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The Institute of Mathematics, Physics and Mechanics
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Bled'11

This issue of *Ars Mathematica Contemporanea* contains a selection of articles presented at the 7th Slovenian Graph Theory Conference (Bled'11), held from June 19 to June 25, 2011, by tradition at Lake Bled, Slovenia. This conference (held every four years) has progressed a long way since the first one in 1991. The number of participants has grown, from just 30 in 1991, to 270 at Bled'11, representing 40 countries and all six continents. The Bled'11 conference was attended by some of the leading researchers in graph theory, as well as many postdocs and talented PhD students. There were nine keynote speakers (Jonathan L. Gross, Wilfried Imrich, Alexander A. Ivanov, László Lovász, Jaroslav Nešetřil, Egon Schulte, Tamás Szőnyi, Martin Škoviera, and Asia Ivić Weiss), plus 213 contributed talks presented within 16 minisymposia, and a general session. The minisymposia brought together researchers from specific fields, ranging across algebraic, algorithmic, geometric, topological, and other aspects of graph theory, and enabled them to present their work and exchange ideas. We know that this led to progress on many open problems, and catalysed many new collaborations. The algebraic minisymposia were dedicated to Henry H. Glover, a very dear and strong collaborator of the Slovenian Algebraic Graph Theory group, who sadly passed away just a few weeks before the conference.

Many special and satellite events were organised during and after the conference. We celebrated the 70th birthdays of Jonathan L. Gross and Wilfried Imrich. A meeting of the International Academy of Mathematical Chemistry was held, as well as the first meeting of the team from the ESF EuroGiGA GReGAS research project, led by Tomaž Pisanski (University of Primorska). Another highlight was the 'Milestones' exhibition, due to Boštjan Kuzman, which presented many important steps in the development of graph theory in Slovenia — from the first lecture notes, scientific results, published papers and doctoral theses, to international collaborations, celebrated publications, editorial positions, establishments of new institutions, scientific journals, and further projects. Immediately after the conference, about 80 participants attended a satellite PhD Summer School — the Algebraic Graph Theory Summer School, held in Rogla, and organised by the University of Primorska, and some participants attended a 'Mathematics meets Art' event in Ljubljana.

In the past, many papers from each Bled conference were published in a special issue of the *Discrete Mathematics* journal. In 2007 however, when *Ars Mathematica Contemporanea* was established, the participants of the 6th Bled conference were given the option of publishing their contributions in a special issue of this new journal. The organisers of the 2011 conference decided that the contributions related to Bled'11 would be published exclusively by *Ars Mathematica Contemporanea*. This is the first such special issue, and contains 15 articles, accepted for publication after a thorough refereeing process. In producing it, we are able to present to the readers of this journal a selected number of the Bled'11 conference contributions, containing high quality results, and establishing starting points for future research. More of these will be published in another special issue in 2014.

Klavdija Kutnar and Primož Šparl
Guest Editors

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On regular and equivelar Leonardo polyhedra

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Abstract

A Leonardo polyhedron is a 2-manifold without boundary, embedded in Euclidean 3-space $\mathbb{E}^3$, built up of convex polygons and with the geometric symmetry (or rotation) group of a Platonic solid and of genus $g \geq 2$. The polyhedra are named in honour of Leonardo's famous illustrations in [19] (cf. also [12]). Only six combinatorially regular Leonardo polyhedra are known: Coxeter's four regular skew polyhedra, and the polyhedral realizations of the regular maps by Klein of genus 3 and by Fricke and Klein of genus 5. In this paper we construct infinite series of equivelar (i.e. locally regular) Leonardo polyhedra, which share some properties with the regular ones, namely the same Schläfli symbols and related topological structure. So the weaker condition of local regularity allows a much greater variety of symmetric polyhedra.

Keywords: Equivelar polyhedron, Leonardo polyhedron, regular polyhedron, genus, Schläfli symbol, symmetry group.

Math. Subj. Class.: 52B15, 52B70

1 Introduction

A *polyhedron* is a compact 2-manifold without boundary embedded in Euclidean 3-space $\mathbb{E}^3$, hence oriented. It is built up of finitely many (planar) convex polygons, any two of which meet, if at all, in a single edge or a single vertex.

If v, e and f denote the number of vertices, edges and faces, respectively, of the polyhedron, then one has the basic Euler-Poincaré equation

$$v - e + f = 2 - 2g = \chi,$$

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where $g \geq 0$ denotes the genus and χ the Euler characteristic. In this paper we do not consider the case of tori ($g = 1$), but only the polyhedra with $g \geq 2$. If all faces of a polyhedron are p-gons, $p \geq 3$, and all vertices q-valent, $q \geq 3$, then the polyhedron is called *locally regular* or *equivelar* and is denoted by its Schläfli symbol $\{p, q\}$ (cf. [3, 15]). We note that the extended Schläfli symbol $\{p, q; g\}$ is also used. A much stronger condition is (global combinatorial) regularity: a polyhedron is called *regular* if its automorphism group acts transitively on its flags (incidence triples of vertex, edge and face).

Regular maps and their groups play a central role in classical complex analysis and algebraic geometry (e.g. Riemann surfaces, automorphic functions, Poincaré model). Hence regular polyhedra can be interpreted as 3D geometric models or visualizations of regular maps, and they are closely related to the Platonic solids.

The geometric (or Euclidean) symmetry group of the polyhedron is the group of isometries of $\mathbb{E}^3$ stabilizing the polyhedron. It is a subgroup of the automorphism group; to be precise, the automorphism group has a (proper, or improper) subgroup that is isomorphic to the geometric symmetry group. For a combinatorially regular polyhedron the geometric symmetry group is, in general, much smaller than the automorphism group; they coincide only in the case of Platonic solids. For any polyhedron with given combinatorial structure we tacitly assume that it has maximal geometric symmetry.

Polyhedra which have the geometric rotation or full symmetry group of a Platonic solid deserve particular interest. They are called *Leonardo polyhedra*, because Leonardo was the first to draw such polyhedra in Luca Pacioli's book [19] in 1500-1503 (see also [6] and [12]). It is easy to check that the polyhedra in this book are neither equivelar nor regular. Leonardo also drew some polyhedra with lower symmetry groups (e.g. dihedral), but we only use the name for Platonic symmetries.

Obviously there are no Leonardo polyhedra of genus $g = 1$, because tori can have at most dihedral symmetry. For similar reasons there are no Leonardo polyhedra with $g = 2$. For $g = 3$ there are some with tetrahedral symmetry.

2 Regular and equivelar Leonardo polyhedra

Regular Leonardo polyhedra seem to be very rare. Only six are known yet. The first four are Coxeter's regular skew polyhedra [5], first discovered by Coxeter in 1937 and partially by Alicia Boole Stott already in 1913 [1]. There is one dual pair of genus $g = 6$, with tetrahedral symmetry and of type $\{4, 6\}$ and $\{6, 4\}$, and one dual pair of genus $g = 73$, with octahedral symmetry and of type $\{4, 8\}$ and $\{8, 4\}$. (In standard notation: $\{4, 6|3\}$, $\{6, 4|3\}$, $\{4, 8|3\}$ and $\{8, 4|3\}$, cf. [5, 15, 21].) The spines of Coxeter's regular skew polyhedra are isomorphic to the 1-skeletons of the regular 4-simplex or the regular 24-cell, i.e. the only self-dual regular 4-polytopes. We note that the term *spine*, borrowed from topology, is meant here as a graph, embedded in $\mathbb{E}^3$, such that its regular neighbourhood [18] is a 3-manifold with boundary, and the boundary of this manifold is just our polyhedron.

Furthermore, there is the polyhedral realization [20] of Felix Klein's regular map of genus 3 with tetrahedral rotation group and of type $\{3, 7\}$. Its dual with non-convex heptagons was recently discovered [13], but here we only consider polyhedra with convex faces.

The sixth regular polyhedron is the realization of the regular map of Fricke and Klein from 1890. This polyhedron was found by Grünbaum and Shephard in 1984 [11], because of its vertex-transitivity. But its regularity was only recently discovered [2]. It is of type

$\{3, 8\}$, with genus 5, and it has octahedral rotation symmetry.

The spines of these last two polyhedra are isomorphic to the 1-skeleton of the tetrahedron or the cube, hence of convex 3-polytopes in both cases. No other regular Leonardo polyhedra are known yet.

In this paper we construct series of equivelar polyhedra, which are related to the previous 6 regular polyhedra:

Theorem 2.1. *There are infinite series of equivelar Leonardo polyhedra with tetrahedral, octahedral and dodecahedral symmetry group and of Schläfli type* $\{3, 7\}$, $\{3, 8\}$, $\{3, 9\}$, $\{4, 6\}$ *and* $\{6, 4\}$, *and whose spine is isomorphic to the 1-skeleton of a convex 3- or 4-polytope.*

Remark 2.2. The result shows that there are infinite series of equivelar polyhedra, which are closely related to the regular Leonardo polyhedra. Only the types $\{4, 8\}$ and $\{8, 4\}$ are missing.

Remark 2.3. In the previous papers [9] and [23] the authors provided infinite series of equivelar Leonardo polyhedra of type $\{4, 6\}$ and $\{6, 4\}$. But these were of very different spatial structure than the six known regular ones, as they are built up of connected shells (like an onion). The search for closer equivelar analogues was one motivation for this paper. The other one was the recently discovered regularity [2] of the Grünbaum-Shephard polyhedron.

Among the polyhedra of "small" Schläfli-type (i.e. those with $p+q < 12$) the equivelar polyhedra of type $\{4, 5\}$ and $\{5, 4\}$ differ from the others, as follows. The only known regular polyhedra of these types have genus 5 and a small symmetry group of order 4, namely $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ (cf. [17] and [14]).

The equivelar Leonardo polyhedra of this type also differ from the others as the following result shows.

Theorem 2.4. *a) Type* $\{4, 5\}$*: There exist four infinite series of equivelar Leonardo polyhedra with the following genera and symmetry groups:*

- $g = 1 + 6k$ *with tetrahedral symmetry group, in two non-isomorphic versions;*
- $g = 1 + 12k$ *with octahedral symmetry group;*
- $g = 1 + 30k$ *with icosahedral symmetry group* $(k = 1, 2, \dots)$.

b) Type $\{5, 4\}$*: There exist equivelar Leonardo polyhedra with the following genera and symmetry groups:*

- $g = 13, 31$ *with tetrahedral symmetry group;*
- $g = 7, 13, 25, 97, 289$ *with octahedral symmetry group;*
- $g = 31, 61, 3601$ *with icosahedral symmetry group.*

In [22] there is a pair of equivelar Leonardo polyhedra of type $\{4, 5\}$ and $\{5, 4\}$ with $g = 7$ and octahedral symmetry, and in Figure 1 we show a new example of type $\{4, 5\}$ with $g = 19$ and icosahedral (rotation) symmetry. This polyhedron consists of an outer and an inner shell, homothetic to each other and positioned concentrically. Both of them are composed of 60 quadrangular faces; besides, there are 20 triangular holes in each. The two

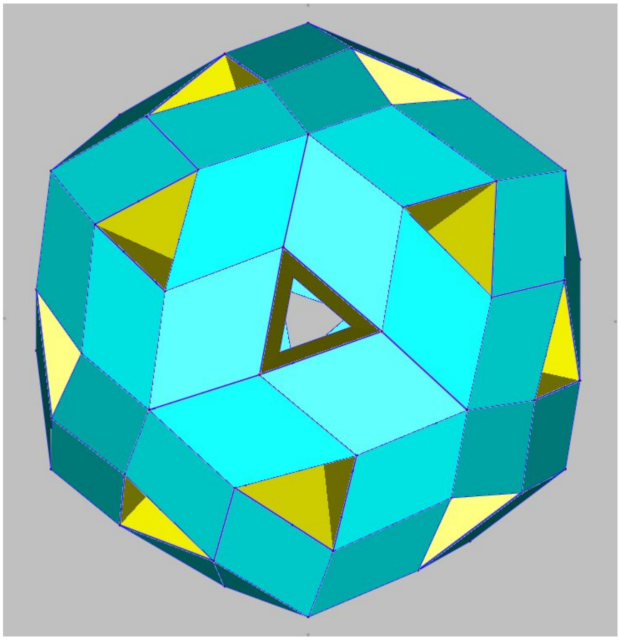


Figure 1: The Leonardo polyhedron of type $\{4, 5; 19\}$; $f = 36(4, 10, 5)$.

shells are joined by triangular prismatic tubes along these holes. Since this polyhedron has shortest non-0-homotopic paths of length 3 and 4, it is not regular. It follows from Conder's list of regular maps [4] that for $g = 19$ there is precisely one regular map of type $\{4, 5\}$. So the only possible Leonardo polyhedron of such type would be a realization of this map, and it is an open question if it exists or not.

We note also that the polyhedron of type $\{5, 4; 7\}$ with octahedral symmetry group was found already in 1983 (see [22], Figure 2). In [9] it is constructed in a slightly different way. The other related types are new and here we show in Figure 2 the example with genus 13.

3 Proof for the existence of equivelar series

In this section we prove Theorems 2.1 and 2.4 by constructing the polyhedra in question.

Proof of Theorem 2.1

CONSTRUCTION for Schläfli type $\{3, 7\}$.

The construction was already done in [16], but without any symmetry assumptions. We split our proof into three parts.

First we show that there are infinitely many simple convex 3-polytopes, i.e. such that all their vertices are 3-valent, for each of the required symmetry groups. The first polytopes of this type are the tetrahedron, the cube and the dodecahedron. They have 4, 8 and 20 3-valent vertices, respectively. We now continue by induction.

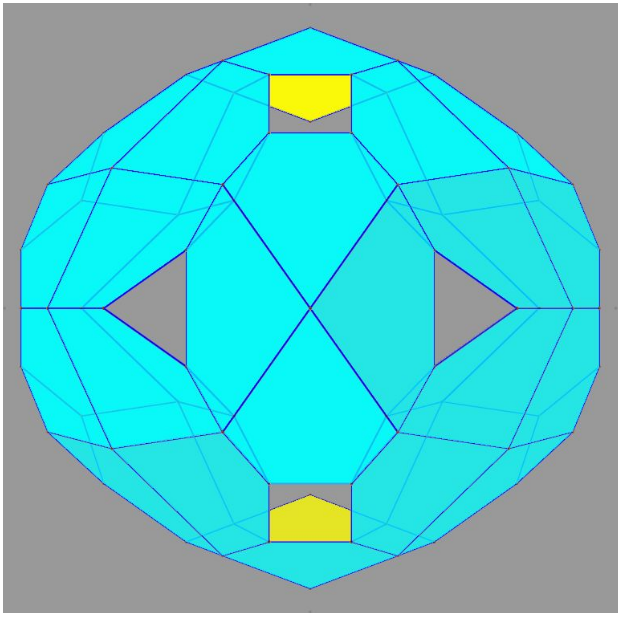


Figure 2: The Leonardo polyhedron of type $\{5, 4; 13\}$; $f = 24(5, 10, 4)$.

For a simple convex polytope with v vertices we cut off these vertices by a plane each, such that no cuts intersect and that the global symmetry is preserved. We obtain a new convex 3-polytope with the same symmetry group and with $3v$ vertices, all of them 3-valent. Clearly this polytope is not equivelar, because it contains faces of different type.

The second step is to construct from each of these polytopes a new one with 5-valent vertices.

For each of the convex polytopes with $3v$ vertices we make the following operation. We shrink all faces by the same factor, such that each face remains in its given affine hull. Hence they are disjoint. Now we take the convex hull of the system of these new polygons, so that we obtain a new convex polytope with 4-valent vertices. Each vertex of the former polytope corresponds to a triangle, so does each edge to a quadrangle, and the vertex-number of the new polytope is $9v$.

We now split the new quadrangles by a diagonal into two triangles in the right order, such that the global symmetry is preserved. More precisely, by this operation the full symmetry group is lost and reduced to the rotation subgroup of index two. We now have a convex 3-polytope with 5-valent vertices and the required symmetry.

The third step leads us to the construction of tunnels and the required polyhedron of type $\{3, 7\}$. First we take the boundary complex of our polytope, and put a smaller copy of this complex into the former one with the same centre and orientation. Then we delete in both objects the shrinked polygons. The remaining faces are all triangles. We now connect any two corresponding holes by tunnels, built up of quadrangles. Again we split each of these quadrangles by a diagonal in the right order, so that the symmetry is preserved. We obtain a polyhedron of Schläfli type $\{3, 7\}$ with the required symmetry. Finally, by a slight rotation of the holes (originating from the shrinked faces), each within its affine hull and to different extent in the two spherical complexes, it is ensured that no adjacent triangles are coplanar (this can also be done in symmetry-preserving way).

CONSTRUCTION for Schläfli types $\{4,6\}$ and $\{6,4\}$.

The construction was already described in [16], although without any symmetry considerations. In order to make the paper self-contained, we sketch the proof for the $\{6,4\}$ series. The dual $\{4,6\}$ series are constructed similarly. Let P be one of the simple 3-polytopes with Platonic symmetry group obtained in the first step of our former proof and let SD be the Schlegel diagram of the 4-prism with base P such that it is in one of the Platonic bases of the prism. All vertices of SD are 4-valent. We take the midpoints of all edges of SD, and then the convex hull of the midpoints of any four edges which are incident to a vertex of SD. Thus to each vertex of SD corresponds a 3-simplex and each vertex of a simplex is shared with a vertex of a neighboring simplex. Now we enlarge each simplex by the same factor $1+\varepsilon$, $\varepsilon > 0$ sufficiently small. We delete those parts of the simplices which lie inside another simplex and obtain a polyhedron of type $\{6,4\}$ with the required symmetry properties.

CONSTRUCTION for Schläfli-types $\{3,8\}$ and $\{3,9\}$.

From each polyhedron of type $\{6,4\}$ we obtain one of type $\{3,8\}$ as follows. In each hexagon one connects a triplet of non-consecutive vertices by segments and obtains a tiling of the hexagon into 4 triangles. If one does this in the right order on the whole $\{6,4\}$ polyhedron, one obtains the required $\{3,8\}$ polyhedron.

The $\{3,9\}$ series is obtained from the $\{4,6\}$ series as follows. Each quadrangle can be divided into two triangles by a diagonal. If one does this in the right order on the whole polyhedron, one obtains the required $\{3,9\}$ polyhedron. The crucial point for this procedure (which was already described in [16]) is the fact that, when applied to any polyhedron of type $\{p,q\}$, the valency q of the vertices is even. $\square$

Proof of Theorem 2.4

CONSTRUCTION for Schläfli-type $\{4,5\}$.

Start from two distinct types of Archimedean polyhedra, P_1 and P_2. P_1 is the truncated octahedron with six square faces and eight hexagonal faces. P_2 is the rhombicuboctahedron, which has 6+12 square faces and eight triangular faces. For the following construction it is crucial that P_1 has only 3-valent vertices, and P_2 only 4-valent vertices. Note that both the octuple of the hexagonal faces of P_1 and the octuple of the triangular faces of P_2 decomposes to two disjoint classes. In each case such a class is a quadruple forming an orbit under the action of the tetrahedral symmetry group (a subgroup of the octahedral symmetry group of these polyhedra). Delete these hexagonal and triangular faces, and denote the complexes obtained in this way by P_1' and P_2', respectively. Take $2k$ ($k \geq 2$) concentric and homothetic copies of P_1'. We call them *shells* of our polyhedron under construction. Now join the holes of the neighbouring shells by hexagonal prismatic tubes. The tubes are arranged so that each intermediate shell is joined to its outer or inner neighbour, in both cases using four tubes and using holes that belong to the same class (but different in the two cases). The innermost shell is joined to the outermost shell. To avoid undesirable contacts, the holes of the outermost shell are shrinked to a suitable size with respect to those of the intermediate shells, while keeping the symmetry. For P_2', the construction is the same with the only difference that here one uses triangular tubes. We obtain two different infinite series of polyhedra, both of the desired Schläfli type, and with the (full) tetrahedral symmetry group and genus $g = 1 + 6k$ $(k = 1, 2, \dots)$.

Consider again the four orbits of faces of the rhombicuboctahedron with respect to the action of the tetrahedral group. Clearly, analogous polyhedra can be constructed, likewise with four orbits of faces with respect to the octahedral and the icosahedral group. In the octahedral case these orbits are 6 squares, 8 regular triangles, 12 rhombi and 24 rectangles, while in the icosahedral case there are 12 regular pentagons, 20 regular triangles, 30 squares and 60 symmetric trapezia. In both cases all the vertices are 4-valent (and the polar dual is such that all the faces are kite-shaped and form two orbits). Now deleting the square and triangular faces in the the octahedral case, and the non-quadrangular faces in the icosahedral case, and applying an analogous construction as above, one obtains the desired infinite series of polyhedra with genus $g = 1 + 12k$ and $g = 1 + 30k$, respectively. The starting member of the icosahedral series is shown in Figure 3.

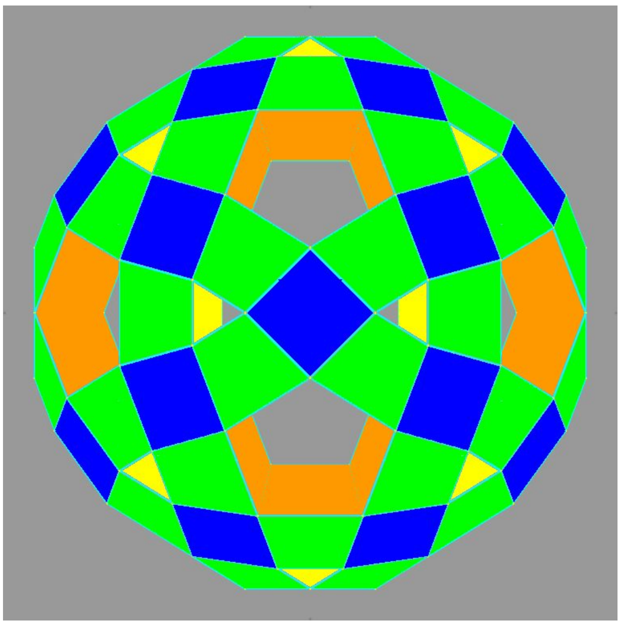


Figure 3: The Leonardo polyhedron of type $\{4, 5; 31\}$; $f = 60(4, 10, 5)$.

CONSTRUCTION for Schläfli-type $\{5, 4\}$.

Let P be a polyhedron satisfying the following conditions:

(1) the symmetry group $G(P)$ of P is equal to the full symmetry group of one of the Platonic solids;

(2) $G(P)$ is transitive on the faces of P;

(3) the faces of P are quadrangles;

(4) each edge of P is contained in one of the mirror planes determined by $G(P)$.

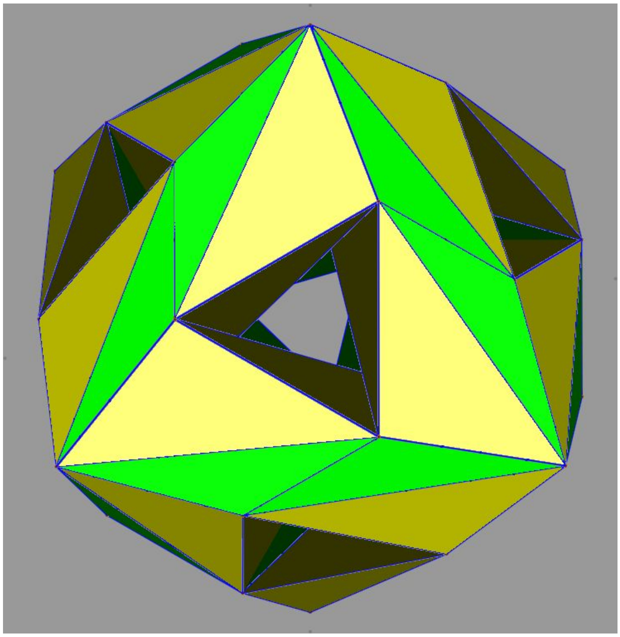
It is easy to see that P is combinatorially equivalent to one of the following five polyhedra (cf. [8]):

- cube;
- rhombic dodecahedron;
- rhombic triacontahedron;
- deltoidal icositetrahedron (dual of the rhombicuboctahedron) (see e.g. Figure 3 in [9];
- deltoidal hexecontahedron (dual of the Archimedean polyhedron called rhombicosidodecahedron) (see e.g. Figure 5 in [9] or Figure 9 in [7]).

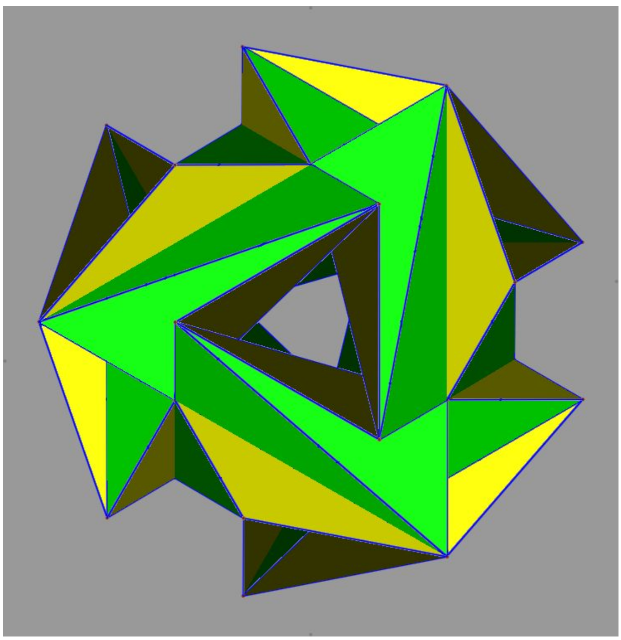
Put on each of the quadrangular faces of P a bipyramid, each pairwise congruent, such that the midpoints of the edges of the face form the basal vertices of the bipyramid. Then enlarge each bipyramid from the centre of its own base by the same factor $1 + \epsilon$, $\epsilon > 0$ sufficiently small. Delete now those parts of the bipyramids which lie inside another bipyramid (along with the original faces of P). One obtains a polyhedron of the desired Schläfli type such that its symmetry group remains the same as that of the example of P we started from. The genus of this polyhedron is $g = f_2(P) + 1$, where $f_2(P)$ is the number of the faces of P; hence the genera in the five cases above are $7, 13, 31, 25$ and 61, respectively. Finally, we note that there is a polyhedron called deltoidal dodecahedron, combinatorially equivalent to the rhombic dodecahedron but with tetrahedral symmetry (a well-known figure in geometric crystallography, see e.g. [10]). It also satisfies the conditions above. Hence starting from it, our construction provides the tetrahedrally symmetric polyhedron with $g = 13$.

The conditions (1–4) above can be suitably modified such that they are satisfied by equivelar polyhedra with quadrangular faces and with spine isomorphic to the 1-skeleton of a regular 4-polytope Q. (Two of these polyhedra are even regular, namely that of type $\{4, 6; 6\}$ and $\{4, 8; 73\}$ [21].) Thus, performing the construction in $\mathbb{E}^4$, then taking a suitable projection to $\mathbb{E}^3$, one obtains Leonardo polyhedra of the following genera: $g = 31$ (Q is the regular 4-simplex), $g = 97$ (Q is either the 4-cube or the regular 16-cell), $g = 289$ (Q is the regular 24-cell) and $g = 3601$ (Q is either the regular 120-cell or the 600-cell). □

In conclusion, we present a sporadic example of a polyhedron of type $\{3, 8; 7\}$, which differs in its structure from those in Theorem 2.1. It is constructed from two solids P and P', such that P' is a non-convex version of the convex 3-polytope P; their boundary are combinatorially equivalent to each other, and have the f-vector $f = (30, 84, 56)$. The size and shape of the bounding polyhedra is adjusted so that deleting a whole 8-element orbit of faces from both, the complexes obtained in this way can be glued together along the holes, thus forming the outer and inner shell of a new polyhedron. This polyhedron has octahedral rotation symmetry; it is shown in Figure 4a, and its inner shell in Figure 4b.



(a) The whole polyhedron.



(b) The inner shell of the polyhedron.

Figure 4: The Leonardo polyhedron of type $\{3, 8; 7\}$; $f = 12(3, 12, 8)$.

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Line graphs and geodesic transitivity*

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Abstract

For a graph Γ, a positive integer s and a subgroup $G \leq \mathrm{Aut}(\Gamma)$, we prove that G is transitive on the set of s-arcs of Γ if and only if Γ has girth at least $2(s-1)$ and G is transitive on the set of $(s-1)$-geodesics of its line graph. As applications, we first classify 2-geodesic transitive graphs of valency 4 and girth 3, and determine which of them are geodesic transitive. Secondly we prove that the only non-complete locally cyclic 2-geodesic transitive graphs are the octahedron and the icosahedron.

Keywords: Line graphs, s-geodesic transitive graphs, s-arc transitive graphs.

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1 Introduction

A geodesic from a vertex u to a vertex v in a graph is a path of shortest length from u to v. In the infinite setting geodesics play an important role, for example, in the classification of infinite distance transitive graphs [11], and in studying locally finite graphs, see for example, [17]. They are also used to model, in a finite network, the notion of visibility in Euclidean space [22]. Here we study transitivity properties on geodesics in finite graphs. Throughout this paper, we assume that all graphs are finite simple and undirected.

Let Γ be a connected graph with vertex set $V(\Gamma)$, edge set $E(\Gamma)$ and automorphism group $\mathrm{Aut}(\Gamma)$. For a positive integer s, an *s-arc* of Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that v_i, v_{i+1} are adjacent and $v_{j-1} \neq v_{j+1}$ for $0 \leq i \leq s-1$, $1 \leq j \leq s-1$; it is an *s-geodesic* if in addition v_0, v_s are at distance s. For $G \leq \mathrm{Aut}(\Gamma)$, Γ is said to be (G,s)-*arc transitive* or (G,s)-*geodesic transitive*, if Γ contains an s-arc or s-geodesic,

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and G is transitive on the set of t-arcs or t-geodesics respectively for all $t \leq s$. Moreover, if $G = \mathrm{Aut}(\Gamma)$, then G is usually omitted in the previous notation. The study of (G, s)-arc transitive graphs goes back to Tutte's papers [18, 19] which showed that if Γ is a (G, s)-arc transitive cubic graph then $s \leq 5$. About twenty years later, relying on the classification of finite simple groups, Weiss [21] proved that there are no $(G, 8)$-arc transitive graphs with valency at least three. The family of s-arc transitive graphs has been studied extensively, see [2, 9, 15, 16, 20]. Here we consider these properties for line graphs.

The *line graph* $L(\Gamma)$ of a graph Γ is the graph whose vertices are the edges of Γ, with two edges adjacent in $L(\Gamma)$ if they have a vertex in common. Our first aim in the paper is to investigate connections between the s-arc transitivity of a connected graph Γ and the $(s-1)$-geodesic transitivity of its line graph $L(\Gamma)$ where $s \geq 2$. A key ingredient in this study is a collection of injective maps $\mathcal{L}_s$, where $\mathcal{L}_s$ maps the s-arcs of Γ to certain s-tuples of edges of Γ (vertices of $L(\Gamma)$) as defined in Definition 2.3. The major properties of $\mathcal{L}_s$ are derived in Theorem 2.4 and the main consequence linking the symmetry of Γ and $L(\Gamma)$ is given in Theorem 1.1, which is proved in Subsection 2.2.

We denote by $\Gamma(u)$ the set of vertices adjacent to the vertex u in Γ. If $|\Gamma(u)|$ is independent of $u \in V(\Gamma)$, then Γ is said to be *regular*. The *girth* of Γ is the length of the shortest cycle; the *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximum distance between two vertices.

Theorem 1.1. *Let Γ be a connected regular, non-complete graph of girth g and valency at least 3. Let $G \leq \mathrm{Aut}(\Gamma)$ and let s be a positive integer such that $2 \leq s \leq \mathrm{diam}(L(\Gamma))+1$. Then G is transitive on the set of s-arcs of Γ if and only if $s \leq \mathsf{g}/2+1$ and G is transitive on the set of $(s-1)$-geodesics of $L(\Gamma)$.*

It follows from a deep theorem of Richard Weiss in [21] that if Γ is a connected s-arc transitive graph of valency at least 3, then $s \leq 7$. This observation yields the following corollary, and its proof can be found in Subsection 2.2.

Corollary 1.2. *Let Γ and g be as in Theorem 1.1 . Let s be a positive integer such that $2 \leq s \leq \mathrm{diam}(L(\Gamma)) + 1$. If $L(\Gamma)$ is $(s-1)$-geodesic transitive, then either $2 \leq s \leq 7$ or $s > \max\{7, \mathsf{g}/2 + 1\}$.*

Note that in a graph, 1-arcs and 1-geodesics are the same, and are called *arcs*. For graphs of girth at least 4, each 2-arc is a 2-geodesic so the sets of 2-arc transitive graphs and 2-geodesic transitive graphs are the same. However, there are also 2-geodesic transitive graphs of girth 3. For such a graph Γ, the subgraph $[\Gamma(u)]$ induced on the set $\Gamma(u)$ is vertex transitive and contains edges. Moreover, if $[\Gamma(u)]$ is complete, then so is Γ. A vertex transitive, non-complete, non-empty graph must have at least 4 vertices and thus valency 4 is the first interesting case.

As an application of Theorem 1.1, we characterise connected non-complete 2-geodesic transitive graphs of girth 3 and valency 4. In this case, $[\Gamma(u)] \cong C_4$ or $2K_2$ for each $u \in V(\Gamma)$. If Γ is s-geodesic transitive with $s = \mathrm{diam}(\Gamma)$, then Γ is called *geodesic transitive*. A graph Γ is said to be *distance transitive* if its automorphism group is transitive on the ordered pairs of vertices at any given distance.

Theorem 1.3. *Let Γ be a connected non-complete graph of girth 3 and valency 4. Then Γ is 2-geodesic transitive if and only if Γ is either $L(K_4) \cong \mathcal{O}$ or $L(\Sigma)$ for a connected 3-arc transitive cubic graph Σ.*

Moreover, Γ is geodesic transitive if and only if $\Gamma = L(\Sigma)$ for a cubic distance transitive graph Σ, namely $\Sigma = K_4$, $K_{3,3}$, the Petersen graph, the Heawood graph or Tutte's 8-cage.

Since there are infinitely many 3-arc transitive cubic graphs, there are therefore infinitely many 2-geodesic transitive graphs with girth 3 and valency 4. Theorem 1.3 is proved in Section 3, and it provides a useful method for constructing 2-geodesic transitive graphs of girth 3 and valency 4 which are not geodesic transitive, an example being the line graph of a triple cover of Tutte's 8-cage constructed in [14]. The line graphs mentioned in the second part of Theorem 1.3 are precisely the distance transitive graphs of valency 4 and girth 3 given, for example, in [4, Theorem 7.5.3 (i)].

A graph Γ is said to be *locally cyclic* if $[\Gamma(u)]$ is a cycle for every vertex u. In particular, the girth of a locally cyclic graph is 3. It was shown in [8, Theorem 1.1] that for 2-geodesic transitive graphs Γ of girth 3, the subgraph $[\Gamma(u)]$ is either a connected graph of diameter 2, or isomorphic to the disjoint union mK_r of m copies of a complete graph K_r with $m \geq 2, r \geq 2$. Thus one consequence of Theorem 1.3 is a classification of connected, locally cyclic, 2-geodesic transitive graphs in Corollary 1.4: for $[\Gamma(u)] \cong C_n$ has diameter 2 only for valencies $n = 4$ or 5, and the valency 5, girth 3, 2-geodesic transitive graphs were classified in [7]. Its proof can be found at the end of Section 3. We note that locally cyclic graphs are important for studying embeddings of graphs in surfaces, see for example [10, 12, 13].

Corollary 1.4. *Let Γ be a connected, non-complete, locally cyclic graph. Then Γ is 2-geodesic transitive if and only if Γ is the octahedron or the icosahedron.*

2 Line graphs

We begin by citing a well-known result about line graphs.

Lemma 2.1. [1, p.1455] *Let Γ be a connected graph. If Γ has at least 5 vertices, then* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(L(\Gamma))$.

The *subdivision graph* $S(\Gamma)$ of a graph Γ is the graph with vertex set $V(\Gamma) \cup E(\Gamma)$ and edge set $\{\{u, e\} | u \in V(\Gamma), e \in E(\Gamma), u \in e\}$. The link between the diameters of Γ and $S(\Gamma)$ was determined in [6, Remark 3.1 (b)]: $\mathrm{diam}(S(\Gamma)) = 2\,\mathrm{diam}(\Gamma) + \delta$ for some $\delta \in \{0, 1, 2\}$. Here, based on this result, we will show the connection between the diameters of Γ and $L(\Gamma)$ in the following lemma.

Lemma 2.2. *Let Γ be a connected graph with $|V(\Gamma)| \geq 2$. Then it holds* $\mathrm{diam}(L(\Gamma)) = \mathrm{diam}(\Gamma) + x$ *for some $x \in \{-1, 0, 1\}$. Moreover, all three values occur, for example, if $\Gamma = K_{3+x}$, then* $\mathrm{diam}(L(\Gamma)) = \mathrm{diam}(\Gamma) + x = 1 + x$ *for each x.*

Proof. Let $d = \mathrm{diam}(\Gamma)$, $d_l = \mathrm{diam}(L(\Gamma))$ and $d_s = \mathrm{diam}(S(\Gamma))$. Let $(x_0, x_2, \ldots, x_{2d_l})$ be a d_l-geodesic of $L(\Gamma)$. Then by definition of $L(\Gamma)$, each edge intersection $x_{2i} \cap x_{2i+2}$ is a vertex v_{2i+1} of Γ and $(x_0, v_1, x_2, \ldots, v_{2d_l-1}, x_{2d_l})$ is a $2d_l$-path in $S(\Gamma)$. Suppose that $(x_0, v_1, x_2, \ldots, v_{2d_l-1}, x_{2d_l})$ is not a $2d_l$-geodesic of $S(\Gamma)$. Then there is an r-geodesic from x_0 to x_{2d_l}, say $(y_0, y_1, y_2, \ldots, y_r)$ with $y_0 = x_0$ and $y_r = x_{2d_l}$, such that $r < 2d_l$. Since both x_0, x_{2d_l} are in $V(L(\Gamma))$, it follows that r is even, and hence $d_{L(\Gamma)}(x_0, x_{2d_l}) = \frac{r}{2} < d_l$ which contradicts the fact that $(x_0, x_2, \ldots, x_{2d_l})$ is a d_l-geodesic of $L(\Gamma)$. Thus

$(x_0, v_1, x_2, \dots, v_{2d_l-1}, x_{2d_l})$ is a $2d_l$-geodesic in $S(\Gamma)$. It follows from [6, Remark 3.1 (b)] that $d_l \le d_s/2 \le d+1$.

Now take a d_s-geodesic $(x_0, x_1, \dots, x_{d_s})$ in $S(\Gamma)$. If $x_0 \in E(\Gamma)$, then $(x_0, x_2, x_4, \dots, x_{2\lfloor d_s/2 \rfloor})$ is a $\lfloor d_s/2 \rfloor$-geodesic in $L(\Gamma)$, so $d_l \ge \lfloor d_s/2 \rfloor \ge d$. Similarly we see that $d_l \ge d$ if $x_{d_s} \in E(\Gamma)$. Finally if both $x_0, x_{d_s} \in V(\Gamma)$, then d_s is even and $d_\Gamma(x_0, x_{d_s}) = d_s/2$. Moreover $(x_1, x_3, \dots, x_{d_s-1})$ is a $(\frac{d_s-2}{2})$-geodesic in $L(\Gamma)$. By [6, Remark 3.1 (b)], $d_s = 2d$, so $d_l \ge \frac{d_s-2}{2} = d-1$. □

2.1 The map $\mathcal{L}_s$

Let Γ be a finite connected graph. For each integer $s \ge 2$, we define a map from the set of s-arcs of Γ to the set of s-tuples of $V(L(\Gamma))$.

Definition 2.3. Let $\mathbf{a} = (v_0, v_1, \dots, v_s)$ be an s-arc of Γ where $s \ge 2$, and for $0 \le i < s$, let $e_i := \{v_i, v_{i+1}\} \in E(\Gamma)$. Define $\mathcal{L}_s(\mathbf{a}) := (e_0, e_1, \dots, e_{s-1})$.

The following theorem gives some important properties of $\mathcal{L}_s$.

Theorem 2.4. *Let $s \ge 2$, let Γ be a connected graph containing at least one s-arc, and let $\mathcal{L}_s$ be as in Definition* 2.3. *Then the following statements hold.*

(1) *$\mathcal{L}_s$ is an injective map from the set of s-arcs of Γ to the set of $(s-1)$-arcs of $L(\Gamma)$. Further, $\mathcal{L}_s$ is a bijection if and only if either $s = 2$, or $s \ge 3$ and $\Gamma \cong C_m$ or P_n for some $m \ge 3, n \ge s$.*

(2) *$\mathcal{L}_s$ maps s-geodesics of Γ to $(s-1)$-geodesics of $L(\Gamma)$.*

(3) *If $s \le \mathrm{diam}(L(\Gamma)) + 1$, then the image $\mathrm{Im}(\mathcal{L}_s)$ contains the set $\mathcal{G}_{s-1}$ of all $(s-1)$-geodesics of $L(\Gamma)$. Moreover, $\mathrm{Im}(\mathcal{L}_s) = \mathcal{G}_{s-1}$ if and only if $\mathrm{girth}(\Gamma) \ge 2s-2$.*

(4) *$\mathcal{L}_s$ is $\mathrm{Aut}(\Gamma)$-equivariant, that is, $\mathcal{L}_s(\boldsymbol{a})^g = \mathcal{L}_s(\boldsymbol{a}^g)$ for all $g \in \mathrm{Aut}(\Gamma)$ and all s-arcs $\boldsymbol{a}$ of Γ.*

Proof. (1) Let $\mathbf{a} = (v_0, v_1, \dots, v_s)$ be an s-arc of Γ and let $\mathcal{L}_s(\mathbf{a}) := (e_0, e_1, \dots, e_{s-1})$ with the e_i as in Definition 2.3. Then each of the e_i lies in $E(\Gamma) = V(L(\Gamma))$ and $e_k \ne e_{k+1}$ for $0 \le k \le s-2$. Further, since $v_j \ne v_{j+1}, v_{j+2}$ for $1 \le j \le s-2$, we have $e_{j-1} \ne e_{j+1}$. Thus $\mathcal{L}_s(\mathbf{a})$ is an $(s-1)$-arc of $L(\Gamma)$.

Let $\mathbf{b} = (u_0, u_1, \dots, u_s)$ and $\mathbf{c} = (w_0, w_1, \dots, w_s)$ be two s-arcs of Γ. Then $\mathcal{L}_s(\mathbf{b}) = (f_0, f_1, \dots, f_{s-1})$ and $\mathcal{L}_s(\mathbf{c}) = (g_0, g_1, \dots, g_{s-1})$ are two $(s-1)$-arcs of $L(\Gamma)$, where $f_i = \{u_i, u_{i+1}\}$ and $g_i = \{w_i, w_{i+1}\}$ for $0 \le i < s$. Suppose that $\mathcal{L}_s(\mathbf{b}) = \mathcal{L}_s(\mathbf{c})$. Then $f_i = g_i$ for each $i \ge 0$, and hence $f_i \cap f_{i+1} = g_i \cap g_{i+1}$, that is, $u_{i+1} = w_{i+1}$ for each $0 \le i \le s-2$. So also $u_0 = w_0$ and $u_s = w_s$, and hence $\mathbf{b} = \mathbf{c}$. Thus $\mathcal{L}_s$ is injective.

Now we prove the second part. Each arc of $L(\Gamma)$ is of the form $\mathbf{h} = (e, f)$ where $e = \{u_0, u_1\}$ and $f = \{u_1, u_2\}$ are distinct edges of Γ. Thus $u_0 \ne u_2$, so $\mathbf{k} = (u_0, u_1, u_2)$ is a 2-arc of Γ and $\mathcal{L}_2(\mathbf{k}) = \mathbf{h}$. It follows that $\mathcal{L}_2$ is onto and hence is a bijection. If $s \ge 3$ and $\Gamma \cong C_m$ or P_n for some $m \ge 3, n \ge s$, then $L(\Gamma) \cong C_m$ or P_{n-1} respectively, and hence for every $(s-1)$-arc $\mathbf{x}$ of $L(\Gamma)$, we can find an s-arc $\mathbf{y}$ of Γ such that $\mathcal{L}_s(\mathbf{y}) = \mathbf{x}$, that is, $\mathcal{L}_s$ is onto. Thus $\mathcal{L}_s$ is a bijection. Conversely, suppose that $\mathcal{L}_s$ is onto, and that $s \ge 3$. Assume that some vertex u of Γ has valency greater than 2 and let $e_1 = \{u, v_1\}, e_2 = \{u, v_2\}, e_3 = \{u, v_3\}$ be distinct edges. Then $\mathbf{x} = (e_1, e_2, e_3)$ is a 2-arc in $L(\Gamma)$ and there is no 3-arc $\mathbf{y}$ of Γ such that $\mathcal{L}_s(\mathbf{y}) = \mathbf{x}$. In general, for $s = 3a + b \ge 4$ with $a \ge 1$ and $b \in \{0, 1, 2\}$, we concatenate a copies of $\mathbf{x}$ to form an $(s-1)$-arc of $L(\Gamma)$: namely $(\mathbf{x}^a)$ if $b = 0$; $(\mathbf{x}^a, e_1)$ if $b = 1$; $(\mathbf{x}^a, e_1, e_2)$ if $b = 2$. This $(s-1)$-arc does not lie in the image of $\mathcal{L}_s$. Thus each vertex of Γ has valency at most 2. If all vertices have valency 2 then $\Gamma \cong C_m$ for some $m \geq 3$, since Γ is connected. So suppose that some vertex u of Γ has valency 1. Since Γ is connected and each other vertex has valency at most 2, it follows that $\Gamma \cong P_n$ for some $n \geq s$.

(2) Let $\mathbf{a} = (v_0, \dots, v_s)$ be an s-geodesic of Γ and let $\mathcal{L}_s(\mathbf{a}) = (e_0, \dots, e_{s-1})$ as above. If $s = 2$, then $\mathcal{L}_s(\mathbf{a})$ is a 1-arc, and hence a 1-geodesic of $L(\Gamma)$. Suppose that $s \geq 3$ and $\mathcal{L}_s(\mathbf{a})$ is not an $(s-1)$-geodesic. Then $d_{L(\Gamma)}(e_0, e_{s-1}) = r < s - 1$ and there exists an r-geodesic $\mathbf{r} = (f_0, f_1, \dots, f_{r-1}, f_r)$ with $f_0 = e_0$ and $f_r = e_{s-1}$. Since $s \geq 3$ and $\mathbf{a}$ is an s-geodesic, it follows that $\{v_0, v_1\} \cap \{v_{s-1}, v_s\} = \emptyset$, that is, e_0 and e_{s-1} are not adjacent in $L(\Gamma)$. Thus $r \geq 2$. Since $\mathbf{r}$ is an r-geodesic, it follows that the consecutive edges f_{i-1}, f_i, f_{i+1} do not share a common vertex for any $1 \leq i \leq r-1$, otherwise $(f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_r)$ would be a shorter path than $\mathbf{r}$, which is impossible. Hence we have $f_h = \{u_h, u_{h+1}\}$ for $0 \leq h \leq r$. Then $(u_1, u_2, \dots, u_r)$ is an $(r-1)$-path in Γ, $\{u_1\} = e_0 \cap f_1 \subseteq \{v_0, v_1\}$ and $\{u_r\} = f_{r-1} \cap e_{s-1} \subseteq \{v_{s-1}, v_s\}$. It follows that $d_\Gamma(v_0, v_s) \leq d_\Gamma(u_1, u_r) + 2 \leq r + 1 < s$, contradicting the fact that $\mathbf{a}$ is an s-geodesic. Therefore, $\mathcal{L}_s(\mathbf{a})$ is an $(s-1)$-geodesic of $L(\Gamma)$.

(3) Let $2 \leq s \leq \mathrm{diam}(L(\Gamma)) + 1$ and $\mathcal{G}_{s-1}$ be the set of all $(s-1)$-geodesics of $L(\Gamma)$. If $s = 2$, then by part (1), each 1-geodesic of $L(\Gamma)$ lies in the image $\mathrm{Im}(\mathcal{L}_2)$, and hence $\mathcal{G}_1 \subseteq \mathrm{Im}(\mathcal{L}_2)$. Now suppose inductively that $2 \leq s \leq \mathrm{diam}(L(\Gamma))$ and $\mathcal{G}_{s-1} \subseteq \mathrm{Im}(\mathcal{L}_s)$. Let $\mathbf{e} = (e_0, e_1, e_2, \dots, e_s)$ be an s-geodesic of $L(\Gamma)$. Then $\mathbf{e}' = (e_0, e_1, e_2, \dots, e_{s-1})$ is an $(s-1)$-geodesic of $L(\Gamma)$. Thus there exists an s-arc $\mathbf{a}$ of Γ such that $\mathcal{L}_s(\mathbf{a}) = \mathbf{e}'$, say $\mathbf{a} = (v_0, v_1, \dots, v_s)$. Since e_s is adjacent to $e_{s-1} = \{v_{s-1}, v_s\}$ but not to $e_{s-2} = \{v_{s-2}, v_{s-1}\}$ in $L(\Gamma)$, it follows that $e_s = \{v_s, x\}$ where $x \notin \{v_{s-2}, v_{s-1}\}$. Hence $\mathbf{b} = (v_0, v_1, \dots, v_s, x)$ is an $(s+1)$-arc of Γ. Further, $\mathcal{L}_{s+1}(\mathbf{b}) = \mathbf{e}$. Thus $\mathrm{Im}(\mathcal{L}_{s+1})$ contains all s-geodesics of $L(\Gamma)$, that is, $\mathcal{G}_s \subseteq \mathrm{Im}(\mathcal{L}_{s+1})$. Hence the first part of (3) is proved by induction.

Now we prove the second part. Suppose first that for every s-arc $\mathbf{a}$ of Γ, $\mathcal{L}_s(\mathbf{a})$ is an $(s-1)$-geodesic of $L(\Gamma)$. Let $\mathrm{g} := \mathrm{girth}(\Gamma)$. If $s = 2$, as $\mathrm{g} \geq 3$, then $\mathrm{g} \geq 2s - 2$. Now let $s \geq 3$. Assume that $\mathrm{g} \leq 2s - 3$. Then Γ has a g-cycle $\mathbf{b} = (u_0, u_1, u_2, \dots, u_{\mathrm{g}-1}, u_{\mathrm{g}})$ with $u_{\mathrm{g}} = u_0$. It follows that $\mathcal{L}_{\mathrm{g}}(\mathbf{b})$ forms a g-cycle of $L(\Gamma)$. Thus the sequence $\mathbf{b}' = (u_0, u_1, \dots, u_s)$ (where we take subscripts modulo g if necessary) is an s-arc of Γ and $\mathcal{L}_s(\mathbf{b}') = (e_0, e_1, \dots, e_{s-1})$ involves only the vertices of $\mathcal{L}_s(\mathbf{b})$. This implies that $d_{L(\Gamma)}(e_0, e_{s-1}) \leq \frac{\mathrm{g}}{2} \leq \frac{2s-3}{2} < s - 1$, that is, $\mathcal{L}_s(\mathbf{b}')$ is not an $(s-1)$-geodesic, which is a contradiction. Thus, $\mathrm{g} \geq 2s - 2$.

Conversely, suppose that $\mathrm{g} \geq 2s - 2$. Let $\mathbf{a} := (v_0, v_1, v_2, \dots, v_s)$ be an s-arc of Γ. Then $\mathcal{L}_s(\mathbf{a}) = (e_0, e_1, e_2, \dots, e_{s-1})$ is an $(s-1)$-arc of $L(\Gamma)$ by part (1). Let $\mathbf{a}' := (v_0, v_1, v_2, \dots, v_{s-1})$. Since $\mathrm{g} \geq 2s - 2$, it follows that $\mathbf{a}'$ is an $(s-1)$-geodesic, and hence by (2), $\mathcal{L}_{s-1}(\mathbf{a}') = (e_0, e_1, e_2, \dots, e_{s-2})$ is an $(s-2)$-geodesic of $L(\Gamma)$. Thus $z = d_{L(\Gamma)}(e_0, e_{s-1})$ satisfies $s - 3 \leq z \leq s - 1$. There is a z-geodesic from e_0 to e_{s-1}, say $\mathbf{f} = (e_0, f_1, f_2, \dots, f_{z-1}, e_{s-1})$. Further, by the first part of (3), there is a $(z+1)$-arc $\mathbf{b} = (u_0, u_1, \dots, u_z, u_{z+1})$ of Γ such that $\mathcal{L}_{z+1}(\mathbf{b}) = \mathbf{f}$ and we have $e_0 = \{u_0, u_1\} = \{v_0, v_1\}$ and $e_{s-1} = \{u_z, u_{z+1}\} = \{v_{s-1}, v_s\}$. There are 4 cases, in columns 2 and 3 of Table 1: in each case there is a given nondegenerate closed walk $\mathbf{x}$ of length $l(\mathbf{x})$ as in Table 1. Thus $l(\mathbf{x}) \geq \mathrm{g} \geq 2s - 2$ and in each case $l(\mathbf{x}) \leq s + z - 1$. It follows that $z \geq s - 1$, and hence $z = s - 1$. Thus $\mathcal{L}_s(\mathbf{a}) = (e_0, e_1, e_2, \dots, e_{s-1})$ is an $(s-1)$-geodesic of $L(\Gamma)$.

(4) This property follows from the definition of $\mathcal{L}_s$. $\square$

Table 1: Four cases of **x**

Case	(u_0, u_1)	(u_z, u_{z+1})	**x**	$l(\mathbf{x})$
1	(v_0, v_1)	(v_{s-1}, v_s)	$(v_{s-1}, v_{s-2}, \ldots, v_2, v_1, u_2, \ldots, u_{z-1}, v_{s-1})$	$s+z-3$
2	(v_0, v_1)	(v_s, v_{s-1})	$(v_s, v_{s-1}, \ldots, v_2, v_1, u_2, \ldots, u_{z-1}, v_s)$	$s+z-2$
3	(v_1, v_0)	(v_{s-1}, v_s)	$(v_{s-1}, v_{s-2}, \ldots, v_2, v_1, u_1, u_2, \ldots, u_{z-1}, v_{s-1})$	$s+z-2$
4	(v_1, v_0)	(v_s, v_{s-1})	$(v_s, v_{s-1}, \ldots, v_2, v_1, u_1, u_2, \ldots, u_{z-1}, v_s)$	$s+z-1$

Remark 2.5. (i) The map $\mathcal{L}_s$ is usually not surjective on the set of $(s-1)$-arcs of $L(\Gamma)$. In the proof of Theorem 2.4 (1), we constructed an $(s-1)$-arc of $L(\Gamma)$ not in $\mathrm{Im}(\mathcal{L}_s)$ for any Γ with at least one vertex of valency at least 3.

(ii) Theorem 2.4 (1) and (3) imply that, for each $(s-1)$-geodesic **e** of $L(\Gamma)$, there is a unique s-arc **a** of Γ such that $\mathcal{L}_s(\mathbf{a}) = \mathbf{e}$. The s-arc **a** is not always an s-geodesic. For example, if Γ has girth 3 and (v_0, v_1, v_2, v_0) is a 3-cycle, then $\mathbf{a} = (v_0, v_1, v_2)$ is not a 2-geodesic but $\mathcal{L}_2(\mathbf{a})$ is the 1-geodesic (e_0, e_1) where $e_0 = \{v_0, v_1\}$ and $e_1 = \{v_1, v_2\}$.

2.2 Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Let Γ be a connected, regular, non-complete graph of girth g and valency at least 3. Then in particular $|V(\Gamma)| \geq 5$, and by Lemma 2.1, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(L(\Gamma))$. Let $G \leq \mathrm{Aut}(\Gamma)$ and let $2 \leq s \leq \mathrm{diam}(L(\Gamma)) + 1$.

Suppose first that G is transitive on the set of s-arcs of Γ. Then by [3, Proposition 17.2], $s \leq \mathrm{g}/2 + 1$. Since $s - 1 \leq \mathrm{diam}(L(\Gamma))$, it follows that $L(\Gamma)$ has $(s-1)$-geodesics and by Theorem 2.4 (3), $\mathrm{Im}(\mathcal{L}_s)$ is the set of $(s-1)$-geodesics of $L(\Gamma)$. On the other hand, by Theorem 2.4 (4), G acts transitively on $\mathrm{Im}(\mathcal{L}_s)$, and hence G is transitive on the set of $(s-1)$-geodesics of $L(\Gamma)$.

Conversely, suppose that $s \leq \mathrm{g}/2 + 1$ and G is transitive on the $(s-1)$-geodesics of $L(\Gamma)$. Then by the last assertion of Theorem 2.4 (3), $\mathrm{Im}(\mathcal{L}_s)$ is the set of $(s-1)$-geodesics, and since $\mathcal{L}_s$ is injective, it follows from Theorem 2.4 (1) and (4) that G is transitive on the set of s-arcs of Γ. $\square$

Proof of Corollary 1.2. Let Γ, g, s be as in Theorem 1.1. Assume that $\mathrm{Aut}(\Gamma)$ is transitive on the $(s-1)$-geodesics of $L(\Gamma)$. If $s > 7$, then by [21], $\mathrm{Aut}(\Gamma)$ is not transitive on the s-arcs of Γ and so by Theorem 1.1, $s > \frac{\mathrm{g}}{2} + 1$. $\square$

3 2-geodesic transitive graphs that are locally cyclic or locally $2K_2$

In this section, we prove Theorem 1.3. The proof uses the notion of a clique graph. A *maximum clique* of a graph Γ is a clique (complete subgraph) which is not contained in a larger clique. The *clique graph* $C(\Gamma)$ of Γ is the graph with vertices the maximum cliques of Γ, and two maximum cliques are adjacent if and only if they have at least one common vertex in Γ.

Proof of Theorem 1.3. Let Γ be a connected non-complete graph of girth 3 and valency 4, and let $A = \mathrm{Aut}(\Gamma)$ and $v \in V(\Gamma)$. Suppose first that Γ is 2-geodesic transitive. Then Γ is arc transitive, and so A_v is transitive on $\Gamma(v)$. Since Γ is non-complete of girth 3, $[\Gamma(v)]$ is neither complete nor edgeless, and so, as discussed before the statement of Theorem 1.3, $[\Gamma(v)] = C_4$ or $2K_2$. If $[\Gamma(v)] \cong C_4$, then it is easy to see that $\Gamma \cong \mathcal{O}$ (or see [4, p.5] or [5]). So we may assume that $[\Gamma(v)] \cong 2K_2$. It follows from [8, Theorem 1.4] that Γ is isomorphic to the clique graph $C(\Sigma)$ of a connected graph Σ such that, for each $u \in V(\Sigma)$, the induced subgraph $[\Sigma(u)] \cong 3K_1$, that is to say, Σ is a cubic graph of girth at least 4 and $C(\Sigma)$ is in this case the line graph $L(\Sigma)$. Moreover, [8, Theorem 1.4] gives that $\Sigma \cong C(\Gamma)$. A cubic graph with girth at least 4 has $|V(\Sigma)| \geq 5$, so by Lemma 2.1, $A \cong \mathrm{Aut}(\Sigma)$. Now we apply Theorem 1.1 to the graph Σ of girth $\mathrm{g} \geq 4$. Since $\Gamma = L(\Sigma)$ is 2-geodesic transitive and $3 \leq \mathrm{g}/2 + 1$, it follows from Theorem 1.1 that Σ is 3-arc transitive. Therefore, Γ is the line graph of a 3-arc transitive cubic graph.

Conversely, if $\Gamma \cong \mathcal{O}$, then it is 2-geodesic transitive, and hence is geodesic transitive as $\mathrm{diam}(\mathcal{O}) = 2$. If $\Gamma = L(\Sigma)$ where Σ is a 3-arc transitive cubic graph, then by Theorem 1.1 applied to Σ with $s = 3$, $L(\Sigma)$ is 2-geodesic transitive. This proves the first assertion of Theorem 1.3.

To prove the second assertion, suppose first that Γ is geodesic transitive. Then Γ is distance transitive, and so by Theorems 7.5.2 and 7.5.3 (i) of [4], Γ is one of the following graphs: $\mathcal{O} = L(K_4)$, $H(2,3) = L(K_{3,3})$, or the line graph of the Petersen graph, the Heawood graph or Tutte's 8-cage. We complete the proof by showing that all these graphs are geodesic transitive. As noted above, $\mathcal{O}$ is geodesic transitive; by [7, Proposition 3.2], $H(2,3)$ is geodesic transitive. It remains to consider the last three graphs.

Let Σ be the Petersen graph and $\Gamma = L(\Sigma)$. Then Σ is 3-arc transitive, and it follows from Theorem 1.1 that Γ is 2-geodesic transitive. By [4, Theorem 7.5.3 (i)], $\mathrm{diam}(\Gamma) = 3$ and $|\Gamma(w) \cap \Gamma_3(u)| = 1$ for each 2-geodesic (u, v, w) of Γ. Thus Γ is 3-geodesic transitive, and hence is geodesic transitive.

Let Σ_1 be the Heawood graph and Σ_2 be Tutte's 8-cage. Then Σ_1 is 4-arc transitive and Σ_2 is 5-arc transitive, and hence by Theorem 1.1, $L(\Sigma_1)$ is 3-geodesic transitive and $L(\Sigma_2)$ is 4-geodesic transitive. By [4, Theorem 7.5.3 (i)], $\mathrm{diam}(L(\Sigma_1)) = 3$ and $\mathrm{diam}(L(\Sigma_2)) = 4$, and hence both $L(\Sigma_1)$ and $L(\Sigma_2)$ are geodesic transitive. $\square$

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. Let Γ be a connected non-complete locally cyclic graph. If Γ is 2-geodesic transitive, then it is regular of valency n say. As discussed in the introduction, $n = 4$ or 5. If $n = 4$, then we proved in Theorem 1.3, that Γ is isomorphic to the octahedron and that the octahedron is indeed 2-geodesic transitive. If $n = 5$, then by [7, Theorem 1.2], Γ is isomorphic to the icosahedron, and this graph is 2-geodesic transitive. $\square$

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Johnson graphs are Hamilton-connected

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Abstract

We prove that the Johnson graphs are Hamilton-connected and apply this to obtain that another family of graphs is Hamilton-connected.

Keywords: Hamilton path, Johnson graph, Hamilton-connected.

