

Reconfiguring vertex colourings of 2-trees

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Abstract

Let H be a graph and let $k \geq \chi(H)$ be an integer. The k -colouring graph of H , denoted $G_k(H)$, is the graph whose vertex set consists of all proper k -vertex-colourings (or simply k -colourings) of H using colours $\{1, 2, \dots, k\}$; two vertices of $G_k(H)$ are adjacent if and only if the corresponding k -colourings differ in colour on exactly one vertex of H . If $G_k(H)$ has a Hamilton cycle, then H is said to have a *Gray code* of k -colourings, and the *Gray code number* of H is the least integer $k_0(H)$ such that $G_k(H)$ has a Gray code of k -colourings for all $k \geq k_0(H)$. Choo and MacGillivray determine the Gray code numbers of trees. We extend this result to 2-trees. A 2-tree is constructed recursively by starting with a complete graph on three vertices and connecting each new vertex to an existing clique on two vertices. We prove that if H is a 2-tree, then $k_0(H) = 4$ unless H is isomorphic to the join of a tree T and a vertex u , where T is a star on at least three vertices, or the bipartition of T has two even parts; in these cases, $k_0(H) = 5$.

Keywords: 2-trees, graph colouring, Gray codes, Hamilton cycles, reconfiguration problems.

Math. Subj. Class.: 05C15, 05C45

1 Introduction

Let H be a graph and k a positive integer. We define a *proper k -vertex-colouring* of H as a function $f: V(H) \rightarrow \{1, 2, \dots, k\}$ for which $f(x) \neq f(y)$ for any $xy \in E(H)$. Since we are concerned only with proper k -vertex-colourings, we use the simpler term *k -colouring*, and refer to $f(x)$ as the *colour* of x . For notation and terminology not defined here, the reader is referred to Bondy and Murty [1].

A graph H has a *Gray code of k -colourings* if it is possible to list all the k -colourings of H in such a way that consecutive colourings in the list (including the last and the first) differ on exactly one vertex of H , and the *Gray code number of H* is the least integer

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$k_0(H)$ for which H has a Gray code of its k -colourings for all $k \geq k_0(H)$. Equivalently, we may define the k -colouring graph of H , denoted $G_k(H)$, to be the graph whose vertices correspond to all k -colourings of H , and whose edges connect two k -colourings of H that differ on exactly one vertex of H . In this context, H has a Gray code of k -colourings if and only if $G_k(H)$ has a Hamilton cycle, and the Gray code number of H is the least integer $k_0(H)$ for which $G_k(H)$ has a Hamilton cycle for all $k \geq k_0(H)$.

The k -colouring graph is an example of a *reconfiguration graph*. Such graphs are often used in the study of what are known as *reconfiguration problems*. Generally, a reconfiguration problem asks: given two (different) feasible solutions to a problem, can one solution be transformed to the other through a sequence of allowable moves, while maintaining feasibility at each stage? In the context of k -colourings, the k -colouring graph is connected if and only if any k -colouring can be reconfigured into any other k -colouring by re-colouring one vertex at a time in such a way that each intermediate colouring is a k -colouring. Recently, reconfiguration problems have been receiving wide attention, and have been studied for various graph colouring problems [2, 3, 4, 10, 15], for dominating sets [12, 13, 18], and for various other graph problems including vertex covers, cliques, and independent sets [14].

The k -colouring graph arises in the context of theoretical physics, where it is the graph of the Glauber dynamics Markov chain; the goal is to find efficient algorithms for almost uniform sampling of k -colourings of graphs [16]. The Glauber dynamics Markov chain converges to the uniform distribution provided that the k -colouring graph is connected, prompting Cereceda, van den Heuvel and Johnson [4] to ask the question: given a graph H and a positive integer k , is $G_k(H)$ connected? They prove that if H has chromatic number $k \in \{2, 3\}$, then $G_k(H)$ is never connected, whereas for $k \geq 4$, there are k -chromatic graphs H for which $G_k(H)$ is connected, and other k -chromatic graphs H for which $G_k(H)$ is not connected. In general, they prove that $G_k(H)$ is connected for all $k \geq \text{col}(H) + 1$, where $\text{col}(H)$, the *colouring number* of H , is defined as $\text{col}(H) := \max\{\delta(G) \mid G \subseteq H\} + 1$. A slightly weaker version of this result is proven in [8].

Choo and MacGillivray [6] initiated the study of Hamilton cycles in k -colouring graphs by proving that the Gray code number of H is well defined, i.e., $G_k(H)$ has a Hamilton cycle for all $k \geq \text{col}(H) + 2$. This gives the upper bound $k_0(H) \leq \text{col}(H) + 2$. Note that if T is a tree, then $\text{col}(T) = 2$ and $G_2(T)$ is disconnected; hence $3 \leq k_0(T) \leq 4$. Choo and MacGillivray [6] determine which trees have $k_0(T) = 3$ and which have $k_0(T) = 4$. In particular, they prove that if T is a tree, then $k_0(T) = 3$ unless $T \cong K_{1,2\ell}$, for $\ell \geq 1$, in which case $k_0(T) = 4$. They also determine the Gray code numbers of complete graphs and cycles. Celaya, Choo, MacGillivray and Seyffarth [3] establish the Gray code numbers of complete bipartite graphs. Haas [11] studies a variation of k -colouring graphs, namely, canonical colouring graphs, and uses techniques developed in [6] to show that canonical colouring graphs of trees have Hamilton cycles.

Given the results for trees and complete graphs, a natural question is to determine the Gray code numbers of k -trees. We use the definition of k -tree given in [9], that is, a k -tree is constructed recursively by starting with a complete graph on $k + 1$ vertices and connecting each new vertex to an existing clique on k vertices (hence a 1-tree is simply a connected acyclic graph). A vertex of degree k in a k -tree is called a *leaf*. Let H be a k -tree. Then it is clear that the chromatic number of H is $\chi(H) = k + 1$, and that $G_{k+1}(H)$ is disconnected. Since every induced subgraph of H has a leaf, $\text{col}(H) = k + 1$. Thus it follows from [6, Theorem 3.4] that $k_0(H) \leq k + 3$, and therefore $k + 2 \leq k_0(H) \leq k + 3$. The problem

is therefore reduced to classifying k -trees into those with Gray code number $k + 2$ and those with Gray code number $k + 3$. The answer appears to be far from trivial. In the current paper, we provide a complete solution for 2-trees; the characterization can be stated in fairly non-technical language, but the proof involves numerous cases and generalizations of the techniques used in [3, 6]. The *join* of graphs G_1 and G_2 , denoted by $G_1 \vee G_2$, is obtained from the disjoint union of G_1 and G_2 by adding all edges between vertices of G_1 and vertices of G_2 .

Theorem 1.1. *If H is a 2-tree then $k_0(H) = 4$, unless $H \cong T \vee \{u\}$ for some tree T and vertex u , where T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.*

The remainder of the paper is devoted to proving this theorem. We first characterize 2-trees of diameter two (Lemma 3.2). We then determine the 2-trees, H , of diameter two for which $G_4(H)$ has a Hamilton cycle (Lemmas 3.3 and 3.5). This is done by considering the structure of $G_3(T)$, where T is a tree (Lemmas A.2 and 3.6). We show that if H is a 2-tree with diameter at least three, then $G_4(H)$ has a Hamilton cycle (Lemmas 6.3 and 6.4). To do so we describe a specific recursive procedure for constructing 2-trees of diameter at least three (Theorem 4.3).

2 Preliminaries

Definition 2.1. Let H be a graph, and let X and Y be disjoint subsets of $V(H)$. We denote by $[X, Y]$ the set of edges of H that have one end in X and the other end in Y .

Definition 2.2. Let H be a graph and L a function that assigns to each vertex $v \in V(H)$ a set of positive integers, $L(v)$, called the *list* of v . An L -colouring of H (also called a *list colouring* of H with respect to L) is a (proper) colouring $c: V(H) \rightarrow \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V(H)$. We define the L -colouring graph of H , denoted $G_L(H)$, to be the graph whose vertex set consists of all L -colourings of H ; two L -colourings are joined by an edge of $G_L(H)$ if they differ in colour on just one vertex of H .

Remark 2.3. If $k \geq \chi(H)$ and $L(v) := \{1, 2, \dots, k\}$ for each $v \in V(H)$, then $G_L(H) \cong G_k(H)$, the k -colouring graph of H .

We use $G \square H$ to denote the Cartesian product of graphs G and H , and Q_n to denote the n -dimensional hypercube, defined as the Cartesian product of n copies of K_2 .

Remark 2.4. Let H_1 and H_2 be vertex disjoint graphs and let L be an assignment of lists to the vertices of $H_1 \cup H_2$. Then

$$G_L(H_1 \cup H_2) = G_L(H_1) \square G_L(H_2).$$

Lemma 2.5. *Let H be a 2-tree with clique $X = \{x_1, x_2, \dots, x_\ell\}$ where $\ell \leq 3$, and let L be an assignment of lists to the vertices of H so that*

1. $|L(x_i)| = 1$ and $L(x_i) \subseteq \{1, 2, 3, 4\}$, $1 \leq i \leq \ell$;
2. $L(x_i) \neq L(x_j)$ for all $1 \leq i \neq j \leq \ell$;
3. $L(v) = \{1, 2, 3, 4\}$ for each $v \in V(H) \setminus X$.

Then $G_L(H)$ has a spanning tree T with $\Delta(T) \leq 4$.

Proof. The proof is by induction on $|V(H)|$. For the basis, $H \cong K_3$ with $V(H) := \{v_1, v_2, v_3\}$. When $\ell = 0$, $G_L(H) = G_4(K_3)$, which has a Hamilton cycle [6, Theorem 4.1], so take T to be a Hamilton path in $G_4(K_3)$. When $\ell = 1$, we may assume, without loss of generality, that $L(v_1) := \{1\}$ and $L(v_2) = L(v_3) := \{1, 2, 3, 4\}$. Then $G_L(H) \cong G_3(K_2)$, which has a Hamilton cycle [6, Theorem 4.1], so take T to be a Hamilton path in $G_3(K_2)$. When $\ell = 2$, we may assume, without loss of generality, that $L(v_1) := \{1\}$, $L(v_2) := \{2\}$, and $L(v_3) := \{1, 2, 3, 4\}$. Then $G_L(H) \cong K_2$, and the result holds. Finally, when $\ell = 3$, we may assume, without loss of generality, that $L(v_1) := \{1\}$, $L(v_2) := \{2\}$, and $L(v_3) := \{3\}$. Then $G_L(H) \cong K_1$, and the result holds.

Now suppose H is a 2-tree with $|V(H)| > 3$ and clique $X = \{x_1, \dots, x_\ell\}$. Since $\ell \leq 3$, there is a leaf $v \in V(H)$ with $v \notin X$, and thus $H - v$ is a 2-tree containing the clique X . It follows by induction that $G_L(H - v)$ has a spanning tree \mathcal{T} with $\Delta(\mathcal{T}) \leq 4$. Let $V(G_L(H - v)) := \{f_0, f_1, \dots, f_{N-1}\}$, and for each $f_j \in V(G_L(H - v))$, let $F_j \subseteq V(G_L(H))$ be the set of L -colourings of H that agree with f_j on $V(H - v)$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of $G_L(H)$. Since v is a leaf of H , there are two ways to extend an L -colouring of $H - v$ to an L -colouring of H , and hence for each j , $0 \leq j \leq N - 1$, $F_j = \{a_j, b_j\}$ is a clique.

For each edge $f_i f_j \in E(\mathcal{T})$, there is a unique vertex $w \in V(H - v)$ for which $f_i(w) \neq f_j(w)$. If $wv \notin E(H)$, then $[F_i, F_j]$ consists of two disjoint edges, and the subgraph of $G_L(H)$ induced by $F_i \cup F_j$ is a 4-cycle. Otherwise, $wv \in E(H)$, so $[F_i, F_j]$ has only one edge, and the subgraph of $G_L(H)$ induced by $F_i \cup F_j$ is a path of length three. In both cases, label the edge $f_i f_j$ in \mathcal{T} with $|[F_i, F_j]|$.

Let S denote the spanning subgraph of $G_L(H)$ corresponding to the spanning tree \mathcal{T} of $G_L(H - v)$ as described above, that is, S has edge set

$$\left(\bigcup_{f_i f_j \in E(\mathcal{T})} [F_i, F_j] \right) \cup \{a_j b_j \mid 0 \leq j \leq N - 1\}.$$

Since $[F_i, F_j]$ is nonempty for each $f_i f_j \in E(\mathcal{T})$, S is connected. Also, since $\Delta(\mathcal{T}) \leq 4$ and $[F_i, F_j]$ contains either one edge or two disjoint edges, $\Delta(S) \leq 5$. Furthermore, since there are only two vertices adjacent to v in H , at most two edges incident to f_i in $G_L(H - v)$ have label ‘1’.

Let S' be the graph obtained from S by deleting the edges $a_j b_j$ for each $f_j \in V(\mathcal{T})$ with $d_{\mathcal{T}}(f_j) = 4$. Then $\Delta(S') \leq 4$, since if $a_j \in V(S)$ has $d_S(a_j) = 5$, then $d_{\mathcal{T}}(f_j) = 4$, and thus $d_{S'}(a_j) = 4$. We also claim that S' is connected. To prove this, it suffices to show that there is a path in S' from a_p to b_p for each $f_p \in V(\mathcal{T})$ with $d_{\mathcal{T}}(f_p) = 4$. Suppose $f_p \in V(\mathcal{T})$ has $d_{\mathcal{T}}(f_p) = 4$. Construct a path, P , in \mathcal{T} starting at f_p , using edges labelled ‘2’, and whose final vertex f_q has $d_{\mathcal{T}}(f_q) < 4$. Such a path exists since \mathcal{T} has no cycles, and each degree four vertex in \mathcal{T} is incident to at least two edges labelled ‘2’. The union

$$\left(\bigcup_{f_i f_j \in E(P)} [F_i, F_j] \right) \cup \{a_q b_q\}$$

gives us a path in S' from a_p to b_p .

Therefore, S' is a connected spanning subgraph of $G_L(H)$, and thus S' has a spanning tree T that is also a spanning tree of $G_L(H)$. Since $\Delta(S') \leq 4$, $\Delta(T) \leq 4$, thus completing the proof of the lemma. \square

The result in Lemma 2.5 is best possible in that $\Delta(T)$ cannot be reduced from four to three, as illustrated in the following example. Let D denote the unique 2-tree of diameter three on six vertices, with vertices labelled as shown in Figure 1(a).

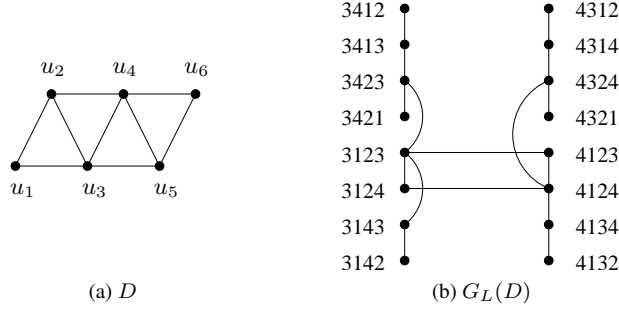


Figure 1: The 2-tree D and $G_L(D)$.

Let $L: V(D) \rightarrow \{1, 2, 3, 4\}$ be defined as follows.

$$\begin{aligned} L(u_1) &:= \{1\}, \\ L(u_2) &:= \{2\}, \text{ and} \\ L(u_i) &:= \{1, 2, 3, 4\} \text{ for } 3 \leq i \leq 6. \end{aligned}$$

If f is an L -colouring of D , then $f(u_1) = 1$ and $f(u_2) = 2$, and thus we may denote the vertices of $G_L(D)$ by strings $ijkl$ where $f(u_3) = i$, $f(u_4) = j$, $f(u_5) = k$ and $f(u_6) = \ell$. Using this convention, $G_L(D)$ is depicted in Figure 1(b). Notice that $G_L(D)$ is unicyclic, and has exactly two nonadjacent vertices of degree four, ‘3123’ and ‘4124’. Thus every spanning tree of $G_L(D)$ has a vertex of degree four.

As part of their proof [6, Theorem 5.5], Choo and MacGillivray prove the following.

Remark 2.6. Let G be a graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$, such that

- (i) $G[F_i]$ is a 4-cycle for each i , $0 \leq i \leq N - 1$;
- (ii) $G[F_{i-1} \cup F_i]$ is isomorphic to either $P_4 \square K_2$ or Q_3 for each i , $1 \leq i \leq N - 1$;
- (iii) if $G[F_{i-1} \cup F_i]$ and $G[F_i \cup F_{i+1}]$ are both isomorphic to $P_4 \square K_2$, then $G[F_{i-1} \cup F_i \cup F_{i+1}]$ is not isomorphic to the graph in Figure 2.

Then G has a Hamilton cycle.

The conditions imply that $[F_{i-1}, F_i] \neq \emptyset$, $1 \leq i \leq N - 1$, and hence there is a function, h , from a spanning subgraph of G to a path $f_0 f_1 \dots f_{N-1}$ of length $N - 1$ defined by $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N - 1$.

In our next lemma, we adapt the result of Choo and MacGillivray to a more general scenario. Suppose G is a graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$ where $G[F_i]$

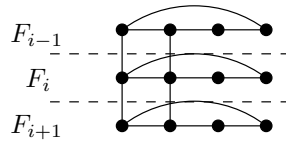


Figure 2: Forbidden subgraph.

contains a spanning cycle for each i , $0 \leq i \leq N-1$ (these cycles form a 2-factor of G). Further, assume that there is a function, h , from a spanning subgraph of G to a tree with vertex set $\{f_0, f_1, \dots, f_{N-1}\}$ such that $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N-1$. The general idea is to choose, for each edge $f_i f_j$ of the spanning tree, appropriate edges from the set $[F_i, F_j]$ of G so that we are able to construct a Hamilton cycle from among these edges and edges of the 2-factor. See Figure 3 for an illustration of this result.

Lemma 2.7. *Let G be a graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$, and let T be a tree with $V(T) := \{f_0, f_1, \dots, f_{N-1}\}$. Suppose there is a function, h , from a spanning subgraph of G to T such that $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N-1$. Furthermore, suppose that for each $f_i f_j \in E(T)$, $0 \leq i, j \leq N-1$, there exist edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ such that*

- (i) if $j \neq k$ and $f_i f_j, f_i f_k \in E(T)$, then $e_{i,j} \neq e_{i,k}$;
- (ii) if $e_{i,j} = ac$ and $e_{j,i} = bd$, then $G[\{a, b, c, d\}]$ contains a 4-cycle;
- (iii) $G[F_i]$ has a Hamilton cycle C_i such that

$$M_i := \{e_{i,j} \mid f_i f_j \in E(T)\} \subseteq E(C_i).$$

Then G has a Hamilton cycle C such that

$$\bigcup_{i=0}^{N-1} (E(C_i) \setminus M_i) \subseteq E(C).$$

Proof. The proof is by induction on N . The result is trivial when $N = 1$.

