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Tight relative t-designs on two shells in hypercubes, and Hahn and Hermite polynomials*

Eiichi Bannai
Faculty of Mathematics, Kyushu University (emeritus), Japan, and
National Center for Theoretical Sciences, National Taiwan University, Taiwan

Etsuko Bannai
National Center for Theoretical Sciences, National Taiwan University, Taiwan

Hajime Tanaka [†]
Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

Yan Zhu [‡]
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

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Abstract

Relative t-designs in the n-dimensional hypercube $\mathcal{Q}_n$ are equivalent to weighted regular t-wise balanced designs, which generalize combinatorial t-(n, k, λ) designs by allowing multiple block sizes as well as weights. Partly motivated by the recent study on tight Euclidean t-designs on two concentric spheres, in this paper we discuss tight relative t-designs in $\mathcal{Q}_n$ supported on two shells. We show under a mild condition that such a relative t-design induces the structure of a coherent configuration with two fibers. Moreover, from this structure we deduce that a polynomial from the family of the Hahn hypergeometric orthogonal polynomials must have only integral simple zeros. The Terwilliger algebra is the main tool to establish these results. By explicitly evaluating the behavior of the zeros of the Hahn polynomials when they degenerate to the Hermite polynomials under an appropriate limit

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process, we prove a theorem which gives a partial evidence that the non-trivial tight relative t-designs in $\mathcal{Q}_n$ supported on two shells are rare for large t.

Keywords: Relative t-design, association scheme, coherent configuration, Terwilliger algebra, Hahn polynomial, Hermite polynomial.

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1 Introduction

This paper is a contribution to the study of *relative t-designs* in Q-polynomial association schemes. In the Delsarte theory [16], the concept of *t-designs* is introduced for arbitrary Q-polynomial association schemes. For the Johnson scheme $J(n,k)$, the t-designs in the sense of Delsarte are shown to be the same thing as the combinatorial t-(n,k,λ) designs. There are similar interpretations of t-designs in some other important families of Q-polynomial association schemes [16, 17, 19, 34, 41]. The concept of relative t-designs is also due to Delsarte [18], and is a relaxation of that of t-designs. Relative t-designs can again be interpreted in several cases, including $J(n,k)$. For the n-dimensional hypercube $\mathcal{Q}_n$ (or the binary Hamming scheme $H(n,2)$) which will be our central focus in this paper, these are equivalent to the weighted regular *t-wise balanced designs*, which generalize the combinatorial t-(n,k,λ) designs by allowing multiple block sizes as well as weights.

The Delsarte theory has a counterpart for the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, established by Delsarte, Goethals, and Seidel [20]. The t-designs in $\mathbb{S}^{n-1}$ are commonly called the *spherical t-designs*, and are essentially the equally-weighted *cubature formulas* of degree t for the spherical integration, a concept studied extensively in numerical analysis. Spherical t-designs were later generalized to *Euclidean t-designs* by Neumaier and Seidel [35] (cf. [21]). Euclidean t-designs are in general supported on multiple concentric spheres in $\mathbb{R}^n$, and it follows that we may think of them as the natural counterpart of relative t-designs in $\mathbb{R}^n$. This point of view was discussed in detail by Bannai and Bannai [3]. See also [7, 8]. The success and the depth of the theory of Euclidean t-designs (cf. [38]) has been one driving force for the recent research activity on relative t-designs in Q-polynomial association schemes; see, e.g., [3, 5, 6, 7, 8, 9, 11, 32, 51, 53, 54].

A relative t-design in a Q-polynomial association scheme $(X,\mathscr{R})$ is often defined as a certain *weighted subset* of the vertex set X, i.e., a pair (Y,ω) of a subset Y of X and a function $\omega\colon Y\to(0,\infty)$. We are given in advance a 'base vertex' $x\in X$, and (Y,ω) gives a 'degree-t approximation' of the shells (or spheres or subconstituents) with respect to x on which Y is supported. See Sections 2 and 3 for formal definitions. Bannai and Bannai [3] proved a Fisher-type lower bound on $|Y|$, and we call (Y,ω) *tight* if it attains this bound. We may remark that t must be even in this case. In this paper, we continue the study (cf. [5, 9, 32, 51, 53]) of tight relative t-designs in the hypercubes $\mathcal{Q}_n$, which are one of the most important families of Q-polynomial association schemes. The Delsarte theory directly applies to the tight relative t-designs in $\mathcal{Q}_n$ supported on one shell, say, the k^{th} shell, as these are equivalent to the tight combinatorial t-(n,k,λ) designs. (We note that the k^{th} shell induces $J(n,k)$.) Our aim is to extend this structure theory to those supported on two shells. We may view the results of this paper roughly as counterparts to (part of) the

E-mail addresses: bannai@math.kyushu-u.ac.jp (Eiichi Bannai), et-ban@rc4.so-net.ne.jp (Etsuko Bannai), htanaka@tohoku.ac.jp (Hajime Tanaka), zhuyan@usst.edu.cn (Yan Zhu)

results by Bannai and Bannai [2, 4] on tight Euclidean t-designs on two concentric spheres.

Let $t = 2e$ be even. In Theorem 5.3, which is our first main result, we show under a mild condition that a tight relative $2e$-design in $\mathcal{Q}_n$ supported on two shells induces the structure of a coherent configuration with two fibers. Moreover, from this structure we deduce that a certain polynomial of degree e, known as a *Hahn polynomial*, must have only integral simple zeros. We note that the case $e = 1$ was handled previously by Bannai, Bannai, and Bannai [5]. The Hahn polynomials are a family of hypergeometric orthogonal polynomials in the Askey scheme [31, Section 1.5], and that their zeros are integral provides quite a strong necessary condition on the existence of such relative $2e$-designs. The corresponding necessary condition for the tight combinatorial $2e$-(n, k, λ) designs from the Delsarte theory was used successfully by Bannai [1]; that is to say, he showed that, for each given integer $e \geqslant 5$, there exist only finitely many non-trivial tight $2e$-(n, k, λ) designs, where n and k (and thus λ) vary. See also [22, 36, 52]. We extend Bannai's method to prove our second main result, Theorem 7.1, which presents a version of his theorem for our case.

The sections other than Sections 5 and 7 are organized as follows. We collect the necessary background material in Sections 2 and 3. Section 3 also includes a few general results on relative t-designs in Q-polynomial association schemes. As in [6, 44], our main tool in the analysis of relative t-designs is the *Terwilliger algebra* [46, 47, 48], which is a non-commutative semisimple $\mathbb{C}$-algebra containing the adjacency algebra. Section 4 is devoted to detailed descriptions of the Terwilliger algebra of $\mathcal{Q}_n$. It is well known (cf. [30, 31]) that the Hahn polynomials (${}_3F_2$) degenerate to the Hermite polynomials (${}_2F_0$) by an appropriate limit process, and a key in Bannai's method above was to evaluate precisely the behavior of the zeros of the Hahn polynomials in this process. In Section 6, we revisit this part of the method in a form suited to our purpose. Our account will also be simpler than that in [1]. In Appendix, we provide a proof of a number-theoretic result (Proposition 7.2) which is a variation of a result of Schur [40, Satz I].

2 Coherent configurations and association schemes

We begin by recalling the concept of coherent configurations.

Definition 2.1. The pair $(X, \mathscr{R})$ of a finite set X and a set $\mathscr{R}$ of non-empty subsets of X^2 is called a *coherent configuration* on X if it satisfies the following (C1) – (C4):

(C1) $\mathscr{R}$ is a partition of X^2.

(C2) There is a subset $\mathscr{R}_0$ of $\mathscr{R}$ such that

$$\bigsqcup_{R \in \mathscr{R}_0} R = \{(x, x) : x \in X\}.$$

(C3) $\mathscr{R}$ is invariant under the transposition $\tau\colon (x, y) \mapsto (y, x)$ $((x, y) \in X^2)$, i.e., $R^\tau \in \mathscr{R}$ for all $R \in \mathscr{R}$.

(C4) For all $R, S, T \in \mathscr{R}$ and $(x, y) \in T$, the number

$$p^T_{R,S} := \bigl|\{z \in X : (x, z) \in R,\ (z, y) \in S\}\bigr|$$

is independent of the choice of $(x, y) \in T$.

Moreover, a coherent configuration $(X, \mathscr{R})$ on X is called *homogeneous* if $|\mathscr{R}_0| = 1$, and an *association scheme* if $R^\intercal = R$ for all $R \in \mathscr{R}$.

Remark 2.2. Suppose that a finite group $\mathfrak{G}$ acts on X, and let $\mathscr{R}$ be the set of the orbitals of $\mathfrak{G}$, that is to say, the orbits of $\mathfrak{G}$ in its natural action on X^2. Then $(X, \mathscr{R})$ is a coherent configuration. Moreover, $(X, \mathscr{R})$ is homogeneous (resp. an association scheme) if and only if the action of $\mathfrak{G}$ on X is transitive (resp. generously transitive, i.e., for any $x, y \in X$ we have $(x^g, y^g) = (y, x)$ for some $g \in \mathfrak{G}$).

Let $(X, \mathscr{R})$ be a coherent configuration as above. For every $R \in \mathscr{R}_0$, let Φ_R be the subset of X such that $R = \{(x, x) : x \in \Phi_R\}$. Then we have

$$\bigsqcup_{R \in \mathscr{R}_0} \Phi_R = X.$$

We call the Φ_R $(R \in \mathscr{R}_0)$ the *fibers* of $(X, \mathscr{R})$. By setting in (C4) either $R \in \mathscr{R}_0$ and $S = T$, or $S \in \mathscr{R}_0$ and $R = T$, it follows that for every $T \in \mathscr{R}$, we have $T \subset \Phi_R \times \Phi_S$ for some $R, S \in \mathscr{R}_0$. In particular, $(X, \mathscr{R})$ is homogeneous whenever it is an association scheme. Let

$$\gamma_{R,S} = \big|\{T \in \mathscr{R} : T \subset \Phi_R \times \Phi_S\}\big| \qquad (R, S \in \mathscr{R}_0).$$

The matrix

$$[\gamma_{R,S}]_{R,S \in \mathscr{R}_0},$$

which is symmetric by (C3), is called the *type* of $(X, \mathscr{R})$.

Let $M_X(\mathbb{C})$ be the $\mathbb{C}$-algebra of all complex matrices with rows and columns indexed by X, and let $V = \mathbb{C}^X$ be the $\mathbb{C}$-vector space of complex column vectors with coordinates indexed by X. We endow V with the Hermitian inner product

$$\langle u, v \rangle = v^\dagger u \qquad (u, v \in V),$$

where † denotes adjoint. For every $R \in \mathscr{R}$, let $A_R \in M_X(\mathbb{C})$ be the adjacency matrix of the graph (X, R) (directed, in general), i.e.,

$$(A_R)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R, \\ 0 & \text{otherwise,} \end{cases} \qquad (x, y \in X).$$

Then (C1) – (C4) above are rephrased as follows:

(A1) $\displaystyle\sum_{R \in \mathscr{R}} A_R = J$ (the all-ones matrix).

(A2) $\displaystyle\sum_{R \in \mathscr{R}_0} A_R = I$ (the identity matrix).

(A3) $(A_R)^\dagger \in \{A_S : S \in \mathscr{R}\}$ $(R \in \mathscr{R})$.

(A4) $\displaystyle A_R A_S = \sum_{T \in \mathscr{R}} p^T_{R,S} A_T$ $(R, S \in \mathscr{R})$.

Let

$$\boldsymbol{A} = \operatorname{span}\{A_R : R \in \mathscr{R}\}.$$

Then from (A2) and (A4) it follows that $\boldsymbol{A}$ is a subalgebra of $M_X(\mathbb{C})$, called the *adjacency algebra* of $(X, \mathscr{R})$. We note that $\boldsymbol{A}$ is semisimple as it is closed under † by virtue of (A3). By (A1), $\boldsymbol{A}$ is also closed under entrywise (or *Hadamard* or *Schur*) multiplication, which we denote by $\circ$. The A_R are the (central) primitive idempotents of $\boldsymbol{A}$ with respect to $\circ$, i.e.,

$$A_R \circ A_S = \delta_{R,S} A_R, \qquad \sum_{R \in \mathscr{R}} A_R = J.$$

Remark 2.3. If $(X, \mathscr{R})$ arises from a group action as in Remark 2.2, then $\boldsymbol{A}$ coincides with the centralizer algebra (or Hecke algebra or commutant) for the corresponding permutation representation $g \mapsto P_g$ $(g \in \mathfrak{G})$ on V, i.e.,

$$\boldsymbol{A} = \{B \in M_X(\mathbb{C}) : BP_g = P_g B \ (g \in \mathfrak{G})\}.$$

A subalgebra of $M_X(\mathbb{C})$ is called a *coherent algebra* if it contains J, and is closed under $\circ$ and †. We note that the coherent algebras are precisely the adjacency algebras of coherent configurations. It is clear that the intersection of coherent algebras in $M_X(\mathbb{C})$ is again a coherent algebra. In particular, for any subset $\boldsymbol{S}$ of $M_X(\mathbb{C})$, we can speak of *the* smallest coherent algebra containing $\boldsymbol{S}$, which we call the *coherent closure* of $\boldsymbol{S}$.

From now on, we assume that $(X, \mathscr{R})$ is an association scheme. As is the case for many examples of association schemes, we write

$$\mathscr{R} = \{R_0, R_1, \dots, R_n\}, \quad \text{where } \mathscr{R}_0 = \{R_0\},$$

and say that $(X, \mathscr{R})$ has n *classes*. We will then abbreviate $p^k_{i,j} = p^{R_k}_{R_i,R_j}$, $A_i = A_{R_i}$, and so on. The adjacency algebra $\boldsymbol{A}$ is commutative in this case, and hence it has a basis $E_0, E_1, \dots, E_n$ consisting of the (central) primitive idempotents, i.e.,

$$E_i E_j = \delta_{i,j} E_i, \qquad \sum_{i=0}^{n} E_i = I.$$

Put differently, $E_0V, E_1V, \dots, E_nV$ are the maximal common eigenspaces (or homogeneous components or isotypic components) of $\boldsymbol{A}$, and the E_i are the corresponding orthogonal projections. Since the A_i are real symmetric matrices, so are the E_i. Note that the matrix $|X|^{-1}J \in \boldsymbol{A}$ is an idempotent with rank one, and thus primitive. We will always set

$$E_0 = \frac{1}{|X|} J.$$

For convenience, we let

$$A_i = E_i := O \text{ (the zero matrix)} \quad \text{if } i < 0 \text{ or } i > n.$$

Though our focus in this paper will be on Q-polynomial association schemes, we first recall the P-polynomial property for completeness. We say that the association scheme $(X, \mathscr{R})$ is *P-polynomial* (or *metric*) with respect to the ordering $A_0, A_1, \dots, A_n$ if there are non-negative integers a_i, b_i, c_i $(0 \leqslant i \leqslant n)$ such that $b_n = c_0 = 0$, $b_{i-1}c_i \neq 0$ $(1 \leqslant i \leqslant n)$, and

$$A_1 A_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \qquad (0 \leqslant i \leqslant n),$$

where b_{-1} and c_{n+1} are indeterminates. In this case, A_1 recursively generates $\boldsymbol{A}$, and hence has $n+1$ distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_n \in \mathbb{R}$, where we write

$$A_1 = \sum_{i=0}^{n} \theta_i E_i. \tag{2.1}$$

We note that $(X, \mathscr{R})$ is P-polynomial as above precisely when the graph (X, R_1) is a *distance-regular graph* and (X, R_i) is the distance-i graph of (X, R_1) $(0 \leqslant i \leqslant n)$. See, e.g., [10, 12, 27, 15] for more information on distance-regular graphs.

We say that $(X, \mathscr{R})$ is *Q-polynomial* (or *cometric*) with respect to the ordering $E_0, E_1, \ldots, E_n$ if there are real scalars a_i^*, b_i^*, c_i^* $(0 \leqslant i \leqslant n)$ such that $b_n^* = c_0^* = 0$, $b_{i-1}^* c_i^* \neq 0$ $(1 \leqslant i \leqslant n)$, and

$$E_1 \circ E_i = \frac{1}{|X|}(b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1}) \qquad (0 \leqslant i \leqslant n), \tag{2.2}$$

where b_{-1}^* and c_{n+1}^* are indeterminates. In this case, $|X|E_1$ recursively generates $\boldsymbol{A}$ with respect to $\circ$, and hence has $n+1$ distinct entries $\theta_0^*, \theta_1^*, \ldots, \theta_n^* \in \mathbb{R}$, where we write

$$|X|E_1 = \sum_{i=0}^{n} \theta_i^* A_i. \tag{2.3}$$

We call the θ_i^* the *dual eigenvalues* of $|X|E_1$. We may remark that $E_1 \circ E_i$, being a principal submatrix of $E_1 \otimes E_i$, is positive semidefinite, so that the scalars a_i^*, b_i^*, and c_i^* are non-negative (the so-called *Krein condition*). The Q-polynomial association schemes are an important subject in their own right, and we refer the reader to [23, 29] and the references therein for recent activity.

Below we give two fundamental examples of P- and Q-polynomial association schemes, both of which come from transitive group actions. See [10, 12, 16] for the details.

Example 2.4. Let v and k be positive integers with $v > k$, and let X be the set of k-subsets of $\{1, 2, \ldots, v\}$. Set $n = \min\{k, v-k\}$. For $x, y \in X$ and $0 \leqslant i \leqslant n$, we let $(x, y) \in R_i$ if $|x \cap y| = k - i$. The R_i are the orbitals of the symmetric group $\mathfrak{S}_v$ acting on X. We call $(X, \mathscr{R})$ a *Johnson scheme* and denote it by $J(v, k)$. The eigenvalues of A_1 are given in decreasing order by

$$\theta_i = (k-i)(v-k-i) - i \qquad (0 \leqslant i \leqslant n),$$

and $J(v, k)$ is Q-polynomial with respect to the corresponding ordering of the E_i (cf. (2.1)).

Example 2.5. Let $q \geqslant 2$ be an integer and let $X = \{0, 1, \ldots, q-1\}^n$. For $x, y \in X$ and $0 \leqslant i \leqslant n$, we let $(x, y) \in R_i$ if x and y differ in exactly i coordinate positions. The R_i are the orbitals of the wreath product $\mathfrak{S}_q \wr \mathfrak{S}_n$ of the symmetric groups $\mathfrak{S}_q$ and $\mathfrak{S}_n$ acting on X. We call $(X, \mathscr{R})$ a *Hamming scheme* and denote it by $H(n, q)$. The eigenvalues of A_1 are given in decreasing order by

$$\theta_i = n(q-1) - qi \qquad (0 \leqslant i \leqslant n),$$

and $H(n, q)$ is Q-polynomial with respect to the corresponding ordering of the E_i (cf. (2.1)). The Hamming scheme $H(n, 2)$ is also known as the n-*cube* (or n-*dimensional hypercube*) and is denoted by $\mathscr{Q}_n$.

Assumption 2.6. *For the rest of this section and in* Section 3*, we assume that* $(X, \mathscr{R})$ *is an association scheme and is Q-polynomial with respect to the ordering* $E_0, E_1, \ldots, E_n$ *of the primitive idempotents.*

In general, for any positive semidefinite Hermitian matrices $B, C \in M_X(\mathbb{C})$, we have (cf. [45])

$$(B \circ C)V = \operatorname{span}(BV \circ CV),$$

where

$$BV \circ CV = \{u \circ v : u \in BV,\, v \in CV\}.$$

Hence it follows from (2.2) that

$$\operatorname{span}(E_1 V \circ E_i V) = \begin{cases} E_{i-1}V + E_iV + E_{i+1}V & \text{if } a_i^* \neq 0, \\ E_{i-1}V + E_{i+1}V & \text{if } a_i^* = 0, \end{cases} \qquad (0 \leqslant i \leqslant n), \tag{2.4}$$

from which it follows that

$$\begin{aligned} \sum_{i=0}^{h} \sum_{j=0}^{k} \operatorname{span}(E_i V \circ E_j V) &= \sum_{i=0}^{h} \sum_{j=0}^{k} \operatorname{span}(\underbrace{E_1 V \circ \cdots \circ E_1 V}_{i \text{ times}} \circ E_j V) \\ &= \sum_{i=0}^{h+k} E_i V \end{aligned} \tag{2.5}$$

for $0 \leqslant h, k \leqslant n$. See also [10, Section 2.8].

We now fix a 'base vertex' $x \in X$. Let

$$X_i = \{y \in X : (x, y) \in R_i\} \qquad (0 \leqslant i \leqslant n).$$

We call the X_i the *shells* (or *spheres* or *subconstituents*) of $(X, \mathscr{R})$ with respect to x. For every i $(0 \leqslant i \leqslant n)$, define the diagonal matrix $E_i^* = E_i^*(x) \in M_X(\mathbb{C})$ by

$$(E_i^*)_{y,y} = \begin{cases} 1 & \text{if } y \in X_i, \\ 0 & \text{otherwise}, \end{cases} \qquad (y \in X).$$

Then we have

$$E_i^* E_j^* = \delta_{i,j} E_i^*, \qquad \sum_{i=0}^{n} E_i^* = I.$$

We call the E_i^* the *dual idempotents* of $(X, \mathscr{R})$ with respect to x. The subspace

$$\boldsymbol{A}^* = \boldsymbol{A}^*(x) = \operatorname{span}\{E_0^*, E_1^*, \ldots, E_n^*\}$$

is then a subalgebra of $M_X(\mathbb{C})$, which we call the *dual adjacency algebra* of $(X, \mathscr{R})$ with respect to x. The *Terwilliger algebra* (or *subconstituent algebra*) of $(X, \mathscr{R})$ with respect to x is the subalgebra $\boldsymbol{T} = \boldsymbol{T}(x)$ of $M_X(\mathbb{C})$ generated by $\boldsymbol{A}$ and $\boldsymbol{A}^*$ [46, 47, 48]. We note that $\boldsymbol{T}$ is semisimple as it is closed under †.

Remark 2.7. If $(X, \mathscr{R})$ arises from a group action as in Remark 2.2, which we recall is generously transitive in this case, then $\boldsymbol{T}$ is a subalgebra of the centralizer algebra for the action of the stabilizer $\mathfrak{G}_x$ of x in $\mathfrak{G}$. The two algebras are known to be equal, e.g., for $J(v,k)$ and $H(n,q)$; see [25, 43].

For every subset Y of X, let $\hat{Y} \in V$ be the characteristic vector of Y, i.e.,

$$(\hat{Y})_y = \begin{cases} 1 & \text{if } y \in Y, \\ 0 & \text{otherwise,} \end{cases} \qquad (y \in X).$$

In particular, $\hat{X}$ denotes the all-ones vector in V. We will simply write $\hat{x}$ for the characteristic vector of the singleton $\{x\}$. With this notation established, we have

$$\hat{X}_i = E_i^* \hat{X} = A_i \hat{x} \qquad (0 \leqslant i \leqslant n),$$

from which it follows that

$$\boldsymbol{T}\hat{x} = \operatorname{span}\{\hat{X}_i : 0 \leqslant i \leqslant n\} = \operatorname{span}\{E_i \hat{x} : 0 \leqslant i \leqslant n\}. \tag{2.6}$$

The $\boldsymbol{T}$-module $\boldsymbol{T}\hat{x}$ is easily seen to be irreducible with dimension $n+1$ (cf. [46, Lemma 3.6]), and is called the *primary* $\boldsymbol{T}$-module.

We define the *dual adjacency matrix* $A_1^* = A_1^*(x) \in M_X(\mathbb{C})$ by (cf. (2.3))

$$A_1^* = |X| \operatorname{diag} E_1 \hat{x} = \sum_{i=0}^{n} \theta_i^* E_i^*. \tag{2.7}$$

Since the θ_i^* are mutually distinct, A_1^* generates $\boldsymbol{A}^*$. Moreover, since

$$A_1^* v = |X| (E_1 \hat{x}) \circ v \qquad (v \in V),$$

it follows from (2.4) that

$$E_i A_1^* E_j = O \quad \text{if } |i-j| > 1 \quad (0 \leqslant i, j \leqslant n). \tag{2.8}$$

Let W be an irreducible $\boldsymbol{T}$-module. We define the *dual support* W_s^*, the *dual endpoint* $r^*(W)$, and the *dual diameter* $d^*(W)$ of W by

$$W_s^* = \{i : E_i W \neq 0\}, \qquad r^*(W) = \min W_s^*, \qquad d^*(W) = |W_s^*| - 1,$$

respectively. We call W *dual thin* if $\dim E_i W \leqslant 1$ $(0 \leqslant i \leqslant n)$. We note that the primary $\boldsymbol{T}$-module $\boldsymbol{T}\hat{x}$ is dual thin, and that it is a unique irreducible $\boldsymbol{T}$-module up to isomorphism which has dual endpoint zero or dual diameter n. The following lemma is an easy consequence of (2.8):

Lemma 2.8 ([46, Lemma 3.12])**.** *With reference to* Assumption 2.6*, write* $A_1^* = A_1^*(x)$*,* $\boldsymbol{A}^* = \boldsymbol{A}^*(x)$*,* $\boldsymbol{T} = \boldsymbol{T}(x)$*. Let* W *be an irreducible* $\boldsymbol{T}$*-module and set* $r^* = r^*(W)$*,* $d^* = d^*(W)$*. Then the following hold:*

1. $A_1^* E_i W \subset E_{i-1}W + E_i W + E_{i+1}W$ $(0 \leqslant i \leqslant n)$.
2. $W_s^* = \{r^*, r^*+1, \ldots, r^*+d^*\}$.
3. $E_i A_1^* E_j W \neq 0$ *if* $|i-j| = 1$ $(r^* \leqslant i, j \leqslant r^* + d^*)$.
4. *Suppose that* W *is dual thin. Then*

$$\sum_{h=0}^{i} E_{r^*+h} W = \sum_{h=0}^{i} (A_1^*)^h E_{r^*} W \qquad (0 \leqslant i \leqslant d^*).$$

In particular, $W = \boldsymbol{A}^* E_{r^*} W$.

3 Relative t-designs in Q-polynomial association schemes

In this section, we develop some general theory on relative t-designs in Q-polynomial association schemes.

Recall Assumption 2.6. Throughout this section, we fix a base vertex $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant n)$, $A_1^* = A_1^*(x)$, $\boldsymbol{A}^* = \boldsymbol{A}^*(x)$, and $\boldsymbol{T} = \boldsymbol{T}(x)$. In Introduction, we meant by a weighted subset of X a pair (Y, ω) of a subset Y of X and a function $\omega\colon Y \to (0, \infty)$. For convenience, however, we extend the domain of ω to X by setting $\omega(y) = 0$ for every $y \in X \backslash Y$. We will also naturally identify V with the set of complex functions on X, so that $\omega \in V$ and $Y = \operatorname{supp} \omega$. In our discussions on relative t-designs, we will often consider the set

$$L = L_Y = \{\ell : Y \cap X_\ell \neq \emptyset\}, \tag{3.1}$$

and say that (Y, ω) is *supported on* $\bigsqcup_{\ell \in L} X_\ell$.

For comparison, we begin with the algebraic definition of t-designs in $(X, \mathscr{R})$ due to Delsarte [16, 17].

Definition 3.1. A weighted subset (Y, ω) of X is called a *t-design* in $(X, \mathscr{R})$ if $E_i \omega = 0$ for $1 \leqslant i \leqslant t$.

Delsarte [18] generalized this concept as follows:

Definition 3.2. A weighted subset (Y, ω) of X is called a *relative t-design* in $(X, \mathscr{R})$ (with respect to x) if $E_i \omega \in \operatorname{span}\{E_i \hat{x}\}$ for $1 \leqslant i \leqslant t$.

Remark 3.3. Delsarte introduced the concept of t-designs for subsets Y of X in [16], i.e., when $\omega = \hat{Y}$, whereas in [17, 18] he mostly considered general (i.e., not necessarily non-negative) non-zero vectors $\omega \in V$ in the discussions on t-designs and relative t-designs. Some facts/results below, such as Examples 3.4 and 3.5, Proposition 3.6, and Theorem 3.8, are still valid for general $\omega \in V$, but the Fisher-type lower bound on $|Y| = |\operatorname{supp} \omega|$ (cf. Theorem 3.9) makes sense only when ω is non-negative.

For the Johnson and Hamming schemes, Delsarte [16, 17, 18] showed that these algebraic concepts indeed have geometric interpretations:

Example 3.4. Let $(X, \mathscr{R})$ be the Johnson scheme $J(v, k)$ from Example 2.4. Then (Y, ω) is a t-design if and only if, for every t-subset z of $\{1, 2, \ldots, v\}$, the sum λ_z of the values

$\omega(y)$ over those $y \in Y$ such that $z \subset y$, is a constant independent of z. On the other hand, (Y, ω) is a relative t-design if and only if the above λ_z depends only on $|x \cap z|$. We note that $(Y, \hat{Y})$ is a t-design if and only if Y is a t-(v, k, λ) design (cf. [13, Chapter II.4]) for some λ.

Example 3.5. Let $(X, \mathscr{R})$ be the Hamming scheme $H(n, q)$ from Example 2.5. Then (Y, ω) is a t-design if and only if, for every t-subset $\mathscr{T}$ of $\{1, 2, \ldots, n\}$ and every function $f \colon \mathscr{T} \to \{0, 1, \ldots, q-1\}$, the sum $\lambda_{\mathscr{T},f}$ of the values $\omega(y)$ over those $y = (y_1, y_2, \ldots, y_n) \in Y$ such that $y_i = f(i)$ $(i \in \mathscr{T})$, is a constant independent of the pair $(\mathscr{T}, f)$. On the other hand, (Y, ω) is a relative t-design if and only if the above $\lambda_{\mathscr{T},f}$ depends only on $|\{i \in \mathscr{T} : x_i = f(i)\}|$, where $x = (x_1, x_2, \ldots, x_n)$. We note that $(Y, \hat{Y})$ is a t-design if and only if the transpose of the $|Y| \times n$ matrix formed by arranging the elements of Y (in any order) is an orthogonal array $\mathrm{OA}(|Y|, n, q, t)$ (cf. [13, Chapter III.6]). For the case $q = 2$, i.e., for $\mathcal{Q}_n$, if we choose the base vertex as $x = (0, 0, \ldots, 0)$, then $(Y, \hat{Y})$ is a relative t-design if and only if Y is a regular t-wise balanced design of type t-(n, L, λ) (cf. [38, Section 4.4]) for some λ, where L is from (3.1), and where we identify the elements of $X = \{0, 1\}^n$ with their supports.

Similar results hold for some other important families of P- and Q-polynomial association schemes; see, e.g., [17, 18, 19, 34, 41].

Proposition 3.6 (cf. [3, Theorem 4.5])**.** *With reference to* Assumption 2.6*, let (Y, ω) be a weighted subset supported on $\bigsqcup_{\ell \in L} X_\ell$. Then we have*

$$\omega|_{\boldsymbol{T}\hat{x}} = \sum_{\ell \in L} \frac{\langle \omega, \hat{X}_\ell \rangle}{|X_\ell|} \hat{X}_\ell, \tag{3.2}$$

where $\omega|_{\boldsymbol{T}\hat{x}}$ denotes the orthogonal projection of ω on the primary $\boldsymbol{T}$-module $\boldsymbol{T}\hat{x}$. Moreover, (Y, ω) is a relative t-design if and only if

$$\langle \omega, v \rangle = \langle \omega|_{\boldsymbol{T}\hat{x}}, v \rangle = \sum_{\ell \in L} \frac{\langle \omega, \hat{X}_\ell \rangle}{|X_\ell|} \langle \hat{X}_\ell, v \rangle$$

for every $v \in \sum_{i=0}^{t} E_i V$.

Proof. Recall (2.6). The first part follows since the $\hat{X}_i$ form an orthogonal basis of $\boldsymbol{T}\hat{x}$ with $\|\hat{X}_i\|^2 = |X_i|$. The second part is also immediate from

$$E_i\omega \in \mathrm{span}\{E_i\hat{x}\} \iff E_i\omega \in \boldsymbol{T}\hat{x} \iff E_i\omega|_{\boldsymbol{T}\hat{x}} = E_i\omega. \qquad \square$$

Remark 3.7. It is clear that $(X_\ell, \hat{X}_\ell)$ is a relative n-design for every $0 \leqslant \ell \leqslant n$. Hence, if (Y, ω) is a relative t-design such that $X_\ell \subset Y$ for some ℓ, and if ω is constant on X_ℓ, then the weighted subset $(Y \backslash X_\ell, (I - E^*_\ell)\omega)$ obtained by discarding X_ℓ from Y is again a relative t-design. This observation is particularly important when applying Theorem 3.8 below; for example, we can always assume that $0 \notin L$.

The following is a slight generalization of Delsarte's Assmus–Mattson theorem for Q-polynomial association schemes [18, Theorem 8.4], and can also be viewed as a variation of [9, Theorem 3.3], which in turn generalizes [28, Proposition 1]. See also [11]. The proof is in fact identical to that of [44, Theorem 4.3], but we include it below because of the potential importance of the result.

Theorem 3.8. *With reference to* Assumption 2.6*, let* (Y, ω) *be a relative* t*-design supported on* $\bigsqcup_{\ell \in L} X_\ell$*. Then* $(Y \cap X_\ell, E^*_\ell \omega)$ *is a relative* $(t - |L| + 1)$*-design for every* $\ell \in L$*.*

Proof. Let $U = (\boldsymbol{T}\hat{x})^\perp$ be the orthogonal complement of $\boldsymbol{T}\hat{x}$ in V, which we recall is the sum of all the non-primary irreducible $\boldsymbol{T}$-modules in V. On the one hand, we have

$$\omega|_U \in \sum_{\ell \in L} E^*_\ell U.$$

Since A^*_1 generates $\boldsymbol{A}^*$ and has at most $|L|$ distinct eigenvalues on this subspace (cf. (2.7)), it follows that

$$\boldsymbol{A}^* \omega|_U = \operatorname{span}\{\omega|_U, A^*_1 \omega|_U, \dots, (A^*_1)^{|L|-1} \omega|_U\}. \tag{3.3}$$

On the other hand, since $E_0 U = 0$, that (Y, ω) is a relative t-design is rephrased as

$$\omega|_U \in \sum_{i=t+1}^{n} E_i U.$$

Hence it follows from (2.8) and (3.3) that

$$\boldsymbol{A}^* \omega|_U \subset \sum_{k=0}^{|L|-1} (A^*_1)^k \sum_{i=t+1}^{n} E_i U \subset \sum_{i=t-|L|+2}^{n} E_i U.$$

In particular, for every $\ell \in L$ we have

$$E^*_\ell \omega|_U \in \sum_{i=t-|L|+2}^{n} E_i U.$$

In other words, $(Y \cap X_\ell, E^*_\ell \omega)$ is a relative $(t - |L| + 1)$-design, as desired. □

Bannai and Bannai [3, Theorem 4.8] established the following Fisher-type lower bound on the size of a relative t-design with t even:

Theorem 3.9. *With reference to* Assumption 2.6*, let* (Y, ω) *be a relative* $2e$*-design* $(e \in \mathbb{N})$ *supported on* $\bigsqcup_{\ell \in L} X_\ell$*. Then*

$$|Y| \geqslant \dim \left(\sum_{\ell \in L} E^*_\ell \right) \left(\sum_{i=0}^{e} E_i V \right).$$

Definition 3.10. A relative $2e$-design (Y, ω) is called *tight* if equality holds above.

Recall from Example 3.5 that the relative t-designs in the hypercubes are equivalent to the weighted regular t-wise balanced designs.

Example 3.11. Let $(X, \mathscr{R})$ be the n-cube $\mathcal{Q}_n$ from Example 2.5. Xiang [51] showed that if $e \leqslant \ell \leqslant n - e$ for every $\ell \in L$, then

$$\dim \left(\sum_{\ell \in L} E^*_\ell \right) \left(\sum_{i=0}^{e} E_i V \right) = \sum_{i=0}^{\min\{|L|-1, e\}} \binom{n}{e-i}. \tag{3.4}$$

We may remark that (cf. [12, Theorem 9.2.1])

$$\dim E_iV = \binom{n}{i} \qquad (0 \leqslant i \leqslant n). \tag{3.5}$$

See also [32] and [6, Theorem 2.7, Example 2.9].

Example 3.12. Consider a symmetric 2-$(n+1,k,\lambda)$ design (cf. [13, Chapter II.6]). Observe that removing a point yields a tight relative 2-design in $\mathcal{Q}_n$ with $L = \{k-1,k\}$. Likewise, taking the complement of every block which contains a given point followed by removing that point gives rise to a tight relative 2-design in $\mathcal{Q}_n$ with $L = \{k, n+1-k\}$. The complement of this is yet another example[1] such that $L = \{k-1, n-k\}$. See [32, Section 3] and [50, Theorem 8]. Note that the weights are constant for these three examples. On the other hand, Bannai, Bannai, and Bannai [5, Theorem 2.2] showed that there is a tight relative 2-design in $\mathcal{Q}_n$ with $L = \{2, n/2\}$ for $n \equiv 6 \,(\mathrm{mod}\, 8)$, provided that a Hadamard matrix of order $n/2+1$ exists. This construction provides examples in which the weights take two distinct values depending on the shells. See also [53].

Example 3.13. Working with the tight 4-$(23,7,1)$ and 4-$(23,16,52)$ designs instead of a symmetric 2-$(n+1,k,\lambda)$ design as in Example 3.12, we obtain four tight relative 4-designs in $\mathcal{Q}_{22}$ with constant weight such that

$$L \in \big\{\{6,7\},\{6,15\},\{7,16\},\{15,16\}\big\}.$$

See [9, Theorem 6.3] and [32, Section 3].

Let (Y,ω) be a tight relative $2e$-design supported on $\bigsqcup_{\ell\in L} X_\ell$. Bannai, Bannai, and Bannai [5, Theorem 2.1] showed that if the stabilizer of x in the automorphism group of $(X,\mathscr{R})$ acts transitively on each of the shells X_i then ω is constant on $Y \cap X_\ell$ for every $\ell \in L$. The next theorem generalizes this result by replacing group actions by combinatorial regularity. Observe that the fibers of the coherent closure of $\boldsymbol{T}$ are in general finer than the shells X_i.

Theorem 3.14. *With reference to* Assumption 2.6*, let* (Y,ω) *be a tight relative* $2e$*-design* $(e \in \mathbb{N})$ *supported on* $\bigsqcup_{\ell\in L} X_\ell$*. For every* $\ell \in L$*, the weight* ω *is constant on* $Y \cap X_\ell$ *provided that* X_ℓ *remains a fiber of the coherent closure of* $\boldsymbol{T}$*.*

Proof. Let (cf. (3.2))

$$D = \operatorname{diag}\omega, \qquad \tilde{D} = \operatorname{diag}\omega|_{\boldsymbol{T}\hat{x}} = \sum_{\ell\in L} \frac{\langle \omega, \hat{X}_\ell\rangle}{|\hat{X}_\ell|} E^*_\ell.$$

Note that $\tilde{D} \in \boldsymbol{T}$. Let F be the orthogonal projection onto BV, where

$$B = \sqrt{\tilde{D}} \sum_{i=0}^{e} E_i \in \boldsymbol{T}.$$

Observe that

$$BV = (BB^\dagger)V,$$

[1] It seems that this construction is missing in [50, Theorem 8].

and that F is written as a polynomial in the Hermitian (in fact, real and symmetric) matrix $BB^{\dagger}$. In particular, $F \in \boldsymbol{T}$.

Since (Y, ω) is tight, we have

$$\dim BV = \dim \sqrt{\tilde{D}} \left(\sum_{\ell \in L} E_{\ell}^{*} \right) \left(\sum_{i=0}^{e} E_i V \right) = |Y|.$$

Let $u_1, u_2, \ldots, u_{|Y|}$ be an orthonormal basis of BV, and let

$$G = \left[u_1 \mid u_2 \mid \cdots \mid u_{|Y|} \right].$$

Then we have

$$F = GG^{\dagger}. \tag{3.6}$$

Let

$$D' = D|_{Y \times Y}, \quad \tilde{D}' = \tilde{D}|_{Y \times Y}, \quad F' = F|_{Y \times Y}, \quad G' = G|_{Y \times \{1,2,\ldots,|Y|\}},$$

where $|_{Y \times Y}$ etc. mean taking corresponding submatrices. Note that these are square matrices, and that D' and $\tilde{D}'$ are invertible. Then it follows that

$$(G')^{\dagger} D' (\tilde{D}')^{-1} G' = I_{|Y|}. \tag{3.7}$$

Indeed, since we may write

$$u_i = \sqrt{\tilde{D}}\, v_i, \quad \text{where } v_i \in \sum_{r=0}^{e} E_r V \qquad (1 \leqslant i \leqslant |Y|),$$

it follows from (2.5) (applied to $h = k = e$) and Proposition 3.6 that the (i, j)-entry of the LHS in (3.7) is equal to

$$(v_i)^{\dagger} D v_j = \langle \omega, v_i \circ \overline{v_j} \rangle = \langle \omega|_{\boldsymbol{T}\hat{x}}, v_i \circ \overline{v_j} \rangle = (v_i)^{\dagger} \tilde{D} v_j = \langle u_j, u_i \rangle = \delta_{i,j},$$

where $^{-}$ means complex conjugate. By (3.6) and (3.7), we have

$$I_{|Y|} = D' (\tilde{D}')^{-1} G' (G')^{\dagger} = D' (\tilde{D}')^{-1} F',$$

so that

$$(D')^{-1} = (\tilde{D}')^{-1} F'. \tag{3.8}$$

In particular, F' is a diagonal matrix.

Now, let $\ell \in L$ and suppose that X_{ℓ} remains a fiber of the coherent closure of $\boldsymbol{T}$. Then the (y, y)-entry of $F \in \boldsymbol{T}$ is constant for $y \in X_{\ell}$ (cf. (A1) and (A2)), and the same is true for $\tilde{D}$. Hence it follows from (3.8) that $\omega(y) = D_{y,y}$ must be constant for $y \in Y \cap X_{\ell}$. This completes the proof. $\square$

4 The Terwilliger algebra of $\mathcal{Q}_n$

For the rest of this paper, we will focus on relative t-designs in the n-cube $\mathcal{Q}_n$ from Example 2.5. We will need detailed descriptions of the Terwilliger algebra of $\mathcal{Q}_n$ and its irreducible modules, and we collect these in this section. Thus, we assume that $(X, \mathscr{R}) = \mathcal{Q}_n$, where $X = \{0,1\}^n$. We again fix a base vertex $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant n)$, $A_1^* = A_1^*(x)$, and $\boldsymbol{T} = \boldsymbol{T}(x)$. The Q-polynomial ordering we consider is the one given in Example 2.5.[2]

Proposition 4.1 (cf. [39, Section I.C]). *We have*

$$\boldsymbol{T} = \operatorname{span}\{E_i^* A_j E_k^* : 0 \leqslant i, j, k \leqslant n\}. \tag{4.1}$$

In particular, $\boldsymbol{T}$ is a coherent algebra.

Proof. The RHS in (4.1) is a subspace of $\boldsymbol{T}$. Recall from Example 2.5 that $\mathcal{Q}_n$ admits the action of $\mathfrak{G} = \mathfrak{S}_2 \wr \mathfrak{S}_n$. The stabilizer $\mathfrak{G}_x$ of x in $\mathfrak{G}$ is isomorphic to $\mathfrak{S}_n$, and it is immediate to see that every orbital of $\mathfrak{G}_x$ is of the form

$$\{(y,z) \in X \times X : (x,y) \in R_i,\ (y,z) \in R_j,\ (z,x) \in R_k\}$$

for some i, j, and k, where the corresponding adjacency matrix is $E_i^* A_j E_k^*$. Hence the RHS in (4.1) agrees with the centralizer algebra for the action of $\mathfrak{G}_x$ on X, which is a coherent algebra; cf. Remark 2.3. Since $\boldsymbol{T}$ is generated by the A_i and the E_i^*, the result follows. □

Lemma 4.2. *For $0 \leqslant i, j, k \leqslant n$, we have $E_i^* A_j E_k^* \neq O$ if and only if*

$$j \in \big\{|i-k|, |i-k|+2, |i-k|+4, \ldots, \min\{i+k, 2n-i-k\}\big\}.$$

Proof. Routine. □

Next we recall basic facts about the irreducible $\boldsymbol{T}$-modules. Let W be an irreducible $\boldsymbol{T}$-module. We define the *support* W_s, the *endpoint* $r(W)$, and the *diameter* $d(W)$ of W by

$$W_s = \{i : E_i^* W \neq 0\}, \qquad r(W) = \min W_s, \qquad d(W) = |W_s| - 1,$$

respectively. We call W *thin* if $\dim E_i^* W \leqslant 1$ $(0 \leqslant i \leqslant n)$.

Theorem 4.3 (cf. [26]). *Let W be an irreducible $\boldsymbol{T}$-module and set $r = r(W)$, $r^* = r^*(W)$, $d = d(W)$, and $d^* = d^*(W)$. Then W is thin, dual thin, and we have*

$$r = r^*, \qquad d = d^* = n - 2r, \qquad W_s = W_s^* = \{r, r+1, \ldots, n-r\}.$$

Moreover, the isomorphism class of W is determined by r.

Remark 4.4. Recall that the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ is defined by the generators $\mathfrak{x}, \mathfrak{y}, \mathfrak{h}$ and the relations

$$\mathfrak{x}\mathfrak{y} - \mathfrak{y}\mathfrak{x} = \mathfrak{h}, \qquad \mathfrak{h}\mathfrak{x} - \mathfrak{x}\mathfrak{h} = 2\mathfrak{x}, \qquad \mathfrak{h}\mathfrak{y} - \mathfrak{y}\mathfrak{h} = -2\mathfrak{y}.$$

[2] If n is even then $\mathcal{Q}_n$ has another Q-polynomial ordering $E_0, E_{n-1}, E_2, E_{n-3}, \ldots$ in terms of the natural ordering; cf. [10, p. 305].

There is a surjective homomorphism $U(\mathfrak{sl}_2(\mathbb{C})) \to \boldsymbol{T}$ such that (cf. [26, Lemma 7.5])

$$\mathfrak{x} \mapsto \sum_{i=1}^{n} E_{i-1}A_1^* E_i, \qquad \mathfrak{y} \mapsto \sum_{i=0}^{n-1} E_{i+1}A_1^* E_i, \qquad \mathfrak{h} \mapsto A_1.$$

Every irreducible $\boldsymbol{T}$-module is then irreducible as an $\mathfrak{sl}_2(\mathbb{C})$-module. We also obtain another surjective homomorphism $U(\mathfrak{sl}_2(\mathbb{C})) \to \boldsymbol{T}$ by interchanging A_1 and A_1^* and replacing the E_i by the E_i^* above; cf. [26, Lemma 5.3].

