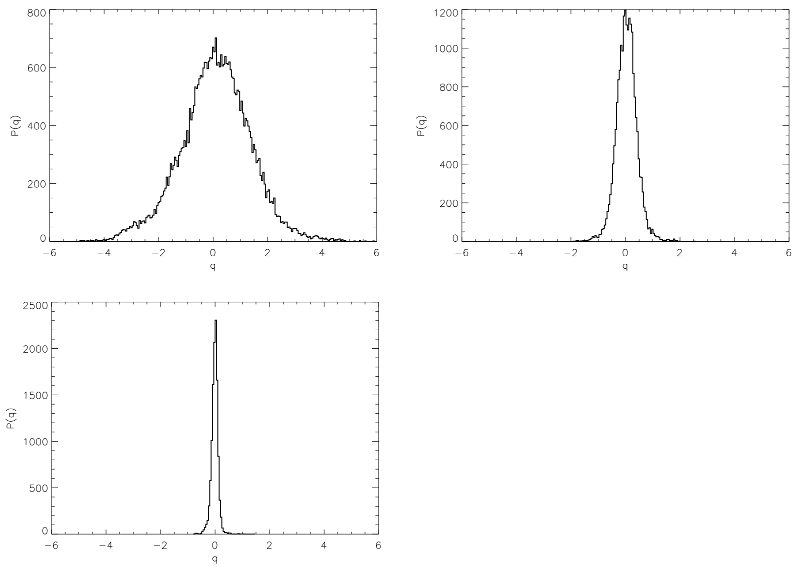

Fig. 2. Distribution P of the topological charge q in the Twisted Eguchi-Kawai model for $N = 25$ and $\beta = 0.78125, 1.5625$ and 3.125.

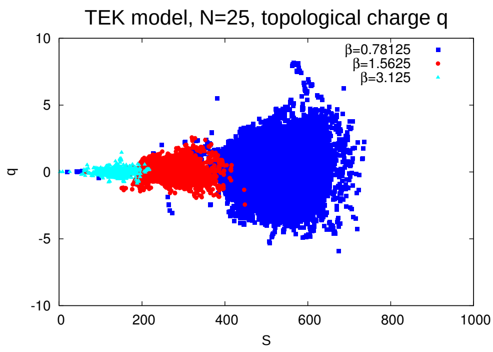

Fig. 3. Scatter plots of the action S divided by β (x-axis) and the topological charge q (y-axis) for the Monte Carlo simulations at $N = 25$ and $\beta = 0.78125, 1.5625$ and 3.125, combining different couplings.

5 Classical solutions

The classical equation of motion can be obtained from the action (10) as [7,3]

$$V_\mu^\dagger (W - W^\dagger) V_\mu = W - W^\dagger \tag{13}$$

where the unitary matrix W is defined by

$$W = \mathcal{Z}_{\nu\mu} V_\mu\, V_\nu\, V_\mu^\dagger\, V_\nu^\dagger \tag{14}$$

The general solutions to this equation can be brought into a block-diagonal form [7]

$$V_\mu = \begin{pmatrix} \Gamma_\mu^{(1)} & & & \\ & \Gamma_\mu^{(2)} & & \\ & & \ddots & \\ & & & \Gamma_\mu^{(k)} \end{pmatrix} \tag{15}$$

by an appropriate SU(N) transformation, where $\Gamma_\mu^{(k)}$ are $n_k \times n_k$ unitary matrices satisfying the 't Hooft-Weyl algebra

$$\Gamma_\mu^{(j)} \Gamma_\nu^{(j)} = Z_{\mu\nu}^{(j)} \Gamma_\nu^{(j)} \Gamma_\mu^{(j)} \tag{16}$$

$$Z_{12}^{(j)} = Z_{21}^{(j)*} = \exp\left(2\pi i \frac{m_j}{n_j}\right) \tag{17}$$

$$m_j = \frac{n_j + 1}{2} \tag{18}$$

An explicit representation is given, for instance, by the clock and shift operators, Q and P

$$\Gamma_1^{(j)} = P_{n_j}\ , \quad \Gamma_2^{(j)} = (Q_{n_j})^{m_j} \tag{19}$$

For each solution, the action and the topological charge can be evaluated as

$$S = 4N\beta \sum_j n_j \sin^2 \left\{ \pi \left(\frac{m_j}{n_j} - \frac{M}{N} \right) \right\} \tag{20}$$

$$q = \frac{N}{2\pi} \sum_j n_j \sin \left\{ 2\pi \left(\frac{m_j}{n_j} - \frac{M}{N} \right) \right\} \tag{21}$$

Note that the topological charge q is not an integer in general. If we require the action to be less than of order N, however, the argument of the sine has to vanish for all j. In that case the topological charge approaches an integer

$$q \simeq N \left(\sum_j m_j - M \right) \tag{22}$$

which is actually a multiple of N.

6 Quantum Monte Carlo versus classical solutions

In the following analysis we compare the classical topological charges taken from Ref. [3] with our quantum Monte Carlo simulation at $N = 25$. In Fig. 4 we plot our data for a cold start from Fig. 3 together with the classical solutions. One sees from the scatter plots that the quantum simulation reaches only small topological numbers. This brings the situation of QCD into mind where one has to apply some cooling or smoothing procedure to damp the quantum fluctuations and get in touch with the integer-valued topological charges. Since a configuration from a cold start is topologically trivial, we constructed classical solutions and started with them. In Fig. 5 we overlay the Monte Carlo histories at $\beta = 1.5625$ starting with $q = -25$ and $q = -50$, respectively, to the scatter plots of the classical topological charges from Ref. [3]. One observes that the equilibrium configurations tend to smaller values of q. Remarkably, the equilibration seems to proceed along a "classical branch".