Let $N > 1$. For each $f_i f_j \in E(T)$, $0 \leq i \leq N-1$, suppose $e_{i,j} \in E(G[F_i])$, $e_{j,i} \in E(G[F_j])$, and C_i (a Hamilton cycle in $G[F_i]$) satisfy conditions (i), (ii) and (iii). Without loss of generality, we may assume that f_{N-1} is a leaf of T , and that $f_{N-1} f_{N-2} \in E(T)$. Let $G' := G - F_{N-1}$ and $T' := T - f_{N-1}$. Using $e_{i,j}$, $e_{j,i}$ and C_i as previously defined, $0 \leq i \leq N-2$, and M_i as defined in (iii) except with M_{N-2} replaced by $M'_{N-2} := M_{N-2} \setminus \{e_{N-2, N-1}\}$, we apply the induction hypothesis to G' . The result is a Hamilton cycle C' in G' such that

$$\left(\bigcup_{i=0}^{N-3} (E(C_i) \setminus M_i) \right) \cup (E(C_{N-2}) \setminus M'_{N-2}) \subseteq E(C'),$$

and $e_{N-2, N-1} \in E(C')$. Let $e_{N-2, N-1} := ac$, $e_{N-1, N-2} := bd$; without loss of generality, $abdca$ is a 4-cycle in $G[\{a, b, c, d\}]$, and hence

$$C := (C' - e_{N-2, N-1}) \cup (C_{N-1} - e_{N-1, N-2}) + \{ab, cd\}$$

is a Hamilton cycle in G with the required property. □

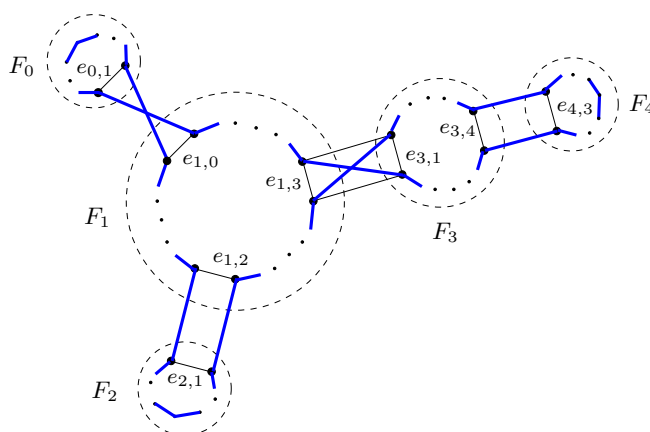


Figure 3: Illustration of Lemma 2.7.

We remark that if $d_T(f_i) = 1$ for some i , then the Hamilton cycle C constructed in Lemma 2.7 contains all except one edge of $E(C_i)$.

3 2-trees of diameter two

In this section we characterize 2-trees H of diameter two in which $G_4(H)$ has a Hamilton cycle. We begin by defining a class of 2-trees that we denote by $T(p, q, r)$ (see Figure 4).

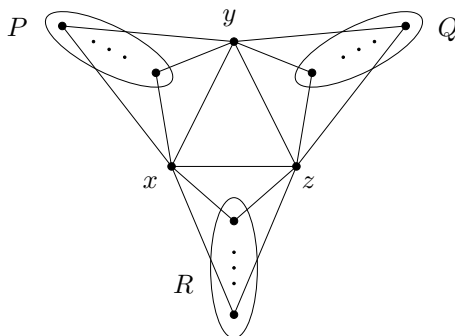


Figure 4: The 2-tree $T(p, q, r)$.

Definition 3.1. Let P , Q , and R be pairwise disjoint independent sets of vertices with $|P| := p$, $|Q| := q$, and $|R| := r$. The graph $T(p, q, r)$ is the graph on $p + q + r + 3$ vertices defined on vertex set $\{x, y, z\} \cup P \cup Q \cup R$ where the subgraph induced by $P \cup Q \cup R$ is an independent set, and

- the subgraph induced by $\{x, y\} \cup P$ is isomorphic to $K_{1,1,p}$;
- the subgraph induced by $\{y, z\} \cup Q$ is isomorphic to $K_{1,1,q}$;
- the subgraph induced by $\{z, x\} \cup R$ is isomorphic to $K_{1,1,r}$.

A *dominating vertex* in a graph is a vertex adjacent to all other vertices of the graph.

Lemma 3.2. *A graph H is a 2-tree of diameter two if and only if H has a dominating vertex or $H \cong T(p, q, r)$ for $p, q, r > 0$.*

Proof. Any 2-tree with a dominating vertex has diameter two, and one can easily verify that $T(p, q, r)$ is a 2-tree of diameter two for any $p, q, r > 0$.

For the converse, suppose H is a 2-tree of diameter two. We proceed by induction on $n := |V(H)|$. When $n = 4$, H is isomorphic to the graph obtained from K_4 by deleting one edge, and has a dominating vertex. Now suppose that $n \geq 5$, and let $u \in V(H)$ be a leaf of H , i.e., $d_H(u) = 2$. By the induction hypothesis, $H - u$ has a dominating vertex, or $H - u \cong T(p', q', r')$ for some $p', q', r' > 0$. If $H - u \cong T(p', q', r')$ for some $p', q', r' > 0$, then let $V(H - u) := \{x, y, z\} \cup P' \cup Q' \cup R'$, where $|P'| := p'$, $|Q'| := q'$, and $|R'| := r'$. Since H has diameter two, u must be adjacent to at least two vertices from $\{x, y, z\}$. However $d_H(u) = 2$, and thus $N_H(u) = \{x, y\}$, $N_H(u) = \{y, z\}$, or $N_H(u) = \{z, x\}$. It follows that $H \cong T(p' + 1, q', r')$, $H \cong T(p', q' + 1, r')$, or $H \cong T(p', q', r' + 1)$, respectively.

Now suppose $H - u$ has a dominating vertex, x . If $ux \in E(H)$, then x is a dominating vertex in H . Otherwise, let y and z denote the neighbours of u in H , and note that $yz \in E(H)$. Since H has diameter two, every vertex in $V(H - u)$ is adjacent to x and at least one of y or z . Let P' be the set of vertices in $H - u$ adjacent to both x and y , R' be the set of vertices in $H - u$ adjacent to both x and z , and suppose $|P'| := p'$ and $|R'| := r'$. Since $H - u$ is a 2-tree, $P' \cup R'$ is an independent set. Therefore, $P' \cup R' \cup \{u\}$ is an independent set in H , and thus $H \cong T(p', 1, r')$. \square

In what follows, we first prove that if $p, q, r > 0$, then $G_4(T(p, q, r))$ has a Hamilton cycle. Let G be a graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$. For each i , $1 \leq i \leq N - 1$, let $S_{i-1} \subseteq F_{i-1}$ and $S'_i \subseteq F_i$ denote the vertices incident to the edges of $[F_{i-1}, F_i]$.

Lemma 3.3. *If $p, q, r > 0$, then $G_4(T(p, q, r))$ has a Hamilton cycle.*

Proof. Let $V(K_3) := \{x, y, z\}$. Suppose $f: V(K_3) \rightarrow \{1, 2, 3, 4\}$ is a 4-colouring of K_3 and $V(G_4(K_3)) := \{f_0, f_1, \dots, f_{N-1}\}$. Since $G_4(K_3)$ has a Hamilton cycle by [6, Theorem 4.1], we may assume that $f_0 f_1 \dots f_{N-1}$ is a Hamilton path in $G_4(K_3)$.

For $0 \leq i \leq N - 1$, let F_i be the set of 4-colourings of $T(p, q, r)$ that agree with f_i on $\{x, y, z\}$. In order to simplify notation, we define $G := G_4(T(p, q, r))$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of G , and $G[F_i] \cong Q_{p+q+r}$, $0 \leq i \leq N - 1$.

Let $st \in [F_{i-1}, F_i]$, where $s \in S_{i-1}$ and $t \in S'_i$ for some $1 \leq i \leq N - 1$. Then $s(u) = t(u)$ for all $u \in V(G) \setminus \{v\}$, where v is one of $\{x, y, z\}$, and $s(v) \neq t(v)$. Thus, $[F_{i-1}, F_i]$ is a set of independent edges. If $v = x$, then $s(u) = s(w)$ for all $u, w \in P$ and for all $u, w \in R$, and $t(u) = t(w)$ for all $u, w \in P$ and for all $u, w \in R$. Thus $G[S_{i-1}] \cong Q_q \cong G[S'_i]$, and $G[S_{i-1} \cup S'_i] \cong Q_{q+1}$. Analogously, if $v = y$, $G[S_{i-1}] \cong Q_r \cong G[S'_i]$ and $G[S_{i-1} \cup S'_i] \cong Q_{r+1}$; and if $v = z$, $G[S_{i-1}] \cong Q_p \cong G[S'_i]$, $G[S_{i-1} \cup S'_i] \cong Q_{p+1}$.

Consider the path $f_{i-1} f_i f_{i+1}$ in $G_4(K_3)$, $1 \leq i \leq N - 2$. Note that

$$f_{i-1}(x), f_i(x), f_{i+1}(x)$$

use at most two different colours; otherwise, there would be only one colour available for y and z , which is impossible since y and z always receive different colours. Analogously, f_{i-1}, f_i, f_{i+1} assign at most two different colours to each of y and z . It follows that if $|[F_{i-1}, F_i]| = |[F_i, F_{i+1}]| = 2$, then $[F_{i-1}, F_i] \cup [F_i, F_{i+1}] \not\cong 2P_3$.

For each edge $f_{i-1}f_i$ with $1 \leq i \leq N-1$, we choose a 4-cycle, B_{i-1} , containing exactly two edges of $[F_{i-1}, F_i]$ as follows.

- For each $|[F_{i-1}, F_i]| = 2$, $G[S_{i-1} \cup S'_i]$ is a 2-cube, so we choose $B_{i-1} := G[S_{i-1} \cup S'_i] = a_{i-1}c_{i-1}d_i b_i a_{i-1}$, where $a_{i-1}, c_{i-1} \in F_{i-1}$ and $b_i, d_i \in F_i$.
- For $i = 1, 2, \dots, N-1$, and $|[F_{i-1}, F_i]| > 2$, first note that $|[F_{i-1}, F_i]| \geq 4$ since $G[S_{i-1} \cup S'_i] \cong Q_n$ for some $n \geq 3$. Thus it is possible to choose edges $a_{i-1}b_i$ and $c_{i-1}d_i$ of $[F_{i-1}, F_i]$ so that $B_{i-1} := a_{i-1}c_{i-1}d_i b_i a_{i-1}$ is edge disjoint from B_{i-2} , and also edge disjoint from B_i if $|[F_i, F_{i+1}]| = 2$ and $i < N-1$.

Let $e_{i,i+1} := G[S_i] \cap B_i$ and $e_{i+1,i} := G[S'_{i+1}] \cap B_i$, $0 \leq i \leq N-2$. Observe that G and the path $f_0 f_1 \dots f_{N-1}$ satisfy conditions (i) and (ii) of Lemma 2.7. Recall that $G[F_i] \cong Q_{p+q+r}$, $0 \leq i \leq N-1$, and $p+q+r \geq 3$. Since any pair of edges of Q_n , $n \geq 2$, is contained in a Hamilton cycle (see [7, Theorem 4.1]), there exist Hamilton cycles C_0 in $G[F_0]$ containing $e_{0,1}$, C_{N-1} in $G[F_{N-1}]$ containing $e_{N-1,N-2}$, and, for $1 \leq i \leq N-2$, C_i in $G[F_i]$ containing $e_{i,i-1}$ and $e_{i,i+1}$. Therefore, by Lemma 2.7, G has a Hamilton cycle. \square

In the case where $p > 0$ and $q = r = 0$, $T(p, q, r)$ has a dominating vertex, and is isomorphic to $K_2 \vee \overline{K}_n$.

Lemma 3.4. *For $n \geq 2$, $G_4(K_2 \vee \overline{K}_n)$ has no Hamilton cycle.*

Proof. Let $H := K_2 \vee \overline{K}_n$ for $n \geq 2$, let $\mathcal{H} := G_4(H)$, and let u, v be the two vertices of H of degree $n+1$. For each $1 \leq i \neq j \leq 4$ let

$$V_{ij} := \{c \in V(\mathcal{H}) \mid c(u) = i \text{ and } c(v) = j\}.$$

Then

$$\{V_{12}, V_{13}, V_{14}, V_{21}, V_{23}, V_{24}, V_{31}, V_{32}, V_{34}, V_{41}, V_{42}, V_{43}\}$$

is a partition of $V(\mathcal{H})$. Note that $[V_{\alpha\beta}, V_{\gamma\delta}] \neq \emptyset$ if and only if $\alpha = \gamma$ or $\beta = \delta$. For $1 \leq i \neq j \leq 4$, let L_{ij} be an assignment of lists to the vertices of H such that $L_{ij}(u) := \{i\}$, $L_{ij}(v) := \{j\}$ and $L_{ij}(x) := \{1, 2, 3, 4\}$ for $x \in V(H - \{u, v\})$. Note that $G_{L_{ij}}(H) \cong \mathcal{H}[V_{ij}] \cong Q_n$, for each $1 \leq i \neq j \leq 4$. Define the three-coloured vertices of \mathcal{H} (that is, the colourings of H with three colours) by $c_{ijk} \in V_{ij}$ such that

$$c_{ijk}(x) := \begin{cases} i, & \text{if } x = u, \\ j, & \text{if } x = v, \\ k, & \text{otherwise.} \end{cases}$$

Each V_{ij} has two such vertices, c_{ijk_1} and c_{ijk_2} , where $k_1, k_2 \in \{1, 2, 3, 4\} \setminus \{i, j\}$. Furthermore, $\mathcal{H} - \{c_{ijk_1}, c_{ijk_2}\}$ is disconnected so that any Hamilton cycle in \mathcal{H} must contain the edges of a Hamilton path of $\mathcal{H}[V_{ij}]$ with endpoints c_{ijk_1} and c_{ijk_2} , for each $1 \leq i \neq j \leq 4$.

By [6, Lemma 2.1] there is no Hamilton path in the n -dimensional cube from $00 \dots 0$ to $11 \dots 1$ whenever n is even. Thus, for n even, there is no Hamilton path in $\mathcal{H}[V_{ij}]$ with endpoints c_{ijk_1} and c_{ijk_2} . Therefore, \mathcal{H} has no Hamilton cycle when n is even.

For n odd, such Hamilton paths exist and must be used in any Hamilton cycle of \mathcal{H} , if one exists. We construct an auxiliary graph A (see Figure 5(a)) where the vertex (i, j)

represents a Hamilton path in $\mathcal{H}[V_{ij}]$ from c_{ijk_1} to c_{ijk_2} . There is an edge in A between (i_1, j_1) and (i_2, j_2) whenever $c_{i_1 j_1 k}$ is adjacent to $c_{i_2 j_2 k}$ in \mathcal{H} . We label the edge between (i_1, j_1) and (i_2, j_2) by the unique element of $\{1, 2, 3, 4\} \setminus (\{i_1, j_1\} \cup \{i_2, j_2\})$ (see Figure 5(b)). Notice that the edges labelled i , $1 \leq i \leq 4$, induce a 6-cycle, and the edges of these four 6-cycles partition $E(A)$.

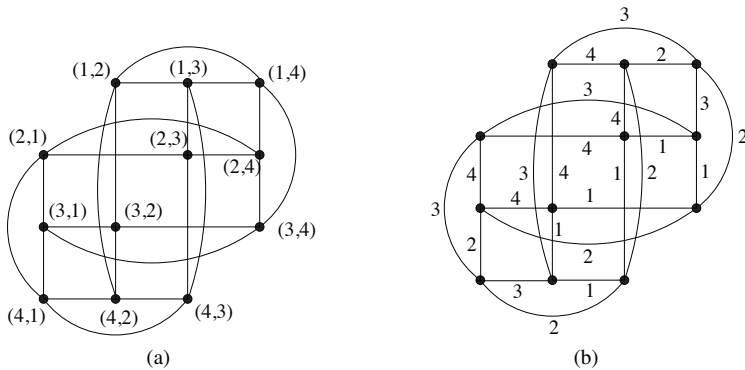


Figure 5: The auxiliary graph A .

A Hamilton cycle in \mathcal{H} corresponds to a Hamilton cycle, \mathcal{C} , in A in which any two consecutive edges of \mathcal{C} have different labels. Such a cycle \mathcal{C} uses a matching of size three from each labelled 6-cycle in A . Without loss of generality, we may assume \mathcal{C} contains horizontal edges of the 6-cycle labelled 1. Now, \mathcal{C} uses horizontal edges from one of the remaining labelled 6-cycles, otherwise, \mathcal{C} contains a K_3 . Regardless of whether \mathcal{C} uses horizontal edges of the 6-cycle labelled by 2, 3 or 4, using vertical edges of the two remaining 6-cycles gives $\mathcal{C} \cong C_4 \cup C_8$, a contradiction. Therefore \mathcal{H} has no Hamilton cycle. \square

Observe that if H is a 2-tree of diameter two having a dominating vertex u , then $H \cong T \vee \{u\}$ for some tree T .

Lemma 3.5. *Let T be a tree on at least two vertices. Then $G_4(T \vee \{u\})$ has a Hamilton cycle unless T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts.*

The proof of this lemma requires a result that we state here, but whose proof is technical and is postponed to the Appendix A.

Lemma 3.6. *Let T be a tree with bipartition (A, B) , where $|A| := \ell$ and $|B| := r$, and let $G_3(T)$ be the 3-colouring graph of T with colours $C = \{1, 2, 3\}$. Define c_{ij} to be the vertex of $G_3(T)$ with $c_{ij}(a) = i$ for all $a \in A$ and $c_{ij}(b) = j$ for all $b \in B$.*

- (1) *If $\ell, r > 0$ are both even, then $G_3(T)$ has no spanning subgraph consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$.*
- (2) *If $\ell > 1$ is odd and $r > 0$ is even, then $G_3(T)$ has a Hamilton path from c_{12} to c_{23} .*
- (3) *If $\ell > 1$ and $r > 1$ are both odd, then $G_3(T)$ has a Hamilton path from c_{12} to c_{13} .*

Proof of Lemma 3.5. Let T be a tree with $|V(T)| \geq 2$ and bipartition (A, B) , where $A := \{x_1, x_2, \dots, x_\ell\}$ and $B := \{y_1, y_2, \dots, y_r\}$. Let $H := T \vee \{u\}$, and let $\mathcal{H} := G_4(H)$ be the 4-colouring graph of H with colours $\{1, 2, 3, 4\}$. For each $1 \leq k \leq 4$, define

$$V_k := \{c \in V(\mathcal{H}) \mid c(u) = k\}.$$

Then $\{V_1, V_2, V_3, V_4\}$ is a partition of $V(\mathcal{H})$. Define L_k to be an assignment of lists with $L_k(u) := \{k\}$ and $L_k(w) := \{1, 2, 3, 4\}$ for $w \in V(T)$. Note that $G_{L_k}(H) \cong \mathcal{H}[V_k] \cong G_3(T)$. Define the three-coloured vertices of \mathcal{H} (that is, the colourings of H with three colours) by $c_{ijk} \in V_k$ so that $c_{ijk}(x) := i$ for all $x \in A$, $c_{ijk}(y) := j$ for all $y \in B$.

Observe

$$\begin{aligned} [V_1, V_2] &= \{c_{341}c_{342}, c_{431}c_{432}\}, & [V_1, V_3] &= \{c_{241}c_{243}, c_{421}c_{423}\}, \\ [V_1, V_4] &= \{c_{231}c_{234}, c_{321}c_{324}\}, & [V_2, V_3] &= \{c_{142}c_{143}, c_{412}c_{413}\}, \\ [V_2, V_4] &= \{c_{132}c_{134}, c_{312}c_{314}\}, & [V_3, V_4] &= \{c_{123}c_{124}, c_{213}c_{214}\}. \end{aligned}$$

Case 1. Suppose $|V(T)| = 2$. Then $T \vee \{u\} \cong K_3$ and $G_4(K_3)$ has a Hamilton cycle by [6, Theorem 4.1].

Case 2. Suppose T is a star with $|V(T)| \geq 3$. Then $G_4(T \vee \{u\}) \cong K_2 \vee \bar{K}_n$ for some $n \geq 2$. By Lemma 3.4, $G_4(T \vee \{u\})$ has no Hamilton cycle.

Case 3. Suppose both $|A|$ and $|B|$ are odd. By Lemma 3.6, there is a Hamilton path in $\mathcal{H}[V_1]$ from c_{421} to c_{431} , in $\mathcal{H}[V_2]$ from c_{432} to c_{132} , in $\mathcal{H}[V_3]$ from c_{134} to c_{124} and in $\mathcal{H}[V_4]$ from c_{123} to c_{423} . The union of these paths with edges $\{c_{431}c_{432}, c_{132}c_{134}, c_{124}c_{123}, c_{423}c_{421}\}$ gives a Hamilton cycle in \mathcal{H} .

