Kovine, zlitine, tehnologije / letnik 28 / številka 6 / strani 539 do 541 / 1994

General Solution of Heating and/or Cooling of Metallurgical Furnace Wall by Means of Jacobi ϑ_3 Function

Generalno rješenje zagrijavanja i/ili hlađenja zida metalurških peći pomoću Jacobijeve ϑ_3 funkcije

Grozdanić Vladimir, Metalurški fakultet, Zagreb

General solution of heating and/or cooling of metallurgical furnace wall has been developed in the paper. Boundary conditions are time-dependent and solution is especially favourable for metallurgical furnaces which operate discontinuously.

Key words: heating, cooling, metallurgical furnace, Jacobi ϑ_3 function

U radu je izvedeno generalno rješenje zagrijavanja i/ili hlađenja zida metalurških peći s time da su granični uvjeti uzeti kao funkcija vremena pa je posebno pogodno za metalurške agregate koji rade diskontinuirano.

Ključne riječi: zagrijavanje, hlađenje, metalurške peći, Jacobijeva ϑ_3 funkcija

Introduction

Heating and/or cooling of metallurgical furnace wall has been an interesting thermotechnical problem particularly regarding the choice of wall furnace material, optimal control of process in the furnace, and adequate financial savings if this two aspects are harmonized. At heating and/or cooling furnace walls have been submitted to different thermal changes which are a priori time-dependent. That is especially important at periodical heating and/or cooling of the wall. So general solution of the problem has been derived with particular case when the furnace walls have been submitted to periodical changes of temperature.

Definition of the problem of heating and/or cooling of furnace wall

At heating and/or cooling of furnace wall Fourier's differential equation of heat conduction[1] is necessary to solve, which in case of homogenous isotropic material has the form:

$$\frac{\delta u(x,t)}{\delta t} = a\frac{\delta^2 u(x,t)}{\delta x^2} \tag{1}$$

Equation (1) is going to be solved for the domain which is illustrated on **figure 1**.

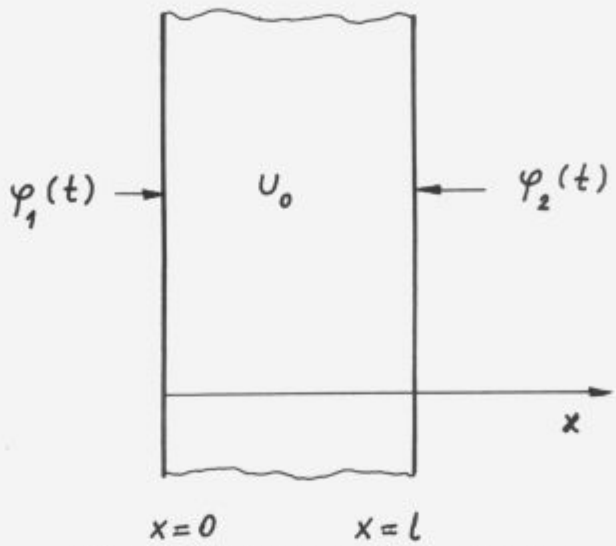

Figure 1: Furnace wall with initial and boundary conditions

Slika 1: Zid peći s početnim i graničnim uvjetima

Initial and boundary conditions are as follows:

$t = 0$	$u(x,0) = u0$	(2)
$t > 0$	$u(0,t) = \varphi_1(t)$	(3)
	$u(l,t) = \varphi_2(t)$	(4)

Solution of the problem

Partial differential equation of heat conduction (1) with adequate initial and boundary conditions (2–4) has been solved by means of Laplace transform[2] which has been defined as:

$$\alpha\{u(x,t)\} \equiv U(x,t) = \int_0^\infty e^{-st} u(x,t)\, dt \tag{5}$$

$$\alpha\left\{\frac{\delta u(x,t)}{\delta t}\right\} = sU(x,s) - u(x,0) \tag{6}$$

$$\alpha\left\{\frac{\delta^2 u(x,t)}{\delta x^2}\right\} = \frac{d^2 U(x,s)}{dx^2} \tag{7}$$

$$\alpha\left\{u(0,t)\right\} = \phi_1(s) \tag{8}$$

$$\alpha\left\{u(l,t)\right\} = \phi_2(s) \tag{9}$$

where

$$\Phi_k(s) = \int_0^\infty e^{-st}\varphi_k(t)dt,\ k = 1,2 \tag{10-11}$$

After applying Laplace transform to (1), the equation adopts the following form:

$$\frac{d^2U(x,s)}{dx^2} - \frac{1}{a}sU(x,s) = -\frac{1}{a}u(x,0) \tag{12}$$

Solution of equation (12) has the form:

$$U(x,s) = C_1(s)\exp\left(x\sqrt{s/a}\right) + C_2(s)\exp\left(-x\sqrt{s/a}\right) + \frac{u(x,0)}{s} \tag{13}$$

Constants C(s) have been defined by means of boundary conditions:

$$U(0,s) = C_1 + C_2 + \frac{u_0}{s} = \Phi_1(s) \tag{14}$$

$$U(l,s) = C_1\exp\left(l\sqrt{s/a}\right) + C_2\exp\left(-l\sqrt{s/a}\right) + \frac{u_0}{s} = \Phi_2(s) \tag{15}$$

and the result is

$$C_1(s) = \frac{\Phi_1(s)\exp\left(-l\sqrt{s/a} - \Phi_2(s) + \frac{u_0}{s}\left[1 - \exp\left(-l\sqrt{s/a}\right)\right]\right)}{\exp\left(-l\sqrt{s/a}\right) - \exp\left(l\sqrt{s/a}\right)} \tag{16}$$

$$C_2(s) = \frac{\Phi_2(s) - \Phi_1(s)\exp\left(l\sqrt{s/a}\right) + \frac{u_0}{s}\left[\exp\left(l\sqrt{s/a}\right) - 1\right]}{\exp\left(-l\sqrt{s/a}\right) - \exp\left(l\sqrt{s/a}\right)} \tag{17}$$

