ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 20 (2021) 69–87
https://doi.org/10.26493/1855-3974.2226.e93
(Also available at http://amc-journal.eu)

On plane subgraphs of complete topological drawings*

Alfredo García Olaverri †, Javier Tejel Altarriba ‡

Departamento de Métodos Estadísticos and IUMA, University of Zaragoza, Spain

Alexander Pilz §

Institute of Software Technology, Graz University of Technology, Austria

Received 22 January 2020, accepted 15 October 2020, published online 17 August 2021

Abstract

Topological drawings are representations of graphs in the plane, where vertices are represented by points, and edges by simple curves connecting the points. A drawing is *simple* if two edges intersect at most in a single point, either at a common endpoint or at a proper crossing. In this paper we study properties of maximal plane subgraphs of simple drawings D_n of the complete graph K_n on n vertices. Our main structural result is that maximal plane subgraphs are 2-connected and what we call *essentially* 3*-edge-connected*. Besides, any maximal plane subgraph contains at least $\lceil 3n/2 \rceil$ edges. We also address the problem of obtaining a plane subgraph of D_n with the maximum number of edges, proving that this problem is NP-complete. However, given a plane spanning connected subgraph of D_n, a maximum plane augmentation of this subgraph can be found in $O(n^3)$ time. As a side result, we also show that the problem of finding a largest compatible plane straight-line graph of two labeled point sets is NP-complete.

Keywords: Graph, topological drawing, plane subgraph, NP-complete problem.

Math. Subj. Class. (2020): 05C10, 68R10

*This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 734922.

†Supported by MINECO project MTM2015-63791-R and Gobierno de Aragón Grant E41-17 (FEDER).

‡Supported by MINECO project MTM2015-63791-R, Gobierno de Aragón Grant E41-17 and project PID2019-104129GB-I00 / AEI / 10.13039/501100011033 of the Spanish Ministry of Science and Innovation.

§Supported by a Schrödinger fellowship of the Austrian Science Fund (FWF): J-3847-N35.

E-mail addresses: olaverri@unizar.es (Alfredo García Olaverri), jtejel@unizar.es (Javier Tejel Altarriba), apilz@ist.tugraz.at (Alexander Pilz)

1 Introduction

In a *topological drawing* (in the plane or on the sphere) of a graph, vertices are represented by points and edges by simple curves connecting the corresponding pairs of points.Usually, we only consider drawings satisfying some natural non-degeneracy conditions, in particular a drawing is called *simple* (or a *good drawing*) if two edges intersect at most in a single point, either at a common endpoint or at a crossing in their relative interior. When all the edges of a topological drawing are straight-line segments, then the drawing is called a *rectilinear drawing* or *geometric graph*.

In this paper we consider only simple topological drawings of the complete graph K_n on n vertices. Simple topological drawings of complete graphs have been studied extensively, mainly in the context of crossing number problems. It is well known that a drawing minimizing the number of crossings has to be simple, and besides, if $n \geq 8$, the drawings of K_n minimizing that crossing number are not rectilinear. We refer the reader to [1, 3, 4] for recent advances on the Harary-Hill conjecture on the minimum number of crossings of drawings of K_n, and to the survey [22] for some variants on this crossing number problem.

The problem of enumerating all the non-isomorphic drawings of K_n has been studied in [2, 12, 13, 18] (two drawings are isomorphic if there is a homeomorphism of the sphere that transforms one drawing into the other).

Let D_n be a simple topological drawing of K_n. Herein, we consider graphs in connection with their drawings, and in particular when addressing subgraphs of K_n we also consider the associated sub-drawing of D_n. We are interested in crossing-free edge sets F in D_n, and we will say that F is a plane subgraph of D_n. Crossing-free edge sets in D_n have attracted considerable attention, in part because problems on embedding graphs on a set of points usually generalize to finding plane subgraphs of D_n. For instance, the problem of computing the maximum number of plane Hamiltonian cycles that a simple drawings D_n can contain, is a generalization of the same problem considering only rectilinear drawings of K_n. And this last is the (open) problem of computing the maximum number of simple n-gons that can be formed on n points in the plane.

There are relatively few results on plane subgraphs of D_n. It is well known that in any drawing D_n of K_n, there are plane subgraphs with $2n-3$ edges, and that there are at most $2n-2$ edges uncrossed by any other edge [6, 8, 19]. Pach, Solymosi, and Tóth [14] showed that any D_n has $\Omega\left(\log^{1/6}(n)\right)$ pairwise disjoint edges. This bound was subsequently improved in [5, 16, 23]. The current best bound of $\Omega(n^{1/2-\epsilon})$ is by Ruiz-Vargas [20]. However, the much stronger conjecture that any simple drawing D_n of K_n contains a plane Hamiltonian cycle remains unproved, although it has been verified for $n \leq 9$, see [2].

In the course of their work on disjoint edges and empty triangles in D_n, Fulek and Ruiz-Vargas [6] showed the following lemma.[1]

Lemma 1.1 (Fulek and Ruiz-Vargas [6]). *Between any plane connected subgraph F of D_n and a vertex v not in F, there exist at least two edges from v to F that do not cross F.*

This result can be used to build large plane subgraphs. For instance, we can begin with F consisting of only one edge, then for each vertex v not in F, we add to F the edges from v to F not crossing F. In this way, we will obtain a maximal plane subgraph: a plane subgraph $\overline{F}$ such that any edge $e \notin \overline{F}$ crosses some edge of $\overline{F}$.

[1] Their lemma is actually more general. It does not require F and v to be elements of a drawing of K_n, but rather of a drawing that contains all edges from v to vertices of F.

In Section 2 of this work, we extend that Lemma 1.1 to arbitrary (not necessarily connected) plane subgraphs. Further, in Section 3, we prove that any plane subgraph of D_n can be augmented to a 2-connected plane subgraph of D_n. A consequence of this result is that maximal plane subgraphs contain at least $\min(\lceil 3n/2 \rceil, 2n-3)$ edges, and this bound is tight. Maximal plane subgraphs of D_n have other interesting properties. For example, we show that, when removing two edges from a maximal plane subgraph, it either stays connected or one of the two components is a single vertex. Another consequence of the previous results is that for every vertex v of a drawing D_n, there is a plane subgraph of D_n consisting of the n-vertex star of edges incident to v, plus the edges of a spanning tree on the $n-1$ vertices of $V \setminus \{v\}$.

The problem setting changes when we want our plane graphs not only to be maximal, but also to contain the maximum number of edges. While for geometric graphs, every maximal plane subgraph is a triangulation and thus also has a maximum number of edges, the situation is different for plane subgraphs of D_n. In Section 4, we will prove that computing a plane subgraph of D_n with maximum number of edges is an NP-complete problem. However, if a connected plane spanning subgraph F is given, we can adapt a classic algorithm from computational geometry to show that a maximum plane augmentation of F can be found in $O(n^3)$ time.

As a side result, we also show that the problem of finding a largest compatible plane graph on two labeled point sets is NP-complete.

Finally, going back to Lemma 1.1, we give an $O(n)$ algorithm to compute all the edges from a vertex v to a plane connected subgraph F that do not cross F.

2 Adding a single vertex

We now discuss a generalization of Lemma 1.1 to arbitrary plane subgraphs. This generalization will also follow independently from Theorem 3.1. Still, the following proposition gives further insight on the position of the uncrossed edges around the vertex v, which might help in the construction of algorithms.

We assume that a simple topological drawing D_n of K_n in the plane is given, with vertex set $V = \{v_1, \dots, v_n\}$. If x_1, x_2 are two points on an edge e of D_n (not necessarily the endpoints of e), by line x_1x_2 we mean the portion of the curve e of the drawing placed between the points x_1 and x_2. For a vertex v, the *star graph* with center v is the subgraph formed by the edges connecting v to all the other vertices. We denote this set of edges by $S(v)$, usually call *rays* to these edges emanating from v, and we suppose that the rays of $S(v)$ are (circularly) clockwise ordered. By the clockwise range $[vp, vq]$ of $S(v)$ we mean the ordered set of rays placed clockwise between vp and vq, including rays vp and vq. When vp or vq or both are not included in that ordered set of rays, we will use $(vp, vq]$, $[vp, vq)$ or (vp, vq), respectively. In the same way, we can define counterclockwise ranges.

In the rest of this section, we suppose F is a given plane subgraph of D_n and v a vertex not in F. In the figures, we use red color for the edges of F, so we usually call them *red edges*. We say a ray vr of $S(v)$ is *uncrossed* if it does not cross any edge of F.

Suppose that the ray vr crosses some edge of F, let $e = pq$ be the first edge of F crossed by vr, and let x be the first crossing point. Without loss of generality, we can suppose that the rays vr, vp, and vq appear in this clockwise order in $S(v)$. See Figure 1.