7 Analyses for large topological charge

The question was how to produce start configurations with different large values of q. One idea was to apply some "cooling procedure" to the topological charge in contrast to the action, being the imaginary part instead of the real part of the plaquette, respectively. The topological charge q is now forced to become larger or smaller every step without posing a condition on the action. In Fig. 6 we start with an equilibrium configuration of $q = -15.7034$. It turns out that the negative value of q is decreasing and configurations of higher action are preferred. In principle, one could use those configurations with different values of q to start a Metropolis Monte Carlo simulation [8].

Another idea was to look at random gauge field configurations. This leads to an ensemble with a large spread in q and large action. In Fig. 7 we present

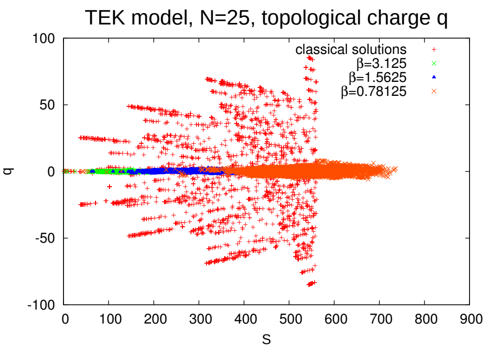

Fig. 4. Scatter plots of the action S divided by β (x-axis) and the topological charge q (y-axis) for Monte Carlo histories at $N = 25$ with cold starts being topologically trivial, $q = 0$. The numbers of the classical topological charges are superimposed.

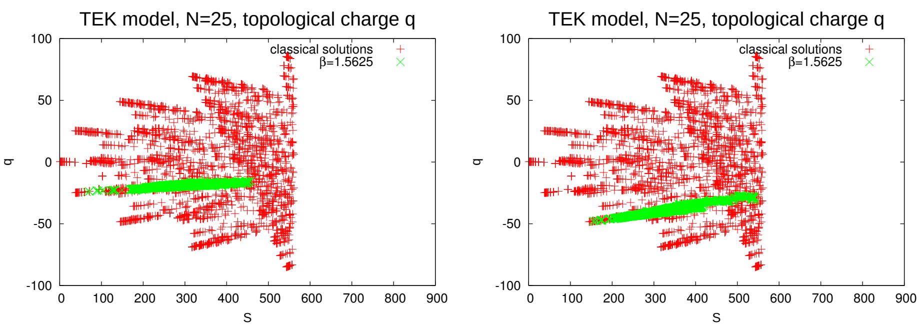

Fig. 5. Scatter plots of the action S divided by β (x-axis) and the topological charge q (y-axis) for Monte Carlo histories at $N = 25$ and $\beta = 1.5625$ with starts at $q = -25$ and -50. The numbers of the classical topological charges are superimposed.

several Metropolis Monte Carlo simulations which were initialized with topological charges in the range $-60 < q < 60$. This preliminary plot is taken from our ongoing research [9].

8 Conclusion and outlook

The diagram of the classical topological charges and the corresponding action from Ref. [3] allows for large values of q. The equilibrium configurations visit only a small part of this charge-action diagram. Thus we constructed classical gauge field configurations with large topological charge and used them as start configuration for quantum simulations. It turned out that the value of the topological charge is decreasing during a Monte Carlo history, preferably along the classical minima. To summarize, our results show that the topological charge is in general suppressed.

The situation is reminiscent of lattice QCD where quantum gauge field configurations are topologically trivial and one needs to apply some smoothing pro-

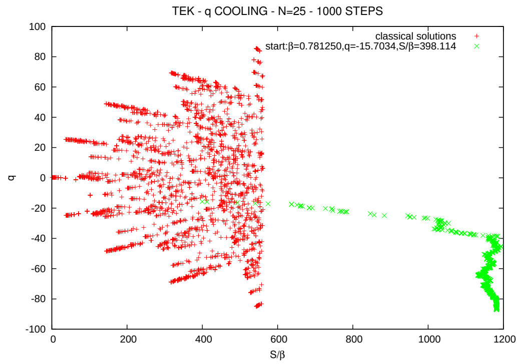

Fig. 6. Scatter plots of the action S divided by β (x-axis) and the topological charge q (y-axis) for "cooling" of the topological charge, starting with an equilibrium configuration at $N = 25$, $\beta = 0.78125$ and $q = -15.7034$. The numbers of the classical topological charges are superimposed.

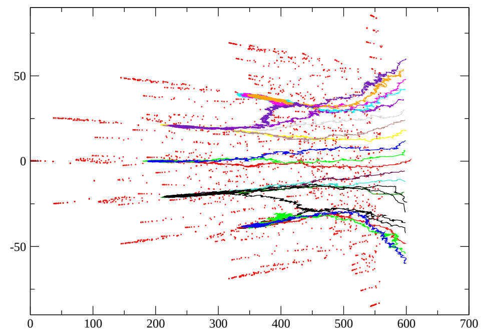

Fig. 7. Scatter plots of the action S divided by β (x-axis) and the topological charge q (y-axis) for Monte Carlo histories at $N = 25$ and $\beta = 1.5625$, starting with several values $-60 < q < 60$. The numbers of the classical topological charges are superimposed.

cedure on the gauge fields to unhide instantons. We adapted cooling techniques known from QCD to the two-dimensional non-commutative U(1) theory. We further performed quantum Monte Carlo simulations for large topological charges. It is desirable to tackle the four-dimensional non-commutative gauge theory in order to obtain a realistic comparison of its topological content with the well-studied topological objects like instantons and monopoles in QCD.

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