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On the inertia of weighted $(k-1)$-cyclic graphs*

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Abstract

Let G_w be a weighted graph. The inertia of G_w is the triple $\text{In}(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w), i_-(G_w), i_0(G_w)$ are, respectively, the number of the positive, negative and zero eigenvalues of the adjacency matrix $A(G_w)$ of G_w including their multiplicities. A simple n-vertex connected graph is called a $(k-1)$-cyclic graph if its number of edges equals $n+k-2$. Let $\theta(r_1, r_2, \dots, r_k)_w$ be an n-vertex simple weighted graph obtained from k weighted paths $(P_{r_1})_w, (P_{r_2})_w, \dots, (P_{r_k})_w$ by identifying their initial vertices and terminal vertices, respectively. Set $\Theta_k := \{\theta(r_1, r_2, \dots, r_k)_w : r_1 + r_2 + \cdots + r_k = n + 2k - 2\}$. The inertia of the weighted graph $\theta(r_1, r_2, \dots, r_k)_w$ is studied. Also, the weighted $(k-1)$-cyclic graphs that contain $\theta(r_1, r_2, \dots, r_k)_w$ as an induced subgraph are studied. We characterize those graphs among Θ_k that have extreme inertia. The results generalize the corresponding results obtained by Tan and Liu in 2013 and Yu et al., 2014.

Keywords: Weighted k-cyclic graph, adjacency matrix, inertia.

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1 The first section

In this paper, we only consider simple weighted graphs on positive weight set. Let G_w be a weighted graph with vertex set $\{v_1, v_2, \dots, v_n\}$, edge set $E(G_w) \neq \emptyset$ and weight set $W(G_w) = \{w(e) > 0, e \in E(G)\}$. The function $w : E(G_w) \to W(G_w)$ is called a weight function of G_w. It is obvious that each weighted graph corresponds to a weight function. The *adjacency matrix* of G_w is defined as the matrix $A(G_w) = (a_{ij})$ such that $a_{ij} = w(v_iv_j)$ if $v_iv_j \in E(G_w)$ and 0 otherwise. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of

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$A(G_w)$ are said to be the eigenvalues of the weighted graph G_w. The *inertia* of G_w is defined to be the triple $\mathrm{In}(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w), i_-(G_w)$ and $i_0(G_w)$ are the numbers of the positive, negative and zero eigenvalues of $A(G_w)$ including multiplicities, respectively. $i_+(G_w)$ and $i_-(G_w)$ are called the *positive, negative index of inertia* (for short, *positive, negative index*) of G_w, respectively. The number $i_0(G_w)$ is called the *nullity* of $A(G_w)$. The nullity and the rank of $A(G_w)$ are also called the nullity and the rank of G_w, and denoted by $\eta(G)$ and $\mathrm{R}(G)$, respectively. Obviously, $\mathrm{R}(G_w) = i_+(G_w) + i_-(G_w)$ and $i_+(G_w) + i_-(G_w) + i_0(G_w) = n$. For convenience, in the whole context, we let G denote the unweighted graph with respect to the weighted graph G_w; G can be also viewed as a trivial weighted graph in which the weight for each edge is 1.

An *induced subgraph* of G_w is an induced subgraph of G having the same weights with those of G_w. For an induced weighted subgraph H_w of G_w, let $G_w - H_w$ be the subgraph obtained from G_w by deleting all vertices of H_w and all incident edges. A m*-cyclic graph* is a simple connected graph in which the number of edges equals the number of vertices plus $m-1$. A weighted path and a weighted cycle of order n are denoted by $(P_n)_w$, $(C_n)_w$, respectively. An isolated vertex is denoted by K_1.

The study of eigenvalues of graph has been received a lot of attention due to its applications in chemistry (see [2, 7, 10, 15] for details). Gregory et al. [8] studied the subadditivity of the positive, negative indices of inertia and developed certain properties of Hermitian rank which were used to characterize the biclique decomposition number. Gregory et al. [9] investigated the inertia of a partial join of two graphs and established a few relations between the inertia and biclique decompositions of partial joins of graphs. Daugherty [3] characterized the inertia of unicyclic graphs in terms of matching number and obtained a linear-time algorithm for computing it. Yu et al. [19] investigated the minimal positive index of inertia among all unweighted bicyclic graphs of order n with pendants, and characterized the bicyclic graphs with positive index 1 or 2. Very recently, it is interesting to see that Marina et al. [1] studied the inertia set of a signed graph in algebraic approach.

The nullity of unweighted graphs has been studied extensively in the literature. Tan and Liu [18] gave the nullity set of unicyclic graphs and characterized the unicyclic graphs with maximum nullity. In addition, Nath and Sarma [17] presented another version of characterization of an acyclic or unicyclic graph to be singular. One of the present authors [13] studied the nullity of graphs with pendant vertices. Fan and Qian [6] characterized the bipartite graphs with the second largest nullity and the regular bipartite graphs with the third largest nullity. Fan and Wang [5] characterized the unicyclic signed graphs of order n with nullity $n-2, n-3, n-4, n-5$, respectively.

Our paper is motivated directly by [4, 11, 13, 19, 20, 21]. On the one hand, Fan et al. [4] studied the nullity of signed bicyclic graph (which is, in fact, the bicyclic graph with edge weight 1 or -1); Li [13] and Hu [11] studied the nullity of unweighted bicyclic graph. On the other hand, Yu et al. [20] characterized all n-vertex weighted uicyclic graphs with positive index 1 or 2; Tan and Liu [21] studied the nullity of unweighted $(k-1)$-cyclic graphs. It is natural and interesting for us to consider the extremal problems on the inertia of weighted $(k-1)$-cyclic graphs, which may generalize the corresponding results obtained in [20, 21].

This paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we define two classes of weighted $(k-1)$-cyclic graph, denoted by Θ_k and $\Gamma_{n,k-1}$. Moreover, we give a method to determine the inertia of a weighted graph in Θ_k.

In Section 4, we characterize all weighted $(k-1)$-cyclic graphs in $\Gamma_{n,k-1}$ having just one or two positive (resp. negative) eigenvalues. In Section 5, we characterize all weighted $(k-1)$-cyclic graphs in $\Gamma_{n,k-1}$ of rank $2,3,4$, respectively.

2 Preliminaries

In this section, we list some lemmas which will be used to prove our main results. Suppose M, N are two Hermitian matrices of order n, if there exists an invertible matrix Q of order n such that $QMQ^T = N$, where Q^T denotes the conjugate transpose of Q, then we say that M is *congruent to* N, denoted by $M \cong N$.