We define the clockwise range R_{cw} of rays centered at v corresponding to the crossing

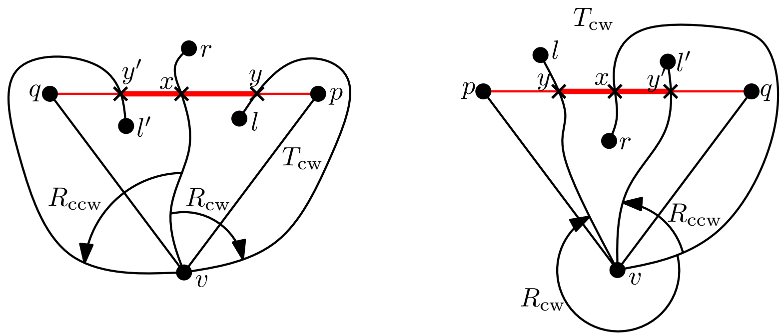

Figure 1: The clockwise and counterclockwise ranges of a first crossing.

x in the following way: if no ray in the clockwise range (vp, vq) crosses the edge pq between x and p, then $R_{\rm cw}$ is the range $(vr, vp]$; otherwise, (some rays in the clockwise range (vp, vq) cross the line xp), $R_{\rm cw}$ is the clockwise range $(vr, vl]$, where vl is the last ray in (vp, vq) crossing the line xp. That implies that if vl crosses xp at the point y, among the intersection points of rays in (vp, vq) with the line xp, the closest to x is y. See Figure 1. Analogously, the range $R_{\rm ccw}$ is defined either as the counterclockwise range $(vr, vq]$ if no edge in the counterclockwise range (vq, vp) crosses the line xq, or as the counterclockwise range $(vr, vl']$, where vl' is the ray in the counterclockwise range (vq, vp) crossing the line xq in a point y' closest to x. By definition, the rays vr, vp, vl, vl', vq appear clockwise in that order around v. Observe that $R_{\rm cw}$ and $R_{\rm ccw}$ are disjoint sets and they are also nonempty, as vp is in $R_{\rm cw}$ and vq is in $R_{\rm ccw}$. The following result generalizes Lemma 1.1.[2]

Proposition 2.1. *Suppose the ray vr first crosses the edge e of F at the point x. Let R_{cw} and R_{ccw} be the clockwise and counterclockwise ranges of rays of v corresponding to that crossing. Then, each one of these two ranges contains an uncrossed ray. As a consequence, $S(v)$ contains at least two uncrossed rays.*

Proof. We prove the statement for $R_{\rm cw}$, the proof for $R_{\rm ccw}$ is identical.

Observe that by definition, no red edge can cross the line vx, and a ray in the clockwise range $(vq, vr]$ cannot cross the line xp. Let y be the crossing point between the red edge $e = pq$ and the ray vl. When $R_{\rm cw}$ is $(vr, vp]$ (i.e., no ray in the clockwise range (vp, vq) crosses xp), then we identify the points p, l and y. The lines vx, xy and yv define a simple closed curve, that divides the plane into two regions $T_{\rm cw}, \overline{T_{\rm cw}}$, where $T_{\rm cw}$ is the region not containing the point q.

From the definition of $T_{\rm cw}$, it follows that a ray containing a point placed in the interior of $T_{\rm cw}$ must be in the range $R_{\rm cw}$. Besides, a red edge can cross the boundary of that region only through the line yv, and hence, if a red edge e crosses yv, one endpoint of e must be inside $T_{\rm cw}$ the other one in $\overline{T_{\rm cw}}$.

The proof is done by induction on $|R_{\rm cw}|$, the number of rays in that range. So, first suppose that the only ray in the range $R_{\rm cw}$ is the ray vp. In this case, $T_{\rm cw}$ is the region bounded by the closed curve vx, xp, pv not containing the point q. This region cannot contain any

[2]Like Lemma 1.1, this result is more general. It does not require F and v to be elements of a drawing of K_n, but rather of a drawing that contains all edges from v to vertices of F.

vertex r' of F, because then vr' would be in R_{cw}, therefore vp must be uncrossed. This proves the base case of the induction.

Now suppose that the proposition has been proved for any clockwise range containing less than $|R_{\text{cw}}|$ rays. Let vr' be the first ray of R_{cw}. Of course, if the ray vr' is uncrossed, the proof is done, so we can suppose that the ray vr' is first crossed by a red edge e' at a point x'. We are going to prove that the clockwise range R'_{cw} corresponding to the crossing x' is strictly contained in R_{cw}. Then, by induction, R'_{cw} contains an uncrossed ray, and thus also R_{cw}. To prove that $R'_{\text{cw}} \subset R_{\text{cw}}$ strictly, it is enough to prove that the clockwise last ray of R'_{cw} is contained in R_{cw}.

Let us first analyze Case A: when the edge e' is precisely the edge e. See Figure 2. In this case, if x' is between x and y, then the clockwise range R'_{cw} corresponding to the

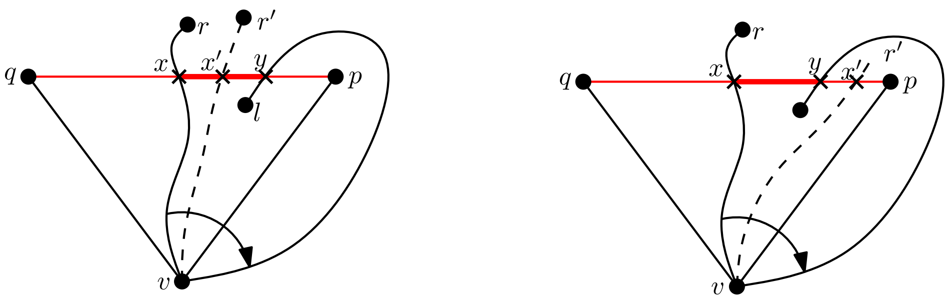

Figure 2: Case A, the ray vr' first crosses the edge $e = pq$.

new crossing point x' is precisely R_{cw} minus its first ray vr'. And if x' is between y and p, then all the points of the line $x'p$ (including point p) are in the interior of the region T_{cw}, therefore the corresponding last ray of R'_{cw} has to be in R_{cw}. Thus, in both subcases is $R'_{\text{cw}} \subset R_{\text{cw}}$ strictly.

Suppose now Case B: when $e' \neq e$. See Figure 3. Clearly, at least one endpoint of e' is

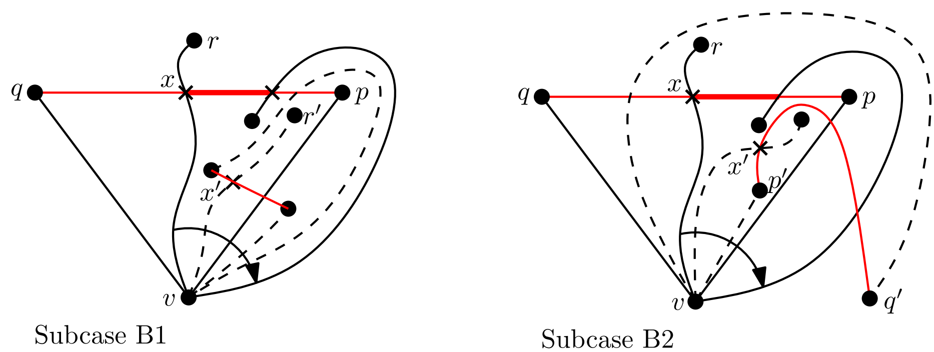

Figure 3: Case B, the ray vr' first crosses an edge $e' \neq e$.

in T_{cw}, as otherwise the ray vl would be crossed twice by e'. Hence, either both endpoints are in T_{cw}, subcase B1, or one of them is in T_{cw} and the other one in $\overline{T_{\text{cw}}}$, subcase B2. In subcase B1, the entire edge $e' = p'q'$ must be inside T_{cw}. Therefore, any ray containing a point of e' must be in R_{cw}. In particular, the last ray of R'_{cw} must be in R_{cw}, and hence, R'_{cw} is strictly contained in R_{cw}.

In subcase B2, an endpoint, p', of e' is inside T_{cw} and the other, q', is in $\overline{T_{\text{cw}}}$. Observe that the ray vp' must be in R_{cw}, however the ray vq' cannot be in R_{cw} because the range (vr, vr') is empty, and a ray in $[vr', vl]$ finishing at q' has to cross the edge e'. See Figure 3. Therefore, the rays vr', vp', vq' appear clockwise around v in this order. Hence, the last ray of R'_{cw} is either vp' or a ray crossing the line $x'p'$. In any case, as the line $x'p'$ is inside T_{cw}, this last ray of R'_{cw} has to be in R_{cw}. This completes the proof. □

3 Structure of maximal plane subgraphs

Let D_n be an arbitrary simple drawing of K_n. In this section, we identify several structural properties of maximal plane subgraphs of D_n, using Lemma 1.1 or Proposition 2.1 as our main tool. Maximal plane subgraphs turn out to be 2-connected. While there are examples of maximal plane subgraphs that are not 3-connected, we elaborate further on the structure, showing that a maximal plane subgraph is either 3-edge-connected or has a vertex of degree 2.