From now on, we fix an orthogonal irreducible decomposition

$$V = \bigoplus_{W \in \Lambda} W \tag{4.2}$$

of the standard module V. In view of Theorem 4.3, let

$$\Lambda_r = \{W \in \Lambda : r(W) = r^*(W) = r\} \qquad (0 \leqslant r \leqslant \lfloor n/2 \rfloor), \tag{4.3}$$

and fix a unit vector $v_W \in E_r W$ for each $W \in \Lambda_r$. Since

$$\dim E_i V = \sum_{W \in \Lambda} \dim E_i W = \sum_{r=0}^{i} |\Lambda_r| \qquad (0 \leqslant i \leqslant \lfloor n/2 \rfloor) \tag{4.4}$$

by Theorem 4.3, it follows from (3.5) that

$$|\Lambda_r| = \binom{n}{r} - \binom{n}{r-1} \qquad (0 \leqslant r \leqslant \lfloor n/2 \rfloor).$$

It is known (cf. [26, Theorem 9.2]) that if $W \in \Lambda_r$ then the vectors

$$E_r^* v_W,\, E_{r+1}^* v_W, \dots,\, E_{n-r}^* v_W \tag{4.5}$$

form an orthogonal basis of W, called a *standard basis* of W. By [26, Lemma 6.6], we also have

$$\|E_i^* v_W\|^2 = \binom{n-2r}{i-r} \|E_r^* v_W\|^2 \qquad (r \leqslant i \leqslant n-r). \tag{4.6}$$

We note that

$$1 = \|v_W\|^2 = \sum_{i=r}^{n-r} \|E_i^* v_W\|^2 = 2^{n-2r} \|E_r^* v_W\|^2. \tag{4.7}$$

For $W, W' \in \Lambda_r$, we observe that the linear map $W \to W'$ defined by

$$E_i^* v_W \mapsto E_i^* v_{W'} \qquad (r \leqslant i \leqslant n-r)$$

is an isometric isomorphism of $\boldsymbol{T}$-modules. Let

$$\breve{E}_r^{i,j} = \frac{2^{n-2r}}{\sqrt{\binom{n-2r}{i-r}\binom{n-2r}{j-r}}} \sum_{W \in \Lambda_r} (E_i^* v_W)(E_j^* v_W)^\dagger \qquad (r \leqslant i, j \leqslant n-r). \tag{4.8}$$

Then we have

$$(\breve{E}_r^{i,j})^\dagger = \breve{E}_r^{j,i} \qquad (r \leqslant i, j \leqslant n-r), \tag{4.9}$$

and from (4.6) and (4.7) it follows that

$$\breve{E}_r^{i,j} \breve{E}_{r'}^{i',j'} = \delta_{r,r'} \delta_{j,i'} \breve{E}_r^{i,j'}$$

for $0 \leqslant r, r' \leqslant \lfloor n/2 \rfloor$, $r \leqslant i, j \leqslant n-r$, and $r' \leqslant i', j' \leqslant n-r'$. By Theorem 4.3 and Wedderburn's theorem (cf. [14, Section 3]), $\boldsymbol{T}$ is isomorphic to the direct sum of full matrix algebras

$$\boldsymbol{T} \cong \bigoplus_{r=0}^{\lfloor n/2 \rfloor} M_{n-2r+1}(\mathbb{C}),$$

and the $\breve{E}_r^{i,j}$ form an orthogonal basis of $\boldsymbol{T}$. See also [24, Section 2]. We note that

$$E_i^* \boldsymbol{T} E_j^* = \operatorname{span}\bigl\{ \breve{E}_r^{i,j} : 0 \leqslant r \leqslant \min\{i, j, n-i, n-j\} \bigr\} \qquad (0 \leqslant i, j \leqslant n). \tag{4.10}$$

We now recall the *Hahn polynomials* [31, Section 1.5]

$$Q_r(\xi; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -\xi, -r, r+\alpha+\beta+1 \\ \alpha+1, -N \end{matrix} \,\middle|\, 1 \right) \in \mathbb{R}[\xi] \quad (0 \leqslant r \leqslant N), \tag{4.11}$$

where

$${}_sF_t\left(\begin{matrix} a_1, \dots, a_s \\ b_1, \dots, b_t \end{matrix} \,\middle|\, c \right) = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_s)_i}{(b_1)_i \cdots (b_t)_i} \frac{c^i}{i!},$$

and

$$(a)_i = a(a+1)\cdots(a+i-1).$$

For $\alpha, \beta > -1$, or for $\alpha, \beta < -N$, we have

$$\begin{aligned} \sum_{\xi=0}^{N} \binom{\alpha+\xi}{\xi} \binom{\beta+N-\xi}{N-\xi} & Q_r(\xi; \alpha, \beta, N) Q_{r'}(\xi; \alpha, \beta, N) \\ &= \delta_{r,r'} \frac{(-1)^r (r+\alpha+\beta+1)_{N+1} (\beta+1)_r r!}{(2r+\alpha+\beta+1)(\alpha+1)_r (-N)_r N!}. \end{aligned} \tag{4.12}$$

Our aim is to describe the entries of the $\breve{E}_r^{i,j}$. In view of (4.9), we will assume for the rest of this section that

$$0 \leqslant i \leqslant j \leqslant n.$$

By Proposition 4.1 and Lemma 4.2, we have

$$E_i^* \boldsymbol{T} E_j^* = \operatorname{span}\bigl\{ E_i^* A_{2\xi+j-i} E_j^* : 0 \leqslant \xi \leqslant \min\{i, n-j\} \bigr\}.$$

Moreover, it follows that (cf. (4.10))

$$\begin{aligned} & E_i^* A_{2\xi+j-i} E_j^* \\ & \quad = \sum_{r=0}^{\min\{i,n-j\}} {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ j-n, -i \end{matrix} \,\middle|\, 1 \right) \frac{\binom{j}{i-\xi}\binom{n-j}{\xi}\binom{j-r}{j-i}\sqrt{\binom{n-2r}{j-r}}}{\binom{j}{i}\sqrt{\binom{n-2r}{i-r}}} \breve{E}_r^{i,j}. \end{aligned} \tag{4.13}$$

This formula can be found in [33, Section 10]. See also [39, 49] for similar calculations.

If $i \leqslant n-j$ then

$$ {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ j-n, -i \end{matrix} \middle| 1 \right) = Q_r(\xi; j-n-1, -j-1, i). $$

Since

$$ \binom{j}{i-\xi}\binom{n-j}{\xi} = (-1)^i \binom{j-n-1+\xi}{\xi}\binom{-j-1+i-\xi}{i-\xi}, $$

it follows from (4.12) (applied to $\alpha = j-n-1$, $\beta = -j-1$, $N = i$) and (4.13) that, for $0 \leqslant r \leqslant i$,

$$ \begin{aligned} &\sum_{\xi=0}^{i} {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ j-n, -i \end{matrix} \middle| 1 \right) E_i^* A_{2\xi+j-i} E_j^* \\ &\qquad = \frac{(-1)^r (r-n-1)_{i+1} (-j)_r r!}{(2r-n-1)(j-n)_r(-i)_r i!} \cdot \frac{(-1)^i \binom{j-r}{j-i}\sqrt{\binom{n-2r}{j-r}}}{\binom{j}{i}\sqrt{\binom{n-2r}{i-r}}} \breve{E}_r^{i,j} \\ &\qquad = \frac{\binom{n}{i}\binom{n-i}{r}\sqrt{\binom{n-2r}{j-r}}}{\left(\binom{n}{r}-\binom{n}{r-1}\right)\binom{n-j}{r}\sqrt{\binom{n-2r}{i-r}}} \breve{E}_r^{i,j}. \end{aligned} $$

Likewise, if $n-j \leqslant i$ then

$$ {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ j-n, -i \end{matrix} \middle| 1 \right) = Q_r(\xi; -i-1, i-n-1, n-j). $$

In this case, since

$$ \binom{j}{i-\xi}\binom{n-j}{\xi}\binom{j}{i}^{-1} = (-1)^{n-j} \binom{-i-1+\xi}{\xi}\binom{i-1-j-\xi}{n-j-\xi}\binom{n-i}{n-j}^{-1}, $$

again it follows from (4.12) (applied to $\alpha = -i-1$, $\beta = i-n-1$, $N = n-j$) and (4.13) that, for $0 \leqslant r \leqslant n-j$,

$$ \begin{aligned} &\sum_{\xi=0}^{n-j} {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ j-n, -i \end{matrix} \middle| 1 \right) E_i^* A_{2\xi+j-i} E_j^* \\ &\qquad = \frac{(-1)^r (r-n-1)_{n-j+1} (i-n)_r r!}{(2r-n-1)(-i)_r(j-n)_r (n-j)!} \cdot \frac{(-1)^{n-j} \binom{j-r}{j-i}\sqrt{\binom{n-2r}{j-r}}}{\binom{n-i}{n-j}\sqrt{\binom{n-2r}{i-r}}} \breve{E}_r^{i,j} \\ &\qquad = \frac{\binom{n}{i}\binom{n-i}{r}\sqrt{\binom{n-2r}{j-r}}}{\left(\binom{n}{r}-\binom{n}{r-1}\right)\binom{n-j}{r}\sqrt{\binom{n-2r}{i-r}}} \breve{E}_r^{i,j}. \end{aligned} $$

In either case, it follows that

$$ \breve{E}_r^{i,j} = \frac{\left(\binom{n}{r}-\binom{n}{r-1}\right)\binom{n-j}{r}\sqrt{\binom{n-2r}{i-r}}}{\binom{n}{i}\binom{n-i}{r}\sqrt{\binom{n-2r}{j-r}}} $$

$$\times \sum_{\xi=0}^{\min\{i,n-j\}} {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ j-n, -i \end{matrix} \middle| 1 \right) E_i^* A_{2\xi+j-i} E_j^* \tag{4.14}$$

for $0 \leqslant i \leqslant j \leqslant n$ and $0 \leqslant r \leqslant \min\{i, n-j\}$.

5 Tight relative $2e$-designs on two shells in $\mathcal{Q}_n$

We retain the notation of the previous sections. In this section, we discuss tight relative $2e$-designs (Y, ω) in $\mathcal{Q}_n$ supported on two shells $X_\ell \sqcup X_m$, i.e., $L = \{\ell, m\}$ (cf. (3.1)). Recall from (3.4) that we have in this case

$$|Y| = \binom{n}{e} + \binom{n}{e-1},$$

but recall also that this is valid under the additional condition that $e \leqslant \ell, m \leqslant n-e$. However, both $(Y \cap X_\ell, E_\ell^* \omega)$ and $(Y \cap X_m, E_m^* \omega)$ are relative $(2e-1)$-designs by Theorem 3.8, so that if $\ell < 2e$ or $\ell > n - 2e$ for example, then $(Y \cap X_\ell, E_\ell^* \omega)$ must be trivial in view of Example 3.5, i.e., $X_\ell \subset Y$ and ω is constant on X_ℓ, and hence $(Y \cap X_m, E_m^* \omega)$ is by itself a relative $2e$-design; cf. Remark 3.7. This shows that the above condition is not a restrictive one. We also note that

Lemma 5.1. *Let (Y, ω) be a relative t-design in $\mathcal{Q}_n$ supported on $\bigsqcup_{\ell \in L} X_\ell$. Then $(Y', A_n \omega)$ is a relative t-design supported on $\bigsqcup_{\ell \in L} X_{n-\ell}$, where $Y' = \{y' : y \in Y\}$, and for every $y \in X$, y' denotes the unique vertex such that $(y, y') \in R_n$.*

Proof. Immediate from $E_i A_n \in \operatorname{span}\{E_i\}$ $(0 \leqslant i \leqslant n)$. □

In view of the above comments, we now make the following assumption:

Assumption 5.2. *In this section, let (Y, ω) be a tight relative $2e$-design $(e \in \mathbb{N})$ in $\mathcal{Q}_n$ supported on two shells $X_\ell \sqcup X_m$, where*

$$e \leqslant \ell < m \leqslant n - \ell \, (\leqslant n - e).$$

Our aim is to show that Y then induces the structure of a coherent configuration with two fibers, and to obtain a necessary condition on the existence of such (Y, ω) akin to Delsarte's theorem on tight $2e$-designs. To this end, we first recall the proof of (3.4) given in [6, Theorem 2.7, Example 2.9] under the above assumption.

For convenience, set

$$E_L^* = E_\ell^* + E_m^*.$$

By (4.2) and (4.3), we have

$$E_L^* \left(\sum_{i=0}^{e} E_i V \right) = \sum_{r=0}^{e} \sum_{W \in \Lambda_r} E_L^* \left(\sum_{i=r}^{e} E_i W \right). \tag{5.1}$$

Let $W \in \Lambda_r$, where $0 \leqslant r \leqslant e$. Recall Theorem 4.3 and also the standard basis (4.5) of W. If $r = e$ then $E_e W$ is spanned by v_W, and hence we have

$$E_L^* E_e W = \operatorname{span}\{E_L^* v_W\}.$$

Note that $E_L^* v_W$ is non-zero by Assumption 5.2, and hence

$$\dim E_L^* E_e W = 1$$

in this case. Suppose next that $0 \leqslant r < e$. On the one hand, since

$$E_L^* \left(\sum_{i=r}^{e} E_i W \right) \subset E_L^* W = E_\ell^* W + E_m^* W,$$

we have

$$\dim E_L^* \left(\sum_{i=r}^{e} E_i W \right) \leqslant 2.$$

On the other hand, it follows from (2.8) that

$$v_W, A_1^* v_W \in E_r W + E_{r+1} W \subset \sum_{i=r}^{e} E_i W, \tag{5.2}$$

and hence

$$E_L^* v_W, E_L^* A_1^* v_W \in E_L^* \left(\sum_{i=r}^{e} E_i W \right).$$

Moreover, we have (cf. (2.7))

$$E_L^* v_W = E_\ell^* v_W + E_m^* v_W, \qquad E_L^* A_1^* v_W = \theta_\ell^* E_\ell^* v_W + \theta_m^* E_m^* v_W,$$

so that these two vectors are non-zero and are linearly independent by Assumption 5.2 and since $\theta_\ell^* \neq \theta_m^*$. It follows that

$$\dim E_L^* \left(\sum_{i=r}^{e} E_i W \right) = 2.$$

Note that in this case we in fact have

$$E_L^* \left(\sum_{i=r}^{e} E_i W \right) = \operatorname{span}\{E_\ell^* v_W, E_m^* v_W\},$$

as

$$E_\ell^* v_W = E_L^* \frac{\theta_m^* I - A_1^*}{\theta_m^* - \theta_\ell^*} v_W, \qquad E_m^* v_W = E_L^* \frac{\theta_\ell^* I - A_1^*}{\theta_\ell^* - \theta_m^*} v_W. \tag{5.3}$$

Combining these comments, we now obtain (3.4) as follows:

$$\begin{aligned}
\dim E_L^* \left(\sum_{i=0}^{e} E_i V \right) &= \sum_{r=0}^{e} \sum_{W \in \Lambda_r} \dim E_L^* \left(\sum_{i=r}^{e} E_i W \right) \\
&= |\Lambda_e| + \sum_{r=0}^{e-1} 2|\Lambda_r| \\
&= \dim E_e V + \dim E_{e-1} V
\end{aligned}$$

$$= \binom{n}{e} + \binom{n}{e-1},$$

where we have used (3.5) and (4.4).

By the above discussions, the set of vectors below forms an orthogonal basis of the subspace (5.1):

$$\left(\bigsqcup_{r=0}^{e-1} \bigsqcup_{W \in \Lambda_r} \{E^*_\ell v_W, E^*_m v_W\} \right) \bigsqcup \left(\bigsqcup_{W \in \Lambda_e} \{E^*_L v_W\} \right).$$

As in the proof of Theorem 3.14, let

$$D = \operatorname{diag} \omega.$$

We next apply $\sqrt{D}$ to the above basis vectors and compute their inner products. First, let $W, W' \in \bigsqcup_{r=0}^{e-1} \Lambda_r$. It is clear that

$$\left\langle \sqrt{D} E^*_\ell v_W, \sqrt{D} E^*_m v_{W'} \right\rangle = \left\langle \sqrt{D} E^*_m v_W, \sqrt{D} E^*_\ell v_{W'} \right\rangle = 0. \tag{5.4}$$

By (5.3), we have

$$(E^*_\ell \overline{v_W}) \circ (E^*_\ell v_{W'}) = E^*_L u,$$

where $^-$ means complex conjugate, and

$$u = \left(\frac{\theta^*_m I - A^*_1}{\theta^*_m - \theta^*_\ell} \overline{v_W} \right) \circ \left(\frac{\theta^*_m I - A^*_1}{\theta^*_m - \theta^*_\ell} v_{W'} \right).$$

Observe that u belongs to $\sum_{i=0}^{2e} E_i V$ by (2.5) (applied to $h = k = e$) and (5.2). Hence, by Proposition 3.6 we have

$$\begin{aligned}
\left\langle \sqrt{D} E^*_\ell v_W, \sqrt{D} E^*_\ell v_{W'} \right\rangle &= \langle \omega, E^*_L u \rangle \\
&= \langle \omega, u \rangle \\
&= \frac{\langle \omega, \hat{X}_\ell \rangle}{|X_\ell|} \langle \hat{X}_\ell, u \rangle + \frac{\langle \omega, \hat{X}_m \rangle}{|X_m|} \langle \hat{X}_m, u \rangle \\
&= \frac{\langle \omega, \hat{X}_\ell \rangle}{|X_\ell|} \langle \hat{X}_\ell, E^*_L u \rangle \\
&= \frac{\langle \omega, \hat{X}_\ell \rangle}{|X_\ell|} \langle E^*_\ell v_W, E^*_\ell v_{W'} \rangle \\
&= \delta_{W,W'} \frac{\langle \omega, \hat{X}_\ell \rangle}{|X_\ell|} \| E^*_\ell v_W \|^2.
\end{aligned} \tag{5.5}$$

Likewise, we have

$$\left\langle \sqrt{D} E^*_m v_W, \sqrt{D} E^*_m v_{W'} \right\rangle = \delta_{W,W'} \frac{\langle \omega, \hat{X}_m \rangle}{|X_m|} \| E^*_m v_W \|^2. \tag{5.6}$$

Next, let $W \in \bigsqcup_{r=0}^{e-1} \Lambda_r$ and $W' \in \Lambda_e$. Then, by the same argument we have

$$\left\langle \sqrt{D} E^*_\ell v_W, \sqrt{D} E^*_L v_{W'} \right\rangle = \left\langle \sqrt{D} E^*_m v_W, \sqrt{D} E^*_L v_{W'} \right\rangle = 0. \tag{5.7}$$

Finally, let $W, W' \in \Lambda_e$. In this case, we have

$$\left\langle \sqrt{D}E_L^* v_W, \sqrt{D}E_L^* v_{W'} \right\rangle = \delta_{W,W'} \left(\frac{\langle \omega, \hat{X}_\ell \rangle}{|\hat{X}_\ell|} \|E_\ell^* v_W\|^2 + \frac{\langle \omega, \hat{X}_m \rangle}{|\hat{X}_m|} \|E_m^* v_W\|^2 \right). \tag{5.8}$$

Since (Y, ω) is a tight relative $2e$-design, it follows from (5.4) – (5.8) that the set of vectors below is an orthogonal basis of the subspace $\sqrt{D}V = \mathrm{span}\{\hat{y} : y \in Y\}$ of dimension $|Y| = \binom{n}{e} + \binom{n}{e-1}$:

$$\left(\bigsqcup_{r=0}^{e-1} \bigsqcup_{W \in \Lambda_r} \left\{ \sqrt{D}E_\ell^* v_W, \sqrt{D}E_m^* v_W \right\} \right) \bigsqcup \left(\bigsqcup_{W \in \Lambda_e} \left\{ \sqrt{D}E_L^* v_W \right\} \right).$$

For convenience, set

$$Y_\ell = Y \cap X_\ell, \qquad Y_m = Y \cap X_m.$$

We will naturally make the following identification by discarding irrelevant entries:

$$\sqrt{D}E_\ell^* V = \mathrm{span}\{\hat{y} : y \in Y_\ell\} \longleftrightarrow \mathbb{C}^{Y_\ell},$$
$$\sqrt{D}E_m^* V = \mathrm{span}\{\hat{y} : y \in Y_m\} \longleftrightarrow \mathbb{C}^{Y_m}.$$

We write

$$\Lambda_r = \left\{ W_r^1, W_r^2, \ldots, W_r^{|\Lambda_r|} \right\} \qquad (0 \leqslant r \leqslant e).$$

For $0 \leqslant r \leqslant e$, define a $|Y_\ell| \times |\Lambda_r|$ matrix H_r^ℓ and a $|Y_m| \times |\Lambda_r|$ matrix H_r^m by

$$H_r^\ell = \left[\sqrt{D}E_\ell^* v_{W_r^1} \;\middle|\; \cdots \;\middle|\; \sqrt{D}E_\ell^* v_{W_r^{|\Lambda_r|}} \right],$$
$$H_r^m = \left[\sqrt{D}E_m^* v_{W_r^1} \;\middle|\; \cdots \;\middle|\; \sqrt{D}E_m^* v_{W_r^{|\Lambda_r|}} \right].$$

We then define a *characteristic matrix* H of (Y, ω) by

$$H = \left[\begin{array}{c|c|c|c|c|c|c} H_0^\ell & \cdots & H_{e-1}^\ell & O & \cdots & O & H_e^\ell \\ \hline O & \cdots & O & H_0^m & \cdots & H_{e-1}^m & H_e^m \end{array} \right].$$

We note that H is a square matrix of size $|Y| = \binom{n}{e} + \binom{n}{e-1}$. By (4.6), (4.7), and (5.4) – (5.8), and since

$$|X_i| = \binom{n}{i} \qquad (0 \leqslant i \leqslant n),$$

we have

$$H^\dagger H = \left(\bigoplus_{r=0}^{e-1} \kappa_r^\ell I_{|\Lambda_r|} \right) \oplus \left(\bigoplus_{r=0}^{e-1} \kappa_r^m I_{|\Lambda_r|} \right) \oplus \kappa_e I_{|\Lambda_e|}, \tag{5.9}$$

where

$$\kappa_r^\ell = \frac{\omega_\ell \binom{n-2r}{\ell-r}}{2^{n-2r}\binom{n}{\ell}}, \quad \kappa_r^m = \frac{\omega_m \binom{n-2r}{m-r}}{2^{n-2r}\binom{n}{m}} \qquad (0 \leqslant r < e),$$

$$\kappa_e = \frac{\omega_\ell \binom{n-2e}{\ell-e}}{2^{n-2e}\binom{n}{\ell}} + \frac{\omega_m \binom{n-2e}{m-e}}{2^{n-2e}\binom{n}{m}},$$

and we abbreviate

$$\omega_\ell = \langle \omega, \hat{X}_\ell \rangle, \qquad \omega_m = \langle \omega, \hat{X}_m \rangle.$$

Let K denote the diagonal matrix on the RHS in (5.9). Then it follows that

$$\begin{aligned} I_{|Y|} &= HK^{-1}H^\dagger \\ &= \left[\begin{array}{c|c} \sum_{r=0}^{e} \frac{1}{\kappa_r^\ell} H_r^\ell (H_r^\ell)^\dagger & \frac{1}{\kappa_e} H_e^\ell (H_e^m)^\dagger \\ \hline \frac{1}{\kappa_e} H_e^m (H_e^\ell)^\dagger & \sum_{r=0}^{e} \frac{1}{\kappa_r^m} H_r^m (H_r^m)^\dagger \end{array} \right], \end{aligned} \tag{5.10}$$

where we write

$$\kappa_e^\ell = \kappa_e^m := \kappa_e \tag{5.11}$$

for brevity. In particular, we have

$$\frac{1}{\kappa_e} H_e^\ell (H_e^m)^\dagger = O. \tag{5.12}$$

Moreover, from (5.9) and (5.10) it follows that

$$\left(\frac{1}{\kappa_r^\ell} H_r^\ell (H_r^\ell)^\dagger\right)\left(\frac{1}{\kappa_{r'}^\ell} H_{r'}^\ell (H_{r'}^\ell)^\dagger\right) = \delta_{r,r'} \frac{1}{\kappa_r^\ell} H_r^\ell (H_r^\ell)^\dagger \qquad (0 \leqslant r, r' < e), \tag{5.13}$$

$$\frac{1}{\kappa_e} H_e^\ell (H_e^\ell)^\dagger = I_{|Y_\ell|} - \sum_{r=0}^{e-1} \frac{1}{\kappa_r^\ell} H_r^\ell (H_r^\ell)^\dagger, \tag{5.14}$$

$$\left(\frac{1}{\kappa_r^m} H_r^m (H_r^m)^\dagger\right)\left(\frac{1}{\kappa_{r'}^m} H_{r'}^m (H_{r'}^m)^\dagger\right) = \delta_{r,r'} \frac{1}{\kappa_r^m} H_r^m (H_r^m)^\dagger \qquad (0 \leqslant r, r' < e), \tag{5.15}$$

$$\frac{1}{\kappa_e} H_e^m (H_e^m)^\dagger = I_{|Y_m|} - \sum_{r=0}^{e-1} \frac{1}{\kappa_r^m} H_r^m (H_r^m)^\dagger. \tag{5.16}$$

Note that the matrices $(\kappa_r^\ell)^{-1} H_r^\ell (H_r^\ell)^\dagger, (\kappa_r^m)^{-1} H_r^m (H_r^m)^\dagger$ $(0 \leqslant r < e)$ are non-zero since H_r^ℓ, H_r^m are non-zero. Likewise, by setting

$$\kappa_r = \sqrt{\kappa_r^\ell \kappa_r^m} \qquad (0 \leqslant r < e)$$

for brevity, we have

$$\left(\frac{1}{\kappa_r^\ell} H_r^\ell (H_r^\ell)^\dagger\right)\left(\frac{1}{\kappa_{r'}} H_{r'}^\ell (H_{r'}^m)^\dagger\right) = \delta_{r,r'} \frac{1}{\kappa_r} H_r^\ell (H_r^m)^\dagger \qquad (0 \leqslant r, r' < e), \tag{5.17}$$

$$\left(\frac{1}{\kappa_r} H_r^\ell (H_r^m)^\dagger\right)\left(\frac{1}{\kappa_{r'}^m} H_{r'}^m (H_{r'}^m)^\dagger\right) = \delta_{r,r'} \frac{1}{\kappa_r} H_r^\ell (H_r^m)^\dagger \qquad (0 \leqslant r, r' < e), \tag{5.18}$$

$$\left(\frac{1}{\kappa_r} H_r^\ell (H_r^m)^\dagger\right)\left(\frac{1}{\kappa_{r'}} H_{r'}^m (H_{r'}^\ell)^\dagger\right) = \delta_{r,r'} \frac{1}{\kappa_r^\ell} H_r^\ell (H_r^\ell)^\dagger \qquad (0 \leqslant r, r' < e), \tag{5.19}$$

$$\left(\frac{1}{\kappa_r} H_r^m (H_r^\ell)^\dagger\right)\left(\frac{1}{\kappa_{r'}} H_{r'}^\ell (H_{r'}^m)^\dagger\right) = \delta_{r,r'} \frac{1}{\kappa_r^m} H_r^m (H_r^m)^\dagger \qquad (0 \leqslant r, r' < e). \tag{5.20}$$

Since the matrices $(\kappa_r^\ell)^{-1}H_r^\ell(H_r^\ell)^\dagger, (\kappa_r^m)^{-1}H_r^m(H_r^m)^\dagger$ $(0 \leqslant r < e)$ are non-zero, it follows from (5.17) – (5.20) that the matrices $(\kappa_r)^{-1}H_r^\ell(H_r^m)^\dagger$ $(0 \leqslant r < e)$ are non-zero and are linearly independent.

It follows from Theorem 3.14 and Proposition 4.1 that ω is constant on each of Y_ℓ and Y_m, from which it follows that

$$D_{y,y} = \omega(y) = \begin{cases} \dfrac{\omega_\ell}{|Y_\ell|} & \text{if } y \in Y_\ell, \\ \dfrac{\omega_m}{|Y_m|} & \text{if } y \in Y_m. \end{cases} \tag{5.21}$$

Hence, by comparing with the formula (4.8) for the matrices $\breve{E}_r^{i,j}$, we have

$$\frac{1}{\kappa_r^\ell}H_r^\ell(H_r^\ell)^\dagger = \frac{\binom{n}{\ell}}{|Y_\ell|}\breve{E}_r^{\ell,\ell}|_{Y_\ell\times Y_\ell} \qquad (0 \leqslant r < e), \tag{5.22}$$

$$\frac{1}{\kappa_e}H_e^\ell(H_e^\ell)^\dagger = \frac{\omega_\ell\binom{n-2e}{\ell-e}}{2^{n-2e}\kappa_e|Y_\ell|}\breve{E}_e^{\ell,\ell}|_{Y_\ell\times Y_\ell}, \tag{5.23}$$

$$\frac{1}{\kappa_r^m}H_r^m(H_r^m)^\dagger = \frac{\binom{n}{m}}{|Y_m|}\breve{E}_r^{m,m}|_{Y_m\times Y_m} \qquad (0 \leqslant r < e), \tag{5.24}$$

$$\frac{1}{\kappa_e}H_e^m(H_e^m)^\dagger = \frac{\omega_m\binom{n-2e}{m-e}}{2^{n-2e}\kappa_e|Y_m|}\breve{E}_e^{m,m}|_{Y_m\times Y_m}, \tag{5.25}$$

$$\frac{1}{\kappa_r}H_r^\ell(H_r^m)^\dagger = \frac{\sqrt{\binom{n}{\ell}\binom{n}{m}}}{\sqrt{|Y_\ell||Y_m|}}\breve{E}_r^{\ell,m}|_{Y_\ell\times Y_m} \qquad (0 \leqslant r < e), \tag{5.26}$$

$$\frac{1}{\kappa_e}H_e^\ell(H_e^m)^\dagger = \frac{\sqrt{\omega_\ell\omega_m\binom{n-2e}{\ell-e}\binom{n-2e}{m-e}}}{2^{n-2e}\kappa_e\sqrt{|Y_\ell||Y_m|}}\breve{E}_e^{\ell,m}|_{Y_\ell\times Y_m}, \tag{5.27}$$

where $|_{Y_\ell\times Y_\ell}$ etc. mean taking corresponding submatrices. From (5.23) and (5.25) it follows that the matrices $(\kappa_e)^{-1}H_e^\ell(H_e^\ell)^\dagger, (\kappa_e)^{-1}H_e^m(H_e^m)^\dagger$ are also non-zero, since each of $\breve{E}_e^{\ell,\ell}|_{Y_\ell\times Y_\ell}, \breve{E}_e^{m,m}|_{Y_m\times Y_m}$ has non-zero constant diagonal entries by (4.14).

Let $\boldsymbol{H}'$ be the set consisting of the $|Y| \times |Y|$ matrices of the form

$$\left[\begin{array}{c|c} \sum_{r=0}^{e} a_r^{\ell,\ell}\frac{1}{\kappa_r^\ell}H_r^\ell(H_r^\ell)^\dagger & \sum_{r=0}^{e-1} a_r^{\ell,m}\frac{1}{\kappa_r}H_r^\ell(H_r^m)^\dagger \\ \hline \sum_{r=0}^{e-1} a_r^{m,\ell}\frac{1}{\kappa_r}H_r^m(H_r^\ell)^\dagger & \sum_{r=0}^{e} a_r^{m,m}\frac{1}{\kappa_r^m}H_r^m(H_r^m)^\dagger \end{array}\right],$$

where $a_r^{\ell,\ell}$ etc. are in $\mathbb{C}$, and we are again using the notation (5.11). By (5.13) – (5.20) and the above comments, $\boldsymbol{H}'$ is a $\mathbb{C}$-algebra with

$$\dim \boldsymbol{H}' = 4e + 2. \tag{5.28}$$

Define

$$S_{\ell,\ell}(Y) = \{j : R_j \cap (Y_\ell \times Y_\ell) \neq \emptyset\},$$

and define $S_{\ell,m}(Y)(= S_{m,\ell}(Y))$ and $S_{m,m}(Y)$ in the same manner. Let $\boldsymbol{H}$ be the set consisting of the $|Y| \times |Y|$ matrices of the form

$$\left[\begin{array}{c|c} \sum_{j\in S_{\ell,\ell}(Y)} b_j^{\ell,\ell}A_j|_{Y_\ell\times Y_\ell} & \sum_{j\in S_{\ell,m}(Y)} b_j^{\ell,m}A_j|_{Y_\ell\times Y_m} \\ \hline \sum_{j\in S_{m,\ell}(Y)} b_j^{m,\ell}A_j|_{Y_m\times Y_\ell} & \sum_{j\in S_{m,m}(Y)} b_j^{m,m}A_j|_{Y_m\times Y_m} \end{array}\right], \tag{5.29}$$

where $b_j^{\ell,\ell}$ etc. are in $\mathbb{C}$. Then $\boldsymbol{H}$ is a $\mathbb{C}$-vector space with

$$\dim \boldsymbol{H} = |S_{\ell,\ell}(Y)| + |S_{\ell,m}(Y)| + |S_{m,\ell}(Y)| + |S_{m,m}(Y)|. \tag{5.30}$$

Note that $\boldsymbol{H}$ is closed under $\circ$. By (5.22) – (5.26) and Proposition 4.1 (or (4.14)), $\boldsymbol{H}'$ is a subspace of $\boldsymbol{H}$.

By (4.14), (5.14), (5.22), and (5.23), we have

$$\begin{aligned} I_{|Y_\ell|} &= \sum_{r=0}^{e-1} \frac{\binom{n}{\ell}}{|Y_\ell|} \breve{E}_r^{\ell,\ell}|_{Y_\ell\times Y_\ell} + \frac{\omega_\ell\binom{n-2e}{\ell-e}}{2^{n-2e}\kappa_e|Y_\ell|}\breve{E}_e^{\ell,\ell}|_{Y_\ell\times Y_\ell} \\ &= \frac{1}{|Y_\ell|}\sum_{\xi=0}^{\min\{\ell,n-\ell\}}\left(\sum_{r=0}^{e-1}\left(\binom{n}{r}-\binom{n}{r-1}\right){}_3F_2\left(\begin{matrix}-\xi,-r,r-n-1\\ \ell-n,-\ell\end{matrix}\,\middle|\,1\right)\right. \\ &\qquad \left.+\frac{\omega_\ell\binom{n-2e}{\ell-e}\left(\binom{n}{e}-\binom{n}{e-1}\right)}{2^{n-2e}\kappa_e\binom{n}{\ell}}{}_3F_2\left(\begin{matrix}-\xi,-e,e-n-1\\ \ell-n,-\ell\end{matrix}\,\middle|\,1\right)\right)A_{2\xi}|_{Y_\ell\times Y_\ell}. \end{aligned} \tag{5.31}$$

Hence it follows that $\{\xi\neq 0 : 2\xi\in S_{\ell,\ell}(Y)\}$ is a set of zeros of the polynomial

$$\begin{aligned} \psi_e^{\ell,\ell}(\xi) &= \sum_{r=0}^{e-1}\left(\binom{n}{r}-\binom{n}{r-1}\right){}_3F_2\left(\begin{matrix}-\xi,-r,r-n-1\\ \ell-n,-\ell\end{matrix}\,\middle|\,1\right) \\ &\quad + \frac{\omega_\ell\binom{n-2e}{\ell-e}\left(\binom{n}{e}-\binom{n}{e-1}\right)}{2^{n-2e}\kappa_e\binom{n}{\ell}}{}_3F_2\left(\begin{matrix}-\xi,-e,e-n-1\\ \ell-n,-\ell\end{matrix}\,\middle|\,1\right)\in\mathbb{R}[\xi]. \end{aligned} \tag{5.32}$$

Note that $\psi_e^{\ell,\ell}(\xi)$ has degree exactly e, from which it follows that

$$|S_{\ell,\ell}(Y)| \leqslant e+1. \tag{5.33}$$

Likewise, we find that $\{\xi\neq 0 : 2\xi\in S_{m,m}(Y)\}$ is a set of zeros of the polynomial

$$\begin{aligned} \psi_e^{m,m}(\xi) &= \sum_{r=0}^{e-1}\left(\binom{n}{r}-\binom{n}{r-1}\right){}_3F_2\left(\begin{matrix}-\xi,-r,r-n-1\\ m-n,-m\end{matrix}\,\middle|\,1\right) \\ &\quad + \frac{\omega_m\binom{n-2e}{m-e}\left(\binom{n}{e}-\binom{n}{e-1}\right)}{2^{n-2e}\kappa_e\binom{n}{m}}{}_3F_2\left(\begin{matrix}-\xi,-e,e-n-1\\ m-n,-m\end{matrix}\,\middle|\,1\right)\in\mathbb{R}[\xi], \end{aligned} \tag{5.34}$$

and hence that

$$|S_{m,m}(Y)| \leqslant e+1. \tag{5.35}$$

Finally, by (4.14), (5.12), and (5.27), we have

$$\begin{aligned} O &= \frac{\sqrt{\omega_\ell\omega_m\binom{n-2e}{\ell-e}\binom{n-2e}{m-e}}}{2^{n-2e}\kappa_e\sqrt{|Y_\ell||Y_m|}}\breve{E}_e^{\ell,m}|_{Y_\ell\times Y_m} \\ &= \frac{\sqrt{\omega_\ell\omega_m}\binom{n-m}{e}\binom{n-2e}{\ell-e}\left(\binom{n}{e}-\binom{n}{e-1}\right)}{2^{n-2e}\kappa_e\sqrt{|Y_\ell||Y_m|}\,\binom{n}{\ell}\binom{n-\ell}{e}} \end{aligned}$$

$$\times \sum_{\xi=0}^{\min\{\ell,n-m\}} {}_3F_2\left(\begin{matrix}-\xi,-e,e-n-1\\ m-n,-\ell\end{matrix}\,\middle|\,1\right) A_{2\xi+m-\ell}|_{Y_\ell\times Y_m}.$$

Hence it follows that $\{\xi : 2\xi+m-\ell \in S_{\ell,m}(Y)\}$ is a set of zeros of the polynomial

$$\psi_e^{\ell,m}(\xi) = {}_3F_2\left(\begin{matrix}-\xi,-e,e-n-1\\ m-n,-\ell\end{matrix}\,\middle|\,1\right) \in \mathbb{R}[\xi], \tag{5.36}$$

and that

$$|S_{\ell,m}(Y)| = |S_{m,\ell}(Y)| \leqslant e. \tag{5.37}$$

By (5.30), (5.33), (5.35), and (5.37), we have

$$\dim \boldsymbol{H} \leqslant 4e+2.$$

Since $\boldsymbol{H}'$ is a subspace of $\boldsymbol{H}$, it follows from (5.28) that $\boldsymbol{H}=\boldsymbol{H}'$. In particular, $\boldsymbol{H}$ is a $\mathbb{C}$-algebra. It is also clear that $\boldsymbol{H}$ is closed under $\dagger$ and contains $J_{|Y|}$. We now conclude that $\boldsymbol{H}$ is a coherent algebra. Note also that equality holds in each of (5.33), (5.35), and (5.37).

To summarize:

Theorem 5.3. *Recall* Assumption 5.2. *With the above notation, the following hold:*

(i) *The set $\boldsymbol{H}$ from* (5.29) *is a coherent algebra of type* $\left[\begin{smallmatrix}e+1 & e\\ e & e+1\end{smallmatrix}\right]$.

(ii) *The sets of zeros of the polynomials $\psi_e^{\ell,\ell}(\xi), \psi_e^{m,m}(\xi)$, and $\psi_e^{\ell,m}(\xi)$ from* (5.32), (5.34)*, and* (5.36) *are given respectively by*

$$\{\xi\neq 0 : 2\xi \in S_{\ell,\ell}(Y)\},\ \{\xi\neq 0 : 2\xi\in S_{m,m}(Y)\},\ \text{and}$$
$$\{\xi : 2\xi+m-\ell\in S_{\ell,m}(Y)\}.$$

In particular, the zeros of these polynomials are integral.

Concerning the scalars ω_ℓ and ω_m appearing in the polynomials $\psi_e^{\ell,\ell}(\xi)$ and $\psi_e^{m,m}(\xi)$, it follows that

Proposition 5.4. *Recall* Assumption 5.2. *The scalars ω_ℓ and ω_m satisfies*

$$\frac{\omega_m}{\omega_\ell} = \frac{\binom{n}{m}\binom{n-2e}{\ell-e}}{\binom{n}{\ell}\binom{n-2e}{m-e}} \cdot \frac{|Y_m| - \binom{n}{e-1}}{|Y_\ell| - \binom{n}{e-1}}.$$

In particular, the weight function ω is unique up to a scalar multiple.

Proof. By comparing the diagonal entries of both sides in (5.31), we have

$$1 = \frac{\psi_e^{\ell,\ell}(0)}{|Y_\ell|} = \frac{1}{|Y_\ell|}\left(\binom{n}{e-1} + \frac{\omega_\ell\binom{n-2e}{\ell-e}\left(\binom{n}{e}-\binom{n}{e-1}\right)}{2^{n-2e}\kappa_e\binom{n}{\ell}}\right).$$

Likewise,

$$1 = \frac{\psi_e^{m,m}(0)}{|Y_m|} = \frac{1}{|Y_m|}\left(\binom{n}{e-1} + \frac{\omega_m\binom{n-2e}{m-e}\left(\binom{n}{e}-\binom{n}{e-1}\right)}{2^{n-2e}\kappa_e\binom{n}{m}}\right).$$

By eliminating κ_e, we obtain the formula for $\omega_m(\omega_\ell)^{-1}$. The uniqueness of ω follows from this and (5.21). $\square$

Example 5.5. Suppose that $e = 1$. In this case, Theorem 5.3(i) was previously obtained by Bannai, Bannai, and Bannai [5, Theorem 2.2 (i)]. Moreover, Theorem 5.3(ii) and Proposition 5.4 are together equivalent to [5, Proposition 4.3].

Example 5.6. Suppose that $e = 2$. Then we have

$$\begin{aligned}\psi_e^{\ell,m}(\xi) &= 1 + \frac{(-\xi)(-2)(1-n)}{(m-n)(-\ell)} + \frac{(-\xi)(1-\xi)(-2)(-1)(1-n)(2-n)}{(m-n)(m-n+1)(-\ell)(1-\ell)2} \\ &= 1 - \frac{2(n-1)\xi}{(n-m)\ell} + \frac{(n-1)(n-2)\xi(\xi-1)}{(n-m)(n-m-1)\ell(\ell-1)}.\end{aligned}$$

From Example 3.13 we find two parameter sets satisfying Assumption 5.2:

n	ℓ	m	ξ
22	6	7	$3, 5$
22	6	15	$1, 3$

The zeros ξ given in the last column are indeed integers. Note that the other two parameter sets in Example 3.13 correspond to the complements of these two; cf. Lemma 5.1. On the other hand, the existence of tight relative 4-designs with the following feasible parameter sets was left open in [9, Section 6]:

n	ℓ	m	ξ
37	9	16	$\frac{1}{14}(71 \pm \sqrt{337})$
37	9	21	$\frac{1}{14}(55 \pm \sqrt{337})$
41	15	16	$\frac{1}{26}(237 \pm \sqrt{1569})$
41	15	25	$\frac{1}{26}(153 \pm \sqrt{1569})$

Here, we are again taking Lemma 5.1 into account. Observe that the zeros ξ are irrational, thus proving the non-existence.

We end this section with a comment on the expressions of the polynomials $\psi_e^{\ell,\ell}(\xi)$ and $\psi_e^{m,m}(\xi)$. We first invoke the following identity which agrees with the formula of the backward shift operator on the dual Hahn polynomials (cf. [31, Section 1.6]):

$$\begin{aligned}&\alpha(N+1)(\alpha+\beta+2r)Q_r(\xi;\alpha-1,\beta,N+1) \\ &\quad= (\alpha+r)(\alpha+\beta+r)(N+1-r)Q_r(\xi-1;\alpha,\beta,N) \\ &\qquad - r(\alpha+\beta+N+1+r)(\beta+r)Q_{r-1}(\xi-1;\alpha,\beta,N).\end{aligned} \tag{5.38}$$

This can be routinely verified by writing the LHS as a linear combination of the polynomials $(1-\xi)_i$ $(0 \leqslant i \leqslant r)$ using

$$(-\xi)_i = (1-\xi)_i - i(1-\xi)_{i-1},$$

and then comparing the coefficients of both sides. Setting $\alpha = \ell - n$, $\beta = -\ell - 1$, and $N = \ell - 1$ in (5.38), it follows that the first term of the RHS in (5.32) is rewritten as follows:

$$\sum_{r=0}^{e-1} \left(\binom{n}{r} - \binom{n}{r-1} \right) {}_3F_2\left(\begin{matrix} -\xi, -r, r-n-1 \\ \ell-n, -\ell \end{matrix} \middle| 1 \right)$$

$$
\begin{aligned}
&= \sum_{r=0}^{e-1} \frac{n!(n-2r+1)}{r!(n-r+1)!} Q_r(\xi;\alpha-1,\beta,N+1) \\
&= \frac{n!}{\ell(n-\ell)} \sum_{r=0}^{e-1} \Bigg(\frac{(\ell-n+r)(r-n-1)(\ell-r)}{r!(n-r+1)!} Q_r(\xi-1;\alpha,\beta,N) \\
&\qquad\qquad - \frac{r(r+\ell-n-1)(r-\ell-1)}{r!(n-r+1)!} Q_{r-1}(\xi-1;\alpha,\beta,N) \Bigg) \\
&= \frac{n!}{\ell(n-\ell)} \cdot (-1) \frac{(\ell-n+e-1)(\ell-e+1)}{(e-1)!(n-e+1)!} Q_{e-1}(\xi-1;\alpha,\beta,N) \\
&= \binom{n}{e-1} \frac{(n-\ell-e+1)(\ell-e+1)}{\ell(n-\ell)} {}_3F_2\!\left(\begin{matrix} 1-\xi, 1-e, e-n-1 \\ \ell-n+1, 1-\ell \end{matrix} \,\middle|\, 1 \right).
\end{aligned}
$$

Likewise, the first term of the RHS in (5.34) is given by

$$
\begin{aligned}
&\sum_{r=0}^{e-1} \left(\binom{n}{r} - \binom{n}{r-1} \right) {}_3F_2\!\left(\begin{matrix} -\xi, -r, r-n-1 \\ m-n, -m \end{matrix} \,\middle|\, 1 \right) \\
&\quad = \binom{n}{e-1} \frac{(n-m-e+1)(m-e+1)}{m(n-m)} {}_3F_2\!\left(\begin{matrix} 1-\xi, 1-e, e-n-1 \\ m-n+1, 1-m \end{matrix} \,\middle|\, 1 \right).
\end{aligned}
$$

6 Zeros of the Hahn and Hermite polynomials

Recall the Hahn polynomials $Q_r(\xi;\alpha,\beta,N)$ from (4.11). Recall also that the zeros of orthogonal polynomials are always real and simple; see, e.g., [42, Theorem 3.3.1]. It is well known that we can obtain the Hermite polynomials as limits of the Hahn polynomials; cf. [30, 31]. In this section, we revisit this limit process and describe the limit behavior of the zeros of the $Q_r(\xi;\alpha,\beta,N)$, in a special case which is suited to our purpose.

Assumption 6.1. *Throughout this section, we assume that $\alpha < -N$ and $\beta < -N$, so that the $Q_r(\xi;\alpha,\beta,N)$ satisfy the orthogonality relation* (4.12)*. We consider the following limit:*

$$\epsilon := -\frac{\alpha+\beta}{\sqrt{\alpha\beta N}} \to +0.$$

We write

$$\alpha = \frac{\alpha_\epsilon}{\epsilon^2}, \qquad \beta = \frac{\beta_\epsilon}{\epsilon^2}, \qquad N = \frac{N_\epsilon}{\epsilon^2},$$

and assume further that

$$\lim_{\epsilon\to+0} \frac{N_\epsilon}{\alpha_\epsilon+\beta_\epsilon} = 0, \qquad \lim_{\epsilon\to+0} \frac{\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon} = \rho \in [0,1].$$

Remark 6.2. We do not require in Assumption 6.1 that $\alpha_\epsilon, \beta_\epsilon$, and N_ϵ are uniquely determined by ϵ. In other words, these are multi-valued functions of ϵ in general (for admissible values of ϵ), but their limit behaviors are uniformly governed by ϵ.

With reference to Assumption 6.1, observe that

$$\lim_{\epsilon\to+0} \alpha_\epsilon = \lim_{\epsilon\to+0} \alpha\epsilon^2 = \lim_{\epsilon\to+0} \frac{\alpha_\epsilon+\beta_\epsilon}{\beta_\epsilon} \cdot \frac{\alpha_\epsilon+\beta_\epsilon}{N_\epsilon} = -\infty.$$

Likewise, we have

$$\lim_{\epsilon\to+0}\beta_\epsilon=-\infty,\qquad \lim_{\epsilon\to+0}N_\epsilon=\frac{1}{\rho(1-\rho)}\in[4,\infty].$$

We will work with the *normalized* (or *monic*) Hahn polynomials:

$$q_r(\xi)=q_r(\xi;\epsilon)=\frac{(\alpha+1)_r(-N)_r}{(r+\alpha+\beta+1)_r}Q_r(\xi;\alpha,\beta,N). \tag{6.1}$$

Their recurrence relation is given by (cf. [31, Section 1.5])

$$\xi q_r(\xi)=q_{r+1}(\xi)+(a_r+b_r)q_r(\xi)+a_{r-1}b_rq_{r-1}(\xi), \tag{6.2}$$

where $q_{-1}(\xi):=0$, and

$$\begin{aligned} a_r&=\frac{(r+\alpha+\beta+1)(r+\alpha+1)(N-r)}{(2r+\alpha+\beta+1)(2r+\alpha+\beta+2)},\\ b_r&=\frac{r(r+\alpha+\beta+N+1)(r+\beta)}{(2r+\alpha+\beta)(2r+\alpha+\beta+1)}. \end{aligned}$$

For convenience, let

$$\lambda_\epsilon=\sqrt{\frac{2(\alpha_\epsilon+\beta_\epsilon+N_\epsilon)}{\alpha_\epsilon+\beta_\epsilon}}.$$

Note that

$$\lim_{\epsilon\to+0}\lambda_\epsilon=\sqrt{2}. \tag{6.3}$$

Consider the polynomial $\tilde{q}_r(\eta;\epsilon)$ in the new indeterminate η defined by

$$\tilde{q}_r(\eta)=\tilde{q}_r(\eta;\epsilon)=q_r\left(\frac{\lambda_\epsilon\eta}{\epsilon}+\frac{\alpha_\epsilon N_\epsilon}{(\alpha_\epsilon+\beta_\epsilon)\epsilon^2}\right)\cdot\frac{\epsilon^r}{(\lambda_\epsilon)^r}\in\mathbb{R}[\eta].$$

Note that $\tilde{q}_r(\eta)$ is also monic with degree r in η. Then (6.2) becomes

$$\eta\tilde{q}_r(\eta)=\tilde{q}_{r+1}(\eta)+\frac{1}{\lambda_\epsilon}\left((a_r+b_r)\epsilon-\frac{\alpha_\epsilon N_\epsilon}{(\alpha_\epsilon+\beta_\epsilon)\epsilon}\right)\tilde{q}_r(\eta)+\frac{a_{r-1}b_r\epsilon^2}{(\lambda_\epsilon)^2}\tilde{q}_{r-1}(\eta). \tag{6.4}$$

It is a straightforward matter to show that

$$\frac{1}{\lambda_\epsilon}\left((a_r+b_r)\epsilon-\frac{\alpha_\epsilon N_\epsilon}{(\alpha_\epsilon+\beta_\epsilon)\epsilon}\right)=-(\mu_\epsilon+r\sigma_\epsilon)\epsilon+O(\epsilon^3), \tag{6.5}$$

$$\frac{a_{r-1}b_r\epsilon^2}{(\lambda_\epsilon)^2}=\frac{r}{2}+O(\epsilon^2), \tag{6.6}$$

where

$$\mu_\epsilon:=\frac{(\alpha_\epsilon-\beta_\epsilon)N_\epsilon}{\lambda_\epsilon(\alpha_\epsilon+\beta_\epsilon)^2},\qquad \sigma_\epsilon:=\frac{(\alpha_\epsilon-\beta_\epsilon)(\alpha_\epsilon+\beta_\epsilon+2N_\epsilon)}{\lambda_\epsilon(\alpha_\epsilon+\beta_\epsilon)^2}$$

are convergent:

$$\lim_{\epsilon\to+0}\mu_\epsilon=0,\qquad \lim_{\epsilon\to+0}\sigma_\epsilon=\frac{1-2\rho}{\sqrt{2}}. \tag{6.7}$$

Recall the *Hermite polynomials* [31, Section 1.13]

$$H_r(\eta) = (2\eta)^r {}_2F_0\left(\begin{matrix} -r/2, -(r-1)/2 \\ - \end{matrix} \,\middle|\, -\frac{1}{\eta^2}\right) \in \mathbb{R}[\eta] \quad (r = 0, 1, 2, \ldots).$$

Their normalized recurrence relation is given by

$$\eta h_r(\eta) = h_{r+1}(\eta) + \frac{r}{2} h_{r-1}(\eta), \tag{6.8}$$

where

$$h_r(\eta) = \frac{H_r(\eta)}{2^r}, \tag{6.9}$$

and $h_{-1}(\eta) := 0$. We also note that

$$\frac{dh_r}{d\eta}(\eta) = r h_{r-1}(\eta), \tag{6.10}$$

and that

$$h_r(-\eta) = (-1)^r h_r(\eta). \tag{6.11}$$

Since $\tilde{q}_0(\eta) = h_0(\eta) = 1$, it follows from (6.4) – (6.8) that

$$\lim_{\epsilon \to +0} \tilde{q}_r(\eta; \epsilon) = h_r(\eta) \tag{6.12}$$

in the sense of coefficient-wise convergence.