Lemma 2.1 ([12]). *Let M, N be two Hermitian matrices of order n satisfying $M \cong N$. Then $i_+(M) = i_+(N)$, $i_-(M) = i_-(N)$ and $i_0(M) = i_0(N)$.*

Let M be a Hermitian matrix. We denote three types of elementary congruence matrix operations (ECMOs) on M as follows:

(1) interchanging i-th and j-th rows of M, while interchanging i-th and j-th columns of M;

(2) multiplying i-th row of M by a non-zero number k, while multiplying i-th column of M by k;

(3) adding i-th row of M multiplied by a non-zero number k to j-th row, while adding i-th column of M multiplied by k to j-th column.

By Lemma 2.1, the ECMOs do not change the inertia of a Hermitian matrix.

Lemma 2.2 ([14]). *Let H_w be an induced subgraph of G_w. Then $i_+(H_w) \leqslant i_+(G_w)$ and $i_-(H_w) \leqslant i_-(G_w)$.*

Lemma 2.3 ([14]). *Let G_w be a weighted graph containing a pendant vertex v with its unique neighbor u. Then $i_+(G_w) = i_+(G_w - u - v) + 1$ and $i_-(G_w) = i_-(G_w - u - v) + 1$.*

The following result is a direct consequence of Lemma 2.3.

Lemma 2.4. *Let $(P_n)_w$ be a weighted path of order n. Then $\mathrm{In}((P_n)_w) = (\frac{n}{2}, \frac{n}{2}, 0)$ if n is even and $(\frac{n-1}{2}, \frac{n-1}{2}, 1)$ otherwise.*

By Lemma 2.4, we can show that the adjacency matrix of $(P_{2k})_w$ is invertible. In fact, let $\{v_1, v_2, \ldots, v_{2k}\}$ be the vertex set of the weighted path $(P_{2k})_w$ such that $v_iv_{i+1} \in E((P_{2k})_w)\,(i = 1, \ldots, 2k-1)$ and $w_{ii} = w(v_{2i-1}v_{2i})\,(i = 1, \ldots, k)$, $w_{i,i+1} = w(v_{2i}v_{2i+1})\,(i = 1, \ldots, k-1)$. Then the adjacency matrix of $(P_{2k})_w$ has the following block form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & \mathbf{0} & \mathbf{0} \\ A_{21} & A_{22} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & A_{k-1,k-1} & A_{k-1,k} \\ \mathbf{0} & \mathbf{0} & \ldots & A_{k,k-1} & A_{k,k} \end{pmatrix}$$

where $A_{ii} = \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix}, (i = 1, \ldots, k)$ and

$$A_{i+1,i}^T = A_{i,i+1} = \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix}, (i = 1, \ldots, k-1).$$

Let $B = (B_{ij})_{i,j=1}^{k}$, where

$$B_{ij} = \begin{cases} \begin{pmatrix} 0 & \dfrac{1}{w_{ii}} \\ \dfrac{1}{w_{ii}} & 0 \end{pmatrix} & \text{if } i = j; \\ \begin{pmatrix} 0 & \dfrac{w_{i,i+1} \cdots w_{j-1,j}}{w_{i,i} \cdots w_{j,j}} \\ 0 & 0 \end{pmatrix} & \text{if } i < j \text{ and } j - i \equiv 0 \pmod 2; \\ \begin{pmatrix} 0 & -\dfrac{w_{i,i+1} \cdots w_{j-1,j}}{w_{i,i} \cdots w_{j,j}} \\ 0 & 0 \end{pmatrix} & \text{if } i < j \text{ and } j - i \equiv 1 \pmod 2; \\ B_{ji}^T, & \text{if } i > j. \end{cases}$$

Lemma 2.5. *Let A and B be the matrices defined as above. Then $AB = I$.*

Proof. Let $C = (C_{ij})_{i,j=1}^{k} = AB$. It suffices to show that $C_{ii} = I_2$ for $i = 1, \dots, k$, where I_2 is the identity matrix of order 2, and $C_{ij} = \mathbf{0}$ if $i \neq j$. Note that the first (resp. last) row of A contains just two non-zero blocks, whereas each of the rest rows of A contains just three non-zero blocks, the proofs are a little different between them. First we consider the cases that $i \neq 1, k$.

If $1 < i = j < k$, then

$$\begin{aligned} C_{ii} &= \sum_{s=1}^{k} A_{is}B_{si} = A_{i,i-1}B_{i-1,i} + A_{ii}B_{ii} + A_{i,i+1}B_{i+1,i} \\ &= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dfrac{w_{i-1}}{w_{i-1,i-1}w_{i,i}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{w_{ii}} \\ \frac{1}{w_{ii}} & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\dfrac{w_{i,i+1}}{w_{ii}w_{i+1,i+1}} & 0 \end{pmatrix} \\ &= I_2. \end{aligned}$$

If $1 < i < j < k$, we distinguish the following three possible cases to prove our result.

Case 1: $j - i \equiv 0 \pmod 2$. In this case, we have

$$\begin{aligned} C_{ij} &= \sum_{s=1}^{k} A_{is}B_{sj} = A_{i,i-1}B_{i-1,j} + A_{i,i}B_{i,j} + A_{i,i+1}B_{i+1,j} \\ &= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{w_{i-1,i} \cdots w_{j-1,j}}{w_{i-1,i-1} \cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{w_{i,i+1} \cdots w_{j-1,j}}{w_{i,i} \cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{w_{i+1,i+2} \cdots w_{j-1,j}}{w_{i+1,i+1} \cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

Case 2: $j-i=1$. In this case, we have

$$\begin{aligned}
C_{ij} &= \sum_{s=1}^{k} A_{is}B_{sj} = A_{i,i-1}B_{i-1,j} + A_{i,i}B_{i,j} + A_{i,i+1}B_{i+1,j} \\
&= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{w_{i-1,i}w_{i,j}}{w_{i-1,i-1}w_{ii}w_{jj}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dfrac{w_{ij}}{w_{i,i}w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{w_{jj}} \\ \dfrac{1}{w_{jj}} & 0 \end{pmatrix} \\
&= \mathbf{0}.
\end{aligned}$$

Case 3: $j-i \equiv 1 \pmod 2$ and $j-i>1$. In this case, we have

$$\begin{aligned}
C_{ij} &= \sum_{s=1}^{k} A_{is}B_{sj} = A_{i,i-1}B_{i-1,j} + A_{i,i}B_{i,j} + A_{i,i+1}B_{i+1,j} \\
&= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{w_{i-1,i}\cdots w_{j-1,j}}{w_{i-1,i-1}\cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dfrac{w_{i,i+1}\cdots w_{j-1,j}}{w_{i,i}\cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{w_{i+1,i+2}\cdots w_{j-1,j}}{w_{i+1,i+1}\cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&= \mathbf{0}.
\end{aligned}$$

For $i=1$ or $i=k$, all the proofs above are still correct if we set the corresponding blocks to be **0** whenever one of its subscripts equals 0 or $k+1$, such as $A_{10}=A_{k,k+1}=\mathbf{0}$.