Theorem 3.1. *A maximal plane subgraph of D_n is spanning and 2-connected.*

Proof. The proof is by induction on n. The result is obviously true for $n \leq 3$. For $n > 3$, assume there exists a maximal plane subgraph $\overline{F}$ that is not 2-connected, and let us see that a contradiction is reached.

We first claim that, under this assumption, $\overline{F}$ has no vertices of degree less than 3. Suppose the contrary, that the vertex v has degree ≤ 2. Let F' be the subgraph of $\overline{F}$ obtained after removing the vertex v, and let $\overline{F'}$ be a maximal plane subgraph (in the drawing $D_n - \{v\}$ of K_{n-1}) containing F'. By the induction hypothesis, $\overline{F'}$ is 2-connected. We observe that v cannot have (in $\overline{F}$) degree less than 2, since applying Lemma 1.1 to v and $\overline{F'}$ would give two edges at v not crossing $\overline{F}$, contradicting the maximality of $\overline{F}$. So suppose v has degree 2. As we assume that $\overline{F}$ is not 2-connected, F' cannot be 2-connected. However, $\overline{F'}$ is 2-connected, and hence there exists an edge e' in $\overline{F'} - F'$. By the maximality of $\overline{F}$, e' must cross at least one edge vw of $\overline{F}$ incident to v. But applying Lemma 1.1 to v and $\overline{F'}$ gives at least two edges incident to v not crossing $\overline{F'}$. These two edges and also vw do not cross $\overline{F}$, contradicting the maximality of $\overline{F}$. Therefore, the claim follows.

Assume now that $\overline{F}$ is not connected. Let C_1, C_2 be two connected components of $\overline{F}$. As all vertices have (in $\overline{F}$) degree at least 3, C_1 cannot be an outerplanar graph, and it has more than one face. Without loss of generality, we can suppose that C_2 is in the unbounded face of C_1. Let v_1 be an interior vertex of C_1, F' the graph obtained from $\overline{F}$ by removing v_1, and f_1 the face of F' containing v_1. The face containing C_2 remains unchanged by the removal of v_1. By induction, F' can be completed to a 2-connected plane graph $\overline{F'}$, and due to the maximality of $\overline{F}$, all the edges in $\overline{F'} - F'$ should be in the face f_1. But then, as C_2 is outside f_1, $\overline{F'}$ could not be connected, a contradiction. Thus, $\overline{F}$ has to be connected.

By a similar reasoning we arrive at our contradiction to $\overline{F}$ not being 2-connected. A *block* is a 2-connected component of a graph, and a *leaf block* is a block with only one cut vertex. Since $\overline{F}$ is not 2-connected, it has at least two leaf blocks B_1 and B_2. As all vertices have degree at least 3, B_1 cannot have all its vertices on the same face. Again, without loss of generality, we can suppose B_2 is in the outer face of B_1, and there is an interior vertex

v_1 of B_1. Removing v_1 from $\overline{F}$, we obtain a plane graph F' that has a face f_1 containing v_1, and F' is contained in a maximal plane graph $\overline{F'}$ that is 2-connected. Again, by the maximality of $\overline{F}$, all the edges in $\overline{F'} - F'$ must be in f_1, implying that B_2 is still a block of $\overline{F'}$, contradicting the fact that $\overline{F'}$ is 2-connected. Hence, $\overline{F}$ must be 2-connected. □

Theorem 3.1 can be used to obtain more properties of maximal plane subgraphs.

Lemma 3.2. *If a maximal plane subgraph $\overline{F}$ of D_n contains a vertex v of degree 2, then the subgraph of $\overline{F}$ obtained after removing v is also maximal in $D_n - \{v\}$.*

Proof. Suppose the contrary. Remove v from $\overline{F}$ to obtain F' and let $\overline{F'}$ be a maximal plane graph containing F'. As $\overline{F}$ is maximal but F' is not, $\overline{F'} - F'$ must contain an edge e' that crosses some edge vw of $\overline{F}$. But by Lemma 1.1 there are at least two edges from v to $\overline{F'}$. These two edges and also vw do not cross $\overline{F}$, contradicting the maximality of $\overline{F}$. □

Proposition 3.3. *Any maximal plane subgraph $\overline{F}$ of D_n with $n \geq 3$ must contain at least* $\min(\lceil 3n/2 \rceil, 2n-3)$ *edges. This bound is tight.*

Proof. Suppose that $n > 3$ and $\overline{F_0} = \overline{F}$ has a vertex v_0 with degree 2. By removing this vertex we obtain another maximal plane graph $\overline{F_1}$ (maximal on $n-1$ points), and if $\overline{F_1}$ is in the same conditions (with at least three vertices and a vertex v_1 of degree 2), by removing v_1 we obtain a new maximal plane graph $\overline{F_2}$, and so on. We finish this process in a step k because either $\overline{F_k}$ only has three points, or all the points of $\overline{F_k}$ have degree at least 3. In the first case, the original graph $\overline{F}$ contains $n = k+3$ vertices and $2k+3$ edges, so $2n-3$ edges. In the second case, $\overline{F}$ must contain at least $2k + \lceil 3(n-k)/2 \rceil$ edges, this amount reaching its minimum value when $k = 0$.

Finally, let us see that the bound is tight. If $2 \leq n \leq 6$, then a straight-line drawing on n points in convex position gives the bound $2n - 3 \leq \lceil 3n/2 \rceil$. If $n > 6$ and n is an even number, a drawing like the one shown in Figure 4 proves that the bound $\lceil 3n/2 \rceil$ is tight. The drawing is done on $n = 2(k+1)$ points in convex position, that clockwise are

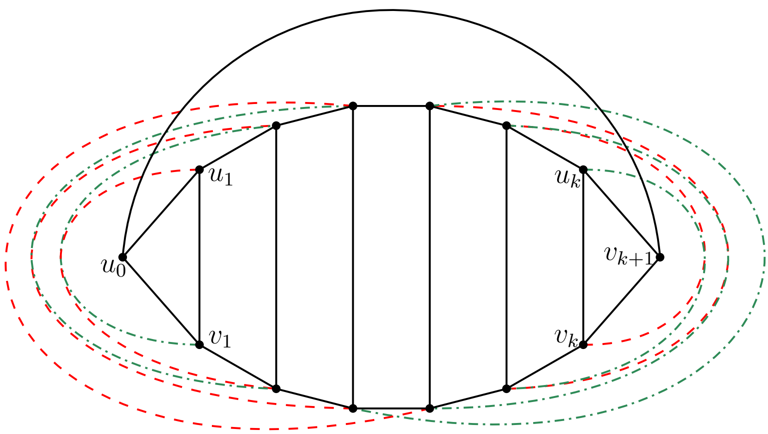

Figure 4: A drawing of K_n. The missing edges should be drawn as straight-line segments inside the convex hull of the set of points. The black edges form a maximal plane subgraph with $\lceil 3n/2 \rceil$ edges.

denoted by $u_0, u_1, u_2, \dots, u_k, v_{k+1}, v_k, \dots, v_2, v_1$. Let C denote the convex hull of that set of points. All the edges of D_n are drawn straight-line except for the $2(k-1)$ edges $u_i v_{i+1}, v_i u_{i+1}, i = 1, \dots, k-1$, and the edge $u_0 v_{k+1}$, that are drawn outside C as shown in Figure 4. Observe that the $2(k-1)$ edges $u_i v_{i+1}, v_i u_{i+1}, i = 1, \dots, k-1$, are the diagonals of the $(k-1)$ quadrilaterals $u_i u_{i+1} v_{i+1} v_i$, with $u_i v_{i+1}$ only crossing $v_i u_{i+1}$ and $u_0 v_{k+1}$, for $i = 1, \dots, k-1$. Clearly, straight-line edges can cross at most once, and the edges placed outside C, by construction, cross at most once. The graph $\overline{F}$ formed by the $2(k+1)$ edges on the boundary of C, the k edges $u_i v_i, i = 1, \dots, k$, and the edge $u_0 v_{k+1}$ is clearly plane and maximal, since the other straight-line edges cross at least one edge $u_i v_i$, and the non-straight-line edges cross the edge $u_0 v_{k+1}$.

If n is odd, we can add to the previous set a point u_0' between u_0 and u_1, very close to segment $u_0 u_1$, but keeping all the $2k+3$ points in convex position. By connecting u_0' with straight lines to the rest of the points, we obtain a simple topological drawing of K_n on this set of $n = 2k+3$ points, and a new maximal plane graph is obtained by adding the edges $u_0 u_0'$,$u_0' u_1$ to the above graph $\overline{F}$. This new maximal plane subgraph also has $\lceil 3n/2 \rceil$ edges. □

We mention another interesting implication of Theorem 3.1. For a vertex v, we can augment the star $S(v)$ to a 2-connected plane graph $\overline{F}$, and since $\overline{F} \setminus \{v\}$ is connected, it contains a spanning tree. So we have

Corollary 3.4. *For each vertex v there exists a spanning tree T_v of $V \setminus \{v\}$, such that the edges of $S(v) \cup T_v$ form a plane subgraph of D_n.*

Our next results are about diagonals on plane cycles. Let $C = (v_1, v_2, \dots, v_k)$ be a plane cycle of D_n. A diagonal of C is an edge of D_n connecting two non-consecutive vertices of C. It was previously known that, even for the case where there are diagonals intersecting both faces of C, there are at least $\lceil k/3 \rceil$ of them not crossing C (cf. [17, Corollary 6.6]). Proposition 3.3, applied to the subdrawing induced by the vertices of C, directly implies the following result.

