

A non-associative incidence near-ring with a generalized Möbius function*

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This paper is dedicated to the memory of John Johnson.

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Abstract

There is a convolution product on 3-variable partial flag functions of a locally finite poset that produces a generalized Möbius function. Under the product this generalized Möbius function is a one sided inverse of the zeta function and satisfies many generalizations of classical results. In particular we prove analogues of Phillip Hall's Theorem on the Möbius function as an alternating sum of chain counts, Weisner's Theorem, and Rota's Crosscut Theorem. A key ingredient to these results is that this function is an overlapping product of classical Möbius functions. Using this generalized Möbius function we define analogues of the characteristic polynomial and Möbius polynomials for ranked lattices. We compute these polynomials for certain families of matroids and prove that this generalized Möbius polynomial has -1 as root if the matroid is modular. Using results from Ardila and Sanchez we prove that this generalized characteristic polynomial is a matroid valuation.

Keywords: Incidence algebra, matroid, Möbius function, valuation.

Math. Subj. Class. (2020): 37K15, 42A99, 60E05, 05A17

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Neasociativni incidenčni približni kolobar s posplošeno Möbiusovo funkcijo*

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Povzetek

Obstaja konvolucijski produkt na parcialnih funkcijah (s 3 spremenljivkami) praporov lokalno končne delno urejene množice, ki da posplošeno Möbiusovo funkcijo. Za ta produkt je ta posplošena Möbiusova funkcija enostranski inverz zeta funkcije in zadošča mnogim posplošitvam klasičnih rezultatov. Dokažemo analogije Phillip Hallovega izreka o Möbiusovi funkciji kot izmenični vsoti verižnih štetij, Weisnerjevega izreka in Rotajevega prečnega izreka. Ključna sestavina teh rezultatov je, da je ta funkcija prekrivajoči se produkt klasičnih Möbiusovih funkcij. Z uporabo te posplošene Möbiusove funkcije definiramo analogne pojme karakterističnega polinoma in Möbiusovih polinomov za rangirane mreže. Te polinome izračunamo za določene družine matroidov in dokažemo, da je -1 ničla tega posplošenega Möbiusovega polinoma, če je matroid modularen. Z uporabo rezultatov Ardile in Sancheza dokažemo, da je ta posplošeni karakteristični polinom matroidna valuacija.

Ključne besede: Incidenčna algebra, matroid, Möbiusova funkcija, valuacija.

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