If $1 \leqslant j < i \leqslant k$, the proof is similar to the case $1 \leqslant i < j \leqslant k$. We omit the procedure here. □

3 The inertia of weighted graphs in Θ_k

For $m \geqslant 1$, a *m-cyclic graph* is a simple connected graph in which the number of edges equals the number of vertices plus $m-1$. Let P_{r_i} be a path of order $r_i\,(r_i \geqslant 2)$ and $\{P_{r_i} | 1 \leqslant i \leqslant k\}$ be the set of $k\,(k \geqslant 2)$ vertex-disjoint paths, where there exists at most one path of order 2. Identify the k initial vertices as u_0 and terminal vertices as v_0, respectively. The resultant graph, denoted by $\theta(r_1, r_2, \ldots, r_k)$, is called a *$\Theta$-graph*. Denote by Θ_k the set of all n-vertex weighted Θ-graphs having form $\theta(r_1, r_2, \ldots, r_k)_w$. Note that any weighted Θ-graph is also a weighted $(k-1)$-cyclic graph. Denote the set of all weighted $(k-1)$-cyclic graphs of order n, which contain a weighted Θ-graph as an induced subgraph, by $\Gamma_{n,k-1}$. In this section, we'll give a method to determine the inertia of weighted graphs in Θ_k.

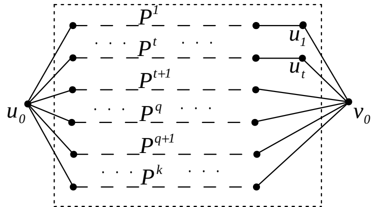

Figure 1: The structure of $\theta(r_1, r_2, \dots, r_k)$

Let $G_w := \theta(r_1, r_2, \dots, r_k)_w$ be a graph of order n. Let n_i be the number of r_j's which satisfy $r_j - 2 \equiv i \pmod 4$, $1 \leqslant j \leqslant k$, $0 \leqslant i \leqslant 3$ and set $t := n_1 + n_3$ and $q := t + n_2$. It is easy to see that $G_w \in \Theta_k$, we arrange the structure of G_w as follows: First come the paths $P_{r_1}, \dots, P_{r_{n_1}}$ with $r_1 \leqslant r_2 \leqslant \dots \leqslant r_{n_1}$ and $r_i \equiv 3 \pmod 4$, $i = 1, 2, \dots, n_1$; next $P_{r_{n_1+1}}, \dots, P_{r_t}$ with $r_{n_1+1} \leqslant r_{n_1+2} \leqslant \dots \leqslant r_t$ and $r_i \equiv 1 \pmod 4$, $i = n_1+1, n_1+2, \dots, t$; then $P_{r_{t+1}}, \dots, P_{r_q}$ with $r_{t+1} \leqslant r_{t+2} \leqslant \dots \leqslant r_q$ and $r_i \equiv 2 \pmod 4$, $i = t+1, t+2, \dots, q$; finally $P_{r_{q+1}}, \dots, P_{r_k}$ with $r_{q+1} \leqslant r_{q+2} \leqslant \dots \leqslant r_k$ and $r_i \equiv 0 \pmod 4$, $i = q+1, q+2, \dots, k$. Let u_i be the neighbor of v_0 in the odd path P_{r_i}, $i = 1, 2, \dots, t$. Let $P^i = u_1^i u_2^i \dots u_{2s_i}^i$ $(1 \leqslant i \leqslant k)$ be the path in P_{r_i} $(1 \leqslant i \leqslant k)$ obtained by deleting u_0, v_0 and u_i if r_i is odd; see Fig. 1. Further on we will label the weight for each edge of G_w according to the following possible cases.

Case 1: $\min\{r_1, r_2, \dots, r_k\} = 4$**.** In this case, partition the vertex set of G_w as follows: $\{u_0\}, V(P^1), \dots, V(P^k), \{u_1, \dots, u_t\}, \{v_0\}$. Let $a_i = w(u_0u_1^i)\,(i = 1, \dots, k)$, $b_i = w(u_iu_{2s_i}^i)\,(i = 1, \dots, t)$, $b_j = w(v_0u_{2s_j}^j)\,(j = t+1, \dots, k)$, $d_i = w(v_0u_i)$ $(i = 1, \dots, t)$, $w_{jj}^i = w(u_{2j-1}^i u_{2j}^i)\,(i = 1, \dots, k; j = 1, \dots, \frac{1}{2}|V(P^i)|)$ and $w_{j,j+1}^i = w(u_{2j}^i u_{2j+1}^i)\,(i = 1, \dots, k; j = 1, \dots, \frac{1}{2}|V(P^i)| - 1)$. Then the adjacency matrix of G_w has the following form:

$$A(G_w) = \left(\begin{array}{c|ccc|ccc|ccc|c}
0 & \alpha_1^T & \dots & \alpha_t^T & \alpha_{t+1}^T & \dots & \alpha_k^T & & \mathbf{0} & & 0 \\ \hline
\alpha_1^T & A_1 & & & & & & \beta_1 & & & \\
\vdots & & \ddots & & & \mathbf{0} & & & \ddots & & \mathbf{0} \\
\alpha_t^T & & & A_t & & & & & & \beta_t & \\ \hline
\alpha_{t+1}^T & & & & A_{t+1} & & & & & & \beta_{t+1}^T \\
\vdots & & \mathbf{0} & & & \ddots & & & \mathbf{0} & & \vdots \\
\alpha_k^T & & & & & & A_k & & & & \beta_k^T \\ \hline
 & \beta_1^T & & & & & & & & & d_1 \\
\mathbf{0} & & \ddots & & & \mathbf{0} & & & \mathbf{0} & & \vdots \\
 & & & \beta_t^T & & & & & & & d_t \\ \hline
0 & & \mathbf{0} & & \beta_{t+1}^T & \dots & \beta_k^T & d_1 & \dots & d_t & 0
\end{array}\right),$$

where $\alpha_i^T = (a_i, 0, \dots, 0)$ and $\beta_i^T = (0, \dots, 0, b_i)$.