Corollary 3.5. *Let $C = (v_1, v_2, \dots, v_k)$ be a plane cycle of D_n, with $k \geq 6$. Then, there exists a set D of diagonals, with $|D| \geq \lceil k/2 \rceil$, such that the subgraph $C \cup D$ is plane.*

It turns out that the structure of the diagonals of a cycle, as shown in the next lemma, is useful for our further results.

Lemma 3.6. *Let $C = (v_1, v_2, \dots, v_k)$ be a plane cycle of D_n, $k \geq 3$, dividing the plane into two faces f_1 and f_2. If there is no diagonal of C entirely in f_1, then all the diagonals of C are entirely in f_2.*

Proof. The proof is by induction on k. For $k < 5$ the statement is obvious, so suppose $k \geq 5$ and consider only the subdrawing D_k induced by the vertices of C. Suppose $C \cup D$ is a maximal plane graph of D_k, so necessarily, D consists of diagonals placed on f_2. Let d be a diagonal of D connecting two vertices at minimum distance on the graph C. Lemma 3.2 implies that in a maximal plane subgraph, vertices with degree 2 cannot be adjacent. Therefore, diagonal d has to connect two vertices at distance 2 on C. Without loss of generality, suppose $d = v_k v_2$ and let Δ be the triangle $v_k v_1 v_2$. Then, the cycle $C_1 = (v_2, v_3, \dots, v_k)$ with $k-1$ vertices has the faces $f_1' = f_1 + \Delta$ and $f_2' = f_2 - \Delta$.

We claim that there cannot be diagonals of C_1 entirely in f_1'. Such a diagonal e entirely in f_1' would have to intersect Δ. Then, adding e to $C \cup \{v_k v_2\}$ and removing all edges crossed by e, we would obtain a plane graph F in which v_1 has degree 0 or 1. By Lemma 1.1, there must be another edge between v_1 and C_1, and this edge would be a diagonal of C entirely in f_1, a contradiction. Thus, by induction, any diagonal $v_i v_j$ of C_1 is entirely in f_2' and hence also in f_2.

It remains to see that the diagonals with endpoint v_1 are also in f_2. By our induction hypothesis, the diagonal $v_2 v_4$ is in f_2' and thus also in f_2. Hence, arguing as before on the cycle $C_3 = (v_1, v_2, v_4, \ldots, v_k)$, we deduce that all the diagonals of C_3 incident to v_1 must be in f_2. So it remains to see that the diagonal $v_1 v_3$ is also in f_2. But $v_3 v_5$ is also in f_2', so it is in f_2, and again applying the same reasoning on the cycle $(v_1, v_2, v_3, v_5, \ldots, v_k)$, all the diagonals of this cycle not incident to v_4 have to be in f_2. □

To prove the next result, we recall some definitions and properties of any 2-connected graph $G = (V, E)$. Two vertices v_1, v_2 are called a *separation pair* of G if the induced subgraph $G \setminus \{v_1, v_2\}$ on the vertices $V \setminus \{v_1, v_2\}$ is not connected. Let $G_1, \ldots, G_l$ be the connected components of $G \setminus \{v_1, v_2\}$, with $l \geq 2$. For each $i \in \{1, \ldots, l\}$, let G_i^* be the subgraph of G induced by $V(G_i) \cup \{v_1, v_2\}$. Observe that G_i^* contains at least one edge incident to v_1 and at least another edge incident to v_2.

Theorem 3.7. *Let $\overline{F}$ be a maximal plane subgraph of D_n, $n \geq 3$. Then, for each separation pair v_1, v_2 of $\overline{F}$, at least one of the subgraphs $\overline{F}_i^*$ must be 2-connected.*

Proof. Suppose that v_1, v_2 is a separation pair of $\overline{F}$, and let $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_l$ be the connected components of $\overline{F} \setminus \{v_1, v_2\}$, $l \geq 2$. Since $\overline{F}$ is 2-connected, the graph $\overline{F} \setminus \{v_2\}$ is connected with v_1 as a cut vertex. As $\overline{F}$ is plane, we can suppose that v_1 is in the outer face of $\overline{F} \setminus \{v_2\}$ (v_2 must be inside that face) and that clockwise around vertex v_1 first there appear the edges from v_1 to some vertices of the component $\overline{F}_1$, then edges connecting v_1 to some vertices of $\overline{F}_2$ and so on. See Figure 5.

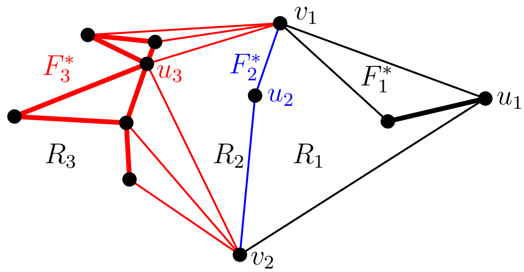

Figure 5: A plane graph with separating pair v_1, v_2 and three subgraphs F_i^*, none of them 2-connected. This plane graph cannot be maximal.

Now suppose that none of the subgraphs $\overline{F}_i^*$ is 2-connected. Then each subgraph $\overline{F}_i^*$ contains at least one cut vertex u_i. Since $\overline{F}_i$ is connected and there exist edges in $\overline{F}_i^*$ incident to v_1 and v_2, vertex u_i is different from v_1 and v_2. On the other hand, a connected component C of $\overline{F}_i^* \setminus \{u_i\}$ must contain at least one of v_1 or v_2 because otherwise, C would be a connected component of $\overline{F} \setminus \{u_i\}$, contradicting that $\overline{F}$ is 2-connected. Therefore, $\overline{F}_i^* \setminus \{u_i\}$ has exactly two components, one containing v_1, the other one containing v_2.

This also implies that the edge v_1v_2 of D_n cannot belong to $\overline{F}$, and that the cut-vertex u_i is in the outer face of $\overline{F}_i^*$ (and hence in the outer face of $\overline{F} \setminus \{v_2\}$) since v_1 and v_2 are in the outer face of $\overline{F} \setminus \{v_2\}$. See Figure 5.

In the graph $\overline{F} \setminus \{v_2\}$, around the vertex v_1, the edges to vertices of $\overline{F_1}$ first appear, then the edges to vertices of $\overline{F_2}$ and so on. Therefore, when we add to that graph the vertex v_2 and all the edges connecting v_2 to each component $\overline{F_i}$ to obtain $\overline{F}$, v_1 and v_2 must be in the faces R_i of $\overline{F}$ defined as the regions placed between the last edge from v_1 to $\overline{F}_i$ and the first edge from v_1 to $\overline{F}_{i+1}$, for $i = 1, \dots, l$, and the vertex u_i must be in the faces R_i and R_{i-1}. However, by the maximality of $\overline{F}$, no edge of D_n is entirely in any of those faces R_i. Then, Lemma 3.6 implies that no point of the edge v_1v_2 of D_n can be inside any face R_i. See Figure 5. Thus, v_1v_2 must begin between two edges v_1v, v_1v' with both v and v' belonging to a common connected component $\overline{F}_i$. However, since u_i belongs to the faces R_{i-1} and R_i, any curve from v_1 to v_2 passes either through the point u_i or through the interior of R_{i-1} or R_i, which contradicts either the simplicity of D_n or Lemma 3.6. Therefore, if none of the subgraphs $\overline{F}_i^*$ is 2-connected, $\overline{F}$ cannot be maximal. □

We call a graph *essentially* 3*-edge-connected* if it stays connected after removing any two edges not sharing a vertex of degree 2 (i.e., the graph either stays connected or one component is a single vertex). Theorem 3.7 implies that a maximal plane subgraph is essentially 3-edge-connected:

Theorem 3.8. *Any maximal plane subgraph, $\overline{F}$, of a simple topological drawing of K_n is essentially* 3*-edge-connected.*

Proof. If the removal of two edges v_1v_2 and $v_1'v_2'$ from the plane subgraph $\overline{F}$ results in two non-trivial components C_1, C_2 (see Figure 6), then v_1, v_2' is a separation pair of $\overline{F}$, that has as induced subgraphs $C_1 \cup \{v_1'v_2'\}$ and $C_2 \cup \{v_1v_2\}$, neither of which is 2-connected. Then, by Theorem 3.7, $\overline{F}$ cannot be maximal. □

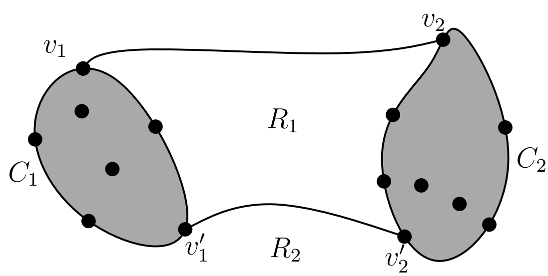

Figure 6: A graph that is not essentially 3-edge-connected. The induced subgraphs of the separation pair v_1, v_2' are subgraph C_1 plus edge $v_1'v_2'$ and subgraph C_2 plus edge v_1v_2. By Lemma 3.6, the edge v_1v_2' of D_n cannot enter either the R_1 face or the R_2 face, which is impossible in any good drawing.