Math. Subj. Class.: 05C70

1 Main Result

The *Johnson graph* $J(n, k)$, $0 \leq k \leq n$, is defined by letting the vertices correspond to the k-subsets of an n-set, where two vertices are adjacent if and only if the corresponding k-subsets have exactly $k - 1$ elements in common. A graph is *Hamilton-connected* if for any pair of distinct vertices u, v there is a Hamilton path whose terminal vertices are u and v. The graph with a single vertex is trivially Hamilton-connected.

In a recent paper [1], I needed a certain graph to be Hamilton-connected. This graph, defined below, contains vertex-disjoint Johnson graphs. The result I needed is embodied in the corollary below.

Theorem 1.1. *The Johnson graph $J(n, k)$ is Hamilton-connected for all $n \geq 1$.*

Proof. For ease of exposition, instead of talking about the vertex corresponding to a subset, we shall simply treat the subsets as if they are vertices so that we use equality notation between vertices and sets. The graphs $J(n, k)$ and $J(n, n - k)$ are isomorphic via the mapping that takes a k-subset to its complement.

The graphs $J(n, 0)$ and $J(n, n)$, $n \geq 1$, are isomorphic to the single vertex K_1 and trivially Hamilton-connected. The graphs $J(n, 1)$ and $J(n, n - 1)$, $n \geq 1$, are isomorphic to the complete graph K_n. Complete graphs certainly are Hamilton-connected.

We proceed by double induction and when considering $J(n, k)$, the induction hypotheses are: $J(m, k')$ is Hamilton-connected whenever $k' < k$ and $m \geq k'$, or $J(m, k)$ is

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Hamilton-connected whenever $m < n$ and $m \geq k$. As noted above, $J(n, 1)$ is Hamilton-connected for all $n \geq 1$. For a fixed k we start with $J(k, k)$ and then proceed by going from $J(n-1, k)$ to $J(n, k)$. Thus, the induction hypotheses make sense.

If $k \leq n \leq 2k-1$, then $n-k < k$ so that $J(n, n-k)$ is Hamilton-connected by hypothesis. This, in turn, implies that $J(n, k)$ is Hamilton-connected because $J(n, k)$ and $J(n, n-k)$ are isomorphic. Thus, it follows that $J(n, k)$ is Hamilton-connected for all n satisfying $k \leq n \leq 2k-1$.

For the remaining cases we need to actually show how to find appropriate Hamilton paths. The symmetric group S_n acts in the obvious way on the vertex set of $J(n, k)$. This action is transitive so that it suffices to find a Hamilton path from the vertex $u = \{1, 2, 3, \ldots, k\}$ to any other vertex. Let $v = \{a_1, a_2, \ldots, a_k\}$ be an arbitrary vertex.

If there is an element x of $\{1, 2, \ldots, n\}$ belonging to neither of the sets, we may relabel elements so that n is missing from both sets. Thus, both k-sets are subsets of $\{1, 2, \ldots, n-1\}$. By induction there is a Hamilton path from u to v in $J(n-1, k)$. Because the vertices that are adjacent along that path also are adjacent in $J(n, k)$, let P' be the corresponding path from u to v in $J(n, k)$. The path P' contains all the vertices corresponding to k-subsets that do not contain n.

Let $w_1 = \{y_1, y_2, \ldots, y_{k-1}, y_k\}$ and $w_2 = \{y_1, y_2, \ldots, y_{k-1}, z_k\}$ be two adjacent vertices on P'. The vertex w_1 is adjacent to the vertex $w_3 = \{y_2, \ldots, y_{k-1}, y_k, n\}$, and the vertex w_2 is adjacent to the vertex $w_4 = \{y_2, \ldots, y_{k-1}, z_k, n\}$.

The subgraph X induced by $J(n, k)$ on all the subsets containing n clearly is isomorphic to $J(n-1, k-1)$. Thus, there is a path from w_3 to w_4 spanning all the vertices of X. Thus, remove the edge of P' between w_1 and w_2, add the edges w_1w_3 and w_2w_4, and then add the path from w_3 to w_4 spanning X. The resulting path is a Hamilton path in $J(n, k)$ with u and v as terminal vertices.

If $n > 2k$, then there always is an element x missing both subsets and the preceding argument establishes that $J(n, k)$ is Hamilton-connected. If $n = 2k$, there is exactly one subset that fails the criterion, namely, the complement of $\{1, 2, \ldots, k\}$. So we need to find a Hamilton path in $J(2k, k)$ from u to its complement.

Consider the k-subsets of $\{1, 2, \ldots, 2k\}$ not containing the element $2k$. The subgraph induced by $J(2k, k)$ on this collection of subsets is isomorphic to the graph $J(2k-1, k)$. It is Hamilton-connected by induction so that there is a Hamilton path from u to $w = \{1, 2, \ldots, k-1, 2k-1\}$. Let P be the copy of this path in $J(2k, k)$.

Now consider all the k-subsets of $\{1, 2, \ldots, 2k\}$ that contain the element $2k$. The subgraph Y' induced on this collection of sets is isomorphic to $J(2k-1, k-1)$ so that it has a spanning path from $\{1, 2, \ldots, k-1, 2k\}$ to $\{k+1, k+2, \ldots, 2k\}$. Because the intermediate terminal vertices on the two paths are adjacent, we have a Hamilton path in $J(2k, k)$ from u to its complement. This completes the proof. □

The corollary below is the real target of this short paper. We need to define a particular graph first. Let $A = \{a_1, a_2, \ldots, a_m\}$ be a non-empty subset of $\{0, 1, 2, \ldots, n\}$ such that the elements are listed in the order $a_1 < a_2 < \cdots < a_m$. We define the graph $QJ(n, A)$ in the following way. For each $a_i \in A$, we include a copy of the Johnson graph $J(n, a_i)$. Thus far the Johnson graphs are vertex-disjoint. We then insert edges between $J(n, a_i)$ and $J(n, a_{i+1})$, for each i, using set inclusion, that is, we join an a_i-subset S_1 and an a_{i+1}-subset S_2 if S_1 is contained in S_2. The graph $QJ(n, A)$ can be pictured as having levels made up of Johnson graphs with edges between successive levels based on set inclusion.

Corollary 1.2. *The graph $QJ(n, A)$ is Hamilton-connected for all $n \geq 1$.*

Proof. If A is a singleton set, then $QJ(n, A)$ is a Johnson graph and the result follows from Theorem 1.1. Hence, we assume that A has at least two elements. Suppose that u and v are two vertices of $QJ(n, A)$ lying at different levels, where u has cardinality a_i, v has cardinality a_j, and $a_i < a_j$. Construct a path starting at u that spans the vertices at level a_i and terminates at an arbitrary vertex u_i at level a_i.

Choose a neighbor u_{i+1} of u_i at level a_{i+1} making certain it is distinct from v if $j = i + 1$. Then add the edge from u_i to u_{i+1} followed by a path spanning the vertices at level a_{i+1}. If v happens to lie at this level make certain the path terminates at v. Otherwise, the path can terminate at any vertex at level a_{i+1}.

It is obvious how to continue this until we have a path starting at u, terminating at v, and spanning all the vertices at levels a_i, a_{i+1} up through level a_j. If this happens to be all the levels of $QJ(n, A)$, then we have found a Hamilton path joining u and v. If we are missing levels, we then continue as follows.

If there are missing levels above level a_j, then remove an edge xy of the current path at level a_j and take distinct neighbors x' and y' of x and y, respectively, at level a_{j+1}. Then extend to a larger path by taking a path joining x' and y' spanning all the vertices at level a_{j+1}. If x and y don't have distinct neighbors at level a_{j+1}, then $a_{j+1} = n$ and the level is the singleton vertex $w = \{1, 2, 3, \ldots, n\}$ which is adjacent to everything at level a_j so that we replace the edge xy of the path with the 2-path xwy.

It is obvious how to continue adding the vertices one level at a time until we finish with the top level. We also can do the analogous extension with the levels below a_i until we achieve a Hamilton path in $Q(n, A)$ that has u and v as terminal vertices.

If u and v are at the same level. Then we start with a path spanning level a_i that has u and v as terminal vertices. We then extend the path through the other levels as above. This completes the proof. □

Corollary 1.2 allows us to process a variety of collections of sets of different cardinalities where we can move from one set to another either by a revolving door operation or restricted inclusions. This is what was required in [1].

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Nonorientable regular maps over linear fractional groups

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Abstract

It is well known that for any given hyperbolic pair (k, m) there exist infinitely many regular maps of valence k and face length m on an orientable surface, with automorphism group isomorphic to a linear fractional group. A nonorientable analogue of this result was known to be true for all pairs (k, m) as above with at least one even entry. In this paper we establish the existence of such regular maps on nonorientable surfaces for all hyperbolic pairs.

Keywords: Regular map, linear fractional group.

Math. Subj. Class.: 05C10, 05C25, 20G99

1 Introduction

A map on a compact, orientable surface is *orientably regular* if the group of all orientation preserving automorphisms of the map is transitive, and hence regular, on darts of the map. A map on a compact, nonorientable surface is *regular* if its automorphism group is transitive, and hence regular, on flags of the map. In either case, such maps have all vertices of the same degree and all faces of the same length; if these quantities are k and m we speak of a map of *type* $\{m, k\}$. The type is said to be *hyperbolic* if $1/k + 1/m < 1/2$. Regarding type, the following basic fact was rediscovered a number of times in the past.

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Theorem 1.1. *For every hyperbolic pair (k, m) there exist infinitely many orientably regular maps of type $\{m, k\}$.*

A brief history of the development around this result together with a list of various proofs can be found in [9]. A particularly important way of proving Theorem 1.1 follows from [8] and implies that all such maps can be chosen to have the orientation preserving automorphism group isomorphic to a linear fractional group over a finite field.

It is quite surprising that a nonorientable analogue of Theorem 1.1 has not been considered. A proof might follow from the study of generation of symmetric and alternating groups by pairs of permutations of given order in [6], but this work does not appear to have a corresponding follow-up and it is not our intention to do so. Instead, motivated by the result of [8] mentioned above, we will be interested in possibilities to prove a stronger form of a nonorientable analogue of Theorem 1.1 for maps with automorphism group isomorphic to a linear fractional group over a finite field. In fact, this has already been done for three quarters of the cases by the third author in [11] where it is shown that for any hyperbolic pair (k, m) with at least one even entry there are infinitely many nonorientable regular maps of type $\{m, k\}$ with automorphism group isomorphic to $PGL(2, F)$ for suitable finite fields F, but the case when both k and m are odd was left untouched.

In this note we fill in this gap, establishing thus the existence of an infinite number of nonorientable regular maps of type $\{m, k\}$ with automorphism group isomorphic to a linear fractional group for any given hyperbolic type $\{m, k\}$. The main results are presented in Section 4, preceded by background information summed up in Section 2 and auxiliary number theoretic results in Section 3.

2 Preliminaries

Foundations of the theory of regular maps have been laid in [4] and [2] and in what follows we just briefly review a few basic facts; for surveys we recommend [7] and [10].

Orientably regular maps of type $\{m, k\}$ can be identified with their orientation preserving automorphism groups and these are in a one-to-one correspondence with the finite groups G presented in the form

$$G = \langle r, s;\ r^k = s^m = (rs)^2 = \ldots = 1\rangle \tag{2.1}$$

where r and s represent a k-fold rotation about a fixed vertex of the map and an m-fold rotation about the centre of a face incident with the vertex. In particular, we require that k, m and 2 are the true orders of r, s, and rs, respectively. Vertices, faces and edges of the orientably regular map $M_{\mathrm{or}}(G)$ corresponding to a presentation of a group G as in (2.1) can be identified with left cosets of the cyclic subgroups $\langle r\rangle$, $\langle s\rangle$ and $\langle rs\rangle$, with incidence determined by non-empty intersection; the group G then acts as the orientation preserving automorphism group of $M_{\mathrm{or}}(G)$ by left multiplication.

Regular maps on nonorientable surfaces are also in a one-to-one correspondence with presentations of finite groups as in (2.1) but satisfying the extra condition that G contains an involution t such that $trt = r^{-1}$ and $tst = s^{-1}$. This time, the nonorientable regular map $M_{\mathrm{nor}}(G)$ corresponding to such a group G has vertices, faces and edges identified with the left cosets of the dihedral subgroups $\langle r, t\rangle$, $\langle s, t\rangle$ and $\langle rt, ts\rangle$. Incidence is again determined by non-empty intersection and G acts as the automorphism group of $M_{\mathrm{nor}}(G)$ by left multiplication.

Thus, if a group G with presentation (2.1) admits an inner automorphism induced by an involution and inverting r and s, the above correspondences allow one to associate two maps with G, namely, $M_{\mathrm{or}}(G)$ and $M_{\mathrm{nor}}(G)$. To remove this ambiguity in what follows, for a group G as in (2.1) we define the map $M(G)$ by letting $M(G) = M_{\mathrm{nor}}(G)$ if G has an inner automorphism induced by an involution, inverting both r and s, and $M(G) = M_{\mathrm{or}}(G)$ if G has no such inner automorphism.

As stated in the Introduction we will be interested in regular maps of a given type with automorphism group isomorphic to a linear fractional group. We begin by recalling the characterisation of such automorphism groups from [8]; for a much more detailed proof we refer to [3].

Proposition 2.1. *Let (k, m) be a hyperbolic pair and let K be an algebraically closed field of a prime characteristic p coprime to km. Let ξ and η be primitive $(\delta k)^{\mathrm{th}}$ and $(\delta m)^{\mathrm{th}}$ roots of unity in K, where $\delta = 1$ if $p = 2$ and $\delta = 2$ if $p > 2$. Let $D = -(\xi^2 + \xi^{-2} + \eta^2 + \eta^{-2})$ and let*

$$R = \pm \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} \quad \text{and} \quad S = \pm(\xi - \xi^{-1})^{-1} \begin{bmatrix} -(\eta + \eta^{-1})\xi^{-1} & D \\ 1 & (\eta + \eta^{-1})\xi \end{bmatrix}$$

be elements of $PSL(2, K)$. Then,

(a) *the orders of R, S, and RS in $PSL(2, K)$ are k, m, and 2, respectively, and*

(b) *if G is a subgroup of $PSL(2, K)$ with presentation (2.1), then there exist primitive $(\delta k)^{\mathrm{th}}$ and $(\delta m)^{\mathrm{th}}$ roots of unity such that G is conjugate to the subgroup generated by the matrices R and S as above.*

It is therefore sufficient to study the groups $G(\xi, \eta) = \langle R, S \rangle$ with R and S as above. Necessary and sufficient conditions for $G(\xi, \eta)$ to give rise to a nonorientable regular map were given in [3]. Here we present an excerpt sufficient for our purposes.

Theorem 2.2. *Let (k, m) be a hyperbolic pair and let K be an algebraically closed field of a prime characteristic p relatively prime to both k and m. Let ξ and η be primitive $(\delta k)^{\mathrm{th}}$ and $(\delta m)^{\mathrm{th}}$ roots of unity in K, where $\delta = 1$ if $p = 2$ and $\delta = 2$ if $p > 2$. Let $e = e(k, m)$ be the smallest positive integer j such that $n \mid (p^j - \varepsilon_n)/\delta$ for each $n \in \{k, m\}$ and some $\varepsilon_n \in \{+1, -1\}$. Then:*

(1) *if e is even, $e = 2f$, then $G(\xi, \eta)$ is isomorphic to $PGL(2, p^f)$ if and only if the quantity $D = -(\xi^2 + \xi^{-2} + \eta^2 + \eta^{-2})$ is not equal to zero, and either* (a) *there is an even entry $n \in \{k, m\}$ and an $\varepsilon \in \{+1, -1\}$ such that n divides $(p^f - \varepsilon)$ but $2n$ does not, while the other entry divides $(p^f - \varepsilon')/2$ for some $\varepsilon' \in \{+1, -1\}$, or* (b) *both k and m are even and for any $n \in \{k, m\}$ there exists an ε'_n such that n is a divisor of $(p^f - \varepsilon'_n)$ but $2n$ is not;*

(2) *$G(\xi, \eta)$ is isomorphic to $PSL(2, p^e)$ if and only if $D \neq 0$ and either e is odd, or the pair (k, m) together with an even e do not satisfy any of the above conditions* (a) *and* (b)*; and*

(3) *if $D \neq 0$, the map $M(G(\xi, \eta))$ is nonorientable if and only if either $e = 2f$ and $G(\xi, \eta) \cong PGL(2, p^f)$, or if $G(\xi, \eta) \cong PSL(2, p^e)$ and D is a square in $GF(p^e)$; in particular, in the last case $M(G(\xi, \eta))$ is always nonorientable if $p = 2$ and both k and m are odd.*

From the last part of this result one sees that to obtain nonorientable regular maps it is, for example, sufficient to make sure that $e = 2f$, $D \neq 0$ and $G(\xi, \eta) \cong PGL(2, p^f)$. In [11] it is shown that this can be guaranteed whenever at least one of k and m is even:

Theorem 2.3. *Let (k, m) be a hyperbolic pair with at least one even entry. Then, there is an infinite number of finite, nonorientable, regular maps of type $\{m, k\}$ with automorphism group isomorphic to $PGL(2, F)$ for suitable finite fields F.*

To be able to extend this result to the case when both k and m are odd, by Theorem 2.2 one can only hope to establish the existence of infinitely many regular maps of type $\{m, k\}$ with automorphism group isomorphic to $PSL(2, F)$ for suitable finite fields F. In particular, by part (3) of Theorem 2.2, to achieve this we need to make sure that for an infinite number of primes p one can select primitive $2k$-th and $2m$-th roots of unity ξ and η in $GF(p^e)$ in such a way that the quantity $D = -(\xi^2 + \xi^{-2} + \eta^2 + \eta^{-2})$ is a square in $GF(p^e)$, where $e = e(k, m)$; note that e depends on p as well but this dependence is not shown in our notation. Observe that if k and m are odd, ξ^2 and η^2 are primitive k-th and m-th roots of unity, respectively. Thus, in this case D has the form $D = -(\omega_k + \omega_m)$ where ω_n denotes the sum of an n-th primitive root of unity and its reciprocal in $GF(p^e)$. In what follows we will investigate such quantities in general, first over the field of complex numbers and subsequently over finite fields by considering factor fields of rings of algebraic integers.

3 Auxiliary results involving complex roots of unity

Let ζ_n denote any primitive n-th root of unity, but this time taken in the field $\mathbb{C}$ of complex numbers unless stated otherwise. It is known that all the primitive n-th roots of unity are conjugate over the rationals $\mathbb{Q}$ and their common minimal polynomial is the n-th cyclotomic polynomial Φ_n of degree $\varphi(n)$, the value of the Euler totient function at n. By ω_n we denote any number of the form $\zeta_n + \zeta_n^{-1}$; these quantities are again conjugate over $\mathbb{Q}$ and their common minimal polynomial will be denoted by $\Psi_n(x)$. It is well known that if $n > 2$, then

$$x^{\varphi(n)/2}\Psi_n(x + x^{-1}) = \Phi_n(x). \tag{3.1}$$

Finally, let U_n denote the set of all primitive n-th roots of unity in $\mathbb{C}$ and let $\overline{U}_n$ stand for the set of all the corresponding quantities ω_n.

We continue with some observations. From the fact that $\Phi_1(x) = x - 1$, $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$ and $\Phi_{pn}(x) = \Phi_n(x^p)$ if $p|n$ and $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$ otherwise, we obtain the following auxiliary result by easy calculations.

Lemma 3.1. *Let $\Phi_n(x)$ be the n-th cyclotomic polynomial. Then, $\Phi_1(1) = 0$, $\Phi_{p^k}(1) = p$ for p prime and $k > 0$, and $\Phi_n(1) = 1$ otherwise. Also, $\Phi_1(-1) = -2$, $\Phi_2(-1) = 0$, $\Phi_{2p^k}(-1) = p$ for p prime and $k > 0$, and $\Phi_n(-1) = 1$ otherwise.* □

With the help of these facts we obtain our basic result on products of the quantities $-(\omega_k + \omega_m)$ for any $\omega_k \in \overline{U}_k$ and $\omega_m \in \overline{U}_m$.

Proposition 3.2. *Let k, m be odd positive integers and let*

$$P(k, m) = \prod_{\omega_k \in \overline{U}_k} \prod_{\omega_m \in \overline{U}_m} -(\omega_k + \omega_m)\,.$$

Then, $P(1, 1) = -4$, $P(k, k)^2 = (-2)^{\varphi(k)}$ for $k \geq 3$, and $P(k, m)^2 = 1$ otherwise.

Proof. Obviously $P(k,m) = P(m,k)$ and we will therefore assume that $k \geq m$ in what follows. The values of $P(k,m)$ for $k,m \leq 2$ are trivial. If $k \geq 3$ and $m = 1$, then $P(k,1) = \prod_{\omega_k \in \overline{U}_k} -(2+\omega_k) = \Psi_k(-2) = (-1)^{-\varphi(k)/2}\Phi_k(-1) = (-1)^{\varphi(k)/2}$ and so $P(k,1)^2 = 1$. For the remaining part of the proof we assume that $k \geq m > 1$.

By the properties of the polynomials

$$\Psi_m(x) = \prod_{\omega_m \in \overline{U}_m} (x - \omega_m)$$

we obtain, for any $\omega_k = \zeta_k + \zeta_k^{-1} \in \overline{U}_k$, the equality

$$\prod_{\omega_m \in \overline{U}_m} -(\omega_k + \omega_m) = \Psi_m(-\omega_k) = (-\zeta_k)^{-\varphi(m)/2}\Phi_m(-\zeta_k)\ .$$

Let U_k' be a subset of U_k of cardinality $\varphi(k)/2$ such that $\overline{U}_k = \{\zeta_k + \zeta_k^{-1};\ \zeta_k \in U_k'\}$. The previous computation then implies that

$$P(k,m) = \prod_{\zeta_k \in U_k'} (-\zeta_k)^{-\varphi(m)/2}\Phi_m(-\zeta_k)\ .$$

Extending the product above from U_k' to U_k means squaring the last equation; combining this with the fact that the product of all the (even number of) k-th primitive roots of unity is equal to 1 we obtain

$$P(k,m)^2 = \prod_{\zeta_k \in U_k} (-\zeta_k)^{-\varphi(m)/2}\Phi_m(-\zeta_k) = \prod_{\zeta_k \in U_k} \Phi_m(-\zeta_k).$$

Invoking the well known identity $\Phi_m(x) = \prod_{d|m}(x^d - 1)^{\mu(m/d)}$, where μ is the Moebius function, we have

$$\Phi_m(-\zeta_k) = \prod_{d|m}(-\zeta_k^d - 1)^{\mu(m/d)}\ .$$

This product is non-zero since both k and m, and hence all the divisors d, are odd and so $(-\zeta_k)^d \neq 1$; note also that the divisors satisfy $d \leq k$ because of the assumption $m \leq k$.

Let us analyze the system of powers $\mathcal{U} = (\zeta_k^d;\ \zeta_k \in U_k)$ appearing in the last equality. For any positive divisor d of m let $n(d) = k/(d,k)$ and $r(d) = \varphi(k)/\varphi(n(d))$; of course, both quantities depend on k as well. It can now be seen that the system $\mathcal{U}$ is a collection, for any d dividing m, of primitive $n(d)$-th roots of unity, each repeated $r(d)$ times. With the help of all these facts together with $\Phi_t(x) = \prod_{\zeta_t \in U_t}(x - \zeta_t)$ evaluated at $x = -1$ we successively obtain

$$P(k,m)^2 = \prod_{\zeta_k \in U_k} \prod_{d|m}(-\zeta_k^d - 1)^{\mu(m/d)} = \prod_{d|m} \prod_{\zeta_k \in U_k} (-1 - \zeta_k^d)^{\mu(m/d)}$$

$$= \prod_{d|m} \prod_{\zeta_{n(d)} \in U_{n(d)}} (-1 - \zeta_{n(d)})^{r(d)\mu(m/d)} = \prod_{d|m}(\Phi_{n(d)}(-1))^{r(d)\mu(m/d)}\ .$$

As all the values of $n(d)$ are odd here, we have $\Phi_{n(d)}(-1) = 1$ if $d < k$ and $\Phi_{n(d)}(-1) = -2$ if $d = k$, where the second possibility occurs if and only if $m = k$ and then $r(d) = \varphi(k)$ and $\mu(m/d) = 1$. We conclude that for $k \geq 3$ we have $P(k,k)^2 = (-2)^{\varphi(k)}$, and $P(k,m)^2 = 1$ if $1 < m < k$. This completes the proof. □

A different and more powerful approach to the investigation of the quantity D from Theorem 2.2 relies on some known facts on algebraic integers in algebraic number fields. We refer to [1] as a suitable introductory reference and recall here just a few basic concepts and results.

Let K be an algebraic number field, that is, an extension of $\mathbb{Q}$ of a finite degree. Let $O = O_K$ be the ring of algebraic integers in K. The ring O is known to be a Dedekind domain, but apart from a few facts the theory of such domains will not be needed. A basic property of O is that every non-zero ideal $J \subset O$ has a finite index $[O : J]$. Without going into too much detail we recall that the the index $[O : J]$ is the *norm* $N(J)$ of J. Another important property of O is that any prime ideal $J \subset O$ is maximal. Thus, for any such J the quotient ring O/J is a finite field and so there exists a unique rational prime p such that $N(J) = p^j$ for some $j \in \{1, 2, \ldots, d\}$, where $d = [K : \mathbb{Q}]$ is the degree of the extension. Further, it is known that K admits exactly d distinct injective homomorphisms $\sigma_1, \ldots, \sigma_d$ into $\mathbb{C}$. The *norm* $N(z)$ of any element $z \in K$ is defined as the product $N(z) = \sigma_1(z) \ldots \sigma_d(z)$; the elements $\sigma_t(z)$, $1 \le t \le d$, are the *conjugates of z over K*. The norm is multiplicative, that is, $N(z_1 z_2) = N(z_1)N(z_2)$ for any $z_1, z_2 \in K$. Norms of elements of O and ideals in O are known to be related by the fact that, for every non-zero algebraic integer $z \in O$, the absolute value of $N(z)$ is equal to the norm of the ideal $(z) \subset O$ generated by z. In particular, the norm of every non-zero element $z \in O$ is a non-zero integer, and it is well known that $|N(z)| = 1$ if and only if z is a unit, that is, an invertible element in the ring O. We will also repeatedly use the fact that if an element $z \in O$ belongs to an ideal I of O, then $N(I)$ divides $N(z)$.

For illustration we present some of the consequences of Proposition 3.2 in the language of algebraic number theory. Let $\alpha = \zeta_{2k}$ and $\beta = \zeta_{2m}$ be complex primitive $2k$-th and $2m$-th roots of unity, respectively, and let $A = -(\alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2})$. In what follows, let K denote the algebraic number field $\mathbb{Q}[\alpha, \beta]$. Since the generators α and β of K are roots of unity in $\mathbb{C}$, every injective homomorphism $\sigma : K \to \mathbb{C}$ is uniquely determined by positive integers i and j, relatively prime to k and m, such that $\sigma(\alpha) = \alpha^i$ and $\sigma(\beta) = \beta^j$. Observe, however, that whether particular i and j give rise to such an injective homomorphism may also depend on α and β and not just on k and m. As before, let $O = O_K$ be the ring of algebraic integers of K. Since α and β themselves are algebraic integers in K, we have $A \in O$; in particular, the norm $N(A)$ in O is an integer.

Lemma 3.3. *If $\alpha \neq \beta$, then A is a unit in O, and if $\alpha = \beta$, then $|N(A)|$ is a power of 2.*

Proof. Observe that all factors in the product $P(k, m)$ in Proposition 3.2 are algebraic integers, with all conjugates of A being among the factors. By the same Proposition we have $P(k, m) = \pm 1$ if $k \neq m$, while $P(k, k)^2 = (-2)^{\varphi(k)}$ for $k \ge 3$. Since algebraic integers have integral norm, it follows that A is a unit in O if $k \neq m$. In the case when $k = m$ and $\alpha = \beta$, the absolute value of the norm of $-2(\alpha^2 + \alpha^{-2})$ is equal to $(-2)^{\varphi(k)/2}$, and therefore for $\alpha \neq \beta$ the absolute value of the norm of A must be 1. □

Returning to our main theme, until the end of this section we will assume that (k, m) is a fixed hyperbolic pair with no restriction on the parity of the two entries. We begin by an elementary observation that will turn out to be crucial later.

Lemma 3.4. *The quantity $A - n^2$ is never a unit in O for any integer $n > 2$.*

Proof. We recall the known fact that K is isomorphic to the cyclotomic field $\mathbb{Q}[\gamma]$, where $\gamma = \cos(\frac{2\pi}{\ell}) + i \sin(\frac{2\pi}{\ell})$ is a primitive ℓ-th root of unity for $\ell = \mathrm{lcm}\{2, k, m\}$. The conjugates of γ over K have the form $\cos(\frac{2\pi}{\ell}j) + i\sin(\frac{2\pi}{\ell}j)$, where $1 \le j < \ell$ and $(j,\ell) = 1$. All the $\varphi(\ell)$ distinct injective homomorphisms $\sigma_t : K \to \mathbb{C}$ preserve the rationals pointwise. Since the explicit form of the conjugates of γ over K implies that $|\sigma_t(A)| < 4$, we have $|\sigma_t(A - n^2)| = |\sigma_t(A) - n^2| \ge n^2 - |\sigma_t(A)| > n^2 - 4$ for any t such that $1 \le t \le \varphi(\ell)$. Thus, by the definition of the norm, for $n > 2$ we have $|N(A - n^2)| > 1$, which means that $A - n^2$ is not invertible in O. □

It is useful to realise that our considerations before Lemma 3.1 did not depend on the parity of k and m and hence we may use them in what follows. Observe that in the general case we want to deal with, the value of A could be equal to zero in K, which happens precisely if $i\beta \in \{\pm\alpha, \pm\alpha^{-1}\}$. If, however, $\{k,m\}$ is a hyperbolic pair, it is easy to see that we can choose α and β avoiding this condition. Keeping to the notation introduced above, for any $n \ge 3$ let $I = I_n$ be a maximal ideal in O containing the element $A - n^2$ and let $p = p_n$ be the characteristic of the field $F = O/I$. Letting $\xi = \alpha + I$, $\eta = \beta + I$, and $D = A + I$, we have:

Lemma 3.5. *If n is relatively prime to $N(A)$, then the element $D = -(\xi^2+\xi^{-2}+\eta^2+\eta^{-2})$ is a non-zero square in F and p does not divide n. Moreover, if p is not a divisor of $2km$, then ξ and η are primitive $2k$-th and $2m$-th root of unity in F.*

Proof. Since $A - n^2 \in I$, that is, $A + I = n^2 + I$, the element $D = A + I$ is a square in F. As $p \in I$ and I is a prime ideal, by our earlier remarks on norms of elements and ideals of the Dedekind ring O the condition $A \in I$ is equivalent to each of the conditions $n^2 \in I$, $n \in I$, and $p|n$. Hence p divides both n and $N(A)$, contrary to our assumption on their relative primeness.

It is obvious that ξ is a $2k$-th root of unity in F. Assume that $\alpha^u - 1 \in I$, where α^u is a primitive c-th root of unity in $\mathbb{C}$ for a proper divisor c of $2k$. As the ideal generated by the algebraic integer $\alpha^u - 1$ is contained in I, the norm of I divides the norm of $\alpha^u - 1$, which implies that the norm of $\alpha^u - 1$ is divisible by p. On the other hand, all conjugates of α^u are c-th primitive roots of unity in $\mathbb{C}$. Arguments analogous to those used in the proof of Proposition 3.2 imply that, up to sign, the norm of $\alpha^u - 1$ is a power of $\Phi_c(1)$. Thus, by Lemma 3.1, c is a power of p, contrary to the assumption that $p \nmid 2k$. It follows that ξ is a primitive $2k$-th root of unity in F. By the same token, η is a primitive $2m$-th root of unity in F. □

By suitably varying the parameter n one obtains an infinite sequence of primes as in Lemma 3.5.

Lemma 3.6. *If $A \neq 0$, then there exists an infinite set of values n and an infinite sequence of prime ideals I_n of O containing the element $A - n^2$ such that the fields O/I_n have pairwise distinct prime characteristic p_n.*

Proof. Referring to the way the primes p_n have been introduced for any $n > 2$, let us define an infinite sequence n_j of integers by letting $n_1 = 2|N(A)| + 1$ and $n_j = \prod_{i=1}^{j-1} p_{n_i}$ for $j > 1$. Applying Lemma 3.5 inductively we deduce that p_{n_j} does not divide n_j for any $j \ge 1$. By our construction, for any $j \ge 2$ the prime p_{n_j} differs from all the previous primes p_{n_i} for $i < j$. □

4 Results

Two immediate consequences in the direction of our interest can be obtained by exploring earlier results. Firstly, there is a much more general version of Theorem 2.2 in which the prime p is not necessarily coprime to k and m, in particular, covering the case when both k and m are equal to p and $G \cong PSL(2, p)$; see Propositions 3.1, 3.2, 4.6 and 6.1 of [3]. In order to avoid a rather long re-statement of these facts we invite the reader to check that part (2) of Proposition 6.1 combined with Proposition 3.1 of [3] imply:

Theorem 4.1. *If p is a prime congruent to 1 mod 4, then there exists a nonorientable regular map of type (p, p) with automorphism group isomorphic to $PSL(2, p)$.* □

Secondly, if both k and m are odd and $p = 2$, part (3) of Theorem 2.2 directly yields the following result, where $e = e(k, m)$ is as introduced in the statement of Theorem 2.2.

Theorem 4.2. *Let (k, m) be a hyperbolic pair consisting of odd entries. Then there is a nonorientable regular map of type $\{m, k\}$ with automorphism group isomorphic to $PSL(2, 2^e)$ for $e = e(k, m)$.* □

Together with the earlier findings this gives at least an existence result of the sought kind on regular maps over linear fractional groups.

Corollary 4.3. *For any hyperbolic pair (k, m) there exists a nonorientable regular map of type $\{m, k\}$ with automorphism group isomorphic to a linear fractional group over a finite field.* □

In the light of Theorem 2.3, the question of existence of an infinite number of such maps of any given type hyperbolic type is settled by the following result. Although we are interested mainly in the case when k and m are odd, we give a more general formulation which yields an alternative proof of Theorem 2 of [11].

Theorem 4.4. *For every hyperbolic pair (k, m) there is an infinite number of finite, nonorientable, regular maps of type $\{m, k\}$ with automorphism group isomorphic to $PSL(2, F)$ or $PGL(2, F)$ for suitable finite fields F.*

Proof. We will refer to the notation introduced in Section 3. For a fixed hyperbolic pair (k, m) and a non-zero $A = -(\alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2})$ with let $p = p_n$ be any prime from Lemma 3.6 relatively prime to $2km$, and let $G = G(\xi, \eta)$ be the corresponding group. By Theorem 2.2, G is isomorphic to $PSL(2, F)$ or $PGL(2, F')$. As D is a square, the corresponding regular map $M(G)$ is nonorientable in both cases. □

We also present two more results based on residue techniques which, although applicable only to a very restricted infinite set of types with both entries odd, may be useful in future investigations.

Theorem 4.5. *Let k and m be prime powers congruent to 3 mod 4. Then, there exist infinitely many nonorientable regular maps of type $\{m, k\}$ with automorphism group isomorphic to $PSL(2, F)$ for suitable finite fields F.*

Proof. Let p be a prime congruent to 1 mod 8 and let $e = \min\{n; k|p^n \pm 1$ and $m|p^n \pm 1\}$. Then, p does not divide $2km$ and the equality from Proposition 3.2 holds also in $GF(p^e)$, with appropriate interpretation of the primitive roots. Since $p \equiv 1$ mod 8, the four elements ± 1 and ± 2 are all quadratic residues in $GF(p)$ and hence also in $GF(p^e)$. By Proposition 3.2, the element $P(k,m)$ is a quadratic residue in $GF(p^e)$; note that in the case $k \neq m$ it would have been sufficient to assume $p \equiv 1 \bmod 4$ to obtain the same conclusion. By our assumptions, both $\varphi(k)/2$ and $\varphi(m)/2$ are odd. The product $P(k,m)$ has therefore an odd number of factors and so at least one of them must be a quadratic residue in $GF(p^e)$. That is, there exist $\omega_k \in \overline{U}_k$ and $\omega_m \in \overline{U}_m$ such that the value $D = -(\omega_k + \omega_m)$ is a square in $GF(p^e)$. This gives, by Theorem 2.2 and the remark after Theorem 2.3, a nonorientable regular map of type $\{m,k\}$. Since there are infinitely many primes p as above, our result follows. □

Theorem 4.6. *Let k and m be odd integers forming a hyperbolic pair such that the number $\varphi(k)\varphi(m)/4$ is even, that is, at least one of k, m is not a prime power congruent to 3 mod 4. If $P(k,m) < 0$, then there are infinitely many finite, nonorientable, regular maps of type $\{m,k\}$ with automorphism group isomorphic to $PGL(2,F)$ for suitable finite fields F.*

Proof. Let $p \equiv 3 \bmod 4$ be a prime such that e is odd (e.g., any $p \equiv \pm 1 \bmod km$). If $k \neq m$, then $P(k,m) = -1$ not only in $\mathbb{C}$ but also in $GF(p^e)$. Similarly if $k = m$, then we have $P(k,m) = (-2)^{\varphi(k)/2}$ both in $\mathbb{C}$ and in $GF(p^e)$. Note that if $k = m$, then $\varphi(k)/2$ is even and $2^{\varphi(k)/2}$ is a quadratic residue in $GF(p)$. As $p \equiv 3 \bmod 4$ and e is odd, the product $P(k,m)$ is a quadratic nonresidue in both $GF(p)$ and $GF(p^e)$. On the other hand, $P(k,m)$ has an even number of factors, and therefore at least one of them is a quadratic residue in $GF(p^e)$. □

5 Remarks

By Theorem 4.4, for any given hyperbolic pair (k,m) there exists an infinite number of nonorientable regular maps of type $\{m,k\}$ with automorphism group isomorphic to linear fractional groups over finite fields. Our approach was based on developing some results obtained in [3] in the course of analysing regular maps over linear fractional groups. The scope of [3] is, however, broader and covers regular hypermaps. For a general theory of hypermaps and their surface representations we recommend [5]. Here we just recall that a finite regular hypermap of type (k,m,l) can be identified with a finite quotient group of the triangle group $T(k,m,l) = \langle r,s,t;\ r^k = s^m = (rs)^l = 1\rangle$. Thus, regular hypermaps are a natural generalisation of regular maps (corresponding to the case when $l = 2$). Facts collected in [8, 3] imply that for any hyperbolic triple (k,m,l), that is, such that $1/k + 1/m + 1/l < 1$, there exist infinitely many regular hypermaps of type (k,m,l) on orientable surfaces, with automorphism group isomorphic to a linear fractional group over a finite field. By the theory developed in [3] combined with the findings in this paper, to establish a nonorientable analogue of this result requires analysing conditions under which the quantity $A' = 4+(\alpha+\alpha^{-1})(\beta+\beta^{-1})(\gamma+\gamma^{-1})-(\alpha+\alpha^{-1})^2-(\beta+\beta^{-1})^2-(\gamma+\gamma^{-1})^2$, where α, β and γ are primitive $2k$-th, $2m$-th, and $2l$-th roots of unity in $\mathbb{C}$, projects onto a non-zero square in a quotient field of the ring of algebraic integers of $\mathbb{Q}[\alpha,\beta,\gamma]$ generated by a suitable prime ideal; note that for $l = 2$ we have $\gamma^2 = 1$ and $\gamma + \gamma^{-1} = 0$ and then A' reduces to the quantity A introduced earlier. In fact, methods of Section 3 can be adapted in an obvious way to construct, for any hyperbolic triple (k,m,l) and for suitable triples (α,β,γ) of primitive roots of unity as above, an infinite number of nonorientable regular hypermaps of type (k,l,m) over linear fractional groups.

A comparison of Theorems 4.4, 4.5 and 4.6 reveals their different nature. Theorem 4.4 is more universal since it applies to all hyperbolic pairs and it is, in essence, constructive, but it yields no information on the corresponding set of primes. On the other hand, Theorems 4.5 and 4.6 apply to a very restricted set of hyperbolic pairs and are, in essence, existential, but the sets of primes for which they guarantee the existence of nonorientable regular maps have positive density in the set of all primes. This leaves the intriguing question of whether it is possible, for any given hyperbolic pair (k, m), to determine all primes p such that there exists a nonorientable regular map of type $\{m, k\}$ with its automorphism group isomorphic to a linear fractional group over a field of characteristic p.

For possible further interest we present a table of values of the product $P(k, m)$ for odd k, m such that $3 \leq k, m \leq 41$. Observe that for $k \neq m$ the table shows negative entries only if both k and m are powers of primes congruent to 3 mod 4. If this observation carries through to all odd k and m, Theorem 4.6 would be applicable only in the case when $k = m$, and the values 5, 13, 25, 29 and 37 show that this Theorem is not void.

3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41
2	1	-1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	1	1
1	-2^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-1	1	-2^3	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	1	1
-1	1	-1	2^3	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	1	1
-1	1	-1	-1	2^5	1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	1	1
1	1	1	1	1	-2^6	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	2^4	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	2^8	1	1	1	1	1	1	1	1	1	1	1	1
-1	1	-1	-1	-1	1	1	1	2^9	1	-1	1	-1	1	-1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	2^6	1	1	1	1	1	1	1	1	1	1
-1	1	-1	-1	-1	1	1	1	-1	1	-2^{11}	1	-1	1	-1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	-2^{10}	1	1	1	1	1	1	1	1
-1	1	-1	-1	-1	1	1	1	-1	1	-1	1	2^9	1	-1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	-2^{14}	1	1	1	1	1	1
-1	1	-1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-2^{15}	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2^{10}	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2^{12}	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-2^{18}	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2^{12}	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2^{20}

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Embeddings of cubic Halin graphs: Genus distributions*

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Abstract

We derive an $O(n^2)$-time algorithm for calculating the *genus distribution* of a given 3-regular *Halin graph* G; that is, we calculate the sequence of numbers $g_0(G)$, $g_1(G)$, $g_2(G), \ldots$ on the respective orientable surfaces S_0, S_1, S_2, $\ldots$. Key topological features are a *quadrangular decomposition* of plane Halin graphs and a new *recombinant-strands* reassembly process that fits pieces together three-at-a-vertex. Key algorithmic features are reassembly along a *post-order traversal*, with just-in-time *dynamic assignment of roots* for quadrangular pieces encountered along the tour.

Keywords: Genus distribution, Halin graph, partitioned genus distribution, gram embedding, outer-planar graph, topological graph theory.

Math. Subj. Class.: 05C10

1 Introduction

A ***Halin graph*** [20] is constructed from a plane tree T with at least four vertices and no 2-valent vertices by drawing a cycle thru the leaves of T in the order they occur in a preorder traversal of T. Any wheel graph W_n (for $n \geq 3$) is a Halin graph. Every Halin graph can be obtained by iterative splitting of the hub of a wheel and of some of the resulting vertices. Although some of the graphs obtained by splitting the hub of a wheel are non-planar, every planar graph so obtained is a Halin graph, since splitting a vertex of a tree yields a tree.

The ***outer cycle*** of a Halin graph is the cycle corresponding to the traversal of the leaves of the ***inscribed tree***. [Since a Halin graph is 3-connected, its planar embedding is unique up to reversal of orientation, as per Whitney's theorem.] In the Halin graph of Figure 1, the outer cycle has length eight.

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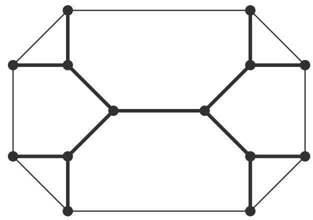

Figure 1: A Halin graph for a 14-vertex tree with 8 leaves.

Genus distributions

DEF. The ***genus distribution*** for graph G is the sequence

$$\gamma_{\text{dist}}(G) : \quad g_0(G), \quad g_1(G), \quad g_2(G), \quad \cdots$$

where $g_i(G)$ denotes the number of embeddings of G in the orientable surface S_i of genus i. In reckoning the number of embeddings of the graph G in the surface S, we regard two embeddings $\iota : G \to S$ and $\iota' : G \to S$ as the *same* if there is an extension of the identity automorphism $1_G : G \to G$ to an orientation-preserving autohomeomorphism $h : S \to S$ such that $\iota \circ h = \iota'$ or, equivalently, if the two embeddings of G correspond to the same *rotation system* (see [19]).

Calculating the genus distribution of a graph requires determining not only its minimum genus and its maximum genus, but also the number of embeddings of every possible genus. Table 1 gives the genus distributions of some familiar graphs, each of which is small enough that its genus distribution can be calculated by hand using *ad hoc* methods.

Table 1: Genus distributions of some familiar graphs.

graph G	$g_0(G)$	$g_1(G)$	$g_2(G)$	$g_3(G)$	$g_4(G)$	$\cdots$
K_4	2	14	0	0	0	...
bouquet B_2	4	2	0	0	0	...
dipole D_3	2	2	0	0	0	...
$K_{3,3}$	0	40	24	0	0	...
$K_2 \times C_3$	2	38	24	0	0	...

The study of genus distributions began with [16]. Some of the early papers, such as [9] and [18], were devoted to calculating genus distributions for all the graphs in a recursively constructible sequence. Other early papers, such as [33] and [7], were concerned with statistical properties of the distribution. A solution to a genus distribution calculation problem can be either a formula or a polynomial-time algorithm.

Lists of some previous papers on genus distributions have appeared in [10], [15], [17], and [29]. Papers published (or written) subsequently include the following: [4], [6], [5], [11], [12], [22], [23], [30], and [31].

Graph amalgamations and bar-amalgamations

In general, ***amalgamating two graphs*** means identifying a subgraph in one of them to an isomorphic subgraph in the other. Figures 2 and 3 illustrate ***vertex-amalgamation*** and

edge-amalgamation, respectively, which are the two simplest kinds of amalgamation of two graphs.

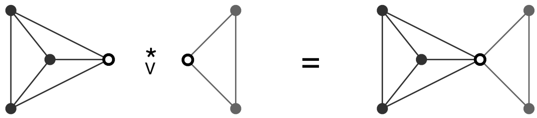

Figure 2: Vertex-amalgamation of two graphs.

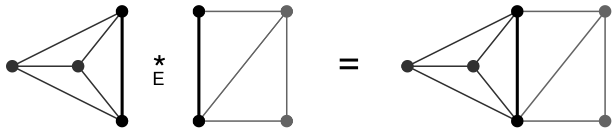

Figure 3: Edge-amalgamation of two graphs.

A ***bar-amalgamation*** of two (disjoint) graphs G and H is obtained by joining a vertex u of G to a vertex v of H with a new edge. It is denoted here by $G \,\overline{*}\, H$. Figure 4 shows a bar-amalgamation.

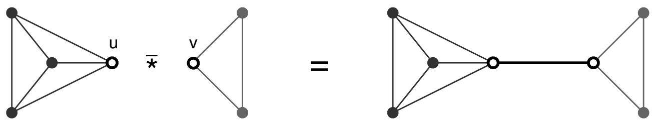

Figure 4: Bar-amalgamation of two graphs.

Proposition 1.1 ([16])**.** *Let G and H be (disjoint) connected graphs, and let u and v be vertices of G and H, respectively. Then*

$$\gamma_{\mathrm{dist}}(G \,\overline{*}\, H) \;=\; deg(u) \cdot deg(v) \cdot \gamma_{\mathrm{dist}}(G) \circ \gamma_{\mathrm{dist}}(H)$$

where $\circ$ means the operation of convolution on two sequences.

Seeking a useful algorithm

The objective herein is to derive a quadratic-time algorithm for calculating the genus distribution of any 3-regular Halin graph. The focus is not merely on proving the existence of such an algorithm, but on developing an algorithm that can by executed (albeit tediously) by hand for graphs with 10-20 vertices and rather quickly by a computer for graphs with a significantly larger number of vertices.

The terminology used here is consistent with [19] and [1]. For additional background (with some terminological differences), see [3], [28], or [37]. All of our ***graph embeddings*** here are cellular and orientable. A graph is taken to be connected, unless one can infer otherwise from the immediate context. Here we refer to a face-boundary walk as an ***fb-walk***.

Thanks to Imran Khan for creating the genus-distribution computer program (based on the Heffter-Edmonds algorithm) used in the course of this research.

2 Known results concerning genus distributions

Although calculating the maximum genus $\gamma_{\max}(G)$ of a graph G is possible in polynomial time [8], calculating the minimum genus $\gamma_{\min}(G)$ is NP-hard [36], and calculating the genus distribution $\gamma_{\mathrm{dist}}(G)$ is clearly at least as hard as calculating the minimum genus. Accordingly, rather few genus distributions are known. A survey of genus distributions, including average genus, is given by [10].

The most familiar such kinds of ladder graphs whose genus distribution formulas are known are as follows:

closed-end ladders [9] (derived 1984) See Figure 5.

circular ladders and Möbius ladders [26] See Figure 6.

Ringel ladders [35] See Figure 7.

By systematic use of iterated amalgamations [29] of double-edge-rooted graphs, self-edge-amalgamations [30], and edge-addition surgery [11], the calculation of formulas for these ladder graphs has been substantially simplified. Moreover, these recently developed techniques have produced quadratic-time algorithms for various generalizations of ladders, in which arbitrary graphs of known *partitioned genus distribution* (see §4) lie between the rungs.

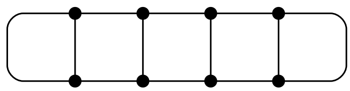

Figure 5: The closed-end ladder L_4.

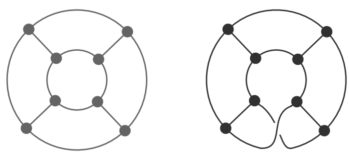

Figure 6: Circular ladder CL_4; and Möbius ladder ML_4.

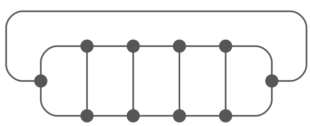

Figure 7: Ringel ladder RL_4.

A recent paper [13] presents a quadratic-time algorithm for the calculation of the genus distribution of any 3-regular outerplanar graph (see Figure 8). It uses a post-order traversal (see §3) and edge-amalgamations [29]. A subsequent paper [30] uses vertex-amalgamations [17] to derive a quadratic-time algorithm for 4-regular outerplanar graphs. Whereas outerplanar graphs are of tree-width 2, Halin graphs are of tree-width 3 (see [2]), which is intuitively a reason for anticipating the necessity for a more complicated analysis. Restrictions to 3-regularity or 4-regularity generally simplify the analysis of a genus distribution problem.

Some genus-distribution deriviations use a formula of Jackson [21] based on the theory of group representations. ***Bouquets***, which are graphs with a single vertex and a number of self-loops (see Figure 10) were the first class to be so derived [18] .

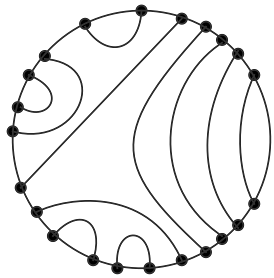

Figure 8: A 3-regular outerplanar graph.

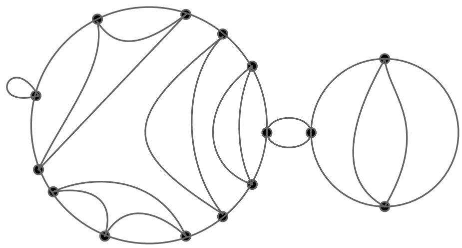

Figure 9: A 4-regular outerplanar graph.

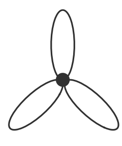

Figure 10: Bouquets B_1, B_2, and B_3.

Another such class is ***dipoles***, which are graphs with two vertices and a number of edges joining them (see Figure 11). Their genus distributions are given by [32] and [24]. Yet another is ***fans***, which are graphs obtained by joining a path graph to a single new vertex (see Figure 12). Their genus distributions were derived by [6].

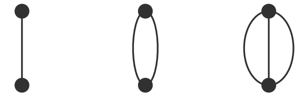

Figure 11: Dipoles D_1, D_2, and D_3.

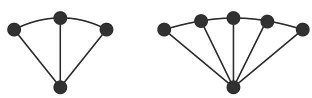

Figure 12: Fans F_3 and F_5.

3 Quadrangulating a plane Halin graph

In deriving the genus distribution of Halin graphs, the critical problem was to invent a new form of decomposition of a plane Halin graph into "atomic" fragments whose genus distributions are known, a new form of amalgamation, and an order of reassembly that reconstructs the Halin graph from the atomic fragments. In this section, we concentrate on the decomposition and the reassembly.

Taking the inscribed tree of a Halin graph as a spanning tree, an edge of a Halin graph is a ***tree-edge*** if it lies in the inscribed tree and a ***cycle-edge*** if it lies on the outer cycle. A ***leaf-edge*** is a tree-edge that is incident at a vertex of the outer cycle. A vertex is called a ***cycle vertex*** if it lies on the outer cycle, or an ***interior tree-vertex*** otherwise.

We regard the vertices and the edges of the given plane Halin graph as **black**. We observe that since H is a Halin graph, there is exactly one cycle edge on each polygonal face of the plane embedding. The decomposition is a 4-step process.

Step 1. In each cycle edge of the Halin graph, insert a red midpoint. This is illustrated in Figure 13.

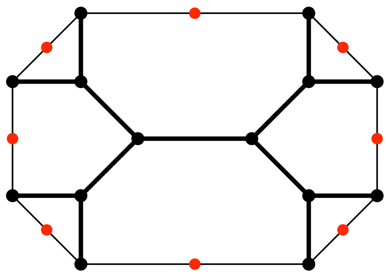

Figure 13: Halin graph plus red midpoints on the exterior cycle.

Step 2. Join each red vertex v to all of the non-leaf vertices on the boundary of the face in whose boundary v lies, as illustrated in Figure 14.

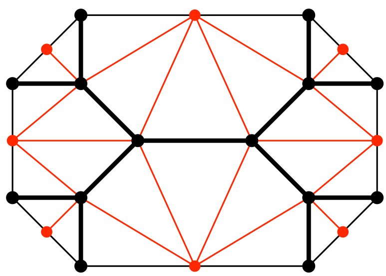

Figure 14: Halin graph plus all of the red edges.

Proposition 3.1. *The red and black edges together triangulate the region inside the exterior cycle of a plane Halin graph G.*

Proof. The black edges create a set of polygons (whose number equals the cycle rank $\beta(G)$). Each of these polygons is triangulated by the red edges. □

Proposition 3.2. *Every black tree edge lies on two of the triangles formed by Steps 1 and 2.*

Proof. Every tree edge lies on two of the polygonal faces of the plane Halin graph (by the Jordan curve theorem). In each of those polygonal faces, it lies on one and only one of the triangles. □

Step 3a. For each black tree edge, we pair the two incident triangles into a quadrangle.

Step 3b. We assign (unseen) colors blue, green, and brown to the tree edges, so as to form a proper edge 3-coloring. This is possible because any tree of maximum degree 3 is edge-3-colorable (via greedy algorithm).

Step 3c. We visibly color each quadrangle with the unseen color of the tree edge that bisects it. The coloring of the quadrangles is a proper 3-coloring of the part of the plane inside the exterior cycle of the Halin graph, because of the way it is induced by the proper 3-edge-coloring of the tree. (This property will not be used, but it is interesting nonetheless.)

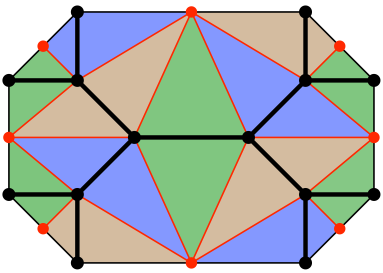

Figure 15: Quadrangulation of a plane Halin graph.

Step 4. Separate the quadrangulated map into quadrangles, and label the interior tree-vertices.

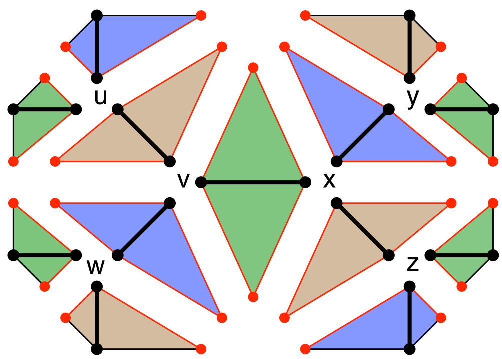

Figure 16: Separated quadrangles of a plane Halin graph.

Reassembling a Halin graph from its quadrangles: a puzzle

The success of our method of calculation the genus distribution in the subsequent sections depends on our ability to reassemble the plane Halin graph from its separated quadrangles in a manner consistent with a puzzle now to be described. The genus distributions of the quadrangular fragments is known, and it will be shown that we can calculate the genus distribution of any graph that can be constructed from quadrangular fragments, according to the rules of this puzzle. After giving the rules for this puzzle, we consider the outcome of three attempts at its solution.

Quadrangulation puzzle for a plane cubic Halin graph $H \to S_0$

1. Each quadrangle Q is regarded as an initial ***fragment***.
2. An ***RR-path*** on a fragment boundary is a 2-path with two red edges, from a red vertex through a black vertex to another red vertex.
3. Initially, all RR-paths are said to be ***live***.
4. A ***legal move*** is initiated by choosing a vertex v such that v is previously unchosen, at least one fragment at v is a quadrangle, and all three RR-paths through v are live RR-paths.

   If these three conditions are satisfied, then the three fragments that meet at v are merged into a single (larger) fragment. If there is more than one live RR-path on the boundary of the merged fragment, then all but one of the live RR-paths are deemed to be ***dead***.
5. You ***LOSE*** if you run out of legal moves before the map is fully reassembled. This happens whenever there occurs an unmerged vertex w such that either there is a dead RR-path through w, or none of the fragments meeting at w is a quadrangle.
6. You ***WIN*** the game by reassembling the plane map.

Attempt 1. Start with a merger at v. There are three live RR-paths on the boundary of the merged fragment. You LOSE, because RR-paths through two of the unmerged vertices u, w, x become dead.

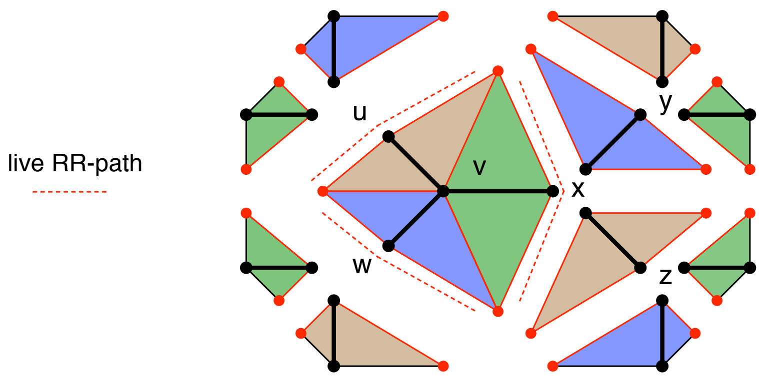

Figure 17: Attempt #1.

Attempt 2. First choose u and then choose v. There are two live RR-paths on the boundary of the merged fragment. You LOSE, because the RR-path through one of the unmerged vertices w, x becomes dead.

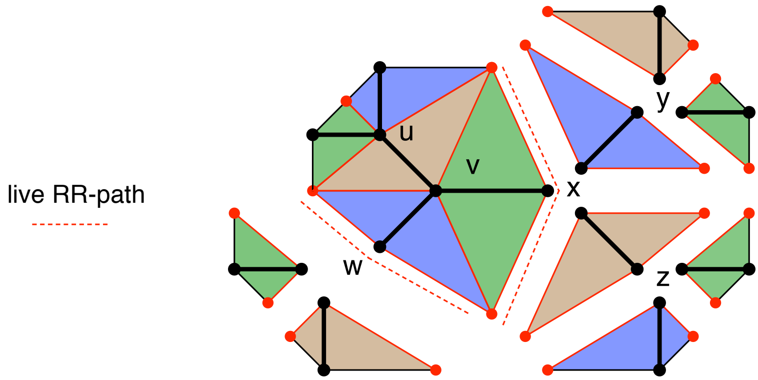

Figure 18: Attempt #2.

Attempt 3. Start with u, w, y, z. You LOSE, since after there is a merger at v or x, there will be no quadrangle at the remaining unmerged vertex.

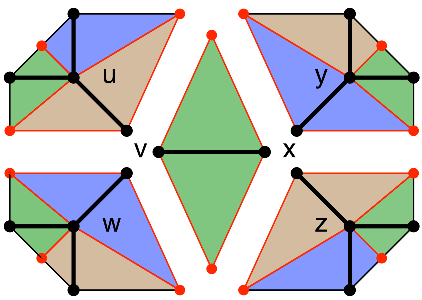

Figure 19: Attempt #3.

Solution: post-order traversal

The ***post-order*** for the vertices of a plane tree is the order produced when one traces the boundary of the only region and calls out the name of a vertex only the *last* time it is visited. For the tree in Figure 20, the post-order is z, y, x, u, v, w.

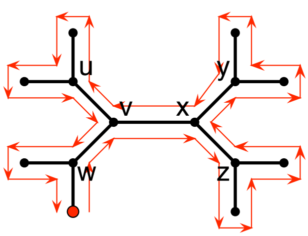

Figure 20: Post-order traversal.

Solution for the quadrangulation puzzle

1. As a root for the inscribed tree of the Halin graph, choose any leaf-vertex. (Must be a leaf to win.)
2. Choose vertices in the order in which they occur on a post-order traversal of the tree.