We now set

$$\tilde{q}_r(\eta; 0) = h_r(\eta),$$

and discuss partial derivatives of $\tilde{q}_r(\eta; \epsilon)$ as a bivariate function of η and ϵ. First, it follows from (6.10) and (6.12) that

$$\lim_{\epsilon \to +0} \frac{\partial \tilde{q}_r}{\partial \eta}(\eta; \epsilon) = \frac{dh_r}{d\eta}(\eta) = r h_{r-1}(\eta). \tag{6.13}$$

Concerning the partial differentiability of $\tilde{q}_r(\eta; \epsilon)$ with respect to ϵ, it follows that

Lemma 6.3. *The function $\tilde{q}_r(\eta; \epsilon)$ is partially right differentiable with respect to ϵ at $(\eta, 0)$, and we have*

$$\frac{\partial \tilde{q}_r}{\partial \epsilon}(\eta; 0) = \frac{r(1-2\rho)}{3\sqrt{2}} \left((r - 1 + \eta^2) h_{r-1}(\eta) - \eta h_r(\eta) \right).$$

Proof. Throughout the proof, we fix $\eta \in \mathbb{R}$ and set

$$\Delta_r(\epsilon) = \Delta_r(\eta; \epsilon) = \frac{\tilde{q}_r(\eta; \epsilon) - h_r(\eta)}{\epsilon}.$$

It follows from (6.4) – (6.8) and (6.12) that

$$\begin{aligned} \eta \Delta_r(\epsilon) &= \Delta_{r+1}(\epsilon) - (\mu_\epsilon + r\sigma_\epsilon)\tilde{q}_r(\eta; \epsilon) + \frac{r}{2}\Delta_{r-1}(\epsilon) + O(\epsilon) \\ &= \Delta_{r+1}(\epsilon) - r\sigma_0 h_r(\eta) + \frac{r}{2}\Delta_{r-1}(\epsilon) + o(1), \end{aligned} \tag{6.14}$$

where we set

$$\sigma_0 := \lim_{\epsilon\to+0} \sigma_\epsilon = \frac{1-2\rho}{\sqrt{2}}$$

for brevity. Since $\tilde{q}_0(\eta;\epsilon) = 1$, we have $\Delta_0(\epsilon) = 0$. Solving the recurrence (6.14) using this initial condition and (6.8), we routinely obtain

$$\Delta_r(\epsilon) = \frac{r(r-1)}{2}\sigma_0 h_{r-1}(\eta) + \frac{r(r-1)(r-2)}{12}\sigma_0 h_{r-3}(\eta) + o(1),$$

where $h_{-1}(\eta) = h_{-2}(\eta) = h_{-3}(\eta) := 0$. It follows that $\tilde{q}_r(\eta;\epsilon)$ is partially right differentiable with respect to ϵ at $(\eta, 0)$:

$$\begin{aligned}\frac{\partial \tilde{q}_r}{\partial \epsilon}(\eta;0) &= \lim_{\epsilon\to+0} \Delta_r(\epsilon)\\ &= \frac{r(r-1)}{2}\sigma_0 h_{r-1}(\eta) + \frac{r(r-1)(r-2)}{12}\sigma_0 h_{r-3}(\eta).\end{aligned}$$

Finally, from (6.8) it follows that

$$\begin{aligned}\frac{\partial \tilde{q}_r}{\partial \epsilon}(\eta;0) &= \frac{r(r-1)}{2}\sigma_0 h_{r-1}(\eta) + \frac{r(r-1)}{6}\sigma_0\big(\eta h_{r-2}(\eta) - h_{r-1}(\eta)\big)\\ &= \frac{r(r-1)}{3}\sigma_0 h_{r-1}(\eta) + \frac{r}{3}\sigma_0\eta\big(\eta h_{r-1}(\eta) - h_r(\eta)\big)\\ &= \frac{r\sigma_0}{3}\left((r-1+\eta^2)h_{r-1}(\eta) - \eta h_r(\eta)\right),\end{aligned}$$

as desired. □

Proposition 6.4. *Recall* Assumption 6.1. *Fix a positive integer e, and let*

$$\begin{aligned}&\xi_{-\lfloor e/2\rfloor} < \cdots < \xi_{-1} < (\xi_0) < \xi_1 < \cdots < \xi_{\lfloor e/2\rfloor},\\ &\eta_{-\lfloor e/2\rfloor} < \cdots < \eta_{-1} < (\eta_0) < \eta_1 < \cdots < \eta_{\lfloor e/2\rfloor}\end{aligned}$$

be the zeros of $q_e(\xi;\epsilon)$ and $h_e(\eta)$ from (6.1) *and* (6.9)*, respectively, where ξ_0 and η_0 appear only when e is odd. Then ξ_i satisfies*

$$\lim_{\epsilon\to+0}\left(\xi_i - \frac{\lambda_\epsilon \eta_i}{\epsilon} - \frac{\alpha_\epsilon N_\epsilon}{(\alpha_\epsilon+\beta_\epsilon)\epsilon^2}\right) = \frac{2\rho-1}{3}\big(e-1+(\eta_i)^2\big)$$

as a function of ϵ, for $i = -\lfloor e/2\rfloor, \dots, -1, (0), 1, \dots, \lfloor e/2\rfloor$.

Proof. Define τ_i by

$$\xi_i = \frac{\lambda_\epsilon(\eta_i+\tau_i)}{\epsilon} + \frac{\alpha_\epsilon N_\epsilon}{(\alpha_\epsilon+\beta_\epsilon)\epsilon^2},$$

so that $\eta_i + \tau_i$ is a zero of $\tilde{q}_e(\eta;\epsilon)$. Then, from (6.12) it follows that

$$\lim_{\epsilon\to+0} \tau_i = 0. \tag{6.15}$$

For the moment, fix i. Then we have

$$0 = \tilde{q}_e(\eta_i+\tau_i;\epsilon) = \tilde{q}_e(\eta_i;\epsilon) + \frac{\partial \tilde{q}_e}{\partial \eta}(\eta_i+\theta\tau_i;\epsilon)\tau_i$$

for some $\theta \in (0,1)$ depending on ϵ. Hence, from (6.13), (6.15), Lemma 6.3, and since

$$\tilde{q}_e(\eta_i; 0) = h_e(\eta_i) = 0,$$

it follows that

$$\begin{aligned}\lim_{\epsilon\to+0} \frac{\tau_i}{\epsilon} &= -\frac{1}{e h_{e-1}(\eta_i)} \lim_{\epsilon\to+0} \frac{\tilde{q}_e(\eta_i;\epsilon)}{\epsilon} \\ &= -\frac{1}{e h_{e-1}(\eta_i)} \frac{\partial \tilde{q}_e}{\partial \epsilon}(\eta_i; 0) \\ &= \frac{2\rho-1}{3\sqrt{2}}\big(e-1+(\eta_i)^2\big),\end{aligned}$$

where we note that $h_e(\eta)$ and $h_{e-1}(\eta)$ have no common zero by the general theory of orthogonal polynomials; see, e.g., [42, Theorem 3.3.2]. By (6.3), we have

$$\lim_{\epsilon\to+0}\left(\xi_i - \frac{\lambda_\epsilon \eta_i}{\epsilon} - \frac{\alpha_\epsilon N_\epsilon}{(\alpha_\epsilon+\beta_\epsilon)\epsilon^2}\right) = \lim_{\epsilon\to+0}\frac{\lambda_\epsilon \tau_i}{\epsilon} = \frac{2\rho-1}{3}\big(e-1+(\eta_i)^2\big).$$

This completes the proof. □

The following is part of the estimates on the zeros of $h_e(\eta)$ used in [1].[3]

Proposition 6.5 ([1, Proposition 13])**.** *Fix a positive integer e, and let the η_i be as in* Proposition 6.4. *Then $\eta_{-i} = -\eta_i$ for all i. Moreover, the following hold:*

1. *If e is odd and $e \geqslant 5$, then $\eta_0 = 0$ and $(\eta_1)^2 < 3/2$.*
2. *If e is even and $e \geqslant 8$, then $(\eta_2)^2 - (\eta_1)^2 < 3/2$.*

Proof. That $\eta_{-i} = -\eta_i$ is immediate from (6.11). We now write $\eta_i = \eta_i^e$ to compare these zeros for different values of e. Then, as an application of Sturm's method, it follows that

$$\sqrt{2e+1}\,\eta_i^e < \sqrt{2e'+1}\,\eta_i^{e'} \qquad (i = 1, 2, \ldots, \lfloor e'/2 \rfloor),$$

whenever $e' < e$ and $e' \equiv e \pmod 2$; see the comments preceding (6.31.19) in [42]. Since

$$h_3(\eta) = \eta^3 - \frac{3}{2}\eta, \qquad h_4(\eta) = \eta^4 - 3\eta^2 + \frac{3}{4},$$

we have

$$\eta_1^3 = \sqrt{\frac{3}{2}}, \qquad \eta_2^4 = \sqrt{\frac{3+\sqrt{6}}{2}}.$$

Hence, for odd $e \geqslant 5$ we have

$$(\eta_1^e)^2 < \frac{7}{2e+1}(\eta_1^3)^2 = \frac{21}{4e+2} < \frac{3}{2},$$

and for even $e \geqslant 8$ we have

$$(\eta_2^e)^2 - (\eta_1^e)^2 < (\eta_2^e)^2 < \frac{9}{2e+1}(\eta_2^4)^2 = \frac{27+9\sqrt{6}}{4e+2} < \frac{3}{2},$$

as desired. □

[3] Bannai [1] worked with the polynomial $\sqrt{2^e}h_e(\eta/\sqrt{2})$. We may remark that the upper bounds $\sqrt{3}$ mentioned in Proposition 13 (i) and (ii) in [1] should both be 3. See also [22, Proposition 2.4].

7 A finiteness result for tight relative $2e$-designs on two shells in $\mathcal{Q}_n$

In this section, we prove that

Theorem 7.1. *For any $\delta \in (0, 1/2)$, there exists $e_0 = e_0(\delta) > 0$ with the property that, for every given integer $e \geqslant e_0$ and each constant $c > 0$, there are only finitely many tight relative $2e$-designs (Y, ω) (up to scalar multiples of ω) supported on two shells $X_\ell \sqcup X_m$ in $\mathcal{Q}_n$ satisfying* Assumption 5.2 *such that*

$$\ell < c \cdot n^{\delta}. \tag{7.1}$$

Our proof is an application of Bannai's method from [1]. We will use the following result, which is a variation of [40, Satz I]:

Proposition 7.2. *For any $\vartheta > 0$ and $\delta \in (0, 1/\vartheta)$, there exists $\mathfrak{k}_0 = \mathfrak{k}_0(\vartheta, \delta) > 0$ such that the following holds for every given integer $\mathfrak{k} \geqslant \mathfrak{k}_0$ and each constant $c > 0$: for all but finitely many pairs $(\mathfrak{a}, \mathfrak{b})$ of positive integers with*

$$\mathfrak{b} < c \cdot \mathfrak{a}^{\delta},$$

the product of $\mathfrak{k}$ consecutive odd integers

$$(2\mathfrak{a} + 1)(2\mathfrak{a} + 3) \cdots (2\mathfrak{a} + 2\mathfrak{k} - 1)$$

has a prime factor which is greater than $2\mathfrak{k} + 1$ and whose exponent in this product is greater than that in

$$(\mathfrak{b} + 1)(\mathfrak{b} + 2) \cdots (\mathfrak{b} + \lfloor \vartheta \mathfrak{k} \rfloor).$$

The proof of Proposition 7.2 will be deferred to the appendix.

We will establish Theorem 7.1 by contradiction:

Assumption 7.3. *We fix $\delta \in (0, 1/2)$. Let $\mathfrak{k}_0 = \mathfrak{k}_0(2, \delta) > 0$ be as in* Proposition 7.2 *(applied to $\vartheta = 2$), and set*

$$e_0 = e_0(\delta) = \max\{2\mathfrak{k}_0, 8\}.$$

We also fix a positive integer $e \geqslant e_0$ and a constant $c > 0$. Throughout the proof, we assume that there exist infinitely many tight relative $2e$-designs (Y, ω) in question.

Let Θ denote the set of triples $(\ell, m, n) \in \mathbb{N}^3$ taken by those (Y, ω) in Assumption 7.3. Recall from Proposition 5.4 that ω is uniquely determined by Y up to a scalar multiple. Moreover, for each $(\ell, m, n) \in \Theta$ there are only finitely many choices for Y. Hence we have

$$|\Theta| = \infty. \tag{7.2}$$

For the moment, we fix $(\ell, m, n) \in \Theta$ and consider the polynomial $\psi_e^{\ell,m}(\xi)$ (which also depends on n) from (5.36). We recall that

$$\psi_e^{\ell,m}(\xi) = Q_e(\xi; \alpha, \beta, N),$$

where

$$\alpha = m - n - 1, \qquad \beta = -m - 1, \qquad N = \ell. \tag{7.3}$$

We note that $\alpha, \beta < -N$ in view of Assumption 5.2. By Theorem 5.3(ii), if we let

$$\xi_{-\lfloor e/2 \rfloor} < \cdots < \xi_{-1} < (\xi_0) < \xi_1 < \cdots < \xi_{\lfloor e/2 \rfloor} \tag{7.4}$$

denote the zeros of $\psi_e^{\ell,m}(\xi)$ (cf. Proposition 6.4), then we have

$$\xi_i \in \{0, 1, \ldots, \ell\} \qquad \text{for all } i. \tag{7.5}$$

We also rewrite $\psi_e^{\ell,m}(\xi)$ as follows:

$$\psi_e^{\ell,m}(\xi) = \sum_{i=0}^{e} s_{e-i}(-1)^i(-\xi)_i,$$

where

$$s_{e-i} = \binom{e}{i} \frac{(e-n-1)_i}{(m-n)_i(-\ell)_i} \qquad (0 \leqslant i \leqslant e).$$

From (7.5) it follows that the polynomial $\psi_e^{\ell,m}(\xi)/s_0$ is monic and integral:

$$\frac{\psi_e^{\ell,m}(\xi)}{s_0} = \sum_{i=0}^{e} \frac{s_{e-i}}{s_0}(-1)^i(-\xi)_i = (\xi - \xi_{-\lfloor e/2 \rfloor}) \cdots (\xi - \xi_{\lfloor e/2 \rfloor}) \in \mathbb{Z}[\xi], \tag{7.6}$$

where the factor $(\xi - \xi_0)$ appears only when e is odd. Since $(-1)^i(-\xi)_i$ is also monic and integral, and has degree i for $0 \leqslant i \leqslant e$, it follows that

$$\frac{s_i}{s_0} = (-1)^i \binom{e}{i} \frac{(n-m-e+1)_i(\ell-e+1)_i}{(n-2e+2)_i} \in \mathbb{Z}\backslash\{0\} \qquad (0 \leqslant i \leqslant e), \tag{7.7}$$

where that these coefficients are non-zero follows from Assumption 5.2.

We now consider the map $f : \Theta \to [0,1]^2$ defined by

$$f(\ell, m, n) = \left(\frac{\ell}{n}, \frac{m}{n}\right) \in [0,1]^2 \qquad ((\ell, m, n) \in \Theta).$$

Recall (7.2). Moreover, from (7.1) it follows that

$$|f^{-1}(a,b)| < \infty \qquad ((a,b) \in [0,1]^2). \tag{7.8}$$

Hence it follows that

$$|f(\Theta)| = \infty,$$

so that $f(\Theta)$ has at least one accumulation point in $[0,1]^2$. Again by (7.1), such an accumulation point must be of the form

$$(0, \rho) \in [0,1]^2.$$

We next show that the parameters α, β, and N from (7.3) satisfy Assumption 6.1 when $f(\ell, m, n) \to (0, \rho)$.

Claim 7.4. *$\ell, m, n-m \to \infty$ as $f(\ell, m, n) \to (0, \rho)$.*

Proof. Since $m, n - m \geqslant \ell$ by Assumption 5.2, it suffices to show that $\ell \to \infty$. Suppose the contrary, i.e., that there is a sequence (ℓ_k, m_k, n_k) $(k \in \mathbb{N})$ of distinct elements of Θ such that

$$\lim_{k\to\infty} f(\ell_k, m_k, n_k) = (0, \rho), \qquad \sup_k \ell_k < \infty.$$

Since the ℓ_k are bounded, it follows from (7.5) and (7.6) that there are only finitely many choices for $\psi_e^{\ell,m}(\xi)/s_0$ when (ℓ, m, n) ranges over this sequence. In particular, there are only finitely many choices for each of the coefficients s_1/s_0 and s_2/s_0, and hence the same is true (cf. (7.7)) for each of

$$\frac{n-m-e+1}{n-2e+2}, \qquad \frac{n-m-e+2}{n-2e+3}.$$

However, it is immediate to see that these distinct scalars in turn determine n and m uniquely, from which it follows that the n_k are bounded, a contradiction. □

Claim 7.5. *$\ell m(n-m)/n^2 \to \infty$ as $f(\ell, m, n) \to (0, \rho)$.*

Proof. If $0 < \rho < 1$ then the result follows from Claim 7.4 and since

$$\frac{m(n-m)}{n^2} \to \rho(1-\rho) > 0.$$

Suppose next that $\rho = 1$. Suppose moreover that there is a sequence (ℓ_k, m_k, n_k) $(k \in \mathbb{N})$ of distinct elements of Θ such that

$$\lim_{k\to\infty} f(\ell_k, m_k, n_k) = (0, 1), \qquad \sup_k \frac{\ell_k m_k(n_k - m_k)}{(n_k)^2} < \infty.$$

Since $m_k/n_k \to 1$, we then have

$$\sup_k \frac{\ell_k(n_k - m_k)}{n_k} < \infty.$$

Let

$$r_k = \frac{(n_k - m_k - e + 1)(\ell_k - e + 1)}{n_k - 2e + 2}, \qquad t_k = \frac{(n_k - m_k - e + 2)(\ell_k - e + 2)}{n_k - 2e + 3}.$$

Then the r_k and the t_k are bounded since

$$r_k \approx t_k \approx \frac{\ell_k(n_k - m_k)}{n_k}$$

by Claim 7.4. From (7.7) it follows that s_1/s_0 and s_2/s_0 are bounded as well, and hence take only finitely many non-zero integral values when (ℓ, m, n) ranges over this sequence. It follows that the r_k and the t_k can assume only finitely many values, and then since $r_k \approx t_k$ we must have $r_k = t_k$ for sufficiently large k. However, it is again immediate to see that $r_k \neq t_k$ for every $k \in \mathbb{N}$, and hence this is absurd. It follows that the result holds when $\rho = 1$.

Finally, suppose that $\rho = 0$. For every $(\ell, m, n) \in \Theta$ we have

$$e\frac{(m-e+1)(\ell-e+1)}{n-2e+2} = \frac{s_1}{s_0} + e(\ell - e + 1),$$

$$\binom{e}{2}\frac{(m-e+1)_2(\ell-e+1)_2}{(n-2e+2)_2}=\frac{s_2}{s_0}+(e-1)(\ell-e+2)\frac{s_1}{s_0}+\binom{e}{2}(\ell-e+1)_2.$$

From (7.7) and Assumption 5.2 it follows that these scalars are non-zero integers. By the same argument as above, but working with these two scalars instead of s_1/s_0 and s_2/s_0, we conclude that the result holds in this case as well. □

By Claims 7.4 and 7.5, it follows that the parameters α, β, and N from (7.3) satisfy Assumption 6.1 when $f(\ell, m, n) \to (0, \rho)$, since

$$-\frac{\alpha+\beta}{\sqrt{\alpha\beta N}} \approx \frac{n}{\sqrt{\ell m(n-m)}}, \qquad \frac{N}{\alpha+\beta} \approx -\frac{\ell}{n}, \qquad \frac{\beta}{\alpha+\beta} \approx \frac{m}{n}.$$

Note that the scalar ρ in Assumption 6.1 agrees with the one used here in this case. Hence we are now in the position to apply the results of the previous section to $\psi_e^{\ell,m}(\xi)$, which is the Hahn polynomial having these parameters.

Claim 7.6. *We have $\rho = 1/2$. In particular, $(0, 1/2)$ is a unique accumulation point of $f(\Theta)$. Moreover, we have $n = 2m$ for all but finitely many $(\ell, m, n) \in \Theta$.*

Proof. Let the ξ_i be as in (7.4). Then from Propositions 6.4 and 6.5 it follows that

$$\xi_i + \xi_{-i} - \xi_j - \xi_{-j} \to \frac{4\rho-2}{3}\big((\eta_i)^2 - (\eta_j)^2\big) \qquad \text{for all } i, j, \tag{7.9}$$

as $f(\ell, m, n) \to (0, \rho)$, where the η_i are the zeros of the monic Hermite polynomial $h_e(\eta)$ from (6.9) as in Proposition 6.4. Recall that $e \geqslant 8$ by Assumption 7.3. Set $(i, j) = (1, 0)$ in (7.9) if e is odd, and $(i, j) = (2, 1)$ if e is even. Then, since

$$\left|\frac{4\rho-2}{3}\right| \leqslant \frac{2}{3},$$

it follows from Proposition 6.5 that the RHS in (7.9) lies in the open interval $(-1, 1)$. However, the LHS in (7.9) is always an integer by (7.5), so that this is possible only when the RHS equals zero, i.e., $\rho = 1/2$. In particular, we have shown that $(0, 1/2)$ is a unique accumulation point of $f(\Theta)$.

Again by (7.5) and (7.9), we then have

$$\xi_i + \xi_{-i} = \xi_j + \xi_{-j} \qquad \text{for all } i, j,$$

provided that $f(\ell, m, n)$ is sufficiently close to $(0, 1/2)$. By the uniqueness of the accumulation point and (7.8), this last condition on $f(\ell, m, n)$ can be rephrased as "for all but finitely many $(\ell, m, n) \in \Theta$." Now, let $\tilde{\xi}$ be the average of the zeros ξ_i of $\psi_e^{\ell,m}(\xi)$. Then the above identity means that the ξ_i are symmetric with respect to $\tilde{\xi}$. Hence, if we write

$$\frac{\psi_e^{\ell,m}(\xi)}{s_0} = \sum_{i=0}^{e} w_{e-i}(\xi - \tilde{\xi})^i,$$

then we have

$$w_{2i-1} = 0 \qquad (1 \leqslant i \leqslant \lceil e/2 \rceil)$$

for all but finitely many $(\ell, m, n) \in \Theta$. On the other hand, using (7.6) and (7.7), we routinely obtain

$$w_3 = \binom{e}{3}(n-2\ell)(n-2m) \\ \times \frac{(\ell-e+1)(m-e+1)(n-\ell-e+1)(n-m-e+1)}{(n-2e+2)^3(n-2e+3)(n-2e+4)}.$$

Hence, by Assumption 5.2, that $w_3 = 0$ forces $n = 2m$. The claim is proved. □

By virtue of Claim 7.6, we may now assume without loss of generality that

$$n = 2m \qquad ((\ell, m, n) \in \Theta),$$

by discarding a finite number of exceptions. Set

$$\mathfrak{k} = \left\lfloor \frac{e}{2} \right\rfloor,$$

and let c' be a constant such that $c' > 2^\delta c$. Note that

$$\mathfrak{k} \geqslant \mathfrak{k}_0 = \mathfrak{k}_0(2, \delta)$$

by Assumption 7.3. Let $(\ell, m, 2m) \in \Theta$. We have

$$c \cdot (2m)^\delta < c' \cdot (m-e+1)^\delta$$

provided that m is large. Hence it follows from Proposition 7.2 (applied to $\vartheta = 2$) and (7.1) that if m is sufficiently large then there is a prime $p > 2\mathfrak{k}+1$ such that

$$\nu_p((2m-2e+3)(2m-2e+5)\cdots(2m-2e+2\mathfrak{k}+1)) > \nu_p((\ell-e+1)_{2\mathfrak{k}}),$$

where $\nu_p(\mathfrak{n})$ denotes the exponent of p in $\mathfrak{n}$. Assuming that this is the case, let i $(1 \leqslant i \leqslant \mathfrak{k})$ be such that

$$\nu_p(2m-2e+2i+1) > 0.$$

Observe that i is unique since $p > 2\mathfrak{k}+1$, so that we have

$$\nu_p(2m-2e+2i+1) > \nu_p((\ell-e+1)_{2\mathfrak{k}}).$$

Moreover, we have

$$\gcd(2m-2e+2i+1, m-e+i+j) = \gcd(2j-1, m-e+i+j) < p$$

for $1 \leqslant j \leqslant i$, from which it follows that

$$\nu_p((m-e+i+1)_i) = 0.$$

By these comments and since

$$2i \leqslant e < p,$$

it follows from (7.7) (with $n = 2m$) that

$$\nu_p\left(\frac{s_{2i}}{s_0}\right) = \nu_p\left(\frac{(m-e+1)_{2i}(\ell-e+1)_{2i}}{(2m-2e+2)_{2i}}\right)$$

$$= \nu_p \left(\frac{(m-e+i+1)_i(\ell-e+1)_{2i}}{2^i(2m-2e+3)(2m-2e+5)\cdots(2m-2e+2i+1)} \right)$$
$$< 0.$$

However, this contradicts the fact that s_{2i}/s_0 is a non-zero integer. Hence we now conclude that Θ must be finite.

The proof of Theorem 7.1 is complete.

ORCID iDs

Hajime Tanaka https://orcid.org/0000-0002-5958-0375
Yan Zhu https://orcid.org/0000-0001-7164-9198

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Appendix A Proof of Proposition 7.2

Our proof of Proposition 7.2 is a slight modification of (the first part of) that of [40, Satz I].

For a positive integer $\mathfrak{n}$, let

$$\chi_{\mathfrak{n}} = 1 \cdot 3 \cdot 5 \cdots (2\mathfrak{n} - 1) = \frac{(2\mathfrak{n})!}{2^{\mathfrak{n}}\mathfrak{n}!}.$$

Observe that the exponent $\nu_p(\chi_{\mathfrak{n}})$ of an odd prime p in $\chi_{\mathfrak{n}}$ is given by

$$\nu_p(\chi_{\mathfrak{n}}) = \sum_{i=1}^{\lfloor \log_p(2\mathfrak{n}) \rfloor} \left(\left\lfloor \frac{2\mathfrak{n}}{p^i} \right\rfloor - \left\lfloor \frac{\mathfrak{n}}{p^i} \right\rfloor \right) = \sum_{i=1}^{\lfloor \log_p(2\mathfrak{n}) \rfloor} \left\lfloor \frac{\mathfrak{n}}{p^i} + \frac{1}{2} \right\rfloor, \tag{A.1}$$

where we have used

$$\lfloor \xi \rfloor + \left\lfloor \xi + \frac{1}{2} \right\rfloor = \lfloor 2\xi \rfloor \qquad (\xi \in \mathbb{R}).$$

Now, let $(\mathfrak{a}, \mathfrak{b})$ be a pair of positive integers with

$$\mathfrak{b} < c \cdot \mathfrak{a}^{\delta}, \tag{A.2}$$

which does not satisfy the desired property about a prime factor; in other words,

$$\nu_p(\chi_{\mathfrak{a}+\mathfrak{k}}) - \nu_p(\chi_{\mathfrak{a}}) \leqslant \nu_p((\mathfrak{b} + \lfloor \vartheta\mathfrak{k} \rfloor)!) - \nu_p(\mathfrak{b}!) \qquad \text{if } p > 2\mathfrak{k} + 1. \tag{A.3}$$

Our aim is to show that $\mathfrak{a}$ is bounded in terms of ϑ, δ, c, and $\mathfrak{k}$, and hence so is $\mathfrak{b}$ by (A.2), from which it follows that there are only finitely many such pairs. (We will specify $\mathfrak{k}_0 = \mathfrak{k}_0(\vartheta, \delta)$ at the end of the proof.) To this end, we may assume for example that

$$\mathfrak{a} > \mathfrak{k}, \qquad c \cdot \mathfrak{a}^{\delta} > \mathfrak{k} + 1. \tag{A.4}$$

Without loss of generality, we may also assume that

$$\mathfrak{b} > \mathfrak{k}, \tag{A.5}$$

for otherwise the pair $(\mathfrak{a}, \mathfrak{k} + 1)$ would also satisfy (A.2) and (A.3).

Let

$$\mathfrak{s} = \frac{\chi_{\mathfrak{a}+\mathfrak{k}}}{\chi_{\mathfrak{k}}\chi_{\mathfrak{a}}} \cdot \frac{\mathfrak{b}!}{(\mathfrak{b} + \lfloor \vartheta\mathfrak{k} \rfloor)!}.$$

Then from (A.3) it follows that

$$\mathfrak{s} \leqslant \prod_{3 \leqslant p \leqslant 2\mathfrak{k}+1} p^{\nu_p(\mathfrak{s})}, \tag{A.6}$$

where the product in the RHS is over the odd primes $p \leqslant 2\mathfrak{k} + 1$, and where

$$\nu_p(\mathfrak{s}) = \nu_p(\chi_{\mathfrak{a}+\mathfrak{k}}) - \nu_p(\chi_{\mathfrak{k}}) - \nu_p(\chi_{\mathfrak{a}}) + \nu_p(\mathfrak{b}!) - \nu_p((\mathfrak{b} + \lfloor \vartheta\mathfrak{k} \rfloor)!).$$

By (A.1), for every odd prime p we have

$$\nu_p(\mathfrak{s}) \leqslant \nu_p(\chi_{\mathfrak{a}+\mathfrak{k}}) - \nu_p(\chi_{\mathfrak{k}}) - \nu_p(\chi_{\mathfrak{a}})$$

$$= \sum_{i=1}^{\lfloor \log_p(2\mathfrak{a}+2\mathfrak{k}) \rfloor} \left(\left\lfloor \frac{\mathfrak{a}+\mathfrak{k}}{p^i} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{\mathfrak{k}}{p^i} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{\mathfrak{a}}{p^i} + \frac{1}{2} \right\rfloor \right).$$

Note that

$$\left\lfloor \xi + \eta + \frac{1}{2} \right\rfloor - \left\lfloor \xi + \frac{1}{2} \right\rfloor - \left\lfloor \eta + \frac{1}{2} \right\rfloor \in \{-1, 0, 1\} \qquad (\xi, \eta \in \mathbb{R}).$$

Hence it follows that

$$\nu_p(\mathfrak{s}) \leqslant \log_p(2\mathfrak{a} + 2\mathfrak{k}) = \frac{\ln(2\mathfrak{a} + 2\mathfrak{k})}{\ln p} \tag{A.7}$$

for every odd prime p. From (A.6) and (A.7) it follows that

$$\ln \mathfrak{s} \leqslant (\pi(2\mathfrak{k} + 1) - 1) \ln(2\mathfrak{a} + 2\mathfrak{k}), \tag{A.8}$$

where $\pi(\mathfrak{n})$ denotes the number of primes at most $\mathfrak{n}$.

On the other hand, we have

$$\mathfrak{s} = \frac{(2\mathfrak{a} + 2\mathfrak{k})!\mathfrak{k}!\mathfrak{a}!}{(\mathfrak{a} + \mathfrak{k})!(2\mathfrak{k})!(2\mathfrak{a})!} \cdot \frac{\mathfrak{b}!}{(\mathfrak{b} + \lfloor \vartheta \mathfrak{k} \rfloor)!}.$$

Using Stirling's formula

$$\ln(\mathfrak{n}!) = \left(\mathfrak{n} + \frac{1}{2} \right) \ln \mathfrak{n} - \mathfrak{n} + \frac{\ln 2\pi}{2} + r_{\mathfrak{n}},$$

where

$$0 < r_{\mathfrak{n}} < \frac{1}{12\mathfrak{n}},$$

we obtain

$$\begin{aligned} \ln \mathfrak{s} > (\mathfrak{a} + \mathfrak{k}) \ln(\mathfrak{a} + \mathfrak{k}) - \mathfrak{k} \ln \mathfrak{k} - \mathfrak{a} \ln \mathfrak{a} + \vartheta \mathfrak{k} - 2 \\ + \left(\mathfrak{b} + \frac{1}{2} \right) \ln \mathfrak{b} - \left(\mathfrak{b} + \vartheta \mathfrak{k} + \frac{1}{2} \right) \ln(\mathfrak{b} + \vartheta \mathfrak{k}). \end{aligned} \tag{A.9}$$

Let

$$\tilde{\mathfrak{a}} = \frac{\mathfrak{a}}{\mathfrak{k}}, \qquad \tilde{\mathfrak{b}} = \frac{\mathfrak{b}}{\mathfrak{k}}.$$

Note that

$$\tilde{\mathfrak{a}}, \tilde{\mathfrak{b}} > 1, \tag{A.10}$$

in view of (A.4) and (A.5). With this notation, we have

$$\begin{aligned} \ln \mathfrak{s} &> \mathfrak{k} \ln \tilde{\mathfrak{a}} + (\tilde{\mathfrak{a}} + 1)\mathfrak{k} \ln\left(1 + \frac{1}{\tilde{\mathfrak{a}}} \right) - \vartheta \mathfrak{k} \ln \mathfrak{k} + \vartheta \mathfrak{k} - 2 \\ &\quad - \vartheta \mathfrak{k} \ln \tilde{\mathfrak{b}} - \left((\tilde{\mathfrak{b}} + \vartheta)\mathfrak{k} + \frac{1}{2} \right) \ln\left(1 + \frac{\vartheta}{\tilde{\mathfrak{b}}} \right) \\ &> \mathfrak{k} \ln \tilde{\mathfrak{a}} - \vartheta \mathfrak{k} \ln \mathfrak{k} - 2 - \vartheta \mathfrak{k} \ln \tilde{\mathfrak{b}} - \left(\vartheta \mathfrak{k} + \frac{1}{2} \right) \ln\left(1 + \frac{\vartheta}{\tilde{\mathfrak{b}}} \right) \end{aligned}$$

$$> (1-\vartheta\delta)\mathfrak{k}\ln\tilde{\mathfrak{a}} - \vartheta\mathfrak{k}\ln c - \vartheta\delta\mathfrak{k}\ln\mathfrak{k} - 2 - \left(\vartheta\mathfrak{k} + \frac{1}{2}\right)\ln(1+\vartheta), \tag{A.11}$$

where the first inequality is a restatement of (A.9), the second follows from

$$0 < \ln(1+\xi) < \xi \qquad (\xi > 0),$$

and the last one follows from (A.2) and (A.10).

Concerning the prime-counting function $\pi(\mathfrak{n})$, it is known that [37, (3.6)]

$$\pi(\mathfrak{n}) < 1.25506\,\frac{\mathfrak{n}}{\ln\mathfrak{n}} \qquad (\mathfrak{n} > 1).$$

By this, (A.8), and (A.10), we have

$$\begin{aligned}\ln\mathfrak{s} &\leqslant (\pi(2\mathfrak{k}+1)-1)\left(\ln\tilde{\mathfrak{a}} + \ln 2\mathfrak{k}\left(1+\frac{1}{\tilde{\mathfrak{a}}}\right)\right)\\ &< \left(1.25506\,\frac{2\mathfrak{k}+1}{\ln(2\mathfrak{k}+1)} - 1\right)(\ln\tilde{\mathfrak{a}} + \ln 4\mathfrak{k}). \end{aligned} \tag{A.12}$$

Combining (A.11) and (A.12), it follows that

$$\begin{aligned}&\left((1-\vartheta\delta)\mathfrak{k} - 1.25506\,\frac{2\mathfrak{k}+1}{\ln(2\mathfrak{k}+1)} + 1\right)\ln\tilde{\mathfrak{a}}\\ &\qquad < \vartheta\mathfrak{k}\ln c + \vartheta\delta\mathfrak{k}\ln\mathfrak{k} + 2 + \left(\vartheta\mathfrak{k}+\frac{1}{2}\right)\ln(1+\vartheta)\\ &\qquad\qquad + \left(1.25506\,\frac{2\mathfrak{k}+1}{\ln(2\mathfrak{k}+1)} - 1\right)\ln 4\mathfrak{k}. \end{aligned} \tag{A.13}$$

If we set

$$\mathfrak{k}_0 = \mathfrak{k}_0(\vartheta,\delta) = \frac{1}{2}\left(\exp\left(\frac{2.51012}{1-\vartheta\delta}\right) - 1\right) > 0$$

for example, then we have

$$(1-\vartheta\delta)\mathfrak{k} - 1.25506\,\frac{2\mathfrak{k}+1}{\ln(2\mathfrak{k}+1)} + 1 \geqslant \frac{1+\vartheta\delta}{2} > 0 \qquad (\mathfrak{k} \geqslant \mathfrak{k}_0).$$

Hence, whenever $\mathfrak{k} \geqslant \mathfrak{k}_0$, it follows from (A.13) that $\ln\mathfrak{a} = \ln\tilde{\mathfrak{a}} + \ln\mathfrak{k}$ is bounded in terms of ϑ, δ, c, and $\mathfrak{k}$, from which and (A.2) it follows that there are only finitely many choices for the pairs $(\mathfrak{a}, \mathfrak{b})$.

This completes the proof of Proposition 7.2.

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Two-distance transitive normal Cayley graphs*

Jun-Jie Huang, Yan-Quan Feng †, Jin-Xin Zhou

School of mathematics and statistics, Beijing Jiaotong University, Beijing 100044, P. R. China

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Abstract

In this paper, we construct an infinite family of normal Cayley graphs, which are 2-distance-transitive but neither distance-transitive nor 2-arc-transitive. This answers a question proposed by Chen, Jin and Li in 2019.

Keywords: Cayley graph, 2*-distance-transitive graph, simple group.*

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1 Introduction

In this paper, all graphs are finite, simple, and undirected. For a graph Γ, let $V(\Gamma), E(\Gamma)$, $A(\Gamma)$ or $\mathrm{Aut}(\Gamma)$ denote its vertex set, edge set, arc set and its full automorphism group, respectively. The graph Γ is called G*-vertex-transitive*, G*-edge-transitive* or G*-arc-transitive*, with $G \leq \mathrm{Aut}(\Gamma)$, if G is transitive on $V(\Gamma), E(\Gamma)$ or $A(\Gamma)$ respectively, and G*-semisymmetric*, if Γ is G-edge-transitive but not G-vertex-transitive. It is easy to see that a G-semisymmetric graph Γ must be bipartite such that G has two orbits, namely the two parts of Γ, and the stabilizer G_u for any $u \in V(\Gamma)$ is transitive on the neighbourhood of u in Γ. An *s-arc* of Γ is a sequence $v_0, v_1, \ldots, v_s$ of $s+1$ vertices of Γ such that v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. If Γ has at least one s-arc and $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs of Γ, then Γ is called (G, s)*-arc-transitive*, and Γ is said to be *s-arc-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive.

For two vertices u and v in $V(\Gamma)$, the *distance* $d(u, v)$ between u and v in Γ is the smallest length of paths between u and v, and the *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximum distance occurring over all pairs of vertices. For $i = 1, 2, \ldots, \mathrm{diam}(\Gamma)$, denote by $\Gamma_i(u)$

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†Corresponding author.

E-mail addresses: 20118006@bjtu.edu.cn (Jun-Jie Huang), yqfeng@bjtu.edu.cn (Yan-Quan Feng), jxzhou@bjtu.edu.cn (Jin-Xin Zhou)

the set of vertices at distance i with vertex u in Γ. A graph Γ is called *distance transitive* if, for any vertices u, v, x, y with $d(u, v) = d(x, y)$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that $(u, v)^g = (x, y)$. The graph Γ is called (G, t)*-distance-transitive* with $G \leq \mathrm{Aut}(\Gamma)$ if, for each $1 \leq i \leq t$, the group G is transitive on the ordered pairs of form (u, v) with $d(u, v) = i$, and Γ is said to be t*-distance-transitive* if it is $(\mathrm{Aut}(\Gamma), t)$-distance-transitive.

Distance-transitive graphs were first defined by Biggs and Smith in [2], and they showed that there are only 12 trivalant distance-transitive graphs. Later, distance-transitive graphs of valencies 3, 4, 5, 6 and 7 were classified in [2, 10, 14, 25], and a complete classification of distance-transitive graphs with symmetric or alternating groups of automorphisms was given by Liebeck, Praeger and Saxl [18]. The 2-distance-transitive but not 2-arc-transitive graphs of valency at most 6 were classified in [4, 16], and the 2-distance-primitive graphs (a vertex stabilizer of automorphism group is primitive on both the first step and the second step neighbourhoods of the vertex) with prime valency were classified in [15]. By definition, a 2-arc-transitive graph is 2-distance-transitive, but a 2-distance-transitive graph may not be 2-arc-transitive; an example is the Kneser graph $KG_{6,2}$, see [16]. Furthermore, Corr, Jin and Schneider [5] investigated properties of a connected $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of girth 4, and they applied the properties to classify such graphs with prime valency. For more information about 2-distance-transitive graphs, we refer to [6, 7].

For a finite group G and a subset $S \subseteq G \setminus \{1\}$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ of the group G with respect to S is the graph with vertex set G and with two vertices g and h adjacent if $hg^{-1} \in S$. For $g \in G$, let $R(g)$ be the permutation of G defined by $x \mapsto xg$ for all $x \in G$. Then $R(G) := \{R(g) \mid g \in G\}$ is a regular group of automorphisms of $\mathrm{Cay}(G, S)$. It is known that a graph Γ is a Cayley graph of G if and only if Γ has a regular group of automorphisms on the vertex set which is isomorphic to G; see [1, Lemma 16.3] and [24]. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called *normal* if $R(G)$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$. The study of normal Cayley graphs was initiated by Xu [27] and has been investigated under various additional conditions; see [8, 22].

There are many interesting examples of arc-transitive graphs and 2-arc-transitive graphs constructed as normal Cayley graphs. However, the status for 2-distance-transitive graphs is different. Recently, 2-distance-transitive circulants were classified in [3], where the following question was proposed:

Question 1.1 ([3, Question 1.2])**.** Is there a normal Cayley graph which is 2-distance-transitive, but neither distance-transitive nor 2-arc-transitive?

In this paper, we answer the above question by constructing an infinite family of such graphs, which are Cayley graphs of the extraspecial p-groups of exponent p of order p^3.

Theorem 1.2. *For an odd prime p, let* $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$ *and* $S = \{a^i, b^i \mid 1 \leq i \leq p - 1\}$. *Then* $\mathrm{Cay}(G, S)$ *is a* 2*-distance-transitive normal Cayley graph that is neither distance-transitive nor* 2*-arc-transitive.*

A *clique* of a graph Γ is a maximal complete subgraph, and the *clique graph* Σ of Γ is defined to have the set of all cliques of Γ as its vertex set with two cliques adjacent in Σ if the two cliques have at least one common vertex. Applying Theorem 1.2, we can obtain the following corollary.

Corollary 1.3. *Under the notation given in* Theorem 1.2*, let* $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ *be the graph with vertex set* $\{\langle a\rangle g \mid g \in G\} \cup \{\langle b\rangle h \mid h \in G\}$ *and with edges all these coset pairs*

$\{\langle a\rangle g, \langle b\rangle h\}$ having non-empty intersection in G. Then $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is the clique graph of $\mathrm{Cay}(G, S)$, and $\mathrm{Cay}(G, S)$ is the line graph of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$. Furthermore, $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is 3-arc-transitive.

The graph $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ was first constructed in [19] as a regular cover of $\mathsf{K}_{p,p}$, where it is said that $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is 2-arc-transitive in [19, Theorem 1.1], but not 3-arc-transitive generally for all odd primes p in a remark after [19, Example 4.1]. However, this is not true and Corollary 1.3 implies that $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is always 3-arc-transitive for each odd prime p. In fact, $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is 3-arc-regular, that is, $\mathrm{Aut}(\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle))$ is regular on the set of 3-arcs of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$. Some more information about the structure and symmetry properties of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ are given in Lemma 3.2.

2 Preliminaries

In this section we list some preliminary results used in this paper. The first one is the well-known orbit-stabilizer theorem (see [9, Theorem 1.4A]).

Proposition 2.1. *Let G be a group with a transitive action on a set Ω and let $\alpha \in \Omega$. Then $|G| = |\Omega||G_\alpha|$.*

The well-known Burnside $p^a q^b$ theorem was given in [12, Theorem 3.3].

Proposition 2.2. *Let p and q be primes and let a and b be positive integers. Then a group of order $p^a q^b$ is soluble.*

The next proposition is an important property of a non-abelian simple group acting transitively on a set with cardinality a prime-power, whose proof depends on the finite simple group classification, and we refer to [13, Corollary 2] or [26, Proposition 2.4].

Proposition 2.3. *Let T be a nonabelian simple group acting transitively on a set Ω with cardinality a p-power for a prime p. If p does not divide the order of a point-stabilizer of T, then T acts 2-transitively on Ω.*

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G with respect to S. Then $R(G)$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$, and $\mathrm{Aut}(G, S) := \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\mathrm{Aut}(\Gamma)$, which fixes 1. Furthermore, $R(G)$ is normalized by $\mathrm{Aut}(G, S)$, and hence we have a semiproduct $R(G) \rtimes \mathrm{Aut}(G, S)$, where $R(g)^\alpha = R(g^\alpha)$ for any $g \in G$ and $\alpha \in \mathrm{Aut}(G, S)$. Godsil [11] proved that the semiproduct $R(G) \rtimes \mathrm{Aut}(G, S)$ is in fact the normalizer of $R(G)$ in $\mathrm{Aut}(\Gamma)$. By Xu [27], we have the following proposition.

Proposition 2.4. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a finite group G with respect to S, and let $\mathsf{A} = \mathrm{Aut}(\Gamma)$. Then the following hold:*

(1) *$N_{\mathsf{A}}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$;*

(2) *Γ is a normal Cayley graph if and only if $\mathsf{A}_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of 1 in A.*

Let Γ be a G-vertex-transitive graph, and let N be a normal subgroup of G. The *normal quotient graph Γ_N* of Γ induced by N is defined to be the graph with vertex set the orbits of N and with two orbits B, C adjacent if some vertex in B is adjacent to some vertex in C in Γ. Furthermore, Γ is called a *normal N-cover* of Γ_N if Γ and Γ_N have the same valency.

Proposition 2.5. *Let Γ be a connected G-vertex-transitive graph and let N be a normal subgroup of G. Suppose that either Γ is an N-cover of Γ_N, or Γ is G-arc-transitive of prime valency and N has at least three orbits on vertices. Then the following statements hold:*

(1) *N is semiregular on $V\Gamma$ and is the kernel of G acting $V(\Gamma_N)$, so $G/N \leq \mathrm{Aut}(\Gamma_N)$;*

(2) *Γ is (G, s)-arc-transitive if and only if Γ_N is $(G/N, s)$-arc-transitive;*

(3) *$G_\alpha \cong (G/N)_\delta$ for any $\alpha \in V\Gamma$ and $\delta \in V(\Gamma_N)$.*

Proposition 2.5 was given in many papers by replacing the condition that Γ is a normal N-cover of Γ_N by one of the following assumptions: (1) N has at least 3-orbits and G is 2-arc-transitive (see [21, Theorem 4.1]); (2) N has at least 3-orbits, G is arc-transitive and Γ has a prime valency (see [20, Theorem 2.5]); (3) N has at least 3-orbits and G is locally primitive (see [17, Lemma 2.5]). The first step for these proofs is to show that for any two vertices $B, C \in V(\Gamma_N)$, the induced subgraph $[B]$ of B in Γ has no edge and if B and C are adjacent in Γ_N then the induced subgraph $[B \cup C]$ in Γ is a matching, which is equivalent to that Γ is a normal N-cover of Γ_N. Then Proposition 2.5(1) - (3) follows from these proofs.