4 Adding the maximum number of edges

Now, assume that a plane subgraph of D_n is given, and we want to add the maximum number of edges keeping plane the augmented graph. Clearly, the decision of adding one edge will in general block other edges from being added. We will see that the complexity of an algorithm solving this problem highly depends on whether the given subgraph is connected or not.

Before talking about algorithms and their complexity we have to talk about what information of the drawing D_n we will need to compute plane subgraphs. For each vertex v, the clockwise cyclic order of edges of $S(v)$ is usually given as a permutation of $V \setminus \{v\}$ (that is to be interpreted circularly) of the second vertices of all edges of $S(v)$. That permutation of $V \setminus \{v\}$ is called the *rotation* of v, and the *rotation system* of a drawing D_n consists of the collection of the rotations of each vertex v of D_n. It is well-known that from the information provided by the rotation system, one can determine whether two edges cross or not, and therefore, that information is enough to compute plane subgraphs. See [7, 11, 15]. From the rotation system, we can also compute (in $O(n^2)$ time) the *inverse rotation system* that, for each vertex v_i and index j, $j \neq i$, gives the position of v_j in the rotation of v_i.

When we say that a drawing D_n is given, we mean that we know the rotation system and the inverse rotation system of D_n. Using these two structures, one can determine whether two edges cross, in which direction an edge is crossed, and in which order two non-crossing edges cross a third one in constant time [11].

Theorem 4.1. *Let F be a connected spanning plane subgraph of D_n. Then there is an $O(n^3)$ time algorithm to augment F to a plane subgraph F' of D_n with the maximum number of edges.*

Proof. As F is plane and thus contains a linear number of edges, we can identify all the edges of D_n not crossed by F in $O(n^3)$ time. This also gives us, for each such edge, the face of F in which it is contained, and we can also compute for each face f of F the set Δ_f of edges of D_n entirely inside f. Clearly, each face of F can be considered independently, adding the maximum number of edges in it.

Let f be a face of F. For simplicity, we assume f to be bounded by a simple cycle $(v_1, \ldots, v_k)$. Other cases can be solved similarly by an appropriate "splitting" of edges having f on both sides. Disregarding D_n, consider the rectilinear drawing $\overline{D_k}$ obtained from k points $p_1, \ldots, p_k$ placed on a circle C, and assign to each edge $p_i p_j$ of $\overline{D_k}$ weight 0 if $v_i v_j$ is in Δ_f, weight 1 otherwise. Observe that two edges of Δ_f cross properly, if and only if, the corresponding 0-weight edges in circle C cross properly. It is well-known that a minimum-weight triangulation in $\overline{D_k}$ can be obtained in $O(k^3)$ time [9] by a dynamic programming algorithm, and this triangulation gives a plane set of 0-weight edges with maximum cardinality. Hence, the corresponding edges of Δ_f form a plane set of edges entirely inside face f with maximum cardinality. □

In contrast to this result, the problem becomes NP-complete when the subgraph F is not connected.

Theorem 4.2. *Given a simple topological drawing D_n of K_n and a cardinality k', it is NP-complete to decide whether there is a plane subgraph that has at least k' edges.*

Proof. We give a reduction from the independent set problem on segment intersection graphs ($\overline{SEG}$ problem), which is known to be NP-complete [10]: Given a set S of s segments in the plane that pairwise either are disjoint or intersect in a proper crossing, and an integer $k > 0$, is there a subset of k disjoint segments?

For each instance of a $\overline{SEG}$ problem, we are going to build, in polynomial time, a drawing D_n of K_n and an integer k' such that the instance of the $\overline{SEG}$ problem has a Yes answer, if and only if, the drawing D_n contains a plane subgraph with k' edges.

Let v_i, t_i be the endpoints of each segment $s_i, i = 1, \ldots, s$, of S. We can suppose that these endpoints are in general position and that their convex hull is a triangle. Thus, for each endpoint v_i, we can find a disc B_i centered at v_i, such that any straight line connecting two endpoints of S different from v_i does not cross B_i.

In each disc B_i, we place two points u_i, w_i very close to the segment v_it_i, in such a way that when connecting the point v_i with straight-line segments to all the other points, the segments v_iu_i, v_it_i, v_iw_i are clockwise consecutive. In other words, the clockwise wedge defined by the half-lines v_iu_i, v_iw_i only contains the endpoint t_i. See Figure 7.

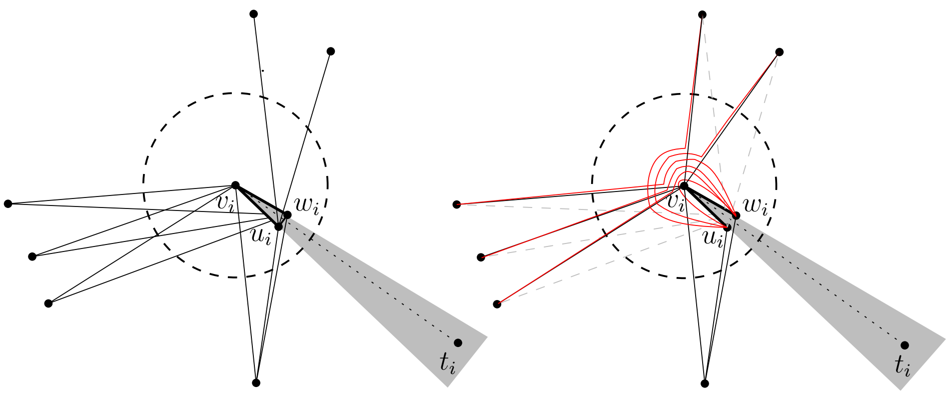

Figure 7: Drawings $\overline{D_n}$ (left) and D_n (right). The gray wedge only contains the endpoint t_i. In D_n, the dashed edges need to take a detour to avoid intersecting the edge u_iw_i twice.

Consider the rectilinear drawing $\overline{D_n}$ obtained by connecting the $n = 4s$ points v_i, u_i, w_i, t_i. In $\overline{D_n}$, maximal plane graphs are triangulations, but we are going to consider only the family Γ of plane triangulations of $\overline{D_n}$ containing the $2s$ edges u_iv_i, w_iv_i. The *weight* of a triangulation of Γ is the number of edges v_it_i that it contains. Clearly, in the set S there are k disjoint segments, if and only if, there is a triangulation in Γ with weight k.

Now, consider the drawing D_n obtained from $\overline{D_n}$ doing the following changes:

For $i = 1, \ldots, s$, only the edges of the star $S(u_i)$ crossing v_iw_i, the edges of the star $S(w_i)$ crossing u_iv_i, and the edge u_iw_i are modified.

Suppose that in $S(u_i)$ after u_iv_i are clockwise the edges $u_ip_1, \ldots, u_ip_k, u_iw_i$, where each u_ip_j has to cross v_iw_i. Let $v_ip_{i_1}, v_ip_{i_2} \ldots, v_ip_{i_k}$ be the clockwise ordered edges of $S(v_i)$ with endpoint one of the vertices p_i. Then, we modify $\overline{D_n}$ by redrawing $u_ip_{i_1}$ following first the line u_iv_i until point v_i, then turning around v_i and following the line $v_ip_{i_1}$, in such a way that in the rotation of u_i the new edge $u_ip_{i_1}$ is placed just before u_iv_i. See Figure 7, right. The new drawing obtained is simple, because no edge crosses both u_iv_i and $v_ip_{i_1}$, edges u_ip_j cannot cross $v_ip_{i_1}$ and none edge of $S(p_{i_1})$ can cross u_iv_i. Moreover, the number of crossings in the edge v_iw_i has decreased by one. We repeat the same process for the edge $u_ip_{i_2}$ (the new edge $u_ip_{i_2}$ is placed just before u_iv_i in the rotation of u_i), then $u_ip_{i_3}$, and so on. The same process can be done with the edges w_iq_j crossing u_iv_i. See Figure 7, right. Finally, we can redraw u_iw_i in the same way, following the edge u_iv_i then turning around v_i following edge v_iw_i. If we do this process for all the edges crossing v_iu_i or u_iw_i, $i = 1, \ldots, s$, at the end we obtain the simple drawing D_n. By construction, in D_n, neither the edges v_iu_i nor the edges v_iw_i are crossed by any other edge.