Case 4. Suppose one of $|A|$ and $|B|$ is even and the other is odd. Without loss of generality, $|A|$ is odd and $|B|$ is even. By Lemma 3.6, there is a Hamilton path in $\mathcal{H}[V_1]$ from c_{231} to c_{341} , in $\mathcal{H}[V_2]$ from c_{342} to c_{412} , in $\mathcal{H}[V_3]$ from c_{413} to c_{123} and in $\mathcal{H}[V_4]$ from c_{124} to c_{234} . The union of these paths with edges $\{c_{341}c_{342}, c_{412}c_{413}, c_{123}c_{124}, c_{234}c_{231}\}$ gives a Hamilton cycle in \mathcal{H} .

Case 5. Suppose both $|A|$ and $|B|$ are even, and suppose \mathcal{H} contains a Hamilton cycle \mathcal{C} . Then $\mathcal{C}[V_1]$ is a spanning subgraph of $\mathcal{H}[V_1] \cong G_3(T)$ consisting of a union of paths whose endpoints are three-coloured vertices of V_1 , contradicting Lemma 3.6. Thus in this case, \mathcal{H} has no Hamilton cycle. \square

4 Constructing 2-trees of diameter at least three

To complete the proof of our main result, we must show that if H' is a 2-tree with diameter at least three, then $k_0(H') = 4$. Naively deleting two leaves with the intention of applying Remark 2.6 may be problematic. For example, let H' be a 2-tree with diameter at least three having leaves x and y of distance at least three. Let $H = H' - \{x, y\}$, $N_{H'}(x) = \{x_1, x_2\}$, and suppose $G_4(H)$ has a Hamilton cycle $f_0 f_1 \cdots f_{N-1} f_0$. Let F_i be the set of 4-colourings of H' that agree with f_i on $V(H)$, $0 \leq i \leq N-1$. In the case that $f_{i-1} f_i$ arises from a colour change on x_1 and $f_i f_{i+1}$ arises from a colour change on x_2 , the subgraph $G[F_{i-1} \cup F_i \cup F_{i+1}]$ is isomorphic to the forbidden subgraph in Figure 2. Because of this we take a more general approach. Suppose a 2-tree H' is obtained from a 2-tree H by repeatedly adding vertices of degree two. Instead of a Hamilton cycle in $G_4(H)$ we take

a spanning tree satisfying certain properties providing the flexibility needed to construct a Hamilton cycle in $G_4(H')$. Our approach does not require $G_4(H)$ to have a Hamilton cycle. To facilitate this procedure, we define nine operations (see Figure 6).

Definition 4.1. Let H be a 2-tree. Then H' is obtained from H by

Operation I if $\{\alpha, \beta, \gamma\} := V(H') \setminus V(H)$, $\{a_1a_2, b_1b_2, c_1c_2\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{a_1, a_2\}$, $N_{H'}(\beta) = \{b_1, b_2\}$, $N_{H'}(\gamma) = \{c_1, c_2\}$.

Operation II if $\{\alpha, \beta, \gamma\} := V(H') \setminus V(H)$, $\{xa, xb, c_1c_2\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{x, a\}$, $N_{H'}(\beta) = \{x, b\}$, $N_{H'}(\gamma) = \{c_1, c_2\}$.

Operation III if $\{\alpha, \beta, \gamma\} := V(H') \setminus V(H)$, $\{ax, xy, yc\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{a, x\}$, $N_{H'}(\beta) = \{x, y\}$, $N_{H'}(\gamma) = \{c, y\}$.

Operation IV if $\{\alpha, \beta, \gamma, \delta\} := V(H') \setminus V(H)$, $\{xy, zw\} \subseteq E(H)$, and $N_{H'}(\alpha) = N_{H'}(\beta) = \{x, y\}$, $N_{H'}(\gamma) = N_{H'}(\delta) = \{w, z\}$.

Operation V if $\{\alpha, \beta, \gamma, \delta\} := V(H') \setminus V(H)$, $\{xy, zw\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta, x, y\}$, $N_{H'}(\beta) = \{\alpha, y\}$, $N_{H'}(\gamma) = N_{H'}(\delta) = \{w, z\}$.

Operation VI if $\{\alpha, \beta, \gamma, \delta\} := V(H') \setminus V(H)$, $\{xy, zw\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta, x, y\}$, $N_{H'}(\beta) = \{\alpha, y\}$, $N_{H'}(\gamma) = \{\delta, w, z\}$, $N_{H'}(\delta) = \{\gamma, z\}$.

Operation VII if $\{\alpha, \beta, \gamma, \delta\} := V(H') \setminus V(H)$, $\{xy, xz\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta, x, y\}$, $N_{H'}(\beta) = \{\alpha, y\}$, $N_{H'}(\gamma) = \{\delta, x, z\}$, $N_{H'}(\delta) = \{\gamma, z\}$.

Operation VIII if $\{\alpha, \beta, \gamma, \delta\} := V(H') \setminus V(H)$, $\{xy, xz\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta, x, y\}$, $N_{H'}(\beta) = \{x, \alpha\}$, $N_{H'}(\gamma) = \{\delta, x, z\}$, $N_{H'}(\delta) = \{\gamma, z\}$.

Operation IX if $\{\alpha, \beta, \gamma, \delta\} := V(H') \setminus V(H)$, $\{xy, xz\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta, x, y\}$, $N_{H'}(\beta) = \{\alpha, y\}$, $N_{H'}(\gamma) = N_{H'}(\delta) = \{x, z\}$.

Remark 4.2. Since each of Operations I through IX can be performed by sequentially adding simplicial vertices of degree two to H , H' is a 2-tree.

Recall that D denotes the unique 2-tree of diameter three on six vertices (Figure 1(a)).

Theorem 4.3. A graph H' is a 2-tree of diameter at least three if and only if $H' \cong D$ or H' can be obtained from a 2-tree H by applying one of Operations I through IX.

Proof. (\Leftarrow): If $H' \cong D$, the result is trivially true. Otherwise, it follows from Remark 4.2 that H' is a 2-tree. Since each operation produces two leaves that are distance at least three apart, H' has diameter at least three.

(\Rightarrow): As already observed, D is the unique 2-tree of diameter three on six vertices. The 2-trees on three, four and five vertices all have diameter less than three. Thus, we may assume that H' is a 2-tree of diameter three and $|V(H')| \geq 7$. Since H' has diameter at least three, there are at least two leaves whose neighbourhoods are vertex disjoint. We consider cases based on the number of edges induced by the neighbourhoods of the leaves of H' , and the number of leaves of H' .

Case 1. First assume that the neighbourhoods of the leaves of H' induce at least three distinct edges.

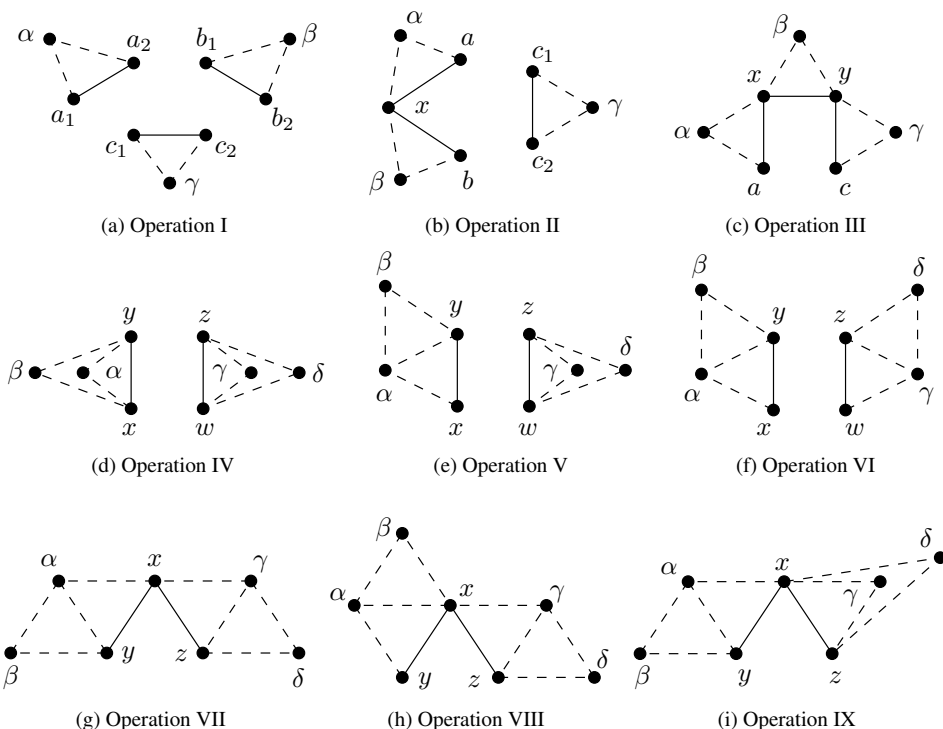


Figure 6: Operations I through IX applied to a graph H by the addition of vertices α, β, γ , and δ where applicable. The solid edges belong to H and the dotted edges are added to construct H' .

If H' has three leaves whose neighbourhoods are pairwise vertex disjoint, then H' contains vertex disjoint edges a_1a_2, b_1b_2, c_1c_2 , and leaves α, β, γ with $N(\alpha) = \{a_1, a_2\}$, $N(\beta) = \{b_1, b_2\}$, $N(\gamma) = \{c_1, c_2\}$. Letting $H := H' - \{\alpha, \beta, \gamma\}$ results in a 2-tree, and applying Operation I to H produces H' .

We may now assume that no three leaves of H' have neighbourhoods that are pairwise vertex disjoint. Choose two leaves α and γ whose neighbourhoods are vertex disjoint, and let $\beta \notin \{\alpha, \gamma\}$ be a leaf such that $N(\beta) \notin \{N(\alpha), N(\gamma)\}$. If $N(\beta)$ intersects exactly one of $N(\alpha)$ and $N(\gamma)$, then we may assume without loss of generality that $|N(\beta) \cap N(\alpha)| = 1$ and $N(\beta) \cap N(\gamma) = \emptyset$. Then H' contains a path of length two, axb and an edge c_1c_2 that is vertex disjoint from axb , such that $N(\alpha) = \{a, x\}$, $N(\beta) = \{x, b\}$, and $N(\gamma) = \{c_1, c_2\}$. Letting $H := H' - \{\alpha, \beta, \gamma\}$ results in a 2-tree, and applying Operation II to H produces H' . Otherwise, $|N(\beta) \cap N(\alpha)| = |N(\beta) \cap N(\gamma)| = 1$, and H' contains a path of length three, c_1xyzc_2 such that $N(\alpha) = \{c_1, x\}$, $N(\beta) = \{x, y\}$ and $N(\gamma) = \{y, c_2\}$. Letting $H := H' - \{\alpha, \beta, \gamma\}$ results in a 2-tree, and applying Operation III to H produces H' .

Case 2. We may now assume that the neighbourhoods of the leaves of H' induce exactly two edges.

Case 2(a). Suppose that H' has at least three leaves. Let leaves β and γ have neighbourhoods that are vertex disjoint, and let $\delta \notin \{\beta, \gamma\}$ be a leaf. Without loss of generality, $N(\delta) = N(\gamma)$.

If there exists a leaf α with $N(\alpha) = N(\beta)$, then $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation IV to H produces H' . Otherwise, no other leaf of H' has the same neighbourhood as β . If we let $N(\beta) := \{\alpha, y\}$, then at least one of $\{\alpha, y\}$ is a leaf of $H' - \beta$; without loss of generality, α is a leaf of $H' - \beta$, and so $d(\alpha) = 3$. It follows that $N(\alpha) = \{\beta, y, x\}$ for some $x \notin \{\alpha, \beta, \gamma, \delta, y\}$, and that $xy \in E(H')$.

Let $N(\gamma) = N(\delta) = \{w, z\}$. We consider two cases depending on $|\{x\} \cap \{w, z\}|$. If $|\{x\} \cap \{w, z\}| = 0$, then $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation V to H produces H' . If $|\{x\} \cap \{w, z\}| = 1$, then $y = w$ or $y = z$, and the two cases are analogous. The graph $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation IX to H produces H' .

Case 2(b). Finally, assume that H' has exactly two leaves, β and δ , with $N(\beta) = \{\alpha, y\}$ and $N(\delta) = \{\gamma, z\}$. Since $d_{H'}(\beta, \delta) \geq 3$, $\{\alpha, y\} \cap \{\gamma, z\} = \emptyset$. We may assume, without loss of generality, that α and γ are leaves in $H' - \{\beta, \delta\}$, and so $d(\alpha) = 3$ and $d(\gamma) = 3$. It follows that $N(\alpha) = \{\beta, y, p\}$ and $N(\gamma) = \{\delta, z, q\}$ for some $p, q \notin \{\beta, \alpha, \delta, \gamma\}$ with $p \neq y, q \neq z$ and $py, qz \in E(H')$.

We consider three cases depending on $|\{p, y\} \cap \{q, z\}|$. If $|\{p, y\} \cap \{q, z\}| = 0$, then $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation VI to H produces H' . If $|\{p, y\} \cap \{q, z\}| = 1$, then either $p = q, p = z$, or $q = y$. In the case $p = q$, $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation VII to H produces H' . In the case $p = z$, $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation VIII to H produces H' . The case $q = y$ is analogous to the case $p = z$. Finally if $|\{p, y\} \cap \{q, z\}| = 2$, then the fact that $y \neq z$ implies that $p = z$ and $q = y$, and hence $H' \cong D$, contradicting the fact that $|V(H')| \geq 7$. \square

5 Operations and the 4-colouring graph

Let H be a 2-tree, and let H' be the 2-tree obtained from H by applying one of the Operations I through IX. As before, let $V(G_4(H)) := \{f_0, f_1, \dots, f_{N-1}\}$, and let $F_j \subseteq V(G_4(H'))$ be the set of 4-colourings of H' that agree with f_j on the vertices of H , $0 \leq j \leq N-1$. For each Operation I through IX, what follows is a description of the subgraph of $G := G_4(H')$ induced by F_i , $0 \leq i \leq N-1$, and also a description of the subgraph of G induced by $F_i \cup F_j$ when $f_i f_j \in E(G_4(H))$, $0 \leq i, j \leq N-1$. Each edge $f_i f_j$ of $G_4(H)$ is also assigned a label to indicate the structure of $G[F_i \cup F_j]$. We remark that for a path $f_i f_j f_k$ of length two in $G_4(H)$, if $f_i(u) \neq f_j(u)$ for some $u \in V(H)$, then $f_j(u) = f_k(u)$.

5.1 Operation I

If H' is obtained from H using Operation I, then there are two choices of colour for each of the vertices α, β , and γ , so for each i , $0 \leq i \leq N-1$, $G[F_i] \cong Q_3$. To simplify the labelling of the vertices of $G[F_i]$, we label the faces of a plane drawing of Q_3 as shown in Figure 7(a), where α_1 and α_2 are the possible colours of vertex α , β_1 and β_2 are the possible colours of vertex β , and γ_1 and γ_2 are the possible colours of vertex γ . Without loss of generality, assume these colour choices are as shown in Figure 7(b). A vertex u of

Q_3 is assigned label $\alpha_i\beta_j\gamma_k$ where $i, j, k \in \{1, 2\}$, and u is incident with the faces labelled α_i , β_j and γ_k (see Figure 7(c)).

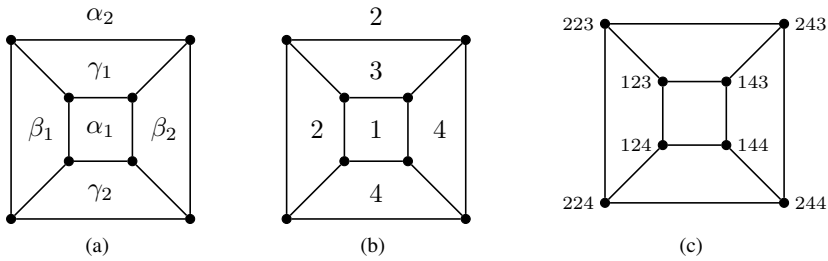


Figure 7: Labelling $G[F_i]$.

The following arguments can be made with sets of colours $\{f_i(a_1), f_i(a_2)\}$, $\{f_i(b_1), f_i(b_2)\}$ and $\{f_i(c_1), f_i(c_2)\}$ each chosen independently as a subset of $\{1, 2, 3, 4\}$. To ease notation, we assume that $f_i(a_1) = 3$, $f_i(a_2) = 4$, $f_i(b_1) = 1$, $f_i(b_2) = 3$, $f_i(c_1) = 1$, and $f_i(c_2) = 2$. Then the colour choices for α are $\{1, 2\}$, for β are $\{2, 4\}$, and for γ are $\{3, 4\}$. As already noted, $G[F_i] \cong Q_3$; assume that $G[F_i]$ has been drawn in the plane and labelled as in Figure 8(a).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. We label each edge $f_i f_j \in E(G_4(H))$ according to the structure of $G[F_i \cup F_j]$.

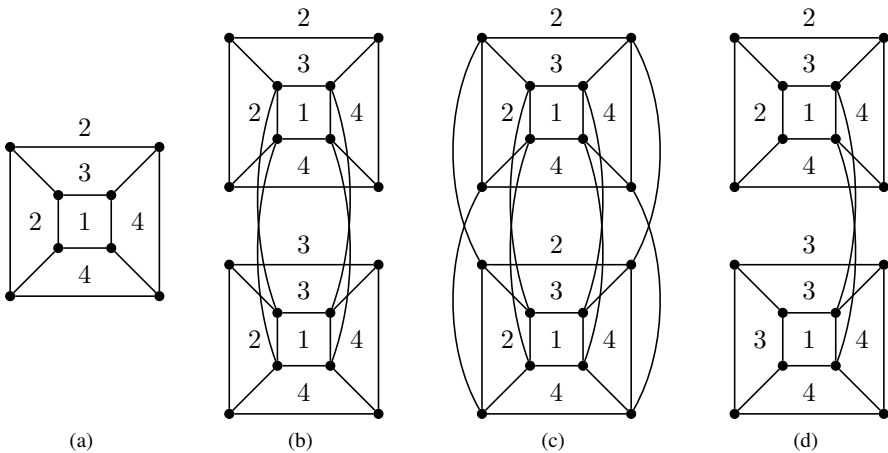


Figure 8: $G[F_i]$ and $G[F_i \cup F_j]$ for Operation I, II and III.

- (i) $f_i(v) \neq f_j(v)$ for $v \in \{a_1, a_2\}$. Without loss of generality, suppose $v = a_1$. Since H is a 2-tree, there is a vertex $a_3 \in V(H)$ such that $H[\{a_1, a_2, a_3\}] \cong K_3$. Observe $f_i(a_3) \in \{1, 2\}$; we may assume $f_i(a_3) = 1$. Then $f_j(a_1) = 2$.¹ Since $f_j(a_2) =$

¹Since $f_j(a_1)$ is uniquely determined there is no f_ℓ with $\ell \neq j$ such that f_i and f_ℓ differ on a_1 .

$f_i(a_2) = 4$, the colours available for α are $\{1, 3\}$; the colours available for β and γ are unchanged. Therefore $[F_i, F_j]$ is a matching of size four between the 4-cycle bounding the α_1 face in $G[F_i]$ and the 4-cycle bounding either the α_1 face or the α_2 face in $G[F_j]$. Thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(b). We label the edge $f_i f_j$ by a-sq. It follows from Footnote 1 that there are at most two edges incident to f_i having label a-sq. Furthermore, we remark that if $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ both have label a-sq, then the four vertices of F_i incident to the edges of $[F_i, F_{j_1}]$ are the same four vertices of F_i that are incident to the edges of $[F_i, F_{j_2}]$, and induce a 4-cycle in $G[F_i]$ that bounds a face with label α_1 or α_2 .