3 Proof Theorem 1.2

For a positive integer n and a prime p, we use $\mathbb{Z}_n$ and $\mathbb{Z}_p^r$ to denote the cyclic group of order n and the elementary abelian group of order p^r, respectively. In this section, we always assume that p is an odd prime, and denote by $\mathbb{Z}_p^*$ the multiplicative group of $\mathbb{Z}_p$ consisting of all non-zero numbers in $\mathbb{Z}_p$. Note that $\mathbb{Z}_p^* \cong \mathbb{Z}_{p-1}$. Furthermore, we also set the following assumptions in this section:

$$G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle,$$
$$S = \{a^i, b^i \mid 1 \leq i \leq p-1\},$$
$$\Gamma = \mathrm{Cay}(G, S),\ \ \mathsf{A} = \mathrm{Aut}(\Gamma),\ \ N = N_{\mathsf{A}}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S),\ \text{ and } \mathbb{Z}_p^* = \langle t\rangle.$$

By Proposition 2.4, $N_{\mathsf{A}}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$, and $R(g)^\delta = R(g^\delta)$ for any $R(g) \in R(G)$ and $\delta \in \mathrm{Aut}(G, S)$. Since $G = \langle S\rangle$, Γ is a connected Cayley graph of valency $2(p-1)$. Let

$$\begin{array}{llll} \alpha\colon a \longmapsto a^t, & b \longmapsto b, & c \longmapsto c^t; \\ \beta\colon a \longmapsto a, & b \longmapsto b^t, & c \longmapsto c^t; \\ \gamma\colon a \longmapsto b, & b \longmapsto a, & c \longmapsto c^{-1}. \end{array}$$

It is easy to check that a^t, b, c^t satisfy the same relations as a, b, c in G, that is, $[a^t, b] = c^t$, $[c^t, a^t] = [c^t, b] = 1$. By the von Dyck's Theorem (see [23, 2.2.1]), α induces an epimorphism from G to $\langle a^t, b, c^t\rangle$, which must be an automorphism of G because $\langle a^t, b, c^t\rangle = G$. Similarly, β and γ are also automorphisms of G.

Lemma 3.1. $\mathrm{Aut}(G, S) = \langle \alpha, \beta, \gamma\rangle \cong (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2$, *and Γ is N-arc-transitive. Furthermore, N has no normal subgroup of order p^2.*

Proof. Since $\mathbb{Z}_p^* = \langle t\rangle$, it is easy to check that $\alpha^{p-1} = \beta^{p-1} = \gamma^2 = 1$, $\alpha\beta = \beta\alpha$ and $\alpha^\gamma = \beta$. Thus $\langle\alpha,\beta,\gamma\rangle \cong (\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})\rtimes\mathbb{Z}_2$. Clearly, $\alpha,\beta,\gamma\in\mathrm{Aut}(G,S)$. To prove $\mathrm{Aut}(G,S) = \langle\alpha,\beta,\gamma\rangle \cong (\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})\rtimes\mathbb{Z}_2$, it suffices to show that $|\mathrm{Aut}(G,S)| \leq 2(p-1)^2$.

Clearly, $\langle\alpha,\beta,\gamma\rangle$ is transitive on S, and hence Γ is N-arc-transitive. Since $G = \langle S\rangle$, $\mathrm{Aut}(G,S)$ is faithful on S. By Proposition 2.1, $|\mathrm{Aut}(G,S)| = |S||\mathrm{Aut}(G,S)_a|$, where $\mathrm{Aut}(G,S)_a$ is the stabilizer of a in $\mathrm{Aut}(G,S)$. Note that $\mathrm{Aut}(G,S)_a$ fixes a^i for each $1\leq i\leq p-1$. Again by Proposition 2.1, $|\mathrm{Aut}(G,S)_a| \leq (p-1)|\mathrm{Aut}(G,S)_{a,b}|$, where $\mathrm{Aut}(G,S)_{a,b}$ is the subgroup of $\mathrm{Aut}(G,S)$ fixing a and b. Since $G = \langle a,b\rangle$, we obtain $\mathrm{Aut}(G,S)_{a,b} = 1$, and then $|\mathrm{Aut}(G,S)| \leq 2(p-1)^2$, as required.

Let $H \leq N$ be a subgroup of order p^2. Since $R(G)$ is the unique normal Sylow p-subgroup of $N = R(G)\rtimes\mathrm{Aut}(G,S)$, we have $H\leq R(G)$, and since $|R(G):H| = p$, we have $H \trianglelefteq R(G)$. Note that the center $C := Z(R(G)) = \langle R(c)\rangle$ and $C\cap H \neq 1$. Thus, $C\cap H = C$ as $|C| = p$, implying $C\leq H$. Since H/C is a subgroup of order p, and $R(G)/C = \langle R(a)C\rangle\times\langle R(b)C\rangle \cong \mathbb{Z}_p^2$, we have $H/C = \langle R(b)C\rangle$ or $\langle R(a)R(b)^iC\rangle$ for some $0\leq i\leq p-1$. It follows that $H = \langle R(b)\rangle\times C$ or $\langle R(ab^i)\rangle\times C$ for some $0\leq i\leq p-1$.

Suppose $H\trianglelefteq N$. Since C is characteristic in $R(G)$ and $R(G)\trianglelefteq N$, we have $C\trianglelefteq N$. Recall that $R(a)^\gamma = R(a^\gamma) = R(b)$. Then $(\langle R(a)\rangle\times C)^\gamma = \langle R(b)\rangle\times C$. This implies that both $\langle R(a)\rangle\times C$ and $\langle R(b)\rangle\times C$ are not normal in N. Thus, $H = \langle R(ab^i)\rangle\times C$ for some $1\leq i\leq p-1$. Since $H\trianglelefteq N$, we have $H^\beta = H$, that is, $\langle R(ab^{ti})\rangle\times C = H^\beta = H = \langle R(ab^i)\rangle\times C$. It follows that $\langle R(ab^{ti})\rangle = \langle R(ab^i)\rangle$ and then $R(ab^{ti}) = R(ab^i)$, which further implies $b^{ti} = b^i$. This gives rise to $p \mid i(t-1)$, and since $(i,p) = 1$, we have $t = 1$, contradicting that $\mathbb{Z}_p^* = \langle t\rangle \cong \mathbb{Z}_{p-1}$. Thus, N has no normal subgroup of order p^2. □

For a positive integer n, n_p denotes the largest p-power diving n. By Lemma 3.1, $\Gamma = \mathrm{Cay}(G,S)$ is N-arc-transitive.

Lemma 3.2. *The clique graph Σ of Γ is a connected p-valent bipartite graph of order $2p^2$, A has a faithful natural action on Σ, and Σ is $R(G)$-semisymmetric and N-arc-transitive. Furthermore, $|\mathsf{A}|_p = p^3$.*

Proof. Recall that $G = \langle a,b,c \mid a^p = b^p = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1\rangle$ and $S = \{a^i, b^i \mid 1\leq i\leq p-1\}$. Then $\Gamma = \mathrm{Cay}(G,S)$ has exactly two cliques passing through 1, that is, the induced subgraphs of $\langle a\rangle$ and $\langle b\rangle$ in Γ. Since $R(G)\leq\mathrm{Aut}(\Gamma)$ is transitive on vertex set, each clique of Γ is an induced subgraph of the coset $\langle a\rangle x$ or $\langle b\rangle x$ for some $x\in G$. Thus, we may view the vertex set of Σ as $\{\langle a\rangle x, \langle b\rangle x \mid x\in G\}$ with two cosets adjacent in Σ if they have non-empty intersection. It is easy to see that $\langle a\rangle x\cap\langle b\rangle y \neq \emptyset$ if and only if $|\langle a\rangle x\cap\langle b\rangle y| = 1$, and any two distinct cosets, either in $\{\langle a\rangle x \mid x\in G\}$ or in $\{\langle b\rangle x\mid x\in G\}$, have empty intersection. Furthermore, $\langle a\rangle$ has non-empty intersection with exactly p cosets, that is, $\langle b\rangle a^i$ for $0\leq i\leq p-1$. Thus, Σ is a p-valent bipartite graph of order $2p^2$. The connectedness of Σ follows from that of Γ.

Clearly, A has a natural action on Σ. Let K be the kernel of A on Σ. Then K fixes each coset of $\langle a\rangle x$ and $\langle b\rangle x$ for all $x\in G$. Since $\langle a\rangle x\cap\langle b\rangle x = \{x\}$, K fixes x and hence $K = 1$. Thus, A is faithful on Σ and we may let $\mathsf{A}\leq\mathrm{Aut}(\Sigma)$.

Note that $R(G)$ is not transitive on $\{\langle a\rangle x, \langle b\rangle x \mid x\in G\}$, but transitive on $\{\langle a\rangle x\mid x\in G\}$ and $\{\langle b\rangle x\mid x\in G\}$. Furthermore, $R(\langle a\rangle)$ fixes $\langle a\rangle$ and is transitive on $\{\langle b\rangle a^i \mid 0\leq i\leq p-1\}$, the neighbourhood of $\langle a\rangle$ in Σ, and similarly, $R(\langle b\rangle)$ fixes $\langle b\rangle$

and is transitive on the neighbourhood $\{\langle a\rangle b^i \mid 0 \leq i \leq p-1\}$ of $\langle b\rangle$ in Σ. It follows that Σ is $R(G)$-semisymmetric. Recall that $N = R(G) \rtimes \mathrm{Aut}(G, S)$ and $\mathrm{Aut}(G, S) = \langle \alpha, \beta, \gamma\rangle$. Since $a^\gamma = b$ and $b^\gamma = a$, γ interchanges $\{\langle a\rangle x \mid x \in G\}$ and $\{\langle b\rangle x \mid x \in G\}$. This yields that Σ is $R(G) \rtimes \langle\gamma\rangle$-arc-transitive and hence N-arc-transitive.

Since Σ is a connected graph with prime valency p, we have $p^2 \nmid |\mathrm{Aut}(\Sigma)_u|$ for any $u \in V(\Sigma)$, and in particular, $p^2 \nmid |\mathsf{A}_u|$. Note that $p \mid |\mathsf{A}_u|$. By Proposition 2.1, $|\mathsf{A}| = |\Sigma||\mathsf{A}_u| = 2p^2|\mathsf{A}_u|$. This implies that $|\mathsf{A}|_p = p^3$. □

Lemma 3.3. $\mathsf{A} = \mathrm{Aut}(\Gamma) = R(G) \rtimes \mathrm{Aut}(G, S)$.

Proof. By Lemma 3.2, $|\mathsf{A}|_p = p^3$, and since $|V(\Gamma)| = p^3$ and A is vertex-transitive on $V(\Gamma)$, the vertex stabilizer A_1 is a p'-group, that is, $p \nmid |\mathsf{A}_1|$. To prove the lemma, by Proposition 2.4 we only need to show that $R(G)\trianglelefteq\mathsf{A}$, and since $R(G)$ is a Sylow p-subgroup of A, it suffices to show that A has a normal Sylow p-subgroup.

Let M be a minimal normal subgroup of A. Then $M = T_1 \times T_2 \cdots \times T_d$, where $T_i \cong T$ for each $1 \leq i \leq d$ with a simple group T. Since $|V(\Gamma)| = p^3$, each orbit of M has length a p-power and hence each orbit of T_i has length a p-power. It follows that $p \mid |T|$. Assume that $|T|_p = p^\ell$. Then $|M|_p = p^{d\ell}$ and $d\ell = 1, 2$ or 3 as $|\mathsf{A}|_p = p^3$.

We process the proof by considering the two cases: M is insoluble or soluble.

Case 1: M is insoluble.

In this case, T is a non-abelian simple group. We prove that this case cannot happen by deriving contradictions. Recall that $d\ell = 1, 2$ or 3.

Assume that $d\ell = 1$. Then $|M|_p = p$. By Lemma 3.2, $M \trianglelefteq \mathsf{A} \leq \mathrm{Aut}(\Sigma)$, and since $|V(\Sigma)| = 2p^2$, M has at least three orbits. Since Σ has valency p, Proposition 2.5 implies that M is semiregular on $V(\Sigma)$ and hence $|M| \mid 2p^2$. By Proposition 2.2, M is soluble, a contradiction.

Assume that $d\ell = 2$. Since $R(G)$ is a Sylow p-subgroup of A and $M \trianglelefteq \mathsf{A}$, $R(G) \cap M$ is a Sylow p-subgroup of M and hence $|R(G) \cap M| = |M|_p = p^2$. Since $R(G) \trianglelefteq N$ and $M \trianglelefteq \mathsf{A}$, $M \cap R(G)$ is a normal subgroup of order p^2 in N, contradicting to Lemma 3.1.

Assume that $d\ell = 3$. Then $(d, \ell) = (1, 3)$ or $(3, 1)$. Since $|M|_p = p^3 = |\mathsf{A}|_p$, we deduce $R(G) \leq M$ and hence M is transitive on Γ.

For $(d, \ell) = (1, 3)$, M is a non-abelian simple group. Since $M_1 \leq \mathsf{A}_1$ is a p'-group, Proposition 2.3 implies that M is 2-transitive on Γ, forcing that Γ is the complete graph of order p^3, a contradiction.

For $(d, \ell) = (3, 1)$, we have $M = T_1 \times T_2 \times T_3$. Then $|M|_p = p^3$, and since $M \trianglelefteq \mathsf{A}$, we derive $R(G) \leq M$. By Lemma 3.2 $M \leq \mathrm{Aut}(\Sigma)$, and Σ is $R(G)$-semisymmetric. Since M has no subgroup of index 2, M fixes the two parts of Σ setwise, and hence Σ is M-semisymmetric. Noting that γ interchanges the two parts of Σ, we have that Σ is $M\langle\gamma\rangle$-arc-transitive. Since γ is an involution, under conjugacy it fixes T_i for some $1 \leq i \leq 3$, say T_1. Then $T_1 \trianglelefteq \langle M, \gamma\rangle$ and by Proposition 2.5, T_1 is semiregular on Σ. This gives rise to $|T_1| \mid 2p^2$, contrary to the simplicity of T_1.

Case 2: M is soluble.

Since $p \mid |M|$, we have $M = \mathbb{Z}_p^d$ with $1 \leq d \leq 3$. If $d = 3$ then A has a normal Sylow p-subgroup, as required. If $d = 2$ then $M \leq R(G) \leq N$ and N has a normal subgroup of order p^2, contrary to Lemma 3.1. Thus, we may let $d = 1$, and since $M \leq R(G)$ and $R(G)$ has a unique normal subgroup of order p that is the center of $R(G)$, we derive that $M = \langle R(c)\rangle$.

Now it is easy to see that the quotient graph $\Gamma_M = \mathrm{Cay}(G/M, S/M)$ with $S/M = \{a^iM, b^iM \mid 1 \le i \le p-1\}$. Note that $G/M = \langle aM\rangle \times \langle bM\rangle \cong \mathbb{Z}_p^2$. Then Γ_M is a connected Cayley graph of order p^2 with valency $2(p-1)$, so Γ is a normal M-cover of Γ_M. By Proposition 2.5, we may let $\mathsf{A}/M \le \mathrm{Aut}(\Gamma_M)$ and Γ_M is A/M-arc-transitive.

Let H/M be a minimal normal subgroup of A/M. Then $H \trianglelefteq \mathsf{A}$ and $H/M = L_1/M \times \cdots \times L_r/M$, where $L_i \trianglelefteq H$ and L_i/M $(1 \le i \le r)$ are isomorphic simple groups. Since $|\Gamma_M| = p^2$, we infer $p \mid |H/M|$ and similarly, $p \mid |L_i/M|$. Let $|L_i/M|_p = p^s$. Then $|H/M|_p = p^{rs}$, and since $|\mathsf{A}/M|_p = p^2$, we obtain that $sr = 1$ or 2.

We finish the proof by considering the two subcases: H/M is insoluble or soluble.

Subcase 2.1: H/M is insoluble.

In this subcase, L_i/M are isomorphic non-abelain simple groups. We prove this subcase cannot happen by deriving contradictions. Recall that $sr = 1$ or 2.

Let $sr = 1$. Then $|H/M|_p = p$, and therefore $|H|_p = p^2$. Since $H \trianglelefteq \mathsf{A}$, $H \cap R(G)$ is a Sylow p-subgroup of H, implying $|H \cap R(G)| = p^2$, and then $R(G) \trianglelefteq N$ yields that $H \cap R(G)$ is a normal subgroup of order p^2 in N, contrary to Lemma 3.1.

Let $rs = 2$. Then $|H/M|_p = p^2$ and $|H|_p = p^3$. This yields $R(G) \le H$ and H is transitive on Γ, so H/M is transitive on $V(\Gamma_M)$. Note that $(r, s) = (1, 2)$ or $(2, 1)$.

For $(r, s) = (1, 2)$, H/M is a nonabelian simple group. By Propostion 2.5, $(H/M)_u$ for $u \in V(\Gamma_M)$ is a p'-group because $H_1 \le \mathsf{A}_1$ is a p'-group, and by Proposition 2.3, H/M is 2-transitive on $V(\Gamma_M)$, forcing that Γ_M is a complete group of order p^2, a contradiction.

For $(r, s) = (2, 1)$, $H/M \cong L_1/M \times L_2/M$, where L_1/M and L_2/M are isomorphic nonabelain simple groups and $|L_i/M|_p = p$. It follows that $|H|_p = p^3$ and $|L_i|_p = p^2$ for $1 \le i \le 2$. Since $H \trianglelefteq \mathsf{A}$, we derive $R(G) \le H$. Note that H has no subgroup of index 2. Since Σ is bipartite, it is H-semisymmetric. Let Δ_1 and Δ_2 be the two parts of Σ. Then $|\Delta_1| = |\Delta_2| = p^2$, and H is transitive on both Δ_1 and Δ_2.

Suppose $(L_1)_u = 1$ for some $u \in V(\Sigma) = \Delta_1 \cup \Delta_2$. By Proposition 2.1, $|L_1| = |u^{L_1}|$, and since $L_1 \trianglelefteq H$ and $|\Delta_1| = |\Delta_2| = p^2$, we derive $|L_1| = p$ or p^2, contrary to the insolubleness of L_1. Thus $(L_1)_u \ne 1$. Since Σ has prime valency p, H_u is primitive on the neighbourhood $\Sigma(u)$ of u in Σ, and since $(L_1)_u \trianglelefteq H_u$, $(L_1)_u$ is transitive on $\Sigma(u)$, which implies that $|(L_1)_u|_p = p$. Since $|L_1|_p = p^2$, each orbit of L_1 on Δ_1 or Δ_2 has length p.

Let $x \in \Delta_1$ and $y \in \Delta_2$ be adjacent in Σ, and let Δ_{11} and Δ_{21} be the orbits of L_1 containing x and y, respectively. Then $|\Delta_{11}| = |\Delta_{21}| = p$. Since $(L_1)_x$ is transitive on $\Sigma(x)$, x is adjacent to each vertex in Δ_{21}, and therefore, each vertex in Δ_{11} is adjacent to each vertex in Δ_{21}, that is, the induced subgroup $[\Delta_{11} \cup \Delta_{21}]$ is the complete bipartite graph $\mathsf{K}_{p,p}$. It follows that $\Sigma \cong p\mathsf{K}_{p,p}$, contrary to the connectedness of Σ.

Subcase 2.2: H/M is soluble.

In this case, $|H| = p^2$ or p^3. Recall that $H \trianglelefteq \mathsf{A}$. If $|H| = p^2$ then $H \le R(G)$ and N has normal subgroup of order p^2, contradicts Lemma 3.1. Thus, $|H| = p^3$ and A has a normal Sylow p-subgroup, as required. This completes the proof. □

Now we are ready to finish the proof.

Proof of Theorem 1.2. By Lemmas 3.1 and 3.3, Γ is a arc-transitive normal Cayley graph. In particular, Γ is 1-distance transitive. Since $S = \{a^i, b^i \mid 1 \le i \le p-1\}$, Γ has girth 3, so it is not 2-arc-transitive.

Recall that $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$. Clearly,

$$\Gamma_1(1) = S = \{a^i, b^i \mid 1 \le i \le p-1\},$$
$$\Gamma_2(1) = \{b^j a^i, a^j b^i \mid 1 \le i, j \le p-1\}.$$

Note that $\mathrm{Aut}(G, S) = \langle \alpha, \beta, \gamma \mid \alpha^{p-1} = \beta^{p-1} = \gamma^2 = 1, \alpha^\beta = \alpha, \alpha^\gamma = \beta\rangle$, where $a^\alpha = a^t, b^\alpha = b, c^\alpha = c^t$, $a^\beta = a, b^\beta = b^t, c^\beta = c^t$, $a^\gamma = b, b^\gamma = a$ and $c^\gamma = c^{-1}$. Then $(ba)^{\alpha^i\beta^j} = b^{t^i}a^{t^j}$, and since $\mathbb{Z}_p^* = \langle t\rangle$, we obtain that $\langle \alpha, \beta\rangle$ is transitive on the set $\{b^j a^i \mid 1 \le i, j \le p-1\}$. Similarly, $\langle\alpha, \beta\rangle$ is transitive on $\{a^j b^i \mid 1 \le i, j \le p-1\}$. Furthermore, γ interchanges the two sets $\{b^j a^i \mid 1 \le i, j \le p-1\}$ and $\{a^j b^i \mid 1 \le i, j \le p-1\}$. It follows that $\mathrm{Aut}(G, S)$ is transitive on $\Gamma_2(1)$ and hence Γ is 2-distance transitive.

Noting that $ab = bac$, we have that $b^{-1}ab = ac \in \Gamma_3(1)$ and $aba = ba^2c \in \Gamma_3(1)$. Also it is easy to see that $(ac)^{\mathrm{Aut}(G,S)} = (ac)^{\langle\alpha,\beta,\gamma\rangle} = \{a^i c^j, b^i c^j \mid 1 \le i, j \le p-1\}$. Now it is easy to see that $ba^2c \notin (ac)^{\mathrm{Aut}(G,S)}$, and since $\mathsf{A}_1 = \mathrm{Aut}(G, S)$ by Proposition 2.4, Γ is not distance-transitive. □

Proof of Corollary 1.3. Recall that Σ is the clique graph of Γ. By the first paragraph in the proof of Lemma 3.2 and the definition of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ in Corollary 1.3, we have $\Sigma = \mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$. Again by Lemma 3.2, Σ is $R(G)$-semisymmetric, and since $|E(\Sigma)| = (2p^2 \cdot p)/2 = p^3 = |R(G)|$, $R(G)$ is regular on the edge set $E(\Sigma)$ of Σ. Thus, the line graph of Σ is a Cayley graph on G.

For a given edge $\{\langle a\rangle x, \langle b\rangle y\} \in E(\Sigma)$, we have $|\langle a\rangle x \cap \langle b\rangle y| = 1$, and then we may identify this edge with the unique element in $\langle a\rangle x \cap \langle b\rangle y$. Note that Σ has valency $2(p-1)$. Then the edge $1 = \langle a\rangle \cap \langle b\rangle$ in Σ is exactly incident to all edges in $S = \{a^i, b^i \mid 1 \le i \le p-1\}$, because $\{a^i\} = \langle a\rangle \cap \langle b\rangle a^i$ and $\{b^i\} = \langle b\rangle \cap \langle a\rangle b^i$. It follows that $\Gamma = \mathrm{Cay}(G, S)$ is exactly the line graph of Σ.

If $\alpha \in \mathrm{Aut}(\Sigma)$ fixes each edge in Σ then α fixes all vertices of Σ, that is, $\mathrm{Aut}(\Sigma)$ acts faithfully on Γ. Thus, we may view $\mathrm{Aut}(\Sigma)$ as a subgroup of $\mathrm{Aut}(\Gamma)$. By Lemmas 3.2 and 3.3, we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Sigma) = R(G) \rtimes \mathrm{Aut}(G, S)$.

Recall that $\mathrm{Aut}(G, S) = \langle\alpha, \beta, \gamma\rangle$ and Σ is arc-transitive. Since $a^\beta = a$, $b^\beta = b^t$ and $c^\beta = c^t$, where $\mathbb{Z}_p^* = \langle t\rangle$, $\langle\beta\rangle$ fixes the arc $(\langle a\rangle, \langle b\rangle)$ in Σ and is transitive on the vertex set $\{\langle a\rangle b^i \mid 1 \le i \le p-1\}$, where $\{\langle a\rangle\} \cup \{\langle a\rangle b^i \mid 1 \le i \le p-1\}$ is the neighbourhood of $\langle b\rangle$ in Σ. Thus, Σ is 2-arc-transitive. Since $a^\alpha = a^t$, $b^\alpha = b$ and $c^\alpha = c^t$, $\langle\alpha\rangle$ fixes the 2-arc $(\langle a\rangle, \langle b\rangle, \langle a\rangle b)$ and is transitive on the vertex set $\{\langle b\rangle a^i b \mid 1 \le i \le p-1\}$, where $\{\langle b\rangle\} \cup \{\langle b\rangle a^i b \mid 1 \le i \le p-1\}$ is the neighbourhood of $\langle a\rangle b$ in Σ. It follows that Σ is 3-arc-transitive. It is easy to see that the number of 3-arcs in Σ equals to $|A| = 2p^3(p-1)^2$, A is regular on the set of 3-arcs of Σ. □

ORCID iDs

Jun-Jie Huang https://orcid.org/0000-0002-9250-0672
Yan-Quan Feng https://orcid.org/0000-0003-3214-0609
Jin-Xin Zhou https://orcid.org/0000-0002-8353-896X

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LDPC codes from cubic semisymmetric graphs*

Dean Crnković, Sanja Rukavina, Marina Šimac †
Department of Mathematics, University of Rijeka,
Radmile Matejčić 2, 51000 Rijeka, Croatia

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Abstract

In this paper we study LDPC codes having cubic semisymmetric graphs as their Tanner graphs. We discuss the structure of the smallest absorbing sets of these LDPC codes. Further, we give the expression for the variance of the syndrome weight of the constructed codes, and present computational and simulation results.

Keywords: LDPC code, cubic graph, semisymmetric graph.

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1 Introduction and preliminaries

Throughout this paper we assume graphs to be finite, simple and connected. For the concepts and notation related to the graph theory and coding theory, we refer the reader to [10] and [15], respectively.

In this paper we use cubic semisymmetric graphs for the construction of LDPC codes. A graph is called a 3-regular graph, i.e. a cubic graph, if every vertex of the graph has the degree equal to three. A graph is edge-transitive (vertex-transitive) if its automorphism group acts transitively on the set of edges (set of vertices). A regular graph is semisymmetric if it is edge-transitive, but not vertex-transitive. It has been proved that every semisymmetric graph is necessarily bipartite with two parts of equal size (see [14]).

Semisymmetric graphs were first studied by Folkman in 1967 (see [12]). He proposed a construction of semisymmetric graphs and constructed the smallest semisymmetric graph with 20 vertices and 40 edges (the Folkman graph). Furthermore, it has been proved that there are no semisymmetric graphs with $2p$ or $2p^2$ vertices for a prime number p.

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†Corresponding author.

E-mail addresses: deanc@math.uniri.hr (Dean Crnković), sanjar@math.uniri.hr (Sanja Rukavina), msimac@math.uniri.hr (Marina Šimac)

A cubic semisymmetric graph is a 3-regular graph which is semisymmetric. A construction of cubic semisymmetric graphs and the (non)existence of graphs with a certain number of vertices have been a subject of many studies. For example, in [20], the existence of the unique cubic semisymmetric graph with $2p^3$ vertices for a prime number p, the Gray graph of order 54, was proved. In [11], the condition for the existence of cubic semisymmetric graphs with $6p^3$ vertices was given, and a construction of such graphs was described. The classification of cubic semisymmetric graphs with at most 768 vertices was given in [4]. All of the listed graphs have girth at least eight.

The dual code $\mathcal{C}^\perp$ of an $[n, k]$ linear code $\mathcal{C}$ is an $[n, n-k]$ code defined by

$$\mathcal{C}^\perp = \left\{ x \in \mathbb{F}_p^n \mid x \cdot y = 0, \ \forall y \in \mathcal{C} \right\},$$

where $\cdot$ is the standard inner product. A generator matrix of the code $\mathcal{C}^\perp$ is called a parity-check matrix of $\mathcal{C}$.

A binary low-density parity-check (LDPC) code is a binary linear code defined by a sparse parity-check matrix H. That is to say, H contains a very small number of nonzero entries. An LDPC code is (w_c, w_r)-regular if the weight of each column is equal to w_c, and the weight of each row is equal to w_r.

LDPC codes can be presented using Tanner graphs, which were introduced by Tanner in [26]. The Tanner graph of an LDPC code is a bipartite graph that consists of two sets of vertices; bit nodes that correspond to codeword bits and check nodes that correspond to parity-check equations. An edge connects a bit node to a check node if that bit is included in the corresponding parity-check equation. If an LDPC code is (w_c, w_r)-regular, the corresponding Tanner graph is a biregular bipartite graph in which vertices are of degree w_c or w_r.

The decoding performance of an LDPC code depends on the structure of the corresponding Tanner graph; the existence of short cycles in the Tanner graph of a code establishes a correlation between iterations in the process of decoding, and therefore, has a negative impact on the bit error rate (BER) performance of the code. The shorter the cycles are, the more significant the effect is. Furthermore, the iterative decoding performance of an LDPC code is related with the existence of certain undesirable substructures of the corresponding Tanner graph. For an AWGN channel, substructures that are called trapping sets, determine error floor performance of an LDPC code. It has been proved that absorbing sets, as a special type of trapping sets, have an important role in the error floor (see [25]).

Various combinatorial structures, including graphs, were used for a construction of LDPC codes without cycles of length four (see, e.g., [6, 16, 17, 23]). In [7], the authors investigated a family of LDPC codes constructed by taking bipartite cubic symmetric graphs as the Tanner graphs. In this paper, we construct LDPC codes from cubic semisymmetric graphs and study the smallest absorbing sets in the corresponding Tanner graphs.

The paper is organized as follows. In Section 2, the construction of the family of LDPC codes using cubic semisymmetric graphs is presented, some properties of the obtained codes are analyzed and the results regarding the code parameters are given. Furthermore, the expression for the variance of the syndrome weight of the constructed LDPC codes is presented. In Section 3, the structure of the smallest absorbing sets is studied. Sections 4 and 5 contain computational and simulation results.

2 LDPC codes constructed from cubic semisymmetric graphs

Let $\mathcal{G}$ be a connected cubic semisymmetric graph with $2n$ vertices, and denote by A its adjacency matrix. Since every semisymmetric graph is bipartite with two parts of equal size, its adjacency matrix can be written as follows

$$A = \begin{bmatrix} 0 & H \\ H^T & 0 \end{bmatrix}, \tag{2.1}$$

where H is an $n \times n$ matrix.

Taking the matrix H as a parity-check matrix, one can construct a $(3,3)$-regular LDPC code $\mathcal{C}_H(\mathcal{G})$ of length n. The dimension of that code is equal to $n - \operatorname{rank}_2(H)$, where $\operatorname{rank}_2(H) = \frac{1}{2}\operatorname{rank}_2(A)$. Furthermore, the density of the parity-check matrix H is equal to $\frac{3}{n}$. For the constructed code $\mathcal{C}_H(\mathcal{G})$, the cubic semisymmetric graph $\mathcal{G}$ is its Tanner graph.

From the fact that semisymmetric graphs are edge-transitive, but not vertex-transitive, it follows that H^T determines another LDPC code $\mathcal{C}_{H^T}(\mathcal{G})$. The code $\mathcal{C}_{H^T}(\mathcal{G})$ is a $(3,3)$-regular LDPC code of length n, and its dimension is equal to $n - \operatorname{rank}_2(H)$ as well.

Let H and H^T be $n \times n$ parity-check matrices of the codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$, respectively. For the code $\mathcal{C}_H(\mathcal{G})$, the bit node graph Γ_b is defined in the following way: vertices of the graph correspond to codeword bits, and two vertices are adjacent if and only if the corresponding bits are included in the same parity-check equation. In other words, two vertices of the graph Γ_b are adjacent if and only if the corresponding bit nodes of the Tanner graph of the code $\mathcal{C}_H(\mathcal{G})$ have a common neighbour. Similarly, the vertices of the check node graph Γ_c correspond to parity-check equations of the code, and two vertices are adjacent if and only if the corresponding parity-check equations have a bit in common. That is to say, two vertices of the graph Γ_c are adjacent if and only if the corresponding check nodes of the Tanner graph of the code $\mathcal{C}_H(\mathcal{G})$ have a common neighbour. Note that the check node graph Γ_c of the code $\mathcal{C}_H(\mathcal{G})$ is the bit node graph of the code $\mathcal{C}_{H^T}(\mathcal{G})$.

Theorem 2.1. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with girth at least six and let H be the parity-check matrix of the code $\mathcal{C}_H(\mathcal{G})$. Then the corresponding bit node graph Γ_b and check node graph Γ_c are 6-regular.*

Proof. Let v be a bit node of the Tanner graph $\mathcal{G}$. The degree of the node v is equal to three, and each of its neighbours is adjacent to another two bit nodes. Using the fact that $\mathcal{G}$ does not have cycles of length four, it follows that v has a common neighbour with exactly six other bit nodes. In other words, the degree of a vertex of the graph Γ_b is equal to six, i.e., the graph Γ_b is 6-regular. In the same way it can be concluded that the graph Γ_c is also 6-regular. □

Theorem 2.2. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with $2n$ vertices and girth at least six. Further, let H be the parity-check matrix of the code $\mathcal{C}_H(\mathcal{G})$ and let Γ_b and Γ_c be the corresponding bit node graph and check node graph, respectively. Matrices T_b and T_c are square $(0,1)$-matrices of order n satisfying $T_b = H^T H - 3I$ and $T_c = HH^T - 3I$ if and only if T_b and T_c are the adjacency matrices of the graphs Γ_b and Γ_c, respectively.*

Proof. Let us consider the $n \times n$ matrix $H^T H = [h_{i,j}]$. The degree of a bit node of the Tanner graph $\mathcal{G}$ of the code $\mathcal{C}_H(\mathcal{G})$ is equal to three, hence $h_{i,i} = 3,\ i \in \{1, \dots, n\}$. An element $h_{i,j},\ i \neq j$, of the matrix H is equal to one or zero depending on whether the corresponding nodes of the graph Γ_b are adjacent or not. Accordingly, $T_b = H^T H - 3I$, where T_b is the adjacency matrix of the graph Γ_b.

Conversely, let $T_b = [t_{i,j}]$ be an $n \times n$ $(0,1)$-matrix with the property that $T_b = H^T H - 3I$. $H^T H$ is a symmetric matrix and, consequently, T_b is also a symmetric matrix such that $t_{i,i} = 0,\ i \in \{1, \dots, n\}$. The girth of the Tanner graph $\mathcal{G}$ is greater than four, so $h_{i,j},\ i \neq j$, is equal to zero or one, and represents the number of common neighbours of the corresponding bit nodes of the Tanner graph $\mathcal{G}$ of the code $\mathcal{C}_H(\mathcal{G})$. It follows that T_b is the adjacency matrix of the graph Γ_b.

An analog statement for the matrix T_c can be formed similarly by observing check nodes of the Tanner graph of the code $\mathcal{C}_H(\mathcal{G})$. □

A clique of a graph G is a complete subgraph of the graph G. The clique number of the graph G, denoted by $\omega(G)$, is the number of vertices in a clique of the largest size in G, i.e. the order of a complete subgraph of G of maximum possible size for G. In the sequel, the clique number of the bit node graph Γ_b and the check node graph Γ_c will be examined.

Lemma 2.3. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph. Further, let $\mathcal{C}_H(\mathcal{G})$ be the corresponding LDPC code and let Γ_b and Γ_c be its bit node and check node graph, respectively. The clique numbers of the graphs Γ_b and Γ_c are at least three.*

Proof. Each check node of the Tanner graph $\mathcal{G}$ is a common neighbour of every pair of its three adjacent bit nodes. Thus, each check node determines the complete graph K_3 as a subgraph of the bit node graph Γ_b. Similarly, each bit node of the Tanner graph determines the complete graph K_3 as a subgraph of the check node graph Γ_c. Hence, $\omega(\Gamma_b), \omega(\Gamma_c) \geq 3$.

□

Lemma 2.4. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with girth greater than six. Further, let $\mathcal{C}_H(\mathcal{G})$ be the corresponding LDPC code and let Γ_b and Γ_c be its bit node and check node graph, respectively. Then the complete graph K_4 is not a subgraph of Γ_b or Γ_c.*

Proof. Suppose that K_4 is a subgraph of the graph Γ_b. Let the bit nodes u_1, u_2, u_3, u_4 be the vertices of K_4. We have the following two possibilities:

(a) One of the check nodes (say v_1) in the corresponding subgraph of the Tanner graph $\mathcal{G}$ has degree three. Let u_1, u_2 and u_3 be the bit nodes adjacent with v_1. Furthermore, let the check node v_2 be a common neighbour of u_1 and u_4. Since u_2 and u_4 are adjacent in Γ_b, they have a common neighbour v_3 in $\mathcal{G}$. Then $u_1 v_1 u_2 v_3 u_4 v_2 u_1$ is a cycle of length six, which is impossible since the girth of the graph $\mathcal{G}$ is greater than six.

(b) The check nodes in the corresponding subgraph of the Tanner graph $\mathcal{G}$ have degrees at most two. Let the check node v_i be a common neighbour of the bit nodes u_1 and u_{i+1}, $i = 1, 2, 3$. Since u_2 and u_4 are adjacent in Γ_b, they have a common neighbour v_4 in $\mathcal{G}$. Then $u_1 v_1 u_2 v_4 u_4 v_3 u_1$ is a cycle of length six, which contradicts the fact that the girth of the graph $\mathcal{G}$ is greater than six.

Analog arguments yield that K_4 is not a subgraph of Γ_c. □

The following theorem is a direct consequence of Lemmas 2.3 and 2.4.

Theorem 2.5. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with girth greater than six. Further, let $\mathcal{C}_H(\mathcal{G})$ be the corresponding LDPC code and let Γ_b and Γ_c be its bit node and check node graph, respectively. Then* $\omega(\Gamma_b) = \omega(\Gamma_c) = 3$.

In the sequel, we discuss the minimum distance of the codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$. The following results from [24] will be used.

Theorem 2.6 ([24, Theorem 3.1])**.** *Let $\mathcal{C}$ be a binary linear code with a parity-check matrix H. Then there exists a codeword in $\mathcal{C}$ with weight w if and only if there are w columns in H whose vector sum is a zero vector.*

Theorem 2.7 ([24, Theorem 3.2])**.** *Let $\mathcal{C}$ be a binary linear code with a parity-check matrix H. Then the minimum distance of the code $\mathcal{C}$ is equal to the smallest number of columns in H whose vector sum is a zero vector.*

The column weight of parity check matrices H and H^T of codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$ is equal to three, and according to Theorem 2.6, the codes are even. Therefore, the minimum distance of the codes is an even number.

Theorem 2.8. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with girth greater than six. Let $d(\mathcal{C}_H(\mathcal{G}))$ and $d(\mathcal{C}_H^T(\mathcal{G}))$ be the minimum distances of the codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$, respectively. Then $d(\mathcal{C}_H(\mathcal{G})) \geq 6$ and $d(\mathcal{C}_H^T(\mathcal{G})) \geq 6$.*

Proof. The column weight of the parity-check matrix H of the code $\mathcal{C}_H(\mathcal{G})$ is equal to three, and since the graph $\mathcal{G}$ does not have cycles of length four, it follows that the minimum distance of the code is at least four (see [13]). Assume that the minimum distance of the code is equal to four. As a consequence of Theorem 2.7, four columns of the parity-check matrix whose sum equals zero exist. Therefore, a set S in the graph $\mathcal{G}$, which consists of four bit nodes such that each pair of the vertices has a different common neighbour in $\mathcal{G}$, exists. Moreover, the set S determines the complete graph K_4 as a subgraph of the bit node graph Γ_b. Using Theorem 2.5, we conclude that the minimum distance of the code is at least six.

Observing check nodes of the Tanner graph of the code $\mathcal{C}_H(\mathcal{G})$, and the check node graph Γ_c, one can prove the statement for the minumum distance of the code $\mathcal{C}_{H^T}(\mathcal{G})$. □

In [7, Theorem 1], the minimum distance of an LDPC code constructed from a bipartite cubic symmetric graph is expressed using the second largest eigenvalue of the adjacency matrix of that graph. In a similar way, using the result given in Theorem 2.8, one can prove the following theorem.

Theorem 2.9. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with $2n$ vertices and girth greater than six. Let λ_2 be the second largest eigenvalue of its adjacency matrix A. Let $d(\mathcal{C}_H(\mathcal{G}))$ and $d(\mathcal{C}_H^T(\mathcal{G}))$ be the minimum distances of the codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$, respectively. Then the following inequalities hold*

$$d \geq \begin{cases} \frac{2}{5}n, & \lambda_2 \leq 2, \\ \frac{2}{9}n, & 2 < \lambda_2 \leq \sqrt{6}, \\ 6, & \sqrt{6} < \lambda_2 < 3, \end{cases}$$

where $d \in \{d(\mathcal{C}_H(\mathcal{G})), d(\mathcal{C}_H^T(\mathcal{G}))\}$.

Remark 2.10. The results given above refer to the LDPC codes constructed from connected cubic semisymmetric graphs with girth greater than six. According to the classification of cubic semisymmetric graphs with at most 768 vertices (see [4]), all such graphs have girth at least eight. Consequently, all of the associated LDPC codes have properties stated above.

Theorem 2.11. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with $2n$ vertices. Then the dimension of the codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$ is at most $n - 2\alpha(\Gamma_b) + 1$, where $\alpha(\Gamma_b)$ is the independence number of the bit node graph Γ_b.*

Proof. The 2-rank of the parity-check matrix of a binary code determines its dimension. The 2-rank of the matrix H is equal to the 2-rank of the matrix H^T and, therefore, it is sufficent to observe the matrix H and the corresponding code $\mathcal{C}_H(\mathcal{G})$. A maximal independent set of Γ_b determines $\alpha(\Gamma_b)$ linearly independent columns of the parity check matrix H. These columns have the property that no two columns have an entry equal to one at the same position. Due to the fact that Γ_b is a 6-regular graph, there are $6\alpha(\Gamma_b)$ ones at different positions within the columns. Therefore, adding any other $\alpha(\Gamma_b) - 1$ columns of the matrix, a set of $2\alpha(\Gamma_b) - 1$ linearly independent columns of the parity check matrix is defined. Hence, 2-rank of the matrix H is at least $2\alpha(\Gamma_b) - 1$.

As a consequence, the dimension of the code is at most $n - 2\alpha(\Gamma_b) + 1$, where n is the length of the code, i.e. the number of vertices of the graph Γ_b, and $\alpha(\Gamma_b)$ is the independence number of the graph Γ. □

2.1 The variance of syndrome weight

To predict a decoding efficiency one can use a channel state information (CSI) (e.g. the crossover probability, a signal-to-noise ratio), which has an important role for communication systems. The estimation (performed prior to decoding) of the crossover probability based on the probability of syndrome weight was proposed in [18] and [27].

The expression for the variance of the syndrome weight of the LDPC codes constructed from bipartite cubic symmetric graphs is given in [7]. In a similar way, one can obtain the expression for the variance of the syndrome weight of the LDPC codes constructed from cubic semisymmetric graphs which is given by

$$Var(w) = \frac{n}{2}\left(7f_6(\rho) - 6f_4(\rho)\right),$$

where the function f_t is defined by $f_t(\rho) = \frac{1-(1-2\rho)^t}{2}$ (see [22]).

3 Absorbing sets

Let $G = G(C)$ be the Tanner graph of an LDPC code C which is determined by an $m \times n$ parity check matrix H. A (κ, τ) trapping set is a set T, that consists of κ bit nodes, having the property that the induced subgraph $G[T]$ has exactly τ check nodes of odd degree. The most harmful trapping sets are those with small sizes and small ratios $\frac{\tau}{\kappa}$. If the Tanner graph of an LDPC code does not have trapping sets with size smaller than the minimum distance of the code, then the error floor of the code is dominated by the minimum distance (see [9]). Let T be a trapping set. If every bit node in $G[T]$ is connected with fewer check nodes of odd degree than check nodes of even degree, then T is called an absorbing set.

Let A be a $(\kappa, \tau)-$ trapping set in the Tanner graph of an $(3, w_r)$ LDPC code. Using simple counting it can be seen that τ is an even number if κ is even, and an odd number if κ is odd.

The results in the sequel refer to the LDPC codes for which the corresponding Tanner graphs have girth at least six. We examine the existence of the smallest absorbing sets in the Tanner graphs of the LDPC codes constructed from the cubic semisymmetric graphs.

Theorem 3.1. *Let the Tanner graph of the LDPC code $\mathcal{C}_H(\mathcal{G})$ be a connected cubic semisymmetric graph $\mathcal{G}$ with girth at least six. Then there is no absorbing set of size smaller than three in the graph $\mathcal{G}$.*

Proof. The proof follows directly from the definition of an absorbing set and the fact that the Tanner graph of the code has no cycles of length four. □

Theorem 3.2. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with girth greater than six, which is the Tanner graph of the LDPC codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$. The Tanner graph $\mathcal{G}$ has no absorbing set of size three.*

Proof. Let A be a $(3, 3)$-absorbing set, which is the only possible structure of an absorbing set of size three in the Tanner graph of the codes (see Figure 1). The proof follows directly from the fact that the absorbing set defines a cycle of length six in the Tanner graph. □

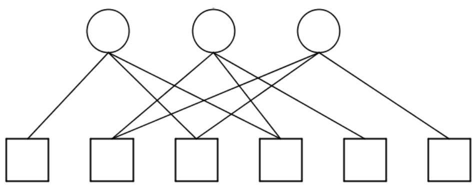

Figure 1: The only possible structure of an absorbing set of size three in the Tanner graph of the LDPC codes $\mathcal{C}_H(\mathcal{G})$. and $\mathcal{C}_{H^T}(\mathcal{G})$.

Theorem 3.3. *Let $\mathcal{G}$ be a connected cubic semisymmetric graph with girth greater than six, which is the Tanner graph of the LDPC codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$. The only possible structure for an absorbing set of size four is $(4, 4)$-absorbing set.*

Proof. Since the size of an absorbing set is an even number, and according to the previous observations, the possible structures for absorbing sets of size four in the Tanner graph of the codes are $(4, 0)$, $(4, 2)$ and $(4, 4)$ absorbing sets (see Figure 2(a), (b) and (c), respectively). The proof follows directly from the fact that $(4, 0)$ and $(4, 2)$ absorbing sets define the complete graph K_4 as a subgraph of the graph Γ_b. □

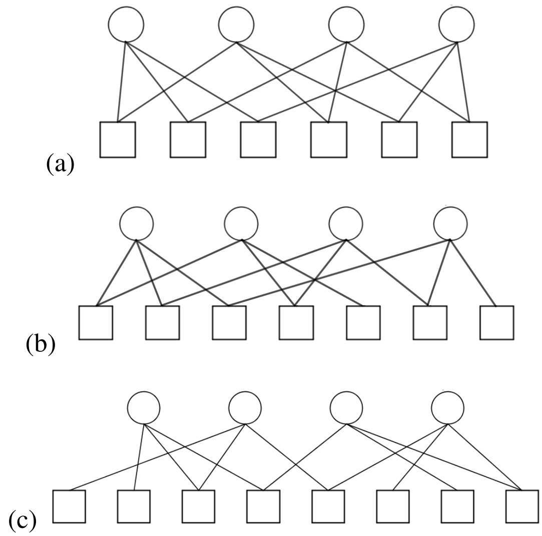

Figure 2: The possible structures of an absorbing set of size four in the Tanner graph of the LDPC codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$.

4 Computational results

Within this section the parameters of the LDPC codes obtained from cubic semisymmetric graphs are presented. For the construction of the cubic semisymmetric graphs we have employed the method presented in [1]. The parameters of the constructed codes can be seen in Table 1. The parameter v denotes the number of vertices of the corresponding cubic semisymmetric graph.

v	**LDPC$_1$**	**LDPC$_2$**	v	**LDPC$_1$**	**LDPC$_2$**
54	$[27, 8, 6]$	$[27, 8, 8]^*$	448	$[224, 33, 32]$	**[224,33,32]**
112	$[56, 12, 14]$	**[56,12,16]**	486	$[243, 2, 162]^*$	$[243, 2, 162]^*$
120	$[60, 14, 8]$	**[60,14,12]**	546	$[273, 5, 130]$	$[273, 5, 130]$
144	$[72, 16, 12]^*$	$[72, 16, 14]^*$	576	$[288, 32, 48]$	$[288, 32, 56]$
216	$[108, 16, 24]$	$[108, 16, 32]$	672	$[336, 47, 14]$	$[336, 47, 42]$
240	$[120, 22, 16]$	$[120, 22, 24]$	702	$[351, 8, 78]$	$[351, 8, 104]^*$
294	$[147, 26, 14]$	$[147, 26, 26]$	720	**[360,10,120]**	**[360,10,120]**
336	$[168, 24, 14]$	$[168, 24, 42]$	784	$[392, 12, 98]$	**[392,12,112]**
378	$[189, 11, 42]$	$[189, 11, 56]$	798	$[399, 5, 190]$	$[399, 5, 190]$
384	$[192, 35, 16]$	$[192, 35, 18]$	864	$[432, 32, 96]$	$[432, 32, 108]$
400	**[200,24,32]**	**[200,24,60]**	882	$[441, 44, 42]$	$[441, 44, 78]$
432	$[216, 24, 48]$	$[216, 24, 60]$	896	$[448, 48, 84]$	**[448,48,100]**

Table 1: The parameters of LDPC codes constructed from cubic semisymmetric graphs with less than 1000 vertices (using the method presented in [1]).

The Tanner graphs of the constructed codes have girth at least eight. The codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$ are isomorphic in the case when the number of vertices of the cubic semisymmetric graph G is $486, 546, 720$ or 798.