Now, let us see that $\overline{D_n}$ has a triangulation of the family Γ of weight k, if and only if, D_n has a plane subgraph of size k', with $k' = 3n - 6 - (s - k) = 11s - 6 + k$. Suppose $\overline{D_n}$ has a triangulation F with weight k. This means that F contains $(s - k)$ edges u_iw_i. By removing from F these u_iw_i edges, we obtain a plane set F' of edges, where no edge of F' has been modified to obtain the drawing D_n. Therefore, the edges of F' also form a plane subgraph in D_n of size $3n - 6 - (s - k) = 11s - 6 + k$.

Conversely, suppose D_n contains a plane subgraph with $3n-6-(s-k)$ edges. Since the edges u_iv_i, w_iv_i are not crossed by any edge of D_n, they must belong to any maximal plane graph of D_n. Therefore, D_n has a plane subgraph F containing all the edges u_iv_i, v_iw_i and of size $k' \geq 3n - 6 - (s - k)$. As the wedge v_iw_i, v_iu_i only contains point t_i, if the edge v_it_i is not in F, then, the face of F containing the edges v_iu_i and v_iw_i cannot be a triangle. But, if a plane graph on n vertices contains more than $(s - k)$ non-triangular faces, its maximum number of edges is $< 3n - 6 - (s - k)$. As a consequence, v_it_i is not in F for at most $(s - k)$ indices i, or equivalently, the plane subgraph F contains at least k edges v_it_i. This means that we can obtain a triangulation of the family Γ of weight k by including k of these non-crossing edges. □

Note that in the straight-line setting, we can always draw a triangulation of the underlying point set, which contains the maximum number of edges. However, this is not the case for simple topological drawings. We were not able to come up with a reduction solving the following problem.

Problem 4.3. What is the complexity of deciding whether a given D_n contains a triangulation, i.e., a plane subgraph whose faces are all 3-cycles?

Our reduction can also be adapted for a related problem on compatible graphs. We leave the realm of general simple topological drawings and consider the following problem in the more specialized setting of geometric graphs (rectilinear drawings). Let $P = \{p_1, \ldots, p_n\}$ and $P' = \{p'_1, \ldots, p'_n\}$ be two sets of points in the plane. A planar graph is *compatible* if it can be embedded on both P and P' in a way that there is an edge p_ip_j if and only if there is an edge $p'_ip'_j$. Saalfeld [21] asked for the complexity of deciding whether two such point sets (with a given bijection between them) have a compatible triangulation. We will say that triangulations $\overline{F}$ of P and $\overline{F'}$ of P' have k' compatible edges when there exists a subset of k' edges p_ip_j of $\overline{F}$, such that their images, edges $p'_ip'_j$, are edges of $\overline{F'}$.

We can show the NP-completeness of the following optimization variant of the problem. (However, as the similar Open Problem 4.3, Saalfeld's problem remains unsolved.)

Theorem 4.4. *Given two point sets $P = \{p_1, \ldots, p_n\}$ and $P' = \{p'_1, \ldots, p'_n\}$ and the indicated bijection between them, as well as a cardinality k', the problem of deciding whether P and P' admit two triangulations with k' compatible edges is NP-complete.*

Proof. We follow the idea of the proof of Theorem 4.2, and use a reduction from the $\overline{SEG}$ problem. Suppose that an instance of the $\overline{SEG}$ problem is given: a set S of s segments in the plane that pairwise either are disjoint or intersect in a proper crossing, and an integer $k > 0$. We will build two sets of points $P = \{p_1, \ldots, p_n\}$ and $P' = \{p'_1, \ldots, p'_n\}$ and obtain an integer k' such that the $\overline{SEG}$ problem has answer Yes if and only if, P and P' admit two triangulations with k' compatible edges.

Let P be the set of $n = 5s$ points formed by the v_i, t_i, u_i, w_i ($i = 1, \dots, s$) points obtained from S as in the above Theorem 4.2, plus s points $\tilde{v}_i$, where each point $\tilde{v}_i$ is placed inside the triangle $v_i u_i w_i$ very close to the point v_i, to the right of the oriented line $v_i t_i$, in such a way that in the wedge defined by the half-lines $\tilde{v}_i u_i, \tilde{v}_i w_i$ the only point of P is t_i, and the wedges $u_i v_i, u_i \tilde{v}_i$ and $w_i \tilde{v}_i, w_i v_i$ do not contain points of P. See Figure 8, left. By construction, any triangulation $\overline{F}$ of the set of points P must contain the edge $v_i \tilde{v}_i$. Observe that if the edge $t_i v_i$ is in $\overline{F}$, then the edge $t_i \tilde{v}_i$ has to be also in $\overline{F}$. Also note that $u_i w_i$ is only crossed by the edges $t_i v_i$ and $t_i \tilde{v}_i$.

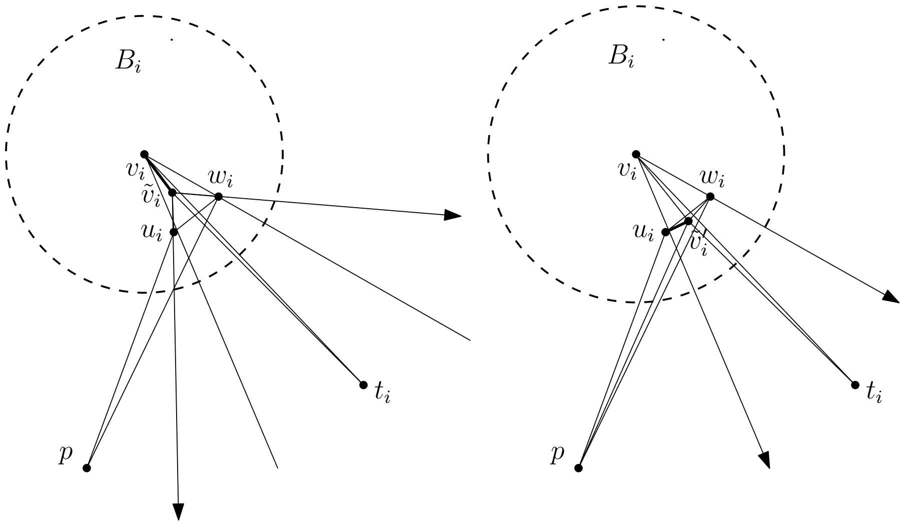

Figure 8: The point sets P (left) and P' (right).

In the same way, let P' be the set of $n = 5s$ points $v_i, t_i, u_i, w_i, \tilde{v}'_i$ ($i = 1, \dots, s$), where now each point $\tilde{v}'_i$ is placed outside the triangle $v_i u_i w_i$, very close to the intersection point of $u_i w_i$ with $v_i t_i$, to the right of the line $v_i t_i$, and satisfying that any clockwise triangle $u_i w_i p$ contains inside the point $\tilde{v}'_i$. See Figure 8, right. The bijection between the points of P and P' is the obvious one, to each point $\tilde{v}_i$ of P corresponds point $\tilde{v}'_i$ of P', for any other point its image is itself.

To prove the statement of the theorem, it is enough to prove the following:

If in the set S there are k disjoint segments, then there are triangulations $\overline{F}$ and $\overline{F'}$ of the sets P and P', respectively, with $k' = 3n - 6 - (s - k)$ compatible edges. And reciprocally, if $\overline{F}$ and $\overline{F'}$ contain $k' = 3n - 6 - (s - k)$ compatible edges, then S contains k disjoint segments.

Suppose first that S contains a set D of k disjoint segments $v_i t_i$. Let P_0 be the set of $4s$ common points of P and P' (all the points v_i, u_i, w_i, t_i). We build a triangulation $\overline{F_0}$ of P_0 in the following way. If $v_i t_i$ is in D, then we include the edges $v_i t_i, v_i u_i, v_i w_i, t_i u_i, t_i w_i$ in $\overline{F_0}$. If $v_i t_i$ is not in D, then we include the edges $u_i v_i, v_i w_i, w_i u_i$ in $\overline{F_0}$. After that, we add edges in an arbitrary way until obtaining a triangulation $\overline{F_0}$ of P_0. Now, to obtain $\overline{F}$ and $\overline{F'}$, we add the points $\tilde{v}_i$ and $\tilde{v}'_i$ to $\overline{F_0}$ and retriangulate the triangular faces where they are. If the edge $v_i t_i$ is in D, then the points $\tilde{v}_i, \tilde{v}'_i$ are both in the triangle $u_i v_i t_i$. So, by adding the point $\tilde{v}_i$ and the three edges $\tilde{v}_i u_i, \tilde{v}_i v_i, \tilde{v}_i t_i$ to $\overline{F_0}$, or the point $\tilde{v}'_i$ and the three edges $\tilde{v}_i'u_i, \tilde{v}_i'v_i, \tilde{v}_i't_i$ we continue with all the edges being compatible. However, if the edge v_it_i is not in D, then the point $\tilde{v}_i$ is in the triangle $u_iv_iw_i$, but the point $\tilde{v}_i'$ is in a triangle $u_iw_ip_i$. Then, we obtain a triangulation $\overline{F}$ of P by adding the edges $\tilde{v}_iu_i, \tilde{v}_iv_i, \tilde{v}_iw_i$, and a triangulation $\overline{F'}$ of P' by adding the edges $\tilde{v}_i'u_i, \tilde{v}_i'p_i, \tilde{v}_iw_i$. Now, the images of the edges $\tilde{v}_iv_i$ of $\overline{F}$, edges $\tilde{v}_i'v_i$, are not in $\overline{F'}$ (there the edges $\tilde{v}_i'p_i$ appear instead). This situation occurs $(s-k)$ times, so the number of compatible edges between $\overline{F}$ and $\overline{F'}$ is $3n-6-(s-k)$.