- (ii) $f_i(v) \neq f_j(v)$ for $v \in \{b_1, b_2\}$. As in (i), we label the edge $f_i f_j$ by b-sq. Note that there are at most two edges incident to f_i having label b-sq. If $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ both have label b-sq, then the four vertices of F_i incident to the edges of $[F_i, F_{j_1}]$ are the same four vertices of F_i that are incident to the edges of $[F_i, F_{j_2}]$, and induce a 4-cycle in $G[F_i]$ that bounds a face with label β_1 or β_2 .
- (iii) $f_i(v) \neq f_j(v)$ for $v \in \{c_1, c_2\}$. As in (i) and (ii), we label the edge $f_i f_j$ by c-sq. Note that there are at most two edges incident to f_i having label c-sq. If $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ both have label c-sq, then the four vertices of F_i incident to the edges of $[F_i, F_{j_1}]$ are the same four vertices of F_i that are incident to the edges of $[F_i, F_{j_2}]$, and induce a 4-cycle in $G[F_i]$ that bounds a face with label γ_1 or γ_2 .
- (iv) $f_i(u) \neq f_j(u)$ for $u \in V(H) \setminus \{a_1, a_2, b_1, b_2, c_1, c_2\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(c). We label the edge $f_i f_j$ by pm.

Table 1: Summary of Operation I.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
a_1, a_2	Figure 8(b)	a-sq
b_1, b_2	Figure 8(b)	b-sq
c_1, c_2	Figure 8(b)	c-sq
$u \in V(H) \setminus \{a_1, a_2, b_1, b_2, c_1, c_2\}$	Figure 8(c)	pm

5.2 Operation II

As with Operation I, $G[F_i] \cong Q_3$ for each i , $0 \leq i \leq N - 1$. We may assume that $f_i(a) = 4$, $f_i(x) = 3$, $f_i(b) = 1$, $f_i(c_1) = 1$, and $f_i(c_2) = 2$. Then the colour choices for α are $\{1, 2\}$, for β are $\{2, 4\}$, and for γ are $\{3, 4\}$. Using the same labelling convention as for Operation I, we assume that $G[F_i]$ is drawn in the plane and labelled as in Figure 8(a).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. We label each edge $f_i f_j \in E(G_4(H))$ according to the structure of $G[F_i \cup F_j]$.

- (i) $f_i(v) \neq f_j(v)$ for $v \in \{a, b, c_1, c_2\}$. This is analogous to Operation I when the colour of one of $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ is changed, and thus $G[F_i \cup F_j]$ is isomorphic

to the graph in Figure 8(b). We label the edge $f_i f_j$ by either a-sq, b-sq or c-sq as in Operation I.

- (ii) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 2$. Then the colours available for α are $\{1, 3\}$ and the colours available for β are $\{3, 4\}$; the colours available for γ are unchanged. Therefore, $[F_i, F_j]$ is a matching of size two where $G[F_i]$ and $G[F_j]$ are edges whose endpoint labels are unchanged, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(d). We label the edge $f_i f_j$ by e.
- (iii) $f_i(u) \neq f_j(u)$ for $u \in V(H) \setminus \{a, b, x, c_1, c_2\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(c). We label the edge $f_i f_j$ by pm.

Table 2: Summary of Operation II.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
a	Figure 8(b)	a-sq
b	Figure 8(b)	b-sq
c_1, c_2	Figure 8(b)	c-sq
x	Figure 8(d)	e
$u \in V(H) \setminus \{a, b, x, c_1, c_2\}$	Figure 8(c)	pm

5.3 Operation III

As with Operations I and II, $G[F_i] \cong Q_3$ for each i , $0 \leq i \leq N - 1$. We may assume that $f_i(x) = 3$, $f_i(y) = 1$, $f_i(a) = 4$, and $f_i(c) = 2$. Then the colour choices for α are $\{1, 2\}$, for β are $\{2, 4\}$, and for γ are $\{3, 4\}$. Using the same labelling convention as for Operation I, we assume that $G[F_i]$ is drawn in the plane and labelled as in Figure 8(a).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. We label each edge $f_i f_j \in E(G_4(H))$ according to the structure of $G[F_i \cup F_j]$.

- (i) $f_i(v) \neq f_j(v)$ for $v \in \{a, c\}$. This is analogous to Operation I when the colour is changed on one of $\{a_1, a_2, c_1, c_2\}$, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(b). We label the edge $f_i f_j$ by either a-sq or c-sq as in Operation I.
- (ii) $f_i(v) \neq f_j(v)$ for $v \in \{x, y\}$. This is analogous to Operation II when the colour of x is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(d). We label the edge $f_i f_j$ by e.
- (iii) $f_i(u) \neq f_j(u)$ for vertex $u \in V(H) \setminus \{a, c, x, y\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(c). We label the edge $f_i f_j$ by pm.

Remark 5.1. We note that for Operations I–III, if $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ have the same label that is not e, and $F_i^1, F_i^2 \subseteq F_i$ are incident to the edges of $[F_i, F_{j_1}], [F_i, F_{j_2}]$, respectively, then $F_i^1 = F_i^2$.

Table 3: Summary of Operation III.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
a	Figure 8(b)	a-sq
c	Figure 8(b)	c-sq
x, y	Figure 8(d)	e
$u \in V(H) \setminus \{a, c, x, y\}$	Figure 8(c)	pm

5.4 Operation IV

We may assume that $f_i(x) = 1$, $f_i(y) = 2$, $f_i(w) = 2$ and $f_i(z) = 3$. Then the pairs of colour available for α and β , respectively, are

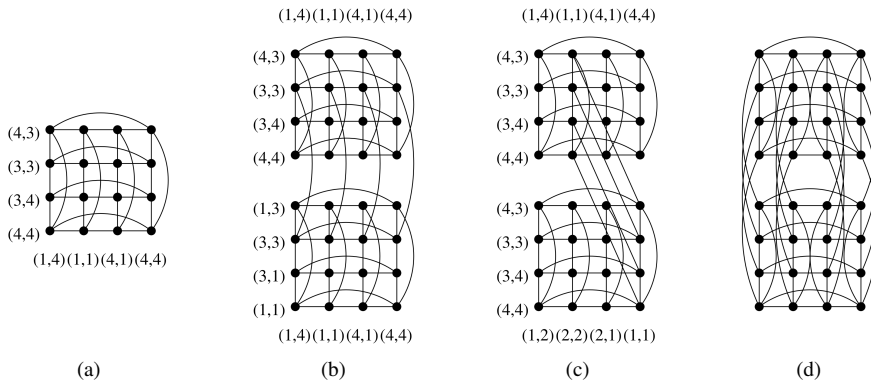
$$\{(4, 3), (3, 3), (3, 4), (4, 4)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(1, 4), (1, 1), (4, 1), (4, 4)\}.$$

Thus the subgraph of G induced by F_i is isomorphic to $C_4 \square C_4$, and we assume that it is drawn as shown in Figure 9(a), with the rows labelled by the pairs of colours available for α and β , respectively, and the columns labelled by the pairs of colours available for γ and δ , respectively.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are three cases to consider.

Figure 9: $G[F_i]$ and $G[F_i \cup F_j]$ for Operation IV.

- (i) $f_i(v) \neq f_j(v)$ for $v \in \{x, y\}$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β , respectively, are

$$\{(1, 3), (3, 3), (3, 1), (1, 1)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 9(b).

- (ii) $f_i(v) \neq f_j(v)$ for $v \in \{w, z\}$. We may assume that $f_j(w) = 4$. Then the pairs of colours available for γ and δ , respectively, are

$$\{(2, 2), (2, 1), (1, 1), (1, 2)\},$$

and the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 9(c).

- (iii) $f_i(u) \neq f_j(u)$ for $u \in V(H) \setminus \{x, y, w, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 9(d).

Table 4: Summary of Operation IV.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x, y	Figure 9(b)	r
z, w	Figure 9(c)	c
$u \in V(H) \setminus \{x, y, z, w\}$	Figure 9(d)	pm

Remark 5.2. We note that for Operation IV, if $f_i f_j \in E(G_4(H))$ has label r and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to α and β . Similarly, if $f_i f_j \in E(G_4(H))$ has label c and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to γ and δ .

5.5 Operation V

We may assume that $f_i(x) = 1$, $f_i(y) = 2$, $f_i(w) = 2$ and $f_i(z) = 3$. Then the pairs of colour available for α and β , respectively, are

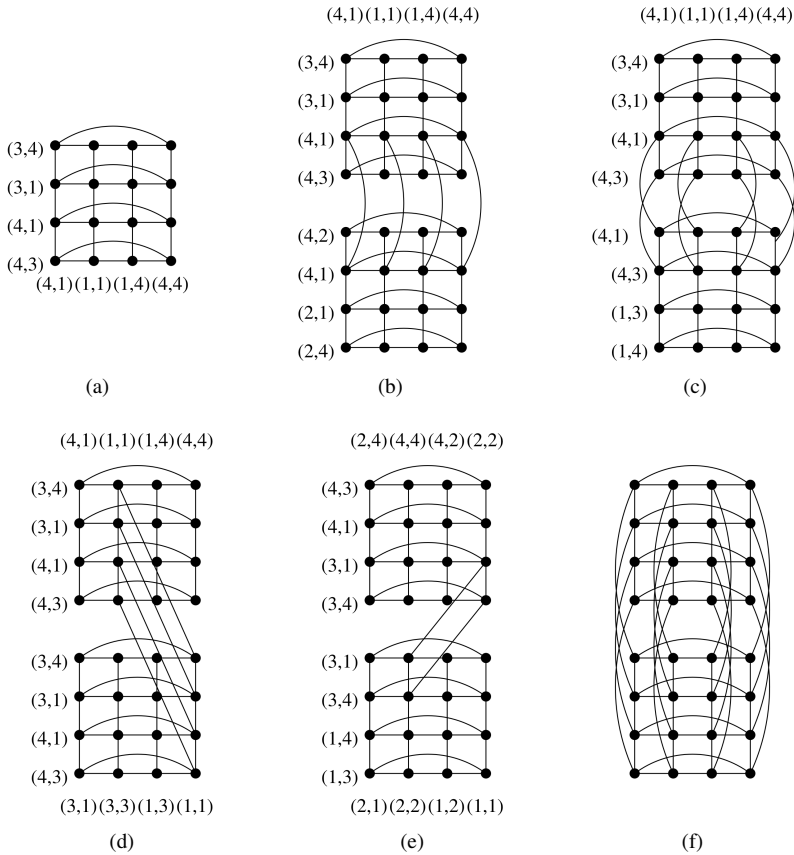
$$\{(3, 4), (3, 1), (4, 1), (4, 3)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(4, 1), (1, 1), (1, 4), (4, 4)\}.$$

Thus the subgraph of $G_4(H')$ induced by F_i is isomorphic to $P_4 \square C_4$, and we assume that it is drawn as shown in Figure 10(a), with the rows labelled by the pairs of colours available for α and β , respectively, and the columns labelled by the pairs of colours available for γ and δ , respectively.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are five cases to consider. Since H is a 2-tree, there are vertices $a, b \in V(H)$ such that $H[\{x, y, a\}] \cong K_3$ and $H[\{w, z, b\}] \cong K_3$. Observe $f_i(a) \in \{3, 4\}$ and $f_i(b) \in \{1, 4\}$. We may assume that $f_i(a) = 4$ and $f_i(b) = 1$. Even though b (respectively, a) could be equal to x or y (respectively, w or z), this does not affect the argument.

Figure 10: Subgraphs induced by $F_i \cup F_j$ for Operations V and IX.

- (i) $f_i(y) \neq f_j(y)$. Then $f_j(y) = 3$ and the pairs of colours available for α and β , respectively, are

$$\{(4, 2), (4, 1), (2, 1), (2, 4)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(b) with appropriate labels.

- (ii) $f_i(x) \neq f_j(x)$. Then $f_j(x) = 3$ and the pairs of colours available for α and β , respectively, are

$$\{(4, 1), (4, 3), (1, 3), (1, 4)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(c).

- (iii) $f_i(z) \neq f_j(z)$. Then $f_j(z) = 4$ and the pairs of colours available for γ and δ , respectively, are

$$\{(3, 3), (1, 3), (1, 1), (3, 1)\},$$

and the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(d).

- (iv) $f_i(w) \neq f_j(w)$. Then $f_j(w) = 4$ and the pairs of colours available for γ and δ , respectively, are

$$\{(3, 3), (1, 3), (1, 1), (3, 1)\},$$

and the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(d).

- (v) $f_i(u) \neq f_j(u)$ $u \in V(H) \setminus \{x, y, w, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(f).

Table 5: Summary of Operation V.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
y	Figure 10(b)	r
x	Figure 10(c)	rr
z, w	Figure 10(d)	c
$u \in V(H) \setminus \{x, y, z, w\}$	Figure 10(f)	pm

We informally refer to the rows and columns of vertices in $G[F_i]$ according to the drawing in Figure 10(a). For Operations IV and VI through IX we use a similar convention.

Remark 5.3. We note that for Operation V, if $f_i f_j \in E(G_4(H))$ has label r, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of row two or three². If $f_i f_j \in E(G_4(H))$ has label rr, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of rows one and two, or rows three and four. If $f_i f_j \in E(G_4(H))$ has label c and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to γ and δ .

5.6 Operation VI

We may assume that $f_i(x) = 1$, $f_i(y) = 2$, $f_i(z) = 3$ and $f_i(w) = 1$. Then the pairs of colour available for α and β , respectively, are

$$\{(3, 4), (3, 1), (4, 1), (4, 3)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 4), (2, 1), (4, 1), (4, 2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \square P_4$, and we assume that it is drawn in the plane as shown in Figure 11(a), with the rows labelled by the pairs of colours available for α and β , respectively, and the columns labelled by the pairs of colours available for γ and δ , respectively.

²Rows are numbered from top to bottom and columns from left to right.

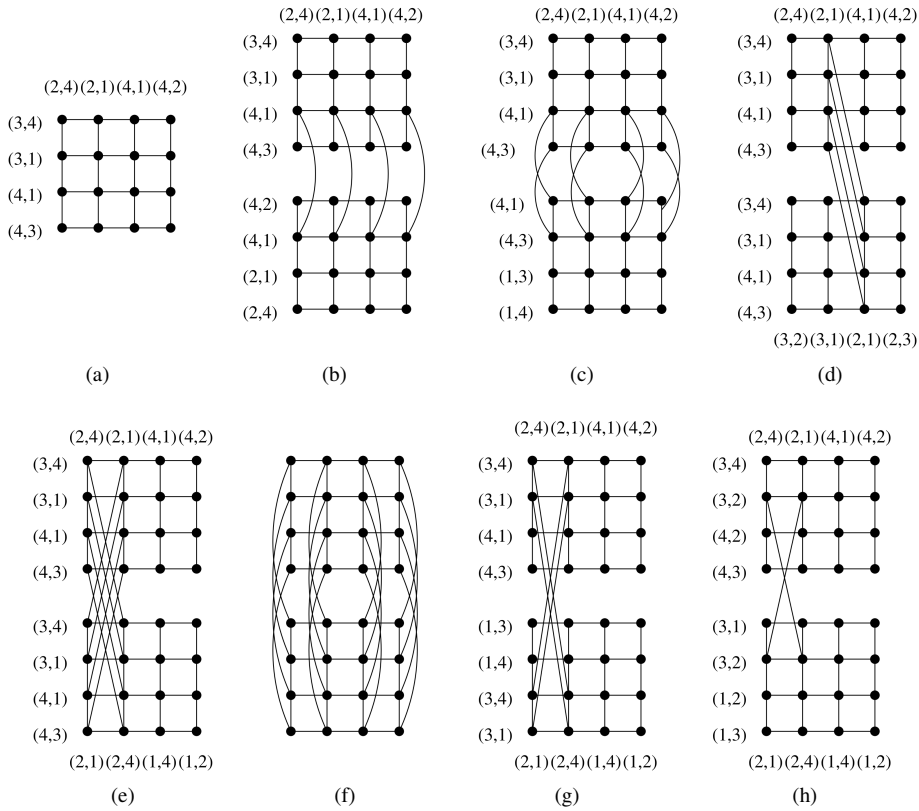


Figure 11: Subgraphs induced by $F_i \cup F_j$ for Operations VI, VII and VIII.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are five cases to consider. Since H is a 2-tree, there are vertices $a, b \in V(H)$ such that $H[\{x, y, a\}] \cong K_3$ and $H[\{w, z, b\}] \cong K_3$. Observe $f_i(a) \in \{3, 4\}$ and $f_i(b) \in \{2, 4\}$. We may assume that $f_i(a) = 4$ and $f_i(b) = 2$. Even though b (respectively, a) could be equal to x or y (respectively, w or z), this does not affect the argument.

- (i) $f_i(y) \neq f_j(y)$. Then $f_j(y) = 3$, and the pairs of colours available for α and β , respectively, are

$$\{(4, 2), (4, 1), (2, 1), (2, 4)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(b).

- (ii) $f_i(x) \neq f_j(x)$. Then $f_j(x) = 3$, and the pairs of colours available for α and β , respectively, are

$$\{(4, 1), (4, 3), (1, 3), (1, 4)\},$$

while the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(c).

- (iii) $f_i(z) \neq f_j(z)$. Then $f_j(z) = 4$ and the pairs of colours available for γ and δ , respectively, are

$$\{(3, 2), (3, 1), (2, 1), (2, 3)\},$$

while the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(d).

- (iv) $f_i(w) \neq f_j(w)$. Then $f_j(w) = 4$ and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 1), (2, 4), (1, 4), (1, 2)\},$$

while the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(e).

- (v) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x, y, z, w\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(f).

Table 6: Summary of Operation VI.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
y	Figure 11(b)	r
x	Figure 11(c)	rr
z	Figure 11(d)	c
w	Figure 11(e)	cc
$u \in V(H) \setminus \{x, y, z, w\}$	Figure 11(f)	pm

Remark 5.4. We note that for Operation VI, if $f_i f_j \in E(G_4(H))$ has label r (respectively, c), then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of row (respectively, column) two or three. If $f_i f_j \in E(G_4(H))$ has label rr (respectively, cc), then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of rows (respectively, columns) one and two or rows (respectively, columns) three and four.

5.7 Operation VII

We may assume that $f_i(x) = 1, f_i(y) = 2$ and $f_i(z) = 3$. Then the pairs of colours available for α and β , respectively, are

$$\{(3, 4), (3, 1), (4, 1), (4, 3)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 4), (2, 1), (4, 1), (4, 2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \square P_4$, and we assume that it is drawn in the plane as shown in Figure 11(a) with rows labelled by the pairs of colours available for α and β , respectively, and the columns labelled by the pairs of colours available for γ and δ , respectively.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. There are four cases to consider.

- (i) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β , respectively, are

$$\{(3, 1), (3, 4), (1, 4), (1, 3)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 1), (2, 4), (1, 4), (1, 2)\}.$$

Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(g).

- (ii) $f_i(y) \neq f_j(y)$. This is analogous to Operation VI when the colour of y is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(b).
- (iii) $f_i(z) \neq f_j(z)$. This is analogous to Operation VI when the colour of z is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(d).
- (iv) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x, y, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(f).

Table 7: Summary of Operation VII.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x	Figure 11(g)	sq
y	Figure 11(b)	r
z	Figure 11(d)	c
$u \in V(H) \setminus \{x, y, z\}$	Figure 11(f)	pm

Remark 5.5. We note that for Operation VII, if $f_i f_j \in E(G_4(H))$ has label r (respectively, c), then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of row (respectively, column) two or three. If $f_i f_j \in E(G_4(H))$ has label sq, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ induces a four-cycle using a degree two vertex of $G[F_i]$.

5.8 Operation VIII

We may assume that $f_i(x) = 1, f_i(y) = 2$ and $f_i(z) = 3$. Then the pairs of colours available for α and β , respectively, are

$$\{(4, 3), (4, 2), (3, 2), (3, 4)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 4), (2, 1), (4, 1), (4, 2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \square P_4$, and we assume that it is drawn in the plane as shown in Figure 11(a) with rows labelled by the pairs of colours available for α and β , respectively,

and the columns labelled by the pairs of colours available for γ and δ , respectively (but not the same labels as in the figure).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$, and there are four cases.