Remark 4.1. Lately, much interest has been devoted to LCD codes, which have an important application in cryptography, in protection against side-channel and fault attacks (see [2]). Self-orthogonal codes can be used to construct quantum error-correcting codes, which can protect quantum information in quantum computations and quantum communications (see [3]).

The LDPC codes marked in bold are self-orthogonal codes, and those labeled with * in Table 1 are LCD codes.

Remark 4.2. Codes $\mathcal{C}_H(\mathcal{G})$ and $\mathcal{C}_{H^T}(\mathcal{G})$ constructed from a cubic semisymmetric graph (CSSG) have the same length and dimension, and, in general, different minimum distance. Thus, the construction gives diversity in code parameters for the same graph, which is not the case for LDPC codes which are constructed in [7] using cubic symmetric graphs (CSGs).

According to the classification of CSSGs with at most 768 vertices (see [4]), all the graphs have girth at least eight, while according to [5] many CSGs have girth equal to six. Moreover, semisymmetric graphs form a wider family than symmetric graphs.

Furthermore, we have compared the parameters of the LDPC codes constructed from CSSGs to the parameters of the LDPC codes constructed from CSGs. The results are shown in Table 2. It can be concluded that, for the same code length, the LDPC codes from CSSGs achieve higher code rate than those constructed using CSGs. When $n = 27$, the code rate is four times greater.

n	**Rate (CSSG)**	**Rate (CSG)**
27	0.296	0.074
56	0.214	$\{0.107, 0.143\}$
60	0.233	$\{0.067, 0.083\}$
72	0.222	$\{0.083, 0.111\}$

Table 2: Rates od LDPC codes constructed from cubic symmetric and semisymmetric graphs with the same length.

5 Simulation results

In this section, we present simulation results of the LDPC codes derived from the cubic semisymmetric graphs, over the additive white gaussian noise (AWGN) channel. We have compared the codes with randomly generated LDPC codes of the same length and dimension and a parity-check matrix with a column weight equal to three. For randomly generated codes we have used the software for LDPC codes available on [21], which employs the construction from [8, 19]. The codes are decoded with the sum-product decoding algorithm and the maximum number of iteration is set to 50. Figures 3 - 6 show the performance of the codes.

Remark 5.1. The LDPC codes that we are aware of were not adequate for the comparison with the LDPC codes obtained in this paper because of the different parameters of the codes. Thus, we have used the best known random construction for LDPC codes. It has been proved in [8] that the construction leads to LDPC codes with performance close to the Shannon limit. Moreover, the best results were obtained in the case of the smallest possible column weight.

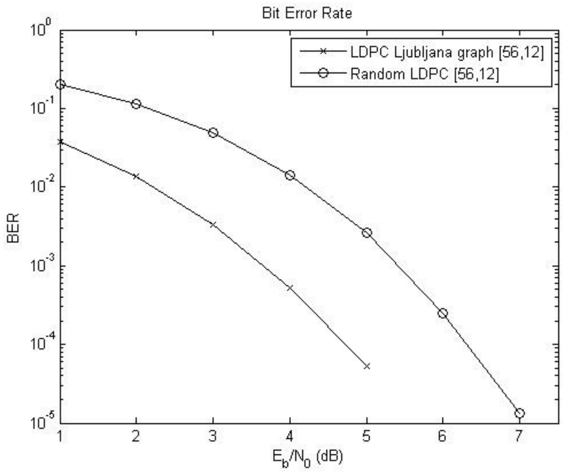

Figure 3: BER performance of the $[56, 12, 16]$ LDPC code derived from the Ljubljana graph.

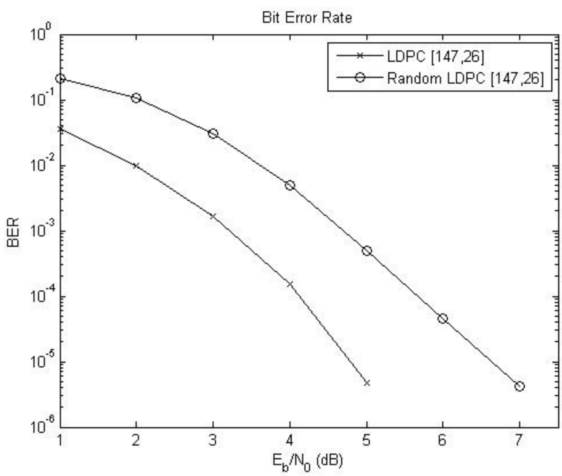

Figure 4: BER performance of the $[147, 26, 26]$ LDPC code derived from the cubic semisymmetric graph with 294 vertices.

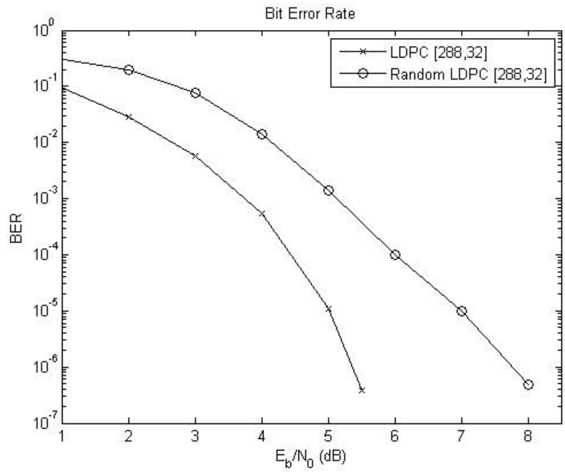

Figure 5: BER performance of the $[288, 32, 56]$ LDPC code derived from the cubic semisymmetric graph with 576 vertices.

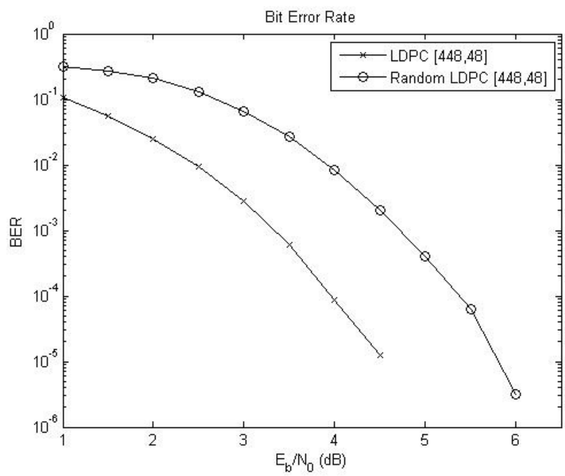

Figure 6: BER performance of the $[448, 48, 100]$ LDPC code derived from the cubic semisymmetric graph with 896 vertices.

The obtained simulation results indicate better BER performance of the codes constructed from the cubic semisymmetric graphs than randomly generated LDPC codes.

ORCID iDs

Dean Crnković https://orcid.org/0000-0002-3299-7859
Sanja Rukavina https://orcid.org/0000-0003-3365-7925
Marina Šimac https://orcid.org/0000-0001-9291-3365

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Paired domination stability in graphs

Aleksandra Gorzkowska

AGH University, Department of Discrete Mathematics, al. Mickiewicza 30, 30-059 Krakow, Poland

Michael A. Henning

Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa

Monika Pilśniak

AGH University, Department of Discrete Mathematics, al. Mickiewicza 30, 30-059 Krakow, Poland

Elżbieta Tumidajewicz *

AGH University, Department of Discrete Mathematics, al. Mickiewicza 30, 30-059 Krakow, Poland, and *Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa*

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Abstract

A set S of vertices in a graph G is a paired dominating set if every vertex of G is adjacent to a vertex in S and the subgraph induced by S contains a perfect matching (not necessarily as an induced subgraph). The paired domination number, $\gamma_{\rm pr}(G)$, of G is the minimum cardinality of a paired dominating set of G. A set of vertices whose removal from G produces a graph without isolated vertices is called a non-isolating set. The minimum cardinality of a non-isolating set of vertices whose removal decreases the paired domination number is the $\gamma_{\rm pr}^-$-stability of G, denoted $\mathrm{st}_{\gamma_{\rm pr}}^-(G)$. The paired domination stability of G is the minimum cardinality of a non-isolating set of vertices in G whose removal changes the paired domination number. We establish properties of paired domination stability in graphs. We prove that if G is a connected graph with $\gamma_{\rm pr}(G) \geq 4$, then $\mathrm{st}_{\gamma_{\rm pr}}^-(G) \leq 2\Delta(G)$ where $\Delta(G)$ is the maximum degree in G, and we characterize the infinite family of trees that achieve equality in this upper bound.

Keywords: Paired domination, paired domination stability.

*Corresponding author.

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1 Introduction

In 1983 Bauer, Harary, Nieminen and Suffel [3] introduced and studied the concept of domination stability in graphs. Stability for other domination type parameters has been studied in the literature. For example, total domination stability, 2-rainbow domination stability, exponential domination stability, Roman domination stability are studied in [1, 2, 12, 15, 16], among other papers. In this paper we study the paired version of domination stability.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Two vertices u and v are *neighbors* if they are adjacent, that is, if $uv \in E$. A *dominating set* of G is a set D of vertices such that every vertex in $V(G) \setminus D$ has a neighbor in D. The minimum cardinality of a dominating set is the *domination number*, $\gamma(G)$, of G. Domination is well studied in the literature. A recent book on domination in graphs can be found in [10]. A small sample of papers on domination critical graphs can be found in [3, 4, 5, 6, 9, 17, 18]. Adopting the notation coined by Bauer et al. [3], the γ^-*-stability* (γ^+*-stability*, resp.) of G, denoted by $\gamma^-(G)$ ($\gamma^+(G)$, resp.), is the minimum number of vertices whose removal decreases (increases, resp.) the domination number. The minimum number of vertices whose removal decreases or increases the domination number is the *domination stability*, $\mathrm{st}_\gamma(G)$, of G, and so $\mathrm{st}_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$.

We refer to a graph without isolated vertices as an *isolate-free graph*. Unless otherwise stated, let G be an isolate-free graph. A *total dominating set*, abbreviated TD-set, of G is a set D of vertices of G such that every vertex, including vertices in the set D, has a neighbor in D. The minimum cardinality of a TD-set of G is the *total domination number*, $\gamma_t(G)$, of G. We call a TD-set of G of cardinality $\gamma_t(G)$ a γ_t*-set of* G. A vertex v is *totally dominated* by a set D in G if the vertex v has a neighbor in D. We refer the reader to the book [14] for fundamental concepts on total domination in graphs. Total domination critical graphs are studied, for example, in [7, 13]. The total version of domination stability was first studied by Henning and Krzywkowski [12].

A *paired dominating set*, abbreviated PD-set, of an isolate-free graph G is a dominating set S of G with the additional property that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). With respect to the matching M, two vertices joined by an edge of M are *paired* and are called *partners* in S. The *paired domination number*, $\gamma_{\mathrm{pr}}(G)$, of G is the minimum cardinality of a PD-set of G. We call a PD-set of G of cardinality $\gamma_{\mathrm{pr}}(G)$ a γ_{pr}*-set of* G. We note that the paired domination number $\gamma_{\mathrm{pr}}(G)$ is an even integer. For a recent survey on paired domination in graphs, we refer the reader to the book chapter [8].

Every PD-set is a TD-set, implying that $\gamma_t(G) \leq \gamma_{\mathrm{pr}}(G)$. A *non-isolating set* of vertices in G is a set $S \subseteq V$ such that the graph $G - S$ is isolate-free, where $G - S$ is the graph obtained from G by removing S and all edges incident with vertices in S. Let $\mathrm{NI}(G)$ denote the set of all non-isolating sets of vertices of G.

Adopting the standard notation for domination stability given in [3, 12], the γ_{pr}^--stability

E-mail addresses: agorzkow@agh.edu.pl (Aleksandra Gorzkowska), mahenning@uj.ac.za (Michael A. Henning), pilsniak@agh.edu.pl (Monika Pilśniak), etumid@agh.edu.pl (Elżbieta Tumidajewicz)

(resp., $\gamma^+_{\rm pr}$-stability) of G, denoted by $\mathrm{st}^-_{\gamma_{\rm pr}}(G)$ (resp., $\mathrm{st}^+_{\gamma_{\rm pr}}(G)$) is the minimum cardinality of a set in $\mathrm{NI}(G)$ whose removal decreases (increases, resp.) the paired domination number. Thus,

$$\mathrm{st}^-_{\gamma_{\rm pr}}(G) = \min_{S\in \mathrm{NI}(G)}\{\,|S|\colon \gamma_{\rm pr}(G-S) < \gamma_{\rm pr}(G)\}$$

and

$$\mathrm{st}^+_{\gamma_{\rm pr}}(G) = \min_{S\in \mathrm{NI}(G)}\{\,|S|\colon \gamma_{\rm pr}(G-S) > \gamma_{\rm pr}(G)\}.$$

If there is no set in $\mathrm{NI}(G)$ whose removal increases the paired domination number, then we define $\mathrm{st}^+_{\gamma_{\rm pr}}(G) = \infty$. For example, $\mathrm{st}^-_{\gamma_{\rm pr}}(P_5) = 1$ while $\mathrm{st}^+_{\gamma_{\rm pr}}(P_5) = \infty$. The *paired domination stability*, $\mathrm{st}_{\gamma_{\rm pr}}(G)$, of G is the minimum cardinality of a set in $\mathrm{NI}(G)$ whose removal increases or decreases the paired domination number. Thus,

$$\mathrm{st}_{\gamma_{\rm pr}}(G) = \min_{S\in \mathrm{NI}(G)}\{\,|S|\colon \gamma_{\rm pr}(G-S) \neq \gamma_{\rm pr}(G)\} = \min\{\mathrm{st}^-_{\gamma_{\rm pr}}(G), \mathrm{st}^+_{\gamma_{\rm pr}}(G)\}.$$

Let G be a graph and let $S \in \mathrm{NI}(G)$. If $\gamma_{\rm pr}(G-S) < \gamma_{\rm pr}(G)$ and $|S| = \mathrm{st}^-_{\gamma_{\rm pr}}(G)$, then we call S a $\mathrm{st}^-_{\gamma_{\rm pr}}$*-set of* G. If $\gamma_{\rm pr}(G-S) > \gamma_{\rm pr}(G)$ and $|S| = \mathrm{st}^+_{\gamma_{\rm pr}}(G)$, then we call S a $\mathrm{st}^+_{\gamma_{\rm pr}}$*-set of* G. If $\gamma_{\rm pr}(G-S) \neq \gamma_{\rm pr}(G)$ and $|S| = \mathrm{st}_{\gamma_{\rm pr}}(G)$, then we call S a $\mathrm{st}_{\gamma_{\rm pr}}$*-set of* G.

Defining the *null graph* K_0, which has no vertices, as a graph, we have the following results due to Bauer et al. [3] and Rad et al. [15] for the γ^--stability of a graph.

Theorem 1.1 ([3, 15])**.** *If G is an isolate-free graph of order n, then the following holds.*

(a) $\mathrm{st}_\gamma(G) \le \delta(G) + 1$.

(b) *If $G \ncong K_n$, then* $\mathrm{st}_\gamma(G) \le n-1$.

Considering the null graph, the paired domination stability of a non-trivial graph is always defined. If G is a graph of order n and $\gamma_{\rm pr}(G) = 2$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = n$ since removing all vertices from the graph G produces the null graph with paired domination number zero.

For notation and graph theory terminology we generally follow [14]. In particular, for $r, s \ge 1$, a *double star* $S(r,s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf-neighbors and the other s leaf-neighbors. A *rooted tree* is a tree T in which we specify one vertex r called the *root*. For each vertex v of T different from r, its *parent* is the neighbor of v on the unique (r,v)-path, while every other neighbor of v is a *child* of v in T. If w is a vertex of T different from v and the (unique) (r,w)-path contains v, then w is a *descendant* of v in T. We note that every child of v is a descendant of v. The *diameter* $\mathrm{diam}(G)$ of G is the maximum distance among all pairs of vertices of G. A *diametral path* in G is a shortest path between two vertices in G of length equal to $\mathrm{diam}(G)$. For an integer $k \ge 1$, $[k] = \{1, \ldots, k\}$.

2 Main results

Our first aim is to show that the paired domination stability of a graph can be very different from its total domination stability studied in [12].

Theorem 2.1. *For $k \ge 1$ an arbitrary integer, the following holds.*

(a) *There exist connected graphs G such that* $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) - \mathrm{st}^-_{\gamma_t}(G) = k$.

(b) *There exist connected graphs H such that* $\mathrm{st}^-_{\gamma_t}(H) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(H) = k$.

Our second aim is to establish properties of paired domination stability in graphs. Thereafter, we establish upper bounds on the paired domination stability and the γ^-_{pr}-stability of a graph. For this purpose, we shall need the following family of trees defined by Henning and Krzywkowski [12]. For integers $k \geq 2$ and $\Delta \geq 2$, the authors in [12] define $T_{k,\Delta}$ as the "graph obtained from the disjoint union of k double stars $S(\Delta - 1, \Delta - 1)$ by adding $k - 1$ edges between the leaves of these double stars so that the resulting graph is a tree with maximum degree Δ." Let $\mathcal{F}_{k,\Delta}$ be the family of all such trees $T_{k,\Delta}$, and let

$$\mathcal{F}_\Delta = \bigcup_{k \geq 2} \mathcal{F}_{k,\Delta}.$$

The following result establishes an upper bound on the γ^-_{pr}-stability of a tree, and characterizes the trees with maximum possible γ^-_{pr}-stability.

Theorem 2.2. *If T is a tree with maximum degree Δ satisfying $\gamma_{\mathrm{pr}}(T) \geq 4$, then the following hold.*

(a) $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta$, *with equality if and only if $T \in \mathcal{F}_\Delta$.*

(b) $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta - 1$, *and this bound is sharp for all $\Delta \geq 2$.*

For general graphs, we establish the following upper bound on the γ^-_{pr}-stability in terms of the maximum degree of the graph.

Theorem 2.3. *If G is a connected graph with $\gamma_{\mathrm{pr}}(G) \geq 4$, then $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) \leq 2\Delta(G)$, and this bound is sharp.*

As an immediate consequence of Theorem 2.3, we have the following upper bound on the paired domination stability of a graph.

Corollary 2.4. *If G is a connected graph with $\gamma_{\mathrm{pr}}(G) \geq 4$, then $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \leq 2\Delta(G)$.*

3 Paired stability versus domination and total stability

In this section, we show that paired domination stability and the domination stability of a graph can be very different. By Theorem 1.1, for every nontrivial graph G, we have $\mathrm{st}_\gamma(G) \leq \delta(G) + 1$. In particular, $\mathrm{st}_\gamma(T) \leq 2$ for every nontrivial tree T. This is in contrast to the paired domination stability, where for any given $\Delta \geq 2$, we show that there exist a family of trees T with maximum degree Δ satisfying $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$.

For $\Delta = 2$, the authors in [12] define $\mathcal{H}_\Delta$ as the family of all paths of order at least 7 and congruent to 3 modulo 4, that is, $\mathcal{H}_\Delta = \{P_n \mid n \equiv 3 \pmod 4 \text{ and } n \geq 7\}$. For integers $\Delta \geq 3$ and $\Delta \geq k \geq 2$, they define $H_{k,\Delta}$ as the graph "obtained from the disjoint union of k double stars $S(\Delta - 1, \Delta - 1)$ by selecting one leaf from each double star and identifying these k leaves into one new vertex" and they define the family

$$\mathcal{H}_\Delta = \bigcup_{k \geq 2} H_{k,\Delta}.$$

We determine next the paired domination stability of a tree in the family $\mathcal{H}_\Delta$.

Proposition 3.1. *For $\Delta \geq 3$, if $T \in \mathcal{H}_\Delta$, then* $\mathrm{st}_{\gamma_{\rm pr}}(T) = 2\Delta - 1$.

Proof. For integers $\Delta \geq k \geq 2$ where $\Delta \geq 3$, consider a tree $T \in \mathcal{H}_{k,\Delta}$. By definition of the family $\mathcal{H}_{k,\Delta}$, the tree T is constructed from the disjoint union of k double stars $S_1, \ldots, S_k$, each isomorphic to $S(\Delta-1, \Delta-1)$, by selecting one leaf from each double star and identifying these k chosen leaves into one new vertex, which we call v_c. Let x_i and y_i be the two central vertices of the double star S_i for $i \in [k]$, where x_i is adjacent to v_c in T. Let $D = \cup_{i=1}^{k} \{x_i, y_i\}$. Since $\Delta \geq 3$, every vertex in D is a support vertex of T, implying that every PD-set in T contains the set D and therefore $\gamma_{\rm pr}(T) \geq |D| = 2k$. Since the set D is a PD-set of T (with the vertices x_i and y_i paired for all $i \in [k]$), we have $\gamma_{\rm pr}(T) \leq |D| = 2k$. Consequently, $\gamma_{\rm pr}(T) = 2k$ and D is the unique $\gamma_{\rm pr}$-set of T.

Let S be a $\mathrm{st}_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}_{\gamma_{\rm pr}}(T)$ satisfying $\gamma_{\rm pr}(T - S) \neq \gamma_{\rm pr}(T) = 2k$. We show that $|S| \geq 2\Delta - 1$. Suppose, to the contrary, that $|S| \leq 2\Delta - 2$. If the set S contains both x_i and y_i for some $i \in [k]$, then since S is a non-isolating set of T every leaf neighbor of x_i and y_i is also in S, implying that $|S| \geq 2\Delta - 1$, a contradiction. Hence, the set S contains at most one of x_i and y_i for every $i \in [k]$. Let D^* be a $\gamma_{\rm pr}$-set of $T - S$, and so $|D^*| \neq 2k$.

Suppose that $v_c \in S$. In this case, if $|S| = 1$, then the paired domination numbers of T and $T - S$ are the same, a contradiction. Hence, $|S| \geq 2$. If neither x_i nor y_i belong to S for some $i \in [k]$, then by the minimality of the non-isolating set S, no vertex of T_i different from v_c belongs to S, and so $|D^* \cap V(T_i)| = 2$. If S contains y_i but not x_i for some $i \in [k]$, then every leaf neighbor of y_i is in S and by the minimality of the set S, no leaf neighbor of x_i belongs to S, and so $|D^* \cap V(T_i)| = 2$. Analogously, if S contains x_i but not y_i for some $i \in [k]$, then $|D^* \cap V(T_i)| = 2$. This is true for all $i \in [k]$, implying that $|D^*| = \sum_{i=1}^{k} |D^* \cap V(T_i)| = 2k$, a contradiction. Hence, $v_c \notin S$.

As observed earlier, the set S contains at most one of x_i and y_i for every $i \in [k]$. If $y_i \in S$ and $y_j \in S$ for some $i, j \in [k]$ where $i \neq j$, then $|S| \geq 2\Delta$, a contradiction. If $y_i \in S$ and $x_j \in S$ for some $i, j \in [k]$ where $i \neq j$, then $|S| \geq 2\Delta - 1$, a contradiction. If $x_i \in S$ and $x_j \in S$ for some $i, j \in [k]$ where $i \neq j$, then $|S| \geq 2\Delta - 2$. In this case, by the minimality of S we have $S = (N[x_i] \cup N[x_j]) \setminus \{v_c, y_i, y_j\}$ and $|S| = 2\Delta - 2$. But then $T - S$ consists of three components, namely two stars isomorphic to $K_{1,\Delta-1}$ and one component belonging to the family $T \in \mathcal{H}_{k-2,\Delta}$ with paired domination number $2(k-2)$. Thus, $\gamma_{\rm pr}(T - S) = 2 + 2 + 2(k-2) = 2k$, a contradiction. Therefore, $\mathrm{st}_{\gamma_{\rm pr}}(T) = |S| \geq 2\Delta - 1$, as claimed.

Conversely, if we take $S = N(x_1) \cup N(y_1) \setminus \{v_c\}$, then $S \in \mathrm{NI}(T)$ and $T - S \in \mathcal{H}_{k-1,\Delta}$. Thus, $\gamma_{\rm pr}(T - S) = 2(k-1) < \gamma_{\rm pr}(T)$, and so $\mathrm{st}_{\gamma_{\rm pr}}(T) \leq \mathrm{st}^-_{\gamma_{\rm pr}}(T) \leq |S| = 2\Delta - 1$. Consequently, $\mathrm{st}_{\gamma_{\rm pr}}(T) = \mathrm{st}^-_{\gamma_{\rm pr}}(T) = 2\Delta - 1$. □

As observed earlier, $\mathrm{st}_\gamma(T) \leq 2$ for every nontrivial tree T. By Proposition 3.1, paired domination stability therefore differs significantly from domination stability. We show next that the paired domination stability and the total domination stability of a graph can also be very different.

Proposition 3.2. *For $k \geq 1$ an integer, there exist trees T such that* $\mathrm{st}^-_{\gamma_{\rm pr}}(T) - \mathrm{st}^-_{\gamma_t}(T) = k$.

Proof. Let $k \geq 1$ be a given integer, and let $T = T_k$ be obtained from a path P_5 given by $v_1v_2v_3v_4v_5$ by attaching k leaf neighbors to each of v_1, v_2 and v_3 (see Figure 1). We note that $\{v_1, v_2, v_3, v_4\}$ is the unique γ_t-set of T and the unique $\gamma_{\rm pr}$-set of T. In particular, $\gamma_t(T) = \gamma_{\rm pr}(T) = 4$. If $S = \{v_5\}$, then the set S is a non-isolating set of T and $\gamma_t(T - S) = |\{v_1, v_2, v_3\}| = 3 < \gamma_t(T)$, implying that $\mathrm{st}^-_{\gamma_t}(T) = 1$.

We show next that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = k + 1$. Let S be a non-isolating set of T such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. We show that $|S| \ge k + 1$. Suppose, to the contrary, that $|S| \le k$. Let D be a $\gamma_{\rm pr}$-set of $T - S$, and so $|D| = \gamma_{\rm pr}(T - S) = 2$. Let L_i denote the set of leaf neighbors of v_i for $i \in [4]$. If $v_i \in S$ for some $i \in [3]$, then S contains all k leaf neighbors of v_i, and so $|S| \ge k + 1$, a contradiction. Hence, $S \cap \{v_1, v_2, v_3\} = \emptyset$. If $\{v_1, v_3\} \subset D$, then $|D| \ge 4$, a contradiction. If $v_1 \notin D$, then $L_1 \subseteq S$, implying that $S = L_1$ and $|S| = k$. However in this case, $\{v_2, v_3, v_4\} \subset D$. If $v_3 \notin D$, then $L_3 \subseteq S$, implying that $S = L_3$ and $|S| = k$. However in this case, $\{v_1, v_2, v_4\} \subset D$. In both cases, $|D| \ge 4$, a contradiction. Therefore, $|S| \ge k + 1$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \ge k + 1$. Conversely, if $S = L_1 \cup L_4$, then S is a non-isolating set of T such that $\gamma_{\rm pr}(T - S) = |\{v_2, v_3\}| < \gamma_{\rm pr}(T)$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S| = k + 1$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = k + 1$. Thus, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) - \mathrm{st}^-_{\gamma_t}(T) = k$. □

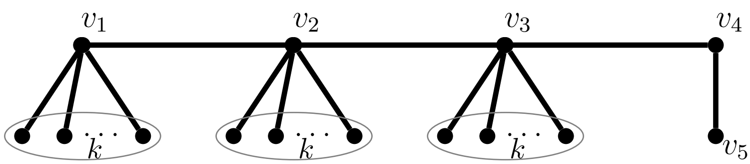

Figure 1: A tree from the family T_k in the proof of Proposition 3.2.

Proposition 3.3. *For $k \ge 1$ an integer, there exist trees T such that $\mathrm{st}^-_{\gamma_t}(T) - \mathrm{st}^-_{\gamma_{\rm pr}}(T) = k$.*

Proof. Let $k \ge 1$ be a given integer, and let $\ell \ge 2k + 1$ be an integer. For $i \in [k]$, let Q_i be obtained from a path $v_{i_1}v_{i_2}v_{i_3}v_{i_4}v_{i_5}$ by attaching ℓ leaf neighbors to each of v_{i_3}, v_{i_4} and v_{i_5}, and let L_{i_3}, L_{i_4} and L_{i_5} be the resulting sets of leaf neighbors of v_{i_3}, v_{i_4} and v_{i_5}, respectively. Let Q be obtained from a path $v_1v_2v_3$ by attaching ℓ leaf neighbors to each of v_1 and v_2, and attaching k leaf neighbors to v_3. Let L_i be the resulting set of leaf neighbors of v_i for $i \in [3]$. Let T be obtained from the disjoint union of the paths $Q, Q_1, \dots, Q_k$ by adding the k edges $v_3v_{i_1}$ for $i \in [k]$. Let A be the set of support vertices of T, and so $|A| = 3(k + 1)$.

Every TD-set of T contains all its support vertices, implying that $\gamma_t(T) \ge |A|$. Since the set A is a TD-set of T, we have $\gamma_t(T) \le |A|$. Consequently, $\gamma_t(T) = |A| = 3(k + 1)$. Every PD-set of T contains the set A and at least one additional vertex from each path Q_i that is a neighbor of v_{i_3} or v_{i_5} for $i \in [k]$, and at least one additional vertex that is a neighbor of v_1 or v_3 since the vertices of every PD-set are paired, implying that $\gamma_{\rm pr}(T) = |A| + k + 1 = 4(k + 1)$.

Let S be a non-isolating set of T such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. If $|S| < k$, then every support vertex of T remains a support vertex of $T - S$, implying that $\gamma_{\rm pr}(T - S) \ge \gamma_{\rm pr}(T)$, a contradiction. Hence, $|S| \ge k$. Conversely, if $S^* = L_3$, then the set $A \setminus \{v_3\}$ of all support vertices of $T - S^*$, together with the vertices v_{i_2} for $i \in [k]$, form a PD-set of $T - S^*$, implying that $\gamma_{\rm pr}(T - S^*) \le 4k + 2 < 4k + 4 = \gamma_{\rm pr}(T)$. Hence, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S^*| = k$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = k$.

We show next that $\mathrm{st}^-_{\gamma_t}(T) = 2k$. Let $A' = A \setminus \{v_3\}$, and so $|A'| = |A| - 1 = 3k + 2$. Let S be a non-isolating set of T such that $\gamma_t(T - S) < \gamma_t(T)$. We show that $|S| \geq 2k$. Suppose, to the contrary, that $|S| \leq 2k - 1$. Let D be a γ_t-set of $T - S$, and so $|D| = \gamma_t(T - S) \leq 3k + 2$. Since $|S| < 2k < \ell$ and each vertex in A' has ℓ leaf neighbors in T, we note that every vertex of A' is a support vertex of $T - S$, implying that $A' \subseteq D$, and so $3k + 2 \geq |D| \geq |A'| = 3k + 2$, implying that $D = A'$. In particular, $v_3 \notin D$, implying that all k leaf neighbors of v_3 belong to S; that is, $L_3 \subseteq S$. If $v_{i_1} \notin S$ for some $i \in [k]$, then in order to totally dominate the vertex v_{i_1}, the vertex $v_{i_2} \in D$, contradicting our earlier observation that $D = A'$. Hence, $v_{i_1} \in S$ for all $i \in [k]$, and so $|S| \geq |L_3| + k = 2k$, a contradiction. Therefore, our original supposition that $|S| \leq 2k - 1$ is incorrect, implying that $|S| \geq 2k$ and $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \geq 2k$. Conversely, if S^* consists of all $2k$ neighbors of v_3 different from v_2 in T, then S^* is a non-isolating set of T such that $\gamma_t(T - S^*) = |A'| < \gamma_t(T)$, implying that $\mathrm{st}^-_{\gamma_t}(T) \leq |S^*| = 2k$. Consequently, $\mathrm{st}^-_{\gamma_t}(T) = 2k$. Thus, $\mathrm{st}^-_{\gamma_t}(T) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = k$. $\square$

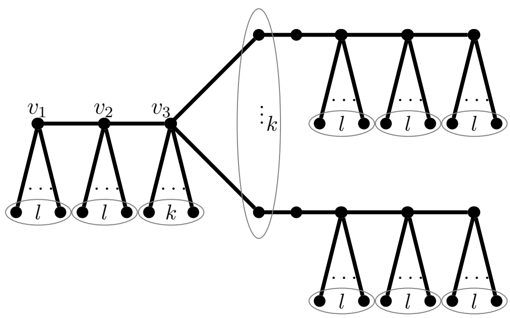

Figure 2: A tree from the family T in the proof of Proposition 3.3.

Theorem 2.1 follows from Propositions 3.2 and 3.3. As further examples, we remark that if P is the Petersen graph, then $\gamma_t(P) = 4$ and $\gamma_{\mathrm{pr}}(P) = 6$. Further, if v is an arbitrary vertex of P, then $\gamma_t(P - v) = 4$, and so $\mathrm{st}^-_{\gamma_t}(P) \geq 2$. Moreover, if S consists of two non-adjacent vertices of P, then $\gamma_t(P - S) = 3$, and so $\mathrm{st}^-_{\gamma_t}(P) \leq 2$. Consequently, $\mathrm{st}^-_{\gamma_t}(P) = 2$. However if v is an arbitrary vertex of P, then $\gamma_{\mathrm{pr}}(P - v) = 4$, implying that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(P) = 1$. Moreover, let G_k be a graph obtained from the Petersen graph by replacing every vertex by a copy of a complete graph K_k for some $k \geq 1$, and adding all edges between two resulting complete graphs that correspond to two vertices of G_k (see Fig. 3). The resulting graph G_k is a $(4k-1)$-regular, $3k$-connected graph that satisfies $\gamma_t(G_k) = 4$ and $\mathrm{st}^-_{\gamma_t}(G_k) = 2k$, and $\gamma_{\mathrm{pr}}(G_k) = 6$ and $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G_k) = k$. This yields the following result.

Proposition 3.4. *For* $k \geq 1$ *an integer, there exists* $(4k - 1)$*-regular,* $3k$*-connected graphs* G *such that* $\mathrm{st}^-_{\gamma_t}(G) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = k$.

4 Properties of paired domination stability

In this section, we present properties of paired domination stability in graphs. We begin with the following property of paired domination in graphs.

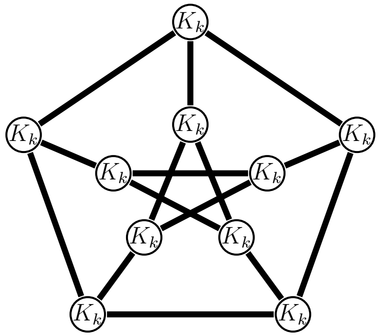

Figure 3: A graph G_k obtained from the Petersen graph by replacing every vertex by K_k.

Proposition 4.1. *Every connected isolate-free graph G contains a spanning tree T such that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$.*

Proof. Since adding edges to a graph cannot increases its paired domination number, if T is an isolate-free spanning subgraph of a graph G, then $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(T)$. Let D be a $\gamma_{\rm pr}$-set of G, and so D is a PD-set of G and $|D| = \gamma_{\rm pr}(G)$. Let M be a perfect matching in the subgraph $G[D]$ induced by D. Let T' be a spanning subgraph of G that consists of the edges in M and for each vertex v outside D, an edge of G that joins v to exactly one vertex of the dominating set D. If the resulting spanning subgraph T' is a tree, then we let $T = T'$. Otherwise, if the resulting spanning subgraph T' is a forest with $\ell \ge 2$ components, then we add $\ell - 1$ edges from the edge set of the graph G between these components, avoiding cycles, to construct a tree, which we call T. Since D is a PD-set in the resulting tree T, we note that $\gamma_{\rm pr}(T) \le |D| = \gamma_{\rm pr}(G)$. Since T is an isolate-free spanning subgraph of G, we have $\gamma_{\rm pr}(T) \ge \gamma_{\rm pr}(G)$. Consequently, T is a spanning tree of G satisfying $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$. □

By our earlier convention, if G is a graph of order n and $\gamma_{\rm pr}(G) = 2$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = n$ since removing all vertices from the graph G produces the null graph with paired domination number zero. We are therefore only interested in the $\gamma^-_{\rm pr}$-stability of graphs with paired domination number at least 4. If G is a graph with $\gamma_{\rm pr}(G) \ge 4$ where x and y are adjacent vertices in G, then $D = V(G) \setminus \{x, y\}$ belongs to the set $\mathrm{NI}(G)$ and $\gamma_{\rm pr}(G - D) = \gamma_{\rm pr}(K_2) = 2 < \gamma_{\rm pr}(G)$. This yields the following result.

Observation 4.2. Every isolate-free graph G of order n with $\gamma_{\rm pr}(G) \ge 4$ satisfies $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le n - 2$.

Proposition 4.3. *If T is a spanning tree of a connected graph G such that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \ge \mathrm{st}^-_{\gamma_{\rm pr}}(G)$.*

Proof. Let S be a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}^-_{\gamma_{\rm pr}}(T)$ such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. Since $\gamma_{\rm pr}(G - S) \le \gamma_{\rm pr}(T - S)$ and $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$, the set S is a non-isolating set of G such that $\gamma_{\rm pr}(G - S) < \gamma_{\rm pr}(G)$. Hence, $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le |S| = \mathrm{st}^-_{\gamma_{\rm pr}}(T)$. □

The following result shows that to determine the $\gamma^-_{\rm pr}$-stability of a graph G, it is not sufficient to only examine spanning trees T of G satisfying $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$.

Proposition 4.4. *For $k \geq 1$ an integer, there exist connected graphs G such that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = k$ for every spanning tree T of G with $\gamma_{\mathrm{pr}}(T) = \gamma_{\mathrm{pr}}(G)$.*

Proof. For $k \geq 1$, let F be obtained from two vertex disjoint copies of $K_{2,k+1}$ by identifying a vertex of degree $k+1$ from each copy. Let u be the resulting identified vertex of degree $2(k+1)$, and let w_1 and w_2 be the two vertices of degree $k+1$ in F. Further, let v_i be a common neighbor (of degree 2) of u and w_i for $i \in [2]$. Let G be obtained from F by adding a leaf neighbor x_i to w_i for $i \in [2]$. Thus, $\mathrm{diam}(G) = 6$ and $x_1w_1v_1uv_2w_2x_2$ is a shortest path in G of length 6. The graph G satisfies $\gamma_{\mathrm{pr}}(G) = 4$. We remark that only connected graphs of $\mathrm{diam}(G) \leq 3$ have $\gamma_{\mathrm{pr}}(G) = 2$. Therefore, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) \geq 3$. Moreover, the set $S = \{w_1, x_1, x_2\}$ is a non-isolating set of minimum cardinality satisfying $\gamma_{\mathrm{pr}}(G - S) = 2 < \gamma_{\mathrm{pr}}(G)$, and so $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = 3$. However, the vertex u must have degree 2 in every spanning tree T of G for which $\gamma_{\mathrm{pr}}(T) = \gamma_{\mathrm{pr}}(G) = 4$, implying that the vertices w_1 and w_2 each have $k+1$ leaf neighbors in T. This implies that every non-isolating set of T that decreases the paired domination number contains at least $k+3$ vertices. The set $S = N_T[w_1]$ is a non-isolating set of minimum cardinality satisfying $\gamma_{\mathrm{pr}}(T - S) = 2 < \gamma_{\mathrm{pr}}(T)$, and so $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq |S| = k+3$. Consequently, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = k+3$, and so $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = k$. □

Proposition 4.5. *If S is a $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}$-set of a connected isolate-free graph G with $\gamma_{\mathrm{pr}}(G) \geq 4$, then $\gamma_{\mathrm{pr}}(G - S) = \gamma_{\mathrm{pr}}(G) - 2$.*

Proof. Let S be a $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}$-set of G. Suppose, to the contrary, that $\gamma_{\mathrm{pr}}(G - S) \leq \gamma_{\mathrm{pr}}(G) - 4$. By the connectivity of G, there exists a vertex $u \in S$ that has a neighbor in the set $V(G) \setminus S$. We now consider the set $S' = S \setminus \{u\}$. Let D be a γ_{pr}-set of $G - S$. If u has a neighbor in D, then D is a γ_{pr}-set of $G - S'$, implying that $\gamma_{\mathrm{pr}}(G - S') \leq |D| = \gamma_{\mathrm{pr}}(G - S) \leq \gamma_{\mathrm{pr}}(G) - 4$, contradicting our choice of the set S. Hence, u has no neighbor in D. Let v be an arbitrary neighbor of u that belongs to $V(G) \setminus S$. The set $D \cup \{u, v\}$ is a PD-set of $G - S'$ with u and v paired, and with the pairings of the vertices of D unchanged from their pairings in $G - S$. Hence, $\gamma_{\mathrm{pr}}(G - S') \leq |D| + 2 \leq \gamma_{\mathrm{pr}}(G) - 2$, once again contradicting our choice of the set S. □

5 Paths and cycles

It is well known (see, for example, [11]) that for $n \geq 3$ we have $\gamma_{\mathrm{pr}}(C_n) = \gamma_{\mathrm{pr}}(P_n) = 2\lceil \frac{n}{4} \rceil$. In this section, we determine the paired domination stability of paths and cycles. The proofs require a detailed case analysis, which is straightforward albeit tedious. We therefore omit the proofs in this section. The γ^-_{pr}-stability of a path P_n and a cycle C_n on n vertices is given by the following result.

Theorem 5.1. *If G is a path P_n, for $n \geq 2$, or a cycle C_n, for $n \geq 3$, then*

$$\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = \begin{cases} 1 & \textit{when } n \equiv 1 \pmod 4 \\ 2 & \textit{when } n \equiv 2 \pmod 4 \\ 3 & \textit{when } n \equiv 3 \pmod 4 \\ 4 & \textit{when } n \equiv 0 \pmod 4. \end{cases}$$

Next we determine the γ^+_{pr}-stability of a path P_n. For $n \leq 10$ with $n \neq 8$ and for $n = 13$, no non-isolating set of vertices in a path P_n exists whose removal increases the paired domination number, and hence, by definition, $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(P_n) = \infty$ for such values of n. It is therefore only of interest to determine the γ^+_{pr}-stability of a path P_n, where $n \geq 8$ and $n \notin \{9, 10, 13\}$.

Theorem 5.2. *For $n \geq 8$ and $n \notin \{9, 10, 13\}$,*

$$\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(P_n) = \begin{cases} 1 & \textit{when } n \pmod 4 \in \{0, 3\} \\ 2 & \textit{when } n \pmod 4 \in \{1, 2\}. \end{cases}$$

As a consequence of Theorems 5.1 and 5.2, the paired domination stability of a path is determined.

Corollary 5.3. *For $n \geq 2$,*

$$\mathrm{st}_{\gamma_{\mathrm{pr}}}(P_n) = \begin{cases} 1 & \textit{when } n \pmod 4 \in \{0, 1, 3\} \textit{ and } n \notin \{3, 4, 7\} \\ 2 & \textit{when } n \equiv 2 \pmod 4 \\ 3 & \textit{when } n \in \{3, 7\} \\ 4 & \textit{when } n = 4. \end{cases}$$

We next consider the γ^+_{pr}-stability of a cycle C_n. As shown in Theorem 5.1, the γ^-_{pr}-stability of a path and a cycle of the same order are equal. This is not always the case for the γ^+_{pr}-stability of a path and a cycle. For example, $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(P_{12}) = 1$ and $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(C_{12}) = 2$. Analogously as in the case of paths, for small values of the order of a cycle the γ^+_{pr}-stability is infinite. Namely, for $n \leq 14$ with $n \neq 12$ and $n = 17$ we have that $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(C_n) = \infty$. The following result determines the γ^+_{pr}-stability of a cycle of large order.

Theorem 5.4. *For $n \geq 12$ and $n \notin \{13, 14, 17\}$,*

$$\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(C_n) = \begin{cases} 2 & \textit{when } n \equiv 0 \pmod 4 \\ 3 & \textit{when } n \pmod 4 \in \{2, 3\} \\ 4 & \textit{when } n \equiv 1 \pmod 4). \end{cases}$$

As a consequence of Theorems 5.1 and 5.4, the paired domination stability of a cycle is determined.

Corollary 5.5. *For $n \geq 3$,*

$$\mathrm{st}_{\gamma_{\mathrm{pr}}}(C_n) = \begin{cases} 1 & \textit{when } n \equiv 1 \pmod 4 \\ 2 & \textit{when } n \pmod 4 \in \{0, 2\} \textit{ and } n \notin \{4, 8\} \\ 3 & \textit{when } n \equiv 3 \pmod 4 \\ 4 & \textit{when } n \in \{4, 8\}. \end{cases}$$

6 Trees

In this section, we first determine the γ_{pr}-stability of trees in the family $\mathcal{F}_\Delta$ and a new family $\mathcal{E}_\Delta$.

Lemma 6.1. *For $\Delta \geq 2$, if $T \in \mathcal{F}_\Delta$, then $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta$.*

Proof of Lemma 6.1. Let T be an arbitrary tree in the family $\mathcal{F}_{k,\Delta}$ for some $k \geq 2$ and $\Delta \geq 2$. We show that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta$. The family $\mathcal{F}_{k,2}$ consists of all paths P_{4k} where $k \geq 2$. Therefore by Theorem 5.1, we have $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 4 = 2\Delta$ for each $T \in \mathcal{F}_{k,2}$, which yields the desired result. Hence, we may assume that $\Delta \geq 3$. We show, by induction on $k \geq 2$, that every tree T in the family $\mathcal{F}_{k,\Delta}$ satisfies $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta$.

Suppose $k = 2$, and so $T \in \mathcal{F}_{2,\Delta}$ (where recall that $\Delta \geq 3$). The tree T can therefore be constructed from two vertex disjoint double stars T_1 and T_2, where $T_i \cong S(\Delta-1, \Delta-1)$ for $i \in [2]$, by selecting leaves w_1 and w_2 of T_1 and T_2, respectively, and adding the edge w_1w_2 to $T_1 \cup T_2$. Let x_i and y_i be the two vertices of T_i that are not leaves, where x_iw_i is an edge. We note that $y_1x_1w_1w_2x_2y_2$ is a path in T. We note that $\gamma_{\mathrm{pr}}(T) = 4$ and the set $\{x_1, x_2, y_1, y_2\}$ is a γ_{pr}-set of T.

Let S be a $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}$-set of G. Thus, S is a set in $\mathrm{NI}(G)$ with $|S| = \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G)$ such that $\gamma_{\mathrm{pr}}(T - S) = 2$. Let R be a γ_{pr}-set of $T - S$, and so R is a minimum PD-set of $T - S$ (of cardinality 2). Since $T[R] = P_2$, we note that $T - S$ is a tree of diameter at most 3. This implies that at most one of x_i and y_{3-i} belong to $T - S$ for $i \in [2]$. Thus, $|S \cap \{x_i, y_{3-i}\}| \geq 1$ for $i \in [2]$.

Suppose that $y_1 \in S$ and $x_2 \in S$. If $x_1 \in S$, then all leaf neighbors of y_1, x_1 and x_2 belong to S, while if $y_2 \in S$, then all leaf neighbors of y_1, y_2 and x_2 belong to S. In both cases, $|S| \geq 3\Delta - 2 > 2\Delta$.

Suppose that $y_1 \in S$ and $x_2 \notin S$. If $y_2 \in S$, then all leaf neighbors of y_1 and y_2 belong to S, implying that $|S| \geq 2\Delta$. If $y_2 \notin S$, then $x_1 \in S$, implying that S contains all leaf-neighbors of y_1 and x_1, and so $|S| \geq 2\Delta - 1$. However if in this case $|S| = 2\Delta - 1$, implying that $\mathrm{diam}(T - S) \geq 4$, a contradiction. Hence, $|S| \geq 2\Delta$.

Suppose that $y_1 \notin S$ and $x_2 \in S$. Since $T - S$ is a tree, $y_2 \in S$ and all leaf neighbors of y_2 and x_2 belong to S, implying that $|S| \geq 2\Delta - 1$. However if in this case $|S| = 2\Delta - 1$, then S contains x_2 and all leaf neighbors of y_1, implying that $\mathrm{diam}(T - S) \geq 4$, a contradiction. Hence, $|S| \geq 2\Delta$. Therefore, in all three cases we have $|S| \geq 2\Delta$, as desired. This proves the base case when $k = 2$.

For the inductive hypothesis, let $k \geq 3$ and assume that if $T' \in \mathcal{F}_{k',\Delta}$ where $2 \leq k' < k$, then $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T') = 2\Delta$. We now consider a tree T in the family $\mathcal{F}_{k,\Delta}$. Therefore, the tree T can be constructed from k vertex disjoint double stars $H_1, \ldots, H_k$, where $H_i \cong S(\Delta - 1, \Delta - 1)$ for $i \in [k]$, by selecting one leaf y_i from each double star H_i and adding $k - 1$ edges between vertices in $\{y_1, \ldots, y_k\}$ in such a way that the resulting graph is a tree with maximum degree Δ. Let w_i and x_i be the two (adjacent) vertices of H_i that are not leaves for $i \in [k]$, where y_i is a leaf neighbor of x_i for $i \in [k]$. We note that $\gamma_{\mathrm{pr}}(T) = 2k$ and the set $\cup_{i=1}^{k}\{w_i, x_i\}$ is the unique γ_{pr}-set of T.