We apply the ECMOs on $A(G_w)$: using $-\alpha_i^T A_i^{-1}$ to multiply the $(i+1)$-th row, then adding it to the first row, we can cancel $\alpha_i^T\,(i = 1, \dots, k)$ in the first row. Similarly, using $-\beta_i^T A_i^{-1}$ to multiply the $(i+1)$-th row, then adding it to $(k+i+1)$-th row if $i \leqslant t$, and adding it to the last row if $t+1 \leqslant i \leqslant k$, we can cancel $\beta_i^T (i = 1, \ldots, k)$. After that, column operations are applied so that each α_i and β_i are reduced to **0**s. By Lemma 2.5, $-\alpha_i^T A_i^{-1} \alpha_i = -\beta_i^T A_i^{-1} \beta_i = 0$ and $c_i = -\alpha_i^T A_i^{-1} \beta_i = -\beta_i^T A_i^{-1} \alpha_i$, where

$$c_i = \begin{cases} -\dfrac{a_i b_i w_{12}^i w_{23}^i \ldots w_{s_i-1,s_i}^i}{w_{11}^i w_{22}^i \ldots w_{s_i,s_i}^i}, & \text{if } |A_i| = 2s_i \equiv 2 \pmod 4; \\ \dfrac{a_i b_i w_{12}^i w_{23}^i \ldots w_{s_i-1,s_i}^i}{w_{11}^i w_{22}^i \ldots w_{s_i,s_i}^i}, & \text{if } |A_i| = 2s_i \equiv 0 \pmod 4. \end{cases}$$

So $A(G_w)$ can be reduced to the following matrix:

$$B = \left(\begin{array}{c|c|c|c|c} 0 & \mathbf{0} & \mathbf{0} & c_1 \ldots c_t & s \\ \hline \mathbf{0} & \begin{matrix} A_1 & & \\ & \ddots & \\ & & A_t \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} A_{t+1} & & \\ & \ddots & \\ & & A_k \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline \begin{matrix} c_1 \\ \vdots \\ c_t \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{matrix} d_1 \\ \vdots \\ d_t \end{matrix} \\ \hline s & \mathbf{0} & \mathbf{0} & d_1 \ldots d_t & 0 \end{array}\right),$$

where $s = \sum_{i=t+1}^{k} c_i$.

Define

$$D = \left(\begin{array}{cc|ccc} 0 & s & c_1 & \ldots & c_t \\ s & 0 & d_1 & \ldots & d_t \\ \hline c_1 & d_1 & & & \\ \vdots & \vdots & & \mathbf{0} & \\ c_t & d_t & & & \end{array}\right). \tag{3.1}$$

After interchanging rows and columns, we get the equivalent matrix of B:

$$\begin{pmatrix} D & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}. \tag{3.2}$$

It follows that

$$\begin{aligned} i_+(G_w) &= i_+(D) + \sum_{j=1}^{k} i_+(A_k) = i_+(D) + \frac{1}{2}\sum_{j=1}^{k} |A_i| \\ &= i_+(D) + \frac{1}{2}\left(\sum_{j=1}^{t}(r_i - 3) + \sum_{j=t+1}^{k}(r_i - 2)\right) \\ &= i_+(D) + \frac{1}{2}\left(\sum_{j=1}^{k}(r_i - 2) - t\right) \\ &= i_+(D) + \frac{1}{2}(n - 2 - t). \end{aligned}$$

Similarly, $i_-(G_w) = i_-(D) + \frac{1}{2}(n-2-t), i_0(G_w) = t + 2 - \mathrm{R}(D)$.

Case 2.: $\min\{r_1, r_2, \ldots, r_k\} = 3$**.** We suppose, without loss of generality, that the first ℓ paths $P_i = u_0u_iv_0$ $(i = 1, \ldots, \ell)$ are of length 3. Partition the vertex of G_w as follows: $\{u_0\}, V(P^{\ell+1}), \ldots, V(P^k), \{u_1, \ldots, u_\ell\}$,$\{u_{\ell+1}, \ldots, u_t\}, \{v_0\}$. Then we label the weight for each edge of G_w as follows: $c_i = w(u_0u_i)\,(i = 1, \ldots, \ell)$, $d_i = w(v_0u_i)\,(i = 1, \ldots, t)$, $a_i = w(u_0u_1^i)\,(i = \ell+1, \ldots, k)$, $b_i = w(u_iu_{2s_i}^i)\,(i = \ell+1, \ldots, t)$, $b_j = w(v_0u_{2s_j}^j)\,(j = t+1, \ldots, k)$ and $w_{jj}^i = w(u_{2j-1}^i u_{2j}^i)\,(i = \ell+1, \ldots, k; j = 1, \ldots, \frac{1}{2}|V(P^i)|)$, $w_{j,j+1}^i = w(u_{2j}^i u_{2j+1}^i)\,(i = \ell+1, \ldots, k; j = 1, \ldots, \frac{1}{2}|V(P^i)| - 1)$. Then the adjacency matrix of G_w has the following form:

$$A(G_w) = \left(\begin{array}{c|c|c|c|c|c}
0 & \alpha_{\ell+1}^T \ldots \alpha_t^T & \alpha_{t+1}^T \ldots \alpha_k^T & c_1 \ldots c_\ell & \mathbf{0} & 0 \\ \hline
\alpha_{\ell+1}^T & A_{\ell+1} & & & \beta_{\ell+1} & \\
\vdots & \ddots & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\
\alpha_t^T & A_t & & & \beta_t & \\ \hline
\alpha_{t+1}^T & & A_{t+1} & & & \beta_{t+1}^T \\
\vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \vdots \\
\alpha_k^T & & A_k & & & \beta_k^T \\ \hline
c_1 & & & & & d_1 \\
\vdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots \\
c_\ell & & & & & d_\ell \\ \hline
 & \beta_{\ell+1}^T & & & & d_{\ell+1} \\
\mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots \\
 & \beta_t^T & & & & d_t \\ \hline
0 & \mathbf{0} & \beta_{t+1}^T \ldots \beta_k^T & d_1 \ldots d_\ell & d_{\ell+1} \ldots d_t & 0
\end{array}\right).$$

After applying ECMOs on the above matrix, we can get a diagonal matrix similar to (3.2), hence the result is still holds in this case.

Case 3: $\min\{r_1, r_2, \dots, r_k\} = 2$. Let $c_{t+1} = w(u_0v_0)$, then we only need to delete the row and the column corresponding to A_{t+1} and replace the upper right and the lower left elements of $A(G_w)$ with c_{t+1}, and the rest arguments are similar.

