

Touching perfect matchings and halving lines

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Abstract

Let V be a set of $2m$ ($1 \leq m < \infty$) points in the plane. Two segments I, J with endpoints in V *cross* if $\text{relint } I \cap \text{relint } J$ is a singleton. A (perfect) *cross-matching* M on V is a set of m segments with endpoints in V such that every two segments in M cross. A *halving line* of V is a line l spanned by two points of V such that each one of the two open half planes bounded by l contains fewer than m points of V . Pach and Solymosi proved that if V is in general position, then V admits a perfect cross-matching iff V has exactly m halving lines. The aim of this note is to extend this result to the general case (where V is unrestricted).

Keywords: Bigraphs, cross-matching, halving lines, perfect matchings.

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1 Introduction, notions and main results

Let V be a set of $2m$ distinct points in the plane $\mathbb{R}^2$ ($1 \leq m < \infty$). By a (perfect geometric) *matching* of V we mean a set $M = \{I_1, \dots, I_m\}$ of m non-degenerate closed line segments whose endpoints are (all) the points of V . The number of matchings of V is

$$(2m-1)!! = \prod_{i=1}^m (2i-1) = \frac{(2m)!}{2^m \cdot m!}.$$

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If V is in general position (no three points on a line), then two distinct segments $I, J \in M$ may be

- (a) disjoint ($I \cap J = \emptyset$),
- (b) or they may *cross*, i.e., share a unique point that lies in the relative interior of both I and J .

When V is unrestricted, two more possibilities arise.

- (c) The unique common point of I and J maybe an interior point of I and an endpoint of J (or vice versa).
- (d) If the four endpoints of I and J are collinear, then I and J may share a line segment. (This includes the possibility that $I \subset \text{relint } J$, or vice versa.)

We shall say that two segments I, J *touch* if they have at least one point in common ($I \cap J \neq \emptyset$). We call M a *simple matching* (SM) if the segments of M are pairwise disjoint.

It is well known and quite easy to show (see [2, Theorem 4.2]) that if V is in general position, then the number $\text{sm}(V)$ of simple matchings on V is bounded from below by the m -th Catalan number C_m , i.e.,

$$\text{sm}(V) \geq C_m = \frac{1}{m+1} \binom{2m}{m}. \quad (1.1)$$

Equality holds for $m = 1$ or when V is the set of vertices of a convex $2m$ -gon. (It can be shown that if V is in general position but not in convex position, then $\text{sm}(V) > C_m$, with only one exception: when $m = 3$ and V consists of the vertices of a convex pentagon P plus a sixth point that lies in the interior of the pentagon formed by the diagonals of P .)

Call M a *cross-matching* (CM) if each two distinct segments of M cross. Let us call M a *touching matching* (TM) if every two segments of M touch.

1.1 Halving lines

Definition 1.1. A line L is a *halving line* of V if each of the two open half-planes L^+, L^- bounded by L contains fewer than m points of V .

This clearly implies that $|L \cap V| \geq 2$, i.e., that the line L is spanned by V . When V is in general position, then necessarily $|L \cap V| = 2$, and $|L^- \cap V| = |L^+ \cap V| = m - 1$. When V is unrestricted we call L a *halving line of order k* if $\max(|L^- \cap V|, |L^+ \cap V|) = m - k$ ($1 \leq k \leq m$). In that case we may assume that, say, $|L^+ \cap V| = m - k$, $|L^- \cap V| = m - k - \varepsilon$, and $|L \cap V| = 2k + \varepsilon$, for some ε , $0 \leq \varepsilon \leq m - k$. (See Figure 1.)

1.2 Halving lines and TMs

If M is a TM on V , I is a segment of M , and $L = \text{aff } I$ is the line spanned by I , then L is a halving line. Indeed, an open half-plane bounded by L contains no endpoint of I , and at most one endpoint of each other segment of M .