Conversely, suppose P and P' contain triangulations $\overline{F}$ and $\overline{F'}$ with $k' = 3n-6-(s-k)$ compatible edges. If $\tilde{v}_it_i$ is not in $\overline{F}$, then the edges $\tilde{v}_iv_i$ and u_iw_i must be both in $\overline{F}$, because the edge u_iw_i can be crossed only by the edges t_iv_i and $t_i\tilde{v}_i$. However, in set P', always the edge $\tilde{v}_i'v_i$ is crossed by the edge u_iw_i. Then, for each index i such that $\tilde{v}_it_i$ is not in $\overline{F}$, one of the edges $\tilde{v}_iv_i$ or u_iw_i of $\overline{F}$ is not in $\overline{F'}$. Therefore, this situation can happen at most $(s-k)$ times, that is, the triangulation $\overline{F}$ must contain at least k edges $\tilde{v}_it_i$. But if k segments $\tilde{v}_it_i$ are disjoint, also their corresponding v_it_i edges are disjoint. Therefore, S has to contain at least k disjoint segments. □

Finally, let us analyze the complexity of augmenting a plane subgraph F of D_n until obtaining a maximal plane subgraph. Since F has $O(n)$ edges, the set of edges of $S(v)$ not crossing F can be trivially found in $O(n^2)$ time. This directly implies an $O(n^3)$ algorithm to obtain a maximal plane graph containing F: For $i = 1, \dots, n$, update F by adding the edges of $S(v_i)$ non-crossing F, not in F. The following result implies that, if F is connected, finding a maximal plane subgraph containing F can be done in $O(n^2)$ time.

Theorem 4.5. *Given a simple topological drawing of K_n, a connected plane subgraph F, and a vertex v, we can find the edges from v to F not crossing F in $O(n)$ time.*

Proof. Notice that as F is a plane graph, we can compute in linear time, for each vertex w the clockwise order of the edges of F incident to w, the faces of F, and for each face f, the clockwise cyclic list of edges and vertices found along its boundary.

Suppose first that the vertex v is not in F, and let vw_1 be the first edge in the rotation of v with one endpoint in F. The algorithm runs in three stages. In the first stage, it starts by finding the edge of F, edge $e_1 = u_0u_1$, that intersects vw_1 closest to v along vw_1. When the first intersection point occurs precisely at the vertex w_1, we take e_1 as the first edge of F that follows, counterclockwise, to w_1v in $S(w_1)$. Using the rotation system and its inverse, this edge e_1 of F can be found in linear time, since $|F| \in O(n)$. It also gives us the face f of F containing the vertex v inside.

For simplicity, let us suppose that f is a bounded face and that the boundary of f is a simple cycle, formed by the edges $e_1 = u_0u_1, e_2 = u_1u_2, e_m = u_{m-1}u_0$. We will later discuss the general case. Notice that if the edges vw_i, vw_j, vw_k are in this clockwise order in $S(v)$, their corresponding first crossing points x_i, x_j, x_k with F are found in a clockwise walk of the boundary of f in that same clockwise order. See the right bottom drawing of Figure 9.

In the second stage, the algorithm simulates a clockwise walk $x_1u_1, u_1u_2, \dots, u_{k-1}u_k, \dots$ of the boundary of f starting at point x_1, the first crossing point of vw_1 with $e_1 = u_0u_1$, and simultaneously a clockwise walk $vw_1, vw_2, \dots, vw_i, \dots$ on the edges of the star $S(v)$, beginning with the edge vw_1. In each step, the algorithm makes progress in at least one of the two walks, by adding the following edge on the boundary to the boundary walk or passing to explore the following edge of $S(v)$. In this process the algorithm will keep a list σ with some of the explored edges of $S(v)$.

In a generic step, the edges $S_i = (vw_1, vw_2, \ldots, vw_i)$ of $S(v)$, and the portion of the boundary of f, $W_k = (x_1u_1, u_1u_2, \ldots, u_{k-1}u_k)$, have been visited, and the two following invariants hold:

(A) The first crossing of the edge vw_i is not on $W_{k-1} = (x_1u_1, u_1u_2, \ldots, u_{k-2}u_{k-1})$ (the walk W_k minus its last edge).

(B) The list σ contains an ordered list $(vu_{i_1}, vu_{i_2}, \ldots, vu_{i_s})$ of the explored edges of $S(v)$ finishing at some of the vertices u_j, $1 \leq j < k$, satisfying:

(B1) All the explored edges of $S(v)$ not placed in σ cross the boundary of f.

(B2) The first crossing point of each edge vu_j of σ with the boundary of f is either u_j or is placed clockwise after u_j.

Initially, if x_1 is an interior point of the edge $e_1 = u_0u_1$, then $W_k = (x_1u_1)$, $S_i = (vw_1, vw_2)$ and the list σ is empty. If x_1 coincides with the vertex u_0, then $W_k = (u_0u_1)$, $S_i = (vw_1, vw_2)$ and the list contains the edge vu_0. In both cases invariants (A) and (B) are satisfied (the walk W_{k-1} is empty or consists of only one vertex).

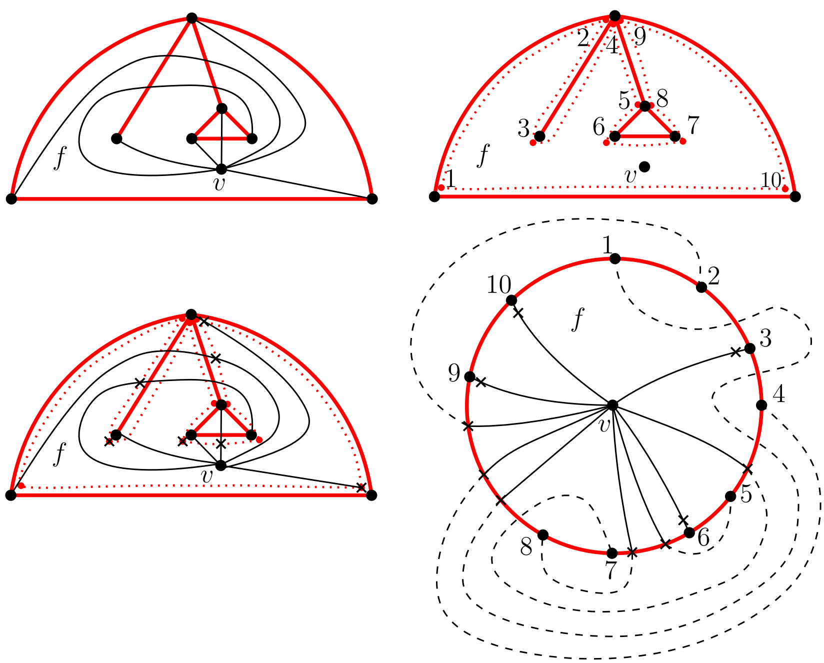

Figure 9: Top left: A vertex v inside the face f. Only the edges vu_i, with u_i incident to the face f, can be uncrossed by F. Top right: A clockwise walk along the boundary of the face f. Bottom left: In a walk along the boundary of f, the first crossing points of the edges of $S(v)$ are found in the same order as the edges of $S(v)$. Bottom right: An equivalent drawing to the top left figure with the boundary of f being a simple cycle. Some vertices, like $(2, 4, 9)$, can correspond to the same vertex of the first drawing.

In this second stage the algorithm proceeds as follows:

- If vw_i crosses the last edge of W_k, edge e_k, or if w_i is not a vertex of f, iterate considering the clockwise successor vw_{i+1} of vw_i in the rotation of v.

  As the first crossing of vw_i must be on the edge e_k or a posterior edge $e_t, t > k$, also the first crossing of vw_{i+1} must be on e_k or a posterior edge. Thus invariant (A) is kept. On the other hand, observe that σ does not change, vw_i must not be included in σ (it crosses f), and W_k is not modified. Therefore invariant (B) is also kept.

- If vw_i does not cross e_k and w_i is a vertex of f, $w_i \neq u_k$, then, add the following edge e_{k+1} on f to W_k, keeping the same edge vw_i of $S(v)$.