Theorem 3.1. *Let $G_w = \theta(r_1, r_2, \dots, r_k)_w$ be a weighted graph of order n. Denote by n_i the number of r_j's which satisfy $r_j - 2 \equiv i \pmod 4$ $(1 \leqslant j \leqslant k, 0 \leqslant i \leqslant 3)$ and let $t = n_1 + n_3$. The matrix D is defined as in* (3.1). *Then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(i_+(D) + \frac{1}{2}(n-2-t), i_-(D) + \frac{1}{2}(n-2-t), t+2-\mathrm{R}(D)\right). \quad (3.3)$$

In particular,

(i) *if $n_1 + n_3 = 0$, $s = 0$, then* $(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}n - 1, \frac{1}{2}n - 1, 2\right)$.

(ii) *if $n_1 + n_3 = 0$, $s \neq 0$, then* $(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}n, \frac{1}{2}n, 0\right)$.

(iii) *if $n_1 n_3 > 0$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}(n-t) + 1, \frac{1}{2}(n-t) + 1, t-2\right).$$

(iv) *if $n_1 + n_3 \neq 0$, $n_1 n_3 = 0$ and $d_i c_t \neq c_i d_t$ holds for some $i \in \{1, 2, \dots, t-1\}$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}(n-t) + 1, \frac{1}{2}(n-t) + 1, t-2\right).$$

(v) *if $n_1 + n_3 \neq 0$, $n_1 n_3 = 0$, $s > 0$ and $d_i c_t = c_i d_t$ holds for $i = 1, 2, \dots, t$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \begin{cases} \left(\frac{1}{2}(n-t), \frac{1}{2}(n-t) + 1, t-1\right), & \text{if } n_1 > 0, n_3 = 0; \\ \left(\frac{1}{2}(n-t) + 1, \frac{1}{2}(n-t), t-1\right), & \text{if } n_3 > 0, n_1 = 0. \end{cases}$$

(vi) *if $n_1 + n_3 \neq 0$, $n_1 n_3 = 0$, $s = 0$ and $d_i c_t = c_i d_t$ holds for $i = 1, 2, \dots, t$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}(n-t), \frac{1}{2}(n-t), t\right).$$

(vii) *if $n_1 + n_3 \neq 0$, $n_1 n_3 = 0$, $s < 0$ and $d_i c_t = c_i d_t$ holds for $i = 1, 2, \dots, t$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \begin{cases} \left(\frac{1}{2}(n-t) + 1, \frac{1}{2}(n-t), t-1\right), & \text{if } n_1 > 0, n_3 = 0; \\ \left(\frac{1}{2}(n-t), \frac{1}{2}(n-t) + 1, t-1\right), & \text{if } n_3 > 0, n_1 = 0. \end{cases}$$

Proof. By the discussion of Cases 1-3 above, the first part of Theorem 3.1 follows directly. Furthermore, by the first part of Theorem 3.1 it is routine to check that (i) and (ii) hold.

(iii) If $n_1 n_3 > 0$, applying ECMOs on D yields the following matrix:

$$\left(\begin{array}{cc|ccc} 0 & s & 0 & \dots & c_t \\ s & 0 & \alpha_1 & \dots & d_t \\ \hline 0 & \alpha_1 & & & \\ \vdots & \vdots & & \mathbf{0} & \\ c_t & d_t & & & \end{array}\right),$$

where $\alpha_i = d_i - \frac{d_t}{c_t} c_i$. Noted that $c_1 > 0$ and $c_t < 0$, hence $\alpha_1 \neq 0$, which implies that $i_+(D) = i_-(D) = 2$ and $\mathrm{R}(D) = 4$. By (3.3), we have $(i_+(G_w), i_-(G_w), i_0(G_w)) = (\frac{1}{2}(n-t), \frac{1}{2}(n-t)+1, t-1)$. By a similar discussion as in the proof of (iii), we can show that (iv) also holds.

(v) In this case, applying ECMOs to D yields the following matrix:

$$\left(\begin{array}{cc|cccc} 0 & s & 0 & \dots & 0 \\ s & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & -\frac{2c_t d_t}{s} \end{array}\right).$$

If $n_1 > 0, n_3 = 0$, then $-\frac{2c_t d_t}{s} < 0$ for $c_t > 0$, hence $i_+(D) = 1, i_-(D) = 2$ and $\mathrm{R}(D) = 3$. In view of (3.3), we have $(i_+(G_w), i_-(G_w), i_0(G_w)) = (\frac{1}{2}(n-t), \frac{1}{2}(n-t)+1, t-2)$. If $n_1 = 0, n_3 > 0$, then $-\frac{2c_t d_t}{s} > 0$ for $c_t < 0$, hence $i_+(D) = 2, i_-(D) = 1$ and $\mathrm{R}(D) = 3$. In view of (3.3), we have $(i_+(G_w), i_-(G_w), i_0(G_w)) = (\frac{1}{2}(n-t)+1, \frac{1}{2}(n-t), t-2)$. By a similar discussion, we can also show that (vi) and (vii) hold.

This completes the proof. □

4 Characterization of weighted graphs in $\Gamma_{n,k-1}$ with small positive (negative) indices

In this section, we'll characterize all the weighted graphs in $\Gamma_{n,k-1}$ with 1 or 2 positive (negative) indices.

Theorem 4.1. *Let $G_w \in \Gamma_{n,k-1}$. Then $i_+(G_w) = 1$ if and only if G_w is one of the following graphs: the weighted graph $\theta(3, \dots, 3)_w$ with $c_k d_i = c_i d_k$, $i = 1, 2, \dots, k$; the weighted graph $\theta(3, \dots, 3, 2)_w$ with $c_{k-1} d_i = c_i d_{k-1}$, $i = 1, 2, \dots, k-1$.*

Proof. The sufficiency follows directly from Theorem 3.1. Here we only show the necessity in what follows.

Note that if $G_w \in \Gamma_{n,k-1}$ with pendants, then assume, without loss of generality, that x is a pendent vertex of G_w. Let $N(x) = \{y\}$ and $G'_w = G_w - \{x, y\}$. It's routine to check that G'_w is not a weighted empty graph, which contradicts to the fact that $i_+(G_w) = 1$.