The connection between the number $h(V)$ of halving lines of V , and the existence of a cross-matching on V , in the case where V is in general position, was established by Pach and Solymosi in [3] as follows: They observed that each point of V lies on at least one halving line, hence $h(V) \geq m$. Then they found that *either* each point of V lies on just one

halving line, $h(V) = m$ and V admits a unique CM, or at least one point of V lies on more than one halving line, $h(V) > m$, and V admits no CM at all. This result was generalized in [1] (see Theorem 1 and Corollary 3 there). In [4] we prove an extremal property of CMs, namely that if V admits a CM M , and M' is another (perfect) geometric matching on V , then the sum of the (Euclidean) lengths of the edges of M' is strictly less than the sum of the lengths of the edges of M . An analogous result holds for TMs. The geometric graph whose edges span (all) the halving lines of its vertex set V (with $|V|$ even and V in general position) is said to be a *bigraph*. We refer to [5] regarding results on bigraphs.

The aim of this note is to extend the result of [3] to arbitrary, unrestricted $2m$ -subsets V of $\mathbb{R}^2$.

In the next section we define the notion of “a halving line at a point $p \in V$ ”, and show that a halving line of order k is a halving line at exactly $2k$ points. We also show that the number of halving lines at any point $p \in V$ is odd, hence ≥ 1 . The main results can be summarized as follows:

Theorem 1.2. *Suppose $L_1, \dots, L_t$ ($t = h(V)$) are all the halving lines of V , with L_i of order k_i ($1 \leq k_i \leq m$, $i = 1, \dots, t$). If for each $p \in V$ there is just one halving line at p , then*

$$\sum_{i=1}^t k_i = m,$$

and the number of TMs of V is precisely

$$\prod_{i=1}^t (k_i!).$$

If, for some $p \in V$, there is more than one halving line at p , then

$$\sum_{i=1}^t k_i > m,$$

and V has no TM.

In particular we have

Corollary 1.3. *The set V has a unique TM iff V has exactly m halving lines, each of order 1. The unique TM is a CM if each of the m halving lines contains just two points of V .*

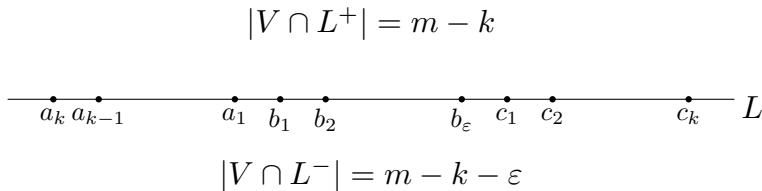


Figure 1: A halving line of order k .

2 Proofs

We start with the definition of “a halving line of V at p ”, where V is a set of $2m$ points in $\mathbb{R}^2$, and $p \in V$. For a point $p \in V$ and a unit vector $\underline{u} = (u_1, u_2)$, denote by $L(p, \underline{u})$ the directed line $\{p + \lambda \underline{u} : \lambda \in \mathbb{R}\}$. (The direction is from small λ to larger λ .) Note that $L(p, -\underline{u})$ is the same line, directed backwards. Define $\underline{u}_+ = (-u_2, u_1)$,

$$\begin{aligned} L(p, \underline{u})_F &= L(p, \underline{u}) + \{\mu \underline{u}_+ : \mu > 0\}, \text{ and} \\ L(p, \underline{u})_B &= L(p, \underline{u}) + \{\mu \underline{u}_+ : \mu < 0\}. \end{aligned}$$

F and B stand for “Front” and “Back”, respectively.

$L(p, \underline{u})_F$ and $L(p, \underline{u})_B$ are the two open half-planes bounded by $L(p, \underline{u})$. Now move the unit vector $\underline{u}$ continuously on the unit circle in counterclockwise direction. Note that $L(p, \underline{u})_F$ and $L(p, \underline{u})_B$ switch when $\underline{u}$ is replaced by $-\underline{u}$. As long as $L(p, \underline{u})$ does not meet $V \setminus \{p\}$, we find that

$$|V \cap L(p, \underline{u})_F| + |V \cap L(p, \underline{u})_B| = |V - \{p\}| = 2m - 1,$$

and therefore one side of $L(p, \underline{u})$ (the “major” side) contains at least m points of V , whereas the other side (the “minor” side) contains at most $m - 1$ points of V .