SOLUTION to puzzle in Figure 16: post-order as shown in Figure 20.

$$z \quad y \quad x \quad u \quad v \quad w$$

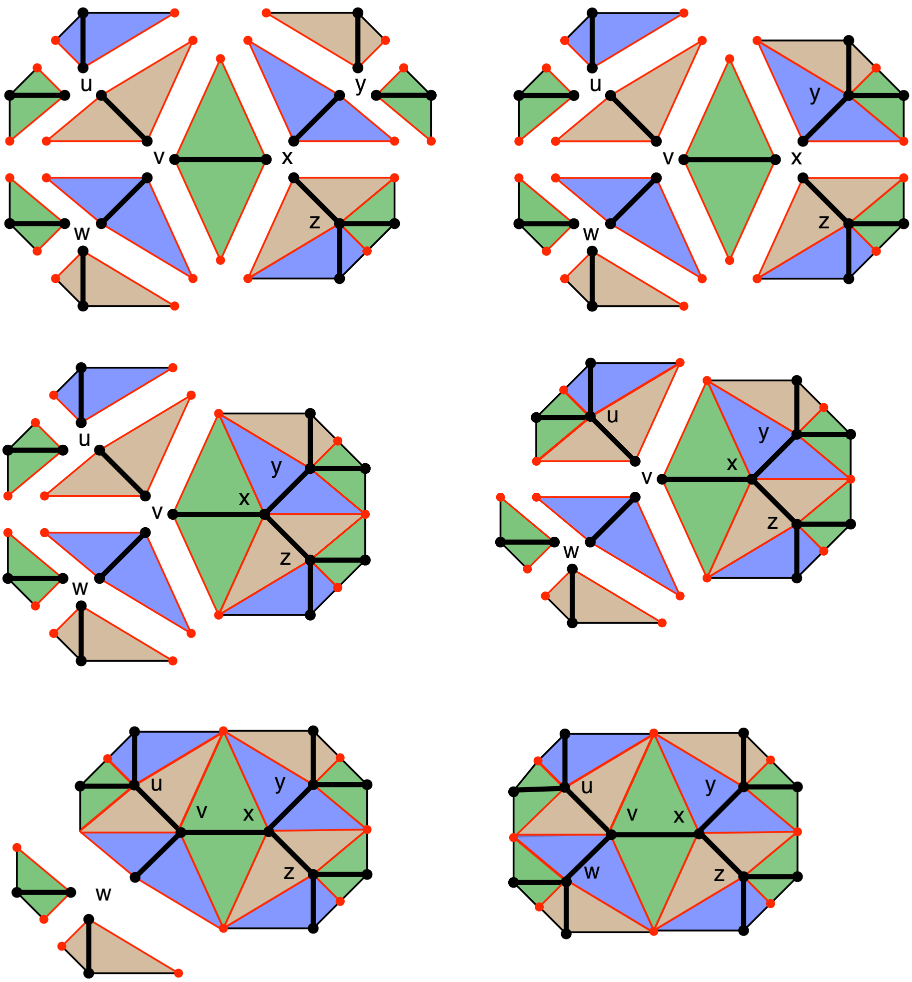

Figure 21: Solving the puzzle with post-order traversal.

Theorem 3.3. *Using the post-order of the interior tree-vertices as the order of merger solves the quadrangle puzzle for any plane cubic Halin graph.*

Proof. When the post-order is used, every RR-path through every vertex that follows the vertices of the fragment remains live. It also ensures that there is at least one quadrangle incident on each of those subsequent vertices. □

REMARK Quadrangulation and using the post-order solves the generalized puzzle for any Halin graph. The generalized algorithm is not presented only because its details are far lengthier than for the 3-regular case.

4 Partials and productions for Halin graphs

When a graph G has one or more of its vertices or edges designated as ***roots***, its genus distribution can be partitioned according to the ways in which face-boundary walks are incident on the roots. The components of such partitions are called ***partials***. A surface-by-surface inventory of the values of the partials is called a ***partitioned genus distribution***. Such partitioning has been a crucial step in most of the calculations of genus distributions.

Here is a general paradigm for calculating of the genus distribution of the graphs in a given graph family $\mathcal{F}$ by various kinds of graph amalgamation. The tricky part is that all of these requirements must be satisfied in coordination with the others.

- Prescribe a set $\mathcal{A}$ of rooted graphs as ***atomic fragments*** and a set $\mathcal{M}$ of ***merging operations***, such that every member of $\mathcal{F}$ can be constructed by iterative application of the merging operations to the atomic fragments. We denote the closure of $\mathcal{A}$ under $\mathcal{M}$ as $\overline{\mathcal{A}}$. Thus, $\mathcal{F} \subseteq \overline{\mathcal{A}}$.
- A procedure is designed to determine, from any graph G in $\overline{\mathcal{A}}$, the sequence of application of operations from $\mathcal{M}$ to atomic fragments and to others constructed earlier in the sequence, by which graph G can be obtained.
- An appropriate set of partials is developed for the rooted graphs in $\overline{\mathcal{A}}$.
- For each operation $\mu \in \mathcal{M}$ there is to be developed a set of rules, called ***productions***, is developed, that prescribe the values of the partials of any graph in $\overline{\mathcal{A}}$ from the values of the partials for the fragments that contribute to its construction under the operation μ.

Example 4.1. For the closed-end ladders and for the other kinds as well, the atomic fragments are doubly edge-rooted cycle graphs. The only operation for closed-end ladders is edge-amalgamation, and the order of application is linear. For the circular ladders and the Möbius ladders, there is an additional operation of self-edge-amalgamation, to be applied last. For the Ringel ladders, the additional operation is edge-addition, to be applied last.

Example 4.2. For the cobblestone walks (see [9]), the atomic fragments are doubly vertex-rooted cycle graphs. The only operation is vertex-amalgamation. The order of application is linear.

Example 4.3. For the 3-regular outerplanar graphs [12], the atomic fragments are doubly edge-rooted cycles. The operations are edge-amalgamation and root-popping on a singly edge-rooted graph. The order of operations is the post-order of a tree. For the 4-regular outerplanar graphs, the atomic fragments are doubly vertex-rooted cycles. The operations are vertex-amalgamation and root-popping. The order of operations is again the post-order of a tree.

Atomic fragments and merging operations for Halin graphs

The atomic fragments for constructing cubic Halin graphs are the quadrangular fragments obtained as in §3. We regard them here as doubly vertex-rooted. We denote this set of atomic fragments by $\mathcal{A}_H$. The only operation is merging three fragments at an interior vertex of the tree, in such a manner that either there is a surviving RR-path through the vertex of the fragment that is last (among the vertices of the fragment) in the post-order, or the Halin graph is fully reassembled.

Order of mergers for Halin graphs

The order of mergers of fragments is according to the post-order of the tree.

Partials for cubic Halin graphs

For a doubly vertex-rooted cubic Halin graph (G, u, v), with the roots u and v inserted at the midpoints of adjacent edges, we split $g_i(G)$ into six partials. Here is what they count:

- dd' Each of the roots u and v lies on two distinct fb-walks. One and only one of these fb-walks traverses both roots.
- dd'' Each of the roots u and v lies on two distinct fb-walks. Both of these fb-walks traverse both roots.
- ds' Root u lies on two distinct fb-walks. One of these fb-walks traverses root v twice.
- sd' Root v lies on two distinct fb-walks. One of these fb-walks traverses root u twice.
- ss^1 A single fb-walks traverses roots u and v twice. The occurrences of each root are consecutive.
- ss^2 A single fb-walks traverses roots u and v twice. The occurrences of the two roots alternate.

These configurations are illustrated in Figure 22.

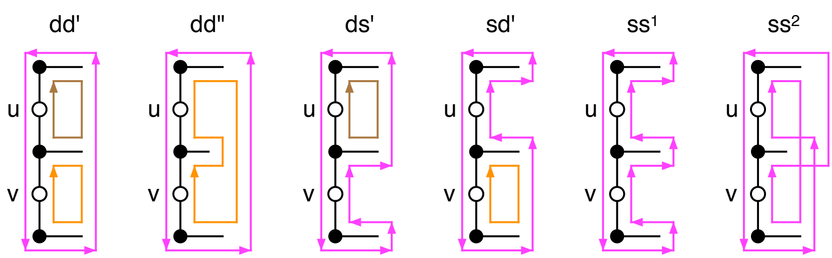

Figure 22: The 6 double-rooted partials for a 3-way pie-merge.

Proposition 4.1. *Let G be any graph that is homeomorphic to a cubic graph, and let its vertex roots u and v be 2-valent endpoints of a pair of edges that are adjacent at a 3-valent vertex. Then the six partials dd', dd'', ds', sd', ss^1, and ss^2 completely partition the genus distribution of G.*

Proof. In every embedding of G, since u and v lie on a pair of edges that are adjacent at a 3-valent vertex, there is necessarily an fb-walk on which both of them occur. Thus, if both roots lie on two different fb-walks, dd' and dd'' are the only possibilities. If one lies on two different fb-walks and the other on only one fb-walk, then ds' and sd' are the only possibilities. If both roots occur twice on the same fb-walk, then either (ss^1) the occurrences of each root are consecutive, or (ss^2) they alternate. □

Productions for cubic Halin graphs

For cubic Halin graphs, we merge three graphs at a time, exactly as for the puzzle, so that one of them is a quadrangle $Q = K_4 - e$, with its two roots inserted at the midpoints of the two quadrangle boundary edges that meet at the vertex to be merged. Envisioning this configuration at a small pie cut into three slices, we call the a 3-way π***-merge***. It is illustrated in Figure 23.

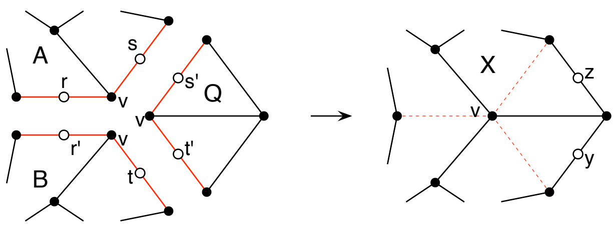

Figure 23: A 3-way π-merge $((A, r, s), (B, t, r'), (Q, s', t')) \to (X, y, z)$ at vertex v.

Proposition 4.2. *In a 3-way π-merge $(A, B, Q) \to X$ at vertex v, each rotation system ρ for X is consistent with exactly two rotation systems for fragment A and exactly two for fragment B.*

Proof. If rotation system ρ is consistent with a given rotation system ρ_A of fragment A, then it is also consistent with the rotation system of A obtained from ρ_A by reversing the rotation at v. A similar observation holds for fragment B. □

Suppose that $p^1, p^2, \ldots, p^s$ is a set of partials or subpartials for a genus distribution. A ***production*** for a given surgical operation that transforms either a graph embedding $X \to S_i$ (or a tuple of graph embeddings) into a set of graph embeddings of the graph Y is a rule of this form:

$$p_i^j(X) \longrightarrow c_1 p^1_{f_1^j(i)}(Y) + \cdots + c_t p^s_{f_s^j(i)}(Y)$$

The left side is called the ***antecedent***, and the right side is called the ***consequent***. The meaning is that the operation transforms a single embedding of graph X of type p^j on the surface S_i into a set of embeddings of the graph Y, of which c_k are of type p^k on the surface $S_{f_k^j(i)}$, for each i, j, and k. A drawing is usually used as an aid in deriving the production and in proving its correctness. The names of the graphs and their roots can be suppressed when there is in context no ambiguity.

Example 4.4. One of the productions for the π-merge of doubly vertex-rooted graphs (A, r, s), (B, t, r') and (Q, s', t') into (X, y, z) is

$$dd'_i(A, r, s) * dd''_j(B, t, r') \longrightarrow 2dd'_{i+j}(X, y, z) + 2ss^2_{i+j+1}(X, y, z)$$

It means that a type-dd' embedding $(A, r, s) \to S_i$ and a type-dd'' embedding $(B, t, r') \to S_j$ combine into two type-dd' embeddings $(X, y, z) \to S_{i+j}$ and two type-ss^2 embeddings $(X, y, z) \to S_{i+j+1}$. The relevant drawing is shown in Figure 24.

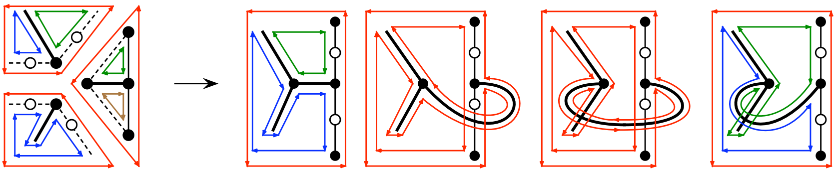

Figure 24: Prod #2: $dd'_i * dd''_j \longrightarrow 2dd'_{i+j} + 2ss^2_{i+j+1}$.

Theorem 4.3. *The following 36 productions are a complete set of rules for calculating the genus distribution of the graph that results from a π-merge of three graphs in $\overline{\mathcal{A}_H}$.*

1. $dd'_i * dd'_j \longrightarrow dd'_{i+j} + 2dd''_{i+j+1} + ss^2_{i+j+1}$.
2. $dd'_i * dd''_j \longrightarrow 2dd'_{i+j} + 2ss^2_{i+j+1}$.
3. $dd'_i * ds'_j \longrightarrow 2dd'_{i+j} + 2ss^2_{i+j+1}$.
4. $dd'_i * sd'_j \longrightarrow 2sd'_{i+j} + 2ss^1_{i+j+1}$.
5. $dd'_i * ss^1_j \longrightarrow 4sd'_{i+j}$.
6. $dd'_i * ss^2_j \longrightarrow 2ds'_{i+j} + 2sd'_{i+j}$.
7. $dd''_i * dd'_j \longrightarrow 2dd'_{i+j} + 2ss^2_{i+j+1}$.
8. $dd''_i * dd''_j \longrightarrow 4dd''_{i+j}$.
9. $dd''_i * ds'_j \longrightarrow 4ds'_{i+j}$.
10. $dd''_i * sd'_j \longrightarrow 4sd'_{i+j}$.
11. $dd''_i * ss^1_j \longrightarrow 4ss^1_{i+j}$.
12. $dd''_i * ss^2_j \longrightarrow 2dd'_{i+j-1} + 2ss^2_{i+j}$.
13. $ds'_i * dd'_j \longrightarrow 2ds'_{i+j} + 2ss^1_{i+j+1}$.
14. $ds'_i * dd''_j \longrightarrow 4ds'_{i+j}$.
15. $ds'_i * ds'_j \longrightarrow 4ds'_{i+j}$.
16. $ds'_i * sd'_j \longrightarrow 4ss^1_{i+j}$.
17. $ds'_i * ss^1_j \longrightarrow 4ss^1_{i+j}$.
18. $ds'_i * ss^2_j \longrightarrow 2ds'_{i+j-1} + 2ss^1_{i+j}$.
19. $sd'_i * dd'_j \longrightarrow 2dd'_{i+j} + 2ss^2_{i+j+1}$.

20. $sd'_i * dd''_j \longrightarrow 4sd'_{i+j}$.
21. $sd'_i * ds'_j \longrightarrow 2dd'_{i+j-1} + 2ss^2_{i+j}$.
22. $sd'_i * sd'_j \longrightarrow 4sd'_{i+j}$.
23. $sd'_i * ss^1_j \longrightarrow 4sd'_{i+j-1}$.
24. $sd'_i * ss^2_j \longrightarrow 2dd'_{i+j-1} + 2ss^2_{i+j}$.
25. $ss^1_i * dd'_j \longrightarrow 4ds'_{i+j}$.
26. $ss^1_i * dd''_j \longrightarrow 4ss^1_{i+j}$.
27. $ss^1_i * ds'_j \longrightarrow 4ds'_{i+j-1}$.
28. $ss^1_i * sd'_j \longrightarrow 4ss^1_{i+j}$.
29. $ss^1_i * ss^1_j \longrightarrow 4ss^1_{i+j-1}$.
30. $ss^1_i * ss^2_j \longrightarrow 4ds'_{i+j-1}$.
31. $ss^2_i * dd'_j \longrightarrow 2ds'_{i+j} + 2sd'_{i+j}$.
32. $ss^2_i * dd''_j \longrightarrow 2dd'_{i+j-1} + 2ss^2_{i+j}$.
33. $ss^2_i * ds'_j \longrightarrow 2dd'_{i+j-1} + 2ss^2_{i+j}$.
34. $ss^2_i * sd'_j \longrightarrow 2sd'_{i+j-1} + 2ss^1_{i+j}$.
35. $ss^2_i * ss^1_j \longrightarrow 4sd'_{i+j-1}$.
36. $ss^2_i * ss^2_j \longrightarrow 2dd''_{i+j-1} + dd'_{i+j-2} + ss^2_{i+j-1}$.

Proof. The correctness of each of these productions is a matter of recombining the strands as prescribed by the π-merge. The 36 figures corresponding to these productions are given by [14]. □

Computational Complexity

Theorem 4.4. *For $|V_A| = k$ and $|V_B| = m$, there is an $O(km)$-time algorithm for calculating the partitioned genus distribution of the resulting graph X of a 3-way pie-merge $(A, B, Q) \to X$ of graphs whose maximum degree is 3.*

Proof. The number of non-zero partials of a cubic graph G with p vertices is in $O(p)$, since the maximum genus cannot exceed $\beta(G)/2$. For each non-zero-valued partial of A and each non-zero-valued partial of B, only one production is applied, and the time for the application of a single production is in $O(1)$. □

Corollary 4.5. *The post-order traversal using the 36 productions corresponding to the six partials yields an $O(n^2)$ algorithm for the genus distribution of a cubic Halin graph with n vertices.*

Proof. Let H have quadrangular fragments $Q_1, \ldots, Q_f$ of respective cardinalities $q_1, \ldots, q_f$. The number of non-zero-valued partials in the π-merge of a k-vertex fragment A with an m-vertex fragment B and a quadrangular fragment Q_i is at most a constant multiple of $k + m$. Since each pair of initial quadrangular fragments is merged only once during the reassembly of the Halin graph, it follows that the number of steps is at most a constant multiple of the sum

$$\sum_{i \neq j} q_i q_j$$

where q_i is the number of non-zero partials of the quadrangular fragment Q_i. However,

$$\sum_{i \neq j} q_i q_j \;<\; (q_1 + q_2 + \cdots + q_f)^2$$

The conclusion follows. □

5 Sample Calculation

In this section, we show the work needed to calculate the genus distribution of the Halin graph of Figure 1.

Merger at z

Graph A ($K_4 - e$):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	0	0	0	0	2	2

Graph B ($K_4 - e$):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	0	0	0	0	2	2

Merged Graph K_4: Use Productions 1, 6, 31, and 36.

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	4	4	4	0	2	14

Merger at y

Merged Graph K_4: Just like the merger at z.

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	4	4	4	0	2	14

Merger at x

Graph A (result from merger at z):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	4	4	4	0	2	14

Graph B (result from merger at y):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	4	4	4	0	2	14

Merged Graph: Use 25 productions (all those without the partial ss^1).

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	40	4	12	12	0	2	70
2	0	16	48	48	32	40	184

Merger at u

Merged Graph: K_4**:** Just like the merger at z.

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	4	4	4	0	2	14

Merger at v

Graph A (result from merger at x):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	40	4	12	12	0	2	70
2	0	16	48	48	32	40	184

Graph B (result from merger at u):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	4	4	4	0	2	14

Merged Graph: Use 30 productions.

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	112	4	28	12	0	2	158
2	544	96	544	352	80	112	1728
3	0	64	448	448	704	544	2208

Merger at w

Graph A (result from merger at v):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	112	4	28	12	0	2	158
2	544	96	544	352	80	112	1728
3	0	64	448	448	704	544	2208

Graph B ($K_4 - e$):

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	0	0	0	0	0	2	2

Merged Graph: final result

i	dd'_i	dd''_i	ds'_i	sd'_i	ss^1_i	ss^2_i	g_i
0	2	0	0	0	0	0	2
1	144	4	60	4	0	2	214
2	1440	224	1632	224	56	144	3720
3	1024	1088	4800	1088	1088	1440	10528
4	0	0	0	0	896	1024	1920

6 Conclusions

We have demonstrated the usefulness of the paradigm given at the beginning of §4 in deriving a practical algorithm for the genus distribution of cubic Halin graphs. To be practical, in the sense intended here, the number of partials needed should be relatively small.

To calculate the genus distribution of a family of graphs, under this paradigm, one first designs a recursive specification of that family, that is, a finite set of base graphs and a finite set of operations whose iterative application can construct any graph in the family. One then derives a set of production rules for obtaining the partitioned genus distribution of the result of the applying any operation from the partitioned genus distributions of the operands.

There are problems whose general solution seems to require exponentially large effort, but which can be solved in polynomial-time for cases in which something is bounded. A familiar result in topological graph theory is that whereas Thomassen [36] proved that determining the minimum genus of a graph is NP-hard, Mohar [27] proved that for every possible orientable surface S_i, there is a linear-time algorithm to decide whether a given graph is embeddable in that surface.

Shortly after the presentation of this paper, the author derived, for any fixed treewidth and maximum degree, a quadratic-time algorithm [15] to calculate the genus distribution of any graph conforming to those bounds. This algorithm is less than practical, since the numbers of partials and productions increase exponentially with the treewidth and the maximum degree.

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Markov chain algorithms for generating sets uniformly at random

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Abstract

In this paper we tackle the problem of generating uniformly at random two representative types of sets with n elements: transitive sets and weakly transitive sets, that is, transitive sets with atoms. A set is said to be *transitive* if any of its elements is also a subset of it; a set is *weakly transitive* if any of its elements, unless disjoint from it, is also a subset of it. We interpret such sets as (weakly) extensional acyclic digraphs—that is, acyclic digraphs whose (non-sink) vertices have pairwise different out-neighborhoods—and employ a Markov chain technique already given for acyclic digraphs. We thus propose Markov chain-based algorithms for generating uniformly at random (weakly) extensional acyclic digraphs with a given number of vertices. The Markov chain is then refined to generate such digraphs which are also simply connected, and digraphs in which the number of arcs is fixed.

Keywords: Extensional digraph, transitive set, set theory, random generation, Markov chain.

Math. Subj. Class.: 03E75, 05C81, 05C20

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1 Introduction

Sets are basic mathematical and computational objects, and for this reason one is sometimes interested—in order to perform tests and evaluate benchmarks, collect statistical data, (dis)prove conjectures, etc.—in generating uniformly at random a set of given size. In this paper we tackle this problem by building on results originally given in the context of graphs.

The sets we consider belong to the standard Zermelo-Fraenkel set theory and thus contain only other sets as elements. We focus on two archetypal sets: *transitive* sets and *weakly transitive sets*, that is, sets s whose elements are also subsets of s, and sets s whose elements, unless disjoint from s, are also subsets of s. A weakly transitive set s can be simply seen as a transitive set with *atoms* (or *ur-elements*), the atoms of s constituting the collection of elements of s disjoint from s. Zermelo-Fraenkel sets stand at the basis of programming languages such as SETL [15], or the more recent $\{\log\}$ [3] and CLP($\mathcal{SET}$) [5], and of the automatic proof-verifier Referee [14].

Since sets are *nested* structures, they are best represented by digraphs: vertices will stand for sets, while the arc relation will correspond to the inverse of the membership relation between them ($\rightarrow\ \equiv\ \ni$). Such digraphs are acyclic, since membership is well-founded, and *weakly extensional*, in the sense that distinct non-sink vertices have distinct collections of out-neighbors. This digraph interpretation was exploited in [13] to give a recurrence relation for the number of weakly transitive sets with n elements, generalizing the result of [12] for transitive sets with n elements.

Under this graph-theoretic interpretation, we show in this paper how a Markov chain based procedure for generating acyclic digraphs, first introduced in [9], can be transferred to our set-theoretic universe. This Markov chain algorithm was already modified in [10] to generate simply connected acyclic digraphs. The random generation of elements from a particular class of acyclic digraphs modeling Bayesian networks was proposed in [6]. Finally, the same approach was used in [2] to generate deterministic acyclic automata. Each of these examples can be seen as a less basic case than the one tackled here.

The idea behind this Markov chain technique is to start with an arbitrary weakly extensional acyclic digraph (*w.e.a.* digraph, for short) on n vertices and apply a certain number T of random local transformations which preserve weak extensionality and acyclicity. The uniformity of the resulting distribution is basically proved by showing that any w.e.a. digraph on n vertices can be thus transformed into any other w.e.a. digraph on n vertices. Like in the acyclic digraph case, we argue that the transformation rules are symmetric and always allow reaching a specific digraph among the collection of w.e.a. digraphs with n vertices. In our case, however, the most natural target digraph for this purpose turns out to be an acyclic tournament on n vertices, that is, the digraph whose interpretation in the universe of sets is the von Neumann numeral of n, the unique transitive set with n elements well-ordered by the membership relation.

We prove here only 'correctness' and defer to future work computational aspects such as estimations for the choice of T or an analysis of the mixing time of the Markov chain [8]. As next research steps we also mention the random generation of hypersets, objects for which the acyclicity requirement is dropped, and weak extensionality is accordingly strengthened by an equality criterion based on the notion of bisimulation [1].

The paper is organized as follows. In Section 2 we give some notation and formally introduce the above-mentioned notions. In Section 3 we put forward a Markov chain for generating weakly extensional acyclic digraphs, while in Section 4 we propose a Markov chain for generating simply connected weakly extensional acyclic digraphs. The latter can be easily adapted to generate extensional acyclic digraphs. Finally, in Section 5, we present a Markov chain for generating w.e.a. digraphs on n vertices and m arcs, $0 \leqslant m \leqslant \binom{n}{2}$, which can also serve to generate acyclic digraphs with a given number of vertices and of arcs.

2 Notation and preliminaries

We consider only finite *simple* digraphs, that is, without parallel arcs or self-loops. Given a digraph D, we denote by $V(D)$ its vertex set and by $E(D)$ the set of its arcs. For any $v \in V(D)$, $N^+(v)$ stands for the set $\{u \in V(D) \mid (v,u) \in E(D)\}$, which is called the *out-neighborhood* of v in D. Similarly, $N^-(v) = \{u \in V(D) \mid (u,v) \in E(D)\}$ is the *in-neighborhood* of v. We will use $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$. A vertex $v \in V(D)$ such that $d^+(v) = 0$ will be called a *sink*; if $d^-(v) = 0$, it will be called a *source*; we denote by $I(D)$ the set of the sinks of D.

A digraph D is said to be *simply connected* if the underlying (undirected) graph of D is connected. If $(i,j) \in E(D)$, we will employ $D - (i,j)$ as a shorthand for the digraph obtained from D by removing the arc (i,j). Analogously, $D + (i,j)$ is the digraph obtained from D by the addition of the arc (i,j).

Definition 2.1. A digraph D is said to be

- *extensional*, if for any distinct $u, v \in V(D)$ it holds that $N^+(u) \neq N^+(v)$;
- *weakly extensional*, if for any distinct vertices $u, v \in V(D) \setminus I(D)$, it holds that $N^+(u) \neq N^+(v)$.

If u, v are distinct vertices of a digraph D having $N^+(u) = N^+(v)$, we say that u and v *collide*. Note that this is not the case if D is acyclic and there is a directed path from u to v. In particular, in an acyclic tournament there are no collisions (a *tournament* is a digraph D such that for any distinct $u, v \in V(D)$ either $(u,v) \in E(D)$ or $(v,u) \in E(D)$ holds, but not both). Moreover, any e.a. digraph is simply connected.

Under the Zermelo-Fraenkel axioms, each set is uniquely characterized by its elements (Extensionality Axiom), and the membership relation is well-founded (Foundation Axiom). The standard universe of sets is von Neumann's cumulative hierarchy, whose subset of hereditarily finite sets is defined as the union, over all natural numbers i, of $\mathbb{V}_i$, where $\mathbb{V}_0 = \emptyset$, each level $\mathbb{V}_{i+1}$ is $\mathcal{P}(\mathbb{V}_i)$, and $\mathcal{P}(\cdot)$ stands for the power-set operator. For example, $\mathbb{V}_1 = \{\emptyset\}$, $\mathbb{V}_2 = \big\{\emptyset, \{\emptyset\}\big\}$, $\mathbb{V}_3 = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}$.

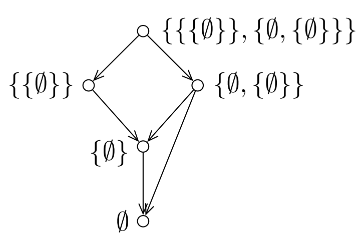


Figure 1: The digraph representation of a transitive set.

To faithfully represent a set x as a digraph, one considers the digraph D_x defined as

$$D_x = (x, \{(u, v) \mid u, v \in x, v \in u\}).$$

See Figure 1 for an example. Since the membership relation between sets is well-founded, such digraphs are acyclic. Moreover, if x is transitive, then D_x is also extensional, while if x is weakly transitive, then D_x is weakly extensional. To see this, observe that if distinct non-sink vertices have the same out-neighborhood, then from the (weak) transitivity assumption they correspond to the same set, contradicting the definition of D_x; see also [13].

Given $n \geqslant 1$, we denote by $\mathcal{W}_n$ the set of all w.e.a. digraphs with vertex set $\{1, \ldots, n\}$, while $\mathcal{W}_n^c$ denotes its subset of simply connected w.e.a. digraphs. Analogously, $\mathcal{W}_{n,m}$ denotes the set of all w.e.a. digraphs with vertex set $\{1, \ldots, n\}$ and m arcs.

Definition 2.2. A *discrete time finite stochastic process* is a sequence $X = (X_t : t \in \mathbb{N})$, where X_t are $\mathcal{S}$-valued random variables and $\mathcal{S}$ is a finite set, called the *state space* of X. We say that X is a *Markov chain* if

$$\forall t \in \mathbb{N},\ \Pr(X_{t+1} = s_{t+1} \mid X_t = s_t, \ldots, X_0 = s_0) = \Pr(X_{t+1} = s_{t+1} \mid X_t = s_t).$$

Moreover, a Markov chain X is said to be *time-homogeneous* if

$$\forall s, s' \in \mathcal{S},\ \exists p_{ss'},\ \forall t \in \mathbb{N},\ \Pr(X_{t+1} = s \mid X_t = s') = p_{ss'}.$$

Definition 2.3. A time-homogeneous Markov chain over the state space $\mathcal{S}$ is said to be:

- *irreducible* iff $\forall s, s' \in \mathcal{S},\ \exists t \in \mathbb{N},\ \Pr(X_t = s' \mid X_0 = s) > 0$;
- *aperiodic* iff $\forall s \in \mathcal{S},\ \gcd\{t \in \mathbb{N} \mid \Pr(X_t = s \mid X_0 = s) > 0\} = 1$;
- *symmetric* iff $\forall s, s' \in \mathcal{S},\ p_{ss'} = p_{s's}$.

A well-known result (see, e.g., [8]) states that any finite, irreducible, aperiodic and symmetric time-homogeneous Markov chain converges toward the uniform distribution on its state space. Therefore, all the Markov chains presented here will be shown to satisfy these three properties.

3 A Markov chain algorithm for generating w.e.a. digraphs

Let M be a Markov chain over $\mathcal{W}_n$, defined in Figure 2. Observe that M differs from the Markov chain of [9] for generating arbitrary acyclic digraphs in the fact that arc deletions and additions are done only provided that the resulting digraph is w.e.a.

Notice that for any $t \in \mathbb{N}$ and any two distinct states $s, s' \in \mathcal{W}_n$, $\Pr(X_{t+1} = s \mid X_t = s') > 0$ if and only if $\Pr(X_{t+1} = s' \mid X_t = s) > 0$. To be more precise, the probability of passing from a state $s \in \mathcal{W}_n$ to any other state $s' \neq s$ is either 0 or $1/n^2$, hence M is symmetric. Moreover, for every $s \in \mathcal{W}_n$ the probability of remaining in s at any $t > 0$ is positive, for example by choosing diagonal pairs (i, i), with $i \in \{1, \ldots, n\}$. Therefore, if M turns out to be irreducible, then it will also be aperiodic.

The initial state of this Markov chain and of the ones given in the next section can be taken to be a linear digraph consisting of a directed path $(n, n-1, \ldots, 1)$. The acyclicity of a digraph on n vertices and m arcs can be established by a depth-first visit, in time $O(n + m)$. To test whether a digraph is (weakly) extensional, the algorithm in [4, Sec. 4] can be used, taking time $O(n + m)$.

Let X_t denote the state of the Markov chain at time t. Suppose a couple of integers (i, j) has been drawn uniformly at random from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

($\mathbf{T}_1$) if $(i, j) \in E(X_t)$ **and**
$X_t - (i, j)$ is w.e., **then** $X_{t+1} = X_t - (i, j)$
else $X_{t+1} = X_t$.
($\mathbf{T}_2$) if $(i, j) \notin E(X_t)$ **and**
$X_t + (i, j)$ is w.e.a., **then** $X_{t+1} = X_t + (i, j)$
else $X_{t+1} = X_t$.

Figure 2: A Markov chain algorithm for generating w.e.a. digraphs.

Lemma 3.1. *Let D be a w.e.a. digraph with $E(D) \neq \emptyset$. There exists an arc $(u, v) \in E(D)$ such that the digraph $D - (u, v)$ is w.e.a.*

Proof. Observe first that there exists $u \in V(D)$ such that $\emptyset \neq N^+(u) \subseteq I(D)$. If this were not the case, then for all u in $V(D)$ with $N^+(u) \neq \emptyset$, there would exist a vertex u' in D with $N^+(u') \neq \emptyset$ such that $(u, u') \in E(D)$. Since the same property holds for u' as well, and as the number of vertices of D is finite, we can find a finite directed cycle, contradicting hence the acyclicity of D.

Let now $U(D)$ be the set of vertices of D with the above property, that is, $U(D) \stackrel{\text{def}}{=} \{u \in V(D) \mid \emptyset \neq N^+(u) \subseteq I(D)\}$. Let $u_0 \in U(D)$ be a vertex of minimum out-degree, i.e., $d^+(u_0) = \min\{d^+(u) : u \in U(D)\}$. Since $N^+(u_0) \neq \emptyset$, let v_0 be an element of $N^+(u_0)$. The arc (u_0, v_0) can be removed and the resulting digraph remains w.e.a. Indeed, since u_0 is among the vertices of minimum out-degree in $U(D)$, in $D - (u_0, v_0)$ it will either be a sink, or it will be the only vertex in $U(D - (u_0, v_0))$ with out-degree $d^+(u) - 1$, hence having its out-neighborhood different from that of any other vertex of $D - (u_0, v_0)$. $\square$

Theorem 3.2 (Irreducibility of M)**.** *Let M be the Markov chain defined over the space $\mathcal{W}_n$ together with the transition rules $\mathbf{T}_1$ and $\mathbf{T}_2$. Given two distinct digraphs D and H in $\mathcal{W}_n$, there exists in M a sequence of transitions $D = D_0 \to D_1 \to \cdots \to D_{p-1} \to D_p = H$, where $p \geqslant 1$ and $D_i \in \mathcal{W}_n$, for all $0 \leqslant i \leqslant p$. Such a sequence exists with length at most $n^2 - n$.*

Proof. Since M is symmetric, it suffices to show that there exists a sequence of transitions from any given w.e.a. digraph $D \in \mathcal{W}_n$ to a fixed element O in $\mathcal{W}_n$. For our purpose here, we will choose O to be the unique totally disconnected digraph, that is, having $E(O) = \emptyset$.

From Lemma 3.1, we get that there exists an arc $(u, v) \in E(D)$ such that $D - (u, v)$ is w.e.a. Using rule ($\mathbf{T}_1$), we can step from D to $D - (u, v)$. Repeating the above argument a finite number of steps, we arrive at O. The number of transitions from D to O is at most $n(n-1)/2$, and this is obtained when D is a tournament. $\square$

4 A Markov chain algorithm for generating e.a. digraphs

Instead of generating e.a. digraphs, we place ourselves in a more general setting, that of generating simply connected w.e.a. digraphs. Afterwards, we will argue that, with minor changes, the proposed Markov chain can generate e.a. digraphs. Let M^c be the Markov chain over $\mathcal{W}_n^c$ whose transitions between states are given in Figure 3. This Markov chain is adapted from [10], with the difference that arc deletions, additions, or reversals are done only if the resulting digraph is simply connected and w.e.a. This simple modification, however, requires a totally new and more involved proof of irreducibility.

Just as in the previous section, the probability of passing from a state $s \in \mathcal{W}_n^c$ to any other state $s' \neq s$ is either 0 or $1/n^2$, implying that M^c is symmetric. Likewise, the aperiodicity of M^c will be implied by its irreducibility and by the fact that for every state in $\mathcal{W}_n^c$ there is a positive probability to remain in that state (by choosing diagonal pairs (i, i)). Even if the two transition rules of M^c are not entirely specular, one can think of M^c as having three basic transitions: (1) removal of an arc, (2) reversal of an arc, (3) addition of an arc. According to this view, M^c is entirely symmetric.

Let X_t denote the state of the Markov chain at time t. Suppose a couple of integers (i, j) has been drawn uniformly at random from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

($\mathbf{T}_1^c$) if $(i, j) \in E(X_t)$ **then**
- (a) **if** $X_t - (i, j)$ is simply connected and w.e., **then** $X_{t+1} = X_t - (i, j)$, **else**
  - (b) **if** $X_t - (i, j) + (j, i)$ is w.e.a., **then** $X_{t+1} = X_t - (i, j) + (j, i)$,
  - (c) **else** $X_{t+1} = X_t$.

($\mathbf{T}_2^c$) if $(i, j) \notin E(X_t)$, **then**
- (a) **if** $X_t + (i, j)$ is w.e.a., **then** $X_{t+1} = X_t + (i, j)$,
- (b) **else** $X_{t+1} = X_t$.

Figure 3: A Markov chain algorithm for generating simply connected w.e.a. digraphs.

To show the irreducibility of the Markov chain M^c, it is useful to partition the vertices of an acyclic digraph according to the maximum length of a directed path to a sink of the digraph. Complying with standard set-theoretic notation, we will make use of the following notion.

Definition 4.1. Given an acyclic digraph D, the *rank* of a vertex $v \in V(D)$ is recursively defined as

$$\mathsf{rk}(v) = 1 + \max\{\mathsf{rk}(u) \ : \ (v, u) \in E(D)\},$$

where $\mathsf{rk}(v) = 0$ if v is a sink.

Clearly, the following lemma holds.

Lemma 4.2. *Given an acyclic digraph D, if $v, u \in V(D)$ and $\mathsf{rk}(v) \neq \mathsf{rk}(u)$, then $N^+(v) \neq N^+(u)$ holds.*

Throughout the subsequent two proofs we employ the following notation: given an acyclic digraph D and a vertex v of D,

$$R(v) \stackrel{\text{def}}{=} \{u \in V(D) \mid u \neq v \text{ and } \mathsf{rk}(u) \leqslant \mathsf{rk}(v)\}.$$

Theorem 4.3 (Irreducibility of M^c). *Let M^c be the Markov chain defined over the space $\mathcal{W}_n^c$ together with the transition rules $\mathbf{T}_1^c$ and $\mathbf{T}_2^c$. Given two distinct digraphs D and H in $\mathcal{W}_n^c$, there exists in M^c a sequence of transitions $D = D_0 \to D_1 \to \cdots \to D_{p-1} \to D_p = H$, where $p \geqslant 1$ and $D_i \in \mathcal{W}_n^c$, for all $0 \leqslant i \leqslant p$. Such a sequence exists with length at most $(3n^2 - 7n + 4)/2$.*

Proof. As before, first we will show that there exists a sequence of transitions from any given w.e.a. digraph $D \in \mathcal{W}_n^c$ to an element $T(D)$ in $\mathcal{W}_n^c$, where $T(D)$ is an acyclic tournament, with the additional property that whenever $\mathsf{rk}(v) > \mathsf{rk}(u)$ in D, $\mathsf{rk}(v) > \mathsf{rk}(u)$ also holds in $T(D)$. Then, given any D and H in $\mathcal{W}_n^c$, we will show that there is a sequence of transitions from $T(D)$ to $T(H)$, completing hence the proof.

To show the former claim, we proceed as follows. Pick a vertex $v \in V(D)$, in decreasing order of rank (when more vertices of the same maximum rank exist, pick an arbitrary one). Apply rule ($\mathbf{T}_2^c$) and add arcs from v to all the vertices $u \in R(v) \setminus N^+(v)$, in decreasing order on the rank of the elements of $R(v) \setminus N^+(v)$. Note that this is possible, first of all, because the addition of an arc (v, u) does not create a cycle in the resulting digraph. Second, observe that the subdigraph of D induced by the vertices $V(D) \setminus R(v)$ is an acyclic tournament. Therefore, an arc addition would create a collision only between v and a vertex $w \in R(v)$. This is however not the case, since after the first addition of such an arc, $\mathsf{rk}(v)$ becomes strictly greater than $\mathsf{rk}(w)$, for all $w \in R(v)$, and Lemma 4.2 guarantees the absence of collisions.

Denote by $T(D)$ the acyclic tournament obtained at the end of this process. Since for any vertex v we have added arcs only to those vertices of rank less than or equal to v, we also have that whenever $\mathsf{rk}(v) > \mathsf{rk}(u)$ in D, the same holds in $T(D)$.[1]

Passing on to the latter point, observe that for any w.e.a. digraph D, since $T(D)$ is a tournament, there are no two distinct elements of the same rank in $T(D)$, and thus $\{\mathsf{rk}(v) : v \in V(T(D))\} = \{0, \ldots, n-1\}$. Hence, to each digraph $T(D)$ we can uniquely associate a linear order $\prec_{T(D)}$ on $V(D)$ defined in the following way: for all $u, v \in V(T(D))$

$$u \prec_{T(D)} v \text{ iff } \mathsf{rk}(u) < \mathsf{rk}(v) \text{ in } T(D).$$

We now show that given two orders $x_0 \prec_{T(D)} x_1 \prec_{T(D)} \cdots \prec_{T(D)} x_{n-1}$ and $y_0 \prec_{T(H)} y_1 \prec_{T(H)} \cdots \prec_{T(H)} y_{n-1}$, where $\{x_i : 0 \leqslant i \leqslant n-1\} = \{y_i : 0 \leqslant i \leqslant n-1\} = \{1, \ldots, n\}$, we can transform $T(D)$ in $T(H)$, applying rule ($\mathbf{T}_1^c$).

Observe first that for any two consecutive elements $x_i \prec_{T(D)} x_{i+1}$ $(0 \leqslant i < n-1)$ it holds that $N^+(x_{i+1}) = N^+(x_i) \cup \{x_i\}$. Therefore, applying rule ($\mathbf{T}_1^c$) on $T(D)$, the arc (x_i, x_{i+1}) cannot be removed (by (a)), but can be reversed (by (b)). In the resulting acyclic tournament $T(D')$, x_i and x_{i+1} have swapped positions, i.e., $x_{i+1} \prec_{T(D')} x_i$. Starting from position $i = 0$ all the way to $i = n-1$, apply the following procedure. If $y_i = x_j$, $(i < j \leqslant n-1)$, then bring x_j to position i by iteratively reversing the arcs (x_j, x_{j-1}), $(x_j, x_{j-2}), \ldots, (x_j, x_i)$.

The maximum number of transitions to pass from D to $T(D)$ is $\binom{n}{2} - (n-1) = (n^2 - 3n + 2)/2$, number obtained when the underlying graph of D is a tree, thus having $n-1$ edges. To pass from $T(D)$ to $T(H)$, $\binom{n}{2}$ transitions are required at most, when all the arcs of $T(D)$ have to be reversed. Hence, to pass between two arbitrary D and H in $\mathcal{W}_n^c$, we need at most $(3n^2 - 7n + 4)/2$ transitions. ◻

[1] However, the converse in general does not hold.

Let us denote by $\mathcal{E}_n$ the set of all e.a. digraphs with vertex set $\{1, \ldots, n\}$. The Markov chain illustrated in Figure 3 can be transformed into an irreducible, aperiodic and symmetric Markov chain, M^e, for the generation of digraphs from $\mathcal{E}_n$. The transitions between two states in M^e are given in Figure 4. The analogue of Theorem 4.3 holds for M^e as well.

Let X_t denote the state of the Markov chain at time t. Suppose a couple of integers (i, j) has been drawn uniformly at random from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

($\mathbf{T}_1^e$) if $(i, j) \in E(X_t)$ **then**
- (a) **if** $X_t - (i, j)$ is extensional, **then** $X_{t+1} = X_t - (i, j)$,
  else
  - (b) **if** $X_t - (i, j) + (j, i)$ is e.a., **then** $X_{t+1} = X_t - (i, j) + (j, i)$,
  - (c) **else** $X_{t+1} = X_t$.

($\mathbf{T}_2^e$) if $(i, j) \notin E(X_t)$, **then**
- (a) **if** $X_t + (i, j)$ is e.a., **then** $X_{t+1} = X_t + (i, j)$,
- (b) **else** $X_{t+1} = X_t$.

Figure 4: A Markov chain algorithm for generating e.a. digraphs.

5 A Markov chain algorithm for generating w.e.a. digraphs with a given number of arcs

A Markov chain M^a for generating w.e.a. digraphs with vertex set $\{1, \ldots, n\}$ and m arcs is given in Figure 5. It will immediately follow from the proof of its irreducibility—Theorem 5.1 below—that M^a can also generate uniformly at random acyclic digraphs with a given number of vertices and a fixed number of arcs, by simply swapping two arcs if the resulting digraph is acyclic. Note that controlling the number of arcs was already considered in the literature: [9] proposed generating acyclic digraphs with a bounded number of arcs, or whose vertices have a bounded degree; [7] proposed generating acyclic digraphs with bounded *induced width*, a complexity measure arising from artificial intelligence.

The probability of passing from a state $s \in \mathcal{W}_{n,m}$ to any other state $s' \neq s$ is either 0 or $1/n^4$, implying that M^a is symmetric. As previously, for any state $s \in \mathcal{W}_{n,m}$ there is a positive probability to remain in s. Our next theorem shows that M^a is indeed irreducible. If $m < n - 1$, the initial state of the Markov chain can be a digraph whose arcs form a directed path of length m. Otherwise, the initial state can be a directed path $(n, n-1, \ldots, 1)$ together with $m - (n - 1)$ arbitrary arcs of the form (i, j), where $i > j$.

Theorem 5.1 (Irreducibility of M^a)**.** *The Markov chain M^a is irreducible.*

Proof. First, if $m = 0$, $\mathcal{W}_{n,m}$ consists only of the totally disconnected digraph. Assuming that $m > 0$, we show that any digraph $D \in \mathcal{W}_{n,m}$ can be transformed, by transitions of M^a, into a digraph $K(D) \in \mathcal{W}_{n,m}$, satisfying the following two properties:

i) for all $v \in V(D)$ such that $\mathsf{rk}(v) > 1$ in $K(D)$, it holds that $N^+(v) = R(v)$ in $K(D)$;

ii) there is only one $v \in V(D)$ such that $\mathsf{rk}(v) = 1$ in $K(D)$.

Let X_t denote the state of M^a at time t. Suppose two pairs of integers (i_1, j_j) and (i_2, j_2) have been drawn uniformly at random and independently from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

if $(i_1, j_1) \in E(X_t)$ **and** $(i_2, j_2) \notin E(X_t)$, **then**
 if $X_t - (i_1, j_1) + (i_2, j_2)$ is w.e.a., **then** $X_{t+1} = X_t - (i_1, j_1) + (i_2, j_2)$,
 else $X_{t+1} = X_t$.

Figure 5: A Markov chain algorithm for generating w.e.a. digraphs on n vertices and m arcs.

To show this, we argue as in the proof of Theorem 4.3, paying particular attention to preserving m arcs at each intermediate step. Observe that if D fails to satisfy i) or ii), then it must own a vertex v such that $N^+(v) \subsetneq R(v)$. Indeed, first note that $N^+(v) \subseteq R(v)$ holds for any vertex v, since D is acyclic. If D does not satisfy i) and $v \in V(D)$ is a vertex with $\mathsf{rk}(v) > 1$ and $N^+(v) \neq R(v)$, then $N^+(v) \subsetneq R(v)$ immediately follows. If D owns two distinct vertices v' and v'' of rank 1, then $v'' \in R(v') \setminus N^+(v')$.

Therefore, as long as D fails to satisfy one of the above conditions i) or ii), apply the following transformation to it. Pick a vertex $v \in V(D)$ inclusion-maximal among the vertices of maximum rank having $N^+(v) \subsetneq R(v)$, and consider all the elements $u \in R(v) \setminus N^+(v)$, in decreasing order on rank. Take $t \in V(D)$ a vertex whose out-neighborhood is inclusion-minimal among the vertices of rank 1. Arguing as in the proof of Lemma 3.1, any arc (t, s) leading from t to an arbitrary sink $s \in V(D)$ can be removed without disrupting the weak extensionality of D. Replace arc (t, s) with arc (v, u) by a transition of M^a.

This is possible, since, on the one hand, the addition of the arc (v, u) does not create a cycle in the resulting digraph. On the other hand, as before, the subdigraph of D induced by the vertices $V(D) \setminus R(v)$ is an acyclic tournament. Therefore, one such arc addition can create a collision only between v and some vertex $w \in R(v)$. This is not the case, since after the first addition of an arc from v to a maximum rank vertex in $R(v)$, $\mathsf{rk}(v)$ becomes strictly greater than $\mathsf{rk}(w)$, for all $w \in R(v)$, and Lemma 4.2 guarantees the absence of collisions. It should be clear that the above procedure stops after a finite number of steps, and that the final digraph satisfies conditions i) and ii).

It remains to show that, given two digraphs D and H in $\mathcal{W}_{n,m}$, there exists a sequence of transitions in M^a from $K(D)$ to $K(H)$. To any acyclic digraph K whose vertices of positive rank have pairwise distinct ranks we can associate a partial order $\prec_K$ in the following way: for all $u, v \in V(K)$,

$$u \prec_K v \text{ iff } \mathsf{rk}(u) < \mathsf{rk}(v) \text{ in } K.$$

For expository purposes, assume that we also order the sinks of K is an arbitrary way so that $\prec_K$ is a linear order on the vertices of K. Therefore, we have to show that we can transform any order $x_0 \prec_{K(D)} x_1 \prec_{K(D)} \cdots \prec_{K(D)} x_{n-1}$ into $y_0 \prec_{K(H)} y_1 \prec_{K(H)} \cdots \prec_{K(H)} y_{n-1}$, where $\{x_i : 0 \leqslant i \leqslant n-1\} = \{y_i : 0 \leqslant i \leqslant n-1\} = \{1, \ldots, n\}$.

Like in the proof of Theorem 4.3, given a digraph $K(D)$ satisfying i) and ii), we show that we can obtain, by transitions of M^a, a digraph $K(D')$, still satisfying i) and ii), and in which a given pair of consecutive elements $x_i \prec_{K(D)} x_{i+1}$, $0 \leqslant i < n-1$, have swapped positions. If such consecutive elements x_i and x_{i+1} are both sinks, then since their order has been imposed arbitrarily, they can be swapped without changing the digraph. Otherwise, we have to consider two cases.

If $\mathsf{rk}(x_{i+1}) > 1$, then $N^+(x_{i+1}) = N^+(x_i) \cup \{x_i\}$. The arc (x_{i+1}, x_i) can be reversed, by the application of the transition of M^a on the arcs (x_{i+1}, x_i) and (x_i, x_{i+1}). Indeed, the resulting digraph K' remains acyclic; K' is also w.e. since, on the one hand, vertices $x_i, x_{i+1}, \dots, x_{n-1}$ induce an acyclic tournament in K', by conditions i) and ii). On the other hand, any non-sink x_j, $0 \leq j < i$, is an out-neighbor of both x_i and x_{i+1}. Moreover, if $\mathsf{rk}(x_{i+1})$ was equal to 2 in $K(D)$ (and hence $\mathsf{rk}(x_i) = 1$), then in K' we may have $N^+(x_i) \neq R(x_i)$. However, it suffices to swap arcs out-going from x_{i+1}, the unique vertex of rank 1 in K', to x_i. The digraph obtained after these transformations satisfies conditions i) and ii), thus is equal to some $K(D')$; the vertices of $K(D')$ have the same ranks as in $K(D)$, with the exception of x_i and x_{i+1} which have swapped ranks.

When however $\mathsf{rk}(x_{i+1}) = 1$, we have that $\mathsf{rk}(x_i) = 0$. Since x_{i+1} is the unique vertex of rank 1, there must be an arc from x_{i+1} to a sink s (s can even be x_i) which can be removed in order to add the arc (x_i, x_{i+1}). After this first arc swap, continue changing all arcs (x_{i+1}, s) into (x_i, s), with s an arbitrary sink. The resulting digraph K' satisfies conditions i) and ii) and is equal to some $K(D')$; moreover, the vertices of $K(D')$ have the same ranks as in $K(D)$, with the exception of x_i and x_{i+1} which, as before, have swapped ranks.

In order to transform $K(D)$ into $K(H)$, apply the following procedure, starting from position $i = 0$ all the way to $i = n - 1$. If $y_i = x_j$, $(i < j \leqslant n - 1)$, then bring x_j to position i by iteratively reversing the arcs (x_j, x_{j-1}), (x_j, x_{j-2}), $\dots$,(x_j, x_i). Finally, change the out-going arcs of the unique vertex of rank 1 so that it has precisely the same out-neighborhood as it has in $K(H)$. □

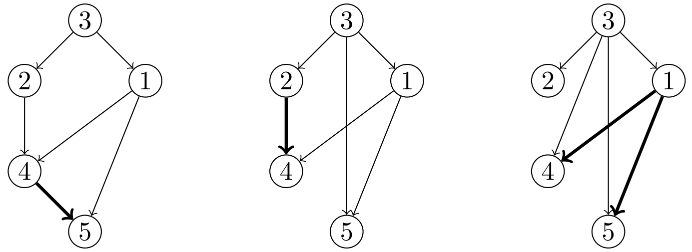
Figure 6 illustrates the transitions indicated by the above proof in order to pass between two digraphs in $\mathcal{W}_{5,6}$.

6 Critical remarks and future work

Although the Markov chains M, M^c and M^e are similar to the Markov chains of [9, 10], the proofs of their irreducibility are different and more involved. In the case of M, the fixed element which can be reached by a chain of transitions from every element D of $\mathcal{W}_n$ is the same as in [9], namely the totally disconnected digraph. However, the arcs of D must be removed in a particular order, respecting the weak extensionality of D. Second, on the one hand, in [10] the fixed element is an arbitrary digraph having a path as underlying graph, which cannot be the case for M^c or M^e since (weak) extensionality would be violated. On the other hand, our proof takes this fixed element to be an acyclic tournament on n vertices, ensuring that the proof proposed here can also show the irreducibility of (a slightly modified version of) the chain of [10]. Lastly, as already noted, the Markov chain M^a can be easily adapted to generate uniformly at random acyclic digraphs on a given number of labeled vertices and a given number of arcs, a result which we have not found in the literature.

Given this dual usability of the Markov chains considered here, and the fact that the acyclic tournament on n vertices (that is, the digraph isomorphic to the von Neumann numeral of n) is a rich structure in which many types of digraphs can be embedded, it would be interesting to give a general characterization of the classes of digraphs whose elements can be generated uniformly at random by these Markov chains.

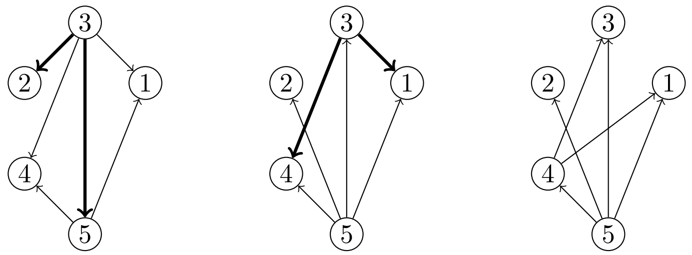
We regard the generation of digraphs on n vertices, possibly having directed cycles,



(a) $(4, 5) \leftrightarrow (3, 5)$

(b) $(2, 4) \leftrightarrow (3, 4)$

(c) $2 \prec 4 \prec \mathbf{5} \prec \mathbf{1} \prec 3$
$(1, 5) \leftrightarrow (5, 1)$
$(1, 4) \leftrightarrow (5, 4)$



(d) $1 \prec 2 \prec 4 \prec \mathbf{5} \prec \mathbf{3}$
$(3, 5) \leftrightarrow (5, 3)$
$(3, 2) \leftrightarrow (5, 2)$

(e) $1 \prec 2 \prec \mathbf{4} \prec \mathbf{3} \prec 5$
$(3, 4) \leftrightarrow (4, 3)$
$(3, 1) \leftrightarrow (4, 1)$

(f) $1 \prec 2 \prec 3 \prec 4 \prec 5$

Figure 6: The sequence of transitions of M^a that transforms $D \in \mathcal{W}_{5,6}$ (Fig. (a)) into $K(D) \in \mathcal{W}_{5,6}$ (Fig. (c)), and then into a digraph $K(H) \in \mathcal{W}_{5,6}$ (Fig. (f))

but deprived of distinct *bisimilar* vertices (the standard interpretation of a *hyperset* [1]) as the next natural step to take. It is interesting to study whether a Markov chain algorithm consisting of three basic operations: addition of an arc, removal of an arc, move of an arc, is irreducible for the class of such digraphs.

As already mentioned in the introduction, the problem of analyzing the mixing times [8] of our proposed Markov chains remains open; this is important from a practical viewpoint.

Finally, since a peculiarity of the sets treated in this paper is to be sets whose elements are themselves sets, it would be interesting to investigate what role our results can play in the theory of random sets ([11], elements taking as values subsets of some space), for example as a tool to generate such objects uniformly.

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Complex parameterization of triangulations on oriented maps

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Abstract

We consider here triangulations of oriented maps having a specified set S of vertices of degree different from 6 and some other vertices of degree 6. Such map can be described by specifying the relative positions between elements of S using Eisenstein integers. We first consider the case of 1 parameter, which corresponds to the Goldberg-Coxeter construction. Then we develop the general theory, the special case of positive curvature studied by Thurston and finally extend the theory to quadrangulations and some other cases. In the last section we expose application of parameterizations to the study of zigzags.

Keywords: Maps, graphs, Groups, parameterizations.

Math. Subj. Class.: 05C10, 57M20

1 Triangulations of oriented maps

By a $(\{v_1, \dots, v_m\}, k)$-map we denote a map on an oriented surface with faces of size k, v_i vertices of degree i and "map" being "sphere" (genus $g = 0$), "torus" ($g = 1$) or "oriented map of genus g". For example, a $(\{v_5 = 2, v_7 = 2, v_6\}, 3)$-torus denotes a triangulation with 2 vertices of degree 5, 7 and an unspecified number of vertices of degree 6. We will be mostly concerned with the description of $(v, 3)$-maps of genus g. Euler relation for them reads as

$$\sum_{j \geq 3} v_j(6 - j) = 12(1 - g). \tag{1.1}$$

The dual of $(\{v_5 = 12, v_6\}, 3)$-spheres are 3-regular plane graphs, whose faces are 5- or 6-gonal. Such graphs, named *fullerenes*, occur in chemistry [13] following the discovery of Buckminsterfullerene (also called truncated icosahedron, soccer ball) in 1985 [22].

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Formula (1.1) can be interpreted as a Gauss-Bonnet formula and $6 - j$ as the curvature of a vertex of degree j. A triangulation is said to be of *positive curvature* if all vertices have non-negative curvature. This implies that the possible vertex degrees belong to $\{3, 4, 5, 6\}$ and the v-vector satisfies $3v_3 + 2v_4 + v_5 = 12$ with v_6 unspecified. All 19 possibilities for (v_3, v_4, v_5) are given below:

$(0,0,12)$	$(0,1,10)$	$(0,2,8)$	$(0,3,6)$	$(0,4,4)$	$(0,5,2)$	$(0,6,0)$
$(1,0,9)$	$(1,1,7)$	$(1,2,5)$	$(1,3,3)$	$(1,4,1)$	$(2,0,6)$	$(2,1,4)$
$(2,2,2)$	$(2,3,0)$	$(3,0,3)$	$(3,1,1)$	$(4,0,0)$		

For the enumeration of 3-regular plane graphs with a specified *face vector* (p_i), i.e. number p_i of faces of size i the program `CPF` by T. Harmuth [2] is very efficient. Another little known program by the same author is `CGF` [19], which can enumerate 3-regular oriented maps of specified genus and face vector. The corresponding program for 4-regular plane graphs is `ENU` by O. Heidemeier [2]. All above mentioned programs are available from [1] and give by duality triangulations and quadrangulations.

The symmetry groups of fullerenes and other plane graphs of positive curvature were determined in [13, 5, 12, 7, 8, 6]. For a given group G of symmetry of a map, denote by $Rot(G)$ the subgroup of index 1 or 2 of G formed by the orientation preserving transformation. The *class of a group* G is the set of groups G' having $Rot(G') = G$. In Table 1 we give the possible groups of $(v, 3)$-spheres of positive curvature by their class $Rot(G)$, where we used Schoenflies nomenclature for point groups. For any class the number of vertices of positive curvature is finite and the number of vertices of degree 6 is unspecified. Since there is essentially only one 6-regular plane triangulation, one sees that the positions of the vertices of positive curvature allow to define the map. We want to encode the positions by complex Eisenstein numbers $z \in \mathbb{Z}[\omega]$ with $\omega = e^{i\pi/3}$.

In Section 2 we describe the simple cases of 1 or 2 parameters. The case of 1 parameter corresponds to the Goldberg-Coxeter construction [9]. In Section 3 we first explain the general theory of complex parameterization of $(v, 3)$-maps on oriented surfaces. Then we explain Thurston's theory [28] which gives stronger results for the case of spheres of positive curvature. Finally we explain the extension to $(v, 4)$-maps, self-dual spheres and $(v, 6)$-spheres.

Applications to zigzags are considered in Section 4. A very basic application of parameterization is for generating maps efficiently provided that the number of parameters is not too high. Another application considered in [11] is for eigenvalue estimation where it was proved that for any interval $[a, b] \subset [-3, 3]$ there is a finite number of graphs of positive curvature having no eigenvalue in I.

We choose to emphasize Eisenstein parameter description but it is of course possible to consider descriptions by integral parameters. This is done in [16, 15] for fullerenes and this allows to write parameterizations for each group, not just rotation subgroup. Another real parameter descriptions, by so called *dihedral angles*, is developed in [26], but it is more suited for describing manifolds than graphs.

2 One and two parameter constructions

2.1 The Goldberg-Coxeter construction

The Goldberg-Coxeter construction takes a $(v, 3)$-map $\mathcal{M}$, two integers $k, l \geq 0$ and returns another $(\{v, v_6\}, 3)$-map $GC_{k,l}(\mathcal{M})$ of the same genus by adding triangles and vertices of

$(\{v_3 = 4, v_6\}, 3)$-spheres

class	all group	# param
D_2	D_2, D_{2d}, D_{2h}	2
T	T, T_d	1

$(\{v_4 = 6, v_6\}, 3)$-spheres

class	all group	# param
C_1	C_1, C_s, C_i	4
C_2	C_2, C_{2v}, C_{2h}	3
D_2	D_2, D_{2d}, D_{2h}	2
D_3	D_3, D_{3d}, D_{3h}	2
D_6	D_6, D_{6h}	1
O	O, O_h	1

$(\{v_5 = 12, v_6\}, 3)$-spheres

class	all group	# param
C_1	C_1, C_s, C_i	10
C_2	C_2, C_{2h}, C_{2v}	6
C_3	C_3, C_{3h}, C_{3v}	4
D_2	D_2, D_{2h}, D_{2d}	4
D_3	D_3, D_{3h}, D_{3d}	3
D_5	D_5, D_{5h}, D_{5d}	2
D_6	D_6, D_{6h}, D_{6d}	2
T	T, T_h, T_d	2
I	I, I_h	1

Table 1: The classes of symmetry groups of $(\{v_a, v_6\}, 3)$-sphere.