- (i) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β , respectively, are

$$\{(1, 3), (1, 2), (3, 2), (3, 1)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 1), (2, 4), (1, 4), (1, 2)\}.$$

Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(h) with appropriate labels.

- (ii) $f_i(y) \neq f_j(y)$. This is analogous to Operation VI when the colour of x is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(c) with appropriate labels.
- (iii) $f_i(z) \neq f_j(z)$. This is analogous to Operation VI when the colour of z is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(d) with appropriate labels.
- (iv) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x, y, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(f).

Table 8: Summary of Operation VIII.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x	Figure 11(h)	e
y	Figure 11(c)	rr
z	Figure 11(d)	c
$u \in V(H) \setminus \{x, y, z\}$	Figure 11(f)	pm

Remark 5.6. We note that for Operation VIII, if $f_i f_j \in E(G_4(H))$ has label c, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of column two or three. If $f_i f_j \in E(G_4(H))$ has label rr, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of rows one and two or rows three and four. If $f_i f_j \in E(G_4(H))$ has label e, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ induces an edge that is the first or last edge of row two or row three.

5.9 Operation IX

We may assume that $f_i(x) = 1, f_i(y) = 2$ and $f_i(z) = 3$. Then the pairs of colours available for α and β , respectively, are

$$\{(4, 3), (4, 1), (3, 1), (3, 4)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 2), (2, 4), (4, 4), (4, 2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \square C_4$, and we assume that it is drawn in the plane as shown in Figure 10(a) with rows labelled by the pairs of colours available for α and β , respectively, and the columns labelled by the pairs of colours available for γ and δ , respectively (but not the same labels as in the figure).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are four cases.

- (i) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β , respectively, are

$$\{(3, 1), (3, 4), (1, 4), (1, 3)\},$$

and the pairs of colours available for γ and δ , respectively, are

$$\{(2, 1), (2, 2), (1, 2), (1, 1)\}.$$

Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(e) with appropriate labels.

- (ii) $f_i(y) \neq f_j(y)$. This is analogous to Operation V when the colour of y is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(b) with appropriate labels.
- (iii) $f_i(z) \neq f_j(z)$. This is analogous to Operation V when the colour of z is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(d) with appropriate labels.
- (iv) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x, y, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(f).

Table 9: Summary of Operation IX.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x	Figure 10(e)	e
y	Figure 10(b)	r
z	Figure 10(d)	c
$u \in V(H) \setminus \{x, y, z\}$	Figure 10(f)	pm

Remark 5.7. We note that for Operation IX, if $f_i f_j \in E(G_4(H))$ has label r and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to α and β . Similarly, if $f_i f_j \in E(G_4(H))$ has label c and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to γ and δ . If $f_i f_j \in E(G_4(H))$ has label e and $e \in [F_i, F_j]$, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ induces an edge that is either the first or last edge in a column, and each colouring corresponding to an end of e assigns the same colour to γ and δ .

6 4-colouring graphs of 2-trees of diameter at least three

Let H be a 2-tree, and let H' be a 2-tree obtained from H by applying one of the Operations I through IX. We prove $G_4(H')$ has a Hamilton cycle.

6.1 Operations I, II and III

We first prove a result about Hamilton cycles in a cube Q_3 that will later be used to show the existence of edges satisfying Lemma 2.7. In this section, we let each face label in Figure 12(a) denote the 4-cycle bounding that face.

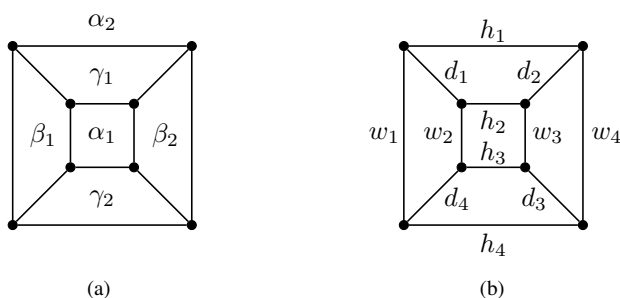


Figure 12: Labelling $G[F_i]$.

We label the six Hamilton cycles of a plane drawing of Q_3 as shown in Figure 13.

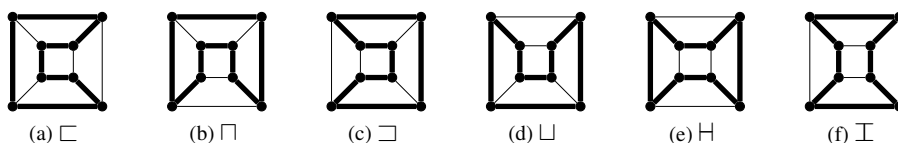


Figure 13: Labels for the Hamilton cycles of Q_3 .

To simplify notation for multisets, we write nZ to mean n copies of Z .

Lemma 6.1. *Let $Q \cong Q_3$ be drawn as in Figure 12(a), and let e be an edge of Q . Let $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_n\}$ be a multiset such that $\mathcal{Z} \subseteq \{2\alpha_1, 2\beta_1, 2\gamma_1, 5Q_3\}$ and $n \leq 5$. That is, each Z_i is an induced subgraph of Q and is either the entire 3-cube or one of the 4-cycles of Q labelled by α_1 , β_1 , or γ_1 . Then there exists a Hamilton cycle in Q containing distinct edges $\{e, e_1, e_2, \dots, e_n\}$ such that $e_i \in E(Z_i)$, for $1 \leq i \leq n$.*

Proof. It is enough to prove the result when $n = 5$ and $\mathcal{Z} = \{2\alpha_1, 2\beta_1, \gamma_1\}$. Note that the Hamilton cycles \square and \square in Q each contain three edges of α_1 , three edges of β_1 , and two edges of γ_1 . When any single edge is deleted from \square or \square , we see that the resulting Hamilton path contains five distinct edges, two from α_1 , two from β_1 and one from γ_1 . Since at least one of \square and \square contains the edge e , either \square or \square contains distinct edges $\{e, e_1, e_2, \dots, e_5\}$ such that $e_i \in E(Z_i)$, for $1 \leq i \leq 5$. \square

Lemma 6.2. Let $Q \cong Q_3$ be drawn as in Figure 12(a) with edges labelled as in Figure 12(b). Assume $\{e, e'\} \subseteq E(Q)$ with $e \in \{w_1, w_2, w_3, w_4\}$. Let $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_n\}$ be a multiset such that $\mathcal{Z} \subseteq \{\alpha_1, \beta_1, 2\gamma_1, 4Q_3\}$ and $n \leq 4$. That is, each Z_i is an induced subgraph of Q and is either the entire 3-cube or one of the 4-cycles of Q labelled by α_1 , β_1 , or γ_1 . Then there exists a Hamilton cycle in Q containing distinct edges $\{e, e', e_1, e_2, \dots, e_n\}$ such that $e_i \in E(Z_i)$, for $1 \leq i \leq n$.

Proof. It is enough to prove the result when $n = 4$ and $\mathcal{Z} = \{\alpha_1, \beta_1, 2\gamma_1\}$ with $Z_1 := \alpha_1$, $Z_2 := \beta_1$, $Z_3 := \gamma_1$ and $Z_4 := \gamma_1$. Let $H := \{h_1, h_2, h_3, h_4\}$, $W := \{w_1, w_2, w_3, w_4\}$ and $D := \{d_1, d_2, d_3, d_4\}$. For each Z_i , we designate a set of candidate edges for e_i as follows (see Table 10).

Table 10: Cases in the proof of Lemma 6.2.

e', e	Cycle in Q	Edges e_i assigned to Z_i			
		Z_1	Z_2	Z_3	Z_4
$e' \in H, e \in \{w_1, w_2\}$	\sqsubset	$\{h_2, h_3\} \setminus \{e'\}$	$\{w_1, w_2\} \setminus \{e\}$	d_2	$\{h_1, h_2\} \setminus \{e'\}$
$e' \in H, e \in \{w_3, w_4\}$	\sqsupset	$\{h_2, h_3\} \setminus \{e'\}$	d_4	d_1	$\{h_1, h_2\} \setminus \{e'\}$
$\{e, e'\} = \{w_1, w_2\}$	\sqcap	w_3	d_4	h_1	h_2
$e, e' \in W, \{e, e'\} \neq \{w_1, w_2\}$	\sqcup	h_3	$\{w_1, w_2\} \setminus \{e, e'\}$	d_1	d_2
$e' \in D, e \in \{w_1, w_4\}$	\sqcup	h_3	$\{d_1, d_4\} \setminus \{e'\}$	h_1	$\{d_1, d_2\} \setminus \{e'\}$
$e' \in D, e \in \{w_2, w_3\}$	\sqcup	$\{w_2, w_3\} \setminus \{e\}$	$\{d_1, d_4\} \setminus \{e'\}$	h_1	$\{d_1, d_2\} \setminus \{e'\}$

As indicated in Table 10, the first case has $e' \in H$ and $e \in \{w_1, w_2\}$. We claim that \sqsubset has the required property. We take $e_3 := d_2$, and as $|\{w_1, w_2\} \setminus \{e\}| = 1$, we take $\{e_2\} := \{w_1, w_2\} \setminus \{e\}$. Observe that the sets $\{h_2, h_3\} \setminus \{e'\}$ and $\{h_1, h_2\} \setminus \{e'\}$ are distinct and non-empty. Thus, we may take $e_1 \in \{h_2, h_3\} \setminus \{e'\}$ and $e_4 \in \{h_1, h_2\} \setminus \{e'\}$ so that $e_1 \neq e_4$. The remaining five cases follow using analogous arguments. \square

Lemma 6.3. Suppose H' is obtained from a 2-tree H by applying one of Operations I, II or III. Then $G_4(H')$ has a Hamilton cycle.

Proof. Case 1. Suppose H' is obtained from H by applying Operation I. By Lemma 2.5, $G_4(H)$ has a spanning tree T with $\Delta(T) \leq 4$. Let $V(T) := \{f_0, f_1, \dots, f_{N-1}\}$ such that f_0 is a leaf, and root T at f_0 , turning T into a branching, \vec{T} , by directing all arcs away from f_0 .

Let $G := G_4(H')$, and let F_i be the set of 4-colourings of H' that agree with f_i on $V(G_4(H))$, $0 \leq i \leq N-1$. Label each $\vec{f_i f_j} \in A(\vec{T})$ with the label of $f_i f_j \in E(G_4(H))$, as described in Section 5, and let $S_{i,j} \subseteq F_i$ denote the vertices incident to the edges of $[F_i, F_j]$.

For each arc $\vec{f_i f_j} \in A(\vec{T})$, we choose edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ satisfying conditions (i) and (ii) of Lemma 2.7 as follows. Suppose $\vec{f_0 f_1} \in A(\vec{T})$. The fact that $G[F_0 \cup F_1]$ is isomorphic to the graph in Figure 8(b) or 8(c) gives us the flexibility to choose $e_{0,1} \in E(G[F_0])$ and $e_{1,0} \in E(G[F_1])$ satisfying (ii) of Lemma 2.7.

Edge choosing procedure. Now suppose for some i , $e_{i,k}$ has been chosen in $G[F_i]$ but $e_{i,j}$ has not yet been chosen for each j where $\vec{f_i f_j} \in A(\vec{T})$. For this i , let $J := \{j \mid$

$\overrightarrow{f_i f_j} \in A(\overrightarrow{T})\}$. We choose edges $e_{i,j}$ and $e_{j,i}$ for $j \in J$ as follows. Assume that $G[F_i]$ is drawn as in Figure 12(a). By Remark 5.1, $S_{i,j_1} = S_{i,j_2}$ whenever $\overrightarrow{f_i f_{j_1}}$ and $\overrightarrow{f_i f_{j_2}}$ have the same label. Without loss of generality, we may assume that an arc $\overrightarrow{f_i f_j}$ with label a-sq has $G[S_{i,j}]$ isomorphic to the 4-cycle α_1 , an arc $\overrightarrow{f_i f_j}$ with label b-sq has $G[S_{i,j}]$ isomorphic to the 4-cycle β_1 , and an arc $\overrightarrow{f_i f_j}$ with label c-sq has $G[S_{i,j}]$ isomorphic to the 4-cycle γ_1 . For each $j \in J$, let $Z_j := G[S_{i,j}]$, and define the multiset $\mathcal{Z} := \{Z_j \mid j \in J\}$. Then each Z_j is either a 3-cube or one of the 4-cycles α_1 , β_1 , or γ_1 . Since f_i is incident to at most two edges with label a-sq, at most two edges with label b-sq, and at most two edges with label c-sq, $\mathcal{Z} \subseteq \{2\alpha_1, 2\beta_1, 2\gamma_1, 5Q_3\}$. Observe $|\mathcal{Z}| \leq 4$ since $\Delta(T) \leq 4$. By Lemma 6.1, using $e := e_{i,k}$, there is a Hamilton cycle C_i in $G[F_i]$ and an edge $e_{i,j} \in E(Z_j)$ for each $Z_j \in \mathcal{Z}$ such that $e_{i,j_1} \neq e_{i,j_2}$ whenever $j_1 \neq j_2$. Thus (i) of Lemma 2.7 is satisfied. Furthermore, for each $j \in J$ there is an edge $e_{j,i} \in E(G[F_j])$ such that $e_{i,j}$ and $e_{j,i}$ satisfy (ii) of Lemma 2.7.

Now suppose for every $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$, $e_{i,j}$ and $e_{j,i}$ have been chosen as above. By construction, (i) and (ii) of Lemma 2.7 are satisfied, and for each $G[F_i]$, $0 \leq i \leq N-1$, the Hamilton cycle C_i satisfies condition (iii) of Lemma 2.7. Therefore, G has a Hamilton cycle.

Case 2. Suppose H' is obtained from H by applying Operation II. Let $\mathcal{H} := G_4(H)$ and $V(\mathcal{H}) := \{f_0, f_1, \dots, f_{N-1}\}$. For each $1 \leq i \leq 4$ let $V_i := \{c \in V(\mathcal{H}) \mid c(x) = i\}$. Then $\{V_1, V_2, V_3, V_4\}$ is a partition of $V(\mathcal{H})$. Let L_i be an assignment of lists with $L_i(x) := \{i\}$ and $L_i(w) := \{1, 2, 3, 4\}$ for $w \in V(\mathcal{H}) \setminus \{x\}$. Note that $G_{L_i}(H) \cong \mathcal{H}[V_i]$ and that $\mathcal{H}[V_1] \cong \mathcal{H}[V_2] \cong \mathcal{H}[V_3] \cong \mathcal{H}[V_4]$. Thus, \mathcal{H} can be partitioned into four copies isomorphic to $G_{L_1}(H)$ with edges between pairs of copies. Furthermore, each edge in $E(\mathcal{H}[V_i])$, $1 \leq i \leq 4$, has label a-sq, b-sq, c-sq or pm, and each edge with one endpoint in V_i and the other endpoint in V_j , $i \neq j$, has label e.

By Lemma 2.5, $\mathcal{H}[V_i]$, $1 \leq i \leq 4$, has a spanning tree T_i with $\Delta(T_i) \leq 4$. Note that $[V_i, V_j] \neq \emptyset$ for $1 \leq i \neq j \leq 4$. Choose one edge from each of $[V_1, V_2]$, $[V_2, V_3]$, and $[V_3, V_4]$. Without loss of generality, suppose the chosen edges are $f_1 f_2 \in [V_1, V_2]$, $f'_2 f_3 \in [V_2, V_3]$, and $f'_3 f_4 \in [V_3, V_4]$ such that $f_i \in V_i$, $1 \leq i \leq 4$. Since $f_1 f_2$, $f'_2 f_3$ and $f'_3 f_4$ each have label e in \mathcal{H} and each vertex of $V(\mathcal{H})$ is incident to at most one edge with label e, the vertices $f_1, f_2, f'_2, f_3, f'_3, f_4$ are distinct. Thus, we may assume that $f'_2 = f_0$ and $f'_3 = f_5$.

Let T be the spanning tree of \mathcal{H} consisting of the union of T_1, T_2, T_3, T_4 , and the edges $\{f_1 f_2, f_0 f_3, f_5 f_4\}$. Then $\Delta(T) \leq 5$ and the only edges of T with label e are $f_1 f_2$, $f_0 f_3$ and $f_4 f_5$. Root T at f_1 , turning T into a branching, \overrightarrow{T} , by directing all arcs away from f_1 . This gives a branching $\overrightarrow{T_i}$ for each T_i , $1 \leq i \leq 4$, and by our choice of labels, f_i is the root of $\overrightarrow{T_i}$.

Let $G := G_4(H')$, and let F_i be the set of 4-colourings of H' that agree with f_i on $V(G_4(H))$, $0 \leq i \leq N-1$. Label each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with the label of $f_i f_j \in E(\mathcal{H})$, and let $S_{i,j} \subseteq F_i$ and $S'_j \subseteq F_j$ denote the vertices incident to the edges of $[F_i, F_j]$.

For each arc $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$, we choose edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ satisfying conditions (i) and (ii) of Lemma 2.7 as follows. For $(i, j) \in \{(1, 2), (0, 3), (5, 4)\}$ (where $\overrightarrow{f_i f_j}$ has label e), let $e_{i,j}$ be the unique edge of $G[S_{i,j}] \subseteq G[F_i]$ and $e_{j,i}$ be the unique edge of $G[S'_j] \subseteq G[F_j]$. For $\overrightarrow{T_1}$ and $\overrightarrow{T_4}$ we apply the edge choosing procedure used in Case 1,

starting with $e_{1,2}$ in $G[F_1]$ for \vec{T}_1 , and $e_{4,5}$ in $G[F_4]$ for \vec{T}_4 . The resulting set of edges

$$\{e_{i,j}, e_{j,i} \mid \overrightarrow{f_i f_j} \in (A(\vec{T}_1 \cup \vec{T}_4) \cup \{\overrightarrow{f_1 f_2}, \overrightarrow{f_5 f_4}\})\}$$

and set of cycles $\{C_i \mid f_i \in V_1 \cup V_4\}$ satisfy Lemma 2.7. For \vec{T}_2 , we let f_M be the parent of f_0 and f_k the parent of f_M . Let \vec{T}_2' be the subtree of \vec{T}_2 obtained by deleting the descendants of f_M and \vec{T}_2'' be the subtree of \vec{T}_2 rooted at f_M . If $M = 2$ then $k = 1$, so $e_{M,k} = e_{2,1}$. Otherwise apply the edge choosing procedure used in Case 1 to \vec{T}_2' , starting with $e_{2,1}$ in $G[F_2]$. This leads to the designation of an edge $e_{M,k}$ in $G[F_M]$.

Let $J := \{j \mid \overrightarrow{f_M f_j} \in A(\vec{T}_2)\}$. We choose edges $e_{M,j}$ and $e_{j,M}$ for $j \in J$ as follows. Without loss of generality, we may assume that an arc $\overrightarrow{f_M f_j}$ with label a-sq has $G[S_{M,j}]$ isomorphic to the 4-cycle α_1 , an arc $\overrightarrow{f_M f_j}$ with label b-sq has $G[S_{M,j}]$ isomorphic to the 4-cycle β_1 , an arc $\overrightarrow{f_M f_j}$ with label c-sq has $G[S_{M,j}]$ isomorphic to the 4-cycle γ_1 , and an arc with label pm has $G[S_{M,j}]$ isomorphic to Q_3 . Let \mathcal{Z} be a multiset consisting of the graphs $Z_j := G[S_{M,j}]$ for $j \in J$.