Now we consider the case that G_w contains no pedants and $i_+(G_w) = 1$. In view of Theorem 3.1,

• $t = 0$ and $s = 0$. In this subcase, we have $i_+(G_w) = \frac{1}{2}n - 1 = 1$ holds for $n = 4$. Then $G_w = \theta(2, 4)_w$ with weighted condition $c_1 w_{11}^2 = a_2 b_2$ for $s = 0$. Note that the weighted graph $\theta(2,4)_w$ with $c_1 w_{11}^2 = a_2 b_2$ is, in fact, the weighted graph $\theta(3,3)_w$ with $c_2 d_i = c_i d_2$, $i = 1, 2$.

• $t = 0$ and $s \neq 0$. In this subcase, we have $n \geqslant 4$, hence $i_+(G_w) = \frac{n}{2} \geqslant 2$.

• $n_1 > 0$ and $n_3 > 0$. In this subcase, we have $n - t \geqslant 4$, hence $i_+(G_w) = \frac{1}{2}(n-t) + 1 \geqslant 3$.

• Just one of n_1 and n_3 is 0, and $d_i c_t \neq c_i d_t$ holds for some $i \in \{1, 2, \ldots, t\}$. In this subcase, we have $n - t \geqslant 2$ if $n_3 = 0$ and $n - t \geqslant 6$ if $n_1 = 0$. Hence $i_+(G_w) = \frac{1}{2}(n-t) + 1 \geqslant 2$.

• Just one of n_1 and n_3 is 0, $s = 0$ and $d_i c_t = c_i d_t$ holds for $i = 1, 2, \ldots, t$. In this subcase, we have $n - t \geqslant 2$ if $n_3 = 0$ and $n - t \geqslant 6$ if $n_1 = 0$. Hence, $i_+(G_w) = 1$ if and only if $n - t = 2$ and $n_3 = 0$. This gives that G_w must be the weighted graph $\theta(3, \ldots, 3)_w$ with $c_k d_i = c_i d_k$ holding for $i = 1, 2, \ldots, k$.

• Just one of n_1 and n_3 is 0, $s > 0$ and $d_i c_t = c_i d_t$ holds for $i = 1, 2, \ldots, t$. In this subcase, we have $n - t \geqslant 2$ if $n_3 = 0$ and $n - t \geqslant 4$ if $n_1 = 0$. Hence, $i_+(G_w) = 1$ if and only if $n - t = 2$ and $n_3 = 0$. This gives that G_w must be the weighted graph $\theta(3, \ldots, 3, 2)_w$ with $c_{k-1} d_i = c_i d_{k-1}$ holding for $i = 1, 2, \ldots, k-1$.

• Just one of n_1 and n_3 is 0, $s < 0$ and $d_i c_t = c_i d_t$ holds for $i = 1, 2, \ldots, t$. In this subcase, we have $n - t \geqslant 4$ if $n_3 = 0$ and $n - t \geqslant 6$ if $n_1 = 0$, which implies that $i_+(G_w) = \frac{1}{2}(n-t) + 1 > 1$.

Hence, we conclude that $i_+(G_w) = 1$ if and only if G_w is the weighted graph $\theta(3, \ldots, 3)_w$ with $c_k d_i = c_i d_k$ holding for $i = 1, 2, \ldots, k$ or, G_w is the weighted graph $\theta(3, \ldots, 3, 2)_w$ with $c_{k-1} d_i = c_i d_{k-1}$ holding for $i = 1, 2, \ldots, k-1$. □

Theorem 4.2. *Let $G_w \in \Theta_k$. Then $i_+(G_w) = 2$ if and only if G_w is one of the following graphs: the weighted graph $\theta(2,4,4)_w$ with $c_1 = \frac{a_2 b_2}{w_{11}^2} + \frac{a_3 b_3}{w_{11}^3}$; the weighted graph $\theta(3, \ldots, 3)_w$ with $d_i c_t \neq c_i d_k$ for some $i \in \{1, 2, \ldots, k\}$; the weighted graph $\theta(3, \ldots, 3, 2)_w$ with $d_i c_{k-1} \neq c_i d_{k-1}$ for some $i \in \{1, 2, \ldots, k-1\}$; the weighted graph $\theta(3, \ldots, 3, 2, 4)_w$ with $c_{k-2} d_i = c_i d_{k-2}$, $i = 1, 2, \ldots, k-2$ and $c_{k-1} w_{11}^k \geqslant a_k b_k$.*

Proof. The sufficiency is clear by Theorem 3.1. To prove the necessity, suppose that $G_w \in \Theta_k$ with $i_+(G_w) = 2$. We proceed by distinguishing the following subcases.

• $t = 0$ and $s = 0$. In this subcase, $i_+(G_w) = \frac{1}{2}n - 1 = 2$, hence we have $n = 6$. Then G_w may be $\theta(2,4,4)_w$, $\theta(2,6)_w$ or $\theta(4,4)_w$. If G_w is the weighted graph $\theta(2,4,4)_w$, then $c_1 w_{11}^2 = a_2 b_2$ for $s = 0$, whereas the s of $\theta(2,6)_w$ is positive and the s of $\theta(4,4)_w$ is negative, which contradicts the assumption that $s = 0$.

• $t = 0$ and $s \neq 0$. In this subcase, $i_+(G_w) = \frac{1}{2}n = 2$, hence we have $n = 4$. Then G_w is just the weighted graph $\theta(2,4)_w$ with $c_1 w_{11}^2 \neq a_2 b_2$. In fact, the weighted graph $\theta(2,4)_w$ with $c_1 w_{11}^2 \neq a_2 b_2$ is also the weighted graph $\theta(3,3)_w$ with $c_k d_i \neq c_i d_k$ for $i = 1, 2$.

• $n_1 > 0, n_3 > 0$. In this subcase, we have $n - t \geqslant 4$. Hence, $i_+(G_w) = \frac{1}{2}(n-t) + 1 \geqslant 3$, which implies that there does not exist such weighted graph G_w.

• Just one of n_1 and n_3 is 0, and $d_i c_t \neq c_i d_t$ holds for some $i \in \{1, 2, \ldots, t\}$. In this subcase, by a similar discussion in the proof of Theorem 4.1, $i_+(G_w) = 2$ holds only if

$n_3 = 0$ in which $i_+(G_w) = \frac{1}{2}(n-t)+1$. So we have $n-t=2$. Hence G_w must be the weighted graph $\theta(3,\ldots,3)_w$ with $d_i c_t \neq c_i d_k$ for some $i \in \{1,2,\ldots,k\}$, or the weighted graph $\theta(3,\ldots,3,2)_w$ with $d_i c_{k-1} \neq c_i d_{k-1}$ for some $i \in \{1,2,\ldots,k-1\}$.