Let U be the graph of order k whose vertices correspond to the k double stars $H_1, \ldots, H_k$ where two vertices are adjacent in U if and only if the corresponding double stars are joined by an edge in T. We call U the underlying graph of T. By construction, the graph U is a tree, noting that T is a tree. Let $V(U) = \{u_1, \ldots, u_k\}$ where u_i is the vertex of U corresponding to the double star H_i for $i \in [k]$. Renaming the double stars if necessary, we may assume that u_1 is a leaf in U, and that H_1 is joined to H_2 in T. Thus, $y_1y_2 \in E(T)$ and $y_1y_j \notin E(T)$ for $j \in [k] \setminus [2]$. We note that $w_1x_1y_1y_2x_2w_2$ is a path in T. Let $T' = T - V(H_1)$. By construction, the tree T' belongs to the family $\mathcal{F}_{k',\Delta}$ where $k' = k - 1 \geq 2$. By induction, we have $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T') = 2\Delta$.

Let S be a $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T)$ such that $\gamma_{\mathrm{pr}}(T - S) \leq \gamma_{\mathrm{pr}}(T) - 2 = 2k - 2$. Let Q be a γ_{pr}-set of $T - S$, and so $|Q| \leq 2k - 2$.

Let $Q' = Q \cap V(T')$ and $S' = S \cap V(T')$. For $i \in [k]$, let $Q_i = Q \cap V(H_i)$ and $S_i = S \cap V(H_i)$. We proceed further with the following claim.

Claim 6.2. $|S| \geq 2\Delta$.

Proof of Claim 6.2. Suppose, to the contrary, that $|S| \leq 2\Delta - 1$.

Subclaim 6.2.1. $|Q_1| \geq 2$.

Proof of Subclaim 6.2.1. Suppose, to the contrary, that $|Q_1| \leq 1$. Suppose that $Q_1 = \emptyset$. In this case, $V(H_1) \setminus \{y_1\} \subseteq S_1$. If $y_1 \in S_1$, then $|S_1| = 2\Delta > |S|$, a contradiction. Hence, $y_1 \notin S_1$, and so $2\Delta - 1 \geq |S| \geq |S_1| = 2\Delta - 1$, implying that $S = S_1$ and $|S| = 2\Delta - 1$. In this case, a $\gamma_{\rm pr}$-set of $T - S$ contains at least one of y_1 and y_2. Since the set $\cup_{i=2}^{k}\{w_i, x_i\}$ is the unique $\gamma_{\rm pr}$-set of T', a $\gamma_{\rm pr}$-set of $T - S$ is therefore not a $\gamma_{\rm pr}$-set of T', and so $\gamma_{\rm pr}(T - S) \geq \gamma_{\rm pr}(T') + 2 = 2(k-1) + 2 = 2k$, a contradiction. Hence, $|Q_1| \geq 1$.

By supposition, $|Q_1| \leq 1$. Consequently, $|Q_1| = 1$, implying that $Q_1 = \{y_1\}$ and $V(H_1) \setminus \{x_1, y_1\} \subseteq S_1$, and so $|S_1| \geq 2\Delta - 2$. If $x_1 \in S_1$, then $|S_1| = 2\Delta - 1$ and we end up in the previous case, which leads to a contradiction. Hence, $x_1 \notin S_1$ and $x_1 \notin Q_1$, implying that $y_2 \in Q$ with the vertices y_1 and y_2 paired in Q, and $|S_1| = 2\Delta - 2$. By supposition, $|S| \leq 2\Delta - 1$. If $|S| = 2\Delta - 2$, then $S = S_1$ and $\gamma_{\rm pr}(T - S) \geq \gamma_{\rm pr}(T') + 2 = 2k$, a contradiction. Hence, $|S| = 2\Delta - 1$, and so the set S contains a vertex $v' \in V(T') \setminus \{y_2\}$. However noting that $\Delta \geq 3$, every non-isolating set of vertices of $T' - y_2$ that decreases the paired domination number cannot contain only one vertex, implying that $\gamma_{\rm pr}(T - S) \geq |\{y_1, y_2\}| + \gamma_{\rm pr}(T' - y_2) = 2 + \gamma_{\rm pr}(T') = 2k$, a contradiction. □

Subclaim 6.2.2. $\{x_1, y_1\} \subseteq Q$.

Proof of Subclaim 6.2.2. Suppose, to the contrary, that $y_1 \notin Q_1$, implying that $S' \in \mathrm{NI}(T')$. Recall that S is a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T and $|S'| \leq |S| \leq 2\Delta - 1$. However, $\mathrm{st}^-_{\gamma_{\rm pr}}(T') = 2\Delta$. Therefore, $\gamma_{\rm pr}(T' - S') \geq \gamma_{\rm pr}(T') = 2(k-1)$. Hence, $\gamma_{\rm pr}(T - S) = \gamma_{\rm pr}(T' - S') + |Q_1| \geq 2(k-1) + 2 = 2k$, a contradiction. Hence, $y_1 \in Q_1$.

Suppose, to the contrary, that $x_1 \notin Q_1$. Thus, all $\Delta - 2$ leaf-neighbors of x_1 belong to the set S_1. By Claim 6.2.1, we have $|Q_1| \geq 2$. Hence, the set Q_1 contains w_1 and one of its leaf-neighbor w_1'. We now consider the set $S^* = S \setminus S_1$. Since $S^* \in \mathrm{NI}(T)$ and $(Q \setminus \{w_1'\}) \cup \{x_1\}$ is a PD-set of $T - S^*$, we have $\gamma_{\rm pr}(T - S^*) \leq |Q| = \gamma_{\rm pr}(T - S)$, contradicting our choice of the set S. Hence, $x_1 \in Q_1$. □

Subclaim 6.2.3. $w_1 \notin Q_1$.

Proof of Subclaim 6.2.3. Suppose, to the contrary, that $w_1 \in Q_1$. Hence, $\{w_1, x_1, y_1\} \subseteq Q_1$, and so $S \cap V(H_1) = \emptyset$ by the minimality of S. Thus, $S = S'$ and therefore $|S'| \leq 2\Delta - 1$.

We show firstly that x_1 and y_1 are paired in Q. Suppose, to the contrary, that x_1 and y_1 are not paired in Q. This implies that $y_2 \in Q$, and that y_1 and y_2 are paired in Q. Suppose that $x_2 \notin S$, implying that $S' \in \mathrm{NI}(T')$. By the minimality of the set Q, we have $x_2 \notin Q$. Thus, the set $Q' \cup \{x_2\}$ is a PD-set of $T' - S'$, and so $|Q'| + 1 = |Q' \cup \{x_2\}| \geq \gamma_{\rm pr}(T' - S') \geq \gamma_{\rm pr}(T') = 2(k-1)$. Hence, $|Q| = |Q_1| + |Q'| \geq 3 + (2k-3) = 2k = \gamma_{\rm pr}(T)$, a contradiction. Hence, $x_2 \in S$. We now consider the set $S^* = S \setminus \{x_2\}$. We note that S^* is a non-isolating set of vertices of T, and the set Q is a PD-set of $T - S^*$. Thus,

$\gamma_{\rm pr}(T - S^*) \le |Q| \le 2k - 2$, which contradicts our choice of the set S. Hence, x_1 and y_1 are paired in Q.

Since x_1 and y_1 are paired in Q, the vertex w_1 is paired with one of its leaf neighbors, say w_1'. By the minimality of Q we note that $Q_1 = \{w_1, w_1', x_1, y_1\}$. If $x_2 \in Q$, then the set $Q \setminus \{w_1', y_1\}$ is a PD-set of $T - S$ (with w_1 and x_1 paired), contradicting the minimality of Q. Hence, $x_2 \notin Q$. This in turn implies that $y_2 \notin Q$. If $y_2 \in S$, then once again we contradict the minimality of Q. Therefore, $y_2 \notin S$. We remark, though, that possibly $x_2 \in S$. Recall that by our earlier observations, $S = S'$.

Let $S'' = S \setminus \{x_2\}$. Thus, if $x_2 \notin S$, then $S'' = S$, while if $x_2 \in S$, then $S'' = S \setminus \{x_2\}$. The set S'' is a non-isolating set of T' such that $|S''| \le |S| \le 2\Delta - 1$. As observed earlier, $y_2 \notin Q'$ and $x_2 \notin Q'$. The set $Q' \cup \{y_2, x_2\}$ is a PD-set of $T' - S''$, implying that $|Q'| + 2 \ge \gamma_{\rm pr}(T' - S'') \ge \gamma_{\rm pr}(T') = 2(k-1)$. Hence, $|Q'| \ge 2k - 4$, and so $|Q| = |Q_1| + |Q'| \ge 4 + (2k-4) = 2k$, contradicting the fact that $|Q| \le 2k - 2$. □

Proof of Claim 6.2*, continued:* By Claim 6.2.3, $w_1 \notin Q_1$. This implies that $Q_1 = \{x_1, y_1\}$. The set S_1 therefore consists of the $\Delta - 1$ leaf neighbors of w_1, and so $|S_1| = \Delta - 1$. This is true for every leaf in the tree U. Hence, if u_i is a leaf in U for some $i \in [k]$, then in the corresponding double star H_i of T we have $Q_i = \{x_i, y_i\}$ and $|S_i| = \Delta - 1$. Further, the set S_i consists of the $\Delta - 1$ leaf neighbors of w_i. In particular, $|Q_1| = 2$ and $|S_1| = \Delta - 1$. Since the underlying tree U of T has order $k \ge 3$, there are at least two leaves in U. Thus, u_p is a leaf in U for some $p \in [k] \setminus \{1\}$, implying that $|Q_p| = 2$ and $|S_p| = \Delta - 1$.

If $|Q_i| \ge 2$ for all $i \in [k]$, then $|Q| \ge 2k$, a contradiction. Hence, $|Q_q| \le 1$ for some $q \in [k]$. By our earlier observations, u_q is not a leaf in the tree U, and so $q \notin \{1, p\}$. If $|Q_q| = 0$, then $\{w_q, x_q\} \subseteq S_q$, and so $|S_q| \ge 2$ (in fact, $|S_q| \ge 2\Delta - 1$) and $|S| \ge |S_1| + |S_p| + |S_q| \ge (\Delta - 1) + (\Delta - 1) + 2 = 2\Delta$, a contradiction. Hence, $|Q_q| = 1$, implying that $Q_q = \{y_q\}$ and $w_q \in S_q$, and so $|S_q| \ge 1$. Since the paired dominating number is an even integer and $|Q| \le 2k$, there exists $r \in [k] \setminus \{1, p, q\}$ such that $|Q_r| = 1$. Therefore, $Q_r = \{y_r\}$ and $|S_r| \ge 1$. Hence, $|S| \ge |S_1| + |S_p| + |S_q| + |S_r| \ge (\Delta - 1) + (\Delta - 1) + 1 + 1 = 2\Delta$, a contradiction. This completes the proof of Claim 6.2. □

Proof of Lemma 6.1*, continued:* By Claim 6.2, we have $|S| \ge 2\Delta$. By our choice of the set S, this implies that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = |S| \ge 2\Delta$. Conversely, if we consider the set $S = V(H_1)$, then $S \in \mathrm{NI}(T)$ satisfies $|S| = 2\Delta$ and $\gamma_{\rm pr}(T - S) = \gamma_{\rm pr}(T') = 2k - 2 < \gamma_{\rm pr}(T)$, and so $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le 2\Delta$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = 2\Delta$. This completes the proof of Lemma 6.1. □

We determine next the $\gamma^+_{\rm pr}$-stability of a tree in the family $\mathcal{F}_\Delta$.

Lemma 6.3. *For $\Delta \ge 2$, if $T \in \mathcal{F}_\Delta$, then* $\mathrm{st}^+_{\gamma_{\rm pr}}(T) \le \Delta - 1$.

Proof. Let T be an arbitrary tree in the family $\mathcal{F}_{k,\Delta}$ for some $k \ge 2$ and $\Delta \ge 2$. We use the same notation as in the proof of Lemma 6.1. In particular, $\gamma_{\rm pr}(T) = 2k$ and H_1 corresponds to a leaf u_1 in the underlying tree U of T. Moreover, y_1y_2 is the edge joining H_1 and H_2 in T. Also, w_i and x_i are the support vertices in the double star H_i and $w_ix_iy_i$ is a path in H_i for $i \in [k]$. Let L be the set of $\Delta - 2$ leaf neighbors of x_1 in T, and let $S = L \cup \{x_1\}$. We resulting set $S \in NI(T)$ and the forest $T - S$ has two components, say F_1 and F_2 where $w_1 \in V(F_1)$ and $y_1 \in V(F_2)$. Moreover, $\gamma_{\rm pr}(T - S) = \gamma_{\rm pr}(F_1) + \gamma_{\rm pr}(F_2) = 2 + 2k > \gamma_{\rm pr}(T)$. Therefore, $\mathrm{st}^+_{\gamma_{\rm pr}}(T) \le |S| = \Delta - 1$. □

Recall that by Proposition 3.1, for $\Delta \geq 3$, if $T \in \mathcal{H}_\Delta$, then $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$. Further we remark that $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(T) = \infty$. We next define another family of trees T with maximum degree Δ such that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$. For integers $\Delta \geq 3$ and $\Delta - 1 \geq k \geq 3$, let $E_{k,\Delta}$ be a graph obtained from the path P_2 with vertices u and v and the disjoint union of $2k$ double stars $S(\Delta - 1, \Delta - 1)$ by selecting one leaf from each double star and identifying half of the selected leaves with the vertex v and the other half of the selected leaves with the vertex u (see Figure 4). Let

$$\mathcal{E}_\Delta = \bigcup_{k \geq 3} E_{k,\Delta}.$$

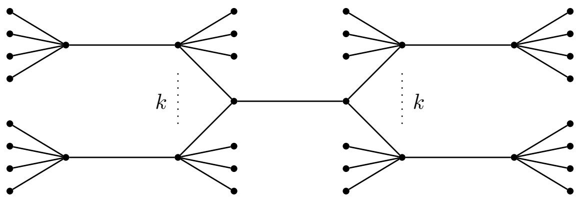

Figure 4: A tree $E_{k,5}$ from the family $\mathcal{E}_5$.

If T is a tree from the family $\mathcal{E}_\Delta$, then $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$. Moreover, if T is isomorphic to the graph $E_{k,\Delta}$, then $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(T) = k(\Delta - 1)$. In contrast to the family $\mathcal{H}_\Delta$, the trees from the family $\mathcal{E}_\Delta$ have finite γ^+_{pr}-stability.

7 Proof of Theorem 2.2

In this section we present a proof of Theorem 2.2, which we restate below.

Theorem 2.2. *If T is a tree with maximum degree Δ satisfying $\gamma_{\mathrm{pr}}(T) \geq 4$, then the following hold.*

(a) $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta$, *with equality if and only if $T \in \mathcal{F}_\Delta$.*

(b) $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta - 1$, *and this bound is sharp for all $\Delta \geq 2$.*

Proof. We first prove the statement given in part (a). Since $\gamma_{\mathrm{pr}}(T) \geq 4$, we have $\Delta \geq 2$. If $\Delta = 2$, then G is a path P_n of order $n \geq 5$. In this case, the family $\mathcal{F}_{k,\Delta} = \{P_n \colon n \equiv 0 \pmod 4 \text{ and } n \geq 8\}$, and Theorem 5.1 and Lemma 6.1 imply the desired result. Suppose, therefore, that $\Delta \geq 3$. The sufficiency of part (a) follows from Lemma 6.1. To prove the necessity, let T be a tree with maximum degree $\Delta \geq 3$ satisfying $\gamma_{\mathrm{pr}}(T) \geq 4$. Let $d = \mathrm{diam}(T)$, and so $d \geq 4$. Let $P \colon v_0 v_1 \ldots v_d$ be a diametral path in G. Thus, v_0 and v_d are leaves in T and $d(v_0, v_d) = \mathrm{diam}(G)$. We now consider the tree T rooted at the vertex v_d. Let D be a γ_{pr}-set of T.

Suppose that there is a child u_1 of v_2 that is a support vertex in T where $u_1 \neq v_1$. Let u_0 be a leaf neighbor of u_1. Since every PD-set of T contains all support vertices, we have $\{v_1, u_1\} \subset D$. Renaming vertices if necessary, we may assume that u_0 and u_1 are paired in D. Thus, if S consists of the vertex u_1 and all leaf neighbors of u_1, then $S \in \mathrm{NI}(T)$ and $\gamma_{\mathrm{pr}}(T - S) \leq |D| - 2 = \gamma_{\mathrm{pr}}(T) - 2$. Hence, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq |S| \leq \Delta < 2\Delta - 1$, and the desired result follows. Assume, therefore, that every child of v_2 different from v_1 is a leaf.

Suppose that there is a $\gamma_{\rm pr}$-set, $D_{2,3}$, of T such that v_2 and v_3 are paired in $D_{2,3}$. Necessarily, $v_1 \in D_{2,3}$ and v_1 is paired in $D_{2,3}$ with one of its leaf neighbors. Let S consist of the vertex v_1 and all of its leaf neighbors. Thus, $S \in {\rm NI}(T)$ and $\gamma_{\rm pr}(T-S) \le |D_{2,3}| - 2 = \gamma_{\rm pr}(T) - 2 < \gamma_{\rm pr}(T)$, implying that ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| \le \Delta < 2\Delta - 1$, once again implying the desired result. Therefore, we may assume that in every $\gamma_{\rm pr}$-set of T the vertices v_2 and v_3 are not paired.

Suppose that there is a $\gamma_{\rm pr}$-set, D_3, of T which contains a neighbor of v_3 different from v_2. In this case, if S consists of the vertex v_2 and all its descendants, then $|S| \le 2\Delta - 1$, $S \in {\rm NI}(T)$ and $\gamma_{\rm pr}(T-S) \le |D_3| - 2 = \gamma_{\rm pr}(T) - 2 < \gamma_{\rm pr}(T)$, noting that the set $D_3 \setminus S$ is a PD-set of $T - S$ and, by the minimality of D_3 we have $|D_3 \cap S| = 2$. Thus, ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| \le 2\Delta - 1$, and the desired result follows. Hence, we may assume that every $\gamma_{\rm pr}$-set of T contains the vertex v_2 but no other vertex in $N[v_3]$. In particular, $N[v_3] \cap D = \{v_2\}$.

Suppose that $d_T(v_1) < \Delta$ or $d_T(v_2) < \Delta$. Thus, $d_T(v_1) + d_T(v_2) \le 2\Delta - 1$. In order to dominate the vertex v_0, we have $v_1 \in D$. By our earlier assumptions, $v_2 \in D$ and every child of v_2 different from v_1 is a leaf. Thus by the minimality of the set D, the vertex v_1 is the only descendant of v_2 that belongs to the set D, and the vertices v_1 and v_2 are paired in D. Hence, if $S = N[v_2] \cup N[v_1]$, then $S \in {\rm NI}(T)$ and $|S| = d_T(v_1) + d_T(v_2) \le 2\Delta - 1$. Further, $D \setminus \{v_1, v_2\}$ is a PD-set of $T - S$, and so $\gamma_{\rm pr}(T-S) \le |D| - 2 = \gamma_{\rm pr}(T) - 2$, implying that ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| \le 2\Delta - 1$, yielding the desired result. Hence, we may assume that $d_T(v_1) = d_T(v_2) = \Delta$.

Suppose that $d_T(v_3) \ge 3$, and let u_2 be a child of v_3 different from v_2. If u_2 is a leaf, then v_3 belongs to every $\gamma_{\rm pr}$-set of T, while if u_2 is not a leaf, then from the structure of the rooted tree T the vertex u_2 can be chosen to belong to some $\gamma_{\rm pr}$-set of T. In both cases, we contradict our earlier assumption that every $\gamma_{\rm pr}$-set of T contains the vertex v_2 but no other vertex in $N[v_3]$. Hence, $d_T(v_3) = 2$. We now let $S = N[v_1] \cup N[v_2]$, and so $S \in {\rm NI}(T)$ and $|S| = d_T(v_1) + d_T(v_2)$. By our earlier observations, $|S| = 2\Delta$ and $\gamma_{\rm pr}(T-S) = \gamma_{\rm pr}(T) - 2 < \gamma_{\rm pr}(T)$, implying that ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta$. This proves the desired upper bound.

We show next that if we have equality in the upper bound in part (a), then $T \in \mathcal{F}_\Delta$. Let ${\rm st}^-_{\gamma_{\rm pr}}(T) = 2\Delta$. By our earlier observations, we have that every child of v_2 different from v_1 is a leaf. Further, $d_T(v_1) = d_T(v_2) = \Delta$ and $d_T(v_3) = 2$. We now re-root the tree T at the vertex v_0, thereby interchanging the roles of v_0 and v_d. Identical arguments as before show that every child of v_{d-2} different from v_{d-1} is a leaf. Further, $d_T(v_{d-1}) = d_T(v_{d-2}) = \Delta$ and $d_T(v_{d-3}) = 2$. In particular, $d \ge 6$.

Suppose that $d = 6$, and so $v_{d-3} = v_3$. In this case, the tree T is determined and $\gamma_{\rm pr}(T) = 4$. Letting $S = (N[v_1] \cup N[v_2]) \setminus \{v_3\}$, we have $S \in {\rm NI}(T)$ and $|S| = d_T(v_1) + d_T(v_2) - 1 = 2\Delta - 1$. Further, $\gamma_{\rm pr}(T-S) = 2 < \gamma_{\rm pr}(T)$. Therefore, ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta - 1$, a contradiction. Hence, $d \ge 7$, and so $v_{d-3} \ne v_3$.

We now consider the tree $T' = T - (N[v_1] \cup N[v_2])$. If $\gamma_{\rm pr}(T') = 2$, then by our earlier observations, we have $d = 7$ and $T' \cong S(\Delta - 1, \Delta - 1)$ where v_{d-1} and v_{d-2} are the two (adjacent) vertices in T' that are not leaves. Therefore, $T \in T_{2,\Delta}$, and so $T \in T_\Delta$. Hence, we may assume that $\gamma_{\rm pr}(T') \ge 4$, for otherwise the desired characterization follows. In particular, $d \ge 8$. As observed earlier, $d_T(v_{d-1}) = d_T(v_{d-2}) = \Delta$, implying that $\Delta(T') = \Delta$ and ${\rm st}^-_{\gamma_{\rm pr}}(T') \le 2\Delta$.

Let D be a $\gamma_{\rm pr}$-set of T. Since every PD-set of T contains the set of support vertices, we note that $v_1, v_2 \in D$. By the minimality of D, no leaf-neighbor of v_1 or v_2 belongs to

D. If $v_3 \in D$, then $v_4 \in D$ (with v_3 and v_4 paired in D). However in this case, we can replace v_3 in D with an arbitrary neighbor of v_4 that does not belong to D. Hence, we can choose the $\gamma_{\rm pr}$-set D of T so that $v_3 \notin D$. The resulting set D when restricted to $V(T')$ is a PD-set of T', implying that $\gamma_{\rm pr}(T') \le |D| - 2 = \gamma_{\rm pr}(T) - 2$. Conversely, every PD-set of T' can be extended to a PD-set of T by adding to it the vertices v_1 and v_2 (with v_1 and v_2 paired), and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2$. Consequently, $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(T') + 2$.

Suppose that $\mathrm{st}^-_{\gamma_{\rm pr}}(T') < 2\Delta$. Let S' be a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T'. Thus, S is a set in $\mathrm{NI}(T')$ with $|S'| = \mathrm{st}^-_{\gamma_{\rm pr}}(T') < 2\Delta$ such that $\gamma_{\rm pr}(T - S') < \gamma_{\rm pr}(T')$. If D' is a $\gamma_{\rm pr}$-set of $T' - S'$, then $D' \cup \{v_1, v_2\}$ is a PD-set of $T - S$, and so $\gamma_{\rm pr}(T - S') \le |D'| + 2 = \gamma_{\rm pr}(T - S') + 2 < \gamma_{\rm pr}(T') + 2 = \gamma_{\rm pr}(T)$. Hence, $S' \in \mathrm{NI}(T)$ and $\gamma_{\rm pr}(T - S') < \gamma_{\rm pr}(T')$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S'| = \mathrm{st}^-_{\gamma_{\rm pr}}(T') < 2\Delta$, a contradiction. Therefore, $\mathrm{st}^-_{\gamma_{\rm pr}}(T') = 2\Delta$.

Hence, the tree T' satisfies $\Delta(T') = \Delta$, $\gamma_{\rm pr}(T') \ge 4$ and $\mathrm{st}^-_{\gamma_{\rm pr}}(T') = 2\Delta$. Proceeding by induction, we have $T' \in \mathcal{F}_\Delta$. Thus, T' is constructed from the disjoint union of k' double stars each isomorphic to $S(\Delta - 1, \Delta - 1)$, by selecting one leaf from each double star and adding $k' - 1$ edges between these selected leaves to produce a tree with maximum degree Δ. The resulting tree T' satisfies $\gamma_{\rm pr}(T') = 2k'$ with the $2k'$ support vertices forming a $\gamma_{\rm pr}$-set of T'.

By construction of T', the tree T' contains the vertex v_4 but not the vertex v_3. Suppose that v_4 is a support vertex in T', implying by construction of T' that v_4 is a vertex of degree Δ in T'. Let $S = (N[v_1] \cup N[v_2]) \setminus \{v_3\}$. We note that $S \in \mathrm{NI}(T)$ and $|S| = 2\Delta - 1$. Let D' be the (unique) $\gamma_{\rm pr}$-set of T', and so D' is the set of $2k'$ support vertices in T'. In particular, we note that $v_4 \in D'$. The set D' is a PD-set of $T - S$, and so $\gamma_{\rm pr}(T - S) \le |D'| = \gamma_{\rm pr}(T') = \gamma_{\rm pr}(T) - 2$. Therefore, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta - 1$, a contradiction. Hence, v_4 is a leaf of T', and so v_4 is a leaf in one of the k' double stars in the construction of T'. Selecting the leaf v_4 from this double star and selecting the leaf v_3 from the double star induced by $N[v_1] \cup N[v_2]$, which is isomorphic to $S(\Delta - 1, \Delta - 1)$, and adding back the edge v_3v_4 we re-construct the tree T, showing that $T \in \mathcal{F}_\Delta$. This completes the proof of part (a).

Part (b) now follows readily from part (a). If $T \in \mathcal{F}_\Delta$ for some $\Delta \ge 2$, then by Lemmas 6.1 and 6.3, we have $\mathrm{st}_{\gamma_{\rm pr}}(T) \le \Delta - 1$. Hence, we may assume that $T \notin \mathcal{F}_\Delta$ for any $\Delta \ge 2$, for otherwise the bound in part (b) is immediate. With this assumption, the upper bound in part (b) follows immediately from part (a) noting that $\mathrm{st}_{\gamma_{\rm pr}}(T) \le \mathrm{st}^-_{\gamma_{\rm pr}}(T) \le 2\Delta - 1$. That the bound is tight for all $\Delta \ge 2$ follows from Proposition 3.1. □

8 Proof of Theorem 2.3

In this section we present a proof of Theorem 2.3, which we restate below.

Theorem 2.3. *If G is a connected graph with $\gamma_{\rm pr}(G) \ge 4$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le 2\Delta(G)$, and this bound is sharp.*

Proof. Let G be a connected graph with $\gamma_{\rm pr}(G) \ge 4$ and let $\Delta = \Delta(G)$. Since $\gamma_{\rm pr}(G) \ge 4$, we have $\Delta \ge 2$. If $\Delta = 2$, then G is a path P_n or a cycle C_n, and by Theorem 5.1, we have $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le 2\Delta$, with equality if and only if $n \equiv 0 \pmod 4$. Assume, therefore, that $\Delta \ge 3$.

Let T be a spanning tree of G such that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$. We note that such a tree exists by Lemma 4.1. Let S be a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}^-_{\gamma_{\rm pr}}(T)$ such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. By Observation 4.2, we have

$|S| = \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \le n-2$. Since $S \in \mathrm{NI}(T)$, every vertex in $T-S$, and therefore in the supergraph $G-S$, has degree at least 1. Hence, $S \in \mathrm{NI}(G)$ and since $\gamma_{\mathrm{pr}}(G-S) \le \gamma_{\mathrm{pr}}(T-S)$, we have $\gamma_{\mathrm{pr}}(G-S) < \gamma_{\mathrm{pr}}(G)$. Thus, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) \le |S| = \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T)$. By Theorem 2.2, we have $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \le 2\Delta(T)$. Noting that $\Delta(T) \le \Delta(G)$, we therefore have that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) \le \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \le 2\Delta(T) \le 2\Delta(G) = 2\Delta$.

To show that the upper bound in Theorem 2.3 is tight, we present a family of graphs with maximum degree Δ and $\gamma_{\mathrm{pr}}(G) \ge 4$ satisfying $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = 2\Delta$. Our first family, $\mathcal{G}_\Delta$, is constructed as follows. For $k \ge 2$ and $\Delta \ge 2$, let $G_{k,\Delta}$ be a graph obtained from k double stars $S(\Delta-1, \Delta-1)$ by choosing two leaves at distance 3 apart in each double star and adding k edges between the chosen leaves in such a way, that every chosen vertex has degree 2 in the resulting graph. Let $\mathcal{G}_\Delta$ be the family of all such graphs $G_{k,\Delta}$ for all $k \ge 2$. The graph $G_{2,6} \in \mathcal{G}_6$, for example, is illustrated in Figure 5. We note that $\gamma_{\mathrm{pr}}(G_{k,\Delta}) = 2k$ and that set of $2k$ vertices of degree Δ is the unique γ_{pr}-set of $G_{k,\Delta}$. Furthermore, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G_{k,\Delta}) = 2\Delta$. $\square$

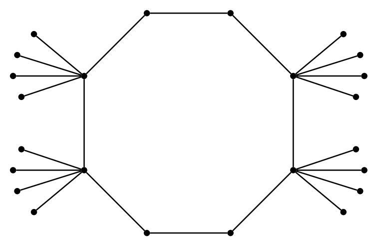

Figure 5: The graph $G_{2,6}$ from a class of graphs $G_{k,\Delta}$.

Recall that by definition we have $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \le \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G)$ for every graph G. Hence, as an immediate consequence of Theorem 2.3 we have Corollary 2.4. Recall its statement.

Corrolary 2.4. *If G is a connected graph with $\gamma_{\mathrm{pr}}(G) \ge 4$, then* $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \le 2\Delta(G)$.

It remains an open problem, however, to determine if the upper bound of Corollary 2.4 is best achievable for all values of possible value of $\Delta(G) = \Delta \ge 2$. If $\Delta = 2$ and G is a path, then $G \cong P_n$ where $n \ge 5$, and $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \le 2\Delta - 2$ by Corollary 5.3. If $\Delta = 2$ and G is a cycle, then $G \cong C_n$ where $n \ge 5$, and $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \le 2\Delta$ by Corollary 5.5, with equality if and only if $G = C_8$. Hence, the only connected graph G with maximum degree $\Delta = 2$ satisfying $\gamma_{\mathrm{pr}}(G) \ge 4$ and $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) = 2\Delta$ is the 8-cycle, namely $G = C_8$. For $\Delta \ge 3$, we do not know of a connected graph G with maximum degree Δ satisfying $\gamma_{\mathrm{pr}}(G) \ge 4$ and $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) = 2\Delta$.

By Corollary 5.5 and Proposition 3.1, for any given $\Delta \ge 2$, there do exists infinite families of connected graphs G with maximum degree Δ satisfying $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) = 2\Delta - 1$. Thus, if the upper bound of Corollary 2.4 can be improved to $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \le 2\Delta - 1$ in the case when $\Delta \ge 3$, then this bound would be tight.

ORCID iDs

Aleksandra Gorzkowska https://orcid.org/0000-0001-5335-7351
Michael A. Henning https://orcid.org/0000-0001-8185-067X

Monika Pilśniak https://orcid.org/0000-0002-3734-7230
Elżbieta Tumidajewicz https://orcid.org/0000-0002-1413-2413

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Notes on weak-odd edge colorings of digraphs*

César Hernández-Cruz

*Facultad de Ciencias, Universidad Nacional Autónoma de México,
Av. Universidad 3000, Circuito Exterior S/N, Ciudad Universitaria, CDMX, México*

Mirko Petruševski †

*Faculty of Mechanical Engineering - Skopje, University Ss Cyril and Methodius,
1000 Skopje, Macedonia*

Riste Škrekovski

FMF, University of Ljubljana, 1000 Ljubljana, Slovenia, and
Faculty of Information Studies, 8000 Novo mesto, Slovenia, and
University of Primorska, FAMNIT, 6000 Koper, Slovenia

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Abstract

A weak-odd edge coloring of a general digraph D is a (not necessarily proper) coloring of its edges such that for each vertex $v \in V(D)$ at least one color c satisfies the following conditions: if $d_D^-(v) > 0$ then c appears an odd number of times on the incoming edges at v; and if $d_D^+(v) > 0$ then c appears an odd number of times on the outgoing edges at v. The minimum number of colors sufficient for a weak-odd edge coloring of D is the weak-odd chromatic index, denoted $\chi'_{\mathrm{wo}}(D)$. It is known that $\chi'_{\mathrm{wo}}(D) \leq 3$ for every digraph D, and that the bound is sharp. In this article we show that the weak-odd chromatic index can be determined in polynomial time. Restricting to edge colorings of D with at most two colors, the minimum number of vertices $v \in V(D)$ for which no color c satisfies the above conditions is the defect of D, denoted $\operatorname{def}(D)$. Surprisingly, it turns out that the problem of determining the defect of digraphs is (polynomially) equivalent to the problem of finding the matching number of simple graphs. Moreover, we characterize the classes of associated digraphs and tournaments in terms of the weak-odd chromatic index and the defect.

Keywords: Digraph, weak-odd edge coloring, weak-odd chromatic index, defective coloring, tournament.

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†Corresponding author.

E-mail addresses: japo@ciencias.unam.mx (César Hernández-Cruz), mirko.petrushevski@gmail.com (Mirko Petruševski), skrekovski@gmail.com (Riste Škrekovski)

1 Introduction

Throughout the article we mainly follow terminology and notation used in [1]. All graphs and digraphs are considered to be finite. Loops, parallel edges and parallel arcs are admissible, i.e., strictly speaking, we consider the general setup of pseudographs and directed pseudographs, but in order to avoid lengthy terminology, we abbreviate these terms to 'graphs' and 'digraphs', respectively.

Cheilaris et al. [4] introduced the notion of *odd coloring* of a hypergraph $\mathcal{H}$ as a vertex coloring such that for every (hyper-)edge $e \in \mathcal{E}(\mathcal{H})$ there is a color c with an odd number of vertices of e colored by c. Restricting to graphs G (and thus to plain edges), the previous coloring notion is merely the usual notion of 'proper' coloring of G. Through interchanging the roles played by vertices and edges, initially motivated by [5, 6], the analogous edge coloring notion for graphs was introduced in [10] as follows. An edge coloring of G is said to be *weak-odd* if it holds that:

(WO) For every vertex $v \in V(G)$ with degree $d_G(v) > 0$, at least one color c appears an odd number of times on the set of edges incident with v.

The additional adjective 'weak' has been included in the name simply in order to differentiate this from the related, already existing, notion of 'odd edge coloring' of graphs, defined in [12]. (The latter notion has stronger requirements for the colors appearing at a vertex.)

Let us clarify that, by definition, any loop at v colored by c contributes 2 to the count of appearances of c on $E_G(v)$ (the set of all edges incident with v). An obvious necessary and sufficient condition for weak-odd edge colorability of graph G is the absence of 'isolated loops', i.e., nonempty trivial components. A weak-odd edge coloring of G using at most k colors is referred to as a *weak-odd k-edge coloring* of G, and such a graph is said to be *weak-odd k-edge colorable*. The *weak-odd chromatic index*, $\chi'_{\mathrm{wo}}(G)$, is the minimum k for which G is weak-odd k-edge colorable. Obviously, apart from 'isolated loops', any other loop addition or removal does not influence the existence nor alters the value of $\chi'_{\mathrm{wo}}(G)$. The following characterization of graphs G in terms of $\chi'_{\mathrm{wo}}(G)$ was obtained in [10]. It makes use of the next two notions. A graph G is *even* (resp. *odd*) if every vertex $v \in V(G)$ has even (resp. odd) degree $d_G(x)$.

Theorem 1.1. *For any connected graph G whose edge set does not consist entirely of loops, it holds that*

$$\chi'_{\mathrm{wo}}(G) = \begin{cases} 0 & \text{if } G = K_1, \\ 1 & \text{if } G \text{ is odd}, \\ 2 & \text{if } G \text{ has even order or is not even}, \\ 3 & \text{if } G \text{ is even, has odd order, and is not } K_1. \end{cases}$$

This paper treats the analogous coloring notion for digraphs. In the next section we further explain the motivation behind the definition of 'weak-odd edge coloring' of digraphs, introduced in [11], and then show that the corresponding coloring index χ'_{wo} can be determined in linear time. We also address a related problem concerning the minimum number of vertices at which an arbitrary 2-edge coloring of a digraph fails at being 'weak-odd', and prove a connection with the problem of determining the matching number of a simple graph. In Section 3, the discussion restricts to two common classes of digraphs, namely, the class of associated digraphs[1] and the class of tournaments. For each of these classes we give a descriptive characterization of their members in terms of χ'_{wo}. In the final section we briefly convey our thoughts on possible further work on the topic of weak-odd edge colorings of digraphs. For the end of this introductory section, we mention some common notions and facts that will be frequently used throughout.

1.1 General terminology and notation

We denote the symmetric difference of sets A, B by $A \oplus B$. The same notation is in use for the symmetric difference of two spanning subgraphs A, B of a ground graph. Given a graph G and an even-sized subset T of $V(G)$, a spanning subgraph H is a *T-join* of G if $d_H(v)$, the degree of v with respect to H, is odd for all $v \in T$ and even for all $v \in V(G) \setminus T$. The symmetric difference of a T'-join and a T''-join of G is a $T' \oplus T''$-join, which yields the following classical result (see [14]): *every connected graph G contains a T-join.* In particular, every even-ordered connected graph G has an odd factor (i.e., a $V(G)$-join).

An *edge coloring* of a graph G (resp. digraph D) with *color set S* is an assignment $E(G) \to S$ (resp. $A(D) \to S$). Every T-join of G can be interpreted as an edge coloring with color set $\{1, 2\}$ such that 1 is used an odd number of times at each $v \in T$ and and even number of times (possibly zero) at each $v \in V(G) \setminus T$.

Given a digraph D and a vertex $v \in V(D)$, the size of the set $\partial_D^-(v)$, of incoming edges at v, (resp. $\partial_D^+(v)$, of outgoing edges at v,) is the *in-degree* $d_D^-(v)$ (resp. *out-degree* $d_D^+(v)$) of the vertex v; we call each of $\partial_D^-(v)$ and $\partial_D^+(v)$ (resp. $d_D^-(v)$ and $d_D^+(v)$) a *semi-cut* (resp. *semi-degree*) of v. Since loops are allowed, let us clarify that $\partial_D^-(v) \cap \partial_D^+(v)$ constitutes the set of loops at v; in other words, any loop at vertex v contributes 1 to each semi-degree of v, i.e., strictly speaking $d_D^-(v)$ and $d_D^-(v)$ are the semi-pseudodegrees of v (the in-pseudodegree and out-pseudodegree, respectively). The sum $d_D(v) = d_D^-(v) + d_D^+(v)$ is the *degree* of v; a vertex of degree 0 is said to be *isolated.* Given a nonisolated vertex v, if $d_D^-(v) = 0$ (resp. $d_D^+(v) = 0$), then v is a *source* (resp. *sink*) of D. Any source or sink is a *peripheral vertex* of D, whereas a nonisolated vertex that is neither a source nor a sink is an *intermediate vertex*. A vertex u is said to *dominate* a vertex v if $v \in \partial_D^+(u)$; equivalently, v is *dominated by u*.

Two graphs or digraphs are considered identical if they are isomorphic to each other. The numbers of vertices and edges of graph or digraph D are denoted by $n(D)$ and $m(D)$; these basic parameters are the *order* and *size* of D, respectively; a graph or digraph of order 1 (resp. size 0) is *trivial* (resp. *empty*).

A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph, and X and Y its partite sets. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. Given a digraph D, its *split* (or *bipartite representation*), $\mathrm{BG}(D)$, is the bipartite graph G whose partite sets V^+, V^- are copies of $V(D)$. For each $v \in V(D)$, there is one vertex $v^+ \in V^+$ and one $v^- \in V^-$. For each directed uv-edge in D, there is an edge with endpoints u^+ and v^- in G. Hence, the degrees of the vertices v^+, v^- in the split of D are precisely the out-degree and in-degree of v in D, respectively; the pair (v^+, v^-) is obtained by *splitting v in regard to D*. The re-identification of each such pair (v^+, v^-) into v results in the so-called *underlying graph* of D, denoted $G(D)$.

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[1]A digraph D is *associated* if for every arc (u, v) of D, the arc (v, u) is also present in D, and the number of loops at any vertex is even.

Furthermore, any balanced bipartite graph G is a split of some digraph D, i.e., can be 'transformed' into D by reversing the described procedure.

The other way around, any graph G can be regarded as a digraph $D(G)$, by replacing each of its edges by two oppositely oriented arcs with the same ends (each loop of G gives rise to two directed loops on the same vertex); this digraph is the *associated digraph* of G. One may also obtain a digraph D from a graph G by replacing each edge by one arc on the same endpoints; such a digraph D is an *orientation* of G. A *tournament* is an orientation of a complete graph. Conversely, the underlying graph $G(D)$ of a digraph D is obtained by 'forgetting orientation'. A *directed path* or *directed cycle* is an orientation of a path or cycle in which each vertex dominates its successor in the sequence.

A digraph D is said to be *strongly connected* (or simply *strong*) if for any pair u, v of its vertices there is a directed uv-path, i.e., a directed path joining vertex u with vertex v. Given a digraph D, every maximal strong subdigraph of D is a *strong component* of D. The *condensation* $C(D)$ of a digraph D is the loopless directed (multi)graph whose vertices correspond to the strong components of D, with any two vertices of $C(D)$ being linked by as many directed edges as there are directed edges in D linking the corresponding strong components, and with the consistent orientation. The *peripheral* strong components of D which correspond to the vertices of $C(D)$ that are sources (resp. sinks), are the *initial* (resp. *terminal*) strong components; the remaining strong components of D are called *intermediate* or *isolated* according to the nature of the corresponding vertices in $C(D)$.

2 Weak-odd edge colorings of digraphs

In the realm of digraphs D, when defining the notion of 'weak-odd edge coloring' two options come to mind. Initially, one may obtain a 'directed version' of the condition (WO) as follows:

($\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}}$) For every vertex $v \in V(D)$, on each nonempty semi-cut of v at least one color c appears an odd number of times.

However, the above 'definition' fails to capture the essence of digraphs since it basically ignores the fact that arc sets $\partial_D^-(v)$ and $\partial_D^+(v)$ are incident with a common vertex (namely, v). Actually, a moment's thought reveals that, if we decide to adopt the initial definition, then this coloring notion for digraphs would be merely a 'disguise' of weak-odd edge coloring of bipartite graphs with equally sized partite sets. Namely, it can readily be seen that an edge coloring of D satisfies condition ($\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}}$) if and only if the induced edge coloring on $\mathrm{BG}(D)$ satisfies condition (WO). Consequently, in view of Theorem 1.1, every digraph D admits a 3-edge coloring obeying ($\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}}$); moreover, three colors are indeed required if and only if at least one component of $\mathrm{BG}(D)$ is a nonempty even graph of odd order. One such example is depicted in Figure 1. Notice that if at least one of the loops is removed from D then the obtained digraph would admit a 2-edge coloring that satisfies condition ($\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}}$).

As noticed, unlike for graphs, in the realm of digraphs the presence of loops may influence the value of the corresponding chromatic index in a nontrivial manner. This is so because any such loop in the digraph is no longer a loop in its split.

A more appropriate definition of the notion of 'weak-odd edge coloring' for digraphs, the one we shall adopt in this study, is obtained as follows. Recall that any graph G can be seen as a digraph, the associated digraph $D(G)$. In the obvious fashion, every edge

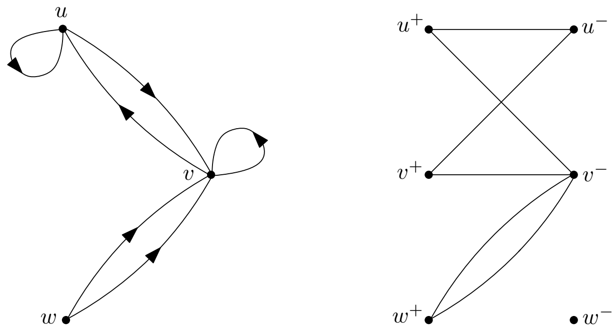


Figure 1: (left) A digraph D that fails condition $(\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}})$ under any 2-edge coloring, and (right) its split $\mathrm{BG}(D)$ with the left-hand (resp. right-hand) partite set representing V^+ (resp. V^-). The nonempty component of $\mathrm{BG}(D)$ is an even graph of odd order.

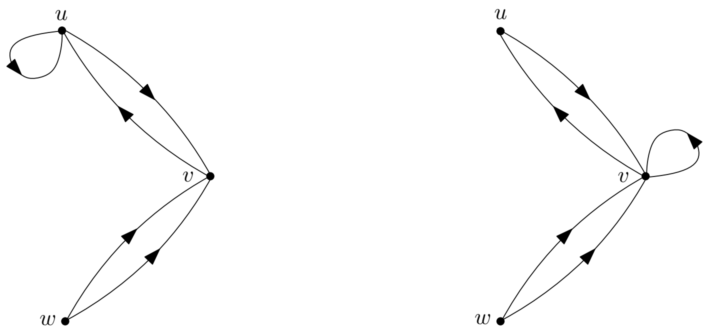


Figure 2: Digraphs D_1 (left) and D_2 (right) are obtained from the digraph in Figure 1 by removing a certain loop (at v and at u, respectively). Both admit 2-edge colorings fulfilling condition $(\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}})$, but none admits a 2-edge coloring satisfying $(\overrightarrow{\mathrm{WO}})$.

coloring φ of G can be interpreted as an edge coloring φ_D of $D(G)$. Notice that if φ is weak-odd then φ_D satisfies the condition $(\overrightarrow{\mathrm{WO}})$ below, which states a stronger requirement than the one imposed by $(\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}})$. This particular reasoning served as the motivation in [11] for defining an edge coloring of a digraph D to be *weak-odd* if:

$(\overrightarrow{\mathrm{WO}})$ For every vertex $v \in V(D)$, at least one color c appears an odd number of times on each nonempty semi-cut of v.

Same as with $(\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}})$, in case v is a sink (resp. source), the condition $(\overrightarrow{\mathrm{WO}})$ amounts to the appearance of c an odd number of times on the incut $\partial_D^-(v)$ (resp. outcut $\partial_D^+(v)$). The difference between $(\overrightarrow{\mathrm{W}}\overrightarrow{\mathrm{O}})$ and $(\overrightarrow{\mathrm{WO}})$ is reflected at the intermediate vertices (cf. Figure 2).

The minimum number of colors sufficient for a weak-odd edge coloring of a digraph D is the weak-odd chromatic index, denoted $\chi'_{\mathrm{wo}}(D)$. A weak-odd edge coloring of D using at most k colors is referred to as a *weak-odd k-edge coloring*, and any such D is said to be *weak-odd k-edge colorable*. Hence, $\chi'_{\mathrm{wo}}(D)$ is the minimum k for which D is weak-odd k-edge colorable.

Interestingly, analogous to graphs, the same upper bound (of three colors) holds for the weak-odd chromatic index of digraphs. Namely, the following was proven in [11].

Theorem 2.1. *Every digraph is weak-odd* 3*-edge colorable.*

As already illustrated through Figures 1 and 2, the bound $\chi'_{\mathrm{wo}}(D) \le 3$ is sharp, i.e., not every digraph is weak-odd 2-edge colorable. Analogous to the setting of graphs, it is quite trivial to characterize which digraphs are weak-odd 1-edge colorable. Indeed, the inequality $\chi'_{\mathrm{wo}}(D) \le 1$ holds if and only if for every vertex $v \in V(D)$ both semi-degrees $d^-_D(v), d^+_D(v)$ are odd or zero. Furthermore, $\chi'_{\mathrm{wo}}(D) = 0$ holds precisely when D is empty. Thus, in order to characterize all digraphs in terms of their weak-odd chromatic index, it suffices to figure out which are the weak-odd 2-edge colorable ones.

2.1 Characterization of weak-odd 2-edge colorability

The *partial split*, $\mathrm{PS}(D)$, of given digraph D is a graph obtained by splitting (in regard to D) only those vertices $v \in V(D)$ for which at least one semi-degree is even (including zero), and then forgetting orientation. In other words, $\mathrm{PS}(D)$ is the graph obtained from $\mathrm{BG}(D)$ by re-identifying each pair (u^+, u^-) for which both $d^+_D(u)$ and $d^-_D(u)$ are odd (cf. Figure 3). In particular, if no vertex of D has only odd-sized semi-cuts, then $\mathrm{PS}(D)$ is the same graph as the split $\mathrm{BG}(D)$; contrarily, if every nonisolated vertex of D has only odd-sized semi-cuts, then $\mathrm{PS}(D)$ is the same as the underlying graph $G(D)$. However, in general, these three graphs related to D differ from each other.