Suppose there exists $\ell \in (J \setminus \{0\})$ such that both $\overrightarrow{f_M f_0}$ and $f_M f_\ell$ have label c-sq. We apply the edge choosing procedure used in Case 1 to \vec{T}_2'' , with chosen edge $e_{M,k}$. The resulting set of edges $\{e_{M,j} \mid j \in J\}$ and cycle C_M satisfy Lemma 2.7. Observe that $e_{M,0}, e_{M,\ell} \in E(\gamma_1)$. Suppose $e_{M,0} = ab$, and let the vertices of $\{[a, b], F_0\}$ incident to F_0 be $\{c, d\}$. If $e_{0,3} = cd$ then exchange $e_{M,0}$ with $e_{M,\ell}$. It now follows that for each $j \in J$ there is an edge $e_{j,M} \in E(G[F_j])$ such that $e_{M,j}$ and $e_{j,M}$ satisfy (ii) of Lemma 2.7, with $e_{0,M} \neq e_{0,3}$.

Otherwise $\overrightarrow{f_M f_0}$ is the only arc in $\{\overrightarrow{f_M f_j} \mid j \in J\}$ labelled c-sq, or $\overrightarrow{f_M f_0}$ has label a-sq, b-sq, or pm. To ensure (i) of Lemma 2.7 is satisfied for $i = 0$, we duplicate $Z_0 \in \mathcal{Z}$ and apply Lemma 6.1. Let $Z_{0'} := Z_0$ and $F_{0'} := F_0$. Observe $|\mathcal{Z} \cup \{Z_{0'}\}| \leq 5$ since $\Delta(T) \leq 4$. Since f_M is incident to at most one edge with label a-sq, at most one edge with label b-sq, and at most two edges with label c-sq, $(\mathcal{Z} \cup \{Z_{0'}\}) \subseteq \{2\alpha_1, 2\beta_1, 2\gamma_1, 5Q_3\}$. By Lemma 6.1 applied to $\mathcal{Z} \cup \{Z_{0'}\}$, and using $e := e_{M,k}$, there is a Hamilton cycle C_M in $G[F_M]$ and edges $e_{M,j} \in E(Z_j)$ for each $j \in (J \cup \{0'\})$ such that $e_{M,j_1} \neq e_{M,j_2}$ whenever $j_1 \neq j_2$. Thus (i) of Lemma 2.7 is satisfied for $i = M$. Furthermore, for each $j \in (J \cup \{0'\})$ there is an edge $e_{j,M} \in E(G[F_j])$ such that $e_{j,M}$ and $e_{M,j}$ satisfy (ii) of Lemma 2.7. Since $e_{M,0'} \neq e_{M,0}$, we have $e_{0,M} \neq e_{0',M}$, and hence, one of $e_{0,M}$ and $e_{0',M}$ is different from $e_{0,3}$. If $e_{0,M} \neq e_{0,3}$ then we ignore $e_{0',M}$ and $e_{M,0'}$; otherwise, redefine $e_{0,M}$ to be $e_{0',M}$ and $e_{M,0}$ to be $e_{M,0'}$ so that (i) of Lemma 2.7 is satisfied for $i = 0$.

Finally, let $G[F_0] \cong Q_3$ be drawn as in Figure 12(a) with edges labelled as in Figure 12(b). Observe that $e = e_{0,3} \in \{w_1, w_2, w_3, w_4\}$ and let $e' := e_{0,M}$. Let $J := \{j \mid \overrightarrow{f_0 f_j} \in A(\vec{T}_2)\}$. We choose edges $e_{0,j}$ and $e_{j,0}$ for $j \in J$ as follows. Without loss of generality, we may assume that an arc $\overrightarrow{f_0 f_j}$ with label a-sq has $G[S_{0,j}]$ isomorphic to the 4-cycle α_1 , an arc $\overrightarrow{f_0 f_j}$ with label b-sq has $G[S_{0,j}]$ isomorphic to the 4-cycle β_1 , an arc $\overrightarrow{f_0 f_j}$ with label c-sq has $G[S_{0,j}]$ isomorphic to the 4-cycle γ_1 , and an arc with label pm has $G[S_{0,j}]$ isomorphic to Q_3 . Let \mathcal{Z} be a multiset consisting of the graphs $Z_j := G[S_{0,j}]$ for $j \in J$. Observe $\mathcal{Z} \subseteq \{\alpha_1, \beta_1, 2\gamma_1, 4Q_3\}$ and $|\mathcal{Z}| \leq 4$. By Lemma 6.2, there exists a Hamilton cycle C_0 in $G[F_0]$ containing distinct edges $\{e, e'\} \cup \{e_{0,j} \mid j \in J\}$ such that

$e_{0,j} \in E(Z_j)$ for $j \in J$. Thus (i) of Lemma 2.7 is satisfied for $i = 0$. Furthermore, for each $j \in J$ there is an edge $e_{j,0} \in E(G[F_j])$ so that $e_{j,0}$ and $e_{0,j}$ satisfy (ii) of Lemma 2.7.

We now apply the edge choosing procedure used in Case 1 to the remaining nodes of \vec{T}_2 . This designates edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$, and cycles C_i satisfying Lemma 2.7. For \vec{T}_3 we apply the argument for \vec{T}_2 giving us edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$, and cycles C_i satisfying Lemma 2.7, for each $\vec{f_i f_j} \in A(\vec{T}_3)$.

Now $e_{i,j}$ and $e_{j,i}$ have been chosen for all $\vec{f_i f_j} \in A(\vec{T})$. By construction, each such $e_{i,j}$ and $e_{j,i}$ satisfy (ii) of Lemma 2.7, and for each $G[F_i]$, $0 \leq i \leq N-1$, the Hamilton cycle C_i satisfies condition (iii) of Lemma 2.7. Furthermore, the collection of chosen edges are all distinct. Therefore, G has a Hamilton cycle.

Case 3. Suppose H' is obtained from H by applying Operation III. Let $\mathcal{H} := G_4(H)$ and $V(H) := \{f_0, f_1, \dots, f_{N-1}\}$. For each $1 \leq i \neq j \leq 4$ let

$$V_{ij} := \{c \in V(\mathcal{H}) \mid c(x) = i \text{ and } c(y) = j\}.$$

Then

$$V := \{V_{12}, V_{13}, V_{14}, V_{21}, V_{23}, V_{24}, V_{31}, V_{32}, V_{34}, V_{41}, V_{42}, V_{43}\}$$

is a partition of $V(\mathcal{H})$. Note that $[V_{\alpha\beta}, V_{\gamma\delta}] \neq \emptyset$ if and only if $\alpha = \gamma$ or $\beta = \delta$. Furthermore, each edge in $E(\mathcal{H}[V_{ij}])$, $1 \leq i \neq j \leq 4$, has label a-sq, c-sq or pm, and each edge with one endpoint in $V_{i_1 j_1}$ and the other endpoint in $V_{i_2 j_2}$, $(i_1, j_1) \neq (i_2, j_2)$, has label e.

Let $\{i, j, k\} \subset \{1, 2, 3, 4\}$. As H is a 2-tree, H is 3-colourable and for each $1 \leq i \neq j \leq 4$, there is a unique vertex $c_{ijk} \in V(\mathcal{H})$ with $c_{ijk}(x) = i$, $c_{ijk}(y) = j$ and $c_{ijk}(w) \in \{i, j, k\}$ for $w \in V(H) \setminus \{x, y\}$.

Consider the ordering

$$(V_{14}, V_{12}, V_{32}, V_{34}, V_{31}, V_{21}, V_{24}, V_{23}, V_{13}, V_{43}, V_{42}, V_{41})$$

of V . For each $V_{ij} \in V \setminus \{V_{41}\}$, suppose $V_{\ell m}$ immediately follows V_{ij} in the list. Then $|\{i, j\} \cup \{\ell, m\}| = 3$, and hence there is a unique $k_{ij} \in \{1, 2, 3, 4\} \setminus (\{i, j\} \cup \{\ell, m\})$ such that $c_{ijk_{ij}} \in V_{ij}$. The ordering of V ensures that for each $V_{ij} \in V \setminus \{V_{42}, V_{41}\}$ with $V_{\ell m}$ immediately following V_{ij} in the list, $k_{ij} \neq k_{\ell m}$. Choose the edge $c_{14}c'_{14}$ from $[V_{14}, V_{12}]$ with endpoint $c_{14} := c_{143} \in V_{14}$; note that $c'_{14} \neq c_{124}$. For each $[V_{ij}, V_{\ell m}]$ where $V_{\ell m}$ immediately follows V_{ij} in the list, there is a unique edge $c_{ij}c'_{ij}$ with endpoint $c_{ij} = c_{ijk_{ij}} \in V_{ij}$.

By Lemma 2.5, $\mathcal{H}[V_{ij}]$, $1 \leq i \neq j \leq 4$, has a spanning tree T_{ij} with $\Delta(T_{ij}) \leq 4$. Let T be the spanning tree of \mathcal{H} with $T_{ij} \subset T$, $1 \leq i \neq j \leq 4$, and $c_{ij}c'_{ij} \in E(T)$, $1 \leq i \neq j \leq 4$ with $(i, j) \neq (4, 1)$. Then $\Delta(T) \leq 5$ and the only edges of T with label e are $c_{ij}c'_{ij}$, $1 \leq i \neq j \leq 4$ with $(i, j) \neq (4, 1)$. Root T at $c_{143} \in V_{14}$, turning T into a branching, \vec{T} , by directing all arcs away from c_{14} . This gives a branching \vec{T}_{ij} for each T_{ij} , $1 \leq i \neq j \leq 4$. Now repeat the argument in Case 2 for each \vec{T}_{ij} , $1 \leq i \neq j \leq 4$. \square

6.2 Operations IV to IX

We introduce some labelled Hamilton cycles of $C_4 \square C_4$, $P_4 \square C_4$ and $P_4 \square P_4$ to be used in Lemma 2.7 to show the existence of a Hamilton cycle in $G_4(H')$, where H' is obtained from a 2-tree H by applying one of Operations IV through IX.

Let \square be the edge labelled Hamilton cycle in $C_4 \square C_4$ shown in Figure 14, with the edges in rows two and four labelled by r , the edges in columns two and four labelled by c , and the remaining edges left unlabelled. Let \sqsubset and \sqsubset' be the edge labelled Hamilton cycles in $P_4 \square C_4$ shown in Figure 14, and let \sqsupset , \sqsupset' and \sqsupset'' be the edge labelled Hamilton cycles in $P_4 \square P_4$ shown in Figure 14.

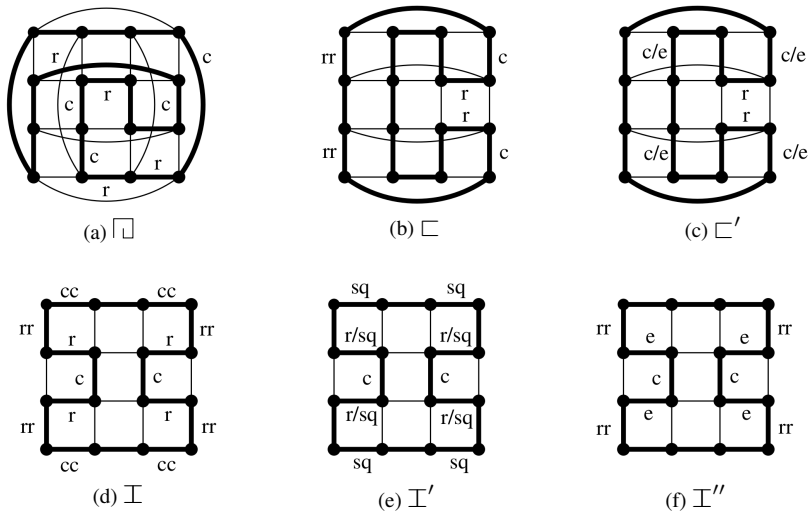


Figure 14: Labeled Hamilton cycles of $C_4 \square C_4$, $P_4 \square C_4$ and $P_4 \square P_4$.

Lemma 6.4. Suppose H' is obtained from a 2-tree H by applying one of Operations IV through IX. Then $G_4(H')$ has a Hamilton cycle.

Proof. By Lemma 2.5, $\mathcal{H} := G_4(H)$ has a spanning tree T with $\Delta(T) \leq 4$. Let $V(T) := \{f_0, f_1, \dots, f_{N-1}\}$ such that f_0 is a leaf, and root T at f_0 , turning T into a branching, \vec{T} , by directing all arcs away from f_0 . Let $G := G_4(H')$, and let F_i be the set of 4-colourings of H' that agree with f_i on $V(\mathcal{H})$, $0 \leq i \leq N-1$. Label each $\vec{f_i f_j} \in A(\vec{T})$ with the label of $f_i f_j \in E(\mathcal{H})$, and let $S_{i,j} \subseteq F_i$ and $S'_j \subseteq F_j$ denote the vertices incident to the edges of $[F_i, F_j]$.

We first traverse \vec{T} using breadth-first search starting at f_0 to construct a drawing of each $G[F_i]$ as shown in Figure 9(a) for Operation IV, Figure 10(a) for Operations V and IX, and Figure 11(a) for Operations VI, VII and VIII, so that each drawing has the following property, denoted (*).

- (*) The rows are labelled by the pairs of colours available for α and β , respectively, and the columns are labelled by the pairs of colours available for γ and δ , respectively. In the case of Operations IV, V and IX, we further assume that the second and fourth columns have γ and δ the same colour. Also, in the case of Operation IV, we assume the second and fourth rows have α and β the same colour.

Start with a drawing of $G[F_0]$ satisfying (*). Assume $\vec{f_i f_j} \in A(\vec{T})$ where $G[F_i]$ has been drawn but $G[F_j]$ has not. We draw $G[F_j]$ satisfying (*) as follows. If $\vec{f_i f_j}$ has label

- **pm**, then draw $G[F_j]$ so that the labels of the rows and columns are in the same order as in $G[F_i]$.
- **r**, then draw $G[F_j]$ so that the column labels of $G[F_j]$ are in the same order as the column labels of $G[F_i]$, and so that the row number of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can be done for Operation IV due to the cyclic structure of $C_4 \square C_4$; furthermore, (*) guarantees that $S_{i,j}$ and S'_j have row number two or four. For Operations V, VI, VII and IX, this can be done by Remarks 5.3, 5.4, 5.5 and 5.7. Thus, one of the two possible labellings for the rows of $G[F_j]$ gives a drawing of $G[F_j]$ such that the row number of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$.
- **c**, then draw $G[F_j]$ so that the row labels of $G[F_j]$ are in the same order as the row labels of $G[F_i]$, and so that the column number of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can always be done using a similar argument as in the case for $\overrightarrow{f_i f_j}$ having label r. For Operations IV, V and IX, (*) guarantees that $S_{i,j}$ and S'_j have column number two or four. For Operations VI, VII and VIII, by Remarks 5.4, 5.5 and 5.6, $S_{i,j}$ and S'_j have column number two or three.
- **rr**, then draw $G[F_j]$ so that the column labels of $G[F_j]$ are in the same order as the column labels of $G[F_i]$, and so that the set of row numbers of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can be done for Operations V, VI and VIII, since by Remarks 5.3, 5.4 and 5.6, the set of row numbers of $S_{i,j}$ and S'_j is either $\{1, 2\}$ or $\{3, 4\}$.
- **cc**, then draw $G[F_j]$ so that the row labels of $G[F_j]$ are in the same order as the row labels of $G[F_i]$, and so that the set of column numbers of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can be done for Operation VI, since by Remark 5.4, the set of column numbers of $S_{i,j}$ and S'_j is either $\{1, 2\}$ or $\{3, 4\}$.
- **sq**, then draw $G[F_j]$ satisfying (*).
- **e**, then draw $G[F_j]$ satisfying (*). Note that for Operation IX, (*) guarantees that $S_{i,j}$ and S'_j belong to column number two or four.

For Operation IV (respectively, V, VI, VII, VIII, and IX), for each i , $0 \leq i \leq N - 1$, let C_i be the Hamilton cycle \square (respectively, \square , \sqcap , \sqcap' , \sqcap'' , and \square') in $G[F_i]$.

We describe how to construct a set of edges

$$\mathcal{E} := \{e_{i,j}, e_{j,i} \mid f_i f_j \in E(T)\}$$

so that for each arc $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$, the edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ satisfy conditions (i) and (ii) of Lemma 2.7, and so that $e_{i,j} \in E(C_i)$ and $e_{j,i} \in E(C_j)$. Start with $\mathcal{E} := \emptyset$. We consider the arcs of \overrightarrow{T} in the following order according to their labels.

- (1) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label e (Operations VIII and IX), $e_{i,j} \in G[F_i]$ and $e_{j,i} \in G[F_j]$ satisfying condition (ii) of Lemma 2.7 are uniquely determined. Note that for Operation VIII, $e_{i,j} \in C_i$ and has label e, and $e_{j,i} \in C_j$ and has label e; this follows by the symmetry of \sqcap'' and the drawings $G[F_i]$ and $G[F_j]$. For Operation IX, $e_{i,j} \in C_i$ and has label c/e, and $e_{j,i} \in C_j$ and has label c/e; this follows from the drawings of $G[F_i]$ and $G[F_j]$, and Remark 5.7. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .

- (2) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label sq (Operation VII), choose $e_{i,j}$ in C_i with label sq and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label sq or label r/sq; this follows from the drawings of $G[F_i]$ and $G[F_j]$, and Remark 5.7. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .
- (3) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label rr (Operations V, VI and VIII), choose $e_{i,j}$ in C_i with label rr and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label rr; this follows from the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .
- (4) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label cc (Operation VI), choose $e_{i,j}$ in C_i with label cc and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label cc; this follows from the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .
- (5) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label r (Operations V, VI and IX), choose $e_{i,j}$ in C_i with label r and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label r; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .

For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label r (Operation VII), choose $e_{i,j}$ in $E(C_i) \setminus \mathcal{E}$ with label r/sq and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. This is possible since Γ' has four edges with label r/sq. Note that $e_{j,i} \in C_j$ and has label r/sq; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .

In the context of Operation IV, let T' be the subgraph of T induced by edges with label r. Each component of T' is a path since each f_i is incident to at most two edges of T with label r. Let P be a component of T' , and assume without loss of generality that $P = f_0 f_1 \cdots f_{m-1}$. For each i , $0 \leq i \leq m-2$, starting at $i = 0$, choose $e_{i,i+1}$ in $E(C_i) \setminus \mathcal{E}$ with label r and $e_{i+1,i}$ in $G[F_{i+1}]$ satisfying condition (ii) of Lemma 2.7. This is possible because the Hamilton cycle \square has two edges labelled r in both rows two and four. Note that $e_{i+1,i} \in C_{i+1}$ and has label r; this follows by the drawings of $G[F_i]$ and $G[F_{i+1}]$. Add $e_{i,i+1}$ and $e_{i+1,i}$ to \mathcal{E} .

- (6) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label c (Operations VI, VII and VIII), choose $e_{i,j}$ in C_i with label c and $e_{j,i}$ in $G[F_j]$ that satisfies condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label c; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .

For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label c (Operation IX), choose $e_{i,j}$ in $E(C_i) \setminus \mathcal{E}$ with label c/e and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. This is possible because the Hamilton cycle \square' has two edges labelled c/e in columns two and four, and f_i is incident to at most one edge in T with label c and at most one edge in T with label e. Note that $e_{j,i} \in C_j$ and has label c/e; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .

In the context of Operations IV and V, let T' be the subgraph of T induced by edges with label c. Each component of T' is a path since each f_i is incident to at most two edges of T with label c. Let P be a component of T' , and assume without loss of generality that $P = f_0 f_1 \cdots f_{m-1}$. For each i , $0 \leq i \leq m-2$, starting at $i = 0$, choose $e_{i,i+1}$ in $E(C_i) \setminus \mathcal{E}$ with label c and $e_{i+1,i}$ in $G[F_{i+1}]$ satisfying condition

- (ii) of Lemma 2.7. This is possible because the Hamilton cycles \square and \sqcap have two edges labelled c in both columns two and four. Note that $e_{i+1,i} \in C_{i+1}$ and has label c ; this follows by the drawings of $G[F_i]$ and $G[F_{i+1}]$. Add $e_{i,i+1}$ and $e_{i+1,i}$ to \mathcal{E} .
- (7) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label pm , choose $e_{i,j}$ in $E(C_i) \setminus \mathcal{E}$ and $e_{j,i}$ in $E(G[F_j]) \setminus \mathcal{E}$ satisfying condition (ii) of Lemma 2.7. Add $e_{i,j}$ and $e_{j,i}$ to \mathcal{E} .