• Just one of n_1 and n_3 is 0, $s = 0$ and $d_i c_t = c_i d_t$ holds for $i = 1,2,\ldots,t$. In this subcase, $i_+(G_w) = \frac{1}{2}(n-t)$. Hence, by a similar discussion in the proof of Theorem 4.1, $i_+(G_w) = 2$ if and only if $n-t = 4$ and $n_3 = 0$, which implies that G_w must be the weighted graph $\theta(3,\ldots,2,4)_w$ with $c_{k-2}d_i = c_i d_{k-2}$ $i = 1,2,\ldots,k-2$ and $c_{k-1}w_{11}^k = a_k b_k$.

• Just one of n_1 and n_3 is 0, $s > 0$ and $d_i c_t = c_i d_t$ holds for $i \in \{1,2,\ldots,t\}$. In this subcase, $i_+(G_w) = \frac{1}{2}(n-t)$. Hence, by a similar discussion in the proof of Theorem 4.1, $i_+(G_w) = 2$ if and only if $n-t = 4$ and $n_3 = 0$, which implies that G_w must be the weighted graph $\theta(3,\ldots,2,4)_w$ with $c_{k-2}d_i = c_i d_{k-2}$ for $i \in \{1,2,\ldots,k-2\}$ and $c_{k-1}w_{11}^k > a_k b_k$.

• Just one of n_1 and n_3 is 0, $s < 0$ and $d_i c_t = c_i d_t$ holds for $i \in \{1,2,\ldots,t\}$. In this subcase, by a similar discussion in the proof of Theorem 4.1, we have $n-t \geqslant 4$ if $n_3 = 0$ and $n-t \geqslant 6$ if $n_1 = 0$. Hence, we have $i_+(G_w) = \frac{1}{2}(n-t)+1 > 2$.

This completes the proof. □

Theorem 4.3. *Let $G_w \in \Gamma_{n,k}$ with pedants. Then $i_+(G_w) = 2$ if and only if $G \cong G^1, G^2, \ldots, G^9$ or G^{10} (see Fig. 2) and the corresponding weighted conditions are as shown in Table 1, where the empty cell means that there is no correlation between the inertia index of G_w and its weight set.*

Table 1: The weighted condition for each $G_w \in \Gamma(n,k)$ with pedants satisfying $i_+(G_w) = 2$.

weighted graph G_w	weighted conditions of G_w
$G_w^1, G_w^2, G_w^3, G_w^4$	
G_w^5	$c_{k-1}d_i = c_i d_{k-1}$ $(1 \leqslant i \leqslant k-1)$
G_w^6, G_w^7	$c_k d_i = c_i d_k$ $(1 \leqslant i \leqslant k)$
G_w^8	$c_{k-1}d_i = c_i d_{k-1}$ $(2 \leqslant i \leqslant k-1)$
G_w^9, G_w^{10}	$c_{k-1}d_i = c_i d_{k-1}$ $(1 \leqslant i \leqslant k-1)$

Proof. It is routine to check that $i_+(G_w^i) = 2$ holds for $i = 1,2,\ldots,10$. To show the converse, suppose that $i_+(G_w) = 2$. Since G_w has at least one pendent x, let $N(x) = \{y\}$ and $G'_w = G_w - \{x,y\} = H_w + pK_1$, where H_w is obtained from G'_w by deleting all the isolated vertices. By Lemma 2.3 we have $2 = i_+(G_w) = i_+(G'_w)+1 = i_+(H_w)+1$. Hence, $i_+(H_w) = 1$. Recall that G_w contains a Θ-graph as an induced subgraph, we conclude that H_w is either isomorphic to a weighted star or one of the weighted graphs described in Theorem 4.1. If H_w is a star, then G must be isomorphic to $G^i, i = 1,2,3,4$. If H_w is the weighted graph $\theta(3,\ldots,3)_w$, then G must be isomorphic to G^i, $i = 5,6,7$ and if H_w is the weighted graph $\theta(3,\ldots,3,2)_w$, then G must be isomorphic to G^i, $i = 8,9,10$.

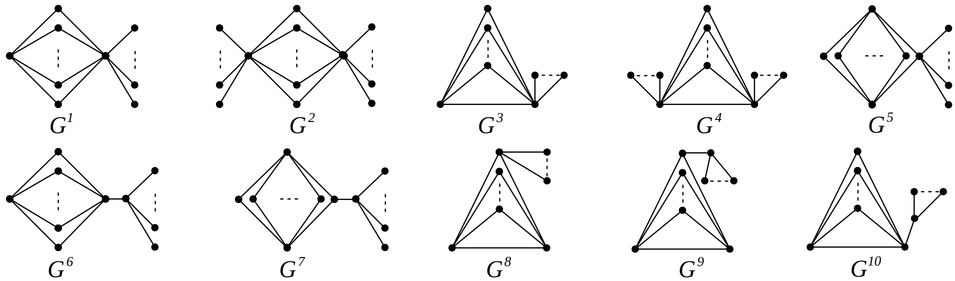

Figure 2: Graphs $G^1, G^2, \ldots, G^9$ and G^{10}.

If G is isomorphic to G^5, without loss of generality, assume that x is adjacent to the internal vertex of the k-th path P_3 (see Fig. 2), so the weighted condition is that $c_{k-1}d_i = c_i d_{k-1}$ holds for $i = 1, 2, \ldots, k-1$. If G is isomorphic to G^6 or G^7, the weighted condition is $c_k d_i = c_i d_k$ for $i = 1, 2, \ldots, k$.

If G is isomorphic to G^8, without loss of generality, assume that x is adjacent to the internal vertex of the first path P_3 (see Fig. 2), so the weighted condition is that $c_{k-1}d_i = c_i d_{k-1}$ holds for $i = 2, 3, \ldots, k-1$. If G is isomorphic to G^9 or G^{10}, the weighted condition is $c_{k-1}d_i = c_i d_{k-1}$ for $i = 1, 2, \ldots, k-1$. □

Similarly, we can have the following theorems:

Theorem 4.4. *Let $G_w \in \Gamma_{n,k-1}$. Then $i_-(G_w) = 1$ if and only if G_w is the weighted $\theta(3, \ldots, 3)_w$ with the weighted condition that $c_k d_i = c_i d_k$ holds for $i = 1, 2, \ldots, k$.*