Constants C_1 and C_2 have been substituted in equation (13) and the solution in terms of Laplace transforms is

$$U(x,s) = \Phi_1(s)F_1(x,s) + \Phi_2(s)F_2(x,s) + \frac{u_0}{s}\left[1 - F_1(x,s) - F_2(x,s)\right] \tag{18}$$

where

$$F_1(x,s) = \frac{\mathrm{sh}(l-x)\sqrt{s/a}}{\mathrm{sh}\,l\sqrt{s/a}} \tag{19}$$

$$F_2(x,s) = \frac{\mathrm{sh}\,x\sqrt{s/a}}{\mathrm{sh}\,l\sqrt{s/a}} \tag{20}$$

Functions $F_1(x,s)$ and $F_2(x,s)$ are transforms of Jacobi ϑ_3 function[3]:

$$\begin{aligned}\alpha^{-1}\left\{F_1(x,s)\right\} &= -\frac{a}{l}\frac{\delta}{\delta x}\vartheta_3\left[\frac{x}{2l},\ \frac{at}{l^2}\right] = \\ &= \frac{2\pi a}{l^2}\sum_{n=1}^{\infty} n\exp\left(-n^2\pi^2 at/l^2\right)\sin\frac{n\pi x}{l}\end{aligned} \tag{21}$$

$$\begin{aligned}\alpha^{-1}\left\{F_2(x,s)\right\} &= \frac{a}{l}\frac{\delta}{\delta x}\vartheta_3\left[\frac{l-x}{2l},\frac{at}{l^2}\right] = \\ &= \frac{2\pi a}{l^2}\sum_{n=1}^{\infty}(-1)^n n\exp\left(-n^2\pi^2 at/l^2\right)\sin\frac{n\pi x}{l}\end{aligned} \tag{22}$$

where is ϑ_3 (v, x) Jacobi ϑ_3 function[4-6]

$$\begin{aligned}\vartheta_3(v,x) &= \sum_{k=-\infty}^{\infty}\exp\left(2k\pi iv - k^2\pi x\right) = \\ &= 1 + 2\sum_{n=1}^{\infty}\exp\left(-n^2\pi^2 x\right)\cos 2n\pi v\end{aligned} \tag{23}$$

By means of Borel theorem (theorem about image of functions composition) and theorem about integration of the original final solution of equation (18) in the real area has been obtained, which represents temperature distribution in furnace wall:

$$\begin{aligned}u(x,t) = &-\frac{a}{l}\int_{\tau=0}^{t}\frac{\delta}{\delta x}\vartheta_3\left[\frac{x}{2l},\frac{a}{l^2}(t-\tau)\right]\varphi_1(\tau)d\tau + \\ &+\frac{a}{l}\int_{\tau=0}^{t}\frac{\delta}{\delta x}\vartheta_3\left[\frac{l-x}{2l},\frac{a}{l^2}(t-\tau)\right]\varphi_2(\tau)d\tau + \\ &+u_0\left\{1 + \frac{a}{l}\int_{\tau=0}^{t}\frac{\delta}{\delta x}\vartheta_3\left[\frac{x}{2l},\frac{a}{l^2}\tau\right]d\tau - \right. \\ &\left. -\frac{a}{l}\int_{\tau=0}^{t}\frac{\delta}{\delta x}\vartheta_3\left[\frac{l-x}{2l},\frac{a}{l^2}\tau\right]d\tau\right\}\end{aligned} \tag{24}$$

In the case when furnace wall has been submitted to periodical temperature changes, for example

$$\varphi_1(t) = \sin bt \tag{25}$$

$$\varphi_2(t) = \cos ct \tag{26}$$

The temperature distribution has the form:

$$\begin{aligned}u(x,t) = &\frac{2\pi a}{l^2}\sum_{k=1}^{\infty}k\sin\frac{\pi kx}{l} \\ &\left\{\exp\left(-\pi^2k^2at\right)\left[\frac{b}{\frac{\pi^4k^4a}{l^4}+b^2} - (-1)^k\frac{\frac{\pi^2k^2a}{l^2}}{\frac{\pi^4k^4a^2}{l^4}+c^2}\right] + \right. \\ &\left. +\frac{\frac{\pi^2k^2a}{l^2}\sin bt - b\cos bt}{\frac{\pi^4k^4a^4}{l^4}+b^2} + (-1)^k\frac{\frac{\pi^2k^2a}{l^2}\cos ct + c\sin ct}{\frac{\pi^4k^4a^2}{l^4}+c^2}\right\} + \\ &+u_0\left\{1 - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[1 - \exp\left(-4\pi^2k^2at/l^2\right)\right]\sin\frac{2\pi kx}{l}\right\}\end{aligned} \tag{27}$$

Conclusion

In real conditions of heating and/or cooling of metallurgical furnace walls boundary conditions have been time-dependent. So general solution of heating and/or cooling has been relatively complicated thermotechnical problem which has been solved in the paper by means of classical methods with Jacobi ϑ_3 function.

List of simbols

a - thermal difusivity
b - constant
c - constant
i - imaginary unit
k - integer number
l - thickness of furnace wall
n - natural number
t - time
u - temperature
x - coordinate
$\Phi1$, $\Phi2$ - functions of time
τ - time
ϑ_3 - Jacobi's function

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