Now $e_{i,j}$ and $e_{j,i}$ have been chosen for all $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$. By construction, (i) and (ii) of Lemma 2.7 are satisfied, and for each $G[F_i]$, $0 \leq i \leq N-1$, the Hamilton cycle C_i satisfies condition (iii) of Lemma 2.7. Therefore, G has a Hamilton cycle. \square

As a consequence of Lemmas 3.3, 3.5, 6.3 and 6.4, we now have our main result.

Theorem 1.1. *If H is a 2-tree then $k_0(H) = 4$, unless $H \cong T \vee \{u\}$ for some tree T and vertex u , where T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.*

As pointed out in Section 1, if H is a k -tree then $k+2 \leq k_0(H) \leq k+3$. For both 1-trees (i.e., trees) and 2-trees, equality can occur in both the upper and lower bound. By [6, Corollary 5.6] and Theorem 1.1, if H is a tree or 2-tree of diameter at least three, then the lower bound holds. We ask if this extends to k -trees, that is, if H is a k -tree with diameter at least three, is it the case that $k_0(H) = k+2$? On a related note, k -trees are a subclass of chordal graphs. We ask if the techniques presented here can be extended to determine the Gray code numbers of other chordal graphs.

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Appendix A 3-colouring graphs of trees

A 2-tree with a dominating vertex u has the form $T \vee \{u\}$ for some tree T . Lemma 3.5 characterizes when $k_0(T \vee \{u\}) = 5$. Its proof requires a technical result (Lemma 3.6) that we prove in this appendix. To do so, we introduce additional terminology along with structural results on the 3-colouring graphs of trees.

Define L_i to be an assignment of lists to the vertices of $T \vee \{u\}$ with $L_i(u) := \{i\}$ and $L_i(w) := \{1, 2, 3, 4\}$ for $w \in V(T)$. Then $G_{L_i}(T \vee \{u\}) \cong G_3(T)$ and thus, $G_4(T \vee \{u\})$ can be partitioned into four copies of $G_3(T)$ with edges between pairs of copies. We prove that when T has even number of vertices then $G_3(T)$ is bipartite. First, we prove a more general result for $G_L(H)$ where H is a connected graph.

Lemma A.1. *Let H be a connected graph on n vertices and L an assignment of lists to the vertices of H such that $L(v) \subseteq \{1, 2, 3\}$ for each $v \in V(H)$. If H is L -colourable and there is a vertex $w \in V(H)$ with $|L(w)| < 3$, then $G_L(H) \subseteq Q_n$. In particular, $G_L(H)$ is a bipartite graph.*

Proof. The proof is by induction on n . Observe that the result is true for $n = 1$ and $n = 2$.

First suppose that $|L(w)| = 1$ and without loss of generality, $L(w) = \{1\}$. In any list colouring of H , each vertex $v \in N_H(w)$ cannot be coloured ‘1’. Thus, let \hat{L} be an assignment of lists of allowable colours defined as

$$\hat{L}(v) := \begin{cases} L(v) \setminus \{1\}, & \text{if } v \in N_H(w), \\ L(v), & \text{otherwise.} \end{cases}$$

Denote the components of $H - w$ by H_1, H_2, \dots, H_N . Then by the inductive hypothesis, for each i , $1 \leq i \leq N$,

$$G_{\hat{L}}(H_i) \subseteq Q_{|V(H_i)|}.$$

Since w must be coloured using the colour ‘1’, we have

$$\begin{aligned} G_L(H) &= G_{\hat{L}}(H - w) \\ &= G_{\hat{L}}\left(\bigcup_{i=1}^N H_i\right) \\ &= \bigcap_{i=1}^N G_{\hat{L}}(H_i), && \text{by Remark 2.4} \\ &\subseteq \bigcap_{i=1}^N Q_{|V(H_i)|}, && \text{by the inductive hypothesis} \\ &= Q_{n-1}. \end{aligned}$$

Next suppose that $|L(w)| = 2$ and without loss of generality, $L(w) := \{1, 2\}$. For $i = 1, 2$, define L_i as $L_i(w) := \{i\}$ and $L_i(v) := L(v)$ for $v \neq w$ with $v \in V(H)$. Also define \hat{L}_i as $\hat{L}_i(w) := \{i\}$ and $\hat{L}_i(v) := \{1, 2, 3\}$ for $v \neq w$ with $v \in V(H)$. Observe that $G_{L_1}(H) \subseteq G_{\hat{L}_1}(H)$ and $G_{L_2}(H) \subseteq G_{\hat{L}_2}(H)$. Furthermore, $G_{\hat{L}_1}(H) \cong G_{\hat{L}_2}(H)$, and hence, $G_L(H) \subseteq G_{\hat{L}_1}(H) \square K_2$. By the preceding argument, $G_{\hat{L}_1}(H) \subseteq Q_{n-1}$, and therefore

$$G_L(H) \subseteq Q_{n-1} \square K_2 = Q_n. \quad \square$$

Lemma A.2. *If T is a tree with an even number of vertices then $G_3(T)$ is bipartite.*

Proof. If $T \cong K_2$, then $G_3(T) \cong C_6$. Let T be a tree on an even number of vertices with $|V(T)| \geq 4$ having bipartition (A, B) . Fix $u \in A, v \in B$ with u adjacent to v in T . Define $\mathcal{H} := G_3(T)$. For each $1 \leq i \neq j \leq 3$ let

$$V_{ij} := \{c \in V(\mathcal{H}) \mid c(u) = i \text{ and } c(v) = j\}.$$

Then $\{V_{12}, V_{13}, V_{23}, V_{21}, V_{31}, V_{32}\}$ is a partition of $V(\mathcal{H})$. Observe each $\mathcal{H}[V_{ij}]$ is connected (this can be shown by induction on $|V(T)|$), and that $[V_{\alpha\beta}, V_{\gamma\delta}] \neq \emptyset$ if and only if $\alpha = \gamma$ or $\beta = \delta$. It follows that $\mathcal{H}_1 = \mathcal{H}[V_{12} \cup V_{13} \cup V_{23} \cup V_{21}]$ is connected, and by Lemma A.1 is bipartite. Similarly, $\mathcal{H}_2 = \mathcal{H}[V_{23} \cup V_{21} \cup V_{31} \cup V_{32}]$ and $\mathcal{H}_3 = \mathcal{H}[V_{31} \cup V_{32} \cup V_{12} \cup V_{13}]$ are connected and bipartite. Denote the two-coloured vertices of \mathcal{H} (that is, the colourings of T with two colours) by $c_{ij} \in V_{ij}$ such that

$$c_{ij}(x) := \begin{cases} i, & \text{if } x \in A, \\ j, & \text{if } x \in B. \end{cases}$$

Suppose the bipartition of each $\mathcal{H}[V_{ij}]$ is (A_{ij}, B_{ij}) where $c_{ij} \in B_{ij}$.

If $|A|$ and $|B|$ are even, then $d_{\mathcal{H}}(c_{ij}, c_{i'j'})$ is even. It follows that

$$(A_{12} \cup A_{13} \cup A_{23} \cup A_{21}, B_{12} \cup B_{13} \cup B_{23} \cup B_{21})$$

is a bipartition of \mathcal{H}_1 ,

$$(A_{23} \cup A_{21} \cup A_{31} \cup A_{32}, B_{23} \cup B_{21} \cup B_{31} \cup B_{32})$$

is a bipartition of \mathcal{H}_2 , and

$$(A_{31} \cup A_{32} \cup A_{12} \cup A_{13}, B_{31} \cup B_{32} \cup B_{12} \cup B_{13})$$

is a bipartition of \mathcal{H}_3 . Hence,

$$\mathcal{A} := \bigcup_{1 \leq i \neq j \leq 3} A_{ij} \quad \text{and} \quad \mathcal{B} := \bigcup_{1 \leq i \neq j \leq 3} B_{ij}$$

are independent, and thus form a bipartition of \mathcal{H} .

If $|A|$ and $|B|$ are both odd, $d_{\mathcal{H}}(c_{ij}, c_{i'j'})$ is even if and only if $i \neq i'$ and $j \neq j'$. It follows that

$$(A_{12} \cup B_{13} \cup A_{23} \cup B_{21}, B_{12} \cup A_{13} \cup B_{23} \cup A_{21})$$

is a bipartition of \mathcal{H}_1 ,

$$(A_{23} \cup B_{21} \cup A_{31} \cup B_{32}, B_{23} \cup A_{21} \cup B_{31} \cup A_{32})$$

is a bipartition of \mathcal{H}_2 , and

$$(A_{31} \cup B_{32} \cup A_{12} \cup B_{13}, B_{31} \cup A_{32} \cup B_{12} \cup A_{13})$$

is a bipartition of \mathcal{H}_3 . Hence,

$$\mathcal{A} = \{A_{12} \cup B_{13} \cup A_{23} \cup B_{21} \cup A_{31} \cup B_{32}\}$$

and

$$\mathcal{B} = \{B_{12} \cup A_{13} \cup B_{23} \cup A_{21} \cup B_{31} \cup A_{32}\}$$

are independent, and thus form a bipartition of \mathcal{H} . □

Definition A.3. A connected bipartite graph with bipartition (A, B) is *Hamilton laceable* if there is a Hamilton path between any $u \in A$ and $v \in B$.

Remark A.4. The following are Hamilton laceable.

1. $P_{2k_1} \square P_{k_2} \square P_{k_3} \square \cdots \square P_{k_n}$, for $n \geq 2$ and $k_i \geq 1$, $1 \leq i \leq n$ [17].
2. Q_n , for $n \geq 1$ [5].

Definition A.5. A B -graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$ is a bipartite graph G with bipartition $(\mathcal{A}, \mathcal{B})$ together with a partition $\{F_0, F_1, \dots, F_{N-1}\}$ of $V(G)$ so that, for $i = 0, 1, \dots, N-1$, $G[F_i]$ is Hamilton laceable.

Lemma A.6. Let G be a B -graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$ and bipartition $(\mathcal{A}, \mathcal{B})$. Suppose for each $i = 1, 2, \dots, N-1$, there is an edge $b_{i-1}a_i$ with $b_{i-1} \in \mathcal{B} \cap F_{i-1}$ and $a_i \in \mathcal{A} \cap F_i$. Then G has a Hamilton path between any vertex in $\mathcal{A} \cap F_0$ and any vertex in $\mathcal{B} \cap F_{N-1}$.

Proof. Let $a_0 \in \mathcal{A} \cap F_0$ and $b_{N-1} \in \mathcal{B} \cap F_{N-1}$. For each F_i , $i = 0, 1, \dots, N-1$, choose a Hamilton path P_i in $G[F_i]$ between b_i and a_i . Then

$$\left(\bigcup_{i=0}^{N-2} \{b_i a_{i+1}\} \right) \cup \left(\bigcup_{i=0}^{N-1} E(P_i) \right)$$

are the edges of a Hamilton path in G between a_0 and b_{N-1} . □

Corollary A.7. Let G be a B -graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$ and bipartition $(\mathcal{A}, \mathcal{B})$ such that $[F_{i-1}, F_i]$ is a set of independent edges and $|[F_{i-1}, F_i]| \geq 2$, $i = 1, 2, \dots, N-1$. If for each $i = 1, 2, \dots, N-1$, the endpoints of any pair of edges in $[F_{i-1}, F_i]$ induces a 4-cycle in G , then G has a Hamilton path between any vertex in $\mathcal{A} \cap F_0$ and any vertex in $\mathcal{B} \cap F_{N-1}$.

Proof. Let $a_0 \in \mathcal{A} \cap F_0$ and $b_{N-1} \in \mathcal{B} \cap F_{N-1}$. For each $[F_{i-1}, F_i]$, $i = 1, 2, \dots, N-1$, choose two edges $b_{i-1}a_i$ and $b'_{i-1}a'_i$. Then $G[\{a_i, a'_i, b_{i-1}, b'_{i-1}\}]$ induces a 4-cycle $a_i a'_i b'_{i-1} b_{i-1} a_i$. Note that either $b_{i-1} \in \mathcal{B}$ or $b'_{i-1} \in \mathcal{B}$. Without loss of generality, suppose $b_{i-1} \in \mathcal{B}$. Then $a_i \in \mathcal{A}$. The result follows by Lemma A.6. □

Definition A.8. An *odd flare* is a tree obtained from $K_{1,t}$, $t \geq 3$ and odd, by a single subdivision of one edge.

Lemma 3.6. Let T be a tree with bipartition (A, B) , where $|A| := \ell$ and $|B| := r$, and let $G_3(T)$ be the 3-colouring graph of T with colours $C = \{1, 2, 3\}$. Define c_{ij} to be the vertex of $G_3(T)$ with $c_{ij}(a) = i$ for all $a \in A$ and $c_{ij}(b) = j$ for all $b \in B$.

- (1) If $\ell, r > 0$ are both even, then $G_3(T)$ has no spanning subgraph consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$.
- (2) If $\ell > 1$ is odd and $r > 0$ is even, then $G_3(T)$ has a Hamilton path from c_{12} to c_{23} .
- (3) If $\ell > 1$ and $r > 1$ are both odd, then $G_3(T)$ has a Hamilton path from c_{12} to c_{13} .

Proof. (1): Suppose $\ell, r > 0$ are both even. By Lemma A.2, $\mathcal{H} := G_3(T)$ is bipartite. Suppose \mathcal{H} has bipartition $(\mathcal{A}, \mathcal{B})$. Without loss of generality, assume $c_{12} \in \mathcal{A}$. Since $d_{\mathcal{H}}(c_{12}, c_{ij})$ is even for each $c_{ij} \in \{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$ and \mathcal{H} is bipartite, we have $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\} \subseteq \mathcal{A}$. By [6, Theorem 5.5] there is a Hamilton cycle in \mathcal{H} , and thus $|\mathcal{A}| = |\mathcal{B}|$. It follows that there is no spanning subgraph of \mathcal{H} consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$ (otherwise $|\mathcal{A}| > |\mathcal{B}|$).

(2): Suppose $\ell > 1$ is odd and $r > 0$ is even. Then $\ell + r \geq 5$. We first prove that $G_3(T)$ has a Hamilton path between c_{12} and c_{23} whenever T is P_5 or any odd flare. If $T \cong P_5$ (with $|\mathcal{A}| = 3, |\mathcal{B}| = 2$), then there is a Hamilton path between c_{12} to c_{23} in $G_3(P_5)$, as described in Figure 15. A 3-colouring f of $P_5 = x_1x_2x_3x_4x_5$ is represented by the string $f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)$; for example, $c_{12} = 12121$ and $c_{23} = 23232$.

12121	21313	32321	13131	23231	31212
12321	21323	32121	12131	13231	31232
12323	21321	32123	32131	13232	31231
12313	31321	12123	32132	13212	21231
12312	31323	13123	12132	13213	21232
32312	31313	23123	13132	23213	21212
31312	32313	23121	23132	21213	23212
21312	32323	13121	23131	31213	23232

Figure 15: A Hamilton path in $G_3(P_5)$ from c_{12} to c_{23} .

Let T be an odd flare on n vertices with u denoting the unique vertex of degree two and let $N_T(u) = \{v, v'\}$ where v is the unique vertex of degree $n - 2$ (Figure 16).

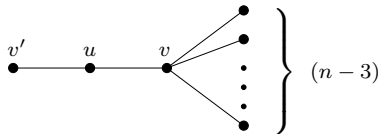


Figure 16: An odd flare T .

Here $|\mathcal{A}| = n - 2$ and $|\mathcal{B}| = 2$. We partition $\mathcal{H} := G_3(T)$ according to the colours of u and v . For each $1 \leq i \neq j \leq 3$, let $V_{ij} := \{c \in V(\mathcal{H}) \mid c(u) = i \text{ and } c(v) = j\}$ and let L_{ij} be an assignment of lists to the vertices of H such that $L_{ij}(u) := \{i\}$, $L_{ij}(v) := \{j\}$ and $L_{ij}(w) := \{1, 2, 3\}$ for $w \in V(H - \{u, v\})$. Note that $G_{L_{ij}}(H) \cong \mathcal{H}[V_{ij}] \cong Q_{n-2}$, for each $1 \leq i \neq j \leq 3$. Let $\mathcal{H}_1 := \mathcal{H}[V_{32} \cup V_{12}] \cong Q_{n-3} \square P_4$, $\mathcal{H}_4 := \mathcal{H}[V_{21} \cup V_{31}] \cong Q_{n-3} \square P_4$, $\mathcal{H}_2 := \mathcal{H}[V_{13}]$, and $\mathcal{H}_3 := \mathcal{H}[V_{23}]$.

For $\{i, j, k\} = \{1, 2, 3\}$, let $d_{ijk} \in V(\mathcal{H})$ denote the vertex with $d_{ijk}(v) = j$, $d_{ijk}(v') = k$, and $d_{ijk}(w) = i$ for all $w \in N_T(v)$. Then $[V_{12}, V_{13}] = \{c_{12}d_{132}, d_{123}c_{13}\}$ and $[V_{23}, V_{21}] = \{c_{21}d_{231}, d_{213}c_{23}\}$.

Claim. For $m \geq 2$, every edge of $Q_m \square P_4$ is in a Hamilton cycle of $Q_m \square P_4$.

The claim follows by induction and the fact that any pair of distinct edges of Q_{m-1} ($m \geq 3$) belongs to a Hamilton cycle of Q_{m-1} (for example, see [7, Theorem 4.1]).

In what follows, we define cycles and paths to construct a Hamilton path from c_{12} to c_{23} in $G_3(T)$ when T is an odd flare on n vertices. See Figure 17 for the case $n = 5$; here,

the labels of the two columns of V_{ij} represent the colour choices for v' and the labels of the rows of V_{ij} represent the colour choices for the two vertices in $N_T(v) \setminus \{u\}$. Let C_1 be a Hamilton cycle of \mathcal{H}_1 containing $c_{12}d_{123}$ and C_4 be a Hamilton cycle of \mathcal{H}_4 containing $d_{213}c_{21}$; these exist by the previous claim. Let $c \in N_{\mathcal{H}_3}(c_{23})$ have $c(v') = 3$ and $d \in V_{13}$ be the unique vertex of \mathcal{H}_2 adjacent to c . By [7, Theorem 4.1], there is a Hamilton cycle C_3 in \mathcal{H}_3 containing the edges cc_{23} and $c_{23}d_{231}$. Observe $d_{\mathcal{H}_2}(d, c_{13})$ is odd. Since \mathcal{H}_2 is Hamilton laceable, there is a Hamilton path P between d and c_{13} (see Figure 17).

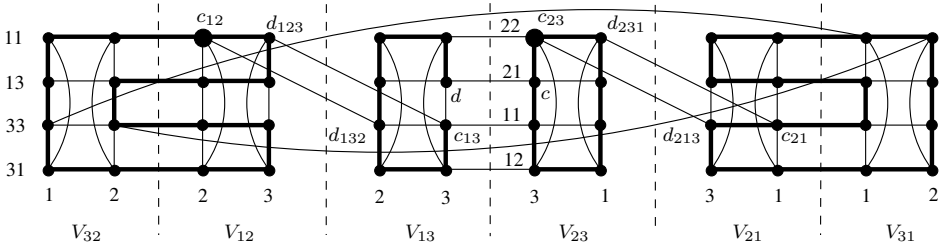


Figure 17: $\mathcal{H} = G_3(T)$, T an odd flare on five vertices, along with C_1 , P , C_3 , and C_4 .