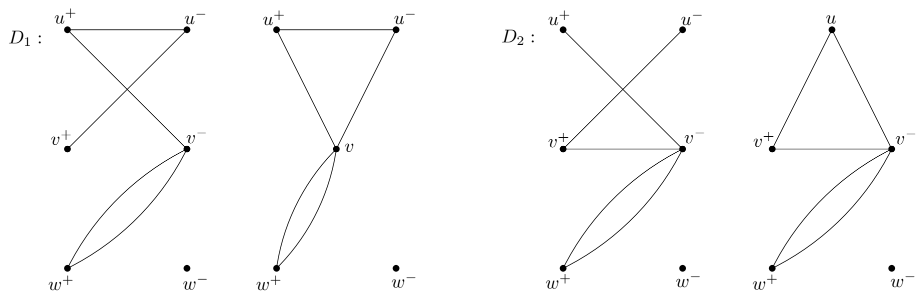


Figure 3: The split $\mathrm{BG}(D_1)$ and partial split $\mathrm{PS}(D_1)$ (left), and the split $\mathrm{BG}(D_2)$ and partial split $\mathrm{PS}(D_2)$ (right), where D_1, D_2 are the digraphs from Figure 2. The induced 3-partition $\{V_1, V_2, V_3\}$ of $V(\mathrm{PS}(D_1))$ consists of $V_1 = \{v\}, V_2 = \{u^+, u^-, w^+, w^-\}$ and $V_3 = \varnothing$, whereas the corresponding 3-partition of $V(\mathrm{PS}(D_2))$ has $V_1 = \{u\}$, $V_2 = \{v^+, v^-, w^+, w^-\}, V_3 = \varnothing$.

We distinguish between three types of vertices in $\mathrm{PS}(D)$ inducing a 3-partition $\{V_1, V_2, V_3\}$ of $V(\mathrm{PS}(D))$:

- the first type of vertices, constituting V_1, are the members of $V(D) \cap V(\mathrm{PS}(D))$, i.e., the vertices u of D having odd semi-degrees $d^+_D(u), d^-_D(u)$.
- the second type of vertices, forming V_2, are the members v of $V(\mathrm{PS}(D)) \backslash V(D)$ that have even degree $d_{\mathrm{PS}(D)}(v)$.
- finally, the third type of vertices, comprising V_3, are the members w of $V(\mathrm{PS}(D)) \backslash V(D)$ that have odd degree $d_{\mathrm{PS}(D)}(w)$.

For simplicity of presentation, on several occasions we shall employ the following ad-hoc terminology and notation. Given a graph G, a vertex $x \in V(G)$ is said to be *even* (resp. *odd*) if $d_G(x)$ is even (resp. odd). The set of even (resp. odd) vertices of G is denoted $EvenV(G)$ (resp. $OddV(G)$). Note that, by the handshake lemma, $OddV(G)$ is always even-sized. Thus, V_3 above equals $OddV(\mathrm{PS}(D))$, whereas $V_1 \cup V_2 = EvenV(\mathrm{PS}(D))$.

Theorem 2.2. *A digraph D is weak-odd 2-edge colorable if and only if for every nonempty component K of* $\mathrm{PS}(D)$ *we have that $V(K) \cap V_2$ is even-sized or $V(K) \cap V_3 \neq \emptyset$.*

Proof. Assuming $\chi'_{\mathrm{wo}}(D) \leq 2$, let $\varphi : A(D) \to \{1, 2\}$ be a weak-odd edge coloring of D and consider the induced edge coloring of $\mathrm{PS}(D)$. Observe that for every vertex $u \in V_1$, each of the colors 1 and 2 is used an even number of times on $E_{\mathrm{PS}(D)}(u)$. Indeed, the edge set $E_{\mathrm{PS}(D)}(u)$ corresponds to the entire cut $\partial_D(u) = \partial^+_D(u) \cup \partial^-_D(u)$; thus, since both constituents in this disjoint union are odd-sized, for every color $c \in \{1, 2\}$ the parities of $|\partial^+_D(u) \cap \varphi^{-1}(c)|$ and $|\partial^-_D(u) \cap \varphi^{-1}(c)|$ are equal. In contrast, for every nonisolated vertex $v \in V_2$, each of the colors 1 and 2 appears an odd number of times on $E_{\mathrm{PS}(D)}(v)$. The reason behind this is that the edge set $E_{\mathrm{PS}(D)}(v)$ corresponds to a nonempty even-sized semi-cut of a vertex in D.

Therefore, if K is a nonempty component of $\mathrm{PS}(D)$ such that $V(K) \subseteq V_1 \cup V_2$, then each of the two color classes induces in K a subgraph K_i ($i \in \{1, 2\}$) such that $OddV(K_i) = V(K) \cap V_2$. Consequently, the intersection $V(K) \cap V_2$ is even-sized.

Arguing in the opposite direction, assume now that every nonempty component K of $\mathrm{PS}(D)$ meets the stated requirements. We construct an assignment $E(K) \to \{1, 2\}$ as follows. If $V(K) \cap V_2$ is even-sized, then define $T = V(K) \cap V_2$. Otherwise, select an odd-sized subset $S_K \subseteq V(K) \cap V_3$ and define $T = (V(K) \cap V_2) \cup S_K$. Either way, T is an even-sized subset of $V(K)$. Therefore, there exists a T-join H of K. Color $E(H)$ with 1 and $E(K) \backslash E(H)$ with 2. Consider the induced edge coloring of D, and observe the following.

(1) *At each vertex $u \in V(D) \cap V_1$, precisely one of the colors $1, 2$ satisfies condition* ($\overrightarrow{\mathrm{WO}}$). Indeed, by construction, each color is used an even number of times on $\partial_D(u)$, and thus has equal parities of appearance on the odd-sized sets $\partial^-_D(u)$ and $\partial^+_D(u)$, respectively.

(2) *At each nonisolated vertex $v \in V(D) \backslash V_1$ such that the vertices $v^+, v^- \in V_2$, colors* 1 *and* 2 *both satisfy condition* ($\overrightarrow{\mathrm{WO}}$). Namely, by construction, color 1 is used an odd number of times on each nonempty semi-cut of v; on the other hand, both $\partial^-_D(v)$ and $\partial^+_D(v)$ are even-sized.

(3) *At each vertex $w \in V(D) \backslash V_1$ such that one of the vertices w^+, w^- belongs in V_2 (and the other in V_3), precisely one of the colors $1, 2$ satisfies condition* ($\overrightarrow{\mathrm{WO}}$). Indeed, if the 'half' of w belonging to V_3 is used in some S_K then color 1 meets *(*$\overrightarrow{\mathrm{WO}}$*)*; otherwise, color 2 does so.

Thus, the digraph D is weak-odd 2-edge colorable. □

For example, the digraph D depicted in Figure 4 is weak-odd 2-edge colorable because the only nonempty component of $\mathrm{PS}(D)$ intersects V_3. With the notation employed in the proof, if we take $E(H)$ to consist of x^+y^-, y^+u, and a uz^- edge, then the induced weak-odd 2-edge coloring of D assigns color 1 to xy, yu and a uz arc, and assigns color 2 to the rest of $A(D)$.

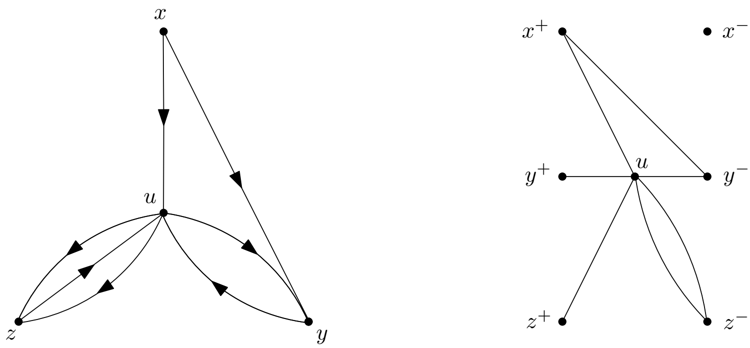


Figure 4: A digraph D (left) and its partial split $\mathrm{PS}(D)$ (right). The induced 3-partition has $V_1 = \{u\}$, $V_2 = \{x^+, x^-, y^-, z^-\}$ and $V_3 = \{y^+, z^+\}$.

Contrarily, the digraph D depicted in Figure 5 is not weak-odd 2-edge colorable, since the induced 3-partition has $V_1 = \{u\}$, $V_2 = \{x^+, x^-, y^+, y^-, z^+, z^-\}$ and $V_3 = \varnothing$, the only nonempty component of its partial split does not intersect V_3 and contains an odd number of vertices from V_2.

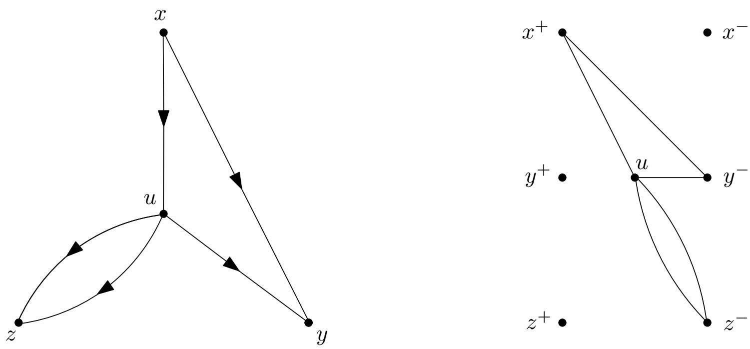


Figure 5: A digraph D (left) and its partial split $\mathrm{PS}(D)$ (right).

The proof of Theorem 2.2 and the fact that the problem of constructing a T-join of any connected graph G for a given even-sized subset T of $V(G)$ is solvable in linear time (see [14]), imply that the decision problem of whether $\chi'_{\mathrm{wo}}(D) \leq 2$ is solvable in linear time; in the affirmative case, a weak-odd 2-edge coloring of D can be found in linear time. Thus, in view of Theorem 2.1 and the subsequent discussion, we conclude the following.

Corollary 2.3. *The weak-odd chromatic index of any digraph D and a corresponding weak-odd $\chi'_{\mathrm{wo}}(D)$-edge coloring can be determined in linear time.*

To end this subsection we point out an infinite family of digraphs having weak-odd chromatic index equal to 3. A digraph is said to be *even* if every vertex $v \in V(D)$ is of even degree $d_D(v)$; in other words, the requirement is that the semi-degrees of v are of equal parity.

Figure 6: Two even digraphs with weak-odd chromatic index 3.

Proposition 2.4. *If an even digraph D has an odd number of peripheral vertices, then* $\chi'_{\mathrm{wo}}(D) = 3$.

Proof. We may assume that D is connected. Consider the partial split $\mathrm{PS}(D)$ of D and the induced 3-partition $\{V_1, V_2, V_3\}$. Obviously, $V_3 = \varnothing$ and the number of isolated vertices of $\mathrm{PS}(D)$ equals the number of peripheral vertices of D. However, this implies that the number of nonisolated vertices of $\mathrm{PS}(D)$ which belong in V_2 is odd. Therefore, an odd number of nonempty components of $\mathrm{PS}(D)$ fail to meet the requirements of Theorem 2.2. □

Notice that out of the two even digraphs depicted in Figure 6, only the left one has an odd number of peripheral vertices; nevertheless, neither is weak-odd 2-edge colorable. This demonstrates that the condition 'an odd number of peripheral vertices' from the statement of Proposition 2.4, although sufficient, is by no means necessary.

2.2 Defective weak-odd 2-edge colorings

The following straightforward result serves as our motivation for the brief discussion within this subsection. In a way, it tells that every graph can be almost weak-odd 2-edge colored.

Proposition 2.5. *Every connected graph G admits a 2-edge coloring such that condition* (WO) *is satisfied at each vertex apart from a prescribed vertex $v \in V(G)$.*

Proof. We construct an even-sized subset T of $V(G)$ as follows. If $n(G)$ is even, then define $T = V(G)$; otherwise, let $T = V(G)\backslash\{v\}$. Since G is connected, consider a T-join H of G. Color $E(H)$ with 1, and the rest of $E(G)$ with 2. The obtained 2-edge coloring of G clearly fulfills condition (WO) at each vertex differing from v. □

One naturally wonders if there exists an analogous result for digraphs that bounds (presumably with 1) the number of 'defective vertices', in regard to condition $(\overrightarrow{\mathrm{WO}})$, for some 2-edge coloring? Unfortunately, in contrast to graphs, there are connected digraphs such that for any 2-edge coloring the condition $(\overrightarrow{\mathrm{WO}})$ fails at an unbounded number of vertices.

To construct examples, consider as a 'gadget digraph' D the right-hand digraph from Figure 6. Observe that no 2-edge coloring of D can fulfill condition $(\overrightarrow{\mathrm{WO}})$ at all vertices excepting the sink (or the source). Thus, by taking any number, say n, of copies of D and identifying their sinks (cf. Figure 7) we obtain a connected digraph D'_n such that under any 2-edge coloring at least n of its vertices are 'defective' in regard to condition $(\overrightarrow{\mathrm{WO}})$. A similar construction using the same gadget graph shows that even strong connectedness comes to no avail in this regard. Namely, take an arbitrary number $n \geq 2$ of copies of D, arrange them in circular order and then identify pairwise the corresponding sink and source of each neighboring copies (cf. Figure 7). Once again, the obtained strong digraph D''_n under any 2-edge coloring has at least n 'defective' vertices in regard to condition $(\overrightarrow{\mathrm{WO}})$. The following question comes to mind: *Does anything change in regard to this problem if we confine to simple digraphs, or even more so, to digraphs with simple underlying graphs?* The next result answers the question in negative.

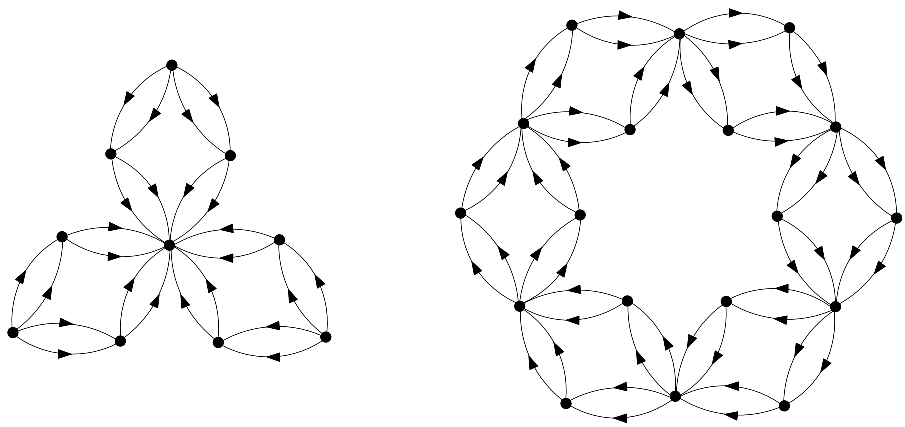


Figure 7: The digraph D'_3 (left), and the digraph D''_6 (right).

Proposition 2.6. *For any given positive integer n, there exists a strongly connected digraph D with simple underlying graph $G(D)$ such that under any 2-edge coloring of D at least n of its vertices are 'defective' in regard to condition* $(\overrightarrow{\mathrm{WO}})$.

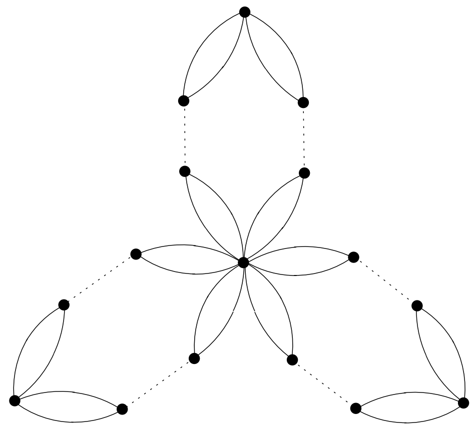
Proof. Simply take D to be a complete subdivision of D''_n. In other words, subdivide (at least once) each arc $e \in A(D''_n)$ and orient the newly formed arcs consistently with e. □

Given a digraph D, let $\mathrm{def}(D)$, the *defect of* D, denote the minimum number of 'defective vertices' in regard to condition $(\overrightarrow{\mathrm{WO}})$ taken over all 2-edge colorings of D. A question that naturally arises is whether this parameter can be effectively determined. As it turns out, the parameter $\mathrm{def}(D)$ is closely related to yet another graph construction, relating a simple graph G_D to each digraph D, which we describe next.

Start from the induced subgraph $\mathrm{BC}(D) \subseteq \mathrm{PS}(D)$ that consists of the 'bad components' of $\mathrm{PS}(D)$ in regard to the requirement of Theorem 2.2; in other words, $V(\mathrm{BC}(D)) = \bigcup_K V(K)$, where the union is taken over all components K of $\mathrm{PS}(D)$ such that $V(K) \cap V_2$ is odd-sized and $V(K) \cap V_3$ is empty.

Thus, the vertex set of G_D consists of vertices v_K corresponding to components K of $\mathrm{BC}(D)$. As for the edge set of G_D, make two distinct vertices $v_{K'}$ and $v_{K''}$ adjacent if the respective bad components K' and K'' contain the 'halves' v^+ and v^- of some vertex

$v \in V(D)$. To exemplify, we shall make use of the digraphs D'_n and D''_n defined above. For the first of these digraphs, it is readily seen that each of the $n+1$ nonempty components of $\mathrm{PS}(D'_n)$ is bad, and that $G_{D'_n} = K_{1,n}$ (cf. Figure 8).



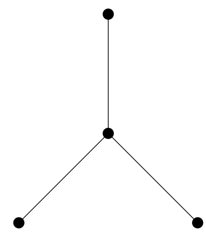


Figure 8: The graph $\mathrm{BC}(D'_3)$ (left) with each dotted line matching the two 'halves' of a splitted vertex of D'_3, and the graph $G_{D'_3} = K_{1,3}$ (right).

Similarly, each of the $2n$ nonempty components of $\mathrm{PS}(D''_n)$ is bad; this time it holds that $G_{D''_n} = C_{2n}$ (cf. Figure 9).

Interestingly, any simple graph G is a realization of some G_D.

Proposition 2.7. *For any simple graph G there exists a digraph D such that $G_D = G$.*

Proof. Let $n = n(G)$ and $m = m(G)$ be the order and size of G, respectively. An *open $2m$-necklace* is a digraph obtained as follows: take a path P of length $2m$, replace each edge $e \in E(P)$ by a pair of parallel edges, and then orient each such pair consistently so that with any natural ordering the vertices become: sink, source, sink, ..., source, sink (cf. Figure 10).

Take n disjoint open $2m$-necklaces, and enumerate them as $D_1, \dots, D_n$. Consider an enumeration of the set $E(G) = \{e_1, \dots, e_m\}$. For each $i \in \{1, \dots, n\}$, fix a natural ordering of the vertices of D_i, and enumerate them accordingly as $e_{0,i}, e^+_{1,i}, e^-_{1,i}, \dots, e^+_{m,i}, e^-_{m,i}$.

Let D be the digraph obtained from $D_1 \uplus \cdots \uplus D_n$ as follows. Take an enumeration of the vertex set $V(G) = \{v_1, \dots, v_n\}$. For each $e_k \in E(G)$, if $e_k = v_i v_j$ with $i < j$, then identify $e^+_{k,i}$ with $e^-_{k,j}$. Observe that, by construction:

- $\mathrm{PS}(D)$ has n nonempty components;
- each such component belongs to $\mathrm{BC}(D)$;
- $G_D = G$.

The last item concludes our proof. □

Recall that a *matching* in a graph is a set of pairwise nonadjacent edges that are not loops. If M is a matching, the two ends of each edge of M are said to be *matched* under M, and each vertex incident with an edge of M is said to be *covered* by M. A *maximum*

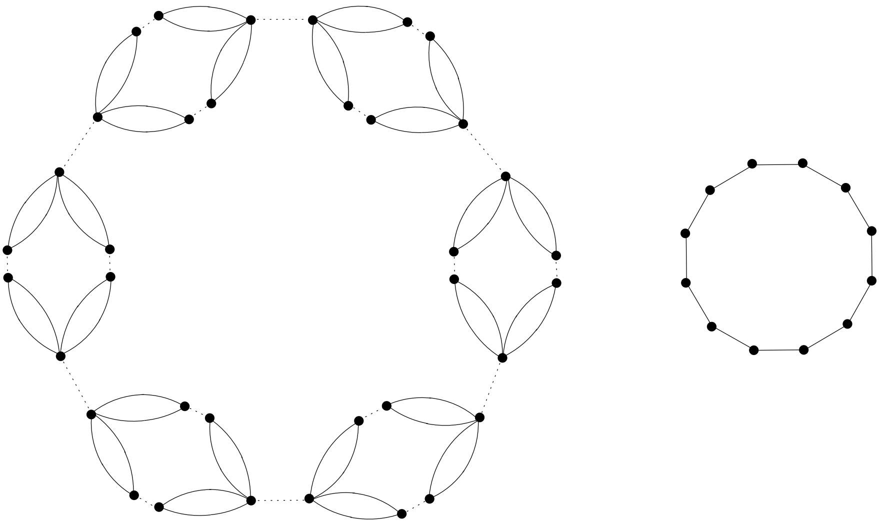


Figure 9: The graph $\mathrm{BC}(D_6'')$ (left) with each dotted line matching the two 'halves' of a splitted vertex of D_6'', and the graph $G_{D_6''} = C_{12}$ (right).

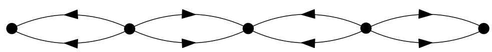


Figure 10: The open 4-necklace.

matching in a given graph covers as many vertices as possible. The *maximum matching problem* is the problem of finding a maximum matching in a given graph G. The number of edges in such a matching is called the *matching number* of G and denoted $\alpha'(G)$. Thanks to the pioneering work of Tutte and Edmonds, the maximum matching problem is known to be solvable in polynomial time. In particular, one of the 1965 papers of Edmonds on polyhedral combinatorics, describes, among other things, the so-called Blossom Algorithm [7] (see also [2], pp. 452)), an $O(n^2m)$ algorithm that finds a maximum matching in any given graph of order n and size m. Over the years, various improvements of the Blossom Algorithm have been found (see, e.g., [14], pp. 422–423).

Our next result establishes a relationship between the defect $\operatorname{def}(D)$ of any given digraph D and the order $n(G_D)$ and matching number $\alpha'(G_D)$ of the corresponding simple graph G_D.

Theorem 2.8. *For every digraph D,* $\operatorname{def}(D) = n(G_D) - \alpha'(G_D)$ *holds.*

Proof of Theorem 2.8. By Theorems 2.1 and 2.2, we may assume that $\chi'_{\mathrm{wo}}(D) = 3$. Take an arbitrary edge coloring φ of D with color set $\{1, 2\}$. For simplicity, we use the same notation φ to denote the inherited edge coloring of $\mathrm{PS}(D)$. Let $\mathrm{PS}(D)_1$ and $\mathrm{PS}(D)_2$ be the spanning subgraphs of $\mathrm{PS}(D)$ whose respective edge sets are the color classes $\varphi^{-1}(1)$ and $\varphi^{-1}(2)$. For every vertex $x \in V(\mathrm{PS}(D))$, we abbreviate $d_{\mathrm{PS}(D)_1}(x)$ to $d_1(x)$, and likewise $d_{\mathrm{PS}(D)_2}(x)$ to $d_2(x)$. Consider the partition $\{V(D) \cap V(\mathrm{PS}(D)), V(D) \backslash V(\mathrm{PS}(D))\}$ of $V(D)$, and observe the following:

- a vertex $u \in V(D) \cap V(\mathrm{PS}(D))$ is 'defective' if and only if both $d_1(u)$ and $d_2(u)$

are odd;

- a vertex $v \in V(D)\backslash V(\mathrm{PS}(D))$ is 'defective' if and only if some $v^{\pm} \in \{v^{+}, v^{-}\}$ is a nonisolated vertex of $\mathrm{PS}(D)$ such that both $d_1(v^{\pm})$ and $d_2(v^{\pm})$ are even (possibly zero); call every such $v^{\pm}$ a 'defective half-vertex' originating from v.

First we show that each bad component of $\mathrm{PS}(D)$ 'contains' at least one defective vertex.

Claim 2.8.1. *Each component K of $\mathrm{BC}(D)$ contains a defective vertex or a defective half-vertex.*

Proof of Claim 2.8.1. Let $K_1 = K \cap \mathrm{PS}(D)_1$ and $K_2 = K \cap \mathrm{PS}(D)_2$. Since K is an even graph, clearly $OddV(K_1) = OddV(K_2)$ and $EvenV(K_1) = EvenV(K_2)$. By the above observations, within $V(K)$, the defective vertices constitute the set $V_1 \cap OddV(K_1)$ and the defective half-vertices constitute the set $V_2 \cap EvenV(K_1)$. In order to show that the union of these two sets is nonempty it suffices to note that

$$(V_1 \cap OddV(K_1)) \cup (V_2 \cap EvenV(K_1)) = (V_2 \cap V(K)) \oplus OddV(K_1)\,, \qquad (2.1)$$

the right-hand side being the symmetric difference of $V_2 \cap V(K)$ and $OddV(K_1)$. Now observe that $V_2 \cap V(K)$ is odd-sized by the assumption that K is bad. And, since $OddV(K_1)$ is even-sized by the handshake lemma, we conclude that $(V_2 \cap V(K)) \oplus OddV(K_1)$ is odd-sized. Thus, the union of (2.1) is indeed nonempty, i.e., K contains a defective vertex or a defective half-vertex. □

We shall establish the desired equality $\mathrm{def}(D) = n(G_D) - \alpha'(G_D)$ by showing that each of the two opposed inequalities $\mathrm{def}(D) \geq n(G_D) - \alpha'(G_D)$ and $\mathrm{def}(D) \leq n(G_D) - \alpha'(G_D)$ holds. In order to demonstrate the former inequality, we will need the following auxiliary result.

Claim 2.8.2. *Let $G[X,Y]$ be a simple bipartite graph such that for each vertex $v \in X$ the degree $d_G(v)$ is at most 2 and for each vertex $w \in Y$ the degree $d_G(w)$ is positive. If $|X| = m$ and $|Y| = n$ then G contains at least $n - m$ pairwise vertex-disjoint 2-paths whose interior vertices belong to X.*

Proof of Claim 2.8.2. Let $p_2(G)$ be the maximum size of a set of pairwise vertex-disjoint 2-paths in G with all interior vertices in X. We prove that $p_2(G) \geq n - m$ by induction on the number $x_2(G)$ of 2-vertices[2] contained in X. If $x_2(G) = 0$ then every vertex $v \in X$ is of degree at most 1. Since every vertex $w \in Y$ is of degree at least 1, we have that

$$n - m = \sum_{w \in Y} 1 - \sum_{v \in X} 1 \leq \sum_{w \in Y} d_G(w) - \sum_{v \in X} d_G(v) = 0 = p_2(G)\,.$$

Assuming $x_2(G) \geq 1$, select a 2-vertex $v_0 \in X$. Define $X' = X\backslash\{v_0\}$ and $Y' = Y\backslash N_G(v_0)$, and let $m' = |X'|$ and $n' = |Y'|$; thus, $m' = m - 1$ and $n' = n - 2$. Note that the induced subgraph $G'[X', Y']$ meets the degree conditions. Since $x_2(G') = x_2(G) - 1$, the inductive hypothesis gives $n' - m' \leq p_2(G')$. Therefore, as clearly $p_2(G') \leq p_2(G) - 1$, we deduce that

$$n - m = (n' - m') + 1 \leq p_2(G') + 1 \leq p_2(G)\,,$$

which completes the inductive argument. □

[2]A vertex v of a graph G is said to be a *d-vertex* if $d_G(x) = d$.

We are ready to show one of the two opposed inequalities stated above.

Claim 2.8.3. $\operatorname{def}(D) \geq n(G_D) - \alpha'(G_D)$.

Proof of Claim 2.8.3. Returning to an arbitrary edge coloring φ of D with color set $\{1, 2\}$, we construct a simple bipartite graph $G[X, Y]$ as follows. Let X be the set of defective vertices in D under φ. Let Y be the set of components of $\mathrm{BC}(D)$. Join a defective vertex v with a bad component K if K contains v or contains a defective half-vertex originating from v. By Claim 1, the obtained graph $G[X, Y]$ meets the requirements of Claim 2. Consequently, there are $n(G_D) - |X|$ pairwise disjoint 2-paths in $G[X, Y]$ whose interiors belong to X. However, this clearly gives a matching in G_D of size $n(G_D) - |X|$; simply for every such 2-path $y_1 x y_2$ from $G[X, Y]$ assign the edge $v_{y_1} v_{y_2}$ to the matching of G_D. We conclude that $\alpha'(G_D) \geq n(G_D) - |X|$. Equivalently, $|X| \geq n(G_D) - \alpha'(G_D)$. The arbitrariness of φ yields the desired inequality. □

In order to complete the proof of Theorem 2.8 we also need to prove the opposite inequality.

Claim 2.8.4. $\operatorname{def}(D) \leq n(G_D) - \alpha'(G_D)$.

Proof of Claim 2.8.4. Consider a maximum matching $M = \{v_{K_{2i-1}} v_{K_{2i}} : 1 \leq i \leq k\}$ in G_D. Returning to $\mathrm{PS}(D)$, for each $i \in \{1, \ldots, k\}$ let $v_i \in V_2$ be a vertex such that $\{v_i^+, v_i^-\}$ intersects both K_{2i-1} and K_{2i}. We color the edges of each nonempty component K of $\mathrm{PS}(D)$ as described below. And for this, we define first an even-sized subset T of $V(K)$:

- If $K \in \{K_1, K_2, \ldots, K_{2k}\}$ then define $T = (V(K) \cap V_2) \backslash \{v_1^+, v_1^-, \ldots, v_k^+, v_k^-\}$. The intersection $V(K) \cap V_2$ is odd-sized and contains precisely one of the vertices $v_1^+, v_1^-, \ldots, v_k^+, v_k^-$; hence, T is even-sized.

- If K is a component of $\mathrm{BC}(D) - \bigcup_{i=1}^{2k} K_i$ then there exists $w_K \in V_2$ such that the intersection $\{w_K^+, w_K^-\} \cap V(K)$ is a singleton; moreover, the 'other half' of $\{w_K^+, w_K^-\}$ does not fall into another component of $\mathrm{BC}(D) - \bigcup_{i=1}^{2k} K_i$, by the maximality of M. Define $T = (V(K) \cap V_2) \backslash \{w_K^+, w_K^-\}$. Again, $V(K) \cap V_2$ is odd-sized and only one of the vertices w_K^+, w_K^- falls inside $V(K) \cap V_2$; consequently, T is even-sized.

- If K is not a component of $\mathrm{BC}(D)$ then (as in the proof of Theorem 2.2) we distinguish between two options: in case $V(K) \cap V_2$ is even-sized, define $T = V(K) \cap V_2$; otherwise, as $V(K) \cap V_3 \neq \emptyset$, select an odd-sized subset $S \subseteq V(K) \cap V_3$ and define $T = (V(K) \cap V_2) \cup S$. Obviously, T is even-sized.

By construction, T is always an even-sized subset of $V(K)$. Therefore, there exists a T-join H of K. Color $E(H)$ with 1 and $E(K) \backslash E(H)$ with 2. After this has been done for every component K of $\mathrm{PS}(D)$, consider the inherited edge coloring of D. The set of its defective vertices is precisely $R = \{v_1, \ldots, v_k\} \cup \{w_K : K$ is a component of $\mathrm{BC}(D) - \bigcup_{i=1}^{2k} K_i\}$. Indeed, by construction we have the following:

- no vertex $u \in V(D) \cap V(\mathrm{PS}(D))$ is defective because one of the colors 1 and 2 has an odd number of appearances on each of the odd-sized semi-cuts of u (as both $d_1(u)$ and $d_2(u)$ are even);

- a vertex $v \in V(D)\backslash V(\mathrm{PS}(D))$ is defective if and only if $v \in R$ because those are the only v's for which some $v^{\pm} \in \{v^+, v^-\}$ is a nonisolated vertex of $\mathrm{PS}(D)$ such that both $d_1(v^{\pm})$ and $d_2(v^{\pm})$ are even (possibly zero).

Thus, the total number of defective vertices equals $n(G_D) - \alpha'(G_D)$, which confirms the desired inequality $\operatorname{def}(D) \leq n(G_D) - \alpha'(G_D)$. □

Proof of Theorem 2.8*, continued:* From Claims 2.8.3 and 2.8.4 it follows that $\operatorname{def}(D) = n(G_D) - \alpha'(G_D)$. □

Let us reconsider our examples. Since $G_{D'_n} = K_{1,n}$ and $G_{D''_n} = C_{2n}$, we have $n(G_{D'_n}) = n + 1$ and $\alpha'(G_{D'_n}) = 1$, whereas $n(G_{D''_n}) = 2n$ and $\alpha'(G_{D''_n}) = n$; thus, in view of Theorem 2.8, $\operatorname{def}(D'_n) = \operatorname{def}(D''_n) = n$.

Note that Theorem 2.8 and Proposition 2.7 combined provide an answer to our previous question about the complexity of finding the defect of a digraph.

Proposition 2.9. *The parameter* $\operatorname{def}(D)$ *can be determined in polynomial time. Moreover, the problem of finding the defect of a digraph is polynomially equivalent to the problem finding the matching number of a graph.*

Another immediate consequence of Theorem 2.8 is the following.

Corollary 2.10. *For every digraph D it holds that*

$$\left\lceil \frac{n(G_D)}{2} \right\rceil \leq \operatorname{def}(D) \leq n(G_D)\,.$$

With all being said, it is clear that, in general, there is no 'directed analogue' of Proposition 2.5, which served as our initial motivation here. In other words, there are digraphs that have arbitrarily many 'defective vertices'.

3 Characterizations in terms of χ'_{wo} and def

We consider two classes of digraphs: the class $\mathcal{AD} = \{D(G) : G \text{ is a pseudograph}\}$ of digraphs $D(G)$ that are associated to graphs G, and the class $\mathcal{T}$ of tournaments.

3.1 Associated digraphs

We shall use here an additional convention hinted in the introduction. Namely, define $\chi'_{\mathrm{wo}}(G) = \infty$ for each graph G that contains 'isolated loops'. The following theorem characterizes the associated digraphs in terms of their weak-odd chromatic index.

Theorem 3.1. *For any connected graph G, it holds that*

$$\chi'_{\mathrm{wo}}(D(G)) = \begin{cases} 0 & \text{if } G = K_1, \\ 1 & \text{if } G \text{ is an odd graph}, \\ 3 & \text{if } G \text{ is an even bipartite graph of odd order} \geq 3, \\ 2 & \text{otherwise}. \end{cases}$$

. Let $D = D(G)$. It always holds that

$$\chi'_{\mathrm{wo}}(D) \leq \chi'_{\mathrm{wo}}(G)\,. \tag{3.1}$$

Indeed, say φ is a weak-odd $\chi'_{\mathrm{wo}}(G)$-edge coloring of G. The accompanying edge coloring φ_D of D assigns to any pair of arcs stemming from an edge e in G color $\varphi(e)$. Consequently, φ_D is a weak-odd $\chi'_{\mathrm{wo}}(G)$-edge coloring of $D(G)$, which settles (3.1).

Thus, the nontrivial part of Theorem 3.1 amounts to showing the following equivalence:

$$\chi'_{\mathrm{wo}}(D) = 3 \quad \Leftrightarrow \quad G \text{ is an even bipartite graph of odd order } n(G) \geq 3\,. \tag{3.2}$$

In view of inequality (3.1) and Theorem 1.1, we may confine to G being an even graph of odd order $n(G) \geq 3$. With that assumption, clearly $\mathrm{PS}(D) = \mathrm{BG}(D)$ and $V_1 = V_3 = \varnothing$. Therefore, by Theorem 2.2, the equality $\chi'_{\mathrm{wo}}(D) = 3$ holds true if and only if some nonempty component of $\mathrm{BG}(D)$ is of odd order. Consequently, the proof of equivalence (3.2) will be complete if we establish the following two assertions.

Claim 3.1.1. *If G bipartite, then $\mathrm{BG}(D) = G \cup G$, i.e., $\mathrm{BG}(D)$ consists of two vertex-disjoint copies of G.*

Claim 3.1.2. *If G is not bipartite, then $\mathrm{BG}(D)$ is connected.*

A moment's reflection reveals that Claim 3.1.1 is implied by the definitions of 'associated digraph' and 'split'. For if $G = G[V_1, V_2]$ is a bipartite graph with bipartition (V_1, V_2), then $\mathrm{BG}(D) = G[V_1^+, V_2^-] \cup G[V_1^-, V_2^+]$, that is, $\mathrm{BG}(D)$ is the disjoint union of two bipartite graphs, with respective bipartitions (V_1^+, V_2^-) and (V_1^-, V_2^+), each of which is isomorphic to G.

As for the demonstration of Claim 3.1.2, let x, y be an arbitrary pair of (not necessarily distinct) vertices of G (and thus of D). In order to show the existence of an x^+-y^- walk in $\mathrm{BG}(D)$, it suffices to find an x-y walk of odd length in G. Indeed, any such walk $W : xv_1v_2\cdots v_{2k}y$ would yield a walk $W^{\pm} : x^+v_1^-v_2^+\cdots v_{2k-1}^-v_{2k}^+y^-$ in $\mathrm{BG}(D)$.

Consider an odd cycle C of G. Let P and Q, respectively, be an x-C and a y-C path in G. Denote by v_x and v_y the (not necessarily distinct) endpoints of P and Q in C. Of the two v_xv_y arcs of C, let L be the one whose length is of opposite parity than the combined length $\ell_P + \ell_Q$ of P and Q. Then $P \cup L \cup Q$ gives rise to a desired x-y walk of odd length.

A similar argument proves the existence of an x^+-y^+ walk in $\mathrm{BG}(D)$; it suffices to find an x-y walk of even length in G, which can be done by using the other v_xv_y arc of C in the previous argument. The existence of x^--y^+ and x^--y^- walks in $\mathrm{BG}(D)$ for an arbitrary pair of vertices x and y in G now follows by symmetry. □

An immediate consequence of Theorem 3.1 and inequality (3.1) is the following.

Corollary 3.2. *If G is a connected graph, then $\chi'_{\mathrm{wo}}(D(G)) \leq \chi'_{\mathrm{wo}}(G)$. Moreover, equality holds unless G is an even nonbipartite graph of odd order.*

Let us characterize the associated digraphs in terms of their defect.

Proposition 3.3. *For any connected graph G, it holds that*

$$\mathrm{def}(D(G)) = \begin{cases} 1 & \text{if } G \text{ is an even bipartite graph of odd order} \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.1, unless G is an even bipartite graph of odd order $n(G) \geq 3$, it holds that $\operatorname{def}(D(G)) = 0$. On the other hand, assuming G is an even bipartite graph of odd order $n(G) \geq 3$, by the proof of Theorem 3.1, $\mathrm{PS}(D) = \mathrm{BG}(D) = G \cup G$. Thus, $\mathrm{BC}(D) = G \cup G$ and $G_{D(G)} = K_2$. Consequently, by Theorem 2.8, $\operatorname{def}(D_G) = 1$. □

Taking into account the established inequality $\operatorname{def}(D(G)) \leq 1$, one naturally wonders if an analogue of Proposition 2.5 holds for all associated digraphs. The following proposition answers this in the positive.

Proposition 3.4. *Every connected associated digraph D admits a 2-edge coloring such that condition $(\overrightarrow{\mathrm{WO}})$ is satisfied at each vertex apart from a prescribed vertex $v \in V(D)$.*

Proof. Let $D = D(G)$. We may assume that G is an even bipartite graph of odd order $n(G) \geq 3$. As already observed in the proof of Proposition 3.3, it holds that $\mathrm{PS}(D) = G \cup G$. Let $T = V(G) \backslash \{v\}$, and take a T-join H of G. Color the edges of $\mathrm{PS}(D)$ with color set $\{1, 2\}$ as follows: in each copy of G, color $E(H)$ by 1 and $E(G) \backslash E(H)$ by 2. The inherited 2-edge coloring of D meets the requirements. □

3.2 Tournaments

In view of Proposition 2.4, there exist tournaments that require three colors for a weak-odd edge coloring; namely, as every tournament of odd order with a single peripheral vertex meets the requirements of the aforementioned proposition, its weak-odd chromatic index equals 3. Our characterization below asserts that those tournaments are the only 'exceptions' to weak-odd 2-edge colorability in the class $\mathcal{T}$. The proof shall make use of the following classical results on tournaments.

Given a digraph D, spanning directed paths and cycles are referred to as *hamiltonian paths* and *hamiltonian cycles*, respectively. Back in 1959, Camion [3] proved that a nontrivial tournament is strong if and only if it contains a hamiltonian cycle. (In fact, this basic result was later on improved, first by Harary and Moser [8], and shortly after by Moon, see, e.g., [9], but for our purposes the initial result of Camion will suffice.) Another basic theorem on tournaments of an even earlier date, due to Rédei [13], is that every tournament (not necessarily strong) has a hamiltonian path. (In fact, Rédei [13] proved that every tournament contains an odd number of hamiltonian paths.)

Theorem 3.5. *For any tournament T, it holds that*

$$\chi'_{\mathrm{wo}}(T) = \begin{cases} 0 & \text{if } T = K_1, \\ 1 & \text{if } T \text{ is nontrivial and every vertex semi-degree is odd or zero}, \\ 3 & \text{if } T \text{ is nontrivial, of odd order, and has just one peripheral vertex}, \\ 2 & \text{otherwise}. \end{cases}$$

Proof. For simplicity of presentation, call every nontrivial tournament of odd order having only one peripheral vertex *bad* and call every other tournament *good*. By Proposition 2.4 and Theorem 2.1, the nontrivial aspect of the proof consists of showing that:

Every good tournament is weak-odd 2-edge colorable.

Consider a good tournament T. If it has two peripheral vertices, then the following furnishes a weak-odd 2-edge coloring: take a hamiltonian path P of T, color $A(P)$ with

1 and the rest of $A(T)$ with 2. Since the initial (resp. terminal) vertex of P is the source (resp. sink) of T, color 1 satisfies condition($\overrightarrow{\text{WO}}$) at all vertices of T. Similarly, if every strong component of T is nontrivial, then a simple construction of a weak-odd 2-edge coloring can be obtained as follows: in every strong component K of T take a hamiltonian cycle C_K, color $\bigcup_K A(C_K)$ with 1 and the rest of $A(T)$ with 2. Then, color 1 meets condition ($\overrightarrow{\text{WO}}$) everywhere.

Hence, we may assume that *there exist a nontrivial peripheral strong component and a trivial strong component of* T. We complete the proof by distinguishing between two cases.

Case 1: *Both peripheral strong components of* T *are nontrivial.* Let K_i and K_t be the initial and terminal strong components of T. There exists a directed K_i-K_t path P in T that passes through every vertex $v \notin V(K_i) \cup V(K_j)$. Indeed, simply take a hamiltonian path in the 'multitournament' $T/\{K_i, K_t\}$, i.e., the directed multigraph obtained from T by contracting $V(K_i)$ and $V(K_t)$ into a pair of new vertices. By the above assumptions, the path P is of length $\ell(P) > 1$. Denote by x and y, respectively, the initial and terminal vertex of P. Thus, the arc $xy \in A(T)\backslash A(P)$. Let C_i and C_t, respectively, be hamiltonian cycles in K_i and K_j. The arc set $A(C_i) \cup A(P) \cup \{xy\} \cup A(C_t)$ induces a spanning subdigraph of T with all semi-degrees odd. By coloring $A(C_i) \cup A(P) \cup \{xy\} \cup A(C_t)$ with 1 and the rest of $A(T)$ with 2 we complete a weak-odd 2-edge coloring of T, because color 1 meets condition ($\overrightarrow{\text{WO}}$) everywhere.

Case 2: *One peripheral strong component of* T *is trivial.* Since T is good, it has even order. We may assume T has a sink, say y. Let K_i be the initial strong component of T, and C_i be a hamiltonian cycle in K_i. If $V(T) = V(K_i) \cup \{y\}$, then we are done by coloring $A(C_i)$ with 1 and the rest of $A(T)$ with 2. Namely, color 2 meets condition ($\overrightarrow{\text{WO}}$) at y, and color 1 takes care of every other vertex.

Assuming $V(T) \neq V(K_i) \cup \{y\}$, take a directed K_i-y path P in T that passes through every vertex $v \notin V(K_i)$ (a hamiltonian path in the 'multitournament' T/K_i, the directed multigraph obtained from T by contracting $V(K_i)$ into a vertex, will do). Let x be the initial vertex of P; thus, $V(C_i) \cap V(P) = \{x\}$. By our latest assumption, the arc $xy \notin A(P)$. Consequently, the arc set $A(C_i) \cup A(P) \cup \{xy\}$ induces a spanning subdigraph of T such that both the semi-degrees of y are even whereas the rest of the semi-degrees are odd. Therefore, as $d_T(y)$ is odd, by coloring $A(C_i) \cup A(P) \cup \{xy\}$ with 1 and the rest of $A(T)$ with 2 we obtain a weak-odd 2-edge coloring of T. Indeed, once again color 2 meets condition ($\overrightarrow{\text{WO}}$) at y, and color 1 takes care of every other vertex. □

Let us characterize the members of the class $\mathcal{T}$ in terms of their defect.

Proposition 3.6. *For any tournament* T*, it holds that*

$$\operatorname{def}(T) = \begin{cases} 1 & \textit{if } T \textit{ is nontrivial, of odd order, and just one vertex semi-degree is zero}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By Theorem 3.5, we may assume that T is nontrivial, of odd order, and just one vertex semi-degree is zero. Say T has a sink y. Apply to $E(T)$ the particular 2-edge coloring(s) constructed for Case 2 in the proof of Theorem 3.5. Observe that condition ($\overrightarrow{\text{WO}}$) is satisfied at each vertex apart from y. □

Therefore, as for the class of associated digraphs, the inequality $\mathrm{def}(T) \le 1$ holds for every tournament T. Our final proposition shows that an analogue of Proposition 2.5 also holds for all tournaments.

Proposition 3.7. *Every tournament T admits a 2-edge coloring such that condition $(\overrightarrow{\mathrm{WO}})$ is satisfied at each vertex apart from a prescribed vertex $v \in V(T)$.*

Proof. Again, we may assume that T is nontrivial, of odd order, and with just one peripheral vertex, say a sink y. Note that $\mathrm{BG}(T)$ has only two components, and moreover, one of those components consists of the isolated vertex y^+. Indeed, by our assumptions, every vertex $w \in V(T)\backslash\{y\}$ dominates y and has $d_T^-(w) > 0$; hence, the component containing y^- also includes both w^+ and w^-. Consequently, $\mathrm{PS}(T)$ has only one nonempty component K and only one empty component $\{y^+\}$. Observe that $V(K) \cap V_2 = V_2\backslash\{y^+\}$ is odd-sized, and $V_3 = \varnothing$. Define an even-sized subset $S \subseteq V(K)$ as follows:

- if $v \in V_1$ then $S = \{v\} \cup (V_2\backslash\{y^+\})$;
- if $v \notin V_1$ then $S = V_2\backslash\{v^-, y^+\}$.

The rest should be clear. We simply take an S-join H of K, and then color $E(H)$ with 1 and $E(K)\backslash E(H)$ with 2. The inherited 2-edge coloring of T meets the requirements. □

4 Concluding remarks and further work

For a graph G (resp. digraph D), an *edge covering* with *color set* S is a mapping that assigns to each edge of G (resp. arc of D) a nonempty subset of S; what distinguishes coverings from colorings is that we allow more than one color per edge (resp. arc). Related notions to weak-odd edge colorings of graphs and digraphs, respectively, are the *weak-odd edge coverings* defined as edge coverings such that conditions (WO) and $(\overrightarrow{\mathrm{WO}})$ are fulfilled. It is known that most of the graphs and digraphs are weak-odd 3-edge colorable. Can a color always be saved by switching to coverings? The answer to this question in the realm of graphs is affirmative. Indeed, the following holds true.

Proposition 4.1. *Any connected graph G whose edge set does not consist entirely of loops, admits a weak-odd 2-edge covering such that the intersection of color classes is contained within a prescribed singleton $\{e\} \subseteq E(G)$.*

Proof. By Theorem 1.1, we may assume that G is a nontrivial even graph of odd order. Subdivide the edge e, and take an odd factor H of the obtained graph. Color $E(G)\cap E(H)$ with 1, $E(G)\backslash(E(H) \cup \{e\})$ with 2, and assign both colors 1 and 2 to the edge e. It is readily seen that the constructed edge covering meets the requirements. □

Following this line of reasoning, we find the next question interesting.

Question 4.2. Does every digraph admit a weak-odd 2-edge covering?

Presuming Question 4.2 answers in positive, define $\mathrm{ovl}(D)$, the *overlapping* of D, to be the minimum possible size of the intersection of the two color classes in an arbitrary weak-odd 2-edge covering of D. In view of the families of digraphs D'_n and D''_n (depicted in Figure 7), it is easily seen that $\mathrm{ovl}(D)$ is not bounded over the class of digraphs; moreover, it can acquire any possible value from the set of naturals. We are tempted to wonder whether this parameter also relates to some 'classical graph parameter', much as like $\operatorname{def}(D)$ relates to the maximum matching number of graphs.

Following the direction explored in Section 3, it may be interesting to characterize other digraph families in terms of their weak-odd chromatic index and their defect. Since tournaments proved to have a nice behavior with respect these parameters, a natural next step is to consider families of digraphs generalizing tournaments.

Three classic generalizations of tournaments that come to mind are semicomplete digraphs, extended tournaments and multipartite tournaments. A digraph is *semicomplete* if it is obtained from a complete graph by replacing each edge uv by the arc (u, v), the arc (v, u) or the pair of arcs (u, v) and (v, u). An *extended tournament* is a digraph obtained from a tournament by blowing up some of its vertices into stable sets. A *multipartite tournament* is an orientation of a complete multipartite graph.