  Invariant (A) is kept, because the first crossing point of vw_i cannot be on W_k. Invariant (B) is also kept, because σ is not modified.

- If vw_i does not cross e_k and $w_i = u_k$, then, add vw_i to the list σ, pass to explore the following edge vw_{i+1} of $S(v)$ and add the following edge e_{k+1} on f to W_k.

  Again, invariant (A) is kept, because the first crossing of vw_{i+1} must be after u_k. On the other hand, the first crossing point of vw_i is either u_k or it is placed after u_k, hence property (B) is kept.

This second stage of the algorithm ends when all the edges of $S(v)$ and f have been explored. The last edge of the boundary of f being either u_0x_1 or $u_{m-1}u_0$. Therefore, at the end, invariant (B) implies that σ will contain the uncrossed edges of $S(v)$ plus some crossed edges vu_i of $S(v)$ satisfying that the first crossing (on the boundary of f) is placed after the endpoint u_i of that edge.

In each step of this stage, a new edge in the boundary of f, a new edge of $S(v)$, or both edges become explored. As the number of edges in f and in $S(v)$ is linear, this second stage of the algorithm runs in $O(n)$ time.

In the third stage, the algorithm repeats counterclockwise the above stage considering only the edges in σ. That means, it explores counterclockwise the boundary of f (in the order x_1u_0, $u_0u_{m-1}, \ldots$, and counterclockwise the edges of $S(v)$ placed in σ (so, in the order $vu_{i_s}, vu_{i_{s-1}}, \ldots$). In this third stage, in linear time, a new list $\overline{\sigma}$ is obtained. By invariant (B1), all the uncrossed edges of $S(v)$ have to be in $\overline{\sigma}$. And by invariant (B2), if vu_i is in $\overline{\sigma}$, its first crossing point cannot be clockwise nor counterclockwise before u_i, so it has to be u_i. Therefore, $\overline{\sigma}$ will contain the uncrossed edges of $S(v)$.

In general, the boundary of face f is not a simple cycle, some edges of f can be incident to f for both sides, so they appear twice in a walk along the boundary of f. However, this general case can be transformed to the previous case by standard techniques, as done in [6] in their proof of the general case of Lemma 1.1. In Figure 9, the bottom right figure shows how to transform the drawing of the top left figure, to obtain an equivalent drawing where the boundary of f is a simple cycle. When the face f is the unbounded face the algorithm is totally analogous.

Finally, let us consider the case when the vertex v is in F. Then, vertex v can be incident to several faces $f_1, \ldots, f_l, l \geq 1$. Again, for simplicity, suppose that the boundary of each one of these faces is a simple cycle. For each face f_i, if vw_{i_1}, vw_{i_2} are the two edges incident to vertex v in f_i, we can compute by the above method the uncrossed edges of $S(v)$ placed inside f_i, using only the edges of $S(v)$ placed clockwise between vw_{i_1} and vw_{i_2}. $\square$

5 Conclusion

In this paper, we considered maximal and maximum plane subgraphs of simple topological drawings of K_n. It turns out that maximal plane subgraphs have interesting structural properties. These insights could be useful in improving the bounds on the number of disjoint edges in any such drawing, continuing this long line of research.

Also, algorithmic questions arise. For example, Proposition 2.1 ensures that there are always two edges connecting a vertex v to a not necessarily connected plane graph F in D_n without crossings. Moreover, the set of edges of $S(v)$ not crossing F can be trivially found in $O(n^2)$ time. This leads to the following question.

Problem 5.1. Given a not necessarily connected plane graph F in D_n, plus a vertex v not in F, can the edges of $S(v)$ incident to but not crossing F be found in $o(n^2)$ time?

ORCID iDs

Alfredo García Olaverri https://orcid.org/0000-0002-6519-1472
Alexander Pilz https://orcid.org/0000-0002-6059-1821
Javier Tejel Altarriba https://orcid.org/0000-0002-9543-7170

References

[1] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, D. McQuillan, B. Mohar, P. Mutzel, P. Ramos, R. B. Richter and B. Vogtenhuber, Bishellable drawings of K_n, *SIAM J. Discrete Math.* **32** (2015), 2482–2492, doi:10.1137/17m1147974.

[2] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, J. Pammer, A. Pilz, P. Ramos, G. Salazar and B. Vogtenhuber, All good drawings of small complete graphs, in: *Proceedings of the 31st European Workshop on Computational Geometry (EuroCG 2015)*, 2015 pp. 57–60.

[3] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos and G. Salazar, Shellable drawings and the cylindrical crossing number of K_n, *Discrete Comput. Geom.* **52** (2015), 743–753, doi:10.1007/s00454-014-9635-0.

[4] M. Balko, R. Fulek and J. Kynčl, Crossing numbers and combinatorial characterization of monotone drawings of K_n, *Discrete Comput. Geom.* **53** (2015), 107–143, doi:10.1007/s00454-014-9644-z.

[5] J. Fox and B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results, *Combinatorica* **29** (2009), 153–196, doi:10.1007/s00493-009-2475-5.

[6] R. Fulek and A. J. Ruiz-Vargas, Topological graphs: empty triangles and disjoint matchings, in: *Proceedings of the Twenty-Ninth Annual Symposium on Computational Geometry (SoCG '13)*, Association for Computing Machinery, New York, NY, USA, 2013 pp. 259–266, doi:10.1145/2462356.2462394.

[7] E. Gioan, Complete graph drawings up to triangle mutations, in: D. Kratsch (ed.), *Graph-Theoretic Concepts in Computer Science*, Springer, Berlin, volume 3787 of *Lecture Notes in Computer Science*, pp. 139–150, 2005, doi:10.1007/11604686_13.

[8] H. Harborth and I. Mengersen, Edges without crossings in drawings of complete graphs, *J. Comb. Theory Ser. B* **17** (1974), 299–311, doi:10.1016/0095-8956(74)90035-5.

[9] G. T. Klincsek, Minimal triangulations of polygonal domains, *Ann. Discrete Math.* **9** (1980), 121–123, doi:10.1016/s0167-5060(08)70044-x.

[10] J. Kratochvíl and J. Nešetřil, INDEPENDENT SET and CLIQUE problems in intersection-defined classes of graphs, *Comment. Math. Univ. Carolinae* **31** (1990), 85–93, `http://dml.cz/dmlcz/106821`.

[11] J. Kynčl, Simple realizability of complete abstract topological graphs in P, *Discrete Comput. Geom.* **45** (2011), 383–399, doi:10.1007/s00454-010-9320-x.

[12] J. Kynčl, Enumeration of simple complete topological graphs, *Euopean. J. Comb.* **30** (2009), 1676–1685, doi:10.1016/j.ejc.2009.03.005.

[13] J. Kynčl, Improved enumeration of simple topological graphs, *Discrete Comput. Geom.* **50** (2013), 727–770, doi:10.1007/s00454-013-9535-8.

[14] J. Pach, J. Solymosi and G. Tóth, Unavoidable configurations in complete topological graphs, *Discrete Comput. Geom.* **30** (2003), 311–320, doi:10.1007/s00454-003-0012-9.

[15] J. Pach and G. Tóth, Which crossing number is it anyway?, *J. Comb. Theory Ser. B* **80** (2000), 225–246, doi:10.1006/jctb.2000.1978.

[16] J. Pach and G. Tóth, Disjoint edges in topological graphs, in: J. Akiyama, E. T. Baskoro and M. Kano (eds.), *Combinatorial Geometry and Graph Theory*, Springer, Berlin, volume 3330 of *Lecture Notes in Computer Science*, pp. 133–140, 2005, doi:10.1007/978-3-540-30540-8_15.

[17] J. Pammer, *Rotation Systems and Good Drawings*, Master's thesis, Graz University of Technology, 2014.

[18] N. H. Rafla, *The Good Drawings D_n of the Complete Graph K_n*, Ph.D. thesis, McGill University, Montreal, 1988.

[19] G. Ringel, Extremal problems in the theory of graphs, in: M. Fiedler (ed.), *Theory of Graphs and Its Applications*, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964 pp. 85–90, proceedings of the Symposium held in Smolenice in June 1963.

[20] A. J. Ruiz-Vargas, Many disjoint edges in topological graphs, *Comput. Geom.* **62** (2017), 1–13, doi:10.1016/j.comgeo.2016.11.003.

[21] A. Saalfeld, Joint triangulations and triangulation maps, in: *Proceedings of the Third Annual Symposium on Computational Geometry (SCG '87)*, Association for Computing Machinery, New York, NY, USA, 1987 pp. 195–204, doi:10.1145/41958.41979.

[22] M. Schaefer, The graph crossing number and its variants: A survey, *Electron. J. Comb.* (2021), #DS21, doi:10.37236/2713.

[23] A. Suk, Disjoint edges in complete topological graphs, *Discrete Comput. Geom.* **49** (2013), 280–286, doi:10.1007/s00454-012-9481-x.