As we change the direction $\underline{u}$, the major side of $L(p, \underline{u})$ will remain (Front or Back) as long as the rotating line $L(p, \underline{u})$ does not meet $V \setminus \{p\}$. We call $L(p, \underline{u}_o)$ a *halving line of V at p* if the major side of $L(p, \underline{u})$ switches (from B to F or vice versa) as $\underline{u}$ passes through $\underline{u}_o$.

Proposition 2.1. *If $L = L(p, \underline{u}_o)$ is a halving line of V at p , then L is a halving line of V .*

Proof. We must show that both open sides of L , $L(p, \underline{u}_o)_F$ and $L(p, \underline{u}_o)_B$, contain fewer than m points of V each. If, say, $|V \cap L(p, \underline{u}_o)_F| \geq m$, then $V \cap L(p, \underline{u})_F \supset V \cap L(p, \underline{u}_o)_F$, and therefore $|V \cap L(p, \underline{u})_F| \geq m$, for all unit vectors $\underline{u}$ sufficiently close to $\underline{u}_o$, on both sides of $\underline{u}_o$, so the major side of $L(p, \underline{u})$ does not switch at $\underline{u} = \underline{u}_o$. $\square$

Proposition 2.2. *For each point $p \in V$, the number of halving lines of V at p is odd (hence ≥ 1).*

Proof. Choose an initial direction $\underline{u}_o$, such that $V \cap L(p, \underline{u}_o) = \{p\}$. Suppose the major side of $L(p, \underline{u}_o)$ is, say, $L(p, \underline{u}_o)_F$. Rotate the line through p counterclockwise by 180° , i.e., move $\underline{u}$ along a semicircle, until we reach $L(p, -\underline{u}_o)$. Now the major side is $L(p, -\underline{u}_o)_B (= L(p, \underline{u}_o)_F)$. We conclude that on the way the major side switched (from F to B or vice versa) an odd number of times. $\square$

Proposition 2.3. *Suppose L is a halving line of V of order k ($1 \leq k \leq m$). Then L is a halving line of V at p for exactly $2k$ points of V .*

Proof. Assume, w.l.o.g., that

$$|V \cap L^-| = m - k - \varepsilon, \quad |V \cap L^+| = m - k, \quad \text{and} \quad |V \cap L| = 2k + \varepsilon,$$

for some $0 \leq \varepsilon \leq m - k$. Label the points of $V \cap L$ in order

$$a_k, a_{k-1}, \dots, a_1, b_1, \dots, b_\varepsilon, c_1, \dots, c_k,$$

as in Figure 1. Fix a point $p \in V \cap L$, and consider a line that rotates counterclockwise through p . As the rotating line passes through the horizontal position (see Figure 1), the major side switches from Above to Below if p is one of the a_i 's, and from Below to Above if p is one of the c_i 's. But if p is one of the b_i 's, then the major side remains Above (at least in a small neighborhood on both sides of the horizontal position). $\square$

Next we show that if L is a halving line of V of order k , as in Figure 1, and M is a TM on V , then M matches the a_i 's with the c_i 's (and vice versa).