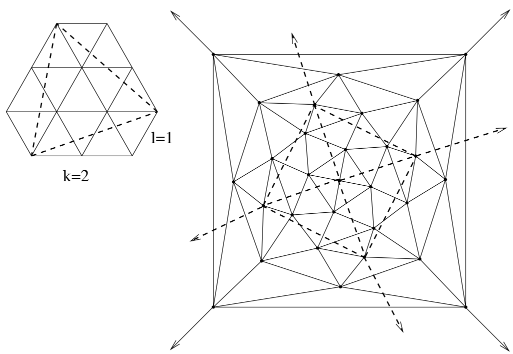


Figure 1: The construction of $GC_{2,1}(Octahedron)$.

degree 6. It works by breaking the triangles of $\mathcal{M}$ into smaller triangles and gluing the pieces together in order to get another triangulation. See an example on Figure 1 and [9] for more details.

If $\mathcal{M}$ has n_T triangles then $GC_{k,l}(\mathcal{M})$ has $n_T(k^2 + kl + l^2)$ triangles. Since $k^2 + kl + l^2 = |k + l\omega|^2$ it makes sense to associate the Eisenstein integer $z = k + l\omega$ to the pair k, l. The parameter symmetry $z \mapsto z\omega^r$ does not change the resulting map. All the cases of 1 parameter in Table 1 are described by the Goldberg-Coxeter construction of one plane graph. For example, if a $(\{v_3 = 4, v_6\}, 3)$-, $(\{v_4 = 6, v_6\}, 3)$-, $(\{v_5 = 12, v_6\}, 3)$-sphere is of symmetry (I, I_h), (O, O_h) or (T, T_d) then it is of the form $GC_{k,l}(Icosahedron)$, $GC_{k,l}(Octahedron)$ or $GC_{k,l}(Tetrahedron)$ [14, 9]. Additionally, $(\{v_4 = 6, v_6\}, 3)$-spheres of symmetry (D_6, D_{6h}) are obtained as $GC_{k,l}(Prism_6^*)$.

2.2 One case of 2 parameters: $(\{v_5 = 12, v_6\}, 3)$-spheres of symmetry D_5

The 5-fold axis of such a sphere has to pass through a vertex of degree 5. There are 5 vertices of degree 5 around it; so, by 5-fold symmetry, 1 complex parameter is needed to

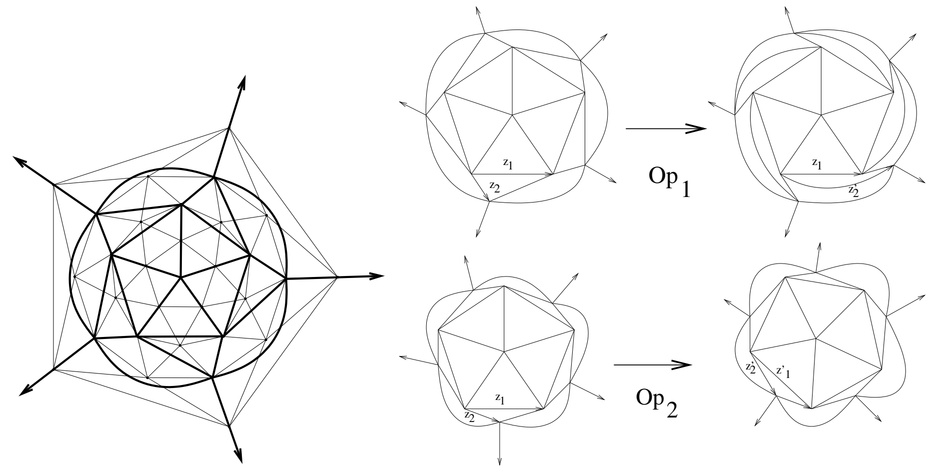


Figure 2: The parameterization of fullerenes of symmetry D_5, D_{5d} or D_{5h} in term of $(z_1, z_2) \in \mathbb{Z}[\omega]^2$ and two parameter operations.

describe them. Around those 5 vertices, there are 5 more vertices, so one more parameter is needed and then the last vertex is uniquely defined.

If one applies the following operations to parameters

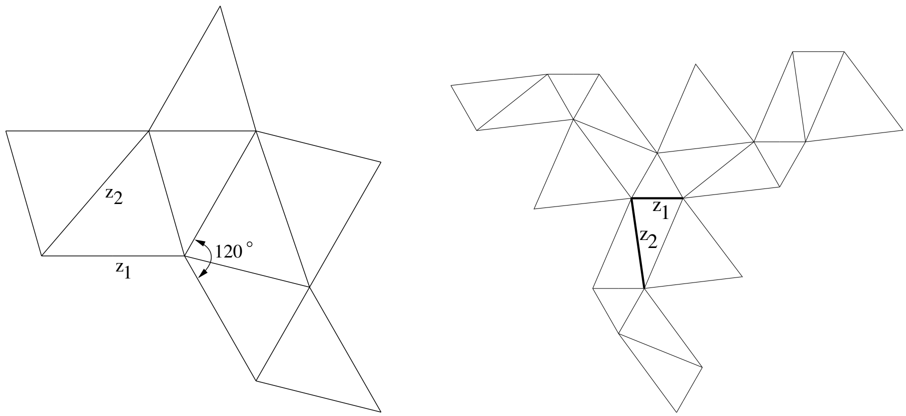
- Operation 1: $(z_1, z_2) \mapsto (z_1, z_1 + z_2)$
- Operation 2: $(z_1, z_2) \mapsto (z_1 + \omega^2 z_2, z_1 - z_2)$
- Operation 3: $(z_1, z_2) \mapsto (z_1, z_2)\omega^r$

then one obtains the same sphere as a result. The group generated by those operations is named *monodromy group*. The number of triangles of S is expressed as $q(z_1, z_2) = 10\{z_1\overline{z_1} + \frac{z_1\overline{z_2} - \overline{z_1}z_2}{\omega - \overline{\omega}}\}$. So, for a given pair (z_1, z_2) a sphere may not exist, for example, if $q(z_1, z_2) < 0$.

2.3 Other two parameters descriptions

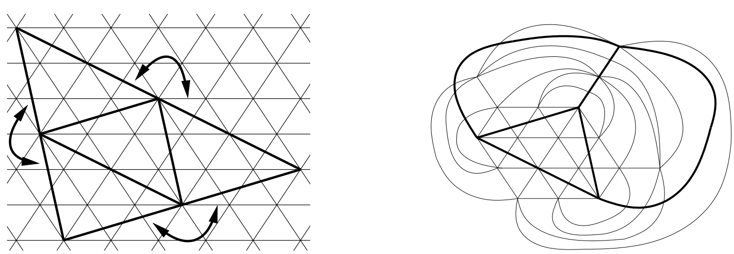
Of course what has been done for fullerene of class D_5 applies just as well for fullerenes of class D_6 and similar simple description are possible for the remaining 2 parameter cases of Table 1 (See Figure 3 for two such cases).

For the $(\{v_3 = 4, v_6\}, 3)$-spheres a very explicit two parameter description is given in Figure 4. Geometrically this corresponds to the fact that any $(\{v_3 = 4, v_6\}, 3)$-sphere is obtained as the quotient of a $(\{v_6\}, 3)$-torus by a group of order 2 leaving invariant exactly 4 vertices. Clearly, the monodromy group is $\mathrm{PSL}(2, \mathbb{Z})$ and the number of triangles is expressed as $\frac{4}{\omega - \overline{\omega}}(z_1\overline{z_2} - \overline{z_1}z_2)$. This description was used in [21] to compute the eigenvalues of dual $(\{v_3 = 4, v_6\}, 3)$-spheres.



$(\{v_4 = 6, v_6\}, 3)$-spheres of symmetry D_2, D_{2h}, D_{2d} $(\{v_5 = 12, v_6\}, 3)$-spheres of symmetry T, T_h and T_d

Figure 3: Two parameters description of two classes of spheres with positive curvature represented on the plane.



4 triangles in $\mathbb{Z}[\omega]$ The triangulation

Figure 4: Description of $(\{v_3 = 4, v_6\}, 3)$-spheres.

3 Parameterization of maps on oriented surfaces

3.1 The general case

The general Eberhard problem is for a given $g \geq 0$ and vector $(v_i)_{3\leq i\leq m, i\neq 6}$ satisfying $\sum_{i=3}^{m}(6-i)v_i = 12(1-g)$ to determine the set $P(v,g)$ of values of v_6 for which there exist a $(\{v, v_6\}, 3)$-oriented map of genus g. It is proved in [20] that $P(v,g)$ is empty only in the case $g = 1$ and $v = \{v_5 = 1, v_7 = 1\}$ and the exact determination of $P(v,g)$ is an active subject of research.

Thus for a given v-vector it is interesting to consider how one can parametrize the $(\{v, v_6\}, 3)$-oriented maps of genus g. Let us call $\mathcal{M}$ such a map, $\widetilde{\mathcal{M}}$ its universal cover, Γ its fundamental group and S the set of vertices of degree different from 6. By adding edges one by one, we can build a triangulation T on $\mathcal{M}$ having S as vertex set. Note that the degree of v in T is a priori not related to the degree of the corresponding vertex in $\mathcal{M}$. Naturally many triangulations are possible and they are mapped on the universal cover $\widetilde{\mathcal{M}}$ to Γ-invariant triangulations. We will see below that one can build a parameterization by Eisenstein integers from a triangulation.

Let us a take a triangulation T and encode it combinatorially. A *directed edge* is an edge from a vertex to another vertex. Every edge e is composed of a directed edge $\overrightarrow{e}$ and its *reversal* $r(\overrightarrow{e})$. The *next operator* n maps a directed edge to the next one in clockwise order around the vertex v in which it is contained. A triangulation T is described by the operators n and r acting on the set of directed edges $DE(T)$. In particular the vertices, edges and faces of T correspond to the orbits of n, r and nr. For a given vertex v of T we denote by $\tilde{v}$ the corresponding vertex in the corresponding $(\{v, v_6\}, 3)$-map. Edges and faces of T will have no direct analogs but homology classes will be mapped to homology classes.

We associate an Eisenstein integer $z_{\overrightarrow{e}} \in \mathbb{Z}[\omega]$ to any directed edge $\overrightarrow{e}$ of T. For any face $f = \{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ we impose the relation

$$z_{\overrightarrow{e_1}} + z_{\overrightarrow{e_2}} + z_{\overrightarrow{e_3}} = 0.$$

If we have an edge $e = \{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ then we impose the consistency relation

$$\omega^{\alpha(\overrightarrow{e_1})} z_{\overrightarrow{e_1}} + \omega^{\alpha(\overrightarrow{e_2})} z_{\overrightarrow{e_2}} = 0.$$

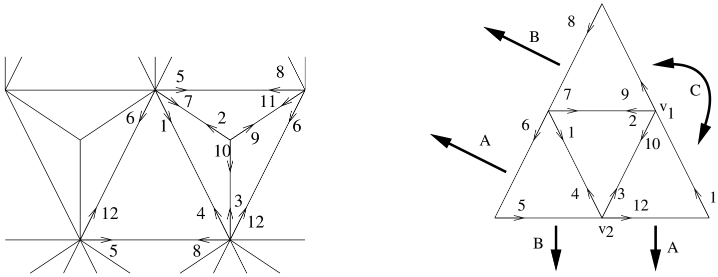
For any vertex v containing the directed edges $(\overrightarrow{e_i})_{1\leq i\leq m}$ we have the relation

$$6 - \deg(\tilde{v}) \equiv \sum_{i=1}^{m} \alpha(\overrightarrow{e_i}) - \alpha(r(\overrightarrow{e_i})) \pmod{6} \tag{3.1}$$

with $\deg(\tilde{v})$ the degree of $\tilde{v}$. We write $D(v) = \deg(\tilde{v})$. If the triangulation T is of genus g then the first homology group $H_1(T)$ is $\mathbb{Z}^{2g}$. For any cycle C composed of directed edges $\{\overrightarrow{e_1}, \dots, \overrightarrow{e_m}\}$ with $\overrightarrow{e}_{i+1} = r(nr)^{\pm 1}(\overrightarrow{e_i})$ we define the cycle sum

$$I(C,\alpha) = \sum_{j} \alpha(\overrightarrow{e_j}) - \alpha(r(\overrightarrow{e_j})). \tag{3.2}$$

This sum depends on the chosen element of the homology class and defines how the orientation is shifted after one moves along C. If one adds 1 to $\alpha(\overrightarrow{e})$ and $\alpha(r(\overrightarrow{e}))$ then the



A $(\{v_3 = 1, v_9 = 1, v_6\}, 3)$-torus | The parameterization

Figure 5: A $(\{v_3 = 1, v_9 = 1, v_6\}, 3)$-torus and its parameterization. The identifications along A, B and C yields the equalities $\omega z_{12} + z_6 = 0$, $\omega z_5 + z_8 = 0$ and $z_9 - z_{11} = 0$. We have $\deg(\tilde{v_1}) = 3$ and $\deg(\tilde{v_2}) = 9$.

resulting map $M(T, D, I, \alpha, z)$ does not change. The same happens if one adds 1 to $\alpha(\overrightarrow{e})$ for $\overrightarrow{e}$ in a face F of the triangulation. In fact if we choose $2g$ basic cycles C_i of T then any two vectors α, α' satisfying Equation (3.1) and $I(C, \alpha) \equiv I(C, \alpha') \pmod 6$ differ by repeated application of above two operations. Equation (3.1) and the cycle sums I allow to find a corresponding function α if it exists, which is not always the case. Henceforth the data of T, D and I determine the class of maps that one can obtain. In Figure 5 we give an example of a parameterization for $(\{v_3 = 1, v_9 = 1, v_6\}, 3)$-torus.

Theorem 3.1. *The parameter spaces of* $(\{v_1, \dots, v_m\}, 3)$*-oriented maps of genus* g *have dimension* $\sum_{3 \leq i \neq 6} v_i - 1 + 2g$ *if all faces have size divisible by* 6 *and* $\sum_{3 \leq i \neq 6} v_i - 2 + 2g$ *otherwise.*

Proof. Let us take such a map and build a triangulation T on it. Let us write $M = \sum_{3 \leq i \neq 6} v_i$. We then construct a spanning tree of $M - 1$ edges on the set of vertices of degree different from 6. Since the map is of genus g we have a basis of $2g$ cycles of the group $H_1(G)$. We add $2g$ edges to the spanning tree and the remaining edges define a tree in the dual map. Once we have defined the position of the $M - 1 + 2g$ edges, we have defined the triangulation uniquely because all other edges can be assigned iteratively. If one of the vertices has a degree not divisible by 6 then its position is defined uniquely once all its neighbors are known and so the dimension decreases by 1 in that case. No other relation exists since one can perturb the remaining parameter and still obtain some corresponding maps. □

Let us call $m(v, g)$ the dimension in the above theorem.

For a given parameterization (T, D, I) we denote by $q_T(z)$ the number of triangles of the obtained triangulation. The number of triangles contained in a face defined by $f = \{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ is

$$q_f(z) = \frac{1}{\omega - \overline{\omega}}(z_{\overrightarrow{e_1}} \overline{z_{\overrightarrow{e_2}}} - \overline{z_{\overrightarrow{e_1}}} z_{\overrightarrow{e_2}}).$$

The function q_T counting the total number of triangles is thus $q_T = \sum q_f$ and it is an Hermitian form. From this, one can deduce that for a fixed v the number of $(\{v, v_6\}, 3)$-oriented maps of genus g with at most n triangles grows like $O\left(n^{m(v,g)}\right)$. Note that [27] gives the more precise estimate $O\left(n^9\right)$ for the number of fullerenes with exactly n triangles. But we cannot have such a statement in the general case because for example $(\{v_3 = 4, v_6\}, 6)$-spheres exist only if n is divisible by 4.

Conjecture 3.2. *If all vertices have degree divisible by* 6 *then the signature of* q_T *is*

$$(n_+, n_-, n_0) = \left(g + \sum_i v_i Fr\left(\frac{6-i}{6}\right), g, m(v,g) - n_+ - n_-\right),$$

otherwise, the signature is

$$(n_+, n_-, n_0) = \left(g - 1 + \sum_i v_i Fr\left(\frac{6-i}{6}\right), m(v,g) - n_+, 0\right)$$

with $Fr(x) = x - \lfloor x \rfloor$.

The conjecture is proved in the case $g = 0$ in [24] and has been checked for the regular maps of genus at most 15 from [3].

For a given triangulation T of a $(\{v, v_6\}, 3)$-oriented map, we choose m independent parameters $z_1, \ldots, z_m \in \mathbb{Z}[\omega]$. The condition of existence of the map $M(T, D, I, \alpha, z)$ is that $q_f(z) > 0$ for all face f of T and this defines the *realizability space*. The *limit realizability space* is the same space with $\mathbb{Z}[\omega]$ replaced by $\mathbb{C}$. The limit realizability space defines a set S in the cone $q_T > 0$. If one approach through a generic point of the boundary of S, which is not in the boundary of $q_T > 0$ then one can rearrange the triangulation and get another triangulation T'. The vertex degree v in T may change but the degree of $\tilde{v}$ remain the same. Similarly the cycles C are mapped to cycles C' and $I(C', \alpha') = I(C, \alpha)$. Hence D and I are intrinsic to the class of triangulations obtained by rearrangements. Moreover, the parameter set (z'_i) of (T', D, I, α') can be expressed linearly in term of the parameter set (z_i) of (T, D, I, α).

For two quadruples (T, D, I, α) and (T', D, I, α') of parameter set (z_i), (z'_i) we say that they are *equivalent* if there is a mapping from T to T' preserving D and I such that (z'_i) can be expressed linearly from (z_i). Such an equivalence preserve edge length, triangle areas and can be extended to adjacent triangulations of T and correspondingly T'. Note that some non-trivial equivalence can have $T = T'$; this is the case for the 2 parameter description considered in Section 2 for which one triangulation suffices. The group of such transformation is the *monodromy group* and is a subgroup of $\mathrm{GL}_m(\mathbb{Z}[\omega])$ leaving invariant the form q_T. It may be that any two triples (T, D, I), (T', D, I) are related by a sequence of such transformation but we have no proof of it and we do not know a counter-example.

3.2 Thurston theory for maps of positive curvature

Let us take one triple (v_3, v_4, v_5) among the 19 triples of possible curvature. By Theorem 3.1 the number of complex parameters needed to describe $(\{v_3, v_4, v_5, v_6\}, 3)$-spheres is

$$m = m(\{v_3, v_4, v_5\}, 0) = v_3 + v_4 + v_5 - 2$$

The monodromy group is denoted by $M(\{v_3, v_4, v_5\}, 3, 0)$ and it preserves the form q, which is of signature $(1, m-1, 0)$. This class of monodromy groups was defined and enumerated in [4, 23, 28] and in particular they are discrete groups. The 19 possible (v_3, v_4, v_5) cases are part of the 94 cases determined there and the form q is the intersection form on $H^1(S^2 - V, L)$ with V a set of $m + 2$ points and L a line bundle on $S^2 - V$.

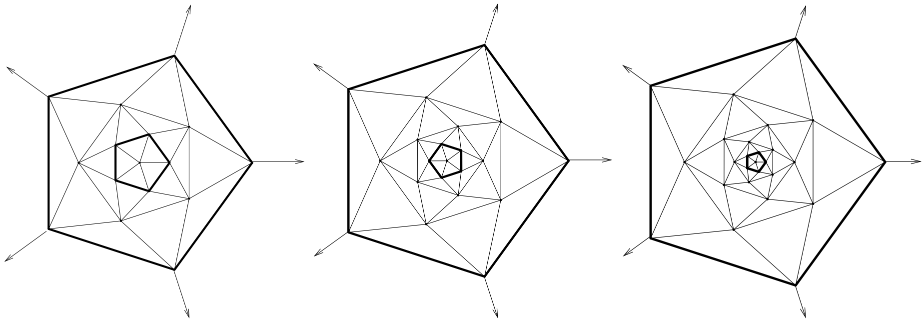


Figure 6: Three fullerenes of symmetry D_5 or more.

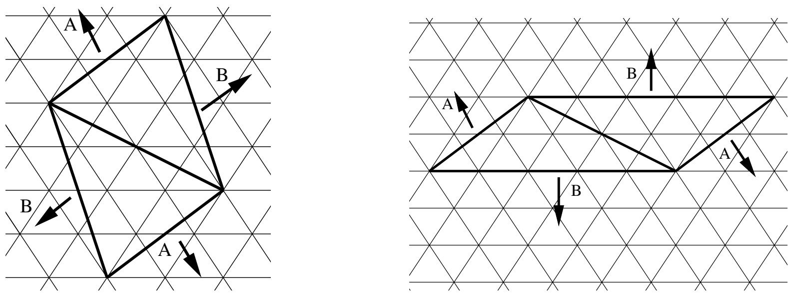


Figure 7: Two representations of a $(\{6\}, 3)$-torus.

In [28] it is proved that if $z \in \mathbb{Z}[\omega]^m$ and $q(z) > 0$ then there exists $f \in M(\{v_3, v_4, v_5\}, 3, 0)$ such that $f(z)$ is realizable as a $(\{v_3, v_4, v_5, v_6\}, 3)$-sphere. Thus $\mathbb{H}^m \cap \mathbb{Z}[\omega]^m$ up to the action of the monodromy group is a parameter space for the $(\{v_3, v_4, v_5, v_6\}, 3)$-spheres. As a consequence, the quotient

$$\mathbb{H}^m / (\mathbb{R}_{>0} \times M_3(\{v_3, v_4, v_5\}, 3, 0))$$

is of finite covolume because the number of $(\{v_3, v_4, v_5, v_6\}, 3)$-spheres is finite for any fixed number of triangles.

In [28] a characterization of the manifolds admitting a cocompact quotient is given. None of those corresponding to $(\{v_3, v_4, v_5, v_6\}, 3)$-spheres are compact. Each of the direction of non-compacity correspond to a partition of the vertices of non-zero curvature into two sets S_i for $i = 1, 2$ each having v_q^i vertices of degree q and satisfying to $3v_3^i + 2v_4^i + v_5^i = 6$. Geometrically those are nanotubes, that is we have two caps C_i with (v_3^i, v_4^i, v_5^i) vertices separated by a number of rings of vertices of degree 6 (see Figure 6).

3.3 Extensions and other cases of parameter descriptions

One extension that can be done relatively simply is to consider vertices of degree 1 or 2. For example, the $(\{v_2 = 3, v_6\}, 3)$-spheres are obtained by applying the Goldberg-Coxeter construction to the sphere reduced to a cycle of length 3 [18].

One example that does not fit exactly into the above scheme is the description of $(\{v_6\}, 3)$-torus by two parameters. It is done by writing down a parallelogram on the plane and identifying the sides. Since all vertices are equivalent in the torus, we have to choose one vertices of degree 6 used in the construction. In Figure 7 we show two equivalent representation of a $(\{v_6\}, 3)$-torus. One of the representation has one horizontal line in the fundamental domain; it is actually always possible to have such a representation and this is the basis of a 3 integral parameters construction [25].

Goldberg-Coxeter construction and complex parameterizations for $(\{v_1, v_2, v_3\}, 6)$-spheres are derived in [6]. The method was to apply the truncation operation to each vertex in order to get a $(\{v_2, v_4, v_6\}, 3)$-sphere for which existing theory could be applied.

Yet another extension is for quadrangulations. The theory extends without difficulty, we are still dealing with triples (T, D, I) but the ring of Eisenstein integers is replaced by the ring of Gaussian integers. For example, for $(\{v_3 = 8, v_4\}, 4)$-sphere, the number of Gaussian integer parameters is 6 and the number for the classes (O, O_h), (D_4, D_{4d}, D_{4h}), (D_3, D_{3d}, D_{3h}), (D_2, D_{2d}, D_{2h}), (C_2, C_{2h}, C_{2v}), (C_1, C_s, C_i), respectively is 1, 2, 2, 3, 4, and 6. As a byproduct of this parameterization, we also get a method for parametrizing self-dual plane graphs with faces of size 3 or 4, see [10] for details.

Finally note that all of the above can be specialized to get parameter description of families of maps having a specific symmetry group G provided that G contains only orientation preserving mappings. This is because the symmetry conditions are translated into linear equalities in the parameters.

4 Zigzag

In an oriented map a *zigzag* is a circuit of edges such that two consecutive share a face and vertex but three do not share a face.

4.1 The Goldberg-Coxeter case

For a triangulation $\mathcal{M}$ we define in [9] a permutation group $Mov(\mathcal{M})$ and two elements L and R. If $gcd(k, l) = 1$ then the lengths of zigzags of $GC_{k,l}(\mathcal{M})$ is computed from the cycle structure of the element $L \odot_{k,l} R$ of $Mov(\mathcal{M})$. This element satisfies the defining relations $L \odot_{1,0} R = L$, $L \odot_{0,1} R = R$ and

$$\begin{cases} L \odot_{k,l} R &= \quad L \quad \odot_{k-ql\ ,\ l} \quad RL^q & \text{if } k - ql \geq 0, \\ L \odot_{k,l} R &= R^q L \quad \odot_{k\ ,\ l-qk} \quad R & \text{if } l - qk \geq 0. \end{cases}$$

If $gcd(k, l) = m > 1$ then every zigzag of $GC_{k/m,l/m}(\mathcal{M})$ corresponds to m zigzags of $GC_{k,l}(\mathcal{M})$ of length multiplied by m. The same method applies as well for 4-regular plane graphs and their *central circuit*, see [9] for an exhaustive description.

4.2 The case of $(\{v_3 = 4, v_6\}, 3)$-spheres

All zigzags of $(\{v_3 = 4, v_6\}, 3)$-spheres are simple and the vector enumerating their lengths is of the form

$$(4s_1)^{m_1}, (4s_2)^{m_2}, (4s_3)^{m_3} \quad \text{with} \quad s_i, m_i \in \mathbb{N} \text{ and } s_i m_i = \frac{n}{4}.$$

This was first established in [17] but there is another way to establish it: Any $(\{v_3 = 4, v_6\}, 3)$-sphere is obtained as a quotient of a $(\{v_6\}, 3)$-torus by a group of order 2 formed

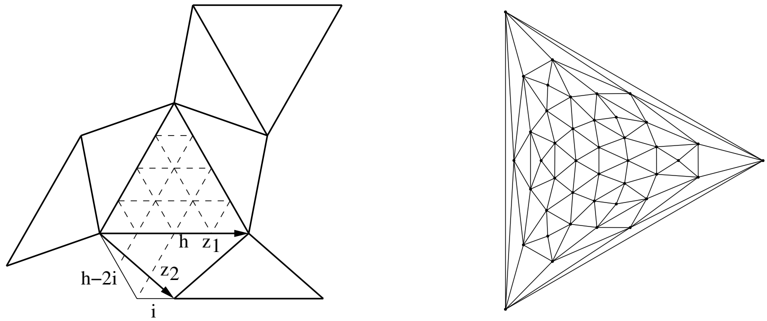


Figure 8: The parameter description of a family of $(\{v_4 = 6, v_6\}, 3)$-sphere with simple zigzags, $z_1 = h\omega$ and $z_2 = h - 2k + k\omega$ and the case $(h, k) = (4, 1)$.

by inversion. The four vertices of degree 3 come from the four invariant vertices of the torus. All zigzags of a $(\{v_6\}, 3)$-torus are partitioned into three parallel classes that cover the vertex set. All the zigzags in a parallel class are of the same length and when passing to the quotient the parallel classes are preserved hence the above result is proved.

The same argument applies for $(\{v_2 = 4, v_4\}, 4)$-spheres and their central circuits [8].

4.3 Other classes

For other classes of maps with more parameters the structure is more complicated and it seems very difficult to obtain simple description of the zigzags of fullerenes. However, for $(\{v_4 = 6, v_6\}, 3)$-spheres we have a simple conjecture for the ones with simple zigzags:

Conjecture 4.1. *All $(\{v_4 = 6, v_6\}, 3)$-spheres with only simple zigzags are:*

- $GC_{k,k}(Octahedron)$ *and*
- *the family of graphs with parameters (m, k) with $n = 4h(2h - 3k)$ triangles whose parameter description is given in Figure 8. The vector enumerating zigzag length is*

$$z = (6h - 6k)^{3h-3k}, (6h)^{h-2k}, (12h - 18k)^k$$

They have symmetry O_h if $k = 0$, D_{6h} if $h = 3k$ and D_{3d} otherwise.

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On the minimum rainbow subgraph number of a graph

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Abstract

We consider the MINIMUM RAINBOW SUBGRAPH problem (MRS): Given a graph G whose edges are coloured with p colours. Find a subgraph $F \subseteq G$ of minimum order and with p edges such that each colour occurs exactly once. This problem is NP-hard and APX-hard.

For a given graph G and an edge colouring c with p colours we define the *rainbow subgraph number* $rs(G, c)$ to be the order of a minimum rainbow subgraph of G with size p. In this paper we will show lower and upper bounds for the rainbow subgraph number of a graph.

Keywords: Edge colouring, rainbow subgraph.

Math. Subj. Class.: 05C15, 05C35

1 Introduction and motivation

We use [2] for terminology and notation not defined here and consider finite and simple graphs only.

Our research was motivated by the following problem from bioinformatics. The problem data consist in a set $\mathcal{G}$ of p *genotypes* $g_1, g_2, \ldots, g_p$ corresponding to p individuals in a population. Each genotype g is a vector with entries in $\{0, 1, 2\}$. Each position where a 2 appears is called *ambiguous* position. For a genotype g we have to determine a pair of haplotypes h_P and h_M (h_P stands for the paternal haplotype and h_M stands for the maternal haplotype), which are binary vectors such that $g = h_P \oplus h_M$.

Given two haplotypes h' and h'', their sum is defined as the vector $g = h' \oplus h''$ with $g[i] = 0$, if $h'[i] = h''[i] = 0$, $g[i] = 1$, if $h'[i] = h''[i] = 1$ and $g[i] = 2$, if $h'[i] \neq h''[i]$.

We say that a set $\mathcal{H}$ of haplotypes *resolves* $\mathcal{G}$ if for every $g \in \mathcal{G}$ there exist $h_1, h_2 \in \mathcal{H}$ such that $g = h_1 \oplus h_2$. Given a set $\mathcal{G}$ of genotypes, the *haplotyping problem* consists

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in finding a set $\mathcal{H}$ of haplotypes that resolves $\mathcal{G}$. In the $Pure\ Parsimony\ Haplotyping\ problem$ ($PPH\ problem$) we are interested in finding a set $\mathcal{H}$ of smallest possible cardinality. If each genotype has at most k ambiguous positions, then we denote this problem by PPH(k). The $PPH\ problem$ has been studied in ([3],[4],[7],[9]).

Matos Camacho et al. [8] have shown that the PPH(k) can be transformed to a graph problem, the MINIMUM RAINBOW SUBGRAPH problem (MRS). Note that this edge-colouring need not be proper.

Definition 1.1 (Rainbow subgraph)**.**
Let G be a graph with an edge-colouring. A subgraph H of G is called *rainbow subgraph* if H does not contain two edges of the same colour.

Definition 1.2 (Minimum Rainbow Subgraph problem (MRS))**.**
Given a graph G, whose edges are coloured with p colours, find a subgraph $F \subseteq G$ of minimum order and with p edges such that each colour occurs exactly once.

For a set $\mathcal{G}$ of p genotypes $g_1, g_2, \ldots, g_p$ we will use p colours $1, 2, \ldots, p$. For each haplotype we introduce a vertex. If two haplotypes h' and h'' resolve a genotype g_i ($g_i = h' \oplus h''$), then the corresponding vertices will be joined by an edge which receives colour i. If a genotype is resolved by two identical haplotypes, then the corresponding vertex is joined by an edge which is called a $loop$.

In this way we construct a graph G, whose edges are coloured with p colours. Note that this is a proper edge colouring (no vertex is incident with two edges of the same colour), since a haplotype h can be used at most once in a pair of haplotypes, which resolves a genotype g. Furthermore, every set $\mathcal{H}$ of haplotypes that resolves $\mathcal{G}$ corresponds to a rainbow subgraph F of G.

It has been shown in [8] that a graph G containing loops can be transformed into a graph G' without loops. Hence in the following we may assume that all graphs have no loops.

Matos Camacho et al. [8] proved the MRS problem to be NP-hard and APX-hard. In [5] it has been shown that the MRS problem remains NP-hard and APX-hard even for graphs with maximum degree 2.

Remark: If we do not consider edge colourings, the analogous problem is known as the $(t, f(t))$ dense subgraph problem ($(t, f(t))$-DSP), which asks whether there is a t-vertex subgraph of a given graph G which has at least $f(t)$ edges. When $f(t) = \binom{t}{2}$, $(t, f(t))$-DSP is equivalent to the well-known t-clique problem (cf. [1]).

2 Lower bounds for the rainbow subgraph number

Definition 2.1. Let G be a graph and c be its edge colouring with p colours. The *rainbow subgraph number* of G (with respect to the colouring c) is defined as the order of its minimum rainbow subgraph of size p, and denoted by $rs(G, c)$ (or $rs(G)$, when the colouring c is clear from the context).

Improved lower bounds for the rainbow subgraph number $rs(G)$ will be of major importance for the design of approximation algorithms with better approximation ratios for the MRS problem (cf. [8, 5]). So far nothing better than the trivial lower bound $rs(G) \geq \frac{2p}{\Delta(G)}$ is known. We can improve this lower bound by counting the number of distinct colours among all edges incident to a vertex.

Definition 2.2. Given an edge colouring of a graph G with colours $1, 2, \ldots, p$, we define $c(e) = i$, if the edge e has colour i for $1 \leq i \leq p$.
Let $cd(v)$ (*colour degree*) denote the number of distinct colours among all edges incident to the vertex v and let $cd(i) = \max\{cd(v) \mid v \in V(G)$ has an incident edge with colour $i\}$ be the *maximum colour degree* for every colour $i, 1 \leq i \leq p$.

Using the maximum colour degrees for all colours we can show the following improved lower bound.

Proposition 2.3. *Let G be a graph, whose edges are coloured with p colours. Then*

$$rs(G) \geq \sum_{i=1}^{p} \frac{2}{cd(i)} \geq \frac{2p}{\Delta(G)}.$$

Proof. Let F be a minimum rainbow subgraph of order $k = rs(G)$. Then

$$rs(G) = k = \sum_{v \in V(F)} \frac{d_F(v)}{d_F(v)} = \sum_{e=uw, e \in E(F)} \frac{1}{d_F(u)} + \frac{1}{d_F(w)} \geq \sum_{i=1}^{p} \frac{2}{cd(i)} \geq \frac{2p}{\Delta(G)}.$$

□

The following example shows that this bound is sharp and improves the lower bound of $\frac{2p}{\Delta(G)}$ significantly.

Example 2.4. For $p \geq 4$ and $\Delta \geq 2$ let $G = K_{1,\Delta} + C_{p-1}$ (where $G + H$ denotes the disjoint union of two graphs G and H). All edges of the cycle C_{p-1} are coloured distinctly, say with colours $1, 2, \ldots, p-1$, and all edges of $K_{1,\Delta}$ are coloured with colour p. Then $rs(G) = p + 1 = p - 1 + 2 = \sum_{i=1}^{p} \frac{2}{cd(i)} > \frac{2p}{\Delta(G)}$.

We can further improve this lower bound by counting the number of distinct colours among all edges incident to the endvertices of an edge. For this purpose we define $q(i) = \min\{\frac{1}{cd(u)} + \frac{1}{cd(w)} \mid uw \in E(G)$ and $c(uw) = i\}$.

Proposition 2.5. *Let G be a graph, whose edges are coloured with p colours. Then*

$$rs(G) \geq \sum_{i=1}^{p} q(i) \geq \sum_{i=1}^{p} \frac{2}{cd(i)} \geq \frac{2p}{\Delta(G)}.$$

Proof. Let F be a minimum rainbow subgraph of order $k = rs(G)$. For every colour $i, 1 \leq i \leq p$, let $u_i w_i$ be an edge such that $\frac{1}{cd(u_i)} + \frac{1}{cd(w_i)} = q(i)$. If $uw \in E(F)$ is an edge with $c(uw) = i$, then $\frac{1}{cd(u)} + \frac{1}{cd(w)} \geq \frac{1}{cd(u_i)} + \frac{1}{cd(w_i)} = q(i) \geq 2 \cdot \frac{1}{\max\{cd(u_i), cd(w_i)\}} \geq \frac{2}{cd(i)}$.
Therefore,

$$rs(G) = k = \sum_{v \in V(F)} \frac{d_F(v)}{d_F(v)} = \sum_{e=uw, e \in E(F)} \frac{1}{d_F(u)} + \frac{1}{d_F(w)} \geq$$

$$\sum_{i=1}^{p} q(i) \geq \sum_{i=1}^{p} \frac{2}{cd(i)} \geq \frac{2p}{\Delta(G)}.$$

□

The following example shows that this bound is sharp and improves the previous two lower bounds significantly.

Example 2.6. Let $G \cong K_{1,p}$ for some $p \geq 2$. Let the edges of G be coloured with p colours. Then $cd(i) = p$ and $q(i) = 1 + \frac{1}{p}$ for $1 \leq i \leq p$. Thus $rs(G) = p + 1 = p \cdot (1 + \frac{1}{p}) = \sum_{i=1}^{p} q(i) > 2 = p \cdot \frac{2}{p} = \sum_{i=1}^{p} \frac{2}{cd(i)} = \frac{2p}{\Delta(K_{1,p})}$.

3 Upper bounds for the rainbow subgraph number

First observe that the trivial upper bound $rs(G) \leq 2p$ is achieved if the rainbow subgraph F is a matching. This upper bound has been improved towards $rs(G) \leq 2p + 1 - \Delta(G)$ by Koch [6] for properly edge-coloured graphs and this bound is sharp. For instance, let $G = K_{1,\Delta} + (p - \Delta)K_2$, where $p \geq \Delta$, and all edges of G are coloured distinctly. Then $rs(G) = 2p + 1 - \Delta(G)$.

Similar to Brooks' Theorem (cf. [2]) we can characterize all graphs achieving this bound.

Theorem 3.1. *Let G be a graph with maximum degree $\Delta \geq 2$, whose edges are properly coloured with p colours. If $rs(G) = 2p + 1 - \Delta(G)$, then G has the following properties:*

1. *G contains a star $K_{1,\Delta}$ with center vertex v_0 and leaves $v_1, \ldots, v_\Delta$ and $G[N(v_0)]$ is edgeless. Let $c(v_0 v_i) = i$ for $1 \leq i \leq \Delta$ and $H_0 \cong G[N[v_0]]$.*
2. *If $p > \Delta$, then let H_i be the subgraph spanned by the edges with colour i for $\Delta + 1 \leq i \leq p$. The subgraphs $H_{\Delta+1}, H_{\Delta+2}, \ldots, H_p$ are pairwise vertex-disjoint and $V(H_0) \cap V(H_i) = \emptyset$ for $\Delta + 1 \leq i \leq p$.*
3. *$E(H_i, H_j) = \emptyset$ for $\Delta + 1 \leq i < j \leq p$ (where $E(H_i, H_j)$ is the set of all edges having one vertex in $V(H_i)$ and the other vertex in $V(H_j)$).*
4. *$E(v_i, H_j) = \emptyset$ for $1 \leq i \leq \Delta$ and $\Delta + 1 \leq j \leq p$ (where $E(v_i, H_j)$ is the set of all edges incident with v_i and a vertex in $V(H_j)$).*
5. *If $uv \in E(H_i)$ for some $\Delta + 1 \leq i \leq p$, then $N(u) \cap N(v) = \emptyset$.*
6. *$N(v_i) \cap N(v_j) = \emptyset$ for $v_i \in V(H_i), v_j \in V(H_j), \Delta + 1 \leq i < j \leq p$.*

Proof.

1. Suppose there is an edge $v_i v_j$ for some $1 \leq i < j \leq \Delta$. If $c(v_i v_j) = k$ for some k with $1 \leq k \leq \Delta, k \neq i, j$, then $rs(G) \leq (\Delta+1) - 1 + (2p - 2\Delta) = 2p - \Delta < 2p + 1 - \Delta$, a contradiction. If $c(v_i v_j) = k$ for some k with $\Delta + 1 \leq k \leq p$, then $rs(G) \leq (\Delta + 1) + (2p - 2\Delta - 2) = 2p - \Delta - 1 < 2p + 1 - \Delta$, a contradiction as well.
2. Suppose there are integers i, j with $\Delta + 1 \leq i < j \leq p$ and two adjacent edges e, f with $c(e) = i, c(f) = j$. Then $rs(G) \leq (\Delta + 1) + (2p - 2\Delta - 1) = 2p - \Delta < 2p + 1 - \Delta$, a contradiction. Suppose there are integers i, j with $1 \leq i \leq \Delta, \Delta + 1 \leq j \leq p$ and two adjacent edges e, f with $c(e) = i, c(f) = j$. Then $rs(G) \leq (\Delta + 1) + (2p - 2\Delta - 1) = 2p - \Delta < 2p + 1 - \Delta$, a contradiction as well.
3. Suppose there is an edge $v_i v_j$ with $v_i \in V(H_i), v_j \in V(H_j), \Delta + 1 \leq i < j \leq p$. Then $c(v_i v_j) = k$ for some $1 \leq k \leq \Delta$. Hence $rs(G) \leq (\Delta + 1) - 1 + (2p - 2\Delta) = 2p - \Delta < 2p + 1 - \Delta$, a contradiction.
4. Suppose there is an edge $v_i v_j$ for two vertices $v_i \in V(H_0)$ and $v_j \in V(H_j), \Delta + 1 \leq j \leq p$. Then $rs(G) \leq (\Delta+1) + (2p - 2\Delta - 1) = 2p - \Delta < 2p + 1 - \Delta$, a contradiction.

5. Suppose there is an edge $uv \in E(H_i)$ for some $\Delta+1 \leq i \leq p$ with $N(u) \cap N(v) \neq \emptyset$. By 3. and 4. we conclude that $N(u) \cap N(v) \cap V(H_0) = \emptyset$. Furthermore, for a vertex $w \in N(u) \cap N(v)$, we have $c(uw) = j, c(vw) = k$ for some $1 \leq j < k \leq \Delta$. Then $rs(G) \leq (\Delta+1)-2+(2p-2\Delta+1) = 2p-\Delta < 2p+1-\Delta$, a contradiction.

6. Suppose $N(v_i) \cap N(v_j) \neq \emptyset$ for two vertices $v_i \in V(H_i), v_j \in V(H_j), \Delta+1 \leq i < j \leq p$. By 3. and 4. we conclude that $N(v_i) \cap N(v_j) \cap V(H_0) = \emptyset$. Furthermore, for a vertex $w \in N(v_i) \cap N(v_j)$, we have $c(uw) = k, c(vw) = l$ for some $1 \leq k < l \leq \Delta$. Then $rs(G) \leq (\Delta+1)-2+(2p-2\Delta+1) = 2p-\Delta < 2p+1-\Delta$, a contradiction.

□

Another upper bound for the rainbow subgraph number follows from an approach presented in [8]. Observe that two adjacent edges of different colours together have three vertices, whereas two edges of different colours in a matching have four vertices. Based on this obervation the following algorithm has been proposed in [8].

Algorithm
Input: A graph G of order n whose edges are coloured with p colours

1. Construct a graph G' with $V(G') = \{v_1, v_2, \ldots, v_p\}$ (v_i corresponds to colour i) and $v_iv_j \in E(G')$ if there exist two adjacent edges $e, f \in E(G)$ with $c(e) = i$ and $c(f) = j$ ($c(x)$ denotes the colour of the edge x).

2. Now compute a maximum matching M of order $\beta(G')$ in G'. This can be done in polynomial time.

3. Next construct a graph H with $V(H) \subseteq V(G)$ as follows: For each matching edge of M choose two adjacent edges in G with these two colours. For each vertex of $V(G')$ not in M choose an edge in G with this colour. In this way we obtain a rainbow subgraph $H \subseteq G$ with $|E(H)| = p$.

Correctness of the algorithm: Edges of the matching correspond to pairs of adjacent edges in the original graph. Colours that are left out by this procedure are added greedily at the end.

Claim 3.2. $|V(H)| \leq 2p - \beta(G')$

Proof. For each matching edge of G' three vertices appear in H. Hence

$$|V(H)| \leq 3\beta(G') + 2(p - 2\beta(G')) = 2p - \beta(G')$$

□

Corollary 3.3. $rs(G) \leq 2p - \beta(G')$.

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A note on domination and independence-domination numbers of graphs*

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Abstract

Vizing's conjecture is true for graphs G satisfying $\gamma^i(G) = \gamma(G)$, where $\gamma(G)$ is the *domination number* of a graph G and $\gamma^i(G)$ is the *independence-domination number* of G, that is, the maximum, over all independent sets I in G, of the minimum number of vertices needed to dominate I. The equality $\gamma^i(G) = \gamma(G)$ is known to hold for all chordal graphs and for chordless cycles of length $0 \pmod 3$. We prove some results related to graphs for which the above equality holds. More specifically, we show that the problems of determining whether $\gamma^i(G) = \gamma(G) = 2$ and of verifying whether $\gamma^i(G) \geq 2$ are NP-complete, even if G is weakly chordal. We also initiate the study of the equality $\gamma^i = \gamma$ in the context of hereditary graph classes and exhibit two infinite families of graphs for which $\gamma^i < \gamma$.

Keywords: Vizing's conjecture, domination number, independence-domination number, weakly chordal graph, NP*-completeness, hereditary graph class,* IDD*-perfect graph.*

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1 Introduction

The closed neighborhood $N_G[v]$ of a vertex in a (finite, simple, undirected) graph G is the set consisting of v itself and its neighbors in the graph. A set A of vertices is said to *dominate* a set B if $B \subseteq \cup\{N_G[a] : a \in A\}$. The minimum size of a set of vertices dominating a set A is denoted by $\gamma_G(A)$. A *dominating set* in a graph G is a set D of vertices that dominates $V(G)$. We write $\gamma(G)$ for $\gamma_G(V(G))$. The concept of domination in graphs has been extensively studied, both in structural and algorithmic graph theory, because of its numerous applications to a variety of areas. Domination naturally arises in facility location

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problems, in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying. The two books [14, 15] discuss the main results and applications of domination in graphs. Many variants of the basic concepts of domination have appeared in the literature. Again, we refer to [14, 15] for a survey of the area, and to [4, 10, 11, 13, 16, 18, 19, 21, 22] for some recent papers on domination and variants thereof.

The *Cartesian product* of two graphs G and H is the graph $G\Box H$ with vertex set $V(G) \times V(H)$ and edge set $\{(u, x)(v, y) : (u, x), (v, y) \in V(G) \times V(H), u = v$ and $xy \in E(H)$, or $x = y$ and $uv \in E(G)\}$. In 1968 Vizing made the following conjecture, according to Brešar et al. [8] "arguably the main open problem in the area of domination theory":

Conjecture 1. For every two graphs G and H, it holds that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

The conjecture is still open and was verified for several specific classes of graphs; see, e.g., [8].

An *independent set* in a graph is a set of pairwise non-adjacent vertices. The *independence-domination number* $\gamma^i(G)$ is the maximum of $\gamma_G(I)$ over all independent sets I in G. The independence-domination number has arisen in the context of matching theory, see, e.g., [2, 20], and was first introduced in the context of domination by Aharoni and Szabó in 2009 [3]. Obviously, $\gamma^i(G) \leq \gamma(G)$, and in general the gap between the two may be large [3]. However, equality holds for:

- *cycles of length* $0 \pmod 3$, and more generally, for graphs that have a set of $\gamma(G)$ vertices with pairwise disjoint closed neighborhoods [17];
- *chordal graphs*, as proved by Aharoni, Berger and Ziv [1] in a result on width and matching width of families of trees.

Recall that a graph G is called *chordal* if it does not contain any induced cycle of length at least 4, and *weakly chordal* if it does not contain any induced cycles of length at least 5 or their complements.

Theorem 2 ([1])**.** For every chordal graph G, it holds that $\gamma^i(G) = \gamma(G)$.

The independence-domination number is related to Vizing's conjecture via the following result proved by Aharoni and Szabó [3]:

Theorem 3 ([3])**.** For every two graphs G and H, it holds that $\gamma(G\Box H) \geq \gamma^i(G)\gamma(H)$.

In particular, Vizing's conjecture is true for chordal graphs. More generally, if G is a graph with $\gamma^i(G) = \gamma(G)$ then $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for every graph H. In a recent survey paper on Vizing's conjecture [8], Brešar et al. asked what other classes of graphs can be found for which $\gamma^i(G) = \gamma(G)$ for every G in the class.

In this note, we prove some results related to graphs for which the independence-domination number coincides with the domination number. First, using a relationship between the independence-domination number and the notion of a dominating clique, we prove that determining whether $\gamma^i(G) = \gamma(G)$ is NP-hard. More specifically, we show that it is NP-complete to determine whether $\gamma^i(G) \geq 2$, as well as to determine whether $\gamma^i(G) = \gamma(G) = 2$. These results, obtained in Section 2, remain valid for weakly chordal graphs.

In Section 3, we turn our attention to graphs in which the equality $\gamma^i = \gamma$ holds in the hereditary sense. We show that this class, which properly contains the class of chordal graphs, is properly contained in the class of graphs in which all induced cycles are of length $0 \pmod 3$. We do this by constructing an infinite family of graphs in which all induced cycles are of length $0 \pmod 3$ but where the independence-domination number is strictly smaller than the domination number. In conclusion, we propose three related problems.

2 The complexity of computing γ^i and testing $\gamma^i = \gamma$

In this section, we study some computational complexity aspects of computing the indepedence-domination number and comparing it to the domination number. We first recall some notions needed in our proofs. For a graph $G = (V, E)$, we denote by $\overline{G}$ its *complement*, that is, the graph with the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G. A *clique* in a graph is a subset of pairwise adjacent vertices. A dominating set that is also a clique is called a *dominating clique*. We assume familiarity with basic notions of computational complexity (see, e.g., [12]).

Theorem 4. Given a weakly chordal graph G, it is NP-complete to determine whether $\gamma^i(G) \geq 2$.

Proof. To show membership in NP, observe that a short certificate for the fact that $\gamma^i(G) \geq 2$ is any independent set I such that for every vertex $v \in V(G)$, it holds that $I \nsubseteq N_G[v]$.

To show hardness, we make a reduction from the problem of determining whether a given weakly chordal graph contains a dominating clique. This is an NP-complete problem, see, e.g., [6]. Clearly, the problem remains NP-complete if we assume that the input graph G does not have a dominating vertex.

Suppose that we are given a weakly chordal graph G without dominating vertices. We compute its complementary graph $H = \overline{G}$. Since H is also weakly chordal, the theorem follows immediately from the claim below.

Claim: G has a dominating clique if and only if $\gamma^i(H) \geq 2$.

For the forward implication, suppose that G has a dominating clique K. We will show that $\gamma^i(H) \geq 2$ by showing that $\gamma_H(K) \geq 2$. Suppose for a contradiction that $\gamma_H(K) = 1$. Then, there exists a vertex $v \in V(H) = V(G)$ such that $K \subseteq N_H[v]$. In particular, v must belong to K, since otherwise in G, vertex v would not have any neighbors in K, contrary to the assumption that K is dominating in G. Since K is independent in H, that facts that $v \in K$ and $K \subseteq N_K[v]$ imply that $K = \{v\}$, that is, v is a dominating vertex in G, which is impossible since we assumed that G has no dominating vertices. Hence, it holds that $\gamma_H(K) \geq 2$ and consequently $\gamma^i(H) \geq 2$.

For the converse implication, suppose that $\gamma^i(H) \geq 2$, and let I be an independent set in H such that $\gamma_H(I) \geq 2$. Clearly, I is a clique in G, and, in fact, a dominating clique: If this were not the case, then there would exist a vertex $v \in V(G) \setminus I$ such that in G, vertex v is not adjacent to any vertex from I. Equivalently, for every $u \in I$, $uv \in E(H)$. But then $\{v\}$ would dominate I in H, contrary to the assumption that $\gamma_H(I) \geq 2$. $\square$

Corollary 5. Given a (weakly chordal) graph G and an integer k, it is NP-hard to determine whether $\gamma^i(G) \geq k$.

Corollary 6. Given a (weakly chordal) graph G and an integer k, it is NP-hard to determine whether $\gamma^i(G) \leq k$.

How difficult it is to determine whether the values of γ^i and γ coincide? Since $\gamma^i(G) \leq \gamma(G)$ holds for every graph G, in order to show that

$$\gamma^i(G) = \gamma(G) = k\,, \tag{2.1}$$

it suffices to argue that $\gamma^i(G) \geq k$ and $\gamma(G) \leq k$. Clearly, for $k = 1$, whether (2.1) holds can be determined in polynomial time: a necessary and sufficient condition for $\gamma^i(G) = \gamma(G) = 1$ is that G has a dominating vertex.

We now show that already for $k = 2$, the problem becomes NP-complete, even for weakly chordal graphs. The proof will also imply intractability of the problem of verifying whether $\gamma^i = \gamma$.

Theorem 7. Given a weakly chordal graph G, it is NP-complete to determine whether $\gamma^i(G) = \gamma(G) = 2$.

Proof. Membership in NP follows from the fact that a short certificate for $\gamma^i(G) = \gamma(G) = 2$ is given by a pair (I, D) where I is an independent set not dominated by any vertex (proving $\gamma^i(G) \geq 2$) and D is a dominating set of size two (proving $\gamma(G) \leq 2$).

To show hardness, we make a reduction from 3-SAT [12]. The reduction is an adaptation of the reduction by Brandstädt and Kratsch [6] used to prove that the dominating clique problem is NP-complete for weakly chordal graphs.

Suppose that we are given an instance to 3-SAT, that is, a Boolean formula φ over variables $x_1, \ldots, x_n$, consisting of m clauses of length 3, say $C_i = x_{i_1}^{\alpha_{i_1}} \vee x_{i_2}^{\alpha_{i_2}} \vee x_{i_3}^{\alpha_{i_3}}$ for $i = 1, \ldots, m$, where $\alpha_{i_j} \in \{0, 1\}$, with the usual interpretation that $x_i^1 = x_i$ and $x_i^0 = \overline{x_i}$. Without loss of generality, we may assume the following properties of the formula:

Property 1: No clause contains both a literal and its negation. (This is because clauses containing both a literal and its negation can be discarded as they will always be satisfied.)

Property 2: There exist two clauses, say C_1 and C_2, that have no literals in common. (If the given formula φ does not have this property, we simply add to it a new clause consisting of three new variables. If necessary, we relabel the clauses.)

Consider the graph H defined as follows:

$$\begin{array}{rcl} V(H) & = & \{x_1, \overline{x_1}, \ldots, x_n, \overline{x_n}\} \cup \{C_1, \ldots, C_m\}\,, \\ E(H) & = & \{x_i^{\alpha_1} x_j^{\alpha_2} \mid 1 \leq i, j \leq n,\ i \neq j,\ \alpha_1, \alpha_2 \in \{0, 1\}\} \cup \\ & & \{x_i^{\alpha} C_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m,\ \alpha \in \{0, 1\},\ x_i^{\alpha} \text{ is a literal in } C_j\}\,. \end{array}$$

We complete the reduction by computing the complementary graph $G = \overline{H}$.

Using Property 1, it is easy to verify that neither H nor G contain an induced cycle of length at least 5, that is, G is weakly chordal. Moreover, the following properties are equivalent:

(i) φ is satisfiable.

(ii) H has a dominating clique.

(iii) $\gamma^i(G) = \gamma(G) = 2$.

(iv) $\gamma^i(G) = \gamma(G)$.

The equivalence between (i) and (ii) has been established in [6].

(ii) implies (iii): Suppose that H has a dominating clique. Since H has no dominating vertex, similar arguments as in the proof of Theorem 4 allow us to conclude that $\gamma^i(G) \geq 2$. Furthermore, by Property 1 and by construction of H, vertices C_1 and C_2 have no common neighbors in H. This implies that $\{C_1, C_2\}$ is a dominating set in G. Therefore $\gamma(G) \leq 2$, and the conclusion follows since $2 \leq \gamma^i(G) \leq \gamma(G) \leq 2$.

Trivially, (iii) implies (iv).

(iv) implies (ii): Suppose that $\gamma^i(G) = \gamma(G)$. Since H has no isolated vertices, G has no dominating vertices. Therefore $\gamma^i(G) = \gamma(G) \geq 2$, and it can be shown, similarly as in the proof of Theorem 4, that H has a dominating clique.

This completes the proof. □

Theorem 8. Given a weakly chordal graph G, it is NP-hard to determine whether $\gamma^i(G) = \gamma(G)$.

Proof. Perform the same reduction as in the proof of Theorem 7 and use the fact that the formula is satisfiable if and only if $\gamma^i(G) = \gamma(G)$. □

3 A hereditary view on $\gamma^i = \gamma$

In this section, we initiate the study of the equality between the domination and independence-domination number of graphs in the context of hereditary graph classes. A graph class is said to be *hereditary* if it is closed under vertex deletions. The family of hereditary graph classes is of particular interest, first of all, since many natural graph properties are hereditary, and second, since hereditary (and only hereditary) classes admit a uniform description in terms of forbidden induced subgraphs. For a set $\mathcal{F}$ of graphs, we say that a graph G is $\mathcal{F}$*-free* if it does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$. The set of all $\mathcal{F}$-free graphs will be denoted by $Free(\mathcal{F})$. Notice that for two sets $\mathcal{F}_1$ and $\mathcal{F}_2$ of graphs, it holds that $Free(\mathcal{F}_1 \cup \mathcal{F}_2) = Free(\mathcal{F}_1) \cap Free(\mathcal{F}_2)$.

Given a hereditary class $\mathcal{G}$, denote by $\mathcal{F}$ the set of all graphs G with the property that $G \notin \mathcal{G}$ but $H \in \mathcal{G}$ for every proper induced subgraph H of G. The set $\mathcal{F}$ is said to be the set of (minimal) forbidden induced subgraphs for $\mathcal{G}$, and $\mathcal{G}$ is precisely the class of $\mathcal{F}$*-free* graphs. The set $\mathcal{F}$ can be either finite or infinite, and many interesting classes of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. Such characterizations can be useful for establishing inclusion relations among hereditary graph classes, and were obtained for numerous graph classes (see, e.g. [7]). The most famous such class is probably the class of *perfect graphs*, for which the forbidden induced subgraph characterization is given by the Strong Perfect Graph Theorem conjectured by Berge in 1961 [5] and proved by Chudnovsky, Robertson, Seymour and Thomas in 2006 [9].

Since Vizing's conjecture holds for graphs G such that $\gamma^i(G) = \gamma(G)$, it would be interesting to determine the largest hereditary class of graphs with this property. Moreover, since recognizing graphs with $\gamma^i = \gamma$ is NP-hard, it would also be interesting to determine whether graphs in which the property $\gamma^i = \gamma$ holds in the hereditary sense can be recognized efficiently. With this motivation in mind, we introduce the class of *independence-domination-domination-perfect graphs*, or shortly, IDD-*perfect graphs*, that is, graphs for which the above equality holds in the hereditary sense:

$$\text{IDD-}\textit{perfect graphs} = \{G : \gamma^i(H) = \gamma(H) \text{ for every induced subgraph } H \text{ of } G\} .$$

We now provide some partial results towards a characterization of IDD-perfect graphs. By Theorem 2, we can immediately relate the class of IDD-perfect graphs to a well studied hereditary subclass of perfect graphs, the class of chordal graphs:

Theorem 9.

$$\textit{Chordal graphs} \subset \text{IDD-}\textit{perfect graphs}\,.$$

Proof. Since every induced subgraph of a chordal graph is chordal, Theorem 2 implies that the class of IDD-perfect graphs contains the class of chordal graphs. This inclusion is proper since chordless cycles of length congruent to $0 \pmod 3$ are IDD-perfect [17] (but not chordal). □

In the rest of this section, we bound the class of IDD-perfect graphs from above, by exhibiting two infinite families of graphs that do not belong to class of IDD-perfect graphs: the chordless cycles of length not congruent to $0 \pmod 3$ and another graph family, which we describe now. For positive integers $k_1, k_2, k_3 > 1$, let F_{k_1,k_2,k_3} denote the graph obtained from the disjoint union of three cycles C_1, C_2 and C_3 where $|V(C_j)| = 3k_j$ as follows: denoting by $(v_1^j, \ldots, v_{3k_j}^j)$ a cyclic order of vertices of C_j, we identify vertex v_1^2 with vertex $v_{3k_1}^1$, vertex v_1^3 with vertex $v_{3k_2}^2$, and vertex v_1^1 with vertex $v_{3k_3}^3$. See Fig. 1 for an example.

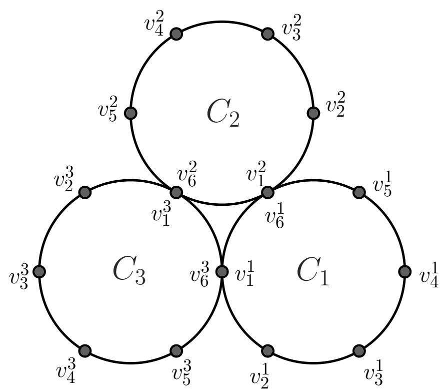

Figure 1: The graph $F_{2,2,2}$

Theorem 10.

$$\text{IDD-}\textit{perfect graphs} \subseteq \textit{Free}\Bigg(\bigcup_{k\geq 1}\{C_{3k+1}, C_{3k+2}\} \cup \bigcup_{k_1,k_2,k_3>1}\{F_{k_1,k_2,k_3}\}\Bigg)\,.$$

Proof. First, we establish the inclusion IDD-*perfect graphs* $\subseteq$ *Free*$\big(\bigcup_{k\geq 1}\{C_{3k+1}, C_{3k+2}\}\big)$. To this end, we show that for every chordless cycle C of order $n = 3k+1$ or $n = 3k+2$ (where k is a positive integer), it holds that $\gamma^i(C) = k$ and $\gamma(C) = k+1$. Let $(v_1, \ldots, v_n)$ be a cyclic order of the vertices of such a cycle C. Observe that for every set $S \subseteq V(C)$ with $|S| \leq k$, it holds that

$$|\cup_{v\in S} N_C(v)| \leq \sum_{v\in S} |N_C[v]| = 3|S| < n\,.$$

Thus, we immediately have $\gamma(C) \geq k+1$. On the other hand, the set

$$\{v_{3j-2} : 1 \leq j \leq k+1\}$$

is dominating, proving $\gamma(C) = k + 1$. Suppose now that I is an independent set in C. We may assume w.l.og. that $v_1 \notin I$. In case that $n = 3k + 2$, we may also assume that $v_n \notin I$. In either case, the set $\{v_{3j} : 1 \le j \le k\}$ is a set of size k dominating I. This shows that $\gamma^i(C) \le k$. Conversely, since the set $I = \{v_{3j} : 1 \le j \le k\}$ is a set of k vertices with pairwise disjoint closed neighborhoods, we have $\gamma^i(C) \ge \gamma^C(I) = |I| = k$. Thus $k = \gamma^i(C) < \gamma(C) = k + 1$ and hence no IDD-perfect graph can contain C as an induced subgraph.