Now,

$$(C_1 - \{c_{12}d_{123}\}) \cup \{d_{123}c_{13}\} \cup P \cup \{dc\} \cup (C_3 - \{c_{23}c, c_{23}d_{231}\}) \\ \cup \{d_{231}c_{21}\} \cup (C_4 - \{c_{21}d_{213}\}) \cup \{d_{213}c_{23}\}$$

is a Hamilton path in \mathcal{H} between c_{12} and c_{23} (see Figure 18 for $n = 5$).

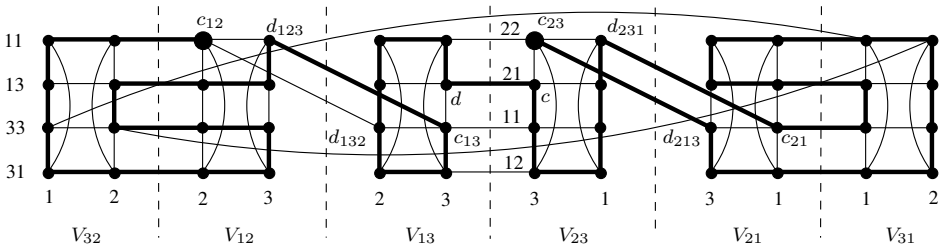


Figure 18: A Hamilton path between c_{12} and c_{23} in $\mathcal{H} = G_3(T)$, T an odd flare on five vertices.

Now suppose T is a tree on $n > 5$ vertices with n odd that is not isomorphic to a star or an odd flare. Then there are leaves $x, y \in V(T)$ with $d_T(x, y) \geq 3$. By choosing leaves x and y so that $d_T(x, y) \geq 3$ is minimum, $T' := T - \{x, y\}$ is not a star. Let $N_T(x) := \{x'\}$ and $N_T(y) := \{y'\}$; since $d_T(x, y) \geq 3$, $x' \neq y'$.

Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By the inductive hypothesis, \mathcal{H}' has a Hamilton path between c'_{12} and c'_{23} . Let $V(\mathcal{H}') := \{f_0, f_1, \dots, f_{N-1}\}$. Since \mathcal{H}' has a Hamilton path, we may assume that $f_0 f_1 \dots f_{N-1}$ is a Hamilton path in \mathcal{H}' between $f_0 = c'_{12}$ and $f_{N-1} = c'_{23}$. Since c'_{12} and c'_{23} differ in colour on at least two vertices, f_0 is not adjacent to f_{N-1} in \mathcal{H}' .

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of \mathcal{H} that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of \mathcal{H} , and $\mathcal{H}[F_i] \cong C_4$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{x', y'\}$, then $|[F_{i-1}, F_i]| = 4$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong Q_3$. Otherwise, f_{i-1} and f_i differ on the colour of x' or y' , implying that $|[F_{i-1}, F_i]| = 2$, the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is a 4-cycle, and $\mathcal{H}[F_{i-1} \cup F_i] \cong P_4 \square K_2$.

Consider the spanning subgraph G of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i] \right).$$

Note that G is a connected B -graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$.

First suppose $c_{23} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{23} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{23} . Now suppose $c_{23} \in \mathcal{A}$. Since $d_G(c_{12}, c_{23})$ is odd, \mathcal{H} must have an edge e with both endpoints in \mathcal{A} or both endpoints in \mathcal{B} . Suppose $e \in [F_p, F_q]$, where $0 \leq p < q \leq N-1$. Since f_0 is not adjacent to f_{N-1} in \mathcal{H}' , either $p \neq 0$ or $q \neq N-1$. Without loss of generality we assume $p \neq 0$. Then $f_p f_q \in E(\mathcal{H}')$, and either

- (i) $|[F_p, F_q]| = 4$ and $\mathcal{H}[F_p \cup F_q] \cong Q_3$, or
- (ii) $|[F_p, F_q]| = 2$, the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_p, F_q]$ is a 4-cycle, and $\mathcal{H}[F_p \cup F_q] \cong P_4 \square K_2$.

In either case, there exists another edge $e' \in [F_p, F_q]$ such that $e := uv$, $e' := u'v'$, $u, u' \in F_p$, and $uvv'u'u$ is a 4-cycle. The choice of $e = uv$ ensures that $u, v \in \mathcal{A}$ or $u, v \in \mathcal{B}$.

Consider the spanning tree \widehat{T} of \mathcal{H}' with edge set

$$E(\widehat{T}) := \{f_{i-1}f_i \mid 1 \leq i \leq q-1\} \cup \{f_{i-1}f_i \mid q+1 \leq i \leq N-1\} \cup \{f_p f_q\}.$$

We define J to be the spanning subgraph of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{f_i f_j \in E(\widehat{T})} [F_i, F_j] \right).$$

Then J is a connected B -graph. Let $(\mathcal{K}, \mathcal{L})$ be the bipartition of J ; we may assume that $c_{12} \in \mathcal{K}$. Since $f_p f_q \in E(\widehat{T})$, $[F_p, F_q] \subseteq E(J)$; in particular, $e = uv$ and $e' = u'v'$ are edges of J , so u and v are in different parts of the partition $(\mathcal{K}, \mathcal{L})$, and thus $c_{23} \in \mathcal{L}$.

Case 1. If $|[F_p, F_{p+1}]| = |[F_p, F_q]| = 2$, then $f_p f_q, f_p f_{p+1} \in E(\mathcal{H}')$ arise from colour changes on x' and y' . Since there are only three possible vertex colours, there is only one possible colour that x' could change to, and only one possible colour that y' could change to; i.e., one of $f_p f_q, f_p f_{p+1}$ arises from a colour change on x' and the other from a colour change on y' . Assuming that $\mathcal{H}[F_p]$ is the 4-cycle $uu'ww'u$, it follows that u, u' are incident to the edges of $[F_p, F_q]$ and that without loss of generality u', w are incident to the edges of $[F_p, F_{p+1}]$.

Let $[F_p, F_q] = \{uv, u'v'\}$ and $[F_p, F_{p+1}] = \{wz, u'z'\}$. Since $uu' \in E(J)$, exactly one of u, u' is in \mathcal{L} .

Case 1(a). First suppose that $u \in \mathcal{L}$. Let J_1 denote the subgraph of J induced by $\cup_{i=0}^p F_i$. Then J_1 is a B -graph with vertex partition $\{F_0, F_1, \dots, F_p\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$ and $u \in \mathcal{L} \cap F_p$. Thus J_1 has a Hamilton path R_1 between c_{12} and u . Note that the proof of Corollary A.7 implies that R_1 can be constructed so as to contain the edge $u'w$.

Let J_2 be the subgraph of J induced by $\cup_{i=q}^{N-1} F_i$. Then J_2 is a B -graph with vertex partition $\{F_q, F_{q+1}, \dots, F_{N-1}\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $v \in \mathcal{K} \cap F_q$ and $c_{23} \in \mathcal{L} \cap F_{N-1}$. Thus J_2 has a Hamilton path R_2 between v and c_{23} .

Finally, let J_3 be the subgraph of J induced by $\cup_{i=p}^{q-1} F_i$. Then by Lemma 2.7, J_3 has a Hamilton cycle C containing the edges uu' , ww' and uw' . Let R_3 be the path between u' and w obtained by deleting u and w' from C .

Now concatenate paths R_1, R_2, R_3 and delete edge $u'w \in R_1$ to form a Hamilton path between c_{12} and c_{23} .

Case 1(b). Now suppose that $u \in \mathcal{K}$. Then $u' \in \mathcal{L}$. Since $p \geq 1$ and $|[F_p, F_{p+1}]| = |[F_p, F_q]| = 2$, we have $|[F_{p-1}, F_p]| = 4$. Let $t \in F_{p-1}$ be such that $tu \in [F_{p-1}, F_p]$. Then $t \in \mathcal{L}$.

Let J_1 denote the subgraph of J induced by $\cup_{i=0}^{p-1} F_i$. Then J_1 is a B -graph with vertex partition $\{F_0, F_1, \dots, F_{p-1}\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$ and $t \in \mathcal{L} \cap F_{p-1}$. Thus J_1 has a Hamilton path R_0 between c_{12} and t . Let R_1 be the concatenation of paths R_0 and $tuw'wu'$.

We define J_2 and R_3 as in Case 1(a). The same argument with v' in place of v gives a Hamilton path R_2 between v' and c_{23} . Now concatenate paths R_1, R_2, R_3 and delete edge $u'w \in R_1$ to form a Hamilton path between c_{12} and c_{23} .

Case 2. Suppose $|[F_p, F_q]| = 4$ and label the 4-cycle of $\mathcal{H}[F_p]$ as $uu'ww'u$. Let J_3 be the subgraph of J induced by $\cup_{i=p}^{q-1} F_i$. Then by Lemma 2.7, J_3 has a Hamilton cycle C . Without loss of generality, suppose C contains the edges uu' , ww' and uw' . Let R_3 be the path between u' and w obtained by deleting u and w' from C .

Let J_1 denote the subgraph of J induced by $\cup_{i=0}^p F_i$. Then J_1 is a B -graph with vertex partition $\{F_0, F_1, \dots, F_p\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$. Thus J_1 has Hamilton paths R'_1 and R''_1 between c_{12} and the two vertices in $\mathcal{L} \cap F_p$. Let R_1 be one of R'_1 and R''_1 such that R_1 contains edge $u'w$. Suppose R_1 is between c_{12} and t . Then $t \in \mathcal{L}$. Let $t' \in F_q$ be such that $tt' \in [F_p, F_q]$. Then $t' \in \mathcal{K}$.

We define J_2 as in Case 1(a). The same argument with t' in place of v gives a Hamilton path R_2 between t' and c_{23} . Now concatenate paths R_1, R_2, R_3 and delete edge $u'w \in R_1$ to form a Hamilton path between c_{12} and c_{23} .

Case 3. Suppose $|[F_p, F_{p+1}]| = 4$. Let J_1 denote the subgraph of J induced by $(\cup_{i=0}^p F_i) \cup (\cup_{i=q}^{N-1} F_i)$. Then J_1 is a B -graph with vertex partition

$$\{F_0, F_1, \dots, F_p, F_q, F_{q+1}, \dots, F_{N-1}\}$$

and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$ and $c_{23} \in \mathcal{L} \cap F_{N-1}$. Thus J_1 has a Hamilton path R between c_{12} and c_{23} . Note that the proof

of Corollary A.7 implies that R can be constructed so as to contain three edges of $\mathcal{H}[F_p]$.

Let J_3 be the subgraph of J induced by $\cup_{i=p+1}^{q-1} F_i$. Then by Lemma 2.7, J_3 has a Hamilton cycle C . Note that C contains three edges of $\mathcal{H}[F_{p+1}]$.

By the Pigeonhole Principle there are $s, t \in F_p$ and $s', t' \in F_{p+1}$ with $st \in R$, $s't' \in C$ such that $ss'tt's$ is a 4-cycle in $\mathcal{H}[F_p \cup F_{p+1}]$. Now $(R \cup C) - \{st, s't'\}$ is a Hamilton path in J between c_{12} and c_{23} .

(3): Finally suppose, $\ell > 1$ and $r > 1$ are both odd. We define $\mathcal{E}_{s,t}^k$ to be the tree obtained from P_k with ends u and v by appending s leaves to u and t leaves to v . The proof is by induction and has the following base cases.

Base Case 1. We first prove that $\mathcal{H} := G_3(T)$ has a Hamilton path between c_{12} and c_{13} when $T \cong \mathcal{E}_{0,0}^{4k+2}$ for $k \geq 0$, that is, when $T \cong P_{4k+2}$. If $T \cong P_2$ then $c_{12}c_{32}c_{31}c_{21}c_{23}c_{13}$ is such a Hamilton path. Suppose $k > 0$ and $T \cong P_{4k+2}$. Let u and v be the leaves of T , $N_T(u) := \{u'\}$, $N_T(v) := \{v'\}$, $N_T(u') := \{u, u''\}$ and $N_T(v') := \{v, v''\}$. Then $T' := T - \{u, u', v, v'\}$ is isomorphic to $P_{4(k-1)+2}$. Let (A', B') denote the bipartition of T' with $A' \subseteq A$, $B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By the inductive hypothesis, \mathcal{H}' has a Hamilton path between c'_{12} and c'_{13} . Let $V(\mathcal{H}') := \{f_0, f_1, \dots, f_{N-1}\}$. Since \mathcal{H}' has a Hamilton path we may assume that $f_0f_1 \cdots f_{N-1}$ is a Hamilton path in \mathcal{H}' between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of \mathcal{H} that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of \mathcal{H} , and $\mathcal{H}[F_i] \cong P_4 \square P_4$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{u'', v''\}$, then $|[F_{i-1}, F_i]| = 16$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong (P_4 \square P_4) \square K_2$. Otherwise, f_{i-1} and f_i differ on the colour of u'' or v'' , implying that $|[F_{i-1}, F_i]| = 8$, and the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is $C_4 \square P_4$.

Consider the spanning subgraph G of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i] \right).$$

Note that by Remark A.4, G is a connected B -graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since \mathcal{H} is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13} .

Base Case 2. Let $T \cong \mathcal{E}_{2s,2t}^{4k+2}$ or $T \cong \mathcal{E}_{2s+1,2t+1}^{4k}$ with $s, t \geq 1$, $k \geq 0$, and $T' \cong P_{4k+2}$ be obtained from T by deleting $2s$ leaves adjacent to u and $2t$ leaves adjacent to v .

Let (A', B') denote the bipartition of T' with $A' \subseteq A$, $B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By Case 1, \mathcal{H}' has a Hamilton path between c'_{12} and c'_{13} . Let $V(\mathcal{H}') := \{f_0, f_1, \dots, f_{N-1}\}$. Since \mathcal{H}' has a Hamilton path we may assume that $f_0f_1 \cdots f_{N-1}$ is a Hamilton path in \mathcal{H}' between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of \mathcal{H} , and $\mathcal{H}[F_i] \cong Q_{2s+2t}$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{u, v\}$, then $|[F_{i-1}, F_i]| = 2^{2s+2t}$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong Q_{2s+2t} \square K_2 \cong Q_{2s+2t+1}$. If f_{i-1} and f_i differ on the colour of u then $|[F_{i-1}, F_i]| = 2^{2t}$, and the subgraph of \mathcal{H} induced by the

endpoints of the edges of $[F_{i-1}, F_i]$ is Q_{2t+1} . Otherwise f_{i-1} and f_i differ on the colour of v implying that $|[F_{i-1}, F_i]| = 2^{2s}$, and the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is Q_{2s+1} .

Consider the spanning subgraph G of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i] \right).$$

Note that by Remark A.4, G is a connected B -graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since \mathcal{H} is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13} .

Base Case 3. Let $T \cong \mathcal{E}_{2s+1,1}^{4k}$ with $s, k \geq 1$, and let $T' \cong P_{4k-2}$ be obtained from T by deleting the $2s+2$ leaves and the vertices u and v . Define u', v' as the leaves of T' so that in T , u' is adjacent to u and v' is adjacent to v .

Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By Case 1, \mathcal{H}' has a Hamilton path between c'_{12} and c'_{13} . Let $V(\mathcal{H}') := \{f_0, f_1, \dots, f_{N-1}\}$. Since \mathcal{H}' has a Hamilton path we may assume that $f_0 f_1 \dots f_{N-1}$ is a Hamilton path in \mathcal{H}' between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of \mathcal{H} , and $\mathcal{H}[F_i] \supseteq P_4 \square P_{2^{2s+2}}, 0 \leq i \leq N-1$. This follows since $G_L(K_{1,2s+1})$ has a Hamilton path $P_{2^{2s+2}}$ where L is an assignment of lists in which vertices of degree one have lists $\{1, 2, 3\}$ and the remaining vertex has list $\{1, 2\}$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{u', v'\}$, then $|[F_{i-1}, F_i]| = 4 \cdot 2^{2s+1}$ and $\mathcal{H}[F_{i-1} \cup F_i] \supseteq (P_4 \square P_{2^{2s+1}}) \square K_2$. If f_{i-1} and f_i differ on the colour of v' then $|[F_{i-1}, F_i]| = 2 \cdot 2^{2s+1}$, and the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ contains $(P_2 \square P_{2^{2s+1}}) \square K_2$. Otherwise f_{i-1} and f_i differ on the colour of u' implying that $|[F_{i-1}, F_i]| = 4 \cdot 2^{2s}$, and the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ contains $(P_4 \square P_{2^{2s}}) \square K_2$.

Consider the spanning subgraph G of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i] \right).$$

Note that by Remark A.4, G is a connected B -graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since \mathcal{H} is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13} .

Induction Step. Now suppose T is a tree with bipartition (A, B) , where $|A| := \ell > 1$ and $|B| := r > 1$ are both odd and T is not isomorphic to any of the graphs in Base Cases 1 to 3. If every pair of leaves $x, y \in A$ or $x, y \in B$ satisfy $d_T(x, y) \leq 2$, then $T \cong \mathcal{E}_{s,t}^k$; since $\ell, r > 1$ are both odd, T is isomorphic to one of the graphs in Base Cases 1 to 3. Thus, there are leaves $x, y \in A$ (or $x, y \in B$) with $d_T(x, y) \geq 3$.

Case 1. If $T - \{x, y\}$ is a star, then T is the graph obtained from $K_{1,2s+1}$, $s \geq 1$, by subdividing two of its edges. Let $u \in V(T)$ be the vertex of degree $2s+1$ and $v \in V(T)$ a leaf adjacent to u . Define $T' := T[\{u, v\}]$ and observe $T' \cong K_2$.

Let $\mathcal{H}' := G_3(T')$ have Hamilton path $f_0 f_1 f_2 f_3 f_4 f_5 := c'_{12} c'_{32} c'_{31} c'_{21} c'_{23} c'_{13}$, where $c'_{ij}(u) = i$ and $c'_{ij}(v) = j$. For $0 \leq i \leq 5$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_5\}$ is a partition of the vertices of \mathcal{H} , and $\mathcal{H}[F_i] \cong P_4 \square P_4 \square Q_{2s-2}$, $0 \leq i \leq 5$. If f_{i-1} and f_i differ on the colour of vertex v , then $|[F_{i-1}, F_i]| = 4 \cdot 4 \cdot 2^{2s-2}$ and $\mathcal{H}[F_{i-1} \cup F_i] = (P_4 \square P_4 \square Q_{2s-2}) \square K_2$. If f_{i-1} and f_i differ on the colour of u then $|[F_{i-1}, F_i]| = 4$, and the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is C_4 .

Consider the spanning subgraph G of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^5 E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^5 [F_{i-1}, F_i] \right).$$

Note that by Remark A.4, the graph $P_4 \square P_4 \square Q_{2s-2}$ is Hamilton laceable since $P_4 \square P_4 \square P_{2s-2}$ is a Hamilton laceable spanning subgraph. Thus, G is a connected B -graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since \mathcal{H} is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_5$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13} .

Case 2. Suppose $N_T(x) := \{x'\}$ and $N_T(y) := \{y'\}$, and that $T' := T - \{x, y\}$ is not a star. Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$.

By the inductive hypothesis, \mathcal{H}' has a Hamilton path between c'_{12} and c'_{13} . Let $V(\mathcal{H}') := \{f_0, f_1, \dots, f_{N-1}\}$. Since \mathcal{H}' has a Hamilton path we may assume that $f_0 f_1 \dots f_{N-1}$ is a Hamilton path in \mathcal{H}' between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of \mathcal{H} , and $\mathcal{H}[F_i] \cong C_4$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{x', y'\}$, then $|[F_{i-1}, F_i]| = 4$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong Q_3$. Otherwise, f_{i-1} and f_i differ on the colour of x' or y' , implying that $|[F_{i-1}, F_i]| = 2$, the subgraph of \mathcal{H} induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is a 4-cycle, and $\mathcal{H}[F_{i-1} \cup F_i] \cong P_4 \square K_2$.

Consider the spanning subgraph G of \mathcal{H} with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i] \right).$$

Note that G is a connected B -graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since \mathcal{H} is bipartite by Lemma A.2 and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13} . \square