Proposition 2.4. *Suppose $V = S \cup T$ is a partition of V into two sets of equal size ($|S| = |T| = m$), and $\text{conv } S \cap \text{conv } T = \emptyset$. If M is a TM of V , then each segment $I \in M$ connects a point of S with a point of T .*

Proof. Assume, on the contrary, that some segment $I \in M$ has both endpoints in S . This leaves (at most) $m - 2$ points of S to be matched to points of T , and thus some other segment $J \in M$ has both endpoints in T . But then $I \cap J \subset \text{conv } S \cap \text{conv } T = \emptyset$. $\square$

Now look again at the halving line L in Figure 1. Define $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_\varepsilon\}$, $C = \{c_1, \dots, c_k\}$, $D_- = B \cup (V \cap L^-)$ and $D_+ = V \cap L^+$ ($|D_-| = |D_+| = m - k$). Applying Proposition 2.4 twice, first with $S = A \cup D_-$, $T = C \cup D_+$, and then with $S' = C \cup D_-$, $T' = A \cup D_+$, we find:

Proposition 2.5. *If M is a TM of V , then each segment $I \in M$ with one endpoint in A has its other endpoint in C (and vice versa), and each segment $J \in M$ with one endpoint in D_- has its other endpoint in D_+ (and vice versa).*

Note also that for *any* permutation θ of $\{1, 2, \dots, k\}$, the intersection of the k segments $[a_i, c_{\theta(i)}]$ ($i = 1, \dots, k$) is the segment $[a_1, c_1]$, that connects the k 'th point of $V \cap L$ from the right with the k 'th point of $V \cap L$ from the left. We call this segment $[a_1, c_1]$ the *central segment* of the halving line L .

Suppose $L_1, \dots, L_t$ ($t = h(V)$) are all the halving lines of V , with L_i of order k_i for $i = 1, \dots, t$. For $p \in V$, denote by $h(p)$ the number of halving lines at p . In view of Propositions 2.1 – 2.3, we have

$$\sum_{i=1}^t k_i = \frac{1}{2} \sum_{p \in V} h(p) \geq m,$$

with equality ($= m$) iff $h(p) = 1$ for all $p \in V$.

Proposition 2.6. *If $h(p) > 1$ for some $p \in V$, then there is no TM on V .*

Proof. Suppose, on the contrary, that V admits a TM M . Let $I = [p, q]$ be a segment in M with one endpoint p . Let L, L' be two different halving lines of V at p ($h(p) > 1$). By Proposition 2.5 we have $q \in L \cap L'$. But $L \cap L' = \{p\}$. $\square$

Assume, from now on, that $h(p) = 1$ for all $p \in V$. Thus $\sum_{i=1}^t k_i = m$. In other words, on each line L_i we can match two disjoint subsets of $V \cap L_i$, each of order k_i , A_i (the k_i “leftmost” points of $V \cap L_i$) and C_i (the k_i “rightmost” points of $V \cap L_i$). L_i is a halving line of V at p iff $p \in A_i \cup C_i$. The sets $A_1, C_1, \dots, A_t, C_t$ form a partition of V . As we have seen in Proposition 2.5, any TM of V will match the points of A_i with those of C_i .

There are $k_i!$ ways to match A_i with C_i , and in each of these matchings, the intersection of the connecting segments is the “central segment” of the halving line L_i . To show that the individual TM’s of $A_i \cup C_i$ on L_i ($i = 1, \dots, t$) yield a TM of V , it suffices to show that the central segments of different halving lines L_i and L_j do meet (assuming, of course, that $h(p) = 1$ for all $p \in V$). This will be done in the next proposition.

Proposition 2.7. *Suppose L is a halving line of V of order k , with $V \cap L$ labelled $a_k, \dots, a_1, b_1, \dots, b_\varepsilon, c_1, \dots, c_k$ as in Figure 1, $A = \{a_k, \dots, a_1\}$, $C = \{c_1, \dots, c_k\}$, and let L' be another halving line of V , of order k' , with $V \cap L'$ labelled similarly: $a'_{k'}, \dots, a'_1, b'_1, \dots, b'_{\varepsilon'}, c'_1, \dots, c'_{k'}$, $A' = \{a'_{k'}, \dots, a'_1\}$, $C' = \{c'_1, \dots, c'_{k'}\}$. If the central segments $[a_1, c_1]$ (of L) and $[a'_1, c'_1]$ (of L') do not meet, then $h(p) > 1$ for some $p \in \{a_1, c_1, a'_1, c'_1\}$.*

Proof. The two distinct lines L, L' cannot be parallel. If they are, and L' lies, say, above L , then the open side L^+ of L includes the closed side $\text{cl } L'^+$ of L' , and therefore $|V \cap L^+| \geq |V \cap \text{cl } L'^+| > m$, which is impossible. Let z be the crossing point of L and L' , and suppose, w.l.o.g., that z misses the central segment $[a_1, c_1]$ of L , and lies to the left of a_1 on L , see Figure 2.