It remains to show that IDD-*perfect graphs* $\subseteq$ *Free*$\left(\bigcup_{k_1,k_2,k_3>1}\{F_{k_1,k_2,k_3}\}\right)$. Equivalently, we must show that for every three integers $k_1, k_2, k_3 > 1$, it holds that $\gamma^i(F_{k_1,k_2,k_3}) < \gamma(F_{k_1,k_2,k_3})$. We will show this in two steps, by computing the exact values of $\gamma^i(F_{k_1,k_2,k_3})$ and $\gamma(F_{k_1,k_2,k_3})$.

Let $F = F_{k_1,k_2,k_3}$ for some $k_1, k_2, k_3 > 1$. First, we show that $\gamma(F) = k_1+k_2+k_3-1$. Consider the set

$$D = \{v^1_{3j-2} : 1 \le j \le k_1\} \cup \{v^2_{3j-1} : 1 \le j \le k_2\} \cup \{v^3_{3j} : 1 \le j \le k_3 - 1\}\,.$$

Then, D is a dominating set of size $k_1+k_2+k_3-1$, showing that $\gamma(F) \le k_1+k_2+k_3-1$. Now, we show that $\gamma(F) \ge k_1 + k_2 + k_3 - 1$. Suppose for a contradiction that D is a dominating set in F with $|D| \le k_1 + k_2 + k_3 - 2$. Clearly, for every $p \in \{1, 2, 3\}$, we have that $|D \cap V(C_p)| \ge k_p$. Moreover, D must contain at least $k_p - 1$ vertices from C_p other than v^p_1 and $v^p_{3k_p}$ since otherwise not all vertices in the set $\{v^1_{3p-2} : 2 \le j \le k_p\}$ can be dominated by D. This implies that $|D \cap \{v^1_1, v^2_1, v^3_1\}| = 1$. We may assume without loss of generality that $D \cap \{v^1_1, v^2_1, v^3_1\} = \{v^1_1\}$. But this implies that $|D \cap V(C_2)| = k_2 - 1$, a contradiction. Hence $\gamma(F) = k_1 + k_2 + k_3 - 1$.

In the rest of the proof, we show that $\gamma^i(F) = k_1 + k_2 + k_3 - 2$. Consider the set

$$I = \{v^1_{3j} : 1 \le j \le k_1\} \cup \{v^2_{3j-2} : 1 \le j \le k_2\} \cup \{v^3_{3j} : 1 \le j \le k_3 - 1\}\,.$$

This is is a set of $k_1 + k_2 + k_3 - 2$ vertices with pairwise disjoint closed neighborhoods. Therefore $\gamma^i(F) \ge |I| = k_1 + k_2 + k_3 - 2$. To see that $\gamma^i(F) \le k_1 + k_2 + k_3 - 2$, we will verify that $\gamma_F(I) \le k_1 + k_2 + k_3 - 2$ for every independent set I in F. Up to symmetry, it is sufficient to consider the following two cases:

- *Case 1:* $v^1_2 \notin I$.
  In this case, the set
  $$D = \{v^1_{3j-2} : 2 \le j \le k_1\} \cup \{v^2_{3j} : 1 \le j \le k_2\} \cup \{v^3_{3j-2} : 2 \le j \le k_3\}$$
  is a set of size $k_1 + k_2 + k_3 - 2$ dominating I.
- *Case 2:* $\{v^1_2, v^1_{3k_1-1}, v^2_2, v^2_{3k_2-1}, v^3_2, v^3_{3k_3-1}\} \subseteq I$.
  In this case, the set
  $$\begin{aligned} D = {} & \{v^1_1, v^2_1, v^3_1\} \cup \{v^1_{3j-1} : 2 \le j \le k_1 - 1\} \cup \{v^2_{3j-1} : 2 \le j \le k_2 - 1\} \cup \\ & \{v^3_{3j-1} : 2 \le j \le k_3 - 1\} \end{aligned}$$
  is a set of size $k_1 + k_2 + k_3 - 3$ dominating I.

This shows that $k_1 + k_2 + k_3 - 2 = \gamma^i(F) < \gamma(F) = k_1 + k_2 + k_3 - 1$ and hence no IDD-perfect graph in can contain $F = F_{k_1,k_2,k_3}$ as an induced subgraph.

This completes the proof. $\square$

Remark. Theorem 10 shows that the class of IDD-perfect graphs is not comparable with the class of perfect graphs. On the one hand, the 9-cycle is an IDD-perfect graph that is not perfect. On the other hand, the 4-cycle is a (bipartite, hence) perfect graph that is not IDD-perfect.

4 Conclusion

We conclude this note with three problems related to results from Section 3.

Problem 1. Determine whether every graph of the form F_{k_1,k_2,k_3} is a *minimal* forbidden induced subgraph for the class of IDD-perfect graphs.

Problem 2. Determine the set of minimal forbidden induced subgraphs for the class of IDD-perfect graphs.

Problem 3. Determine the computational complexity of recognizing IDD-perfect graphs.

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On D. G. Higman's note on regular 3-graphs

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Abstract

We introduce the notion of a t-graph and prove that regular 3-graphs are equivalent to cyclic antipodal 3-fold covers of a complete graph. This generalizes the equivalence of regular two-graphs and Taylor graphs. As a consequence, an equivalence between cyclic antipodal distance regular graphs of diameter 3 and certain rank 6 commutative association schemes is proved. New examples of regular 3-graphs are presented.

Keywords: Antipodal graph, association scheme, distance regular graph of diameter 3*, Godsil-Hensel matrix, group ring, Taylor graph, two-graph.*

Math. Subj. Class.: 05E30, 05B20, 05E18

1 Introduction

This paper is mainly a clarification of [6] — a short draft written by Donald Higman in 1994, entitled “A note on regular 3-graphs”.

The considered generalization of two-graphs was introduced by D. G. Higman in [5]. As in the famous correspondence between two-graphs and switching classes of simple graphs, t-graphs are interpreted as equivalence classes of an appropriate switching relation defined on weights, which play the role of simple graphs.

In his note Higman uses certain association schemes to characterize regular 3-graphs and to obtain feasibility conditions for their parameters. Specifically, he provides a graph theoretic interpretation of a weight and from the resulted graph he constructs a rank 4 symmetric association scheme and a rank 6 fission of it. Furthermore, he proves that rank 6 schemes with parameters as in his construction are equivalent to regular 3-graphs.

During our redetermination of the structure constants of the rank 6 scheme an error in [6] was detected, this miscalculation led Higman to a false restriction on the parameters

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of regular 3-graphs. Our first contribution is the correction of this mistake (see Subsection 5.2). The second contribution is a proof (see Section 4) that in the case of regular 3-graphs, the graph defined by a weight in its switching class is a distance regular cover of the complete graph. Moreover, it is a cyclic antipodal distance regular (ADRG) 3-fold cover of the complete graph in the sense of Godsil and Hensel in [2]. This provides a further restriction on the parameters of regular 3-graphs.

Altogether, in Section 4 and in Section 5 we establish a one-to-one correspondence between regular 3-graphs, cyclic ADRGs of diameter 3 and certain rank 6 association schemes. As a consequence, we provide a new characterization of cyclic antipodal distance regular 3-fold covers of the complete graph in terms of association schemes.

To keep the length of this paper reasonable we did not include all necessary preliminaries. In particular, we assume some knowledge of distance regular graphs, specifically, antipodal distance regular graphs of diameter 3. Also, we assume the reader is familiar with association schemes, in particular, the intersection algebra of an association scheme and its character-multiplicity table. An interested reader may find a more comprehensive consideration of all the diverse links exposed below as well as suggestions for further research in [7].

2 Two-graphs and t-graphs

2.1 Two-graphs and regular two-graphs

Two-graphs have roots originating in diverse areas of combinatorics, geometry and group theory, thus leading to different manifestations in the literature, such as: switching classes of graphs, sets of equidistant points in elliptic geometry, sets of equiangular lines in Euclidean geometry, binary maps of triples with vanishing coboundary, and double coverings of complete graphs (see the celebrated survey [12]). Our focus will be on the last two interpretations and the connection between them. We start with the classical definition and the classical viewpoint of two-graphs as switching classes of simple graphs.

Let X be a set of n elements called *vertices*. For $m \in \mathbb{N}$ denote by $X^{\{m\}}$ the set of all m-subsets of X.

Definition 2.1. A set $\Delta \subseteq X^{\{3\}}$ is a *two-graph* if every 4-subset of X contains an even ($\in \{0, 2, 4\}$) number of members of Δ.

Typically we use the notation (X, Δ) for a two-graph, and call Δ the set of *odd triples*.

Definition 2.2. A two-graph (X, Δ) is called *regular* if every 2-subset $\{x, y\} \in X^{\{2\}}$ is contained in the same number of triples from Δ.

The most famous view of two-graphs is related to a special equivalence relation that is defined on the set of simple (undirected, no loops) graphs. First we remind the reader how to get a two-graph from a graph:

Let $\Gamma = (V, E)$ be a simple graph. The set of triples $\{u, v, w\}$ of vertices, such that the induced subgraph $\Gamma|_{\{u,v,w\}}$ has an odd number of edges, forms a two-graph.

Next, to define the equivalence relation we consider the operation of switching a graph with respect to a set of vertices.

Definition 2.3. Let $X \subseteq V$ be a subset of vertices of a simple graph $\Gamma = (V, E)$. *Switching with respect to* X means interchanging the adjacencies and non-adjacencies between X and its complement $V \setminus X$.

As a more appropriate setting to work with the operation of switching, J. J. Seidel proposed an alternative matrix representation of a simple graph:

Definition 2.4. The *Seidel adjacency matrix* $S = (s_{i,j})$ of a graph $\Gamma = (V, E)$ is a $\{0, -1, 1\}$-matrix having:

$$s_{i,j} = \begin{cases} 0 & i = j, \\ -1 & \{i, j\} \in E, \\ 1 & \{i, j\} \notin E. \end{cases}$$

In this notation, if the graph Γ' is obtained from Γ by switching with respect to $X \subseteq V$, then its Seidel adjacency matrix S' is obtained from S via a similarity transformation by a diagonal matrix having $\{-1, 1\}$ on its diagonal. Explicitly:

$$S' = DSD,$$

where $D_{i,i} = -1 \iff i \in X$.

As was implied above, switching is an equivalence relation on the set of all simple graphs of order n sharing the same prescribed vertex set. Furthermore we note that switching equivalent graphs give rise to the same two-graph, and have the same Seidel spectrum, thus allowing us to define the *eigenvalues* and their *multiplicities* of a two-graph. To sum up we have:

Theorem 2.5. *There is a 1-1 correspondence between two-graphs and switching classes of graphs.*

Theorem 2.6. *A two-graph is regular if and only if it has two distinct (Seidel) eigenvalues $\rho_1 > 0 > \rho_2$, such that $\rho_1\rho_2 = 1 - |X|$.*

The following is an alternative definition of a two-graph. We call it *the cohomological definition* for reasons that will be clear soon.

Definition 2.7. Let U_2 be the group of square roots of unity. A set $\Delta \subseteq X^{\{3\}}$ is a *two-graph* if the function:

$$f : X^{\{3\}} \longrightarrow U_2$$

defined by

$$f(x) = -1 \iff x \in \Delta,$$

satisfies:

$$f(\{x, y, z\}) \cdot f(\{x, y, t\}) \cdot f(\{x, z, t\}) \cdot f(\{y, z, t\}) = 1$$

for any $\{x, y, z, t\} \in X^{\{4\}}$.

Functions satisfying the equation in the above definition are called 3-cocycles (see below).

It is clear that the two definitions are equivalent. Furthermore, we may refer to either (X, Δ), Δ or the function f as the two-graph.

2.2 The connection with double covers of complete graphs

Two-graphs were originally introduced by Graham Higman to study 2-transitive representations of certain sporadic groups, in his description he used antipodal 2-fold covers of complete graphs. In [16], Taylor and Levingston established a one-to-one correspondence between two-graphs and antipodal 2-fold covers of complete graphs. This correspondence will be described in a more general setting with all details in the next section. Meanwhile we give an overview for the case of two-graphs.

Let Γ be a graph with n vertices in the switching class of the two-graph f and let S_Γ be the Seidel adjacency matrix of Γ. Then by inserting a 2×2 matrix in the place of each entry of S_Γ according to the following rule:

$$0 \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad 1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad -1 \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we obtain a $2n \times 2n$ $\{0,1\}$-matrix which is the usual adjacency matrix of the corresponding 2-fold cover of K_n. The converse construction is done in a similar manner: substituting each 2×2 block of the adjacency matrix of a 2-fold cover of K_n (writing it in a suitable ordering of the vertices) with an element of $\{0, 1, -1\}$.

A 2-fold cover of K_n, when it is also distance regular, is called a *Taylor graph*, these are distance regular graphs with intersection array

$$\{k, \mu, 1; 1, \mu, k\}.$$

In the above mentioned correspondence, Taylor graphs correspond to regular two-graphs. This will be a particular case of our more general result later on.

2.3 Generalizing two-graphs

Considering the cohomological definition of two-graphs, two very natural generalizations arise:

- t-cocycles into U_2
  (functions $f : X^{\{t\}} \longrightarrow U_2$ with a similar property as for two-graphs);
- 3-cocycles into U_t
  (functions $f : X^{\{3\}} \longrightarrow U_t$, where U_t is the group of t-th roots of unity).

Historically, the first of these was indeed the first to be considered. The first appearance of the term t-graph as a t-cocycle over U_2 is due to D. Higman's generalization (see [4]) of E. Shult's graph extension theorem (see [13]). Other sources of this (design theoretical) generalization can be found in Mielants [11] or in [1]. In this case, a regular t-graph is a t-cocycle into U_2 which is also a t-design. Here just few examples are known: regular 3-graphs on 8 and 12 points and a regular 5-graph on 12 points (see [10]). Our interest in the current presentation is the second way to generalize two-graphs, i.e. 3-cocycles into U_t. This direction was examined by D. Higman, and the main source of this is [5]. We begin with introducing some very basic elements of cohomology theory, in which terms t-graphs are defined.

2.4 Some cohomology

Let X be a finite set with $|X| = n$. Let ζ be a primitive root of unity of order t, and let $U_t = < \zeta >$ denote the cyclic group of t^{th} roots of unity generated by ζ.

Let $x = (x_1, x_2, \dots, x_p) \in X^p$. A function $f : X^p \longrightarrow U_t$ is called a *p-cochain* if:

(i) $f(x) = \mathbf{1}$ (the identity element of U_t) for all $x \in X^p$ such that $x_i = x_j$ for some $1 \le i \ne j \le p$,

(ii) if y results from x by interchanging x_i and x_j for some $1 \le i \ne j \le p$ then $f(y) = (f(x))^{-1}$.

The set of all p-cochains together with pointwise multiplication forms a group denoted by $C^p_{\cdot}(X, U_t)$. Define the *coboundary operator*:

$$\delta : C^p_{\cdot}(X, U_t) \longrightarrow C^{p+1}_{\cdot}(X, U_t)$$

by

$$\delta f(x) = \prod_{i=0}^{p} \sigma^i(f(\widehat{x_i}))$$

where $\widehat{x_i} \in X^p$ is obtained from $x \in X^{p+1}$ by deleting the i^{th} coordinate x_i, and σ is the inverse operation of U_t.

For $e \in X$ and $p \ge 1$ we have the group homomorphism

$$\Delta_e : C^p_{\cdot}(X, U_t) \longrightarrow C^{p-1}_{\cdot}(X, U_t)$$

defined by

$$\Delta_e f(x) = f(e, x)$$

for $x \in X^{p-1}$.

Define the set of *p-coboundaries*:

$$B^p_{\cdot}(X, U_t) = \left\{ \delta f \mid f \in C^{p-1}_{\cdot}(X, U_t) \right\},$$

and the set of *p-cocycles*:

$$Z^p_{\cdot}(X, U_t) = \{ f \in C^p_{\cdot}(X, U_t) \mid \delta f = \mathbb{1} \}.$$

Here $\mathbb{1}$ is the identity cochain in $C^{p+1}_{\cdot}(X, U_t)$. It is routine to check that $\delta^2 f = \mathbb{1}$ for any $(p-1)$-cochain f, and thus the coboundary of any $(p-1)$-cochain is a p-cocycle. Two $(p-1)$-cochains have the same p-cocycle as their coboundary if and only if their quotient is a $(p-1)$-cocycle. Thus, p-cocycles correspond to cohomology classes of $(p-1)$-cochains, as a generalization of Seidel switching we call the cohomology classes *switching classes*.

Along the considered generalization of two-graphs and regular two-graphs we define:

Definition 2.8. A *t-graph* is a 3-cocycle into U_t.

Definition 2.9. A t-graph is called *regular* if for every pair $x, y \in X$, the number of $z \in X \setminus \{x, y\}$ such that $f(x, y, z) = \alpha$ depends only on $\alpha \in U_t$. This number is denoted $m(\alpha)$.

It is easy to check that in case $t = 2$, the definition of a 3-cocycle into U_2 is compatible with the characterization given in Definition 2.7, and that the above definition of regularity is compatible with Definition 2.2.

2.5 Weights

According to Higman, a *weight on X* with values in U_t is a 2-cochain $w \in C^2_\cdot(X, U_t)$, from this point onward we will call them simply weights. Thus t-graphs are the coboundaries of weights. A weight w can be represented as a $n \times n$ matrix W with entries from U_t where:

$$(W)_{x,y} = w(x,y).$$

Then W has $\mathbf{1}$ on its diagonal, and $W^* = W$, where W^* is obtained from W by transposing and inverting each entry. We will investigate the matrix representation of a weight with much more detail in the next section where we will focus on the case $t = 3$.

Another way to represent a weight is as an antipodal t-fold cover of K_n.

Definition 2.10. Let $w : X^2 \longrightarrow U_t$ be a weight on X. To each element $x \in X$ we associate t vertices $x_1, x_2, ..., x_t$ and define a graph $\Gamma_w = (V, E)$ on the resulting set V of $t|X|$ vertices by

$$\{x_i, y_j\} \in E \Longleftrightarrow w(x,y) = \zeta^{j-i}.$$

The resulting graph is a t-fold cover of the complete graph K_n, and if $w(x,y) = \zeta^i$ then the set of edges between $x_1, x_2, ..., x_t$ and $y_1, y_2, ..., y_t$ forms a perfect matching which is given by the i^{th} power of the permutation matrix of $(1, 2, ..., t)$. Permuting $x_1, x_2, ..., x_t$ according to some power of the permutation $(1, 2, ..., t)$ amounts to a change of w in its switching class.

3 Regular 3-graphs

From now on we focus on the case $t = 3$. We will prove that the situation for regular 3-graphs generalizes the case of regular two-graphs. In particular, regular 3-graphs are in $1-1$ correspondence with regular (cyclic) $(n, 3, c_2)$-covers.

3.1 Main conventions

Let $w : X^2 \longrightarrow U_3$ be a weight on X and $|X| = n$. The coboundary δw of w is a 3-graph $\Phi \in Z^3_\cdot(X, U_3)$ on X. Assume that Φ is regular. Recall that this means that for every pair x, y of distinct elements of X and $\alpha \in U_3$, the number $m(\alpha)$ of $z \in X \setminus \{x, y\}$ such that $\Phi(x, y, z) = \alpha$ is independent of the choice of x and y.

Denote

$$a := m(\mathbf{1}),$$
$$b := m(\zeta) = m(\zeta^2).$$

We call (n, a, b) the *parameters* of the regular 3-graph Φ. We obtain the first restriction on the parameters by simple counting. Fix two vertices $x, y \in X$, then:

$$|X \setminus \{x,y\}| = |\{z \in X \mid \Phi(x,y,z) = 1\}| + |\{z \in X \mid \Phi(x,y,z) = \zeta\}| + \left|\left\{z \in X \;\middle|\; \Phi(x,y,z) = \zeta^2\right\}\right|.$$

Thus we have

$$n - 2 = a + 2b.$$

The corresponding graph Γ_w is a 3-fold cover of K_n with exactly 3 types of matchings between fibres:

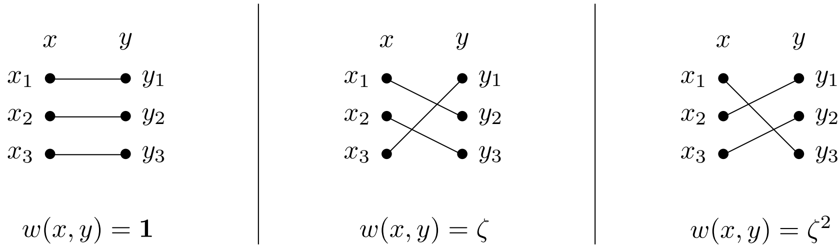

Figure 1: Matchings between fibres of Γ_w

The following subsection serves to remind and fix notation about matrices over the integral group ring, which is the setting in which we characterize regular 3-graphs.

3.2 Matrices over group rings

Let T be a finite group. The elements of the *integral group ring* $\mathbb{Z}[T]$ are expressions of the form

$$\sum_{g\in T} a_g g$$

where $a_g \in \mathbb{Z}$. The ring operations are:

$$\left(\sum_{g\in T} a_g g\right) + \left(\sum_{g\in T} b_g g\right) = \sum_{g\in T} (a_g + b_g) g,$$
$$\left(\sum_{g\in T} a_g g\right) \cdot \left(\sum_{h\in T} b_h h\right) = \sum_{g,h\in T} (a_g \cdot b_h) gh.$$

Following the notation of Klin and Pech in [8], for a subset $M \subseteq T$ define the *simple quantity* $\underline{M} \in \mathbb{Z}[T]$:

$$\underline{M} = \sum_{m\in M} 1 \cdot m.$$

When $M = \{g\}$ we will slightly abuse notation and write $\underline{g}$ instead of $\underline{\{g\}}$. The multiplicative identity of $\mathbb{Z}[T]$ is $\underline{\mathbf{1}}$ where $\mathbf{1}$ is the identity of T. The *adjoint* of $\sum\limits_{g\in T} a_g g$ is

$$\left(\sum_{g\in T} a_g g\right)^* = \sum_{g\in T} a_g g^{-1}.$$

The set of $n \times n$ matrices with entries from $\mathbb{Z}[T]$ is denoted by $\mathbb{Z}[T]^{n\times n}$. This set together with usual addition and multiplication of matrices forms a ring with identity. Moreover, $\mathbb{Z}[T]^{n\times n}$ forms a $\mathbb{Z}[T]$-module, and for a matrix $A = (a_{i,j}) \in \mathbb{Z}[T]^{n\times n}$ we can define the *adjoint* $A^* \in \mathbb{Z}[T]^{n\times n}$, where

$$(a_{i,j})^* = a^*_{j,i}.$$

Recall, that a matrix A is called *self-adjoint* if $A = A^*$.

3.3 Godsil-Hensel matrices

Let Γ be a connected cover of some graph Δ and consider the group T of all automorphisms of Γ that fix each fibre of Γ setwise. Then T acts semi-regularly on $V(\Gamma)$ (cf. [2, Sec. 7]), and in particular on each fibre of Γ. The group T is called the *voltage group* of Γ. If T acts regularly on each fibre, then Γ is called a *regular cover* of Δ.

In [2] Godsil and Hensel studied regular covers in general and in particular gave a characterization of regular antipodal distance regular covers of complete graphs. For this purpose they defined certain matrices over the integral group ring $\mathbb{Z}[T]$, that we will introduce below in the notation used by Klin and Pech in [8].

Let $A = (a_{i,j}) \in \mathbb{Z}[T]^{n\times n}$ be a matrix such that $a_{i,j} \in (\{\underline{g} \mid g \in T\} \cup \{0\})$, all elements on the diagonal are equal to 0, and such that A is self-adjoint. Then to A we can associate two graphs:

1) the underlying graph Δ_A with vertex set $V(\Delta_A) = \{1, \dots, n\}$ and edge set $E(\Delta_A) = \{\{i, j\} \mid a_{i,j} \neq 0\}$,

2) the derived graph Γ^A with vertex set $V(\Gamma^A) = \{1, 2, \dots, n\} \times T$ and edge set $E(\Gamma^A) = \{\{(i, g), (j, h)\} \mid a_{i,j} \neq 0, \text{ and } \underline{g} \cdot a_{i,j} = \underline{h}\}$.

Such matrices, when defining connected covers with voltage group T, are called *covering matrices*. When Δ_A is a complete graph K_n, and Γ^A is an (n, r, c_2)-cover of Δ_A then the matrix A is called the *Godsil-Hensel matrix* of the cover.

Theorem 3.1. *Let T be a finite group and let A be a covering matrix of order n over T. Then A is the Godsil-Hensel matrix of a regular antipodal (n, r, c_2)-cover of K_n with voltage group T if and only if*

$$A^2 = (n-1)I + (n-2-rc_2)A + c_2\underline{T}(J-I). \tag{3.1}$$

4 Main results

Throughout this section we let Φ denote a regular 3-graph, w a weight such that $\delta w = \Phi$. Let Γ_w be the antipodal 3-fold cover of K_n defined by w and let W be the matrix representation of w.

Lemma 4.1. *Let Φ be a regular 3-graph and let w be a weight with $\delta w = \Phi$. Then W satisfies:*

$$\begin{aligned} W^2 &= nI + \left((a+2)\underline{\mathbf{1}} + b\underline{\zeta, \zeta^2}\right)(W - I) && (4.1)\\ &= nI + (a+2-b)\underline{\mathbf{1}}(W-I) + b\underline{U_3}(J-I). && (4.2)\end{aligned}$$

Proof. We calculate $(W^2)_{x,y}$. For $x = y$ we have:

$$\begin{aligned}(W^2)_{x,x} &= \sum_{z\in X}(W)_{x,z}\cdot(W)_{z,x}\\ &= \sum_{z\in X}\underline{w(x,z)}\cdot\underline{w(z,x)}\\ &= \sum_{z\in X}\underline{w(x,z)}\cdot\underline{w(x,z)^{-1}}\\ &= \sum_{z\in X}\underline{\mathbf{1}} = n\underline{\mathbf{1}}.\end{aligned}$$

For $x \neq y$ we have:

$$\begin{aligned}(W^2)_{x,y} &= \sum_{z\in X}(W)_{x,z}\cdot(W)_{z,y}\\ &= \sum_{z\in X} w(x,z)\cdot w(z,y)\\ &= \sum_{z\in X}\delta w(y,x,z)\cdot w(x,y)\\ &= \left(\sum_{z\in X}\delta w(y,x,z)\right)\cdot\underline{w(x,y)}\\ &= \big((m(\mathbf{1})+2)\underline{\mathbf{1}} + m(\zeta)\underline{\zeta} + m(\zeta^2)\underline{\zeta^2}\big)\cdot\underline{w(x,y)}\\ &= ((a+2)\underline{\mathbf{1}} + b\underline{\zeta,\zeta^2})\cdot\underline{w(x,y)}.\end{aligned}$$

Summing up we get Equation (4.1).

Using:

$$b\underline{\zeta,\zeta^2}(W-I) = b\underline{U_3}J - b\underline{U_3}I - b\underline{\mathbf{1}}(W-I)$$

we get Equation (4.2). □

Proposition 4.2. *Every regular* 3*-graph with parameters* (n,a,b) *defines a cyclic* $(n,3,b)$*-cover.*

Proof. We prove that the matrix $C = W - I$ is the Godsil-Hensel matrix of the cyclic cover Γ_w. We use Equation (4.2) to prove that C satisfies the condition of Theorem 3.1.

$$\begin{aligned}C^2 = (W-I)^2 &= W^2 - 2W + I\\ &= W^2 - 2(W-I) - I\\ &= nI + ((a+2)\underline{\mathbf{1}} + b\underline{\zeta,\zeta^2})(W-I) - 2(W-I) - I\\ &= (n-1)I + (a\underline{\mathbf{1}} + b\underline{\zeta,\zeta^2})(W-I)\\ &= (n-1)I + (a-b)\underline{\mathbf{1}}(W-I) + b\underline{U_3}(J-I)\\ &= (n-1)I + (a-b)\underline{\mathbf{1}}C + b\underline{U_3}(J-I).\end{aligned}$$

Plugging in the values

$$c_2 = b,\quad a_1 = a,\quad r = 3$$

we obtain Equation (3.1) in Theorem 3.1. □

The converse is proved similarly:

Proposition 4.3. *Every cyclic $(n, 3, c_2)$-cover defines a regular 3-graph with parameters (n, a_1, c_2).*

Proof. Let A be the Godsil-Hensel matrix of a cyclic $(n, 3, c_2)$-cover. We show that $W = A + I$ is the matrix representation of a weight w in the switching class of a regular 3-graph. We have:

$$\begin{aligned} W^2 = (A+I)^2 &= A^2 + 2A + I \\ &= (n-1)I + (n-2-rc_2)A + c_2\underline{T}(J-I) + 2A + I \\ &= nI + (n-rc_2)A + c_2\underline{T}(J-I) \\ &= nI + (n-rc_2)(W-I) + c_2\underline{T}(J-I). \end{aligned}$$

For the values

$$b = c_2, \quad a = a_1, \quad r = 3$$

W satisfies Equation (4.2) in Lemma 4.1. □

To complete the picture we prove:

Proposition 4.4. *There is a $1-1$ correspondence between regular 3-graphs and cyclic $(n, 3, c_2)$-covers.*

Proof. Let w and w' be weights into U_3. All that needs to be shown is:

$$\delta w = \delta w' \iff \Gamma_w \cong \Gamma_{w'}.$$

As was explained after Definition 2.10, the switching of a weight w is interpreted as a cyclic permutation within the fibres of the corresponding cover Γ_w, thus switching equivalent weights yield isomorphic covers. The converse is straightforward. □

As a consequence, using Theorem 9.2 of Godsil and Hensel in [2], we obtain a restriction on the parameter set of a regular 3-graph.

Corollary 4.5. *If (n, a, b) are the parameters of a regular 3-graph then $3|n$.*

Proof. Since Γ_w is a cyclic $(n, 3, b)$-cover, then by Theorem 9.2 in [2] we have $3|n$. □

5 Higman's note: clarification and corrections

5.1 Regular 3-graphs and association schemes

Higman's first step in [6] is to define $\Gamma_w = (V, E)$, an antipodal 3-fold cover of K_n, with fibre set X. He then constructs a rank 4 symmetric association scheme from Γ_w, this association scheme is (in a different ordering than the one that appears in [6]) the metric association scheme of the ADRG Γ_w. Higman's key observation is the fact that this rank 4 association scheme admits a rank 6 fission by orienting all the non-edges of Γ_w. We present this construction.

Construction 5.1. Define:

$$\begin{aligned}
R_0 &= \mathrm{Id}_V,\\
R_1 &= \left\{\left(x_i, x_{i+1 \pmod 3}\right) \;\middle|\; i = 1,2,3, x \in X\right\},\\
R_2 &= \left\{\left(x_i, x_{i+2 \pmod 3}\right) \;\middle|\; i = 1,2,3, x \in X\right\},\\
R_3 &= E,\\
R_4 &= \left\{(x_i, y_j) \;\middle|\; i = 1,2,3, \{x_{i+1 \pmod 3}, y_j\} \in E\right\},\\
R_5 &= \left\{(x_i, y_j) \;\middle|\; i = 1,2,3, \{x_{i+2 \pmod 3}, y_j\} \in E\right\}.
\end{aligned}$$

Remark 5.2. Notice that the relations R_1, R_2, R_4, R_5 are anti-symmetric, $R_1 = R_2^t$ and $R_4 = R_5^t$; Also $S_1 = R_1 \cup R_2$ is the "distance 3" relation and $S_3 = R_4 \cup R_5$ is the "distance 2" relation with respect to $\Gamma_w = (V, E)$.

Proposition 5.3 (Higman). *$\mathbb{A}_6(\Gamma) := \left(V, \{R_i\}_{i=0}^5\right)$ is an association scheme.*

Proof. We calculate the intersection matrices of $\mathbb{A}_6(\Gamma)$. For example, we compute p_{44}^4: let $(x_i, y_j) \in R_4$ and suppose $w(x_i, y_j) = \zeta$ (we may assume so due to switching), thus $j = i + 1 \pmod 3$. We count the number of $z_k \in V$ such that $(x_i, z_k) \in R_4$ and $(z_k, y_j) \in R_4$: there are 3 types of $z \in X$ which contain such a z_k:

- $k = i \Longrightarrow \begin{matrix} w(x,z) = \zeta^2 \\ w(z,y) = \zeta \end{matrix} \Longrightarrow \delta w(x,y,z) = \zeta \cdot \zeta \cdot \zeta^2 = \zeta,$
- $k = i + 1 \pmod 3 \Longrightarrow \begin{matrix} w(x,z) = \mathbf{1} \\ w(z,y) = \mathbf{1} \end{matrix} \Longrightarrow \delta w(x,y,z) = \zeta \cdot \mathbf{1} \cdot \mathbf{1} = \zeta,$
- $k = i - 1 \pmod 3 \Longrightarrow \begin{matrix} w(x,z) = \zeta \\ w(z,y) = \zeta^2 \end{matrix} \Longrightarrow \delta w(x,y,z) = \zeta \cdot \zeta^2 \cdot \zeta = \zeta.$

Thus $\delta w(x,y,z) = \zeta \iff z$ is one of the above 3 types, hence $p_{44}^4 = b$.
In the same manner we obtain the intersection matrices $\{B_i\}_{i=0}^5$:

$$B_0 = \begin{bmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix}, B_1 = \begin{bmatrix} 0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0 \end{bmatrix}, B_2 = \begin{bmatrix} 0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\n-1&0&0&a&b&b\\0&n-1&0&b&a&b\\0&0&n-1&b&b&a \end{bmatrix}, B_4 = \begin{bmatrix} 0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&n-1&0&b&a&b\\0&0&n-1&b&b&a\\n-1&0&0&a&b&b \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&n-1&b&b&a\\n-1&0&0&a&b&b\\0&n-1&0&b&a&b \end{bmatrix}. \qquad \square$$

It turns out that the existence of such a rank 6 association scheme is a sufficient condition:

Proposition 5.4 (Higman)**.** *Every rank* 6 *association scheme with parameters as in the construction of* $\mathbb{A}_6(\Gamma_w)$ *arises from a regular* 3*-graph.*

Proof. Let $(V, \{R_i\}_{i=0}^5)$ be an association scheme with parameters (and notation) as in Construction 5.1. Define $T = R_0 \cup R_1 \cup R_2$, then T is an equivalence relation on V with equivalence classes of size 3. Denote $X = V/T$. We give a labeling of the elements of V as a 3-fold cover of $K_{|X|}$, and then we verify that in this cover we only have matchings of the 3 types shown in Figure 1. Let $a \in X$ be any fibre, and label its elements by a_1, a_2, a_3 so that $(a_1, a_2) \in R_1$. Then

$$\left(a_i, a_{i+1 \pmod 3}\right) \in R_1$$

for $i = 1, 2, 3$. We now label the elements of each fibre $x \neq a$ in X by x_1, x_2, x_3 so that

$$(a_i, x_i) \in R_3$$

for $i = 1, 2, 3$. To prove that (V, R_3) is a 3-fold cover of $K_{|X|}$ with matchings of the 3 permitted types, we prove two things:

(1) $(x_i, x_{i+1 \pmod 3}) \in R_1$ for all $x \in X$ and $i = 1, 2, 3$,

(2) there is no matching such that $(x_i, y_j) \in R_3$ and $(x_j, y_i) \in R_3$, where $i \neq j$.

Proof of (1): Assume that $(x_i, x_{i+1}) \in R_2$. Let k be such that $(a_i, x_{i+1}) \in R_k$. Then we have:

$$(x_i, a_i) \in R_3, \qquad (x_i, x_{i+1}) \in R_2, \qquad (x_{i+1}, a_i) \in R_{k'}.$$

Therefore:

$$p^3_{2k'} \neq 0.$$

Examining column 3 in the matrix B_2, we deduce that $k' = 4$, which means that $k = 5$. Also, we have:

$$(a_i, x_{i+1}) \in R_k, \qquad (a_i, a_{i+1}) \in R_1, \qquad (a_{i+1}, x_{i+1}) \in R_3.$$

This implies that:

$$p^k_{13} \neq 0.$$

Examining row 3 in the matrix B_1, we deduce that $k = 4$, which is a contradiction.

Proof of (2): Assume that $(x_i, y_j) \in R_3$ and $(x_j, y_i) \in R_3$ for some $i \neq j$. Let k be such that $(x_i, y_i) \in R_k$. W.l.o.g we may assume that $j = i + 1 \pmod 3$. Then we have:

$$(x_i, y_i) \in R_k, \qquad (x_i, x_{i+1}) \in R_1, \qquad (x_{i+1}, y_i) \in R_3.$$

Therefore:

$$p^k_{13} \neq 0.$$

Examining row 3 in the matrix B_1 we see $k = 4$. Also, we have:

$$(y_i, x_i) \in R_{k'}, \qquad (y_i, y_{i+1}) \in R_1, \qquad (y_{i+1}, x_i) \in R_3.$$

Thus we obtain:

$$p_{13}^{k'} \neq 0,$$

which implies that $k' = 4$ and $k = 5$, a contradiction.

It follows that all the matchings of the graph (V, R_3) are of the 3 types shown in Figure 1, and we can define a weight w on X by $w(x,x) = \mathbf{1}$ and $w(x,y) = \mathbf{1}, \zeta$ or ζ^2 for $x \neq y$ according to as the matching is of the first, second or third type. It is straightforward to verify that δw is regular. $\square$

5.2 Characterization and feasibility conditions

We now sum up the results of the previous sections with our characterization of regular 3-graphs. Notice that the equivalence of (ii) and (iii) is a characterization of cyclic $(n, 3, c_2)$-covers in terms of association schemes.

Corollary 5.5. *Let Γ be an antipodal 3-fold cover of K_n. The following are equivalent:*

(i) Γ *defines a regular 3-graph with parameters (n, a, b);*

(ii) Γ *is a cyclic $(n, 3, b)$-cover;*

(iii) $\mathbb{A}_6(\Gamma)$ *is an association scheme.*

Using this characterization we would like to obtain feasibility restrictions on the parameters (n, a, b) of regular 3-graphs. We begin by calculating the character-multiplicity tables of $\mathbb{A}_4(\Gamma_w)$ and $\mathbb{A}_6(\Gamma_w)$. We used the well-known computer software Mathematica to calculate these tables, the program code is presented in [7].

The character-multiplicity table of $\mathbb{A}_4(\Gamma_w)$ is:

$$\begin{bmatrix} 1 & 2 & n-1 & 2(n-1) \\ 1 & 2 & -1 & -2 \\ 1 & -1 & \alpha & -\alpha \\ 1 & -1 & \beta & -\beta \end{bmatrix} \begin{bmatrix} 1 \\ n-1 \\ z_1 \\ z_2 \end{bmatrix}.$$

Here:

- α and β are the roots of $x^2 - (a-b)x - (n-1) = 0$,
- $z_1 = \frac{2n\beta}{\beta - \alpha}$,
- $z_2 = 2n - z_1 = \frac{2n\alpha}{\alpha - \beta}$.

If $z_1 = z_2 = n$ then we have $\alpha = -\beta$, and $\alpha, \beta = \pm\sqrt{n-1}$.

Otherwise, $z_2 - z_1$ is a non-zero integer, and we have:

$$z_2 - z_1 = 2n\left(\frac{\alpha}{\alpha - \beta} - \frac{\beta}{\beta - \alpha}\right) = 2n\left(\frac{\alpha + \beta}{\alpha - \beta}\right).$$

This means that $\alpha - \beta = \sqrt{(a-b)^2 + 4(n-1)}$ is rational, i.e. $(a-b)^2 + 4(n-1)$ is a square, which implies that α and β are rational algebraic integers, and thus are integers.

The character-multiplicity table of $\mathbb{A}_6(\Gamma_w)$ is:

$$\begin{bmatrix} 1 & 1 & 1 & n-1 & n-1 & n-1 \\ 1 & \zeta & \zeta^2 & \alpha & \alpha\zeta & \alpha\zeta^2 \\ 1 & \zeta^2 & \zeta & \alpha & \alpha\zeta^2 & \alpha\zeta \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \zeta & \zeta^2 & \beta & \beta\zeta & \beta\zeta^2 \\ 1 & \zeta^2 & \zeta & \beta & \beta\zeta^2 & \beta\zeta \end{bmatrix} \begin{bmatrix} 1 \\ z_1/2 \\ z_1/2 \\ n-1 \\ z_2/2 \\ z_2/2 \end{bmatrix}.$$

Here:

- $\alpha\zeta$ and $\alpha\zeta^2$ are the roots of $x^2 + \alpha x + \alpha^2 = 0$,
- $\beta\zeta$ and $\beta\zeta^2$ are the roots of $x^2 + \beta x + \beta^2 = 0$.

Remark 5.6. In Higman's note appeared the equations:

- $x^2 - \alpha x + \left(\frac{3(n-1)}{2} + \alpha^2\right) = 0,$
- $x^2 - \beta x + \left(\frac{3(n-1)}{2} + \beta^2\right) = 0,$

which led him to the false conclusion that n must be odd. These equations are the result of a miscalculation of the intersection matrices B_3 and B_5 of $\mathbb{A}_6(\Gamma_w)$ (compare these matrices from our paper with those from the note [6]).

Summing up all the considered restrictions we obtain:

Proposition 5.7. *Necessary conditions for the set (n, a, b) of parameters of a regular 3-graph are:*

(i) $n = a + 2b + 2$,

(ii) $3|n$,

(iii) The roots α and β of the equation $x^2 - (a-b)x - (n-1) = 0$ are integers,

(iv) $\alpha - \beta$ divides $n\alpha$.

Proof. Item (i) appears in the beginning of Section 3. Item (ii) is Corollary 4.5. Item (iii) comes from the latter analysis of the character-multiplicity table of $\mathbb{A}_4(\Gamma_w)$, and (iv) is just the integrality of the multiplicity $\frac{z_2}{2} = \frac{n\alpha}{\alpha-\beta}$ of $\mathbb{A}_6(\Gamma_w)$. □

These feasibility conditions provide a list of just 64 feasible parameter sets with $n \leq 1000$. We refer to [7] for the complete list and details about known constructions for some of them.

5.3 The symplectic example

D. G. Higman provided an infinite family of regular 3-graphs which is described briefly below.

In [5], Higman considers a more general cohomological setting, and presents several group theoretic examples of regular 3-cocycles (here cochains are functions into a monoid with the appropriate conditions). These examples are mainly extensions of examples by

D. E. Taylor in [15]. We mention one of them. In this example, we consider weights with values in the additive group of the field $GF(q)$, thus we will use additive notation: C^2_+ instead of $C^2_\cdot$, δ_+ instead of $\delta.$ etc.

Let V be a $2m$-dimensional vector space over $GF(q)$. Let B be a non-degenerate alternating bilinear form on V. Then $B \in C^2_+(V, GF(q))$ is a weight on V with values in $GF(q)$. In case q is a prime, this 3-cocycle is a q-graph. To see that in this case $\Phi = \delta_+ B$ is a regular q-graph we consider the symplectic group $Sp(2m, q)$. It acts transitively on the non-zero vectors of V, thus the subgroup $H := VSp(2m, q)$ of the affine group on V acts 2-transitively on the vectors of V. The coboundary $\Phi = \delta_+ B$ is invariant under translations and is therefore invariant under the action of H on V. This provides an infinite family of regular q-graphs for every prime q.

6 New constructions

The equivalence of regular 3-graphs with parameters (n, a, b) and cyclic $(n, 3, b)$-covers provides a rich source of new examples of regular 3-graphs.

In their recent paper [8], Klin and Pech present a construction of cyclic $(m^2, 3, \frac{m^2}{3})$-covers from generalized Hadamard matrices of order m over the cyclic group of order 3; the set of such matrices is denoted by $\mathrm{gH}(U_3, m)$. Their method takes as input any generalized Hadamard matrix $H \in \mathrm{gH}(U_3, m)$ and produces a so-called *skew* generalized Hadamard matrix $W \in \mathrm{gH}(U_3, m^2)$ of order m^2; such matrices correspond to cyclic $(m^2, 3, \frac{m^2}{3})$-covers, this is the Godsil-Hensel matrix of the cover.

We used classifications of generalized Hadamard matrices with suitable parameters (see [3], [9] and [14]) to construct all the corresponding non-isomorphic cyclic covers using the Klin-Pech method, which provide different regular 3-graphs. A summary of our new constructions of regular 3-graphs:

- 1 new example with parameters $(36, 10, 12)$,
- 1 new example with parameters $(45, 19, 12)$ (exceptional),
- 1 new example with parameters $(81, 25, 27)$,
- 1 new example with parameters $(144, 46, 48)$,
- 28 new examples with parameters $(324, 106, 108)$.

For the complete list of feasible parameter sets with $n \leq 1000$, and details about the above examples see [7].

7 Extension to regular t-graphs with $t \geq 4$

The theory outlined in this paper can be extended to regular t-graphs with any $t \geq 4$ only if we impose certain restrictions on the parameters of the regular t-graph. For example, when

t is odd, the parameters of a regular t-graph are:

$$\begin{aligned} &n, \\ &m(1), \\ &m(\zeta) = m(\zeta^{t-1}), \\ &\vdots \\ &m(\zeta^{t-1/2}) = m(\zeta^{t+1/2}). \end{aligned}$$

A graph Γ_w defined by a regular t-graph will be distance regular only if most parameters of the regular t-graph are equal. Explicitly, in the case that t is odd we demand:

$$m(\zeta) = m(\zeta^2) = \cdots = m(\zeta^{t-1/2}).$$

In this case, these will also be cyclic covers since for any t we have

$$C_{S_t}(C_t) \cong C_t.$$

Here we use the notation $C_t \leq S_t$ for the cyclic group $C_t = \langle (1, 2, \ldots, t) \rangle$.

Higman's theory also extends to regular t-graphs with $t \geq 4$ in the case of equal parameters (as described above). The construction of $\mathbb{A}_4(\Gamma_w)$ is exactly the same, and it has a rank $2t$ refinement which completely determines the weight w (analogously to $\mathbb{A}_6(\Gamma_w)$ in the case of regular 3-graphs). Thus, the extension of our theory to $t \geq 4$ is described schematically in Figure 2:

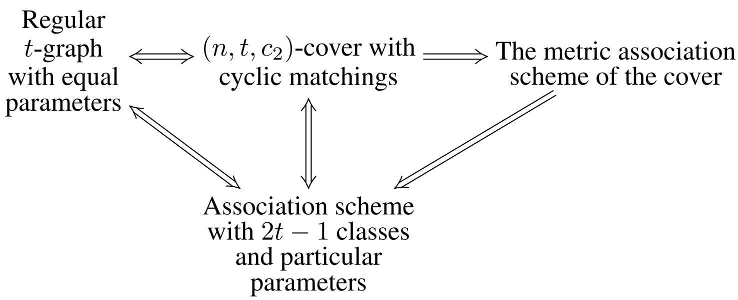

Figure 2: Extension to $t \geq 4$

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Some properties of the Zagreb eccentricity indices

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Abstract

The concept of Zagreb eccentricity (E_1 and E_2) indices was introduced in the chemical graph theory very recently [5, 12]. The first Zagreb eccentricity (E_1) and the second Zagreb eccentricity (E_2) indices of a graph G are defined as

$$E_1 = E_1(G) = \sum_{v_i \in V(G)} e_i^2$$

and

$$E_2 = E_2(G) = \sum_{v_i v_j \in E(G)} e_i \cdot e_j \, ,$$

where $E(G)$ is the edge set and e_i is the eccentricity of the vertex v_i in G. In this paper we give some lower and upper bounds on the first Zagreb eccentricity and the second Zagreb eccentricity indices of trees and graphs, and also characterize the extremal graphs.

Keywords: Graph, first Zagreb eccentricity index, second Zagreb eccentricity index, diameter, eccentricity.

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1 Introduction

Mathematical chemistry is a branch of theoretical chemistry using mathematical methods to discuss and predict molecular properties without necessarily referring to quantum mechanics [1, 8, 14]. Chemical graph theory is a branch of mathematical chemistry which applies graph theory in mathematical modeling of chemical phenomena [6]. This theory has an important effect on the development of the chemical sciences.

Topological indices are numbers associated with chemical structures derived from their hydrogen-depleted graphs as a tool for compact and effective description of structural formulas which are used to study and predict the structure-property correlations of organic compounds. Molecular descriptors are playing significant role in chemistry, pharmacology, etc. Among them, topological indices have a prominent place [13]. One of the best known and widely used is the connectivity index, χ, introduced in 1975 by Milan Randić [11]. The Randić index is one of the most famous molecular descriptors and the paper in which it is defined is cited more than 1000 times. The first M_1, and the second M_2, Zagreb indices (see [2],[3],[4],[7],[9] and the references therein) are defined as:

$$M_1 = M_1(G) = \sum_{v_i \in V(G)} d_i^2$$

and

$$M_2 = M_2(G) = \sum_{v_i v_j \in E(G)} d_i \cdot d_j .$$

where d_i is the degree of the vertex $v_i \in V(G)$ in G.

Let $G = (V, E)$ be a connected simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. Also let d_i be the degree of the vertex v_i, $i = 1, 2, \ldots, n$. For vertices $v_i,\ v_j \in V(G)$, the distance $d_G(v_i, v_j)$ is defined as the length of the shortest path between v_i and v_j in G. The eccentricity of a vertex is the maximum distance from it to any other vertex,

$$e_i = \max_{v_j \in V(G)} d_G(v_i, v_j) .$$

The maximum eccentricity over all vertices of G is called the diameter of G and denoted by d.

The invariants based on vertex eccentricities attracted some attention in Chmistry. In an analogy with the first and the second Zagreb indices, M. Ghorbani et al. and D. Vukičević et al. define the first E_1, and the second, E_2, Zagreb eccentricity indices by [5, 12]

$$E_1 = E_1(G) = \sum_{v_i \in V(G)} e_i^2 \tag{1.1}$$

and

$$E_2 = E_2(G) = \sum_{v_i v_j \in E(G)} e_i \cdot e_j . \tag{1.2}$$

where $E(G)$ is the edge set and e_i is the eccentricity of the vertex v_i in G.

Let $G = (V(G), E(G))$. If $V(G)$ is the disjoint union of two nonempty sets $V_1(G)$ and $V_2(G)$ such that every vertex in $V_1(G)$ has degree r and every vertex in $V_2(G)$ has degree

s, then G is (r,s)-semiregular graph. When $r = s$, is called a regular graph. As usual, $K_{a,b}$ $(a+b=n)$, P_n and $K_{1,n-1}$ denote respectively the complete bipartite graph, the path and the star on n vertices. A vertex of a graph is said to be pendent if its neighborhood contains exactly one vertex. An edge of a graph is said to be pendent if one of its vertices is a pendent vertex. Now we calculate

$$E_1(P_n) = \begin{cases} \frac{1}{12}\,(n-1)(7n^2-2n) & \text{if } n \text{ is even} \\ \frac{1}{12}\,(n-1)(7n^2-2n-3) & \text{if } n \text{ is odd.} \end{cases} \tag{1.3}$$

and

$$E_2(P_n) = \begin{cases} \frac{1}{12}\,n(7n^2-21n+20) & \text{if } n \text{ is even} \\ \frac{1}{12}\,(n-1)(7n^2-14n+3) & \text{if } n \text{ is odd.} \end{cases} \tag{1.4}$$

Also we have

$$E_1(K_{1,n-1}) = 4n-3 \text{ and } E_2(K_{1,n-1}) = 2n-2.$$

Denote by $\tilde{T}_n$, is a tree of order n with maximum degree $n-2$. We have $E_1(\tilde{T}_n) = 9n-10$, $E_2(\tilde{T}_n) = 6n-8$.

In this paper we give some lower and upper bounds on the first Zagreb eccentricity and the second Zagreb eccentricity indices of trees and graphs, and also characterize the extremal graphs.

2 Lower and upper bounds on Zagreb eccentricity indices

We now give lower and upper bounds on the Zagreb eccentricity indices of trees.

Theorem 2.1. Let T be a tree with n vertices. Then

$$\text{(i)} \qquad E_1(K_{1,n-1}) \leq E_1(T) \leq E_1(P_n) \tag{2.1}$$

$$\text{and (ii)} \qquad E_2(K_{1,n-1}) \leq E_2(T) \leq E_2(P_n). \tag{2.2}$$

Moreover, the left hand side (right hand side, respectively) equality holds in (2.1) and (2.2) if and only if $G \cong K_{1,n-1}$ ($G \cong P_n$, respectively).

Proof. Upper bound: If T is isomorphic to path P_n, then the right hand side equality holds in (2.1) and (2.2). Otherwise, $T \ncong P_n$. Let d be the diameter of tree T. Then there exists a path $P_{d+1}:\ v_1v_2\ldots v_{d+1}$ of length d in T. Thus we have the eccentricity of a vertex v_i in tree T,

$$e_i = \max\{d_G(v_i,v_1),\ d_G(v_i,v_{d+1})\}\,.$$

Since T is a tree, both vertices v_1 and v_{d+1} are pendent vertices. Thus we have $e_i \leq d$ for each $v_i \in V(G)$. Since $T \ncong P_n$, let v_k $(k \neq 1,\ d+1)$ be a vertex of degree one, adjacent to vertex v_j in T. We transform T into another tree T^* by deleting the edge $v_k\,v_j$ and join the vertices v_{d+1} and v_k by an edge. Then the longest path $P_{d+2}:\ v_1v_2\ldots v_{d+1}v_k$ of length $d+1$ in T^*. Let the vertex eccentricities be $e_1^*,\ e_2^*,\ldots,e_n^*$ in T^*. Therefore we have $e_t^* = \max\{d_G^*(v_t,v_1),\ d_G^*(v_t,v_k)\} = \max\{d_G(v_t,v_1),\ d_G(v_t,v_{d+1})+1\} \geq \max\{d_G(v_t,v_1),\ d_G(v_t,v_{d+1})\} = e_t$ (as $d_G^*(v_t,v_k) = d_G(v_t,v_{d+1})+1$) for $t \neq k$ whereas $e_k^* = d+1 > d \geq e_k$ ($d_G^*(v_i,v_j)$ is the length of the shortest path between vertices

v_i and v_j in T^*). So we have $e^*_r\, e^*_s \geq e_r\, e_s$ for $v_r v_s \neq v_k v_j,\ v_k v_{d+1}$ and $e^*_k\, e^*_{d+1} = d(d+1) > d^2 \geq e_k\, e_j$. Using above result we get

$$E_1(T^*) - E_1(T) = \sum_{v_i \in V(T^*)} e_i^{*2} - \sum_{v_i \in V(T)} e_i^2 \geq e_k^{*2} - e_k^2 > 0$$

and

$$E_2(T^*) - E_2(T) = \sum_{v_r\, v_s \in E(T^*)} e^*_r\, e^*_s - \sum_{v_r\, v_s \in E(T)} e_r\, e_s \geq e^*_k\, e^*_{d+1} - e_k\, e_j > 0.$$

Therefore we have

$$E_i(T^*) > E_i(T), \ \ i = 1, 2.$$

By the above described construction we have increased the value of $E_i(T)$, $i = 1, 2$. If T^* is the path, we are done. If not, then we continue the construction as follows. Next we choose one pendent vertex ($\neq\ v_1,\ v_k$) from T^*, etc. Repeating the procedure sufficient number of times, we arrive at a tree in which the maximum degree 2, that is, we arrive at path P_n.

Lower bound: If T is isomorphic to star $K_{1,n-1}$, then the left hand side equality holds in (2.1) and (2.2). If T is isomorphic to $\tilde{T}_n$, then the left hand side inequality is strict in (2.1) and (2.2). Otherwise, $T \ncong K_{1,n-1}\,, \tilde{T}_n$. Suppose that a path $P_{d+1}\,:\ v_1 v_2 \ldots v_{d+1}$ of length d in T, where d is the diameter of T. Without loss of generality, we can assume that $d_2 \geq d_d$ (the degree of vertex v_2 is greater than or equal to the degree of vertex v_d). Now choose v_i to be an arbitrary maximum degree vertex, unless v_d has maximum degree, in which case v_i is chosen to be v_2. We transform T into another tree $\hat{T}$ by deleting the edge $v_d\, v_{d+1}$ and join the vertices v_i and v_{d+1} by an edge. Let the vertex eccentricities be $\hat{e}_1,\ \hat{e}_2, \ldots, \hat{e}_n$ in tree $\hat{T}$. Similarly, as before we obtain $\hat{e}_t \leq e_t$ for all $t = 1, 2, \ldots, n$. Using above we get

$$E_1(\hat{T}) - E_1(T) = \sum_{v_i \in V(\hat{T})} \hat{e}_i^2 - \sum_{v_i \in V(T)} e_i^2 \leq 0$$

and

$$E_2(\hat{T}) - E_2(T) = \sum_{v_r\, v_s \in E(\hat{T})} \hat{e}_r\, \hat{e}_s - \sum_{v_r\, v_s \in E(T)} e_r\, e_s \leq 0.$$

Therefore we have

$$E_i(\hat{T}) \leq E_i(T), \ \ i = 1, 2.$$

By the above described construction we have non-increased the value of $E_i(T)$, $i = 1, 2$. If $\hat{T}$ is to the tree $\tilde{T}_n$, we are done. If not, then we continue the construction as follows. Next we choose one pendent vertex from longest path in $\hat{T}$ such that its adjacent vertex is not maximum degree vertex. Now we delete that pendent edge and join the pendent vertex to the maximum degree vertex, etc. Repeating the procedure sufficient number of times, we arrive at a tree in which the maximum degree $n-2$, that is, we arrive at tree $\tilde{T}_n$. This completes the proof. $\square$

We now give lower and upper bounds on the Zagreb eccentricity indices of bipartite graph.

Theorem 2.2. Let G be a connected bipartite graph of order n with bipartition $V(G) = U \cup W$, $U \cap W = \emptyset$, $|U| = p$ and $|W| = q$. Then

$$\text{(i)} \qquad E_1(K_{p,q}) \leq E_1(G) \leq E_1(P_n) \tag{2.3}$$

$$\text{and (ii)} \qquad E_2(K_{p,q}) \leq E_2(G) \leq E_2(P_n). \tag{2.4}$$

Moreover, the left hand side (right hand side, respectively) equality holds in (2.3) and (2.4) if and only if $G \cong K_{p,q}$ ($G \cong P_n$, respectively).

Proof. If G is isomorphic to a complete bipartite graph $K_{p,q}$, then the left hand side equality holds in (2.3) and (2.4). Otherwise, $G \ncong K_{p,q}$. If we add an edge in G, then each vertex eccentricity will non-increase. Thus we have $e_i(G+e) \leq e_i(G)$. Using this property, one can see easily that $E_1(G) \geq E_1(K_{p,q} \backslash \{e\}) > E_1(K_{p,q})$ and $E_2(G) \geq E_2(K_{p,q} \backslash \{e\}) > E_2(K_{p,q})$, where e is any edge in $K_{p,q}$.

Let T be a spanning tree of connected bipartite graph G. Then by the above property, $E_1(G) \leq E_1(T)$ and $E_2(G) \leq E_2(T)$. Using this result with Theorem 2.1, we get the right hand side inequality in (2.3) and (2.4). Moreover, the right hand side equality holds in (2.3) and (2.4) if and only if $G \cong P_n$. This completes the proof. □

In [10], Hua et al. proved the following result in Theorem 3.1.

Lemma 2.3. Let G be a connected graph with $e_i = n - d_i$ for any vertex $v_i \in V(G)$. If $G \ncong P_4$, then $e_i \leq 2$ for any vertex $v_i \in V(G)$.

We now give some relation between first Zagreb index and the first Zagreb eccentricity index of graphs.

Theorem 2.4. Let G be a connected graph of order n with m edges. Then

$$E_1(G) \leq M_1(G) - 4mn + n^3, \tag{2.5}$$

where $M_1(G)$ is the first Zagreb index in G. Moreover, the equality holds in (2.5) if and only if $G \cong P_4$ or $G \cong K_n$ or G is isomorphic to a $(n-1, n-2)$-semiregular graph.

Proof. If $G \cong P_4$, then the equality holds in (2.5). Otherwise, $G \ncong P_4$. Now,

$$\begin{aligned} E_1(G) = \sum_{v_i \in V(G)} e_i^2 \leq \sum_{v_i \in V(G)} (n-d_i)^2 \;\; & \text{as } e_i \leq n - d_i \\ = M_1(G) - 4mn + n^3 \;\; & \text{as } M_1(G) = \sum_{v_i \in V(G)} d_i^2, \sum_{v_i \in V(G)} d_i = 2m. \end{aligned}$$

First part of the proof is over.

Now suppose that equality holds in (2.5). Then $e_i = n - d_i$ for all $v_i \in V(G)$. By Lemma 2.3, we conclude that $e_i \leq 2$ for any vertex $v_i \in V(G)$ as $G \ncong P_4$. Since $e_i = n - d_i$ for any vertex $v_i \in V(G)$, we must have $d_i = n-1$ or $n-2$ for any vertex $v_i \in V(G)$, that is, $G \cong K_n$ or G is isomorphic to a $(n-1, n-2)$-semiregular graph.

Conversely, one can see easily that (2.5) holds for P_4 or K_n or $(n-1, n-2)$-semiregular graph. □

Remark 2.5. $(n-1, n-2)$-semiregular graph is obtained by deleting i independent edges from K_n, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

We now give some relation between first Zagreb index, second Zagreb index and the second Zagreb eccentricity index of graphs.