Theorem 4.5. *Let $G_w \in \Theta_k$. Then $i_-(G_w) = 2$ if and only if G_w is one of the following graphs: the weighted graph $\theta(3, \ldots, 3, 2)_w$ with an arbitrary weighted condition; the weighted graph $\theta(2, 4, 4)_w$ with weighted condition $c_1 = \frac{a_2 b_2}{w_{11}^2} + \frac{a_3 b_3}{w_{11}^3}$; the weighted graph $\theta(3, \ldots, 3)_w$ with the weighted condition that $d_i c_k \neq c_i d_k$ holds for some $i \in \{1, 2, \ldots, k\}$; the weighted graph $\theta(3, \ldots, 3, 2, 4)_w$ with the weighted condition that $c_{k-2}d_i = c_i d_{k-2}$ holds for $i = 1, 2, \ldots, k-2$ and $c_{k-1}w_{11}^k \leqslant a_k b_k$; the weighted graph $\theta(3, \ldots, 3, 4)_w$ with the weighted condition that $c_{k-1}d_i = c_i d_{k-1}$ holds for $i = 1, 2, \ldots, k-1$.*

Theorem 4.6. *Let $G_w \in \Gamma_{n,k-1}$ with pedants. Then $i_-(G_w) = 2$ if and only if G_w is one of the following graphs: the weighted graph G_w has G^1 (resp. G^2, G^3, G^4) as its unweighted graph and its weight set is arbitrary; the weighted graph G_w has G^5 as its unweighted graph satisfying the weighted condition $c_{k-1}d_i = c_i d_{k-1}$, $i = 1, 2, \ldots, k-1$; the weighted graph G_w has G^6 (resp. G^7) as its unweighted graph satisfying the weighted condition $c_k d_i = c_i d_k$, $i = 1, 2, \ldots, k$.*

5 Weighted graphs in $\Gamma_{n,k-1}$ with rank 2, 3, or 4

In this section, we characterize all the weighted $(k-1)$-cyclic graphs in $\Gamma_{n,k-1}$ with rank 2, 3, 4, respectively.

Theorem 5.1. *Let $G_w \in \Gamma_{n,k-1}$. Then $\mathrm{R}(G_w) = 2$ if and only if G_w is the weighted $\theta(3,\ldots,3)_w$ with the weighted condition $c_k d_i = c_i d_k$ holding for $i = 1, 2, \ldots, k$.*

Proof. Let $G_w \in \Gamma_{n,k-1}$, $i_+(G_w) \geqslant 1$ and $i_-(G_w) \geqslant 1$ since it contains P_2 as an induced subgraph. Then $r(G_w) = 2$ if and only if $i_+(G_w) = i_-(G_w) = 1$. By Theorems 4.1–4.6, we know G_w must be the weighted $\theta(3,\ldots,3)_w$ satisfying the weighted condition that $c_k d_i = c_i d_k$ for any $1 \leqslant i \leqslant k$. □

Theorem 5.2. *Let $G_w \in \Gamma_{n,k-1}$. Then $\mathrm{R}(G_w) = 3$ if and only if G_w is the weighted $\theta(3,\ldots,3,2)_w$ with the weighted condition that $c_{k-1} d_i = c_i d_{k-1}$ holds for $i = 1, 2, \ldots, k-1$.*

Proof. Let $G_w \in \Gamma_{n,k-1}$, $i_+(G_w) \geqslant 1$ and $i_-(G_w) \geqslant 1$ since it contains P_{2w} as an induced subgraph. Then $\mathrm{R}(G_w) = 3$ if and only if $i_+(G_w) = 1, i_-(G_w) = 2$ or $i_+(G_w) = 2, i_-(G_w) = 1$. Note that either $i_+(G_w)$ or $i_-(G_w)$ equals 1, hence by Theorems 4.1 and 4.4 we know G_w must be the weighted graph $\theta(3,\ldots,3)_w$ satisfying $c_k d_i = c_i d_k$ for $1 \leqslant i \leqslant k$. □

Theorem 5.3. *Let $G_w \in \Theta_k$. Then $\mathrm{R}(G_w) = 4$ if and only if G_w is one of the following graphs: the weighted graph $\theta(2,4,4)_w$ with weighted condition $c_1 = \frac{a_2 b_2}{w_{11}^2} + \frac{a_3 b_3}{w_{11}^3}$; the weighted graph $\theta(3,\ldots,3)$ with the weighted condition that $d_i c_k \neq c_i d_k$ holds for some $i \in \{1, 2, \ldots, k\}$; the weighted graph $\theta(3,\ldots,3,2)_w$ with the weighted condition that $d_i c_{k-1} \neq c_i d_{k-1}$ holds for some $i \in \{1, 2, \ldots, k-1\}$; the weighted graph $\theta(3,\ldots,3,2,4)_w$ with the weighted condition that $c_{k-2} d_i = c_i d_{k-2}$ holds for $i = 1, 2, \ldots, k-2$ and $c_{k-1} w_{11}^k = a_k b_k$.*

Proof. Let G_w be a weighted $(k-1)$-cyclic graph, it is routine to check that $i_+(G_w) \geqslant 1$ and $i_-(G_w) \geqslant 1$. Then $\mathrm{R}(G_w) = 4$ if and only if $(i_+(G_w), i_-(G_w)) = (1,3)$ or $(i_+(G_w), i_-(G_w)) = (3,1)$ or $(i_+(G_w), i_-(G_w)) = (2,2)$. If one of $i_+(G_w)$ and $i_-(G_w)$ equals 1, by Theorems 4.1 and 4.4, G_w must be the weighted graph $\theta(3,\ldots,3)_w$ or $\theta(3,\ldots,3,2)_w$. In this case, by Theorems 4.1, 4.2, 4.4 and 4.5 we know the rank of such graph G_w is no less than 3. Hence, it should only consider that $(i_+(G_w), i_-(G_w)) = (2,2)$. In this case, based on Theorems 4.2 and 4.5, $(i_+(G_w), i_-(G_w)) = (2,2)$ if and only if G_w is one of the weighted graphs characterized in the above result. □

Similarly, we can have the following theorem:

Theorem 5.4. *Let $G_w \in \Gamma_{n,k-1}$ with pedants. Then $\mathrm{R}(G_w) = 4$ if and only if $G \cong G^1, \ldots, G^7$, what's more, the weighted condition of G_w^1 (resp. G_w^2, G_w^3, G_w^4) is arbitrary; G_w^5 satisfies the weighted condition that $c_{k-1} d_i = c_i d_{k-1}$ holds for $i = 1, 2, \ldots, k-1$; while G_w^6 (resp. G_w^7) satisfies the weighted condition that $c_k d_i = c_i d_k$ holds for $i = 1, 2, \ldots, k$.*

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