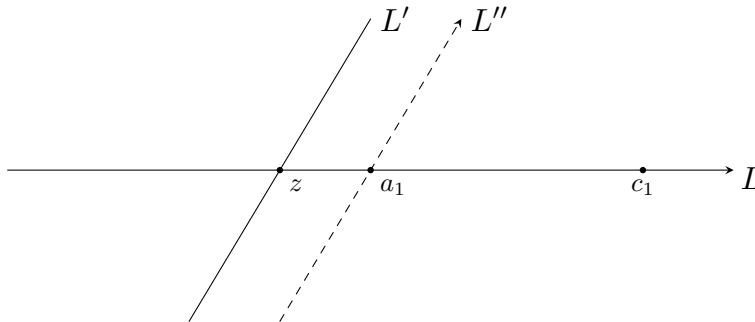


Figure 2: Proof of Proposition 2.7.

Consider a directed line that rotates counter-clockwise through a_1 . As it passes through L (directed from left to right), the major side of V switches from Front to Back. As it reaches L'' (parallel to L'), or any direction sufficiently close to that of L' , the major side of V is again Front, since the open half-plane to the left of L'' includes the closed half-plane to the left of L' , which in turn contains at least $m + k'$ points of V . Thus, there must have been another switch from Back to Front on the way, or, in other words, $h(a_1) > 1$. $\square$

3 Algorithmic aspects

The insights gained in the earlier sections of this note can be used to devise an algorithm that decides whether a set $P \subset \mathbb{R}^2$ ($|P| = 2m$) admits a TM, and to find a TM (or all TMs) if one exists. The algorithm is conceptually simple, and seems to be also computationally quite effective, though not as efficient as the one proposed in [3] (m^2 vs. $m \log m$).

Step 1: Find the point $p_0 = (x_0, y_0) \in P$ that is the first in P with respect to the lexicographic order of points $(x, y) \in \mathbb{R}^2$. p_0 is a vertex of the convex hull $[P] = \text{conv } P$.

Step 2: Calculate the slopes of the $2m - 1$ segments $[p_0, p]$ ($p \in P \setminus \{p_0\}$), arrange them in non-decreasing order and find the median slope (this can be shared by several

segments, of course). This slope determines the (unique) halving line L of P at p_0 . Find the number of points of P that lie below L , on L and above L , and order the points of $P \cap L$ lexicographically. This enables us to determine the order k of the halving line L , and the sets A, C consisting of the first (resp. last) k points of $P \cap L$. These are the $2k$ points $p \in P \cap L$ such that L is a halving line at p . Erase these $2k$ points, and call the remaining set P' ($|P'| = 2(m - k)$). If $P' = \emptyset$, stop. Otherwise, return to Step 1 with P replaced by P' .

To see that this really works, we make the following observations:

- (A) If P admits a TM M , then M contains k segments (on L) that connect points of A with points of C . The rest of M is a TM of $P' (= P \setminus (A \cup C))$. Moreover, if $\tilde{L}$ is any halving line of P other than L , of order $\tilde{k}$, then removal of $A \cup C$ leaves $\tilde{L}$ a halving line of P' of the same order $\tilde{k}$. This is clear when the central segments of L and of $\tilde{L}$ meet at a point that is interior to the central segment $[a_1, c_1]$ of L . In that case we lose k points on each side of $\tilde{L}$.