Theorem 2.6. Let G be a connected graph of order n with m edges. Then

$$E_2(G) \leq M_2(G) - nM_1(G) + mn^2, \tag{2.6}$$

where $M_1(G)$ is the first Zagreb index, $M_2(G)$ is the second Zagreb index in G. Moreover, the equality holds in (2.6) if and only if $G \cong P_4$ or $G \cong K_n$ or G is isomorphic to a $(n-1, n-2)$-semiregular graph.

Proof. Now,

$$\begin{aligned} E_2(G) &= \sum_{v_i v_j \in E(G)} e_i \cdot e_j \\ &\leq \sum_{v_i v_j \in E(G)} (n-d_i)(n-d_j) \ \text{ as } e_i \leq n-d_i \text{ and } e_j \leq n-d_j \\ &= \sum_{v_i v_j \in E(G)} \Big(n^2 + d_i d_j - (d_i+d_j)n\Big) \\ &= M_2(G) - nM_1(G) + mn^2. \end{aligned}$$

First part of the proof is over. Moreover, one can see easily that the equality holds in (2.6) if and only if $G \cong P_4$ or $G \cong K_n$ or G is isomorphic to a $(n-1, n-2)$-semiregular graph, by the proof of Theorem 2.4. □

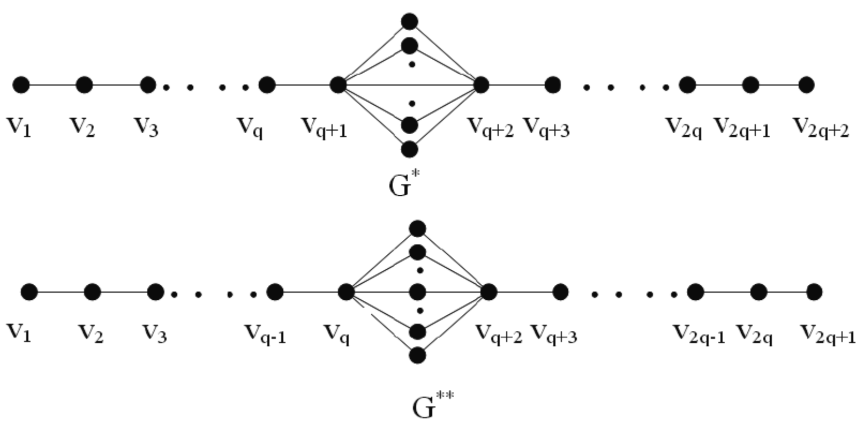

Figure 1: Graphs G^* and G^{**}.

Let $K^1_{2,a-2}$ be a connected graph of order a obtained from the complete bipartite graph $K_{2,a-2}$ with the vertices of degree $a-2$ are adjacent. Denote by G^*, is a connected graph of order n, obtained from $K^1_{2,n-2q-2}$ by attaching two paths P_{q+1} to two of its vertices of degree $n-2q-1$. Let Γ_1 be the class of graphs $H_1 = (V,E)$ such that H_1 is connected graph of diameter d $(d = 2q+1)$ with $V(G^*) = V(H_1)$ and $E(G^*) \subseteq E(H_1)$.

Let $K^2_{3,a-2}$ be a connected graph of order $a+1$ obtained from the complete bipartite graph $K_{2,a-2}$ with the vertices of degree $a-2$ are adjacent to a new vertex. Denote by G^{**}, is a connected graph of order n, obtained from $K^2_{3,n-2q-1}$ by attaching two paths P_q to two of its vertices of degree $n-2q$. Let Γ_2 be the class of graphs $H_2 = (V,E)$ such that H_2 is connected graph of diameter d $(d = 2q+2)$ with $V(G^*) = V(H_2)$ and $E(G^*) \subseteq E(H_2)$.

We now give another lower bound on $E_1(G)$ in terms of n, d and also characterize the extremal graphs.

Theorem 2.7. Let G be a connected graph of order n with diameter d. Then

$$E_1(G) \geq \begin{cases} \frac{1}{12}\,(3nd^2+6nd+3n+4d^3+3d^2-4d-3) & \text{if } d+1 \text{ is even} \\ \frac{d}{12}\,(3nd+4d^2+9d+2) & \text{if } d+1 \text{ is odd} \end{cases} \tag{2.7}$$

with equality holding if and only if $G \cong P_n$ or $G \in \Gamma_1$ or $G \in \Gamma_2$.

Proof. Since G has diameter d, G contains a path P_{d+1}: $v_1\, v_2\, \ldots,\, v_{d+1}$. Moreover, $n \geq d+1$ and $e_i \geq \lceil \frac{d}{2} \rceil$, $i = 1\,,2\,,\ldots\,,n$. If $n = d+1$, then $G \cong P_n$ and the equality holds in (2.7). Otherwise, $n > d+1$. By (1.3), we get

$$\sum_{i=1}^{d+1} e_i^2 = \begin{cases} \frac{d}{12}\,(7d^2+12d+5) & \text{if } d+1 \text{ is even} \\ \frac{d}{12}\,(7d^2+12d+2) & \text{if } d+1 \text{ is odd.} \end{cases} \tag{2.8}$$

Since $e_i \geq \left\lceil \frac{d}{2} \right\rceil$, $i = 1\,,2\,,\ldots\,,n$, using above result, we get

$$\begin{aligned} E_1(G) &= \sum_{i=1}^{d+1} e_i^2 + \sum_{i=d+2}^{n} e_i^2 \\ &\geq \begin{cases} \frac{d}{12}\,(7d^2+12d+5) + (n-d-1)\lceil \frac{d}{2} \rceil^2 & \text{if } d+1 \text{ is even} \\ \frac{d}{12}\,(7d^2+12d+2) + \frac{1}{4}\,(n-d-1)d^2 & \text{if } d+1 \text{ is odd,} \end{cases} \end{aligned} \tag{2.9}$$

from which we get the required result (2.7). First part of the proof is over.

Now suppose that equality holds in (2.7) with $n > d+1$. From equality in (2.9), we get

$$e_i = \left\lceil \frac{d}{2} \right\rceil \quad \text{for } i = d+2\,, d+3\,, \ldots\,, n.$$

Using above result we conclude that all the vertices $v_{d+2}\,, v_{d+3}\,, \ldots\,, v_{n-1}$ and v_n are adjacent to vertices v_q and v_{q+2} (when $d = 2q$), or v_{q+1} and v_{q+2} (when $d = 2q+1$). Hence $G \in \Gamma_1$ or $G \in \Gamma_2$.

Conversely, one can see easily that (2.7) holds for path P_n or graph G, where $G \in \Gamma_1$ or $G \in \Gamma_2$. $\square$

We now give another lower bound on $E_2(G)$ in terms of m, d and also characterize the extremal graphs.

Theorem 2.8. Let G be a connected graph of order n with diameter d. Then

$$E_2(G) \geq \begin{cases} \frac{1}{12}\left(3md^2+6md+4d^3-6d^2-4d+3m+6\right) & \text{if } d+1 \text{ is even} \\ \frac{d}{12}\left(3md+4d^2-4\right) & \text{if } d+1 \text{ is odd} \end{cases} \tag{2.10}$$

with equality holding if and only if $G \cong P_n$ or $G \in \Gamma_1$ or $G \in \Gamma_2$.

Proof. By (1.3), we get

$$\sum_{v_iv_j \in E(P_{d+1})} e_i\, e_j = \begin{cases} \frac{d+1}{12}\left(7d^2-7d+6\right) & \text{if } d+1 \text{ is even} \\ \frac{d}{12}\left(7d^2-4\right) & \text{if } d+1 \text{ is odd.} \end{cases} \tag{2.11}$$

Since $e_i \geq \left\lceil \frac{d}{2} \right\rceil$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} E_2(G) &= \sum_{v_iv_j \in E(P_{d+1})} e_i\, e_j + \sum_{v_iv_j \in E(G\backslash P_{d+1})} e_i\, e_j \\ &\geq \begin{cases} \frac{d+1}{12}\left(7d^2-7d+6\right)+(m-d)\lceil \frac{d}{2} \rceil^2 & \text{if } d+1 \text{ is even} \\ \frac{d}{12}\left(7d^2-4\right)+\frac{1}{4}(m-d)d^2 & \text{if } d+1 \text{ is odd,} \end{cases} \end{aligned}$$

from which we get the required result (2.10). Rest of the proof is similar as Theorem 2.7. □

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Consensus strategies for signed profiles on graphs

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Abstract

The median problem is a classical problem in Location Theory: one searches for a location that minimizes the average distance to the sites of the clients. This is for desired facilities as a distribution center for a set of warehouses. More recently, for obnoxious facilities, the antimedian was studied. Here one maximizes the average distance to the clients. In this paper the mixed case is studied. Clients are represented by a profile, which is a sequence of vertices with repetitions allowed. In a *signed* profile each element is provided with a sign from $\{+, -\}$. Thus one can take into account whether the client prefers the facility (with a $+$ sign) or rejects it (with a $-$ sign). The graphs for which all median sets, or all antimedian sets, are connected are characterized. Various consensus strategies for signed profiles are studied, amongst which Majority, Plurality and Scarcity. Hypercubes are the only graphs on which Majority produces the median set for all signed profiles. Finally, the antimedian sets are found by the Scarcity Strategy on e.g. Hamming graphs, Johnson graphs and halfcubes.

Keywords: Plurality strategy, median, majority rule, Hamming graph, Johnson graph, halfcube.

Math. Subj. Class.: 05C99, 05C12, 90B80

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1 Introduction

Most of the facility location problems in the literature are concerned with finding locations for desirable facilities. The goal there is to minimize a distance function between facilities and the demand sites (clients). One way to model this is using a network, see for instance [23, 24, 17]. In the discrete case one uses graphs, and clients and facilities are to be positioned on vertices.

One may formulate such location problems also in terms of achieving a consensus amongst the clients. Thus it becomes a problem in Consensus Theory. This approach has been fruitful in many other applications, e.g. in social choice theory, clustering, and mathematical biology, see for instance [7, 16, 15, 22].

From the view point of median consensus the classical result of Goldman [12] is very interesting: to find the median in a tree, just move to the majority of the clients. In [20], this majority strategy was formulated for arbitrary graphs. It was proved that majority strategy finds all medians for any set of clients if and only if the graph is a so-called median graph. Clients are termed as profiles in the language of graph theory, defined as a sequence of vertices in which vertices are allowed to repeat.

The class of median graphs comprises that of the trees as well as that of the hypercubes and grids. It allows a rich structure theory [18, 13, 21] and has many and diverse applications, see, for. e.g., [14], for median type consensus. In the majority strategy we compare the two ends of an edge v and w: if we are at v and at least half of the clients are strictly nearer to w than to v, then we move to w. One could relax the requirement for making a move as follows: one may move to w if there are at least as many clients closer to w than to v. Note that in the latter case less than half may actually be closer to w because there are many clients having equal distance to v and w. This idea of relaxing the majority strategy is formalized as plurality strategy in [4]. Other consensus strategies known as Condorcet, hill climbing and Steepest ascent hill climbing strategies were also proposed in [4]. There it is proved that the plurality, hill climbing and steepest ascent hill climbing strategies starting at an arbitrary vertex for arbitrary profiles will always return the median set of the profile if and only if the graph has connected medians.

However just as important are the problems dealing with the location of undesirable or obnoxious facilities, such as nuclear reactors, garbage dumps or water purification plants, see [9, 10, 11]. Here the criterion for optimality is maximizing the sum of the distances from the location of the obnoxious facility to the locations of the clients. The problem is known as the antimedian problem.

In general any two subgraphs may appear as antimedian and median sets, respectively, for clients located at all vertices without repetitions, with the distance between them being arbitrary, see for instance [2]. It is possible that facilities which are undesirable for some clients may be desirable for some other clients. For example, assume the problem of locating a beer parlour in a human habitat area. Some of the inhabitants may consider it as a desirable facility where as some others may consider it as undesirable facility. One way to formulate such problem is to associate a sign with the clients indicating whether the facility is desirable or undesirable to the client. In this paper we are concentrating on methods to solve such problems. For this a more general concept called signed profiles is introduced and is formally defined in the next section. In Section 3, the equivalence of rational weight functions and signed profiles are established, and the relationship between the median and antimedian sets for signed profiles is obtained. In Section 4, various consensus strategies are formulated, amongst which Majority, Plurality and Scarcity strategy, and it is shown that all these consensus strategies are pairwise distinct for signed profiles, as it has already been known for the usual profiles. We show that, for signed profiles, the hypercubes are the only graphs on which Majority produces the median set for any signed profile in Section 5. Finally, for Scarcity, we study various classes of graphs, on which this strategy produces the antimedian set for any signed profile.

2 Preliminaries

Let $G = (V, E)$ be a finite, connected, simple graph with vertex set V and edge set E. The distance function of G is denoted by d, where $d(u, v)$ is the length of a shortest u, v-path. We call a subset W of V *connected* if it induces a connected subgraph in G. The *interval* $I(u, v)$ between two vertices u and v consists of all vertices on shortest u, v-paths, that is:

$$I(u, v) = \{x \mid d(u, x) + d(x, v) = d(u, v)\}.$$

A *profile* on G is a finite sequence $\pi = (x_1, x_2, \ldots, x_k)$ of vertices of G. The *length* of π is the number $k = |\pi|$. Note that, π being a sequence, multiple occurrences are allowed. In this paper we extend the concept of profile: a *signed profile* is a profile where a sign from $\{+, -\}$ is added to each element. We write the sign of element x_i as s_i. Thus a signed profile is a sequence $\pi = (s_1x_1, s_2x_2, \ldots, s_kx_k)$. We call x_i an *element* of π and s_i its *sign*. Note that with this usage, a vertex occurring k times in a profile occurs as k *different* elements in a profile. For an element x of π we denote its sign also by s_x. For computational reasons, we identify a sign s also with the number $s1 = +1$ or -1, and talk about $+1$ or -1 as a sign. Thus we can take the sum of signs. As we will see below, a signed profile with all signs being $+1$, plays the role of the usual profile without signs. We call such a profile a *positive profile*. If all signs are -1, then the profile is *negative*. Since all our profiles are signed, we call a signed profile just a profile, and omit the adjective 'signed', except in the statements of lemmas and theorems (to avoid confusion with similar lemmas and theorems in the literature). A profile obtained from π by changing each s_i by $-s_i$ is denoted by $-\pi$. The *size* of a profile π is defined as

$$\|\pi\| = \sum_{i=1}^{k} s_i.$$

So, for positive profiles we have $\|\pi\| = |\pi|$, and for negative profiles we have $\|\pi\| = -|\pi|$.

For an edge uv in G, we denote by π_{uv} the subprofile of π consisting of the elements of π strictly closer to u than to v, and by π_u^v the subprofile of all elements at equal distance form u and v. Note that a profile, by definition, has a positive length. However, for subprofiles we allow the empty subprofile. For instance, a graph is bipartite if and only if the subprofile π_u^v is empty for any edge uv and any profile π.

In the literature we find such concepts as remoteness, median and antimedian of positive profiles, for e.g., see, [14] and [20]. These are all very natural and the definitions are in accordance with our intuition. Because the definitions for signed profiles are basically the same, we use the same terminology here.

The *remoteness* of a vertex v to a profile π is defined as

$$D(v, \pi) = \sum_{i=1}^{k} s_i d(x_i, v).$$

A permutation of the elements in a profile does not change remoteness. Because we are only interested in the remoteness to profiles, we will consider two profiles as the same if they can be obtained from each other by permuting the elements. We write the concatenation of two profiles π and ρ as $\pi\rho$. Thus, for any edge uv, we can write $\pi = \pi_{uv}\pi_u^v\pi_{vu}$.

A vertex minimizing $D(v, \pi)$ is called a *median* of the profile. The set of all medians of π is the *median set* of π and is denoted by $M(\pi)$. A vertex maximizing $D(v, \pi)$ is called an *antimedian* of the profile. The set of all antimedians of π is the *antimedian set* of π and is denoted by $AM(\pi)$. The reader has to keep in mind that the effects of the signs might just be contra-intuitive. For instance, if π is a negative profile, then a median of π is an antimedian of the positive profile $-\pi$.

A vertex x such that $D(x, \pi) \leq D(y, \pi)$, for all neighbors y of x, is a *local median* of π. The set of all local medians is denoted by $M_{loc}(\pi)$. If $D(x, \pi) \geq D(y, \pi)$, for all neighbors y of x, then x is a *local antimedian* of π. The set of all local antimedians is denoted by $AM_{loc}(\pi)$.

Let $\pi = (s_1x_1, s_2x_2, \ldots, s_kx_k)$ be a profile, then we have

$$D(v, -\pi) = \sum_{i=1}^{k} -s_i d(x_i, v) = -\sum_{i=1}^{k} s_i d(x_i, v) = -D(v, \pi).$$

From this observation we deduce that, by replacing a profile π by its opposite $-\pi$, the roles of (local) medians and (local) antimedians are exchanged. So we have $M(\pi) = AM(-\pi)$, etcetera. We single out one fact that we need in the sequel.

When the profile is of the form $(-x, +x)$ the median set is equal to the antimedian set and is the entire vertex set of the graph. But this is not the only case when the median and antimedian set are equal. For example, consider a positive profile π on a hypercube containing each vertex once. In this case the remoteness is constant and hence the median and antimedian set are the same and coincide with the entire vertex set of the graph. It can also be noted that the situation is the same for $-\pi$. In general, for such positive and negative profiles on so called *distance balanced graphs* both the median sets and antimedian sets coincide. The case for such positive profiles on the class of distance balanced graphs is proved in [5]. The same situation holds for some special even profiles (both positive and negative) in some other class of graphs, see for instance [1].

Lemma 2.1. *Let G be a connected graph and π a signed profile on G. Then, for any two adjacent vertices u, v in G,*

$$\|\pi_{uv}\| \leq \|\pi_{vu}\| \text{ if and only if } D(u, \pi) \geq D(v, \pi).$$

Proof. Since uv is an edge in G, we can ignore π_u^v in the following computation.

$$\begin{aligned} D(u,\pi) - D(v,\pi) = & \\ &= \sum_{x\in\pi_{uv}} s_x d(u,x) + \sum_{x\in\pi_{vu}} s_x d(u,x) - \sum_{x\in\pi_{uv}} s_x d(v,x) - \sum_{x\in\pi_{vu}} s_x d(v,x) \\ &= \sum_{x\in\pi_{uv}} s_x d(u,x) + \sum_{x\in\pi_{vu}} s_x d(u,x) - \sum_{x\in\pi_{uv}} s_x (d(u,x)+1) - \\ &\quad \sum_{x\in\pi_{vu}} s_x (d(u,x)-1) \\ &= \|\pi_{vu}\| - \|\pi_{uv}\|. \end{aligned}$$

From this the assertion follows immediately. □

3 Remoteness with respect to arbitrary weight functions

The concept of remoteness function and hence of medians and antimedians can also be studied with respect to weight functions defined on the vertex set of a graph. This was studied by Bandelt and Chepoi in [6] for non-negative weight functions in the case of medians. The equivalence of non-negative weight functions and positive profiles and hence the corresponding equivalence of the remoteness function and medians of non-negative weight functions and positive profiles are established in [4].

In this section, we establish that the same conclusion follows for arbitrary weight functions and signed profiles.

A *weight function* on G is a mapping f from V to the set of real numbers. Note that we now allow negative weights. We say that f has a *local minimum* at $x \in V$ if $f(x) \leq f(y)$, for every y adjacent to x. It has a *local maximum* if $f(x) \geq f(y)$, for every y adjacent to x. The remoteness function with respect to the weight function f is the function D_f from V to the set of real numbers defined as:

$$D_f(v) = D(v, f) = \sum_{x \in V} d(v, x) f(x).$$

Note that D_f is a weight function on G as well. A *local median* of f is a vertex u such that D_f has a local minimum at u. A *local antimedian* is a vertex at which D_f attains a maximum. The set of all local medians of a weight function f is denoted by $M_{loc}(f)$. The set of all local antimedians is denoted by $AM_{loc}(f)$. A *median* of f is a vertex u such that D_f has a global minimum at u. Similarly, an *antimedian* of f is a vertex at which D_f attains a maximum. The *median set* $M(f)$ of f is the set of all medians of f. The *antimedian set* $AM(f)$ of f is the set of all anti-medians of f.

Let f be a weight function on a graph G and let $-f$ be the weight function defined in the obvious way: its value at x is $-f(x)$. Then clearly, we have $D(v, f) = -D(v, -f)$, for any vertex v in G. In the sequel we make use of the following obvious facts.

Remark 3.1. Let f be an arbitrary weight function defined on the vertex set of a graph G. Then replacing f with $-f$ interchanges the roles of local maxima (minima) of f with local minima (maxima) of $-f$, and hence also interchanges the roles of both local and global medians (antimedians) of f with local and global antimedians (medians) of $-f$, respectively.

Let π be a profile on G. Then the *weight function associated with* π is the function f_π with $f_\pi(x) = \sum s_i$, where the summation is taken over the occurrences of vertex x. If x does not occur in π, then we set $f(x) = 0$. The following lemma follows immediately from the definitions. Note that, for any integer-valued weight function f, there are infinitely many profiles having f as their associated weight function.

Lemma 3.2. *Let G be a connected graph, and let π be a signed profile with associated weight function f_π. Then $D(v, \pi) = D(v, f_\pi)$, and hence $M(f_\pi) = M(\pi)$, and $AM(f_\pi) = AM(\pi)$, and $M_{loc}(f_\pi) = M_{loc}(\pi)$, and $AM_{loc}(f_\pi) = AM_{loc}(\pi)$, for every v in V.*

Let f be a weight function on a connected graph G. For a positive real number t, we define tf to be the weight function with $(tf)(x) = t \times f(x)$. Then we have $M(tf) =$

$M(f)$ and $M_{loc}(tf) = M_{loc}(f)$. Also we have $AM(tf) = AM(f)$ and $AM_{loc}(tf) = AM_{loc}(f)$. Finally, D_{tf} has a strict local minimum (maximum) at a vertex u if and only if D_f has a strict local minimum (maximum) at u. The following lemma is obvious.

Lemma 3.3. *Let g be rational weight function on a connected graph G. Then there is a signed profile π on G such that $f_\pi = tg$ for some positive integer t.*

In other words, antimedians (medians) of signed profiles are exactly antimedians (medians) of rational weight functions. The same holds for local antimedians (medians). Next we show that real-valued weight functions may be replaced by rational-valued weight functions, and thus by profiles, when one wants to compute antimedian (median) sets. We only present the proofs for the antimedian case. This is the one that we need in Sections 4 and 5. The case for the median sets is similar to that in [4], except that one has to take into account the signs. The next two Lemma's are the signed version of Lemma's 5 and 6 in [4]. The proofs are easy adaptations of those in [4]. Because they are short and prepare the way for Proposition 3.6, we include the proofs of the signed versions.

Lemma 3.4. *Let G be a connected graph, and let f be a weight function on G such that D_f has a local maximum at vertex u, which is not a global maximum. Then there is a weight function g such that D_g has a strict local maximum at u, which is not a global maximum. Furthermore if f is rational, then g may also be taken as a rational function.*

Proof. First note that, for any two vertices x and y, we have $d(x, y) < n = |V|$. Let $D(u, f) = \epsilon_1$. Let D_f have a global maximum at z, that is, $D(z, f) = \epsilon > \epsilon_1$. Let $\epsilon_2 = \epsilon - \epsilon_1$. Now define the function g as follows.

$$g(v) = \begin{cases} f(v) & if \quad v \neq u \\ f(u) - \frac{\epsilon_2}{n} & if \quad v = u. \end{cases}$$

Then $D(u, g) = D(u, f)$, because in these sums the values $f(u)$ and $g(u)$ of the functions at u are multiplied by $d(u, u) = 0$. For any vertex v adjacent to u, we have

$$D(v, g) = D(v, f) - \frac{\epsilon_2}{n} < D(v, f) \leq D(u, f) = D(u, g).$$

So D_g has a strict local maximum at u. Furthermore,

$$D(z, g) = D(z, f) - d(u, z)\frac{\epsilon_2}{n} > D(z, f) - \epsilon_2 = D(u, f) = D(u, g).$$

So g has a strict local maximum at u that is not a global maximum. Also if f is rational, then ϵ_2 is rational. So g is also rational.

Lemma 3.5. *Let G be a connected graph with the property that, for each rational weight function g, every local maximum of D_g is also a global maximum. Then the same property holds for any real-valued weight function f on G.*

Proof. Assume that for some real-valued weight function f there is a local maximum for D_f, at some vertex u that is not a global maximum. In view of the preceding lemma, we may assume that D_f has a strict local maximum at u. Let D_f have a global maximum at z, and let

$$\epsilon_1 = \min\{D(u, f) - D(x, f) \mid x \text{ adjacent to } u\},\ \epsilon_2 = D(z, f) - D(u, f),$$

$$\epsilon = \frac{\min(\epsilon_1, \epsilon_2)}{n^2}.$$

Now choose a rational weight function g such that $g(v) < f(v)$ and $f(v) - g(v) < \epsilon$, for all v. Then, for any vertex x adjacent to u, we have $D(u,g) > D(u,f) - \epsilon \times n^2 \geq D(u,f) - \epsilon_1 \geq D(x,f) > D(x,g)$. So u is a local maximum for D_g. Moreover, we have $D(z,g) > D(z,f) - \epsilon \times n^2 \geq D(z,f) - \epsilon_2 \geq D(u,f) > D(u,g)$. So u is not a global maximum for D_g, which is a contradiction. □

Graphs with connected median sets for non-negative weight functions were characterized in [6]. Using an analogous approach, we now are able to characterize graphs with connected antimedian and median sets for arbitrary weight functions. Before stating the result, we define basic concepts used in the following lines. A subgraph G of a graph H is an isometric subgraph if $d_G(u,v) = d_H(u,v)$ for all vertices u, v in G. We call a subset S of the vertex set of G a level set with respect to an integer λ if $S = \{x \in V(G) : D_f(x) \geq \lambda\}$.

Proposition 3.6. *For a graph G and any arbitrary weight function defined on the vertex set of G the following conditions are equivalent*

(i) $AM_{loc}(f) = AM(f)$ *for all weight functions* f*;*
(ii) *all level sets* $\{x : D_f(x) \geq \lambda\}$ *induce isometric subgraphs;*
(iii) *all antimedian sets* $AM(f)$ *induce isometric subgraphs;*
(iv) *all antimedian sets* $AM(f)$ *are connected.*

Proof. The implications $(ii) \Rightarrow (iii), (iii) \Rightarrow (iv)$ are trivial.

Next we prove $(iv) \Rightarrow (i)$. Let f be a weight function. Assume to the contrary that there exists a local antimedian z of f that is not an antimedian. Let y be an antimedian. Amongst such pairs y, z, we may choose y and z such that $d(y,z)$ is as small as possible. Our aim is to find two vertices u and v with $d(u,v) = 2$ and a weight function f' such that $AM(f') = \{u, v\}$. So f' does not have a connected antimedian set.

Consider the interval $I(y,z)$. Because of the minimality of $d(y,z)$, we have $D_f(y) > D_f(x)$ for all x in $I(y,z)$ distinct from y. Since z is a local antimedian, we have $D_f(z) \geq D_f(x)$, for any neighbor x of z, in particular for any neighbor x of z in $I(y,z)$. This implies that $d(y,z) \geq 2$. Hence, going from y to z within $I(y,z)$, we will encounter two vertices u, v such that $d(y,u) = d(y,v) - 2$, $d(z,u) = d(z,v) + 2$, $d(u,v) = 2$, with the properties that $D_f(u) > D_f(x)$ and $D_f(v) \geq D_f(x)$, for any common neighbor x of u and v. Note that these common neighbors of u and v are precisely the vertices in $I(u,v)$ distinct from u and v.

If there is any common neighbor x of u and v such that $D_f(v) = D_f(x)$, then we have $D_f(y) \geq D_f(u) > D_f(v)$. If $D_f(v) > D_f(x)$ for all common neighbors of u and v, then we compare $D_f(u)$ and $D_f(v)$. If $D_f(u) \geq D_f(v)$, then again we have $D_f(y) > D_f(v)$. If $D_f(v) > D_f(u)$, then we have $D_f(y) > D_f(v) > D_f(u)$. In this case we interchange the names of u and v. In all cases we end up with two vertices u and v at distance 2 with

$$D_f(y) \geq D_f(u) \geq D_f(v) \geq D_f(x),$$

for all common neighbors x of u and v, such that, additionally,

$$D_f(y) > D_f(v) \text{ and } D_f(u) > D_f(x),$$

for all common neighbors x of u and v.

We set $\mu_1 = \frac{D_f(u)-D_f(v)}{2}$. So $\mu_1 \geq 0$, and $D_f(v) = D_f(u) - 2\mu_1$. We set $\mu_2 = D_f(y) - D_f(v)$. Then $\mu_2 \geq \mu_1$ and $\mu_2 > 0$. Note that for any x in V, we have

$$D_f(v) \geq D_f(x) - \mu_2.$$

We construct the new weight function f' from f as follows

$$f'(x) = \begin{cases} f(x) - (\mu_1 + \mu_2) & \text{if } x = v \\ f(x) - (\mu_2) & \text{if } x = u \\ f(x) & \text{otherwise.} \end{cases}$$

Straightforward computation now yields

$$\begin{aligned} D_{f'}(u) &= D_f(u) - 2(\mu_1 + \mu_2) \\ &= D_f(v) - 2\mu_2 \\ &= D_{f'}(v); \end{aligned}$$

and for any vertex x in $I(u, v)$ distinct from u and v:

$$D_{f'}(x) = D_f(x) - \mu_1 - 2\mu_2 < D_f(u) - \mu_1 - 2\mu_2 = D_{f'}(u) - \mu_2 < D_{f'}(u);$$

and for any vertex x outside the interval (recall that $\mu_1 \leq \mu_2$):

$$D_{f'}(x) \leq D_f(x) - 3\mu_2 - \mu_1 \leq D_f(u) - 2\mu_2 - \mu_1 < D_{f'}(u).$$

Thus $AM(f') = \{u, v\}$, and hence the antimedian set of f' is not connected. This impossibility proves this implication.

It remains to prove that $(i) \Rightarrow (ii)$. Let $AM_{loc}(f) = AM(f)$ for all weight functions f. Assume to the contrary that the level set $S = \{x \mid D_f(x) \geq \lambda\}$ corresponding to λ is not isometric. Hence there exist two vertices u, v such that no shortest u, v-path lies completely inside S. Obviously, we can select u, v in S such that $D_f(x) < \lambda$ for any x in $I(u, v)$, distinct from u and v. Without loss of generality we may assume $D_f(u) \leq D_f(v)$. Set $D_f(z) - D_f(u) = \mu_1$, where z is an antimedian, and set $\epsilon = \min\{D_f(u) - D_f(w) \mid w \in I(u, v)\}$. Note that, since $D_f(u) > \lambda > D_f(w)$, for w in $I(u, v)$, we have $\epsilon > 0$. Let $\mu_2 = \frac{\epsilon}{d(u,v)}$. Define a weight function f' such that

$$f'(x) = \begin{cases} f(x) - \mu_1 & \text{if } x = u \\ f(x) - (\mu_1 + \mu_2) & \text{if } x = v \\ f(x) & \text{otherwise.} \end{cases}$$

Straightforward computation now yields

$$\begin{aligned} D_{f'}(v) &= D_f(v) - d(u, v)\mu_1 \\ &> D_f(u) - d(u, v)(\mu_1 + \mu_2) \\ &= D_{f'}(u) \end{aligned}$$

and for any vertex w in $I(u, v)$ distinct from u and v:

$$\begin{aligned} D_{f'}(w) < D_f(w) - d(u, v)\mu_1 \leq D_f(u) - \epsilon_2 - d(u, v)\mu_1 = \\ D_f(u) - d(u, v)(\mu_1 + \mu_2) = D_{f'}(u) \end{aligned}$$

and for any other vertex x:

$$D_{f'}(x) < D_f(x) - (d(u,v)+1)\mu_1 < D_f(z) - \mu_1 - d(u,v)\mu_1 = \\ D_f(u) - d(u,v)\mu_1 = D_{f'}(u).$$

This implies that v is the unique antimedian of f', while u is a local antimedian, which is not an antimedian vertex. This contradicts the assumption, by which the proof is complete. □

Above we established the equivalence of real-valued weight functions, rational-valued weight functions, and signed profiles with respect to medians etcetera. The next theorem is now an easy consequence of the previous results.

Theorem 3.7. *Let G be a connected graph. Then the following conditions are equivalent.*

(i) *The antimedian set $AM(f)$ is connected, for all weight functions f on G.*
(ii) $AM(f) = AM_{loc}(f)$*, for all weight functions f on G.*
(iii) *The median set $M(f)$ is connected, for all weight functions f on G.*
(iv) $M(f) = M_{loc}(f)$*, for all weight functions f on G.*
(v) $AM(f) = AM_{loc}(f)$*, for all rational weight functions f on G.*
(vi) $AM(\pi) = AM_{loc}(\pi)$*, for all signed profiles π on G.*
(vii) $M(f) = M_{loc}(f)$*, for all rational weight functions f on G.*
$(viii)$ $M(\pi) = M_{loc}(\pi)$*, for all signed profiles π on G.*

Proof. (i) up to (iv) are equivalent by Proposition 3.6, and Remark 3.1 .

$(ii) \Rightarrow (v)$ follows trivially.

$(v) \Rightarrow (ii)$ follows from Lemma 3.5.

$(v) \Rightarrow (vi)$: Let π be a signed profile on G. Now consider its associated weight function f_π. By Lemma 3.2, we have $D(v, f_\pi) = D(v, \pi)$. Since D_{f_π} cannot have any local maximum that is not a global maximum, D_π also cannot have any local maximum that is not a global maximum.

$(vi) \Rightarrow (v)$: Let g be any rational weight function on G. By Lemma 3.3, there is a positive integer t and a signed profile π such that $f_\pi = tg$. By Lemma 3.2, $D_{f_\pi} = D_\pi$, and, as observed above, D_{f_π} has a local maximum that is not a global maximum if and only if D_g have a local maximum that is not a global maximum. So D_g cannot have a local maximum that is not a global maximum.

The equivalence of (vii) and $(viii)$ with the other statements follows similarly. □

4 Consensus Strategies

If one wants to find the median set of a positive profile in a tree, then there exists a simple strategy formulated by Goldman [12] already in 1971. It reads as follows. When at vertex u, consider neighbor v of u. If there is a majority of the profile closer to v than to u, then move to v. In [20] this Majority Strategy was formulated for arbitrary graphs. There it was proved that the Majority Strategy produces the median set for any positive profile starting at any vertex if and only if the graph is a median graph, Theorem 4.1 below. For more details, we refer the reader to [20, 4]. A connected graph G is called a *median graph*, if every triple of vertices in G has a unique median. One of the main reasons underlying this result is that the structure of median sets is very nice in median graphs. In [4] four other related consensus strategies for positive profiles are studied. Antimedian sets are not so well-structured. So one cannot expect such deep results for signed profiles. But it is still possible to obtain some nice and unexpected results. Below we present a number of consensus strategies for signed profiles similar to the Majority Strategy from [18]. They are analogues of those in [4], but now formulated for signed profiles.

In all the strategies below the *input* is a connected graph G, a profile π, and an *initial vertex* at which the strategy starts. There are two possibilities: one gets stuck at a vertex, or it is possible to visit vertices more than once. In the latter case the strategy could get into a loop, so the stopping rule must be more sophisticated here. In all cases, the *output* after stopping is the single vertex where one gets stuck or the set of vertices visited at least twice. Steps 1, 3 and $4(i)$ below are the same for all strategies, so we list these only in the first instance. In all other instances we only list Step 2, describing when one moves to a neighbor, and Step $4(ii)$, the stopping rule when one does not get stuck.

Majority Strategy
1. Start at the initial vertex.
2. [MoveMS] If we are in u and v is a neighbor of u with $\|\pi_{vu}\| \geq \frac{1}{2}\|\pi\|$,
then we *move* to u.
3. We move only to a vertex already visited if there is no alternative.
4. We stop when
(i) we are stuck at a vertex u *or*
(ii) **[TwiceMS]** we have visited vertices at least twice, and
for each vertex u visited at least twice and each neighbor v of u,
either $\|\pi_{vu}\| < \frac{1}{2}\|\pi\|$ or v is also visited at least twice.

Before presenting the other strategies we quote the main theorem from [20]. This theorem has been the motivation for studying such strategies on graphs. It also shows the special role of median graphs within the Class of All Graphs. Due to the structure theory of median graphs, the equivalence of (ii) and (iii) on median graphs in the theorem is not surprising. But otherwise it would not have been something one would expect at first sight.

Theorem 4.1. [Majority Theorem] *Let G be a graph. Then the following conditions are equivalent.*

(i) *G is a median graph.*
(ii) *The Majority Strategy produces the median set $M(\pi)$ from any initial vertex, for each positive profile π on G.*
(iii) *The Majority Strategy produces the same set from any initial vertex, for each positive profile on G.*

In the majority strategy one moves towards majority. A slightly different point of view is to move away from minority. This seems to be the same, but it is not, as we will see below. This latter strategy is known as the Condorcet Strategy.

Condorcet Strategy
2. [MoveCS] If we are in u and v is a neighbor of u with $\|\pi_{uv}\| \leq \frac{1}{2}\|\pi\|$,
then we *move* to v.
4. (ii) **[TwiceCS]** we have visited vertices at least twice, and
for each vertex u visited at least twice and each neighbor v of u,
either $\|\pi_{uv}\| > \frac{1}{2}\|\pi\|$ or v is also visited at least twice.

In non-bipartite graphs the subprofile π_u^v of π, for an edge uv, is not always empty. From the viewpoint of voting, one might say that the elements of π_u^v abstain from voting when the choice is between u and v. So these may be ignored when the question is whether to move from u to v. This is the idea behind the Plurality Strategy. Note that on bipartite graphs Majority and Plurality coincide.

Plurality Strategy
2. [MovePS] If we are in u and v is a neighbor of u with $\|\pi_{vu}\| \geq \|\pi_{uv}\|$,
then we *move* to v.
4. (ii) **[TwicePS]** we have visited vertices at least twice, and
for each vertex u visited at least twice and each neighbor v of u,
either $\|\pi_{vu}\| < \|\pi_{uv}\|$ or v is also visited at least twice.

The next two strategies were introduced to find a (local) minimum based on a heuristic function in a search graph. They are also known as *Hill Climbing* and *Steepest Ascent Hill Climbing*, respectively.

Ascent Strategy
2. [MoveAS] If we are in u and v is a neighbor of u with $D(v, \pi) \leq D(u, \pi)$,
then we *move* to v.
4. (ii) **[TwiceAS]** we have visited vertices at least twice, and
for each vertex u visited at least twice and each neighbor v of u,
either $D(v, \pi) > D(u, \pi)$ or v is also visited at least twice.

Steepest Ascent Strategy
2. [MoveSAS] If we are in u and v is a neighbor of u with $D(v, \pi) \leq D(u, \pi)$, and
$D(v, \pi)$ is minimum among all neighbors of u, then we *move* to v.
4. (ii) **[TwiceSAS] = [TwiceAS]**.

The next simple Lemma is an analogue of Lemma 1 in [4] for signed profiles with the same conclusion. Note that the Plurality and Ascent strategy produce the same output for signed profiles on any connected graph. On bipartite graphs both coincide with Majority.

Lemma 4.2. *Let G be a connected graph and π a signed profile on G. Plurality Strategy makes a move from vertex v to vertex u if and only if $D(u, \pi) \leq D(v, \pi)$.*

Proof. The assertion follows immediately from the following computation:

$$\begin{aligned}
D(v,\pi) - D(u,\pi) &= \\
&= \sum_{x\in\pi_{vu}} s_x d(v,x) + \sum_{x\in\pi_{uv}} s_x d(v,x) - \sum_{x\in\pi_{vu}} s_x d(u,x) - \sum_{x\in\pi_{uv}} s_x d(u,x) \\
&= \sum_{x\in\pi_{vu}} s_x d(v,x) + \sum_{x\in\pi_{uv}} s_x d(v,x) - \sum_{x\in\pi_{vu}} s_x (d(v,x)+1) - \\
&\quad \sum_{x\in\pi_{uv}} s_x (d(v,x)-1) \\
&= \|\pi_{uv}\| - \|\pi_{vu}\|.
\end{aligned}$$

□

The various strategies are quite similar. But on general graphs they are all different. We present some examples to show this. The first example shows that Plurality, Condorcet and Majority strategies are pairwise distinct. Consider the profile $\pi = (+a, +b, -c, +d, +e, +f)$ on the graph shown in Figure 1. We have $\|\pi_{uv}\| = 1$, $\|\pi_{vu}\| = 0$, $\|\pi_{va}\| = \|\pi_{vb}\| = \|\pi_{vd}\| = \|\pi_{ve}\| = 2$ $\|\pi_{vc}\| = 4$, $\|\pi_{av}\| = \|\pi_{bv}\| = \|\pi_{dv}\| = \|\pi_{ev}\| = 1$, $\|\pi_{cv}\| = -1$, $\|\pi_{ua}\| = \|\pi_{ub}\| = \|\pi_{ud}\| = \|\pi_{ue}\| = \|\pi_{uf}\| = 3$, $\|\pi_{uc}\| = 5$, $\|\pi_{au}\| = \|\pi_{bu}\| = \|\pi_{du}\| = \|\pi_{eu}\| = \|\pi_{fu}\| = 1$, $\|\pi_{cu}\| = -1$.

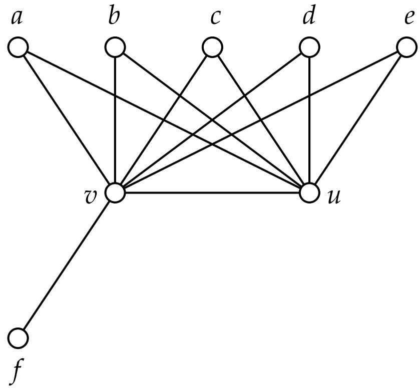

Figure 1: Consensus strategies differing on a graph

Apply all the strategies starting at u. Using Majority we may not move to any of its neighbors, so we are stuck at u. Thus the outcome of Majority is $\{u\}$. Note that we have $\|\pi_{ux}\| \leq \frac{1}{2}\|\pi\|$, for any neighbor x of u other than c. So, if we use Condorcet, then we can move to any of its neighbors except c. Note also that from a, b, d and e a move to either u or v is allowed, but from v we can move only to u. Thus using Condorcet we may move along u, a, b, d, e, v. Hence the output of the Condorcet Strategy is $\{u, a, b, d, e, v\}$. When we use Plurality, then we can move only to v and we get stuck at v. Hence the output of Plurality is $\{v\}$. Ascent and Steepest Ascent strategies also produce the output $\{v\}$.

It is shown in [4] that Steepest Ascent is essentially different from the other strategies for positive profiles. Note that the other strategies might make a move from u as soon as they find a neighbor v of u that satisfies the condition for a move, while Steepest Ascent has to check *all* neighbors of u before it can make a move. For a comparison of efficiencies of these strategies, see [3].

The following example shows that the first four strategies might not even find the median vertex, even if the graph is bipartite. Consider the complete bipartite graph $K_{2,5}$ with vertices a, b and $1, 2, 3, 4, 5$, where two vertices are adjacent if and only if one is a 'letter' and the other is a 'numeral'. Now take the profile $\pi = (+b, +1, +1, +1, +2, +2, +2, +3, +3, +3, +4, -5)$. Then we have $D(a, \pi) = 11$, $D(b, \pi) = 9$, $D(4, \pi) = 21$, $D(5, \pi) = 17$ and $D(i, \pi) = 13$, for $i = 1, 2, 3$. Take 1 as initial vertex and assume that we check its neighbors in alphabetical order. Then Majority, Condorcet, Plurality and Ascent strategies move to a and get stuck there, whereas Steepest Ascent moves to b and thus finds the median vertex of π.

It was already shown in [4] that Plurality produces the median set for any positive profile if and only if all median sets in the graph are connected. Hence Plurality produces the antimedian set for negative profile in such graphs. Moreover the antimedian sets are connected for all negative profiles in such graphs.

In the case of finding antimedian sets one would want to have the "converse" of the above strategies, that is, apply the strategy on $-\pi$ instead of π. Because we are working with signed profiles these are strategies in their own right. We list them below, with their appropriate names.

Minority Strategy
Minority applied to π is identical with Majority applied to $-\pi$.

Scarcity Strategy
Scarcity applied to π is identical with Plurality applied to $-\pi$.

Descent Strategy
Descent applied to π is identical with Ascent applied to $-\pi$.

Steepest Descent Strategy
Steepest Descent applied to π is identical with Steepest Ascent applied to $-\pi$.

It is interesting to note that Scarcity produces the antimedian set in hypercubes. Recall that the *n-dimensional hypercube* Q_n, the *n-cube* for short, has the $0, 1$-vectors of length n as its vertices, two vertices being adjacent if the corresponding vectors differ in exactly one coordinate. Take any i with $1 \leq i \leq n$. Let $Q^0_{n,i}$ be the $(n-1)$-dimensional subcube consisting of the vertices with a 0 as i-th coordinate, and let $Q^1_{n,i}$ be the complementary subcube consisting of the vertices with a 1 as i-th coordinate. For a profile π, let π^0_i be the subprofile of π in $Q^0_{n,i}$, and let π^1_i to the subprofile of π in $Q^1_{n,i}$. Let W be the set of vertices in Q_n, for which π has a *signed minority* in each coordinate. That is, x lies in W if and only if x has a 0 in the i-th coordinate when $\|\pi^1_i\| > \|\pi^0_i\|$, and a 1 when $\|\pi^0_i\| > \|\pi^1_i\|$, and a 0 or 1 when $\|\pi^0_i\| = \|\pi^1_i\|$. This is precisely the antimedian set of π. It is also a subcube of dimension d, where d is the number of coordinates, for which $\|\pi^0_i\| = \|\pi^1_i\|$.

Proposition 4.3. *Scarcity strategy produces the antimedian set on a hypercube for any signed profile.*

Proof. Take any i with $1 \leq i \leq n$. Take any vertex u in $Q^0_{n,i}$, and let v be its neighbor in $Q^1_{n,i}$. Then we have $\pi_{uv} = \pi^0_i$ and $\pi_{vu} = \pi^1_i$. So, if $\|\pi^0_i\| \geq \|\pi^1_i\|$, then we move from $Q^0_{n,i}$ to $Q^1_{n,i}$. And if $\|\pi^0_i\| > \|\pi^1_i\|$, then we never move back to $Q^0_{n,i}$. So Scarcity moves to the set of vertices W in Q_n, for which π has a signed minority in each coordinate. This is precisely the antimedian set of π. □

In general Scarcity will not always produce an antimedian. For example when we use Scarcity moves in a tree we will always stuck at leaf nodes, as we can see in the following lines. Consider a tree T with at least three leaves, and a positive profile π of length k on T. Take any leaf v that occurs less than $\frac{1}{2}k$ times in π, and let u be the neighbor of v in T. Note that $d(v, x) = d(u, x) + 1$ for any $x \neq v$ in T. Obviously we will move to v from u, but we will never move back to v using Scarcity. But v need not be the antimedian of π. If the profile is "close" to u, then obviously antimedian will be at some other leaves far away.

Next we prove an analogue of the main Theorem (Theorem 8 in [4]) for positive profiles in the case of signed profiles.

Theorem 4.4. *The following are equivalent for a connected graph G.*

(i) *The Scarcity Strategy produces $AM(\pi)$ from any initial vertex, for all signed profiles π on G.*
(ii) *$AM(\pi)$ is connected, for all signed profiles π on G.*
(iii) *$AM(\pi) = AM_{loc}(\pi)$, for all signed profiles π on G.*
(iv) *Descent Strategy produces $AM(\pi)$ from any initial vertex, for all signed profiles π on G.*
(v) *Steepest Descent Strategy produces $AM(\pi)$ from any initial vertex, for all signed profiles π on G.*
(vi) *Scarcity, Descent, and Steepest Descent Strategy each produce the same set from any initial vertex, for all signed profiles.*

Proof. $(i) \Rightarrow (ii)$: Suppose the antimedian set is not connected for some profile π. Then let u and v be two vertices in different components of $AM(\pi)$. Now, if Scarcity starts at u, then it cannot reach vertex v, because a move from an antimedian vertex to a non-antimedian vertex is not possible by Lemma 4.2. So the set computed by Scarcity will not include u, which is a contradiction.

$(ii) \Rightarrow (iii)$: This follows from Theorem 3.7.

$(iii) \Rightarrow (iv)$: Starting at any vertex, Descent always finds a local maximum. Since this local maximum is also global, it follows that Descent always reaches an antimedian, and since the antimedian set is connected, Descent finds all antimedian vertices.

$(iv) \Rightarrow (i)$: Assume that Descent finds the antimedian set. This means that Descent will move to an antimedian starting from any vertex and finds all the other antimedians. The same moves will be made by Scarcity, by Lemma 4.2. Hence Scarcity will compute the antimedian set correctly.

$(iii) \Rightarrow (v)$ follows similarly as $(iii) \Rightarrow (iv)$.

$(v) \Rightarrow (ii)$ follows from the fact that Steepest Descent finds a local maximum and does move from antimedian to antimedian but does not move from an antimedian to a non-antimedian.

$(i) \Rightarrow (vi)$ is obvious.

$(vi) \Rightarrow (i)$ follows from the fact that, starting from an antimedian, Scarcity can produce only a set of antimedians which includes the initial vertex. So starting from any vertex it produces the same set if and only if the produced set is actually $AM(\pi)$. The same argument works for Descent and Steepest Descent. □

5 The Majority Strategy for Signed Profiles

In this section a characterization for hypercubes is obtained as the graphs for which Majority always produces the median set for any signed profile. Before stating the result, we need a few facts from the theory of median graphs as developed in [18]. A median graph is bipartite, and does not contain $K_{2,3}$ as induced subgraph. This implies that any two vertices at distance 2 have either one or two common neighbors. It is proved in [18] that a graph G is a hypercube if and only if it is a median graph in which any two vertices at distance 2 have exactly two common neighbors. For any vertex w in a graph G, we write $N_i(w) = \{x \mid d(x, w) = i\}$, and $N_{>i}(w) = \bigcup_{j>i} N_j(w)$.

Proposition 5.1. *Let AM be the antimedian function on a median graph G. Then $AM(\pi)$ is connected for every signed profile π if and only if G is a hypercube.*

Proof. If G is a hypercube, then Proposition 4.3 gives us the required result.

Conversely, let G be a median graph for which the antimedian set is connected, for any signed profile. Let u and v be two vertices at distance 2, and let w be a common neighbor of u and v. Due to the above mentioned characterization of hypercubes in [18] we have to prove that there exists a unique common neighbor of u and v different from w. Consider the profile $\pi = (+w)$ of length 1. Note that, for any x in $N_j(w)$, we have $D_f(x) = j$. So $N_{>0}(w) = V - w$ is a level set with respect to π. Due to Proposition3.6 any level set of π induces an isometric subgraph. So, within $V - w$, the vertices u and v have distance 2 as well, that is, there is a common neighbor z in $V - x$. Since a median graph is bipartite and does not contain $K_{2,3}$, this neighbor is unique. □

The next theorem is an analogue of the majority theorem for signed profiles which turns out to be a new characterization of hypercubes.

Theorem 5.2. *A graph G is a hypercube if and only if the Majority Strategy, starting from any initial vertex, produces the median set for any signed profile on G.*

Proof. If G is a hypercube, then, by Proposition 4.3, Scarcity produces the antimedian set for any signed profile. So, since the hypercube is bipartite, Minority produces the antimedian set for any signed profile, whence Majority produces the median set for any signed profile.

Conversely, assume that Majority produces the median set for any signed profile. Then it also produces the median set for any positive profile. So, by Theorem 4.1, the graph is a median graph. Hence, by Proposition 5.1, the graph is a hypercube. □

6 Graphs for which Scarcity produces the Antimedian Set for any Signed Profile

In this section we discuss some graph classes for which Scarcity always produces the antimedian set for any signed profile.

First we consider the Hamming graphs. Let $k_1, \ldots, k_n$ be positive integers, and let V be the Cartesian product

$$\Pi_{i=1}^{n}\{0, 1, \ldots, k_i - 1\}.$$

The *Hamming graph* $H_{k_1,\ldots,k_n}$ is the graph with vertex set V, in which two vertices are joined by an edge if and only if the corresponding vectors differ in exactly one coordinate. The properties for Hamming graphs needed here probably all belong now to folklore, but could also be found in [18, 19], where they were characterized for the first time. The set of vertices in $H = H_{k_1,\ldots,k_n}$ having a in the i-th position of the corresponding vector is denoted as $H^a_{k_1,\ldots,k_n,i}$, or simply as H^a_i. For a profile π, we denote its subprofile contained in H^a_i by π^a_i.

Let π be a profile on $H = H_{k_1,\ldots,k_n}$. Fix a position i, for which $k_i \geq 2$, and let a and b be distinct elements in $\{0, \ldots, k_i - 1\}$. Let u be a vertex in H^a_i, and let v be its neighbor in H^b_i. Then $\pi_{uv} = \pi^a_i$ and $\pi_{vu} = \pi^b_i$. Note that, if u is in $AM(\pi)$, then we have $\|\pi^a_i\| \leq \|\pi^b_i\|$. This holds for every b in $\{0, \ldots, k_i - 1\}$ distinct from a. In this case we say that there is a *signed minority* at a in position i. Clearly, an antimedian vertex in H is a vertex with a signed minority in each coordinate. Let m_i be the number of elements in $\{0, \ldots, k_i - 1\}$ with a signed minority, for $i = 1, \ldots, n$. Then the antimedian set of π induces a subgraph isomorphic to $H_{m_1,\ldots,m_n}$. Obviously, any antimedian set is connected.

Proposition 6.1. *Starting form any vertex Scarcity Strategy produces the antimedian set on a Hamming graph for any signed profile.*

Proof. Let π be a (signed) profile on the Hamming graph $H = H_{k_1,\ldots,k_n}$. Take any i with $1 \leq i \leq n$ and any vertex u in H_i^a, and let v be its neighbor in H_i^b with $b \neq a$. If $\|\pi_i^a\| \geq \|\pi_i^b\|$, then we move from H_i^a to H_i^b. And if $\|\pi_i^a\| > \|\pi_i^b\|$, then we never move back to H_i^a. So Scarcity moves to the set of vertices W in H for which π has a signed minority in each coordinate. This is precisely the antimedian set of π. □

Graphs which admit a scale-λ, ($\lambda \geq 2$) embedding into a hypercube is called an ℓ_1 graph. The Johnson graphs and half cubes are important classes of ℓ_1 graphs which occur as hosts for isometric embeddings of graphs, [8]. Next we consider the Johnson graphs followed by half cubes. The *Johnson graph* $J_{n,k}$ has as vertices the k-element subsets of $\{1, 2, \ldots, n\}$, and two vertices are adjacent if and only if their intersection has size $k - 1$. In other words the vertices 'differ' in exactly one element. Some special Johnson graphs are: $J_{n,1}$ is the complete graph on n vertices, $J_{n,2}$ is the n-triangular graph, and $J_{n,3}$ is n-tetrahedral graph. Since each vertex u in $J_{n,k}$ corresponds to a k-element subset X of $\{1, 2, \ldots, n\}$, we represent u with the vector $[u_1, \ldots, u_n]$, where

$$u_i = \begin{cases} 1; & i \in X, \\ 0; & i \notin X \end{cases}$$

Clearly the total number of 1's in each vector representation is k. Moreover adjacent vertices differ in two positions. Note that mapping these vectors to the corresponding vectors in a hypercube Q_n corresponds to a so-called scale-2 embedding, that is, two vertices at distance d in the Johnson graph are mapped onto vertices at distance $2d$ in the hypercube, for any two vertices. Since below the antimedian sets in more than one graph will be considered, we denote the antimedian set of π in G also by $AM(\pi, G)$, and so forth.

Proposition 6.2. *Let G be a Johnson graph. Then $M(\pi)$ and $AM(\pi)$ are also Johnson graphs.*

Proof. Assume that $G = J(n, k)$. Consider the scale-2 embedding of G in to the hypercube Q_n. Let π be a profile in G, and let $M(\pi, Q_n)$ be isomorphic to Q_r. Without loss of generality we may assume that, for all the vertices $u = [u_1, \ldots, u_n]$ in this subcube, the coordinates at positions $r + 1$ up to n are all the same, and that in the remaining positions $1, \ldots, r$ values 0 and 1 are taken. Let m be the total number of 1's, in positions $r+1, \ldots, n$.

We analyze the properties of median sets in G by considering two cases.

Case 1. $M(\pi, Q_n) \cap G \neq \emptyset$.
Clearly $M(\pi, G)$ induces a subgraph isomorphic to $J_{r,(k-m)}$.

Case 2. $M(\pi, Q_n) \cap G = \emptyset$.
In this case we have either $m < k - r$ or $m > k$. Clearly, if $m < k - r$, we get a vertex in G by changing a minimum number of coordinates, say p, from the vertex in $M(\pi, Q_n)$ having 1s in positions $1, \ldots, r$.

Similarly, when $m > k$, we get a vertex in G with a minimum number of changes, by selecting the vertex with 0's in positions $1, \ldots, r$. Since we are looking for a median set in G, we select the positions, in such a way that the change in remoteness is minimum. Thus we select p coordinate positions with smaller signed majority values. If the signed majority values are distinct we get a single vertex in G. Otherwise we make a selection among, say p' positions. In this case the subgraph induced by the vertices of G thus obtained will be isomorphic to $J_{p',p}$.

Since the remoteness is same for all vertices in the median set we get the same result independent of the vertex selected. Hence we get a subgraph that is a Johnson graph as the median set.

With similar arguments, by taking the signed minority values at coordinate positions, we can prove that antimedian sets also induce some Johnson graph. This completes the proof.

□

From the above theorem we have the following corollary.

Corollary 6.3. *Let G be a Johnson graph. Then $M(\pi)$ and $AM(\pi)$ are connected, for any signed profile π in G.*

From the above Corollary and Theorem 4.4 we have:

Corollary 6.4. *Starting from any vertex on a Johnson graph Scarcity strategy produces the antimedian set for any signed profile.*

Next, we consider halfcubes. The vertex set of a halfcube is the subset of the vertices of the hypercube Q_n with an even (respectively, odd) number of ones in their vector representation. Two vertices are adjacent when they differ in exactly two positions, see [8]. Halfcubes also admit a scale-2 embedding into the corresponding hypercube.

Theorem 6.5. *Let G be a halfcube, then $M(\pi, G)$ and $AM(\pi, G)$ are connected for any signed profile π in G.*

Proof. Let Q_n be the hypercube of dimension n in which G is scale-2 embedded. Let π be an arbitrary profile in G and $\|\pi\| = k$. Note that by applying the Majority rule for the given profile π of the halfcube embedded into hypercube Q_n (looking as the vertices of a hypercube), we get the median of π in Q_n which will be a sub-hypercube, say Q_r. We analyze the property of $M(\pi, G)$ by considering the following two cases separately.
Case 1. $M(\pi, Q_n)$ is a hypercube Q_r of dimension at least one.
Clearly Q_r has half vertices in the corresponding halfcube - call this set X. Set X forms a halfcube in G, hence X is connected. Since the graph G is scale-2 embedded the remoteness in G is obtained by dividing the corresponding remoteness in Q_n by 2, we get $M(\pi, G) = X$, as we follow the signed Majority rule on π.
Case 2. $M(\pi, Q_n)$ in Q_n contains exactly one vertex say x.
If x belongs to G, then clearly $M(\pi, G) = \{x\}$ as the case may be and hence we are done.

So assume that x is not in G. Note that $x = (x^1, \ldots, x^d)$ can be obtained from the signed Majority rule among coordinates of the profile π. Let m_i, $1 \leq m_i \leq n$ be the signed majority at each position. Let $m = \min\{m_1, \ldots, m_n\}$. Clearly if for any vertex y obtained by changing any single i^{th} coordinate of x, the remoteness changes by $2m_i - k$, where $\|\pi\| = k$. This change in remoteness is minimum for coordinates having signed

Majority value m. Hence $M(\pi, G)$ is precisely the set of vertices obtained from G by changing any coordinate of x, having minimum signed Majority m_i. These vertices are all adjacent to x, and hence forms a clique in G. Thus $M(\pi, G)$ is connected for any signed profile.

With similar arguments and by taking m as maximum($m_1, \ldots, m_n$), where each m_i is signed minority, we can prove that $AM(\pi, G)$ is also connected for any profile, which completes the proof. □

From the proof of the above theorem, we have the following corollary.

Corollary 6.6. *Let G be a halfcube, then $M(\pi, G)$ and $AM(\pi, G)$ induce a halfcube in G or a clique, for any profile π in G.*

From Theorem 6.5 and Theorem 4.4 we have:

Corollary 6.7. *Starting from any arbitrary vertex in a halfcube Scarcity Strategy always produce antimedian set for any signed profile*

7 Concluding remarks

In this paper, we have proved that the classes of graphs in which the consensus strategies Scarcity, Descent and Steepest Descent will always produce the antimedians for any arbitrary signed profile is precisely the class of graphs with connected antimedians. This class of graphs is characterized in terms of (local) medians and (local) antimedians of (rational) weight functions. Also, we proved that, among the median graphs, the hypercubes are precisely the graphs with connected antimedians for an arbitrary signed profile. Moreover, we presented some classes on which Scarcity produces the antimedian set for any signed profile. An intriguing question remains: Which classes of graphs have connected antimedians for arbitrary signed profiles?

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Quasi m-Cayley circulants

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Abstract

A graph Γ is called a *quasi m-Cayley graph on a group* G if there exists a vertex $\infty \in V(\Gamma)$ and a subgroup G of the vertex stabilizer $\mathrm{Aut}(\Gamma)_\infty$ of the vertex ∞ in the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ, such that G acts semiregularly on $V(\Gamma)\setminus\{\infty\}$ with m orbits. If the vertex ∞ is adjacent to only one orbit of G on $V(\Gamma)\setminus\{\infty\}$, then Γ is called a *strongly quasi* m*-Cayley graph on* G. In this paper complete classifications of quasi 2-Cayley, quasi 3-Cayley and strongly quasi 4-Cayley connected circulants are given.

Keywords: Arc-transitive, circulant, quasi m*-Cayley graph.*

Math. Subj. Class.: 05C15, 05C10

1 Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected, and groups are finite. Given a graph Γ we let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ be the set of its vertices, edges, arcs and the automorphism group of Γ, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectively.

Let G be a finite group with identity element 1, and let $S \subset G\setminus\{1\}$ be such that $S^{-1} = S$. We define the *Cayley graph* $Cay(G, S)$ on the group G with respect to the connection set S to be the graph with vertex set G, in which two vertices $x, y \in G$ are adjacent if and only if $x^{-1}y \in S$. A *circulant* of order n is a Cayley graph on a cyclic group of order n.