The case when the common point of these two central segments is an endpoint, say a_1 , of $[a_1, c_1]$, is shown in Figure 3. (The reason why C is included in $\tilde{L}^+$ and not in $\tilde{L}^-$, is given below.)

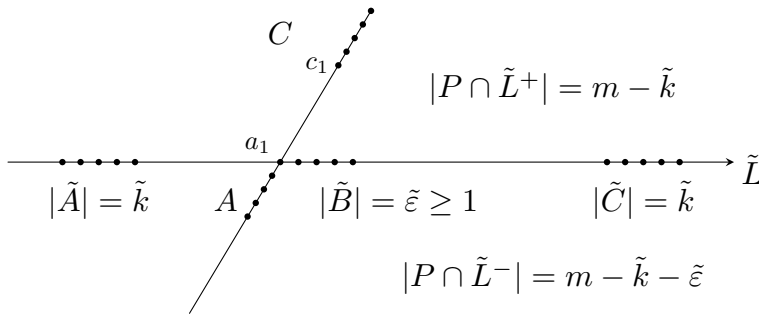


Figure 3: Two central segments whose common point is an endpoint in one of them.

Since M matches $P \cap \tilde{L}^+$ with $(P \cap \tilde{L}^-) \cup \tilde{B}$ and a_1 ($a_1 \in \tilde{B}$ and $a_1 \in A \subset P \cap L$) with some point of C (Proposition 2.5), $C \subset P \cap \tilde{L}^+$ (as in Figure 3). Thus, removing $A \cup C$ will reduce $|P \cap \tilde{L}^+|$ by k to $(m - k) - \tilde{k}$, $P \cap \tilde{B}$ by 1 to $\tilde{\varepsilon} - 1$ (≥ 0 , since $a_1 \in \tilde{B}_1$), and $|P \cap \tilde{L}^-|$ by $k - 1$ to $(m - k) - \tilde{k} - (\tilde{\varepsilon} - 1)$.

- (B) If M' is a TM of P' , and N is a matching of A to C (on $P \cap L$), then $M = M' \cup N$ is a TM of P iff the central segment $[a_1, c_1]$ of L meets the central segment of each halving line of P' . Thus, if applying our algorithm to P' we find that P' has no TM, then the same holds for P . If P' does admit a TM, then P has a TM iff the central segment of L meets the central segment of each halving line of P' . To check this, we may need $O(m^2)$ operations.

References

- [1] A. Andrzejak, B. Aronov, S. Har-Peled, R. Seidel and E. Welzl, Results on k -sets and j -facets via continuous motion, in: R. Janardan (ed.), *Proceedings of the Fourteenth Annual Symposium on Computational Geometry*, New York, NY, pp. 192–199, 1998, doi:10.1145/276884.276906,

proceedings of the 14th ACM Symposium on Computational Geometry (SCG '98) held in Minneapolis, MN, June 07 – 10, 1998.

- [2] A. García, M. Noy and J. Tejel, Lower bounds on the number of crossing-free subgraphs of K_N , *Comput. Geom.* **16** (2000), 211–221, doi:10.1016/s0925-7721(00)00010-9.
- [3] J. Pach and J. Solymosi, Halving lines and perfect cross-matchings, in: B. Chazelle, J. E. Goodman and R. Pollack (eds.), *Advances in Discrete and Computational Geometry*, American Mathematical Society, Providence, RI, volume 223 of *Contemporary Mathematics*, pp. 245–249, 1999, doi:10.1090/conm/223/03141, proceedings of the AMS-IMS-SIAM Joint Summer Research Conference on Discrete and Computational Geometry: Ten Years Later held at Mount Holyoke College, South Hadley, MA, July 14 – 18, 1996.
- [4] M. A. Perles, H. Martini, Y. S. Kupitz, H. Last and R. Pinchasi, Cross-matchings and circuits have maximal length, in preparation.
- [5] G. J. Simmons and R. C. Entringer, Some properties of components of bigraphs, *Period. Math. Hungar.* **3** (1973), 167–174, doi:10.1007/bf02018472.