In this paper we consider quasi-semiregular actions on graphs, a natural generalization of semiregular actions on graphs, which have been an active topic of research in the last decades (see, for example, [1, 2, 3, 4, 5, 8, 9, 11]). Following [7] we say that a group G acts

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quasi-semiregularly on a set X if there exists an element ∞ in X such that G fixes ∞, and the stabilizer G_x of any element $x \in X\backslash\{\infty\}$ is trivial. The element ∞ is called the point at infinity. A graph Γ is called *quasi m-Cayley on G* if the group G acts quasi-semiregularly on $V(\Gamma)$ with m orbits on $V(\Gamma)\backslash\{\infty\}$. If G is cyclic and $m = 1$ (respectively, $m = 2$, $m = 3$ and $m = 4$) then Γ is said to be *quasi circulant* (respectively, *quasi bicirculant*, *quasi tricirculant* and *quasi tetracirculant*). In addition, if the point at infinity ∞ is adjacent with only one orbit of G_∞ then we say that Γ is a *strongly quasi m-Cayley graph* on G.

Quasi m-Cayley graphs were first defined in 2011 by Kutnar, Malnič, Martinez and Marušič [7], who showed which strongly quasi $m-$Cayley graphs are strongly regular graphs.

In this paper, we consider which circulants are also quasi m-Cayley graphs. Our main results are stated in the following three theorems.

Theorem 1.1. Let Γ be a quasi 2-Cayley graph of order n which is also a connected circulant. Then either Γ is isomorphic to the complete graph K_n, or $n \equiv 1 \pmod 4$ is a prime and Γ is isomorphic to the Paley graph $P(n)$. Moreover, Γ is a quasi bicirculant.

Theorem 1.2. Let Γ be a connected circulant. Then Γ is also a quasi 3-Cayley graph if and only if either $\Gamma = K_n$, or replacing Γ with its complement if necessary, $\Gamma \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$. Moreover, Γ is a quasi tricirculant.

Theorem 1.3. Let Γ be a connected circulant. Then Γ is a strongly quasi 4-Cayley graph on a group G if and only if $\Gamma \cong C_9$ or $\Gamma \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all fourth powers in $\mathbb{Z}_n\backslash\{0\}$, and n is a prime such that $n \equiv 1 \pmod 4$. Moreover, Γ is a quasi tetracirculant.

The paper is organized as follows. In Section 2 we recall the classification of connected arc-transitive circulants. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorems 1.2 and 1.3.

2 Arc-transitive circulants

We begin this section with the following lemma:

Lemma 2.1. Let Γ be a connected vertex-transitive strongly quasi $m-$Cayley graph. Then Γ is arc-transitive.

Proof. Since Γ is vertex transitive, it is sufficient to prove that there exists a vertex v such that the stabilizer $\mathrm{Aut}(\Gamma)_v$ acts transitively on the neighborhood of v. It is obvious that if we choose the point at infinity for v, this condition is satisfied. $\square$

The previous lemma implies that we can somehow restrict our study to the connected arc-transitive circulants, therefore it is important to understand the structure of such graphs.

To state the classification of connected arc-transitive circulants, which has been obtained independently by Kovács [6] and Li [10], we need to recall certain graph products and the concept of normal Cayley graphs.

The *wreath (lexicographic) product* $\Sigma[\Gamma]$ of a graph Γ by a graph Σ is the graph with vertex set $V(\Sigma) \times V(\Gamma)$ such that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(\Sigma)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(\Gamma)$. For a positive integer b and a graph

Σ, denote by $b\Sigma$ the graph consisting of b vertex-disjoint copies of the graph Σ. The graph $\Sigma[\overline{K_b}]-b\Sigma$ is called the *deleted wreath (deleted lexicographic) product* of Σ and $\overline{K_b}$, where $\overline{K_b} = bK_1$.

Let $\Gamma = Cay(G, S)$ be a Cayley graph on a group G. Denote by $\mathrm{Aut}(G, S)$ the set of all automorphisms of G which fix S setwise, that is,

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) | S^\sigma = S\}.$$

It is easy to check that $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\Gamma)$ and that it is contained in the stabilizer of the identity element $1 \in G$. It follows from the definition of Cayley graph that the left regular representation G_L of G induces a regular subgroup of $\mathrm{Aut}(\Gamma)$. Following Xu [12], $\Gamma = Cay(G, S)$ is called a *normal Cayley graph* if G_L is normal in $\mathrm{Aut}(\Gamma)$, that is, if $\mathrm{Aut}(G, S)$ coincides with the vertex stabilizer $1 \in G$. Moreover, if Γ is a normal Cayley graph, then $\mathrm{Aut}(\Gamma) = G_L \rtimes \mathrm{Aut}(G, S)$.

Proposition 2.1. [6, 10] Let Γ be a connected arc-transitive circulant of order n. Then one of the following holds:

(i) $\Gamma \cong K_n$;

(ii) $\Gamma = \Sigma[\overline{K}_d]$, where $n = md$, $m, d > 1$ and Σ is a connected arc-transitive circulant of order m;

(iii) $\Gamma = \Sigma[\overline{K}_d] - d\Sigma$, where $n = md$, $d > 3$, $gcd(d, m) = 1$ and Σ is a connected arc-transitive circulant of order m;

(iv) Γ is a normal circulant.

In Section 3 and 4 two lemmas (that show that arc-transitive circulants described in Proposition 2.1(ii) and (iii) are not strongly quasi k-Cayley graphs) will be needed.

Lemma 2.2. Let Γ be an arc-transitive circulant, described in Proposition 2.1(ii). Then Γ is not a strongly quasi k-Cayley graph for any $k \in \mathbb{N}$.

Proof. We have $\Gamma = \Sigma[\overline{K}_d]$, where $n = md$, $m, d > 1$ and Σ is a connected arc-transitive circulant of order m. Suppose that Γ is a strongly quasi k-Cayley graph on a group G. Then $val(\Gamma) = (n-1)/k = (md-1)/k$. On the other hand, since $\Gamma = \Sigma[\overline{K}_d]$, we have $val(\Gamma) = val(\Sigma) \cdot d$. These two facts combined together imply that $d(m - k \cdot val(\Sigma)) = 1$, and so $d = 1$, a contradiction. □

Lemma 2.3. Let Γ be an arc-transitive circulant, described in Proposition 2.1(iii). Then Γ is not a strongly quasi k-Cayley graph for any $k \in \mathbb{N}$.

Proof. We have $\Gamma = \Sigma[\overline{K}_d] - d\Sigma$, where $n = md$, $d > 3$, $gcd(d, m) = 1$, and Σ is an arc-transitive circulant of order m. Suppose that Γ is also a strongly quasi k-Cayley graph on a group G. By [10, Theorem 1.1] the m copies of the graph $\overline{K_d}$ form an imprimitivity block system $\mathcal{B}$ for $\mathrm{Aut}(\Gamma)$. Clearly the block $B \in \mathcal{B}$ containing the point at infinity, that is, the trivial orbit of G, is fixed by G. This implies that $|G|$ divides $d-1$. On the other hand, since the valency of Γ is $|G|$, we have $|G| \geq d-1$. Combining these results we obtain $|G| = d-1$. Thus, connectedness of Γ implies that $m = 2$. However, then there are $2d-1$ vertices in Γ different from the point at infinity, and they cannot be divided into k orbits of size $d-1$ for any natural number k. Therefore, there are no strongly quasi k-Cayley graphs amongst the graphs from Proposition 2.1(iii) for any natural number $k \geq 1$. □

Lemma 2.4. Let Γ be an arc-transitive circulant, described in Proposition 2.1(iv). If Γ is also a strongly quasi m-Cayley graph on a group G, then the order of Γ has at most $m+1$ divisors.

Proof. Let $\Gamma = Cay(\mathbb{Z}_n, S)$ be a normal circulant. Let $A = \text{Aut}(\Gamma)$. Since Γ is a normal Cayley graph, $A \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n, S)$. We may, without loss of generality, assume that the point at infinity corresponds to the vertex $0 \in \mathbb{Z}_n$, and so $G \leq \text{Aut}(\mathbb{Z}_n, S) \leq \text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Therefore, $G \lesssim \mathbb{Z}_n^*$. Since G has m orbits on $\mathbb{Z}_n \backslash \{0\}$, then $\text{Aut}(\mathbb{Z}_n)$ has at most m orbits on $\mathbb{Z}_n \backslash \{0\}$, and at most $m+1$ orbits on $\mathbb{Z}_n$. Elements in the same orbit of $\text{Aut}(\mathbb{Z}_n)$ are clearly of the same order in $\mathbb{Z}_n$. There exist an element in $\mathbb{Z}_n$ of order d, if and only if d divides n. Therefore the number of divisors of n, denoted by $\tau(n)$, is not greater than $m+1$, i.e. $\tau(n) \leq m+1$. □

3 Quasi 2-Cayley graphs

In this section the connected circulants are considered. In particular, connected circulants that are also quasi 2-Cayley graphs are classified (see Theorem 1.1). If a graph Γ of order n is a quasi 2-Cayley graph on a group G, which is not a strongly quasi 2-Cayley graph, then it is isomorphic to the complete graph K_n. Namely, in such a graph, the point at infinity ∞ is adjacent to both nontrivial orbits of G, and thus it is adjacent to all the vertices different from ∞. Consequently, we can conclude that Γ has valency $|V(\Gamma)| - 1$, and so Γ is a complete graph. In order to classify all connected circulants that are also quasi 2-Cayley graphs it therefore suffices to characterize strongly quasi 2-Cayley graphs that are also connected circulants, we do this in Theorem 3.1.

Theorem 3.1. Let Γ be a connected circulant. Then Γ is also a strongly quasi 2-Cayley graph if and only if Γ is isomorphic to the Paley graph $P(p)$, where p is a prime such that $p \equiv 1 \pmod 4$. Moreover, Γ is a quasi bicirculant.

Proof. Let Γ be the Paley graph $P(p)$, where p is a prime, such that $p \equiv 1 \pmod 4$. It is well known that the Paley graphs are connected arc-transitive circulants, and, as was observed in [7], they are also strongly quasi 2-Cayley graphs.

Conversely, let Γ be a connected circulant $Cay(\mathbb{Z}_n, S)$ of order n not isomorphic to the complete graph K_n, which is also a strongly quasi 2-Cayley graph on a group G. Then $|G| = (n-1)/2$ and Γ is of valency $(n-1)/2$. Lemma 2.1 tells us that Γ is an arc-transitive graph, and moreover Proposition 2.1, Lemma 2.2 and Lemma 2.3 combined together imply that Γ is a normal circulant. The theorem now follows from the three claims below.

CLAIM 1: n is an odd prime.

It is obvious that n must be odd, since 2 divides $n-1$. By Lemma 2.4 we have that $\tau(n) \leq 3$. Thus we have the following two possibilities for n:

- $n = p$, where p is a prime;
- $n = p^2$, where p is a prime.

Suppose that the latter case hold. Let $A = \text{Aut}(\Gamma)$. Since Γ is a normal Cayley graph, we have $A \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n, S)$. We may, without loss of generality, assume that the point at infinity corresponds to the vertex $0 \in \mathbb{Z}_n$, and so $G \leq \text{Aut}(\mathbb{Z}_n, S) \leq \text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Therefore, $\mathbb{Z}_n^*$ contains a subgroup G of order $(n-1)/2$. Since $|\mathbb{Z}_n^*| \leq n-1$ and $|G|$

divides $|\mathbb{Z}_n^*|$ we obtain that $|\mathbb{Z}_n^*| = n-1$ or $(n-1)/2$. Since, by assumption, n is not a prime, we have $|\mathbb{Z}_n^*| = (n-1)/2$. This gives in the following equation

$$\frac{p^2-1}{2} = p(p-1)$$

which has the unique solution $p = 1$, a contradiction.

CLAIM 2: $n \equiv 1 \pmod 4$.

Since $S = -S$, and no element in $\mathbb{Z}_n$ can be its own inverse, we have that the number of elements in S is even, and since $|S| = \frac{n-1}{2}$, we have $n \equiv 1 \pmod 4$.

CLAIM 3: Γ is isomorphic to the Paley graph $P(n)$.

By Claim 1, n is a prime. Therefore the group $\mathbb{Z}_n^*$ is cyclic, and thus since G is a subgroup of $\mathbb{Z}_n^*$, G is cyclic as well. By [6, Remark 2], we have $\mathrm{Aut}(\Gamma) = \{g \mapsto g^\sigma + h \mid \sigma \in K, h \in \mathbb{Z}_n\}$, for a suitable group $K < \mathrm{Aut}(\mathbb{Z}_n)$, and S is the orbit under K of a generating element of $\mathbb{Z}_n$, that is, $S = Orb_K(g)$ for some generating element g of $\mathbb{Z}_n$. Now we have that $\mathrm{Aut}(\Gamma)_0 = \{g \mapsto g^\sigma + h \mid \sigma \in K, h \in \mathbb{Z}_n : 0^\sigma + h = 0\} = \{g \mapsto g^\sigma \mid \sigma \in K\} \cong K$. So we see that $G \lesssim K$. Since $S = Orb_K(g) \lesssim Orb_G(g)$, and $|S| = |Orb_G(g)|$ we have that $S \cong Orb_G(g)$, which gives us that $S \cong G$ (taking $g = 1$). Now, since G is the index 2 subgroup of the cyclic group $\mathbb{Z}_n^*$, G is of the form $G = \langle x^2 \rangle$ where x generates $\mathbb{Z}_n^*$. Therefore G consists of all squares in $\mathbb{Z}_n^*$ and $S \cong G$, implying that Γ is isomorphic to the Paley graph $P(n)$ as claimed.

It is obvious that G must be cyclic, so the graph Γ is in fact a quasi bicirculant. $\square$

Proof of Theorem 1.1: It follows from Theorem 3.1 and the paragraph preceding it. $\square$

In general, if Γ is a vertex transitive quasi 2-Cayley graph on a group G, not isomorphic to the complete graph, then it is a strongly regular graph of a rank 3 group. Namely, the orbits of G are contained in the orbits of the stabilizer of the $\mathrm{Aut}(\Gamma)_\infty$ and since there are just two nontrivial orbits of G, then there are exactly two nontrivial orbits of the $\mathrm{Aut}(\Gamma)_\infty$ which in fact must coincide with the orbits of G. Therefore $\mathrm{Aut}(\Gamma)$ must be a rank 3-group, and the graphs of the rank 3 groups are strongly regular graphs.

4 Quasi 3-Cayley and 4-Cayley graphs

In this section we will deal with the question which connected circulants are also quasi 3-Cayley graph or strongly quasi 4-Cayley graphs. We first consider the case of strongly quasi 3-Cayley graphs.

Theorem 4.1. Let Γ be a connected circulant. Then Γ is also a strongly quasi 3-Cayley if and only if $\Gamma \cong Cay(\mathbb{Z}_n, S)$ where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$. Moreover, Γ is a quasi tricirculant.

Proof. Let $\Gamma = Cay(\mathbb{Z}_p, S)$ where $p \equiv 1 \pmod 3$ is a prime and S is the set of all non-zero cubes in $\mathbb{Z}_p$. Since p is a prime, it is well known that $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$ is a cyclic group of order $p-1$. Let $G = \langle a^3 \rangle$, where a is a generating element of $\mathbb{Z}_p^*$. Then G consists of all non-zero cubes in $\mathbb{Z}_p$, and $|G| = \frac{p-1}{3}$. The action of G on $\mathbb{Z}_p$ defined by $x^g = g \cdot x$ gives G as the subgroup of $\mathrm{Aut}(\Gamma)$. The group G acts quasi-semiregularly on $\mathbb{Z}_p$ with $0 \in \mathbb{Z}_p$

as the point at infinity. Namely, it is easy to check that $G_0 = G$, and that the stabilizer of any element $x \in \mathbb{Z}_p \backslash \{0\}$ is trivial. Since $|G| = \frac{p-1}{3}$, it follows that G has 3 orbits on $\mathbb{Z}_p \backslash \{0\}$, and therefore Γ is a quasi 3-Cayley graph. Since one of the orbits of G is the set S, the point at infinity is adjacent to only one orbit of G, so Γ is in fact a strongly quasi 3-Cayley graph. By the construction Γ is an arc-transitive circulant since $G \leq \mathrm{Aut}(\Gamma)_0$ acts transitively on the set of vertices adjacent to the vertex 0.

Conversely, let Γ be a connected circulant of order n, which is also a strongly quasi 3-Cayley graph on a group G. Then $|G| = \frac{n-1}{3}$. From Lemma 2.1 we have that Γ is arc-transitive, and therefore Proposition 2.1, Lemma 2.2 and Lemma 2.3 combined together imply that Γ is a normal circulant. Therefore, we can assume that $\Gamma = Cay(\mathbb{Z}_n, S)$, and that $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n)$, implying that $\frac{n-1}{3} | \varphi(n)$, where $\varphi(n)$ is the Euler totient function.

Claim 1: n is a prime number.

Let

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_t^{k_t},$$

be a canonic factorization of a positive integer n. From Lemma 2.4, we have $\tau(n) \leq 4$. Now we can calculate

$$\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_t + 1).$$

We have the following possibilities for n:

- $n = p$,
- $n = p^2$;
- $n = p^3$;
- $n = pq$,

where p and q are different primes.

If $n = p^2$, then the only solution of $\frac{n-1}{3} | \varphi(n)$ is $p = 2$ and $n = 4$. However, if $n = 4$, the graph Γ is of valency 1, so it is not a connected graph.

If $n = p^3$, then there is no solution of the above equation.

If n is a product of two different primes, then we have $|\mathbb{Z}_n^*| = (n-1)/3$ or $2(n-1)/3$. In the first case $\mathbb{Z}_n^* \cong G$, so $\mathbb{Z}_n^*$ acts semiregularly on $\mathbb{Z}_n \backslash \{0\}$, and it is not difficult to see that this is not the case for $n = pq$. If $|\mathbb{Z}_n^*| = 2(n-1)/3$, then we obtain the following equation

$$(p-1)(q-1) = \frac{2(pq-1)}{3}.$$

The only solutions in natural numbers of the above equation are $(p, q) \in \{(4, 7), (5, 5), (7, 4)\}$, so there are no two different primes p, q satisfying the given equation.

Having in mind all the written above, we conclude that n is a prime.

Claim 2: Γ is isomorphic to the Cayley Graph $Cay(\mathbb{Z}_n, S)$, where S is set of all non zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$.

Similarly as in the previous section, it can be shown that $G \cong S$. Since G is an index 3 subgroup of $\mathbb{Z}_n^*$, we have $G = \langle x^3 \rangle$, where x is a generating element of $\mathbb{Z}_n^*$. It follows that G consists of all cubes in $\mathbb{Z}_n^*$, so Γ is isomorphic to $Cay(\mathbb{Z}_n, S)$, where S is the set of all non zero cubes in $\mathbb{Z}_n$ and $n \equiv 1 \pmod 3$ is a prime. It is obvious from the mentioned above, that the group G must be cyclic, therefore, Γ is in fact a quasi tricirculant. $\square$

Proof of the Theorem 1.2: Let Γ be a connected circulant of order n, which is also a quasi 3-Cayley on a group G. The point at infinity is adjacent to all three nontrivial orbits of G, if and only if Γ is isomorphic to K_n. If the point at infinity is adjacent to just one nontrivial orbit of G, then Γ is a strongly quasi 3-Cayley graph, therefore, Theorem 4.1 gives us the desired result. If the point at infinity is adjacent to two nontrivial orbits of G, then we consider the complement $\Sigma = \overline{\Gamma}$ of the graph Γ. The graph Σ is a quasi 3-Cayley graph on G, and actually it is a strongly quasi 3-Cayley graph on G. Since Σ is the complement of a circulant it is also a circulant. Suppose that Σ is not connected. Then, since it is vertex-transitive, it is the disjoint union of some isomorphic graphs. The point at infinity is adjacent to one orbit of G, so the connected components of Σ must have at least $1 + \frac{n-1}{3}$ points. Therefore $n = k \cdot n_1$, where k is the number of connected components, and n_1 is the number of points in each of the components. We have noticed that $n_1 \geq 1 + \frac{n-1}{3}$, thus $k \leq 2$. If $k = 1$ then Σ is connected. Suppose that $k = 2$. Then there are two connected components of Γ, say Γ_1 and Γ_2, each containing $n/2$ points. Suppose that $\infty \in \Gamma_1$. Let Δ_1, Δ_2 and Δ_3 be a nontrivial orbits of G, and let the point at infinity be adjacent to Δ_1. Then $\Delta_1 \subset \Gamma_1$. Since Γ_1 and Γ_2 have the same size, it means that at least one of Δ_2 and Δ_3 have points both in Γ_1 and Γ_2. Suppose that $u, v \in \Delta_2$, such that $u \in \Gamma_1$ and $v \in \Gamma_2$. Since u and v are in the same orbit of G then there exist $g \in G$ which maps u to v. However, g fixes ∞, and consequently g fixes Γ_1, a contradiction.

Having in mind all the written above, we see that Σ is a connected circulant, which is also a strongly quasi 3-Cayley graph. Therefore we have the desired result. $\square$

We will continue this section with the proof of Theorem 1.3.

Proof of Theorem 1.3: Let $\Gamma = C_9$. Then $\Gamma \cong Cay(\mathbb{Z}_9, \{\pm 1\})$. Then the group $G = \{1, -1\} \subset \mathbb{Z}_9^*$ acts quasi semiregularly on $\mathbb{Z}_9$ with 0 as the point at infinity.

Let $\Gamma \cong Cay(\mathbb{Z}_p, S)$, where S is the set of all fourth powers in $\mathbb{Z}_p \backslash \{0\}$, and p is a prime such that $p \equiv 1 \pmod 4$. Define $G = \langle a^4 \rangle$, where a is some generating element of $\mathbb{Z}_p^*$, which is cyclic in this case. We have that G acts quasi-semiregularly on $\mathbb{Z}_p^*$ with 0 as the point at infinity. Since $|G| = \frac{p-1}{4}$, it follows that G has 4 orbits on $\mathbb{Z}_p \backslash \{0\}$, and therefore Γ is a quasi 4-Cayley graph. It is also easy to see that 0 is adjacent to only one orbit of G on $\mathbb{Z}_p \backslash \{0\}$, therefore Γ is a strongly quasi 4-Cayley graph. By the construction, Γ is a connected arc-transitive circulant.

Conversely, let Γ be a connected circulant of order n which is also a strongly quasi 4-Cayley graph on a group G. Then $|G| = (n-1)/4$. Using Lemma 2.1 we have that Γ is arc-transitive, and so Proposition 2.1, Lemma 2.2 and Lemma 2.3 combined together imply that Γ is a normal circulant. Therefore, we can assume that $\Gamma = Cay(\mathbb{Z}_n, S)$, and that $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n)$, implying that

$$\frac{n-1}{4} | \varphi(n). \tag{4.1}$$

Using Lemma 2.4 we obtain $\tau(n) \leq 5$. So we have the following possibilities:

- $n = p$,
- $n = p^2$,
- $n = p^3$,
- $n = p^4$,
- $n = pq$,

where p and q are different primes.

If $n = p^2$, then the only solution of (4.1) is $n = 9$. In this case, the valency of Γ is $(9-1)/4 = 2$, so $\Gamma \cong C_9$.

In the cases when $n = p^3$, and $n = p^4$ there is no prime satisfying (4.1).

When $n = pq$, we have that $(p-1)(q-1) = \alpha \cdot (pq-1)/4$, where $\alpha \in \{1, 2, 3\}$. If $\alpha = 1$, then we have $\mathbb{Z}_n^* = G$, so $\mathbb{Z}_n^*$ must act semiregularly on $\mathbb{Z}_n \backslash \{0\}$, which is not the case. If $\alpha = 2$, then there are no two different primes satisfying $(p-1)(q-1) = (pq-1)/2$, and finally, when $\alpha = 3$, we have that $n = 5 \cdot 13$ is the only possibility. In this case, Γ is a connected arc-transitive circulant on 65 vertices, which has valency 16. Since G is an index 3 subgroup of $\mathbb{Z}_{65}^* \cong \mathbb{Z}_4 \times \mathbb{Z}_{12}$, then we can calculate $G \cong \{\pm 1, \pm 8, \pm 12, \pm 14, \pm 18, \pm 21, \pm 27, \pm 31\}$, and we can see that G does not act semiregularly on $\mathbb{Z}_{65} \backslash \{0\}$. Namely, the non identity element $21 \in G$ fixes the point $13 \in \mathbb{Z}_{65} \backslash \{0\}$.

Assume now that n is a prime. Similarly as in the proof of Theorem 3.1, we obtain $G \cong S$, and therefore, since G is an index 4 subgroup of $\mathbb{Z}_n^*$, we have $G = \langle x^4 \rangle$, where x is some generating element of $\mathbb{Z}_n^*$. Therefore, $\Gamma \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all fourth powers in $\mathbb{Z}_n \backslash \{0\}$.

From the mentioned above, it is clear that G is a cyclic group, so Γ is in fact a quasi tetracirculant. $\square$

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2-Groups that factorise as products of cyclic groups, and regular embeddings of complete bipartite graphs

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Abstract

We classify those 2-groups G which factorise as a product of two disjoint cyclic subgroups A and B, transposed by an automorphism of order 2. The case where G is metacyclic having been dealt with elsewhere, we show that for each $e \geq 3$ there are exactly three such non-metacyclic groups G with $|A| = |B| = 2^e$, and for $e = 2$ there is one. These groups appear in a classification by Berkovich and Janko of 2-groups with one non-metacyclic maximal subgroup; we enumerate these groups, give simpler presentations for them, and determine their automorphism groups.

Keywords: Regular map, complete bipartite graph, product of cyclic groups.

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1 Introduction

Groups that factorise as products of isomorphic cyclic groups have been studied for over fifty years [4, 7, 9, 10, 11]. In several recent papers [5, 6, 13, 14, 15, 16, 17, 18] these groups have emerged as an important tool for the classification of regular embeddings of complete bipartite graphs in orientable surfaces. They also arise naturally in the theory of finite p-groups, for example in the recent classification by Berkovich and Janko [1, Chapter 87] of 2-groups with a unique non-metacyclic maximal subgroup. Our aim in this paper is to demonstrate some connections between these two problems by showing that a certain class of non-metacyclic 2-groups play an important role in both situations. As a consequence, we are able to give more information and simpler presentations for some of the groups described by Berkovich and Janko.

As shown in [15], the problem of classifying orientably regular embeddings of complete bipartite graphs $K_{n,n}$ is closely related to that of determining those groups G that factorise as a product AB of two cyclic groups $A = \langle a\rangle$ and $B = \langle b\rangle$ of order n such that $A\cap B = 1$ and there is an automorphism of G transposing the generators a and b. Such groups are called *isobicyclic*, or n*-isobicyclic* if we wish to specify the value of n (see [15]). We will call (G, a, b) an *isobicyclic triple*, and (a, b) an *isobicyclic pair* for G.

A result of Itô [9] shows that an isobicyclic group G, as a product of two abelian groups, must be metabelian. In particular it is solvable, so it satisfies Hall's Theorems, which extend Sylow's Theorems from single primes to sets of primes. This fact, together with results of Wielandt [19] on products of nilpotent groups, allows one to reduce the classification of n-isobicyclic groups to the case where n is a prime power (see [13] for full details).

When n is an odd prime power, a result of Huppert [7] implies that G must be metacyclic. When n is a power of 2, however, Huppert's result does not apply, and indeed for each $n = 2^e \geq 4$ there are non-metacyclic n-isobicyclic groups. In this paper we will study all n-isobicyclic groups where $n = 2^e$, our main goal being to give a complete description of the corresponding isobicyclic triples (G, a, b).

In order to state our main result, let us define

$$G_1(e,f) = \langle h, g \mid h^{2^e} = g^{2^e} = 1,\ h^g = h^{1+2^f}\rangle \tag{1.1}$$

where $f = 2, \ldots, e$, and

$$G_2(e;k,l) = \langle a, b \mid \begin{array}{l} a^n = b^n = [b^2, a^2] = 1,\ [b,a] = a^2b^{-2}(a^{n/2}b^{n/2})^k, \\ (b^2)^a = b^{-2}(a^{n/2}b^{n/2})^l,\ (a^2)^b = a^{-2}(a^{n/2}b^{n/2})^l\rangle \end{array} \tag{1.2}$$

where $n = 2^e \geq 4$ and $k, l \in \{0, 1\}$, with $k = l = 0$ when $n = 4$. In fact, it is easily seen that this last group $G_2(2; 0, 0)$ has a simplified presentation

$$G_2(2;0,0) = \langle a, b \mid a^4 = b^4 = [a^2, b] = [b^2, a] = 1,\ [b, a] = a^2b^2\rangle. \tag{1.3}$$

Our main result shows that if $n = 2^e$ then every n-isobicyclic group has one of the above two presentations.

Theorem 1.1. Let G be an n-isobicyclic group where $n = 2^e \geq 4$. Then either

(i) G is metacyclic, and $G \cong G_1(e, f)$ for some $f \in \{2, \ldots, e\}$, or

(ii) G is not metacyclic, in which case either $G \cong G_2(2; 0, 0)$, or $e \geq 3$ and $G \cong G_2(e; k, l)$ where $k, l \in \{0, 1\}$. In the latter case there are, up to isomorphism, just three groups for each e, with $G_2(e; 0, 1) \cong G_2(e; 1, 1)$.

The metacyclic groups $G_1(e,f)$ were treated in detail in [5]; for instance, it was shown there that, up to automorphisms of G, one can take the isobicyclic pair to have the form $a = g^r$ and $b = g^r h$, where r is an odd integer such that $1 \leq r \leq 2^{e-f}$. This paper is therefore devoted to the non-metacyclic groups $G_2(e;k,l)$.

These groups $G_2(e;k,l)$ have recently arisen in a purely group-theoretic context. In [1, Chapter 87] Berkovich and Janko, having classified the minimal non-metacyclic 2-groups (i.e. those with all their maximal subgroups metacyclic), then classify those 2-groups with a unique non-metacyclic maximal subgroup. Clearly such a group requires at most three generators (two to generate a metacyclic maximal subgroup, and one more outside it). The 3-generator groups of this type are relatively easy to deal with, and Berkovich and Janko devote most of their analysis to the 2-generator groups. In Corollary 87.13 they show that any such group factorises as a product of two cyclic groups, and conversely in Theorem 87.22 they show that any non-metacyclic group which factorises in this way (and is therefore a 2-generator group) has a unique non-metacyclic maximal subgroup. Their analysis of the 2-generator groups depends on considering the different possibilities for the commutator subgroup, and one part of the classification (essentially Theorem 87.19, see also [12, Theorem 4.11]) is as follows:

Theorem 1.2. (Berkovich and Janko) Let G be a 2-generator 2-group with exactly one non-metacyclic maximal subgroup. Assume that $G' \cong C_{2^r} \times C_{2^{r+1}}$ where $r \geq 2$. Then

$$G = \langle a, x \mid \begin{array}{l} a^{2^{r+2}} = 1,\ [a,x] = v,\ [v,a] = b,\ v^{2^{r+1}} = b^{2^r} = [v,b] = 1, \\ v^{2^r} = z,\ b^{2^{r-1}} = u,\ x^2 \in \langle u, z\rangle \cong C_2 \times C_2,\ b^x = b^{-1}, \\ v^x = v^{-1},\ b^a = b^{-1},\ a^4 = v^{-2} b^{-1} w,\ w \in \langle u, z\rangle \,\rangle \end{array} \tag{1.4}$$

with $|G| = 2^{2r+4}$ and $G' = \langle b\rangle \times \langle v\rangle \cong C_{2^r} \times C_{2^{r+1}}$.

One should regard (1.4) as giving sixteen presentations for each r, since there are four possibilities for each of x^2 and w in the Klein four-group $\langle u, z\rangle$. In Theorem 4.2, we will show that the groups $G_2(e;k,l)$ for $e \geq 4$ are exactly those groups G in Theorem 1.2 for which $x^2 = z^k$ and $w = z^l$ for some $k, l \in \{0, 1\}$, with $e = r + 2$. As noted by Janko in [12, p. 315], the classification problem is not completely solved since some pairs of presentations define isomorphic groups. Indeed Theorem 1.1 shows that for each $r \geq 2$ there are, up to isomorphism, just three groups presented by (1.4) with $x^2 = z^k$ and $w = z^l$, those with $l = 1$ and $k = 0, 1$ being isomorphic to each other. As a consequence of Theorem 4.2, in (1.2) we give slightly more transparent presentations for these groups, showing that each is an extension of its Frattini subgroup $\Phi(G) \cong C_{2^{r+1}} \times C_{2^{r+1}}$ by $C_2 \times C_2$: the roles of a, b and $a^{n/2}b^{n/2}$ in (1.2) are played by a, ax and the central involution z in (1.4). Moreover, all our structural results proved in Section 2 for the groups $G_2(e;k,l)$ apply to these groups G. For instance, we show that they are all metabelian, of exponent 2^e and nilpotence class e. In classifying all isomorphisms between the groups $G_2(e;k,l)$, we also determine their automorphisms; in particular, we show that for each $e \geq 3$, $\mathrm{Aut}\, G_2(e;k,l)$ has order 2^{4e-3} or 2^{4e-4} as $l = 0$ or 1.

Section 3 begins a structural analysis of isobicyclic 2-groups in general, while Section 4 is devoted specifically to non-metacyclic isobicyclic groups G. We show that if $n = 2^e$ then either G has a cyclic derived subgroup, in which case $e = 2$ and $G \cong G_2(2;0,0)$, or G has a derived group generated by two elements, in which case $e \geq 3$ and G is isomorphic to one of the three non-isomorphic groups of the form $G_2(e;k,l)$. This proves part (ii) of

Theorem 1.1, and since part (i) is dealt with in [5], it completes the proof of that theorem. In Section 5 we apply results from the preceding sections to the classification of regular embeddings of complete bipartite graphs $K_{n,n}$ where n is a power of 2.

A completely different proof of Theorem 1.1(ii) has already been given in [6]; it proceeds by induction on e, based on the fact that if $n = 2^e$ then any n-isobicyclic group has an m-isobicyclic quotient where $m = 2^{e-1}$. However, the main purpose of that paper was not to study these groups for their own sake, but rather to enumerate them and to obtain sufficient information about them to determine the corresponding graph embeddings. Here we present an alternative proof, designed to shed more light on the internal structure of these groups, and on how they are related to more general classes of 2-groups.

2 Non-metacyclic groups $G_2(e; k, l)$

In this section we analyse properties of the non-metacyclic groups $G_2(e; k, l)$ appearing in Theorem 1.1. Throughout this section we write $G(k, l)$, or simply G, instead of $G_2(e; k, l)$. For brevity we also write $n = 2^e$ and $m = n/2 = 2^{e-1}$.

It is useful to note that each group G has a Frattini subgroup $\Phi = \Phi(G) = \langle a^2\rangle \times \langle b^2\rangle \cong C_m \times C_m$, with $G/\Phi \cong C_2 \times C_2$ (see [6, Prop. 2.1]). It therefore has three maximal subgroups, namely $\langle\Phi, a\rangle = \Phi \cup \Phi a$, $\langle\Phi, b\rangle = \Phi \cup \Phi b$ and $\langle\Phi, ab\rangle = \Phi \cup \Phi ab$.

Lemma 2.1. The following properties hold in $G = G(k, l)$.

(i) The elements a^m, b^m, and $z = a^m b^m$ are central involutions of G.

(ii)

$$b^j a^i = \begin{cases} a^i b^j, & \text{for } i \text{ and } j \text{ even,} \\ a^i b^{-j} z^{lj/2}, & \text{for } i \text{ odd and } j \text{ even,} \\ a^{-i} b^j z^{li/2}, & \text{for } i \text{ even and } j \text{ odd,} \\ a^{-i} b^{-j} z^{k+l(i+j)/2}, & \text{for } i \text{ and } j \text{ odd.} \end{cases}$$

(iii) The element $g = a^i b^j$ has order

$$|g| \begin{cases} \text{dividing } m, & \text{for } i \text{ and } j \text{ even,} \\ \text{equal to } n, \text{ with } g^m = a^m, & \text{for } i \text{ odd and } j \text{ even,} \\ \text{equal to } n, \text{ with } g^m = b^m, & \text{for } i \text{ even and } j \text{ odd,} \\ \text{dividing } 4, \text{ and equal to 2 if } k = l = 0, & \text{for } i \text{ and } j \text{ odd.} \end{cases}$$

(iv) The group G is isobicyclic, that is, $G = \langle a\rangle\langle b\rangle$, where $|a| = |b| = 2^e$ and $\langle a\rangle \cap \langle b\rangle = 1$, and there is an involutory automorphism of G interchanging a and b.

(v) $G' = \langle a^2 b^{-2} z^k\rangle \times \langle a^4 z^l\rangle$ with $\langle a^2 b^{-2} z^k\rangle \cong C_m$ and $\langle a^4 z^l\rangle \cong C_{m/2}$.

(vi) G is not metacyclic.

(vii) G has nilpotence class e, with upper central series $1 = Z_0 < Z_1 < \cdots < Z_e = G$ where $Z_i = \langle a^{2^{e-i}}\rangle\langle b^{2^{e-i}}\rangle$ for $i = 0, 1, \ldots, e$.

Proof. Some of these results were proved in [6]; for completeness we give proofs here.

If we define $z = a^m b^m$, the defining relations for G in (1.2) take the form

$$a^n = b^n = [b^2, a^2] = 1,\ [b, a] = a^2 b^{-2} z^k,\ (b^2)^a = b^{-2} z^l,\ (a^2)^b = a^{-2} z^l, \tag{2.1}$$

where $k, l \in \{0, 1\}$ with $k = l = 0$ if $n = 4$.

(i) Since m is even, and $[a^2, b^2] = 1$, the involutions a^m and b^m commute; they are distinct, so their product z is also an involution. Since z commutes with a^2 and b^2, and either m is divisible by 4 or $l = 0$, we have $(b^m)^a = ((b^2)^a)^{m/2} = (b^{-2}z^l)^{m/2} = b^{-m}z^{lm/2} = b^{-m} = b^m$, so b^m is an element of the centre $Z(G)$ of G. Similarly, $a^m \in Z(G)$, so $z \in Z(G)$.

(ii) Now we compute $b^j a^i$. Define $c = [b, a] = b^{-1}a^{-1}ba$. If both i and j are even, then $b^j a^i = a^i b^j$. If i is odd and j is even, then since $i - 1$ is even we have

$$b^j a^i = (b^j a^{i-1})a = a^{i-1}b^j a = a^i((b^2)^a)^{j/2} = a^i b^{-j} z^{lj/2}.$$

If i is even and j is odd, then

$$b^j a^i = b^{-1}(b^{j+1}a^i) = (b^{-1}a^i b)b^j = ((a^2)^b)^{i/2}b^j = a^{-i}b^j z^{li/2}.$$

If both i and j are odd, then

$$\begin{aligned} b^j a^i &= b^{j-1}abca^{i-1} = b^{j-1}ab(b^{-2}a^2z^k)a^{i-1} = a(b^{j-1})^a(a^{i+1})^b b^{-1}z^k \\ &= ab^{1-j}z^{l(j-1)/2}a^{-i-1}z^{l(i+1)/2}b^{-1}z^k = a^{-i}b^{-j}z^{k+l(i+j)/2}. \end{aligned}$$

(iii) If i and j are both even then (ii) implies that

$$g^2 = (a^i b^j)^2 = a^i(b^j a^i)b^j = a^{2i}b^{2j} \in \langle a^4 \rangle \times \langle b^4 \rangle \cong C_{m/2} \times C_{m/2},$$

so $g^m = 1$. If i is odd and j is even then (ii) gives

$$g^2 = a^i(b^j a^i)b^j = a^{2i}b^{-j}z^{lj/2}b^j = a^{2i}z^{lj/2},$$

so $\langle g^4 \rangle = \langle a^{4i} \rangle = \langle a^4 \rangle$; thus $\langle g^{2^r} \rangle = \langle a^{2^r} \rangle$ for all $r \geq 2$, so $|g| = |a| = n$ with $g^m = a^m$. The proofs in the other two cases are similar.

(iv) The formulæ in (ii) show that every element of G can be expressed in the form $a^i b^j$, so $G = \langle a \rangle \langle b \rangle$. In order to see that $\langle a \rangle \cap \langle b \rangle = 1$, note that a^i and b^j lie in distinct cosets of Φ unless i and j are both even; in this case the fact that $\Phi = \langle a^2 \rangle \times \langle b^2 \rangle$ ensures that $\langle a \rangle \cap \langle b \rangle = 1$. The defining relations of G are equivalent to those obtained by transposing a and b, so this transposition can be extended to an automorphism α of order 2 of G. Hence G is an n-isobicyclic group.

(v) Since $G = \langle a, b \rangle$, G' is the normal closure $\langle c^g \mid g \in G \rangle$ in G of the commutator $c = [b, a]$. We will show that this is the subgroup $M := \langle c, c^a \rangle$. Since $c = [b, a] = a^2b^{-2}z^k$ we have $c^a = (a^2b^{-2}z^k)^a = a^2b^2z^{k+l}$, and conjugation by a transposes these two generators of M since $[c, a^2] = 1$. Similarly, conjugation by b transposes the generators c and $(c^a)^{-1}$ of M, so M is normal in G and hence $M = \langle c^g \mid g \in G \rangle = G'$. Thus G' has generators $c = a^2b^{-2}z^k = a^{km+2}b^{km-2}$ and $c^a c = a^4z^l = a^{lm+4}b^{lm}$; these generate disjoint cyclic groups of orders m and $m/2$, so

$$G' = \langle a^2b^{-2}z^k \rangle \times \langle a^4z^l \rangle \cong C_m \times C_{m/2}.$$

(vi) For $e \geq 3$ the fact that G' is not cyclic immediately implies that G is not metacyclic. In the case $e = 2$ it is easily seen that the only cyclic normal subgroups of G are contained in Φ, and these do not have cyclic quotients.

(vii) This follows by induction on e, using the facts that $Z(G) = \{1, a^m, b^m, z\}$ (a simple consequence of (ii)), that $G/Z(G) \cong G(e-1; 0, 0)$, and that $G(2; 0, 0)$, as presented in (1.3), clearly has class 2. $\square$

Proposition 2.2. Each isomorphism $\sigma\colon G(k_1, l_1) \to G(k, l)$ is given by setting $a_1^\sigma = a^i b^j$ and $b_1^\sigma = a^f b^h$, where

(i) $k_1 \equiv k + \frac{l(f+h-i-j)}{2} \pmod 2$,

(ii) $l_1 = l$, and

(iii) either i and h are odd and j and f are even, or i and h are even and j and f are odd.

Moreover, each choice of the parameters i, j, f and h satisfying the above conditions determines an isomorphism $G(k_1, l_1) \to G(k, l)$.

Proof. Recall that

$$G = G(k,l) = \langle a, b \mid \begin{array}{l} a^n = b^n = [b^2, a^2] = 1,\ [b,a] = a^2 b^{-2} z^k, \\ (b^2)^a = b^{-2} z^l,\ (a^2)^b = a^{-2} z^l \rangle \end{array}$$

with $z = a^m b^m$, and define

$$G_1 = G(k_1, l_1) = \langle a_1, b_1 \mid \begin{array}{l} a_1^n = b_1^n = [b_1^2, a_1^2] = 1,\ [b_1, a_1] = a_1^2 b_1^{-2} z_1^{k_1}, \\ (b_1^2)^{a_1} = b_1^{-2} z_1^{l_1},\ (a_1^2)^{b_1} = a_1^{-2} z_1^{l_1} \rangle, \end{array}$$

where $z_1 = a_1^m b_1^m$.

An isomorphism $\sigma\colon G_1 \to G$ is uniquely determined by an assignment

$$a_1 \mapsto a_2 = a^i b^j, \qquad b_1 \mapsto b_2 = a^f b^h$$

for some integers i, j, f and h such that a_2 and b_2 generate G and satisfy the defining relations of G_1, when substituted for a_1 and b_1.

Now a_1 has order n, whereas Lemma 2.1(iii) shows that a_2 has order less than n if i and j are both even or both odd. We may therefore restrict attention to mappings σ for which i and j have opposite parity, that is, $a_2 \in \Phi a \cup \Phi b$. A similar argument shows that $b_2 \in \Phi a \cup \Phi b$. If a_2 and b_2 are both in Φa, or both in Φb, they are both contained in a maximal subgroup $\Phi \cup \Phi a$ or $\Phi \cup \Phi b$ of G and hence cannot generate G. They therefore lie in distinct cosets Φa and Φb, and by composing σ with the automorphism α of G transposing a and b if necessary, we may assume that $a_2 \in \Phi a$ and $b_2 \in \Phi b$, that is, i and h are odd while j and f are even. This ensures that a_2 and b_2 generate G, since none of the three maximal subgroups of G contains both of them.

For any $g \in G$ we have $g^2 \in \Phi \cong C_m \times C_m$, so a_2 and b_2 satisfy the first three relations $a_2^n = b_2^n = [b_2^2, a_2^2] = 1$ for G_1.

Now σ sends $z_1 = a_1^m b_1^m$ to $a_2^m b_2^m$. Since i is odd and j is even, we have $a_2^m = a^m$ by Lemma 2.1(iii). Similarly $b_2^m = b^m$, so σ sends z_1 to $a^m b^m = z$.

We can now consider the fourth relation. Straightforward calculations give

$$[b_2, a_2] = [a^f b^h, a^i b^j] = a^{2i} b^{-2h} z^{k + l(h-i)/2}$$

and

$$a_2^2 b_2^{-2} z^{k_1} = a^{2i} b^{-2h} z^{k_1+l(j-f)/2},$$

so we require

$$k_1 \equiv k + \frac{l(f+h-i-j)}{2} \pmod 2,$$

giving condition (i) of the Lemma.

For the fifth relation, we have

$$(b_2^2)^{a_2} = ((a^f b^h)^2)^{a^i b^j} = ((b^{2h} z^{lf/2})^{a^i b^j} = b^{-2h} z^{l+lf/2}$$

and

$$b_2^{-2} z^{l_1} = b^{-2h} z^{l_1 - lf/2};$$

since f is even and h is odd we require $l_1 = l$. Similar arguments show that the sixth and final relation is also equivalent to this, so we have condition (ii).

Conditions (i) and (ii) are necessary and sufficient conditions for σ to be an isomorphism, in the case where $a_2 \in \Phi a$ and $b_2 \in \Phi b$, that is, i and h are even while j and f are odd. For the case where $a_2 \in \Phi b$ and $b_2 \in \Phi a$ we can compose σ with α, transposing i with j, and f with h; this gives condition (iii) of the Lemma, leaving conditions (i) and (ii) unchanged. □

Corollary 2.3. For each $e \geq 3$ we have $G(1,1) \cong G(0,1)$ while $G(0,0)$, $G(1,0)$ and $G(0,1)$ are pairwise non-isomorphic.

Proof. From Proposition 2.2 we immediately deduce that $G(k,0) \ncong G(k',1)$ for any k and k', and that $G(0,0) \ncong G(1,0)$. Furthermore, taking $i = 3$, $j = f = 0$ and $h = 1$ in the definition of σ, we get an isomorphism from $G(0,1)$ to $G(1,1)$. □

Corollary 2.4. The automorphisms of $G(k,l)$ are given by $\sigma : a \mapsto a^i b^j,\ b \mapsto a^f b^h$ where

(i) either i and h are odd and j and f are even, or i and h are even and j and f are odd, and

(ii) $i + j \equiv f + h \pmod 4$ if $l = 1$.

Proof. This follows immediately from Proposition 2.2, with $k_1 = k$ and $l_1 = l$. □

By counting choices of $i, j, f, h \in \mathbb{Z}_n$ satisfying the conditions of Corollary 2.4, we deduce that $|\mathrm{Aut}\, G(k,l)| = n^4/8$ or $n^4/16$ as $l = 0$ or 1.

3 The derived group of an isobicyclic 2-group

In this section we begin an analysis of the structure of an isobicyclic 2-group. Let (G, a, b) be an n-isobicyclic triple where $n = 2^e \geq 4$. As before, let $c = [b, a]$ and let Φ denote the Frattini subgroup $\Phi(G)$ of G. Let $\mho_i(G) = \langle g^{2^i} \mid g \in G \rangle$, and let $K_i(G) = [G, G, \cdots, G]$ (i times); in particular, $K_2(G) = G'$. Each of these subgroups $\Phi(G), \mho_i(G)$ and $K_i(G)$ is a characteristic subgroup of G.

The following properties of G follow from more general known results.

Lemma 3.1. Let (G, a, b) be a non-abelian n-isobicyclic triple where $n = 2^e \geq 4$, and let $A = \langle a \rangle$ and $B = \langle b \rangle$. Then the following hold.

(i) The derived group G' is abelian (see [9]).

(ii) $G'/(G' \cap A)$ is isomorphic to a subgroup of B (see [3, Corollary C]).

(iii) G is metacyclic if and only if $G/\Phi(G')K_3(G)$ is metacyclic (see [2] or [8, Hilfssatz III.11.3]).

Lemma 3.2. [5, Lemma 3.1] Let G be an isobicyclic 2-group of exponent $2^e \geq 4$. Then G has a central series $1 = Z_0 < Z_1 < Z_2 < \cdots < Z_e = G$ of subgroups $Z_i = \langle a^{2^{e-i}} \rangle \langle b^{2^{e-i}} \rangle$ of order 2^{2i}. Moreover, $\mho_i(G) = Z_{e-i}$ and $Z_i/Z_{i-1} \cong C_2 \times C_2$ for each $i \in \{1, 2, \ldots, e\}$. In particular, for every element $g \in Z_i$ we have $|g| \leq 2^i$.

Outline proof. We proved this result as Lemma 3.1 of [5], so we simply outline the argument here. By a result of Douglas [4] and Itô [10] (see also [8, VI.10.1(a)]), the core of A in G is nontrivial. Since $\langle a^{2^{e-1}} \rangle$ is the unique minimal normal subgroup of A it is therefore normal in G, and hence central. The same applies to $\langle b^{2^{e-1}} \rangle$, so these two disjoint subgroups generate a central subgroup $Z_1 \cong C_2 \times C_2$. Now apply the same argument to the isobicyclic group G/Z_1, and iterate. □

Lemma 3.3. Let (G, a, b) be a non-abelian n-isobicyclic triple where $n = 2^e \geq 4$, and let $A = \langle a \rangle$ and $B = \langle b \rangle$. Then G has the following properties.

(i) There exist an odd integer $d < 2^e$ and integers u and v such that $0 \leq u < v \leq e$, $c = [b, a] = a^{d2^u} b^{-d2^u}$ and $G' = \langle c \rangle \times \langle a^{2^v} \rangle = \langle c \rangle \times \langle b^{2^v} \rangle$. In particular, G' is cyclic if $v = e$. Moreover, $[a^{2^u}, b^{2^v}] = [b^{2^u}, a^{2^v}] = 1$.

(ii) $\langle c \rangle \cap A = \langle c \rangle \cap B = 1$, $|c| = 2^{e-u}$, and for each integer j such that $0 \leq j \leq e - u$ there exists an odd integer h such that $c^{2^j} = a^{h2^{u+j}} b^{-h2^{u+j}}$.

(iii) Either $G' = \langle c \rangle$, or $G' = \langle c, c^a \rangle = \langle c, c^b \rangle$ with $c^a = c^s a^{t2^v}$ and $c^b = c^s b^{-t2^v}$ where s and t are odd.

Proof. Since $G = AB$, each element can be written as $a^i b^j$, and since $A \cap B = 1$ this representation is unique. Let $c = [b, a] = a^r b^w$. Since the automorphism α interchanges a and b, we have

$$b^r a^w = [b^\alpha, a^\alpha] = [a, b] = [b, a]^{-1} = b^{-w} a^{-r}.$$

Therefore $w \equiv -r \pmod{2^e}$ and so $c = a^r b^{-r}$. We can write $r = d2^u$ where d is odd, $d < 2^e$ and $0 \leq u \leq e$. Similarly, for every integer j there is an integer k such that $c^j = a^k b^{-k}$. In particular, $\langle c \rangle \cap A = \langle c \rangle \cap B = 1$, as claimed in (ii).

Let the cyclic group $G' \cap A$ be generated by a^{2^v}, where $v \leq e$. Applying α gives $G' \cap B = \langle b^{2^v} \rangle$. Since $G = \langle a, b \rangle$, Lemma III.1.11 of [8] implies that $G' = \langle c^g \mid g \in G \rangle$. By Lemma 3.1(i), G' is abelian, so c is an element of G' of maximal order. Since $\langle c \rangle \cap A = \langle c \rangle \cap B = 1$, we see that $\langle c \rangle \times \langle a^{2^v} \rangle \leq G'$. By Lemma 3.1(ii), $G'/\langle a^{2^v} \rangle$ is cyclic. Since $\langle c \rangle \cap A = 1$ again, the image of c in $G'/\langle a^{2^v} \rangle$ has order $|c|$, so it is an element of $G'/\langle a^{2^v} \rangle$ of maximal order and therefore generates $G'/\langle a^{2^v} \rangle$. This gives $G' = \langle c \rangle \times \langle a^{2^v} \rangle$ and hence, by applying α, $G' = \langle c \rangle \times \langle b^{2^v} \rangle$.

From $a^{2^v} c = c a^{2^v}$ and $c = a^{d2^u} b^{-d2^u}$ we see that $[b^{d2^u}, a^{2^v}] = 1$. Hence it follows that $[\langle b^{d2^u} \rangle, a^{2^v}] = 1$, and in particular, since d is odd, $[b^{2^u}, a^{2^v}] = 1$. By symmetry, $[a^{2^u}, b^{2^v}] = 1$.

Since $\langle c \rangle = \langle a^{d2^u} b^{-d2^u} \rangle \leq \mho_u(G) = Z_{e-u}$, Lemma 3.2 shows that $|c| \leq 2^{e-u}$. Since c is an element of maximal order in G', we have $2^{e-v} = |a^{2^v}| \leq |c| \leq 2^{e-u}$, so $u \leq v$. This proves (i), apart from the inequality $u \neq v$, which follows later.

To prove (ii), let L denote the subgroup $G'\langle b^{2^u}\rangle$. Then

$$L = \langle a^{d2^u} b^{-d2^u}, b^{2^v}\rangle\langle b^{2^u}\rangle = \langle a^{2^u}, b^{2^u}\rangle = \langle a^{2^u}\rangle\langle b^{2^u}\rangle = Z_{e-u}.$$

Computing the order

$$2^{2(e-u)} = |L| = |G'||\langle b^{2^u}\rangle|/|G' \cap \langle b^{2^u}\rangle| = |c|2^{e-v}2^{e-u}/2^{e-v} = |c|2^{e-u},$$

we see that $|c| = 2^{e-u}$.

For each $j = 0, 1, \ldots, e-u$ we have $c^{2^j} \in Z_{e-(u+j)}$, so $c^{2^j} = a^{h2^{u+j}} b^{-h2^{u+j}}$ for some integer h. Since $|c^{2^j}| = 2^{e-(u+j)}$, it follows that $c^{2^j} \notin Z_{e-(u+j+1)}$, so h is odd. This proves (ii).

We now consider (iii). Since $c^a \in G' = \langle c\rangle \times \langle a^{2^v}\rangle$, either $c^a \in \langle c\rangle$, or $c^a = c^s a^{t2^q}$ for some integers s, t and q where t is odd and $q \geq v$. In the former case we have $G' = \langle c\rangle$, satisfying (iii); we may therefore assume the latter, in which case we also have $c^b = c^s b^{-t2^q}$. Define $M = \langle c, c^a\rangle$, so $M = \langle c\rangle \times \langle a^{2^q}\rangle$. From the preceding paragraph we know that $c^{2^{q-u}} = a^{h2^q} b^{-h2^q}$ for some odd h. Therefore

$$b^{2^q} \in \langle b^{h2^q}\rangle = \langle c^{-2^{q-u}} a^{h2^q}\rangle \leq \langle c^{2^{q-u}}, a^{2^q}\rangle \leq M,$$

which implies that $M = \langle c\rangle \times \langle b^{2^q}\rangle$. Now $M^a = \langle c, a^{2^q}\rangle^a = M$ and $M^b = \langle c, b^{2^q}\rangle^b = M$, so $M^g = M$ for each $g \in G$. In particular, $c^g \in M$ for each $g \in G$. Therefore $G' = \langle c^g \mid g \in G\rangle = M$. In other words, $q = v$, that is $c^a = c^s a^{t2^v}$ where t is odd.

We now show that $u \neq v$. Recall that $G' = \langle c\rangle \times \langle a^{2^v}\rangle = \langle c\rangle \times \langle b^{2^v}\rangle$, so $Z_{e-v} = \langle a^{2^v}\rangle \times \langle b^{2^v}\rangle \leq G'$. On the other hand $c = a^{d2^u} b^{-d2^u}$, so $G' \leq \langle a^{2^u}\rangle\langle b^{2^u}\rangle = Z_{e-u}$. Now suppose that $u = v$, so $G' = Z_{e-v}$, with $e - v > 0$ since G is non-abelian. By Lemma 3.2, the subgroup $G'/Z_{e-(v+1)} = Z_{e-v}/Z_{e-(v+1)}$ is central in $G/Z_{e-(v+1)}$. We have seen that $G' = \langle c, c^a\rangle$, so $c^a Z_{e-(v+1)} = cZ_{e-(v+1)}$ since $c \in G'$. Thus

$$G'/Z_{e-(v+1)} = \langle c, c^a\rangle/Z_{e-(v+1)} = \langle c\rangle/Z_{e-(v+1)}$$

is cyclic, contradicting the fact that $Z_{e-v}/Z_{e-(v+1)} \cong C_2 \times C_2$ by Lemma 3.2. Thus $u < v$, completing the proof of (i).

Finally, since $c^a = c^s a^{t2^v} \in G' = \langle c\rangle \times \langle a^{2^v}\rangle$ we have

$$2^{e-u} = |c| = |c^a| = |c^s a^{t2^v}| = \max\{|c^s|, |a^{t2^v}|\},$$

with $|a^{t2^v}| = 2^{e-v} < 2^{e-u}$ since $v > u$, so $|c^s| = |c|$ and hence s must be odd. □

The next result uses the parameter u, where c has order 2^{e-u}, to distinguish between metacyclic and non-metacyclic n-isobicyclic groups G.

Lemma 3.4. Let (G, a, b) be a non-abelian n-isobicyclic triple where $n = 2^e \geq 4$. With u and v defined as in Lemma 3.3, the following statements hold.

(i) If $u \geq 2$ then G is metacyclic, with $2 \leq u < v = e$ and $G' = \langle c\rangle = \langle (ab^{-1})^r\rangle \cong C_{2^{e-u}}$.

(ii) If $u < 2$ then G is non-metacyclic, with $u = 1$, $v = 2$ and $G' = \langle a^2b^{-2}\rangle \times \langle a^4\rangle = \langle a^2b^{-2}\rangle \times \langle b^4\rangle$, where $\langle a^2b^{-2}\rangle \cong C_{2^{e-1}}$ and $\langle a^4\rangle \cong \langle b^4\rangle \cong C_{2^{e-2}}$.

In particular, if G is non-metacyclic and G' is cyclic, then $e = 2$ and $G' = \langle a^2b^{-2}\rangle \cong C_2$.

Proof. By Lemma 3.3, $G' = \langle c\rangle \times \langle a^{2^v}\rangle$ where $c = [b, a] = a^r b^{-r}$ with $r = d2^u$ for an odd integer d and some integers u and v such that $0 \le u < v \le e$.

(a) We first consider the case where G' is cyclic, so that $v = e$. By Lemma 3.3, $G' = \langle c\rangle$ and hence $c^a = c^s$ for some s, which must be odd. By applying α, which inverts c, we also have $c^b = c^s$. It follows that $c^{ab^{-1}} = c$. Moreover, $[c, a] = [c, b] = c^{s-1} \in \langle c^2\rangle$, which means that the image $c\langle c^2\rangle$ of c in $G/\langle c^2\rangle$ is a central involution in that group. (As a characteristic subgroup of $\langle c\rangle = G'$, $\langle c^2\rangle$ is normal in G.) Now we have

$$\begin{aligned}(ab^{-1})^2 &= ab^{-1}ab^{-1} = a^2(a^{-1}ba)^{-1}b^{-1} = a^2(bc)^{-1}b^{-1}\\ &= a^2c^{-1}b^{-2} \equiv a^2b^{-2}c \pmod{\langle c^2\rangle}\end{aligned}$$

and

$$a^2b^2 = abac^{-1}b = bac^{-1}ac^{-1}b \equiv baab \equiv babac^{-1} \equiv b^2ac^{-1}ac^{-1} \equiv b^2a^2 \pmod{\langle c^2\rangle}.$$

(a1) Suppose that $u \ge 2$, as in (i). Then $r/2$ is even, so

$$\begin{aligned}(ab^{-1})^r &= ((ab^{-1})^2)^{r/2} \equiv (a^2b^{-2}c)^{r/2} \equiv (a^2b^{-2})^{r/2}c^{r/2}\\ &\equiv (a^2b^{-2})^{r/2} \equiv a^rb^{-r} \equiv c \pmod{\langle c^2\rangle}.\end{aligned}$$

Thus $(ab^{-1})^r$ is an odd power of c, so $\langle (ab^{-1})^r\rangle = \langle c\rangle$ and $|(ab^{-1})^r| = |c| = 2^{e-u}$ by Lemma 3.3. Since $G = \langle a, b\rangle$, the quotient group $G/G' = G/\langle c\rangle$ is generated by the images $\overline{a}$ and $\overline{ab^{-1}}$ of a and ab^{-1} in this group. Now $\langle c\rangle \cap A = 1$ by Lemma 3.3(ii), so $\overline{a}$ has order $|a| = 2^e$. Since $(ab^{-1})^r \in \langle c\rangle$, we see that $\overline{ab^{-1}}$ has order dividing r, and hence dividing 2^u. But $G/\langle c\rangle$ is an abelian group of order $|G|/|\langle c\rangle| = 2^{2e}/2^{e-u} = 2^{e+u}$, so $\overline{ab^{-1}}$ must have order 2^u with $G/G' = \langle\overline{a}\rangle \times \langle\overline{ab^{-1}}\rangle \cong C_{2^e} \times C_{2^u}$.

Since $(ab^{-1})^r$ is an odd power of c we have $\langle (ab^{-1})^r\rangle = \langle c\rangle$, so the cyclic subgroup $H := \langle ab^{-1}\rangle$ contains G' with index 2^u. Since the image of H in G/G' has order 2^u, and G' has order 2^{e-u}, it follows that H has order 2^e. Since H contains G' it is a normal subgroup of G. Thus $AH = HA$ is a subgroup of G, and since it contains both a and b we have $G = AH$, so G is metacyclic. This proves (i) in the case where G' is cyclic.

(a2) Now suppose that G' is cyclic and $u = 0$. Then $G' = \langle c\rangle$ has order $2^{e-u} = 2^e$ by Lemma 3.3(ii). Since $G' \cap A = 1$ by Lemma 3.3(ii) we have $|G'A| = |G'||A| = 2^{2e} = |G|$, so $G = G'A$ and G/G' is cyclic. But then G/Φ is cyclic and hence so is G, a contradiction. Hence $u \neq 0$.

We therefore have $u = 1$, so $r = 2d$, giving $c = a^{2d}b^{-2d}$, where d is odd. By Lemma 3.3(ii), $|c| = 2^{e-1}$, and since $G = \langle a, ab^{-1}\rangle$ we have $G/G' = G/\langle c\rangle \cong C_{2^e} \times C_2$.

(a3) Suppose first that G is metacyclic. Huppert gives the general form for a metacyclic p-group in [8, III.11.2]; taking $p = 2$ we have

$$G = \langle g, h \mid h^{2^i} = 1,\ g^{2^j} = h^{2^k},\ h^g = h^q\rangle$$

with $0 \le k \le i$, $q^{2^j} \equiv 1 \pmod{2^i}$ and $2^k(q-1) \equiv 0 \pmod{2^i}$. Thus G has a normal subgroup $H = \langle h\rangle \cong C_{2^i}$ with $G/H \cong C_{2^j}$, so $|G| = 2^{i+j}$ and hence $i + j = 2e$. Since $|h| = 2^i$ and G has exponent $n = 2^e$ we have $i \le e$ and hence $j \ge e$. Since $|g| = 2^{i+j-k}$ we have $i + j - k \le e$ and hence $k \ge e$. But $k \le i$, so $i = j = k = e$. Thus

$$G = \langle g, h \mid g^n = h^n = 1,\ h^g = h^q\rangle$$

for some q satisfying $1 < q < n$. Now G', being cyclic, is generated by $[h, g] = h^{q-1}$. We are assuming that $u = 1$, so $G' \cong C_{2^{e-1}}$ and hence $q \equiv 3 \pmod 4$.

Each element of G has the form $g^i h^j$ for a unique pair $i, j \in \mathbb{Z}_n$. By using the relation $(h^j)^{g^i} = h^{jq^i}$, we obtain

$$\begin{aligned}(g^i h^j)^m &= g^{im}(h^j)^{g^{i(m-1)}}(h^j)^{g^{i(m-2)}} \dots (h^j)^{g^i} h^j \\ &= g^{im} h^{j(q^{i(m-1)}+q^{i(m-2)}+\dots+q^i+1)} \\ &= g^{im} h^{j(q^{im}-1)/(q^i-1)}\end{aligned}$$

for all $m \geq 1$. Let $m = n/2 = 2^{e-1}$. If i is even then $g^{im} = 1$ and $q^i \equiv 1 \pmod 4$; if $2^k \parallel q^i - 1$ then $2^{k+e-1} \parallel q^{im} - 1$, so $2^{e-1} \parallel (q^{im} - 1)/(q^i - 1)$ and hence $(g^i h^j)^m = h^{jm}$. If i is odd then $g^{im} = g^m$ and $q^i \equiv 3 \pmod 4$; if $2^k \parallel q^i + 1$ (so $k \geq 2$) then $2^{k+e-1} \parallel q^{im} - 1$, and $2 \parallel q^i - 1$, so $2^e \mid (q^{im} - 1)/(q^i - 1)$ and $(g^i h^j)^m = g^m$. Thus

$$(g^i h^j)^m = \begin{cases} h^{jm}, & \text{for } i \text{ even,} \\ g^m, & \text{for } i \text{ odd,} \end{cases}$$

so $g^i h^j$ has order n if and only if i or j is odd, that is, $g^i h^j \notin \Phi = \langle g^2, h^2 \rangle$.

If a and b are an isobicyclic pair for G then they have order n, so they are not elements of Φ. Since they generate G, they are in different cosets of Φ, namely $g\Phi$, $h\Phi$ or $gh\Phi$. The subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$ are disjoint, so $a^m \neq b^m$; hence these two cosets cannot be $g\Phi$ and $gh\Phi$ (otherwise $a^m = g^m = b^m$), so one of them must be $h\Phi$, say $a \in h\Phi$. Then $A\Phi = H\Phi$, so $H^\alpha \Phi = B\Phi = g\Phi$ or $gh\Phi$, giving $(h^\alpha)^m = g^m \neq h^m$ and hence $H^\alpha \cap H = 1$. Since H is a normal subgroup of G, so is H^α. Hence $G = H^\alpha \times H$, which is abelian, contradicting the assumption. Thus G cannot be metacyclic.

(a4) Now suppose that G is non-metacyclic, with G' cyclic and $u = 1$ as before. We consider the subgroup $N := \langle c, ab^{-1} \rangle$ of G; this is abelian since $c^{ab^{-1}} = c$, and it is normal in G since it contains $G' = \langle c \rangle$. Note that N is the preimage in G of $\langle \overline{ab^{-1}} \rangle \leq G/G'$. In the abelian group $G/G' = G/\langle c \rangle$ we have $(\overline{ab^{-1}})^{2d} = \overline{a^{2d} b^{-2d}} = \overline{c} = \overline{1}$, which means that $\overline{ab^{-1}}$ is of order 2. Since $|c| = 2^{e-1}$, we have $|N| = 2^e$. Since $\langle N, a \rangle = \langle a, b \rangle = G$, we deduce that $G = N \rtimes \langle a \rangle$. Since G is not metacyclic, N can not be cyclic and so $N \cong C_{2^{e-1}} \times C_2$. Let c' be an involution of N different from $c^{2^{e-2}}$, so that $N = \langle c \rangle \times \langle c' \rangle$. Then the conjugacy action of a on N is defined by $c^a = c^s$ and $(c')^a = c^{j2^{e-2}} c'$, where $j = 0$ or 1, and s is odd. Now $G = \langle c, c', a \rangle$ with $[c, c'] = 1$, so $G' = \langle [c, a]^g, [c', a]^g \mid g \in G \rangle \leq \langle c^2, c^{j2^{e-2}} \rangle$. Since $G' = \langle c \rangle$, we see that $j2^{e-2}$ must be odd, so $j = 1$ and $e = 2$, giving $|G| = 16$. Since $v = e$ we have $v = 2$, and we have proved (ii) in the case where G' is cyclic. (Note that $a^4 = b^4 = 1$ in this case.)

(b) We now consider the case where G' is not cyclic, that is, $v < e$. This immediately implies that G is not metacyclic. Recall that $u < v$. Since $G' = \langle c \rangle \times \langle a^{2^v} \rangle$, we have $\Phi(G') = \langle c^2 \rangle \times \langle a^{2^{v+1}} \rangle$. Moreover, by Lemma 3.3(iii) we have $c^a = c^s a^{t2^v}$ for some odd integers s and t. Then $a^{t2^v} = c^{-s+1}[c, a] \in L := \Phi(G')K_3(G)$, which implies that $a^{2^v} \in L$. Since L is a characteristic subgroup of G, it also contains $b^{2^v} = (a^{2^v})^\alpha$, and hence it contains the subgroup $Z_{e-v} = \langle a^{2^{e-v}} \rangle \langle b^{2^{e-v}} \rangle$.

Suppose that G/Z_{e-v} is metacyclic. Since $G/L \cong (G/Z_{e-v})/(L/Z_{e-v})$, it follows that G/L is metacyclic. Then Lemma 3.1(iii) implies that G is metacyclic, which is a contradiction. Therefore G/Z_{e-v} is non-metacyclic.

Now G/Z_{e-v} is an isobicyclic 2-group. Since it is non-metacyclic, and its derived group $(G/Z_{e-v})' = G'/Z_{e-v} \cong C_{2^{v-u}}$ is cyclic, it follows from part (a2) of this proof that G/Z_{e-v} has order 16, with $|G'/Z_{e-v}| = 2$. Thus $a^4 \in Z_{e-v} = \langle a^{2^v}\rangle\langle b^{2^v}\rangle$, so $v = 2$. Since $G'/Z_{e-v} \cong C_{2^{v-u}}$ and $|G'/Z_{e-v}| = 2$, we deduce that $v - u = 1$, so $u = 1$. We have $G' = \langle c\rangle \times \langle a^{2^v}\rangle = \langle c\rangle \times \langle b^{2^v}\rangle$ with $c = a^{d2^u}b^{-d2^u} = a^{2d}b^{-2d}$ for some odd d, so $G' = \langle a^2b^{-2}\rangle \times \langle a^4\rangle = \langle a^2b^{-2}\rangle \times \langle b^4\rangle$, with first and second factors cyclic of orders 2^{e-1} and 2^{e-2} as required for (ii).

The final statement in the Lemma is an immediate consequence of (i) and (ii). □

4 Non-metacyclic isobicyclic 2-groups

The following theorem characterises non-metacyclic isobicyclic 2-groups.

Theorem 4.1. Let (G, a, b) be a non-metacyclic n-isobicyclic triple with $n = 2^e \geq 4$.
(i) If $e = 2$ then

$$\begin{aligned} G &= \langle a, b \mid a^4 = b^4 = [a^2, b] = [b^2, a] = 1, [b, a] = a^2b^2\rangle \\ &\cong G_2(2; 0, 0). \end{aligned}$$

(ii) If $e \geq 3$ then

$$\begin{aligned} G &= \langle a, b \mid a^n = b^n = [b^2, a^2] = 1,\ [b, a] = a^2b^{-2}(a^{n/2}b^{n/2})^k, \\ &\qquad (b^2)^a = b^{-2}(a^{n/2}b^{n/2})^l,\ (a^2)^b = a^{-2}(a^{n/2}b^{n/2})^l\rangle \\ &\cong G_2(e; k, l) \end{aligned}$$

where $k, l \in \{0, 1\}$.

Proof. (i) If $e = 2$ then $c = a^2b^{-2} = a^2b^2$ is an involution commuting with both a and b, so c is central in G. Thus $[a, b^2] = [b, a^2] = 1$ and it follows that $G \cong G_2(2; 0, 0)$.

(ii) Let $e \geq 3$. By Lemma 3.4(ii) we see that $v = 2$ and $u = 1$, so

$$G' = \langle c\rangle \times \langle a^4\rangle = \langle a^2b^{-2}\rangle \times \langle a^4\rangle \cong C_{2^{e-1}} \times C_{2^{e-2}}$$

where $c = [b, a] = a^{2d}b^{-2d}$ for some odd d. By Lemma 3.3(iii) we have

$$c^a = c^s a^{4t} \quad \text{and} \quad c^b = c^s b^{-4t} \tag{4.1}$$

for some odd s and t. We will determine d, s and t up to group automorphisms.

By Lemma 3.4(ii), $[a^4, b^2] = [a^2, b^4] = 1$; since $d - 1$ is even, this implies that

$$\begin{aligned} c^{a^2b^{-2}} &= b^2a^{-2}a^{2d}b^{-2d}a^2b^{-2} = b^2a^{2(d-1)}b^{-2d}a^2b^{-2} \\ &= a^{2(d-1)}b^{-2(d-1)}a^2b^{-2} = a^{2d}b^{-2d} = c, \end{aligned}$$

so $c^{a^2} = c^{b^2}$. Since

$$c^{a^2} = (c^a)^a = (c^s a^{4t})^a = (c^a)^s a^{4t} = c^{s^2} a^{4t(s+1)}$$

and

$$c^{b^2} = (c^b)^b = (c^s b^{-4t})^b = (c^b)^s b^{-4t} = c^{s^2} b^{-4t(s+1)},$$

we see that $a^{4t(s+1)} = b^{-4t(s+1)}$. However, $A \cap B = 1$, so $a^{4t(1+s)} = 1$ and hence $c^{a^2} = c^{s^2}$. Since t is odd, we have $s \equiv -1 \pmod{2^{e-2}}$. In what follows, we set $s = -1 + l2^{e-2}$; since $|c| = 2^{e-1}$ we can assume that $l = 0$ or 1. Then $s^2 \equiv 1 \pmod{2^{e-1}}$, and because $|c| = 2^{e-1}$ we have $c^{a^2} = c^{s^2} = c$. Thus a^2 commutes with $c = a^{2d}b^{-2d}$, and hence with b^{2d}; since d is odd we therefore have

$$[a^2, b^2] = 1. \tag{4.2}$$

Using equation (4.1) we see that

$$\begin{aligned} b^{-1}a^{2j}b &= ((a^b)^2)^j = ((ac^{-1})^2)^j = (ac^{-1}ac^{-1})^j = (a^2(c^a)^{-1}c^{-1})^j \\ &= (a^2a^{-4t}c^{-(s+1)})^j = a^{2(1-2t)j}c^{-j(s+1)} \end{aligned} \tag{4.3}$$

for each positive integer j. By taking $j = 2^{e-2}$ we deduce that the involution $a^{2^{e-1}}$ is central in G, and the same holds for $b^{2^{e-1}}$. In what follows we set $z = a^{2^{e-1}}b^{2^{e-1}} = c^{2^{e-2}}$. By equations (4.1) and (4.3) we have

$$c^a = c^{-1}a^{4t}z^l, \quad c^b = c^{-1}b^{-4t}z^l, \quad b^{-1}a^{2j}b = a^{2(1-2t)j}z^{lj}. \tag{4.4}$$

From $c^b = (a^{2d}b^{-2d})^b$ and equation (4.4) we have

$$a^{-2d}b^{2d}b^{-4t}z^l = a^{2(1-2t)d}z^{ld}b^{-2d},$$

so

$$a^{4(1-t)d} = b^{4(d-t)}$$

and hence

$$(1-t)d \equiv d - t \equiv 0 \pmod{2^{e-2}}.$$

Solving these equations gives

$$d \equiv t \equiv 1 \pmod{2^{e-2}}.$$

Writing $d = 1 + k2^{e-2}$ where $k = 0$ or 1, and using $b^{2^e} = 1$, we see from these two congruences that the relations $c = a^{2d}b^{-2d}$, $c^a = c^s a^{4t}$ and $c^b = c^s b^{-4t}$ can be respectively rewritten as

$$[b,a] = a^{2+k2^{e-1}}b^{-2+k2^{e-1}},\ (b^2)^a = a^{l2^{e-1}}b^{-2+l2^{e-1}},\ (a^2)^b = a^{-2+l2^{e-1}}b^{l2^{e-1}} \tag{4.5}$$

where $k, l \in \{0, 1\}$. By combining the relations in (4.5) with the fact that $a^{2^e} = b^{2^e} = [a^2, b^2] = 1$, we see that G satisfies all the defining relations of $G_2(e; k, l)$ in (1.2). Thus G is an epimorphic image of $G_2(e; k, l)$, and since these two groups have the same order, they are isomorphic. $\square$

Recall that Theorem 1.2, of Berkovich and Janko, states that a 2-generator 2-group with exactly one non-metacyclic maximal subgroup, and with a derived group isomorphic to $C_{2^r} \times C_{2^{r+1}}$ for some $r \geq 2$, has a presentation of the form (1.4). Here we consider a subset of these groups, namely those for which x^2 and w are powers of the central involution z. For each $r \geq 2$, and for each pair $k, l \in \{0, 1\}$, let $G = G(k, l)$ denote the group given by the presentation (1.4) with $x^2 = z^k$ and $w = z^l$, that is,

$$\begin{aligned} G(k,l) = \langle a, x \mid\ & a^{2^{r+2}} = 1,\ [a,x] = v,\ [v,a] = b,\ v^{2^{r+1}} = b^{2^r} = [v,b] = 1, \\ & v^{2^r} = z,\ b^{2^{r-1}} = u,\ x^2 = z^k,\ b^x = b^{-1}, \\ & v^x = v^{-1},\ b^a = b^{-1},\ a^4 = v^{-2}b^{-1}w,\ w = z^l \rangle. \end{aligned} \tag{4.6}$$

Our aim is to show that G is isomorphic to the group $G_2 = G_2(e; k, l)$, where $e = r + 2$. To avoid notational confusion, let us present G_2 as

$$G_2(e;k,l) = \langle a_1, b_1 \mid \ a_1^{2^e} = b_1^{2^e} = [b_1^2, a_1^2] = 1,\ [b_1, a_1] = a_1^2 b_1^{-2}(a_1^{2^{e-1}} b_1^{2^{e-1}})^k, \\ (b_1^2)^{a_1} = b_1^{-2}(a_1^{2^{e-1}} b_1^{2^{e-1}})^l,\ (a_1^2)^{b_1} = a_1^{-2}(a_1^{2^{e-1}} b_1^{2^{e-1}})^l \rangle. \tag{4.7}$$

Theorem 4.2. For each $e = r + 2 \geq 4$ there is an isomorphism from $G(k, l)$ to $G_2(e; k, l)$ sending the generators a and x of $G(k, l)$ to a_1 and $a_1^{-1}b_1$ in $G_2(e; k, l)$.

Proof. We first show that the map $a \mapsto a_1,\ x \mapsto a_1^{-1}b_1$ extends to a homomorphism $G \to G_2$. We map the other elements of G appearing in (4.6) into G_2 by

$$v \mapsto a_1^{-2}b_1^2 z_1^k, \quad b \mapsto b_1^{-4} z_1^l, \quad z \mapsto z_1, \quad u \mapsto b_1^{2^{e-1}} \quad \text{and} \quad w \mapsto z_1^l$$

where $z_1 = a_1^{2^{e-1}} b_1^{2^{e-1}}$. We need to show that the defining relations for G in (4.6) are satisfied when a, x, v, b, z, u and w are replaced with their images in G_2. This is a routine matter, using the properties of G_2 proved in Section 2, so we will simply illustrate it in a typical case, namely the relation $x^2 = z^k$. For this we need to show that $(a_1^{-1}b_1)^2 = z_1^k$ in G_2. Using Lemma 2.1(ii) and the fact that z_1 is in the centre of G_2, we have

$$(a_1^{-1}b_1)^2 = a_1^{-1}(b_1 a_1^{-1})b_1 = a_1^{-1}(a_1 b_1^{-1} z_1^k)b_1 = z_1^k,$$

as required. The other cases are similar, so the mapping extends to a homomorphism $\theta : G \to G_2$. This is an epimorphism since a_1 and $a_1^{-1}b_1$ generate G_2.

We proved in [6, Prop. 2.1] that $|G_2| = 2^{2e}$, so $|G| \geq 2^{2e}$. The defining relations

$$v^{2^{r+1}} = b^{2^r} = [v, b] = 1$$

for G show that $\langle v, b\rangle$ has order at most 2^{2r+1}. The relations

$$x^2 = z^k\ (= v^{2^r k}),\ v^x = v^{-1},\ b^x = b^{-1}$$

show that $\langle v, b\rangle$ is a normal subgroup of index at most 2 in $\langle v, b, x\rangle$, so the latter group has order at most 2^{2r+2}. Finally the relations

$$a^4 = v^{-2}b^{-1}z^l,\ b^a = b^{-1},\ v^a = vb,\ x^a = xv^{-1}$$

show that $\langle v, b, x\rangle$ is a normal subgroup of index at most 4 in $\langle v, b, x, a\rangle = G$, so $|G| \leq 2^{2r+4} = 2^{2e}$. Thus $|G| = |G_2|$, so θ is an isomorphism. □

This confirms the assertions in [1, 12] that the groups $G(k, l)$ have order 2^{2r+4}, a fact which is not immediately apparent from the presentation (4.6).

5 Regular embeddings of $K_{n,n}$ where $n = 2^e$

A *map* $\mathcal{M}$ is a cellular embedding of a connected graph K in a closed orientable surface. It is *(orientably) regular* if the group $\mathrm{Aut}(\mathcal{M})$ of all orientation-preserving automorphisms of the embedding acts regularly on the oriented edges (darts) of K.

It was shown in [15, Section 2] that every regular embedding $\mathcal{M}$ of a complete bipartite graph $K_{n,n}$ determines an n-isobicyclic triple (G, a, b). Here G is the subgroup $\mathrm{Aut}_0(\mathcal{M})$

of index 2 in Aut($\mathcal{M}$) leaving the bipartition of $K_{n,n}$ invariant. The generators a and b rotate a chosen edge $e = uv$ around its incident vertices u and v to the next edge, following the orientation of the surface around u or v. The automorphism of G transposing a and b is induced by conjugation by the map automorphism reversing e. Conversely, every n-isobicyclic triple (G, a, b) arises in this way, with (G_1, a_1, b_1) and (G, a, b) giving isomorphic maps if and only if there is an isomorphism $G_1 \to G$ sending a_1 to a and b_1 to b (see [15] or [14, Proposition 2]). Thus an isobicyclic group G may have inequivalent pairs a, b leading to non-isomorphic maps.

The following characterisation of regular embeddings of $K_{n,n}$, where $n = 2^e$ and $\mathrm{Aut}_0(\mathcal{M})$ is non-metacyclic, was proved in [6]. Here we give a different proof, using the structure of non-metacyclic isobicyclic 2-groups described in earlier sections.

Theorem 5.1. For each $n = 2^e \geq 8$ there are exactly four non-isomorphic regular embeddings $\mathcal{M}$ of $K_{n,n}$ for which $\mathrm{Aut}_0(\mathcal{M})$ is non-metacyclic; these correspond to the four isobicyclic triples (G, a, b), where $G = G_2(e; k, l)$ and $k, l \in \{0, 1\}$. There is exactly one regular embedding $\mathcal{M}$ of $K_{4,4}$ for which $\mathrm{Aut}_0(\mathcal{M})$ is non-metacyclic; this map corresponds to the isobicyclic triple (G, a, b) where $G = G_2(2; 0, 0)$.

Proof. If $e = 2$ the result follows directly from Theorem 4.1(i). We may therefore assume that $e \geq 3$, so by Theorem 4.1(ii) there are at most four isomorphism classes of isobicyclic triples, corresponding to the four presentations $G_2(e; k, l)$ where $k, l \in \{0, 1\}$. By Corollary 2.3, the groups $G_2(e; 0, 0)$, $G_2(e; 1, 0)$ and $G_2(e; 0, 1)$ are mutually non-isomorphic, and hence so are the corresponding isobicyclic triples. To complete the classification it is enough to show that the triples corresponding to the isomorphic groups $G_2(e; 0, 1)$ and $G_2(e; 1, 1)$ are not equivalent. If there is an isomorphism from $G_2(e; 1, 1) = \langle a_1 \rangle \langle b_1 \rangle$ to $G_2(e; 0, 1) = \langle a \rangle \langle b \rangle$ taking a_1 to a and b_1 to b then condition (i) of Proposition 2.2 gives $1 = k_1 \equiv k = 0 \pmod 2$, a contradiction. Hence there are four non-isomorphic maps, as claimed. $\square$

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Sharp spectral inequalities for connected bipartite graphs with maximal Q-index

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Abstract

The Q-index of a simple graph is the largest eigenvalue of its signless Laplacian. As for the adjacency spectrum, we will show that in the set of connected bipartite graphs with fixed order and size, the bipartite graphs with maximal Q-index are the double nested graphs. We provide a sequence of (in)equalities regarding the principal eigenvector of the signless Laplacian of double nested graphs and apply these results to obtain some lower and upper bounds for their Q-index. In the end, we give some computational results in order to compare these bounds.

Keywords: Double nested graph, signless Laplacian, largest eigenvalue, spectral inequalities.

Math. Subj. Class.: 05C50

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1 Introduction

Let $G = (V, E)$ be a (simple) graph, of order $\nu = |V|$ and size $\epsilon = |E|$. The signless Laplacian of G is defined to be the matrix $Q = A + D$, where $A(= A(G))$ is the adjacency matrix of G, while $D(= D(G))$ is the diagonal matrix of its vertex degrees. The largest eigenvalue (or spectral radius) of Q is usually called the *Q-index* of G, and is denoted by $\kappa(= \kappa(G))$. Much interest has been paid recently to this very important spectral invariant. Let us recall that

$$\Delta + 1 \leqslant \kappa \leqslant 2(\nu - 1)\,, \tag{1.1}$$

where Δ denotes the maximal vertex degree of the graph, with equality for stars, on the lower bound, and complete graphs, on upper bound [7].

In 2007, Cvetković, Rowlinson, and Simić [6] conjectured that

$$\kappa \leqslant \nu - 1 + \bar{d}\,, \tag{1.2}$$

where $\bar{d}$ is the average (vertex) degree of a graph. Later Feng and Yu [9] proved that (1.2) is true (cf. also [1]). Many other bounds on Q-index for arbitrary (connected) graphs can be found in [8].

We will now describe in brief the structure of a connected double nested graph (or DNG for short). It was first considered in [3, 4] and, independently, under the name of *chain graph*, in [5], in studying graphs whose least eigenvalue is minimal among the connected (bipartite) graphs of fixed order and size. The vertex set of any such graph G consists of two colour classes (or co-cliques). To specify the nesting, both of them are partitioned into h non-empty cells $\bigcup_{i=1}^{h} U_i$ and $\bigcup_{i=1}^{h} V_i$, respectively; all vertices in U_s are joined (by cross edges) to all vertices in $\bigcup_{k=1}^{h+1-s} V_k$, for $s = 1, 2, \ldots, h$. Denote by $N_G(w)$ the set of neighbors of a vertex w. Hence, if $u' \in U_{s+1}$, $u'' \in U_s$, $v' \in V_{t+1}$, and $v'' \in V_t$, then $N_G(u') \subset N_G(u'')$ and $N_G(v') \subset N_G(v'')$. This claim makes precise the double nesting property. Observe that $1 \leqslant s, t \leqslant h$.

If $m_s = |U_s|$ and $n_s = |V_s|$, with $s = 1, 2, \ldots, h$, then G is denoted by

$$DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)\,.$$

Note that G is connected whenever $m_1, n_1 > 0$. Additionally, if some of the remaining parameters are equal to zero, we again get a DNG with a smaller value of h. Thus, throughout we assume that all these parameters are greater than zero.

We now introduce some notation to be used later on. Let

$$M_s = \sum_{i=1}^{s} m_i \quad \text{and} \quad N_t = \sum_{j=1}^{t} n_j\,, \quad \text{for } 1 \leqslant s, t \leqslant h.$$

Thus G is of order $\nu = M_h + N_h$ and size $\epsilon = \sum_{s=1}^{h} m_s N_{h+1-s}$. Observe that N_{h+1-s} is the degree of a vertex $u \in U_s$; the degree of a vertex $v \in V_t$ is equal to M_{h+1-t}. We also set $M_{s,t} = M_t - M_{s-1}$ and, additionally, $M_{1,t} = M_t$.

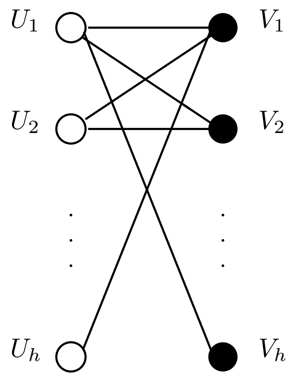

Figure 1: The structure of a double nested graph.

2 Extremal bipartite graphs

Let G be a bipartite graph with colour classes U and V. First, we state the main result of this section.

Theorem 2.1. *If G is a graph for which $\kappa(G)$ is maximal among all connected bipartite graphs of order ν and size ϵ, then G is a DNG with all pendant edges attached to a common vertex.*

Theorem 2.1 means that double nested graphs play the same role among bipartite graphs (with respect to the signless Laplacian index) as nested split graphs among non-bipartite graphs. The same classes of graphs appear as extremal with respect to the adjacency spectra as well, i.e., in the class of all connected (resp. all connected bipartite) graphs of fixed order and size, those with maximal radius with respect to the adjacency matrix are NSGs (resp. DNGs). The proof of Theorem 2.1 is based on the following lemmas, the first of which is taken from [6]. Recall that there exists a unique unit eigenvector corresponding to $\kappa(G)$ having only positive entries; this eigenvector is called the principal eigenvector of G.

Lemma 2.2. *Let G' be the graph obtained from a connected graph G by rotating the edge rs around r to the non-edge position rt. Let $\mathbf{x} = (x_1, x_2, \ldots, x_\nu)^T$ be the principal eigenvector of G. If $x_t \geqslant x_s$ then $\kappa(G') > \kappa(G)$.*

The next lemma will be very helpful when we find a bridge in a graph whose index is assumed to be maximal. Given two rooted graphs $P(= Pu)$ and $Q(= Qv)$ with u and v as roots, let G be the graph obtained from the disjoint union $P \dot{\cup} Q$ by adding the edge uv. Let G' be the graph obtained from the coalescence of Pu and Qv by attaching a pendant edge at the vertex identified with u and v.

Lemma 2.3. *With the above notation, if P and Q are two non-trivial connected graphs then $\kappa(G) < \kappa(G')$.*

Proof. Let $(x_1, x_2, \ldots, x_\nu)^T$ be the principal eigenvector of G. Without loss of generality, we may suppose that $x_u \leqslant x_v$. Let $N_P(u)$ be the neigbourhood of u in P; since P is non-trivial, $N_P(u) \neq \emptyset$. Now G' is obtained from G by replacing the edges uw, with $w \in N_P(u)$ by the edges vw and, therefore, $\kappa(G) < \kappa(G')$, by Lemma 2.2, as required. □

In what follows we assume that G has maximal index among the connected bipartite graphs of fixed order and size.

Lemma 2.4. *Let G be a graph satisfying the above assumptions, and let $\mathbf{x} = (x_1, \ldots, x_\nu)^T$ be the principal eigenvector of G. If v and w are vertices in the same colour class such that $x_v \geqslant x_w$, then $\deg(v) \geqslant \deg(w)$.*

Proof. Let U, V be the colour classes of G. Assuming that v and w are vertices in V such that $x_v \geqslant x_w$ and $\deg(v) < \deg(w)$, then $\deg(w) > 1$ and there exists $u \in U$ such that $v \not\sim u \sim w$. By Lemma 2.3, we may rotate uw to uv to obtain a graph G' such that $\kappa(G') > \kappa(G)$. If uw is a bridge, then $\deg(u) = 1$ and, again by Lemma 2.3, G' is necessarily connected; but now the maximality of $\kappa(G)$ is contradicted and the proof is complete. □

From now on we take the colour classes to be $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, with $x_{u_1} \geqslant x_{u_2} \geqslant \cdots \geqslant x_{u_m}$ and $x_{v_1} \geqslant x_{v_2} \geqslant \cdots \geqslant x_{v_n}$. By Lemma 2.4, this ordering coincides with the ordering by degrees in each colour class. In the next lemma we note some consequences of those facts.

Lemma 2.5. *Let G be a graph satisfying the above assumptions including those on vertex ordering. Then*

(i) the vertices u_1 and v_1 are adjacent;

(ii) u_1 is adjacent to every vertex in V, and v_1 is adjacent to every vertex in U;

(iii) if the vertex u is adjacent to v_k, then u is adjacent to v_j, for all $j < k$, and if the vertex v is adjacent to u_k, then v is adjacent to u_j, for all $j < k$.

Proof. First we consider bridges in G. By Lemma 2.3, all bridges are pendant edges. By Lemma 2.2, all pendant edges are attached at the same vertex, and this vertex w is such that x_w is maximal. Without loss of generality, $x_{u_1} \geqslant x_{v_1}$ and $w = u_1$. It follows that the result holds if G is a tree and, consequently, G is a star. Accordingly, we suppose that G is not a tree.

To prove (i), suppose by way of contradiction that $u_1 \not\sim v_1$. Then v_1 is adjacent to some vertex $u \in U$, and uv_1 is not a bridge. By Lemma 2.2, we may rotate v_1u to v_1u_1 to obtain a connected bipartite graph G' such that $\kappa(G') > \kappa(G)$, contradicting the maximality of $\kappa(G)$.

To prove (ii), suppose that u is a vertex of U not adjacent to v_1. Then $u \neq u_1$ by (i), uv is not a bridge, and u is adjacent to some vertex v in V other than v_1. Now we can rotate uv to uv_1 to obtain a contradiction as before. Secondly, suppose that v is a vertex of V not adjacent to u_1. Then $v \neq v_1$ by (i), again vu_1 is not a bridge, and a rotation about v yields a contradiction.

To prove (iii), suppose that $u \in U$, $u \sim v_k$ and $u \not\sim v_j$ for some $j < k$. Now $u \neq u_1$ by (ii), and so uv_k is not a bridge. Then we can rotate uv_k to uv_j to obtain a contradiction. Finally, suppose that $v \in V$, $v \sim v_k$ and $v \not\sim u_j$ for some $j < k$. In this case, vu_k is not a bridge because $k > 1$, and the rotation of vu_k to vu_j yields a contradiction.

The proof is now finished. □

Taking into account Lemma 2.5 and the definition of a DNG the first part of Theorem 2.1 follows. It remains only to prove that all cut-edges in the observed DNG are pendant edges attached to a common vertex. This easily comes from Lemma 2.3.

3 Q-eigenvectors of DNGs

Here we consider the principal eigenvector of the signless Laplacian of DNGs. In this section (and in the next one, if not told otherwise) we will assume that

$$\mathbf{x} = (x_1, \ldots, x_\nu)^T$$

is a Q-eigenvector of G with all positive entries, which is usually normalized, i.e.,

$$\sum_{i=1}^{\nu} x_i = 1\,.$$

The entries of $\mathbf{x}$ are also called the weights of the corresponding vertices. We first observe that all vertices within the sets U_s or V_t, for $1 \leqslant s, t \leqslant h$, have the same weights, since they belong to the same orbit of G. Let $x_u = a_s$, if $u \in U_s$, while $x_v = b_t$, if $v \in V_t$.

From the eigenvalue equations for κ, applied to any vertex from U_s or V_t, we get

$$\kappa a_s = N_{h+1-s} a_s + \sum_{j=1}^{h+1-s} n_j b_j\,, \quad \text{for } s = 1, \ldots, h, \tag{3.1}$$

and

$$\kappa b_t = M_{h+1-t} b_t + \sum_{i=1}^{h+1-t} m_i a_i\,, \quad \text{for } t = 1, \ldots, h. \tag{3.2}$$

By normalization we have

$$\sum_{i=1}^{h} m_i a_i + \sum_{j=1}^{h} n_j b_j = 1\,, \tag{3.3}$$

and, from (3.1), we easily get

$$a_s = \frac{1}{\kappa - N_{h+1-s}} \sum_{j=1}^{h+1-s} n_j b_j\,, \quad \text{for } s = 1, \ldots, h. \tag{3.4}$$

From (3.2) we have

$$b_t = \frac{1}{\kappa - M_{h+1-t}} \sum_{i=1}^{h+1-t} m_i a_i, \quad \text{for } t = 1, \ldots, h, \tag{3.5}$$

and therefore, using (3.3), we have

$$a_s = \frac{1}{\kappa - N_{h+1-s}} \left(1 - \sum_{i=1}^{h} m_i a_i - \sum_{j=h+2-s}^{h} n_j b_j \right), \quad \text{for } s = 1, \ldots, h, \tag{3.6}$$

or, using (3.2) for $t = 1$,

$$a_s = \frac{1}{\kappa - N_{h+1-s}} \left(1 - (\kappa - M_h) b_1 - \sum_{j=h+2-s}^{h} n_j b_j \right) , \qquad \text{for } s = 1, \dots, h.$$

Similarly,

$$b_t = \frac{1}{\kappa - M_{h+1-t}} \left(1 - (\kappa - N_h) a_1 - \sum_{i=h+2-t}^{h} m_i a_i \right) , \qquad \text{for } t = 1, \dots, h.$$

Setting $a_{h+1} = b_{h+1} = 0$ and $N_0 = 0$, from (3.4) and (3.6), together with (3.3), we get successively

$$(\kappa - N_{h-s}) a_{s+1} - (\kappa - N_{h+1-s}) a_s = -n_{h+1-s} b_{h+1-s}, \qquad \text{for } s = 1, \dots, h-1,$$

and

$$(\kappa - n_1) a_h = n_1 b_1, \qquad \text{for } s = h.$$

Since all components of $\mathbf{x}$ are positive and $\kappa \geqslant \Delta + 1$ (1.1), it comes

$$a_{s+1} \leqslant a_s, \qquad \text{for } s = 1, \dots, h-1, \tag{3.7}$$

and

$$b_{t+1} \leqslant b_t, \qquad \text{for } t = 1, \dots, h-1. \tag{3.8}$$

Furthermore, by setting $s = h$ in (3.2), we obtain

$$(\kappa - m_1) b_h = m_1 a_1. \tag{3.9}$$

Moreover, substituting $s = 1$ in (3.1) and $t = 1$ in (3.2) and applying in (3.3) we get

$$(\kappa - N_h) a_1 + (\kappa - M_h) b_1 = 1,$$

and finally

$$a_s = \frac{1}{\kappa - N_{h+1-s}} \left((\kappa - N_h) a_1 - \sum_{j=h+2-s}^{h} n_j b_j \right) . \tag{3.10}$$

Next we focus our attention on bounding a_i's and b_j's.

Lemma 3.1. *For any $s = 1, \dots, h$, we have*

$$\frac{N_{h+1-s} b_{h+1-s}}{\kappa - N_{h+1-s}} \leqslant a_s \leqslant \frac{N_{h+1-s} b_1}{\kappa - N_{h+1-s}} . \tag{3.11}$$

Proof. From (3.4), we have

$$a_s = \frac{1}{\kappa - N_{h+1-s}} \sum_{j=1}^{h+1-s} n_j b_j .$$

Therefore, (3.11) immediately follows since b_j's are strictly decreasing, from (3.8). □

Lemma 3.2. *For any* $s = 1, \dots, h$,

$$a_s \leqslant a_1 \left(1 - \frac{N_{h+2-s,h}}{\kappa - N_{h+1-s}} \left(1 + \frac{m_1}{\kappa - m_1}\right)\right), \tag{3.12}$$

Proof. The inequality (3.12) follows from (3.10), since b_i's are strictly decreasing, bearing in mind (3.9) as well. □

Lemma 3.3. *For any* $s = 1, \dots, h$,

$$a_s \geqslant \frac{a_1}{\kappa - N_{h+1-s}} \left(1 - \sum_{i=1}^{s-1} \frac{n_{h+1-i} M_i}{\kappa - M_i}\right).$$

Proof. By induction on s. For $s = 1$, the inequality holds trivially. Assume next that

$$a_s \geqslant \frac{a_1}{\kappa - N_{h+1-s}} \left(1 - \sum_{i=1}^{s-1} \frac{n_{h+1-i} M_i}{\kappa - M_i}\right),$$

for $s > 1$. Then

$$\begin{aligned}
a_{s+1} &= \frac{1}{\kappa - N_{h-s}} \sum_{j=1}^{h-s} n_j b_j \\
&= \frac{1}{\kappa - N_{h-s}} \left((\kappa - N_{h+1-s}) a_s - N_{h+1-s} b_{h+1-s}\right) \\
&\geqslant \frac{a_1}{\kappa - N_{h-s}} \left(1 - \sum_{i=1}^{s-1} \frac{n_{h+1-i} M_i}{\kappa - M_i}\right) - \frac{N_{h+1-s} M_s a_1}{(\kappa - N_{h-s})(\kappa - M_s)} \\
&= \frac{a_1}{\kappa - N_{h-s}} \left(1 - \sum_{i=1}^{s} \frac{n_{h+1-i} M_i}{\kappa - M_i}\right).
\end{aligned}$$

This ends the proof. □

Lemma 3.4. *For any* $s = 1, \dots, h$, *we have*

$$a_s \leqslant \frac{b_1}{\kappa - N_{h+1-s}} \left(N_{h+1-s} - \frac{\kappa f_{h+1-s}}{(\kappa - n_1)(\kappa - M_s)}\right), \tag{3.13}$$

where

$$f_{h+1-s} = \sum_{j=1}^{h+1-s} n_j M_{h+2-j,h}.$$

Proof. From (3.4) and (3.12) applied to b_j, we get

$$\begin{aligned}
a_s &= \frac{1}{\kappa - N_{h+1-s}} \sum_{j=1}^{h+1-s} n_j b_j \\
&\leqslant \frac{1}{\kappa - N_{h+1-s}} \sum_{j=1}^{h+1-s} n_j b_1 \left(1 - \frac{M_{h+2-j,h}}{\kappa - M_{h+1-j}} \left(1 + \frac{n_1}{\kappa - n_1}\right)\right) \\
&\leqslant \frac{b_1}{\kappa - N_{h+1-s}} \left(N_{h+1-s} - \frac{\kappa f_{h+1-s}}{(\kappa - n_1)(\kappa - M_s)}\right).
\end{aligned}$$

The proof is now complete. □

4 Some bounds on the Q-index of a DNG

In this section we obtain some upper and lower bounds on the Q-index of DNGs using the eigenvalue and the matrix technique. We also emphasize that our main goal is to consider the estimation of the Q-index of large graphs. Before we proceed, we provide the following observations.

First, if $h = 1$ we get a complete bipartite graph K_{m_1,n_1}, whose Q-index is equal to $m_1 + n_1 = \nu$ [2]. Furthermore, since the Q-index of an arbitrary graph increases by inserting edges (cf. [6]), we have

$$\kappa \leqslant \nu, \tag{4.1}$$

for any (not necessarily connected) DNG.

Otherwise, if $h > 1$ is fixed but the graph size is not, using the same previous arguments, the maximal Q-index would appear in $DNG(m_1, 1, \ldots, 1; n_1, 1, \ldots, 1)$. The computational results suggest this will happen when $|m_1 - n_1| \leqslant 1$. So, these cases are not interesting for our research and, therefore, we will assume that $h > 1$ and the size is fixed.

4.1 Eigenvalue technique

Now we establish some bounds for the Q-index of DNGs using the eigenvalue technique. We start with lower bounds.

Proposition 4.1. *If G is a connected DNG, then*

$$\kappa \geqslant \max_{1\leqslant k\leqslant h} \{M_{h+1-k} + N_k\}.$$

Proof. On the one hand, from (3.2), we get

$$b_k = \frac{1}{\kappa - M_{h+1-k}} \sum_{i=1}^{h+1-k} m_i a_i \geqslant \frac{M_{h+1-k} a_{h+1-k}}{\kappa - M_{h+1-k}},$$

since, from (3.7), a_i's are decreasing. On the other hand, from (3.1), we get

$$a_{h+1-k} = \frac{1}{\kappa - N_k} \sum_{j=1}^{k} n_j b_j \geqslant \frac{N_k b_k}{\kappa - N_k},$$

since b_j's are decreasing, from (3.8). From the last two inequalities we get

$$\kappa(\kappa - (M_{h+1-k} + N_k)) \geqslant 0,$$

which is equivalent to

$$\kappa \geqslant M_{h+1-k} + N_k.$$

□

In particular, for $k = h$ and $k = 1$, we obtain the following corollary.

Corollary 4.2. *If G is a connected DNG, then*

$$\kappa \geqslant m_1 + N_h \quad \textit{and} \quad \kappa \geqslant n_1 + M_h.$$

Proposition 4.3. *If G is a connected DNG, then*

$$\kappa \geqslant \frac{1}{2}\left(t+\frac{\epsilon}{N_h}+\sqrt{\left(t-\frac{\epsilon}{N_h}\right)^2+4\hat{e}_h^*}\right),$$

where

$$t=\frac{\sum_{i=1}^h m_i N_{h+1-i}^3}{\sum_{i=1}^h m_i N_{h+1-i}^2} \quad \textit{and} \quad \hat{e}_h^*=\sum_{i=1}^h m_i \frac{N_{h+1-i}^2}{N_h}.$$

Proof. Let $\mathbf{y}=(y_1,\ldots,y_\nu)^T$ be a vector whose components are indexed by the vertices of G, and let $y_u=N_{h+1-i}$ if $u\in U_i$, for some $i\in\{1,\ldots,h\}$, or, otherwise, $y_v=q=\kappa-t$, for some t, if $v\in V_j$ for some $j\in\{1,\ldots,h\}$. Substituting y into the Rayleigh quotient (see, e.g., [8, p. 49]) we obtain

$$\kappa \geqslant \frac{2\sum_{i=1}^h m_i N_{h-1+i}^2 q+\sum_{i=1}^h m_i N_{h+1-i}^3+\sum_{i=1}^h n_i M_{h-1+i} q^2}{\sum_{i=1}^h m_i N_{h+1-i}^2+N_h q^2}$$

due to Rayleigh's principle which reads $\dfrac{y^TQy}{y^Ty}\leqslant\kappa$. Since $q=\kappa-t$, we get

$$N_h q^3+(N_h t-\epsilon)q^2-\sum_{i=1}^h m_i N_{h+1-i}^2 q \geqslant \sum_{i=1}^h m_i N_{h+1-i}^3 - t\sum_{i=1}^h m_i N_{h+1-i}^2.$$

Choosing

$$t=\frac{\sum_{i=1}^h m_i N_{h+1-i}^3}{\sum_{i=1}^h m_i N_{h+1-i}^2},$$

and having in mind that $N_1\leqslant t\leqslant N_h$, we immediately get a quadratic inequality in q and the proof is concluded. □

Proposition 4.4. *If G is a connected DNG, then*

$$\kappa \leqslant \frac{1}{2}\left(\nu+\sqrt{\nu^2-4(M_hN_h-\epsilon)}\right). \tag{4.2}$$

Proof. From (3.1), with $s=h$, and from (3.3), recalling (3.11), we get

$$(\kappa-M_h)b_1=\sum_{i=1}^h m_i a_i \leqslant \sum_{i=1}^h m_i \frac{N_{h+1-i}}{\kappa-N_{h+1-s}} b_1.$$

Then, we obtain

$$(\kappa-M_h)(\kappa-N_h)\leqslant\epsilon,$$

and, therefore, from the quadratic inequality

$$\kappa^2-(M_h+N_h)\kappa+M_hN_h-\epsilon\leqslant 0,$$

we obtain $\kappa_1\leqslant\kappa\leqslant\kappa_2$ where κ_1 and κ_2 are the solutions of the associated quadratic equality, and this completes the proof. □

The next two bounds improve the bound (4.2). We recall that f_{h+1-i} is defined in Lemma 3.4.

Proposition 4.5. *If G is a connected DNG, then*

$$\kappa \leqslant \frac{1}{2}\left(\nu + \sqrt{\nu^2 - 4(M_h N_h - \epsilon')}\right) ,$$

where

$$\epsilon' = \epsilon - \frac{\nu(\nu - N_h)}{(\nu - n_1)^2(\nu - m_1)} \sum_{i=1}^{h} m_i f_{h+1-i} .$$

Proof. As in the proof of Proposition 4.4, we have

$$(\kappa - M_h)b_1 = \sum_{i=1}^{h} m_i a_i .$$

Using (3.12), we get

$$\kappa - M_h \leqslant \sum_{i=1}^{h} \frac{m_i}{\kappa - N_{h+1-i}} \left(N_{h+1-i} - \frac{\kappa f_{h+1-i}}{(\kappa - n_1)(\kappa - M_i)}\right),$$

and therefore

$$(\kappa - M_h)(\kappa - N_h) \leqslant \epsilon - \frac{\kappa(\kappa - N_h)}{(\kappa - n_1)^2(\kappa - m_1)} \sum_{i=1}^{h} m_i f_{h+1-i} .$$

Taking into account that $\kappa \leqslant \nu$, from Proposition 4.4 it follows

$$(\kappa - M_h)(\kappa - N_h) \leqslant \epsilon',$$

and the proof ends. □

The next result may be proved in a similar way.

Proposition 4.6. *If G is a connected DNG, then*

$$\kappa \leqslant \frac{1}{2}\left(\nu + \sqrt{\nu^2 - 4(M_h N_h - \epsilon'')}\right) ,$$

where

$$\epsilon'' = \epsilon - \frac{\kappa'(\kappa' - N_h)}{(\kappa' - n_1)^2(\kappa' - m_1)} \sum_{i=1}^{h} m_i f_{h+1-i},$$

for

$$\kappa' = \frac{1}{2}\left(\nu + \sqrt{\nu^2 - 4(M_h N_h - \epsilon')}\right) .$$

4.2 Matrix technique

The partition

$$V = \bigcup_{k=1}^{h} U_k \cup \bigcup_{k=1}^{h} V_k \tag{4.3}$$

is *equitable* since every vertex in U_i and every vertex in V_i have the same number of neighbors in U_j and V_j, for all $i, j \in \{1, 2, \ldots, h\}$. Let A_D be the *signless Laplacian divisor* matrix of a $DNG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ with respect to the equitable partition (4.3). The matrix A_D has the following form:

$$A_D = \left(\begin{array}{ccccc|ccccc}
N_h & & & & & n_1 & n_2 & \cdots & n_{h-1} & n_h \\
& N_{h-1} & & & & n_1 & n_2 & \cdots & n_{h-1} & \\
& & \ddots & & & \vdots & \vdots & \iddots & & \\
& & & N_2 & & n_1 & n_2 & & & \\
& & & & N_1 & n_1 & & & & \\
\hline
m_1 & m_2 & \cdots & m_{h-1} & m_h & M_h & & & & \\
m_1 & m_2 & \cdots & m_{h-1} & & & M_{h-1} & & & \\
\vdots & \vdots & \iddots & & & & & \ddots & & \\
m_1 & m_2 & & & & & & & M_2 & \\
m_1 & & & & & & & & & M_1
\end{array}\right),$$

where the non-mentioned entries are to be read as zero. Setting

$$N = \begin{pmatrix}
n_1 & n_2 & \cdots & n_{h-1} & n_h \\
n_1 & n_2 & \cdots & n_{h-1} & \\
\vdots & \vdots & \iddots & & \\
n_1 & n_2 & & & \\
n_1 & & & &
\end{pmatrix}, \qquad M = \begin{pmatrix}
m_1 & m_2 & \cdots & m_{h-1} & m_h \\
m_1 & m_2 & \cdots & m_{h-1} & \\
\vdots & \vdots & \iddots & & \\
m_1 & m_2 & & & \\
m_1 & & & &
\end{pmatrix},$$

$D_1 = \operatorname{diag}(N_h, \ldots, N_1)$, and $D_2 = \operatorname{diag}(M_h, \ldots, M_1)$, A_D can be rewritten in the compact block form

$$A_D = \begin{pmatrix} D_1 & N \\ M & D_2 \end{pmatrix}.$$

In order to obtain more bounds we set

$$P = \begin{pmatrix} 0 & xI \\ I & 0 \end{pmatrix},$$

for some $x \neq 0$. Since the matrices

$$PA_DP^{-1} = \begin{pmatrix} D_2 & xM \\ x^{-1}N & D_1 \end{pmatrix}$$

and A_D are similar, they have the same index. We choose x such that the sum in the first row and the $(h+1)$-th row are equal. It leads to $M_h x^2 - (N_h - M_h)x - N_h = 0$, i.e., $x = \frac{N_h}{M_h}$. By Frobenius Theorem [11, Theorem 3.1.1], we have

$$\min_{1 \leqslant i \leqslant n} R_i \leqslant \kappa \leqslant \max_{1 \leqslant i \leqslant n} R_i,$$

where R_i stands for the sum of elements in the i-th row of PA_DP^{-1}. Using this, we get the following result.

Proposition 4.7. *If G is a connected DNG, then*

$$\min\left\{n_1\left(\frac{N_h}{M_h}+1\right), n_1\left(\frac{M_h}{N_h}+1\right)\right\} \leqslant \kappa \leqslant N_h + M_h = \nu. \tag{4.4}$$

Clearly, the upper bound does not provide a decisive progress in our quest (recall (4.1)). We will establish some more interesting improvements next.

Let R_i be the sum of the entries in row i of the matrix A_D. It is easy to confirm that

$$\begin{aligned} R_i &= 2N_{h-i+1}, \quad \text{for } i \in \{1,\dots,h\} \\ &= 2M_{h-i+1}, \quad \text{for } i \in \{h+1,\dots,2h\}, \end{aligned}$$

and, therefore,

$$\max R_i = \max\{2N_h, 2M_h\}.$$

By Frobenius Theorem

$$\min\{2n_1, 2m_1\} \leqslant \kappa \leqslant \max\{2N_h, 2M_h\}. \tag{4.5}$$

Here the upper bound does not make any (general) improvement since $\max\{2N_h, 2M_h\} \geqslant \nu$ (compare (4.1)), so next we use the result of Minc (see [10]), which for the matrix A_D reads:

$$\min_i \frac{\sum_{j=1}^{h}(A_D)_{ij}R_j}{R_i} \leqslant \kappa \leqslant \max_i \frac{\sum_{j=1}^{h}(A_D)_{ij}R_j}{R_i}.$$

Proposition 4.8. *If G is a connected DNG, then*

$$\min\{n_1 + M_h, m_1 + N_h\} \leqslant \kappa \leqslant \max\left\{\frac{\epsilon}{N_h} + N_h, \frac{\epsilon}{M_h} + M_h\right\}. \tag{4.6}$$

The bounds (4.6) obviously improve both (4.4) and (4.5), but the lower bound is still rough comparing with Corollary 4.2.

5 Computational results

In this final section, we provide several examples which can help to gain a better insight into the quality of the bounds obtained in the previous section.

We compute the lower bounds of Propositions 4.1 and 4.3, and the upper bounds from Propositions 4.4, 4.5, 4.6, and 4.8. One observes that the bound from Proposition 4.1 is always integral. The number of vertices in the corresponding DNG is also given in every example since it makes another upper bound (cf. (4.1)). It can be easy checked that the lower bounds from Corollary 4.2 and Propositions 4.7 and 4.8, all of them having simple expressions, are rough in some cases, and therefore they are not considered in our examples. We also compute the relative errors in each case.

Example 5.1. First we consider a randomly chosen DNG with small number of vertices and some larger DNGs derived from the previous one:

$G_1 = DNG(2, 2, 5, 3; 2, 3, 1, 1)$;
$G_2 = DNG(10, 10, 25, 15; 10, 15, 5, 5)$;
$G_3 = DNG(200, 200, 500, 300; 200, 300, 100, 100)$;

	Prop. 4.3	Prop. 4.1	κ	Prop. 4.8	Prop 4.6	Prop 4.5	Prop 4.4	ν
G_1	13.6785 -12.57%	14 -10.52%	15.6451	16.7500 7.06%	17.0210 8.79%	17.0550 9.01%	17.4530 11.56%	19 21.44%
G_2	68.3923 -12.57%	70 -10.52%	78.2257	83.7500 7.06%	85.1052 8.79%	85.2749 9.01%	87.2649 11.56%	95 21.44%
G_3	1367.8452 -12.57%	1400 -10.52%	1564.5133	1675.0000 7.06%	1702.1030 8.79%	1705.4985 9.01%	1745.2987 11.56%	1900 21.44%

Notice that $\kappa(G_2)$ (resp. $\kappa(G_3)$) is very close to $5\kappa(G_1)$ (resp. $100\kappa(G_3)$); we get $5\kappa(G_1) - \kappa(G_2) \approx 10^{-7}$. Since the similar fact holds for all bounds obtained (compare the corresponding propositions), we get the same results for the relative errors.

Example 5.2. Here we consider the DNGs obtained from G_1 by multiplying some of its parameters:

$H_1 = DNG(2000, 2, 5, 3; 2, 3, 1, 1000)$;
$H_2 = DNG(2000, 2, 5, 3; 2, 3, 1000, 1)$;
$H_3 = DNG(2000, 2, 5, 3; 2, 3000, 1, 1)$;
$H_4 = DNG(2000, 2, 5, 3; 2000, 3, 1, 1)$;

	Prop. 4.3	Prop. 4.1	κ	Prop. 4.8	Prop 4.6	Prop 4.5	Prop 4.4	ν
H_1	3006.0284 $-8 \cdot 10^{-6}$%	3006 $-1 \cdot 10^{-3}$%	3006.0287	3011.0164 0.17%	3008.2682 0.07%	3008.2960 0.08%	3012.6750 0.22%	3016 0.33%
H_2	3008.0175 $-5 \cdot 10^{-6}$%	3007 -0.03%	3008.0177	3012.0104 0.13%	3009.8145 0.06%	3009.8323 0.06%	3013.3388 0.18%	3016 0.27%
H_3	5010.9908 $-5 \cdot 10^{-7}$%	5009 -0.04%	5010.9909	5010.9980 $1 \cdot 10^{-4}$%	5011.7199 0.15%	5011.7199 0.15%	5012.2008 0.02%	5014 0.06%
H_4	4014.9731 $-3 \cdot 10^{-6}$%	4010 -0.12%	4014.9732	4014.9866 $3 \cdot 10^{-4}$%	4014.9800 $2 \cdot 10^{-4}$%	4014.9800 $2 \cdot 10^{-4}$%	4014.9933 $5 \cdot 10^{-4}$%	4015 $7 \cdot 10^{-4}$%

In this example all bounds are (more or less) close to the exact value of Q-index. We already pointed that the bounds obtained in Propositions 4.5 and 4.6 are the improvements of the one obtained in Proposition 4.4. This example shows that the bound from Proposition 4.8 is incomparable to them. In opposition to the previous example, here Proposition 4.1 gives a better estimation than Proposition 4.3.

Example 5.3. The parameters of the following DNGs are obtained by multiplying the parameters of G_1 by $1, 10, 100$ or 1000 ad hoc.

$I_1 = DNG(2, 2, 5, 3; 2000, 300, 10, 1)$;
$I_2 = DNG(2, 2, 5, 3; 2, 30, 100, 1000)$;
$I_3 = DNG(2000, 200, 50, 30; 2000, 300, 10, 1)$;

	Prop. 4.3	Prop. 4.1	κ	Prop. 4.8	Prop 4.6	Prop 4.5	Prop 4.4	ν
I_1	2255.0867 -2.65%	2314 -0.10%	2316.3632	2322.5716 0.26%	2322.5716 0.27%	2322.5733 0.27%	2322.5737 0.27%	2323 0.29%
I_2	1118.5026 -1.37%	1134 $-6 \cdot 10^{-5}$%	1134.0007	1134.3799 0.03%	1134.4002 0.04%	1134.4000 0.04%	1134.4002 0.04%	1144 0.88%
I_3	4562.6064 -0.28%	4550 $-1 \cdot 10^{-3}$%	4562.6584	4563.2717 0.01%	4563.1367 0.01%	4563.1369 0.01%	4563.6312 0.02%	4591 0.62%

Taking into account the lower bound of Proposition 4.3, one can conclude that its deviation from the exact value is expected for I_1 (and other similar graphs). Note that Proposition 4.6 will often give better bound than Proposition 4.5, but not always – see graphs I_2 or J_2 in the next example.

Example 5.4. Finally, we consider the extensions of the original graphs:

$$J_1 = DNG(2,2,5,3,2,3,1,1;2,3,1,1,2,2,5,3);$$
$$J_2 = DNG(20000,2,5,3,10,10,10,10;2,3,1,10000,10,10,10,10);$$
$$J_3 = DNG(2,2,5,3,1,1,1,1;2000,300,10,1,1,1,1,1);$$

	Prop. 4.3	Prop. 4.1	κ	Prop. 4.8	Prop 4.6	Prop 4.5	Prop 4.4	ν
J_1	23.1888 -15.56%	23 -16.24%	27.4601	29.0526 5.80%	32.3032 17.64%	32.3022 17.64%	32.8203 19.52%	38 38.38%
J_2	30065.9176 $-6 \cdot 10^{-7}$%	30046 -0.07%	30065.9178	30080.9446 0.05%	30072.6747 0.02%	30072.7003 0.02%	30085.9668 0.07%	30096 0.10%
J_3	2313.0140 -0.62%	2324 -0.15%	2327.5409	2330.8445 0.14%	2330.8452 0.14%	2330.8452 0.14%	2330.8456 0.14%	2331 0.15%

The bounds obtained will be used in a forthcoming research regarding the graphs with maximal Q-index and fixed (but high) orders and also a fixed particular size. We also remark that the results could be also compared to the corresponding bounds obtained for the adjacency spectra.

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Henry H. Glover (April 10, 1935 – May 31, 2011): Some remarks on his research in mathematics

Henry Glover obtained his Ph.D. in Mathematics in 1967 from the University of Michigan. The topic of his dissertation was *On Embedding and Immersing Manifolds in Euclidean Space*. After a short period as an Assistant Professor at the University of Minnesota, he joined the faculty of Mathematics at the Ohio-State University, where he worked for the rest of his professional career. Thanks to his outgoing personality, he had from the beginning a diverse group of collaborators, which led him to tackle problems in *Fixed Point Theory* and questions on *Graphs*. When during the early 1970's localization techniques in topology were developed, he made immediate use of the new tool to obtain interesting results on vector fields on manifolds. At the same time, he advanced his work on graph theory, culminating in a classification of the irreducible graphs for the projective plane. After about 1980, he began a productive collaboration on questions in geometric group theory, dealing with group cohomology involving the mapping class group as well as the automorphism group of a free group. His deep understanding of the geometry of *Teichmüller Space* and *Outer Space* where crucial at this point. Toward the end of his life he got again more involved in graph theory, with a new group of collaborators, investigating Hamiltonian cycles in Cayley graphs.

Henry Glover made substantial contributions to a wide field of mathematical topics. The many people who had the opportunity to collaborate with him will not forget his generosity and the pleasure of discussing with him mathematical ideas, politics and the philosophy of life in general. His love of art, symmetries of patterns and architecture were part of his complex personality, and that, along with his tremendous sense of loyalty, made him a great and fascinating friend.

Thank you Henry!

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Columbus, Ohio 43210
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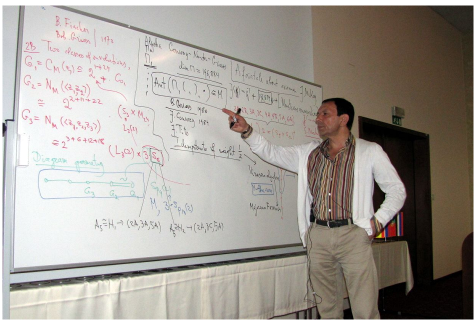

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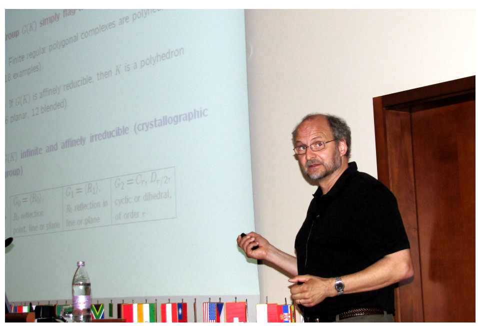

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