Problem 4.3. Characterize the families of semicomplete digraphs, extended tournaments and multipartite tournaments in terms of their weak-odd chromatic index.

We think that the following question should be addressed before stating the analogous problem for the characterization in terms of the defect.

Question 4.4. Is there a constant c such that $\operatorname{def}(D) \leq c$ for every digraph D such that

- D is semicomplete?
- D is an extended tournament?
- D is a multipartite tournament?

A positive answer for Question 4.4 would open the door to consider the following problem.

Problem 4.5. Characterize the families of semicomplete digraphs, extended tournaments and multipartite tournaments in terms of their defect.

ORCID iDs

César Hernández-Cruz https://orcid.org/0000-0002-5867-3801
Mirko Petruševski https://orcid.org/0000-0001-8048-7418
Riste Škrekovski https://orcid.org/0000-0001-6851-3214

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A-trails of embedded graphs and twisted duals*

Qi Yan †

School of Mathematics, China University of Mining and Technology, Xuzhou 221116, P. R. China, and
School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China

Xian'an Jin ‡

School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China

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Abstract

Kotzig showed that every connected 4-regular plane graph has an A-trail—an Eulerian circuit that turns either left or right at each vertex. However, this statement is not true for Eulerian plane graphs and determining if an Eulerian plane graph has an A-trail is NP-hard. The aim of this paper is to give a characterization of Eulerian embedded graphs having an A-trail. Andersen et al. showed the existence of orthogonal pairs of A-trails in checkerboard colourable 4-regular graphs embedded on the plane, torus and projective plane. A problem posed in their paper is to characterize Eulerian embedded graphs (not necessarily checkerboard colourable) which contain two orthogonal A-trails. In this article, we solve this problem in terms of twisted duals. Several related results are also obtained.

Keywords: Embedded graphs, twisted duals, Eulerian, A-trails, checkerboard colourable.

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1 Introduction

A *cellularly embedded graph* is a graph G embedded in a surface Σ such that every connected component of $\Sigma - G$ is a 2-cell, called a *face*. We use the term *embedded graph* loosely to mean any of three equivalent representations of graphs in surfaces: cellularly embedded graphs, ribbon graphs and arrow presentations. We shall move from one to another

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E-mail addresses: qiyan@cumt.edu.cn (Qi Yan), xajin@xmu.edu.cn (Xian'an Jin)

freely and refer the reader to [5, 6, 10] for details. A *quasi-tree* is an embedded graph with exactly one boundary component (or face). A *bouquet* is an embedded graph with exactly one vertex. They are geometric duals to each other. A *quasi-tree bouquet* is a bouquet that is also a quasi-tree.

In this article, all graphs will be finite, connected, but not necessarily simple. A graph is said to be *Eulerian* if the degree of each of its vertices is even. A graph is *bipartite* if its vertex set can be partitioned into two nonempty subsets X and Y so that every edge has one end in X and one end in Y. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. Note that bipartite graphs might have multiple edges but not loops. A *star* is a bipartite graph $G[X, Y]$ with $|X| = 1$ or $|Y| = 1$. A *parallel graph*, also known as a generalized theta graph especially for 3 or more edges, is a special star $G[X, Y]$ with $|X| = 1$ and $|Y| = 1$. A *walk* in a graph between vertices v_0 and v_k is a sequence of vertices and edges $(v_0, e_1, v_1, e_2, \cdots, e_k, v_k)$, where v_{i-1} and v_i are the endvertices of the edge e_i. A *trail* is a walk with no repeated edge and *circuit* is a trail with $v_0 = v_k$. An *A-trail* [7] in an Eulerian embedded graph G is an eulerian circuit such that every two consecutive edges in the circuit are adjacent in the rotation of the common vertex. A *Petrie walk* in an embedded graph G is such a walk that when traveling along it, we alternatingly turn to the left edge and to the right edge of the current edge in the cyclic rotation around the common vertex. It is obvious that an Eulerian Petrie walk is a special kind of A-trails.

Kotzig [7] showed that every 4-regular plane graph has an A-trail, and sufficient conditions were discovered for the existence of A-trails in 2-connected plane graphs in [2]. The existence of A-trails has been studied almost exclusively in the case of graphs embedded in the plane, the projective plane and the torus. In this paper, we shall characterize general Eulerian embedded graphs having an A-trail in terms of twisted duals. Let $G = (V(G), E(G))$ be an embedded graph. We denote by G^* and $G^\times$ the *geometric dual* and *Petrial* [14] of G, respectively. Let $A \subseteq E(G)$. We denote by $G^{\delta(A)}$ and $G^{\tau(A)}$ the partial dual [4] and partial Petrial of G with respect to A, respectively. Particularly, if $A = E(G)$, then $G^{\delta(E(G))} = G^*$ and $G^{\tau(E(G))} = G^\times$. Partial duality and partial Petriality are further combined together to form twisted duality [1, 5]. The twisted duality has the scope to develop the understanding of a wide variety of graph theoretical problems. We refer the reader to [5, 6] for the details. Petrie walks have some very interesting properties, see [8, 12]. They also play an important role in the design of CMOS VLSI circuits, where it is convenient if the graph representing a circuit has an Eulerian Petrie walk. Žitnik [13] gave a characterization of 4-regular plane graphs with Eulerian Petrie walks. Since Eulerian Petrie walks are a special kind of A-trails, we also give a characterization of Eulerian embedded graphs having an Eulerian Petrie walk. We first obtain the following theorem.

Theorem 1.1. *Let G be an Eulerian embedded graph. Then G has an A-trail if and only if there exists $B \subseteq E(G)$ such that the underlying graph of $(G^{\tau(B)})^*$ is a star. In particular, G has an Eulerian Petrie walk if and only if the underlying graph of $(G^\times)^*$ is a star.*

Andersen, Bouchet and Jackson [3] characterized the 4-regular plane graphs which contain two orthogonal A-trails, that is to say two A-trails for which no subtrail of length 2 appears in both A-trails. They also discussed the corresponding problem for checkerboard colourable graphs embedded in the projective plane and the torus. And they posed the following problem.

Problem 1.2 ([3])**.** We do not have any characterization of 4-regular graphs in the projective plane and the torus having two orthogonal A-trails which we know to be valid also for graphs with no 2-face colouring.

In this paper, we give a general definition of two orthogonal A-trails in Eulerian embedded graphs. We say that two A-trails are *orthogonal* if these two trails have different transitions at each vertex with degree greater than 2. We consider the above problem for general Eulerian embedded graphs which may have high genus and are not necessarily checkerboard colourable and characterize the Eulerian embedded graphs which have two orthogonal A-trails or Eulerian Petrie walks in terms of twisted duals as follows.

Theorem 1.3. *Let G be an embedded graph. Then G has two orthogonal A-trails if and only if there exists $B \subseteq E(G)$ such that the underlying graph of $(G^{\tau(B)})^*$ is a parallel graph. In particular, G has two orthogonal Eulerian Petrie walks if and only if the underlying graph of $(G^{\times})^*$ is a parallel graph.*

In the case of 4-regular embedded graphs, we have the following theorem.

Theorem 1.4. *Let H be a 4-regular embedded graph. Then H has two orthogonal A-trails if and only if there exists $D \subseteq E(H)$ such that $H^{\tau(D)}$ is a medial graph of a quasi-tree bouquet. Particularly, H has two orthogonal Eulerian Petrie walks if and only if $H^{\times}$ is a medial graph of a quasi-tree bouquet.*

2 Preliminaries

In this section we introduce the notions and the tools, which we will need in further Sections 3 and 4. We use standard notations $V(G), E(G)$ and $F(G)$ to denote the sets of vertices, edges, and faces, respectively, of a cellularly embedded graph G and $v(G) = |V(G)|, e(G) = |E(G)|$ and $f(G) = |F(G)|$, respectively. We denote by $d(v)$ the degree of a vertex v in G, i.e. the number of half-edges incident with v. We give a brief review of ribbon graphs referring the reader to [5, 6] for further details.

Definition 2.1 ([6]). A *ribbon graph* $G = (V(G), E(G))$ is a (possibly non-orientable) surface with boundary, represented as the union of two sets of topological discs, a set $V(G)$ of vertices, and a set $E(G)$ of edges such that

1. the vertices and edges intersect in disjoint line segments, we call them *common line segments* as in [9];

2. each such common line segment lies on the boundary of precisely one vertex and precisely one edge;

3. every edge contains exactly two such common line segments.

Let G be a ribbon graph, $v \in V(G)$ and $e \in E(G)$. By deleting the common line segments from the boundary of v, we obtain $d(v)$ disjoint line segments, called *vertex line segments*. By deleting common line segments from the boundary of e, we obtain two disjoint line segments, called *edge line segments*. See Figure 1 for an example of these concepts. We think of each edge line segment having two *half-edge line segments*. It is obvious that every edge disc contains four half-edge line segments which correspond to the four flags incident on that edge. For any vertex line segment, there are exactly two half-edge line segments incident with it as shown in Figure 2.

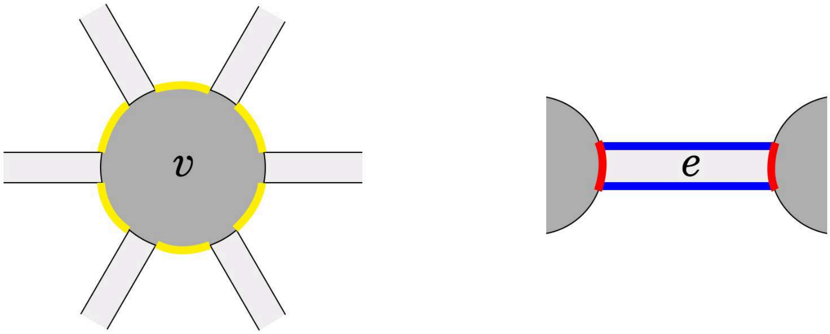

Figure 1: Vertex line segments (yellow), common line segments (red) and edge line segments (blue).

Let G be a ribbon graph and $A \subseteq E(G)$. Then the *partial Petrial*, $G^{\tau(A)}$, of G with respect to A is the ribbon graph obtained from G by adding a half-twist to each of the edges in A. Let $Orb_{(\tau)}(G) = \{G^{\tau(A)} | A \subseteq E(G)\}$ denote the set of all partial Petrials of G. Let H be an arrow presentation and $B \subseteq E(H)$. Then the *partial dual*, $H^{\delta(B)}$, of H with respect to B is the arrow presentation obtained as follows. For each $e \in B$, suppose α and β are the two arrows labelled e in the arrow presentation of H. Draw a line segment with an arrow on it directed from the head of α to the tail of β, and a line segment with an arrow on it directed from the head of β to the tail of α. Label both of these arrows e and delete α and β and the arcs containing them. This process is illustrated locally at a pair of arrows in Figure 3. Let $Orb_{(\delta)}(H) = \{H^{\delta(B)} | B \subseteq E(H)\}$ denote the set of all partial duals of H.

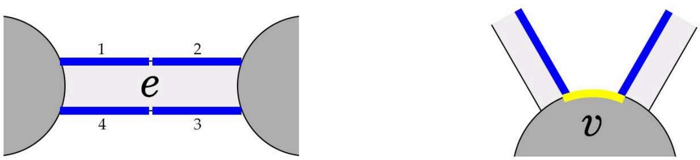

Figure 2: Four half-edge line segments of e, one of the vertex line segments of v and its incident two half-edge line segments.

Let $\mathcal{B} =< \delta, \tau | \delta^2, \tau^2, (\delta\tau)^3 >$.

Definition 2.2 ([5])**.** Let G be a ribbon graph. The ribbon graph H is called a twisted dual (briefly, twual) of G if it can written in the form

$$H = G^{\Pi_{i=1}^{6} \xi_i(A_i)},$$

where the A_i's partition $E(G)$ and the ξ_i's are the six elements of $\mathcal{B}$.

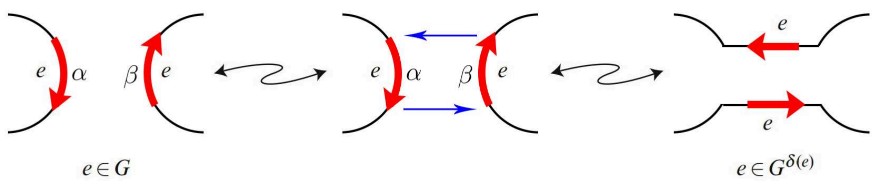

Figure 3: Taking the partial dual of an edge in an arrow presentation.

If G is cellularly embedded in Σ, we construct its *medial graph* G_m in the embedded surface by placing a vertex on each of its edges, and for each face f with boundary $e_1, e_2, \cdots, e_{d(f)}$, drawing $d(f)$ edges $\{e_1, e_2\}, \cdots, \{e_{d(f)}, e_1\}$ inside the face f along the boundary of f. It is obvious that G_m is also cellularly embedded in Σ. In particular, the medial graph of an isolated vertex is a free loop. G is *checkerboard colourable* if the faces of G can be properly 2-coloured. A *checkerboard colouring* of G is a particular proper 2-colouring of the faces of G. Throughout we will use red and blue to refer to the two colours used in a checkerboard colouring. It is obvious that there is a correspondence between checkerboard colouring and *face boundary colouring*, so we shall move from one to another freely. Note that in a checkerboard coloured 4-regular embedded graph G, the *redface graph* G_R of G is the embedded graph constructed by placing one vertex in each red face and adding an edge between two of these vertices whenever the corresponding faces meet at a vertex of G. The *blueface graph* G_B is constructed analogously by placing vertices in the blue faces.

A *Petrie walk* in an embedded graph G is such a walk that when traveling along it, we alternatingly turn to the left edge and to the right edge of the current edge in the cyclic rotation around the common vertex. We shall only consider closed Petrie walks, for which this condition holds also for the last and the first edge of the walk. Petrie walks are sometimes also called *left-right paths*. An example of a Petrie walk is shown in Figure 4, where the dotted curve indicates the order of edges in the walk. Note that the boundary components of faces of the Petrie dual are exactly Petrie walks of the original embedding of G as shown in Figure 5.

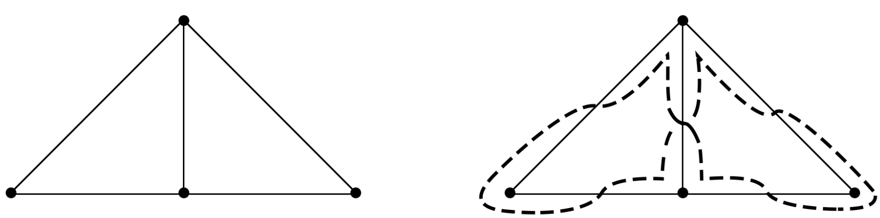

Figure 4: The curve representing a Petrie walk.

Let v be a vertex of G. A *transition* at v is a partition of the half-edges incident to v into

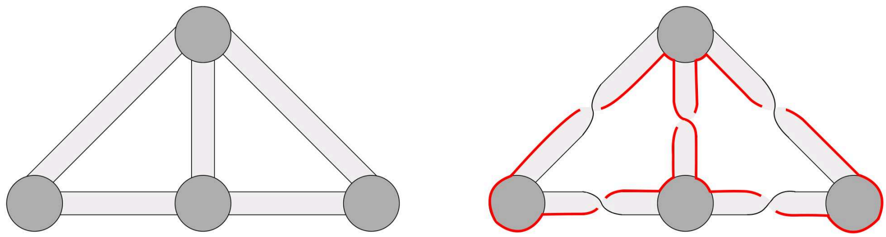

Figure 5: A Petrie walk corresponding to a face of the Petrie dual of the original embedded graph.

pairs. A transition T at v is *smooth* if T only pairs half-edges adjacent in the cyclic order at v given by the embedding of G. A *transition system* of G is a choice of a transition at every vertex of G. A *smooth transition system* is a transition system such that every transition is smooth. It is obvious that we can induce a circuit decomposition of G by a transition system T. Similarly, any circuit decomposition C of G recovers a transition system of G by pairing half-edges traversed consecutively in a circuit of C. A circuit decomposition of G is smooth if it induces a smooth transition system. In particular, an Eulerian circuit that induces a smooth transition system is called an A-trail [11]. Note that there are precisely two disjoint smooth transitions for any vertex v of G with $d(v) \geq 4$ and two edges are consecutive if this is indicated by a curve as shown in Figure 6. We say that two A-trails

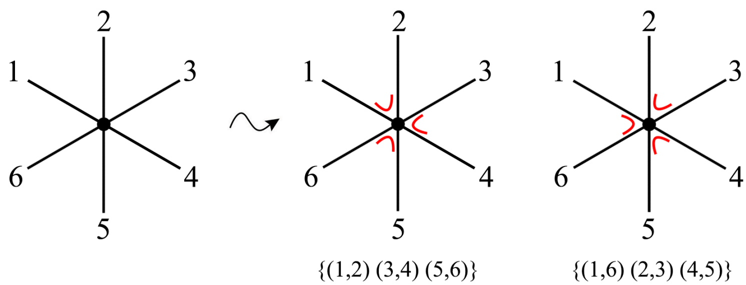

Figure 6: Performing exactly two smooth transitions locally at a vertex.

(or smooth transition systems) of an Eulerian embedded graph G are *orthogonal* if the two trails (or smooth transition systems) have different smooth transitions at each vertex v of G with $d(v) \geq 4$.

Most of the representations of embedded graphs are ribbon graphs in this paper, so we introduce the relation between smooth transition systems and ribbon graphs. Let G be an Eulerian ribbon graph. For every $v \in V(G)$, we assign the colours red and blue to all half-edge line segments and vertex line segments of v such that the colours of vertex line segments are alternating red and blue in the vertex boundary of v and every vertex line segment and two half-edge line segments incident with it assign the same colour. We call this a *checkerboard colouring of half-edge and vertex line segments* of G (see Figure 7 for example). If G is checkerboard coloured of half-edge and vertex line segments, then for any edge disc, there are exactly two cases as shown in Figure 8. The edge is called *consistent* if two half-edge line segments of one of edge line segments have the same colour, and is called *inconsistent* otherwise. Let T be a smooth transition system of G. It is obvious that T induces a checkerboard colouring of half-edge and vertex line segments for any vertex of G as shown in Figure 9. We call this a *canonical checkerboard colouring of half-edge and vertex line segments* by T. An example of a canonical checkerboard colouring of half-edge and vertex line segments by the smooth transition system is given in Figure 10.

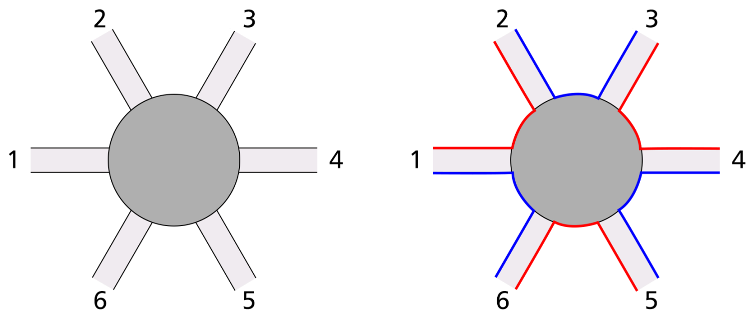

Figure 7: A checkerboard colouring of half-edge and vertex line segments at a vertex.

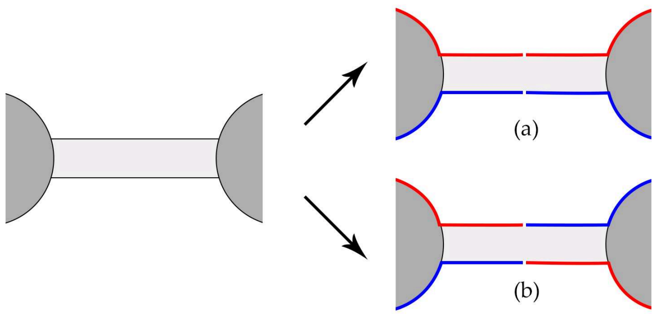

Figure 8: (a) consistent edge (b) inconsistent edge.

3 Main results and proofs

Now we give some characterizations of Eulerian embedded graphs having an A-trail or Eulerian Petrie walk. Kotzig [7] showed that every 4-regular plane graph has an A-trail. We note that this result can be extended to any 4-regular embedded graph.

Lemma 3.1. *Any* 4*-regular embedded graph always has an* A*-trail.*

Proof. Let T be any smooth transition system of a 4-regular embedded graph H. Note that each smooth transition system corresponds to a specific family of edge-disjoint cycles

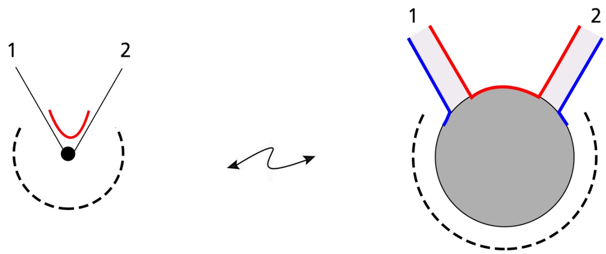

Figure 9: The relation between smooth transition and checkerboard colouring of half-edge and vertex line segments at a vertex.

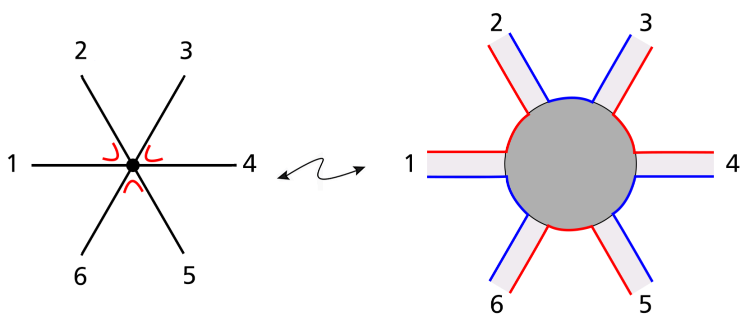

Figure 10: An example of a canonical checkerboard colouring of half-edge and vertex line segments by the smooth transition system.

in H. The number of edge-disjoint cycles in H generated by T will be denoted by $c(T)$. If $c(T) = 1$, then this completes the proof. Otherwise, there exists $v \in V(H)$ such that four half-edges $1, 2, 3, 4$ incident with v are not in the same cycle. Assume that the transition at v is $(1, 2), (3, 4)$. Then we change the smooth transition of v from $(1, 2), (3, 4)$ to $(1, 4), (2, 3)$. Hence, half-edges $1, 2, 3, 4$ are in the same edge-disjoint cycle as shown in Figure 11. Repeating the above process, we obtain a smooth transition system which corresponds to an A-trail. □

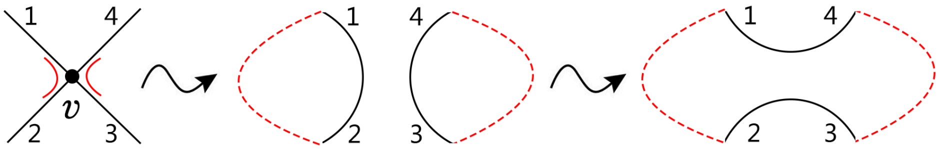

Figure 11: Proof of Lemma 3.1.

Lemma 3.1 can not be further generalized to any Eulerian embedded graph and an example is given in Figure 12. Note that taking partial petrial does not affect cyclic order

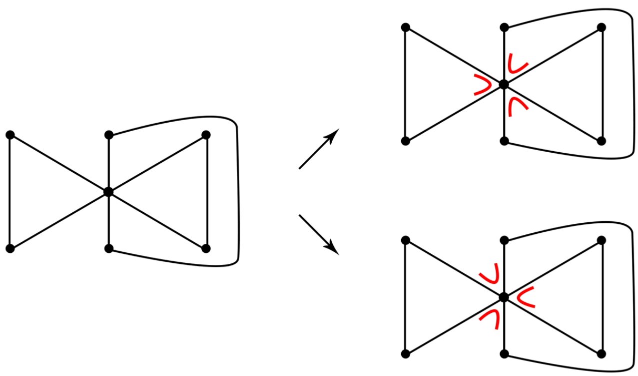

Figure 12: An Eulerian embedded graph which does not have an A-trail.

of half-edges at vertices, we have the following lemma.

Lemma 3.2. *Let G be an Eulerian embedded graph. Then*

(1) *G has an A-trail if and only if every partial Petrial of G has an A-trail.*

(2) *G has two orthogonal A-trails if and only if every partial Petrial of G has two orthogonal A-trails.*

Theorem 1.1. *Let G be an Eulerian embedded graph. Then G has an A-trail if and only if there exists $B \subseteq E(G)$ such that the underlying graph of $(G^{\tau(B)})^*$ is a star. In particular, G has an Eulerian Petrie walk if and only if the underlying graph of $(G^{\times})^*$ is a star.*

Proof. ($\Rightarrow$) Let G be a ribbon graph. If G has an A-trail, then there exists a smooth transition system T of G corresponding to the A-trail. We get a canonical checkerboard colouring of half-edge and vertex line segments according to T. Suppose that B is the set of its inconsistent edges. If the A-trail is an Eulerian Petrie walk, then $B = E(G)$. Thus $G^{\tau(B)}$ obtains a face boundary colouring. It induces a checkerboard colouring with only one red face. Then $(G^{\tau(B)})^*$ is bipartite with $|X| = 1$ or $|Y| = 1$, hence the underlying graph of $(G^{\tau(B)})^*$ is a star.

($\Leftarrow$) Since the underlying graph of $(G^{\tau(B)})^*[X, Y]$ is a star, we assume that $X = \{v\}$. Note that v corresponds to a face of $G^{\tau(B)}$. We assign the colour red to this face and colour blue to other faces of $G^{\tau(B)}$. This is a checkerboard colouring of $G^{\tau(B)}$. Suppose that it is a canonical checkerboard colouring of half-edge and vertex line segments of $G^{\tau(B)}$. Let T be the corresponding smooth transition system. It follows that T induces an A-trail of $G^{\tau(B)}$. Thus, G has an A-trail by Lemma 3.2. If $B = E(G)$, then this is a checkerboard colouring of $G^\times$. Hence, the boundary of the redface of $G^\times$ is an Eulerian Petrie walk of G. □

Remark 3.3. According to Theorem 1.1, suppose that G is an embedded graph whose underlying graph is a star. If $H \in Orb_{(\tau)}(G^*)$, then H has an A-trail. Particularly, if $H = (G^*)^\times$, then H has an Eulerian Petrie walk.

Corollary 3.4. *Let H be a 4-regular embedded graph. Then H has an Eulerian Petrie walk if and only if $H^\times$ is a medial graph of a bouquet.*

Proof. By a similar argument as in the proof of Theorem 1.1, $H^\times$ can obtain a checkerboard colouring with only one red face. Hence, the number of vertices of the redface graph $(H^\times)_R$ is exactly one, that is a bouquet. Therefore, $H^\times$ is a medial graph of a bouquet. Conversely, let G be a bouquet and $H^\times$ is the medial graph of G. We give G_m a checkerboard colouring where the red faces contain the vertices of G. Then the number of red faces is exactly one. Hence, $(G_m)^*$ is a star. Thus, $(G_m)^\times$ has an Eulerian Petrie walk by Remark 3.3. Since $G_m = H^\times$, we can see that H has an Eulerian Petrie walk. □

Theorem 1.3. *Let G be an embedded graph. Then G has two orthogonal A-trails if and only if there exists $B \subseteq E(G)$ such that the underlying graph of $(G^{\tau(B)})^*$ is a parallel graph. In particular, G has two orthogonal Eulerian Petrie walks if and only if the underlying graph of $(G^\times)^*$ is a parallel graph.*

Proof. ($\Rightarrow$) Assume that T and T' are two orthogonal smooth transition systems recovering from the two orthogonal A-trails of a ribbon graph G. Then we get a canonical checkerboard colouring of half-edge and vertex line segments according to T. Suppose that B is the set of its inconsistent edges. If the A-trail is an Eulerian Petrie walk, then $B = E(G)$. By the same argument as in the proof of Theorem 1.1, there is a face boundary colouring of $G^{\tau(B)}$ such that the number of colour red face boundaries is exactly one. Note that T' corresponds to the blue face boundaries of $G^{\tau(B)}$. It follows that the number of colour blue face boundary is also exactly one. Hence, the vertex set of $(G^{\tau(B)})^*$ can be partitioned into two subsets X and Y with $|X| = |Y| = 1$. Thus, the underlying graph of $(G^{\tau(B)})^*$ is a parallel graph.

($\Leftarrow$) Since the underlying graph of $(G^{\tau(B)})^*$ is a parallel graph with vertices v and w. Note that the number of face boundaries of $G^{\tau(B)}$ are two, which correspond to v and w, respectively. We give one colour red and another colour blue. Hence, this is a checkerboard colouring of $G^{\tau(B)}$. Let T and T' be the corresponding two orthogonal smooth transition systems to the red face boundary and the blue face boundary, respectively. It is obvious that T and T' induce two orthogonal A-trails of $G^{\tau(B)}$. Thus, G has two orthogonal A-trails by Lemma 3.2. If $B = E(G)$, then this is a checkerboard colouring of $G^{\times}$. Note that the redface and blueface of $G^{\times}$ are both Eulerian Petrie walks of G. Hence, G has two orthogonal Eulerian Petrie walks. □

Remark 3.5. According to Theorem 1.3, suppose that G is an embedded graph whose underlying graph is a parallel graph. If $H \in Orb_{(\tau)}(G^*)$, then H has two orthogonal A-trails. In particular, if $H = (G^*)^{\times}$, then H has two orthogonal Eulerian Petrie walks.

Theorem 1.4. *Let H be a 4-regular embedded graph. Then H has two orthogonal A-trails if and only if there exists $D \subseteq E(H)$ such that $H^{\tau(D)}$ is a medial graph of a quasi-tree bouquet. Particularly, H has two orthogonal Eulerian Petrie walks if and only if $H^{\times}$ is a medial graph of a quasi-tree bouquet.*

Proof. ($\Rightarrow$) By the same argument as in the proof of Theorem 1.3, there exists $D \subseteq E(H)$ such that there is a checkerboard colouring of $H^{\tau(D)}$ which the number of colour red face and blue face are both exactly one, that is, the number of vertices of $(H^{\tau(D)})_R$ and $(H^{\tau(D)})_B$ are both exactly one. Note the number of face of $(H^{\tau(D)})_R$ is also one, since $(H^{\tau(D)})_R$ and $(H^{\tau(D)})_B$ are geometric duals. Therefore, $(H^{\tau(D)})_R$ is a quasi-tree bouquet. Hence, $H^{\tau(D)}$ is a medial graph of a quasi-tree bouquet.

($\Leftarrow$) Let G be a quasi-tree bouquet and $H^{\tau(D)}$ be the medial graph of G. Since G and G_m embedded in the same surface, we have $v(G) - e(G) + f(G) = v(G_m) - e(G_m) + f(G_m)$ by Euler characteristic. Note that $v(G_m) = e(G), e(G_m) = 2v(G_m) = 2e(G)$. Hence, $f(G_m) = v(G) + f(G) = 2$. Thus, $(G_m)^*$ is a parallel graph since G_m is checkerboard colourable and $f(G_m) = 2$. Then G_m has two orthogonal A-trails by Theorem 1.3, that is, $H^{\tau(D)}$ has two orthogonal A-trails. Hence, H has two orthogonal A-trails by Lemma 3.2. If $D = E(H)$, then $(H^{\times})^*$ is a parallel graph. It follows that H has two orthogonal Eulerian Petrie walks by Remark 3.5. □

Remark 3.6. According to Theorem 1.4, suppose that G is a quasi-tree bouquet. If $H \in Orb_{(\tau)}(G_m)$, then H has two orthogonal A-trails. In particular, if $H = (G_m)^{\times}$, then H has two orthogonal Eulerian Petrie walks.

Lemma 3.7. *Let G be an embedded graph. Then $E(G)$ can be partitioned into two edge disjoint spanning quasi-trees if and only if G is the partial dual of a quasi-tree bouquet.*

Proof. Suppose A and A^c are the edge sets of two spanning quasi-trees which partition $E(G)$, then $G^{\delta(A)}$ is a bouquet since the vertex boundaries of $G^{\delta(A)}$ correspond to the face boundaries of $(V(G), A)$ which is a spanning quasi-tree. Thus, $G^{\delta(A^c)}$ is also a bouquet by the similar discussion. Note that $G^{\delta(A)} = (G^{\delta(A^c)})^*$. Hence, $G^{\delta(A)}$ is a quasi-tree bouquet, that is, G is the partial dual of a quasi-tree bouquet. Conversely, there exists $A \subseteq E(G)$ such that $G^{\delta(A)}$ is a quasi-tree bouquet. Then $G^{\delta(A^c)}$ is also a bouquet. It follows that $(V(G), A)$ and $(V(G), A^c)$ are both spanning quasi-trees, that is, $E(G)$ can be partitioned into two edge disjoint spanning quasi-trees. □

Lemma 3.8 ([5]). *Let G be an embedded graph. Then*

$$Orb_{(\delta)}(G) = \{H | G_m \textit{ and } H_m \textit{ are partial Petrials}\}.$$

Corollary 3.9. *Let H be a checkerboard coloured 4-regular embedded graph. Then H has two orthogonal A-trails if and only if the edges of the redface graph H_R can be partitioned into two edge disjoint spanning quasi-trees.*

Proof. $E(H_R)$ can be partitioned into two edge disjoint spanning quasi-trees if and only if H_R is the partial dual of a quasi-tree bouquet by Lemma 3.7, that is, there exists $D \subseteq E(H)$ such that $H^{\tau(D)}$ is a medial graph of a quasi-tree bouquet by Lemma 3.8, if and only if H has two orthogonal A-trails by Theorem 1.4. □

Corollary 3.10. *Let H be a checkerboard coloured 4-regular orientable embedded graph which has two orthogonal A-trails. Then $v(H)$ is even.*

Proof. $E(H_R)$ can be partitioned into two edge disjoint spanning quasi-trees by Corollary 3.9. Denote these two spanning quasi-trees by $G_1 = (V(H_R), E_1)$, $G_2 = (V(H_R), E_2)$, respectively. Obviously, G_1 and G_2 are both orientable. Then $v(H_R) - |E_1| + 1 = 2 - 2g(G_1)$ and $v(H_R) - |E_2| + 1 = 2 - 2g(G_2)$, where $g(G_1)$ and $g(G_2)$ are the genera of G_1 and G_2, respectively. Hence, $|E_1| + |E_2| = 2v(H_R) + 2g(G_1) + 2g(G_2) - 2$. It follows that $e(H_R)$ is even, that is, $v(H)$ is even. □

Remark 3.11. Andersen, Bouchet and Jackson [3] obtained the same results as Corollaries 3.9 and 3.10 for graphs embedded in the plane, the projective plane and the torus.

4 Quasi-tree bouquets

Theorem 1.4 and Lemma 3.7 show that quasi-tree bouquets are an important class of ribbon graphs. In this section, we give a brief characterization of them.

We start by recalling some necessary statements. A ribbon graph is *non-orientable* if it contains a ribbon subgraph that is homeomorphic to a Möbius band, and is *orientable* otherwise. An edge e of a ribbon graph is a *loop* if it is incident with exactly one vertex. A loop is non-orientable if together with its incident vertex it forms a Möbius band, and is orientable otherwise. Two loops e and f are *interlaced* if they are met in the cyclic order $efef$ when travelling round the boundary of a vertex. A loop e at the vertex of a bouquet G is *trivial* if there is no loop in G which interlaces with e. The *signed rotation* of a bouquet is a cyclic ordering of the half-edges at the vertex and if the edge is orientable, then we give the same sign to the corresponding two half-edges, and give the different signs otherwise. See Figure 13 for an example. Suppose $P = p_1p_2 \cdots p_k$ is a *string*. Then we call $-P = (-p_k) \cdots (-p_2)(-p_1)$ the *inverse* of P. *Operations 1* and *2* are the moves on bouquets defined in Figure 14 and Figure 15, respectively. *Operation 3* is deleting a pair of interlaced orientable loops and *operation 4* is deleting a trivial non-orientable loop as shown in Figure 16 and Figure 17, respectively. Note that operations 1, 2, 3 and 4 do not change the number of boundary components.

Theorem 4.1. *Let G be a bouquet. Then G is a quasi-tree bouquet if and only if there is a sequence of operations 1, 2, 3 and 4 which change the signed rotation of G to be empty.*

Proof. The sufficiency is easily verified. To prove the necessity, the result is easily verified when $E(G) = \emptyset$, so assume that this is not the case. Then there are following two cases.
Case 1: If there exists a non-orientable loop r, we assume that the signed rotation of G is $IrJ(-r)P$. By a sequence of operation 2, we have $I(-J)r(-r)P$. Hence, we get a signed rotation $I(-J)P$ by operation 4.

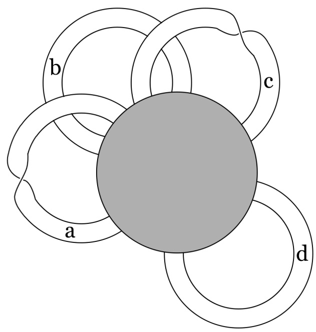

Figure 13: The signed rotation of the bouquet is $ab(-a)cb(-c)dd$.

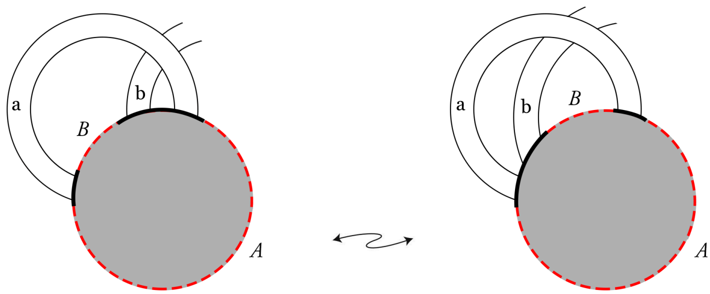

Figure 14: Operation 1. Change the signed rotation from $AaBba$ to $AabBa$.

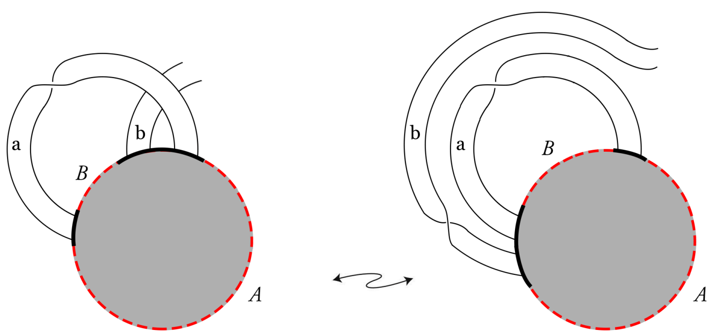

Figure 15: Operation 2. Change the signed rotation from $AaBb(-a)$ to $A(-b)aB(-a)$.

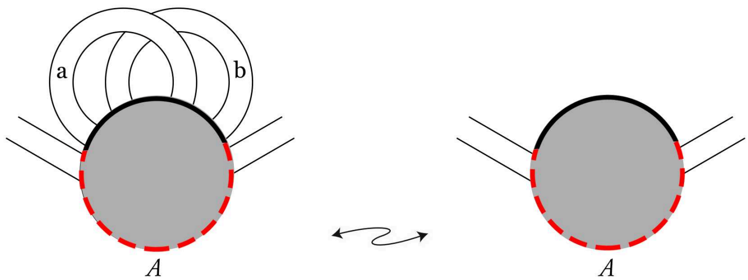

Figure 16: Operation 3. Change the signed rotation from $Aabab$ to A.

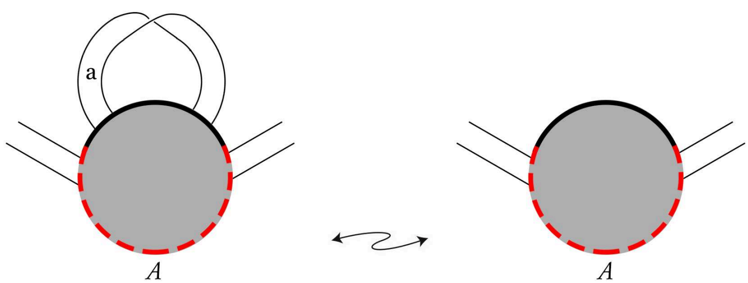

Figure 17: Operation 4. Change the signed rotation from $Aa(-a)$ to A.

Case 2: Otherwise, there exists a pair of interlaced orientable loops e, f. Assume that the signed rotation of G is $IeJfPeQfM$. By a sequence of operation 1, we have

$$IeJfPeQfM \Leftrightarrow IefPeQfJM \Leftrightarrow IefeQPfJM \Leftrightarrow IQPefefJM.$$

Then by operation 3, we get a signed rotation $IQPJM$.

Note that both Case 1 and Case 2 induce a shorter signed rotation. By repeating the above operations, we can reduce the signed rotation of G until it is empty. □

Remark 4.2. A bouquet with signed rotation $AxyB(-x)(-y)C$ is not a quasi-tree bouquet since $AxyB(-x)(-y)C \Leftrightarrow Axx(-B)y(-y)C$.

We now present an algorithm to get the number of boundary components of any bouquet G in terms of signed rotations.

Algorithm 1 Calculate the number of boundary components of any bouquet.

1: **Input** The signed rotation R of a bouquet G.
2: Let $a := 1$.
3: **Step 1**. If $R = \emptyset$, then stop and output the number of boundary components of G is a.
4: **Step 2**. Otherwise, there are three cases.
5: **if** there exists i such that $R = AiiB$ **then**
6: Let $R := AB, a := a + 1$. Return to Step 1.
7: **else if** there exists r such that $R = IrJ(-r)P$ **then**
8: Let $R := I(-J)P, a := a$. Return to Step 1.
9: **else**
10: Find a pair of interlaced loops e, f such that $R = IeJfPeQfM$.
Let $R := IQPJM, a := a$. Return to Step 1.

Example 4.3. A bouquet with signed rotation 13214234 is not a quasi-tree bouquet. Since there is a pair of interlaced loops $1, 3$, we get a shorter signed rotation 4224. A bouquet with signed rotation

$$182(-1)34325(-4)756867$$

is a quasi-tree bouquet. Since

$$182(-1)34325(-4)756867 \overset{1,(-1)}{\longrightarrow} (-2)(-8)34325(-4)756867 \overset{(-2),2}{\longrightarrow}$$

$$(-3)(-4)(-3)85(-4)756867 \overset{(-3),(-4)}{\longrightarrow} 85756867 \overset{8,7}{\longrightarrow} 6565 \overset{6,5}{\longrightarrow} \emptyset.$$

Proposition 4.4. *Let G be an orientable bouquet. If G is a quasi-tree bouquet, then $e(G)$ is even.*

Proof. It follows immediately from Euler's formula. □

Proposition 4.5. *Let G be a quasi-tree bouquet. If there exists $A \subseteq E(G)$ such that $(V(G), A)$ and $(V(G), A^c)$ are plane graphs, then $|A| = |A^c|$.*

Proof. Note that the vertex boundaries and face boundaries of $G^{\delta(A)}$ correspond to the face boundaries of $(V(G), A)$ and $(V(G), A^c)$, respectively. It follows that $v(G^{\delta(A)}) = |A|+1$ and $f(G^{\delta(A)}) = |A^c|+1$, that is, $G^{\delta(A)}$ is a plane graph. Then $E(G^{\delta(A)})$ can be partitioned into two edge disjoint spanning trees by Lemma 3.7. Hence,

$$e(G^{\delta(A)}) = 2(v(G^{\delta(A)}) - 1) = 2|A| = e(G) = |A| + |A^c|,$$

that is, $|A| = |A^c|$. □

Example 4.6. A bouquet G with signed rotation 121324356465 is not a quasi-tree bouquet. This follows from the fact that $(G, \{1, 3, 5, 6\})$ and $(G, \{2, 4\})$ are plane graphs, but $|\{1, 3, 5, 6\}| \neq |\{2, 4\}|$.

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Generalised dihedral CI-groups

Ted Dobson *

University of Primorska, UP IAM, Muzejski trg 2, SI-6000 Koper, Slovenia, and *University of Primorska, UP FAMNIT, Glagoljaşka 8, SI-6000 Koper, Slovenia*

Mikhail Muzychuk

Department of Mathematics, Ben-Gurion University of the Negev, Israel

Pablo Spiga

Dipartimento di Matematica e Applicazioni, University of Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy

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Abstract

In this paper, we find a strong new restriction on the structure of CI-groups. We show that, if R is a generalised dihedral group and if R is a CI-group, then for every odd prime p the Sylow p-subgroup of R has order p, or 9. Consequently, any CI-group with quotient a generalised dihedral group has the same restriction, that for every odd prime p the Sylow p-subgroup of the group has order p, or 9.

Keywords: CI-group, DCI-group, generalised dihedral, Cayley isomorphism.

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1 Introduction

Let R be a finite group and let S be a subset of R. The *Cayley digraph* of R with connection set S, denoted $\mathsf{Cay}(R, S)$, is the digraph with vertex set R and with (x, y) being an arc if and only if $xy^{-1} \in S$. Now, $\mathsf{Cay}(R, S)$ is said to be a *DCI-graph* (here *CI* stands for *Cayley isomorphic while the D stands for directed)*, if whenever $\mathsf{Cay}(R, S)$ is isomorphic to $\mathsf{Cay}(R, T)$, there exists an automorphism φ of R with $S^\varphi = T$. Clearly,

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E-mail addresses: ted.dobson@upr.si (Ted Dobson), muzychuk@bgu.ac.il (Mikhail Muzychuk), pablo.spiga@unimib.it (Pablo Spiga)

$\mathsf{Cay}(R, S) \cong \mathsf{Cay}(R, S^{\varphi})$ for every $\varphi \in \mathrm{Aut}(R)$ and hence, loosely speaking, for a DCI-graph $\mathsf{Cay}(R, S)$ deciding when a Cayley digraph over R is isomorphic to $\mathsf{Cay}(R, S)$ is theoretically and algorithmically elementary, but computationally efficient only if $\mathrm{Aut}(R)$ is small; that is, the solving set for $\mathsf{Cay}(R, S)$ is reduced to simply $\mathrm{Aut}(R)$ (for the definition of a solving set see for example [24, 26]). The group R is a *DCI-group* if $\mathsf{Cay}(R, S)$ is a DCI-graph for every subset S of R. Moreover, R is a *CI-group* if $\mathsf{Cay}(R, S)$ is a DCI-graph for every inverse-closed subset S of R. Thus every DCI-group is a CI-group.

After roughly 50 years of intense research, the classification of DCI- and CI-groups is still open. The current state of the art in this problem is as follows. There exist two rather short lists of candidates for DCI- and CI-groups and it is known that every DCI- and every CI-group must be a member of the corresponding list, see for instance [20]. Showing that a candidate on the lists of possible DCI- or CI-groups is actually a DCI- or CI-group, though, takes a considerable amount of effort. Just to give an example, the recent paper of Feng and Kovács [15] is a tour de force that shows that elementary abelian groups of rank 5 are DCI-groups.

In this paper we find an unexpected new restriction on which generalised dihedral groups are CI-groups, and significantly shorten the list of candidates for CI-groups.

Definition 1.1. Let A be an abelian group. The *generalised dihedral* group $\mathrm{Dih}(A)$ over A is the group $\langle A, x \mid a^x = a^{-1}, \forall a \in A\rangle$. A group is called generalised dihedral if it is isomorphic to $\mathrm{Dih}(A)$ for some A. When A is cyclic, $\mathrm{Dih}(A)$ is called a dihedral group.

Our main result is the following.

Theorem 1.2. *Let* $\mathrm{Dih}(A)$ *be a generalised dihedral group over the abelian group A. If* $\mathrm{Dih}(A)$ *is a* CI-*group, then, for every odd prime p the Sylow p-subgroup of A has order p, or* 9. *If* $\mathrm{Dih}(A)$ *is a* DCI-*group, then, in addition, the Sylow* 3-*subgroup has order* 3.

Generalised dihedral groups are amongst the most abundant members in the list of putative CI-groups. The importance of Theorem 1.2 is the arithmetical condition on the order of such groups, which greatly reduces even further the list of candidates for CI-groups. We believe that every generalised dihedral group satisfying this numerical condition on its order is a genuine CI-group. (This is in line with the partial result in [8].) Additionally, this result further reduces to two other groups on the list, whose definitions we now give.

Definition 1.3. Let A be an abelian group such that every Sylow p-subgroup of A is elementary abelian. Let $n \in \{2, 4, 8\}$ be relatively prime to $|A|$. Set $E(A, n) = A \rtimes \langle g\rangle$, where g has order n and $a^g = a^{-1}, \forall a \in A$.

Note that $E(A, 2) = \mathrm{Dih}(A)$. The groups $E(A, 4)$ and $E(A, 8)$ have centres Z_1 and Z_2 of order 2 and 4, respectively, and $E(A, 4)/Z_1 \cong E(A, 8)/Z_2 \cong \mathrm{Dih}(A)$. Babai and Frankl [2, Lemma 3.5] showed that a quotient of a (D)CI-group by a characteristic subgroup is a (D)CI-group, while the first author and Joy Morris [7, Theorem 8] showed that a quotient of a (D)CI-group is a (D)CI-group. Applying either result and Theorem 1.2 we have the following.

Corollary 1.4. *If $E(A, 4)$ or $E(A, 8)$ is a* CI-*group, then, for every odd prime p the Sylow p-subgroup of A has order p or* 9. *If $E(A, n), n \in \{2, 4, 8\}$ is a* DCI-*group, then, in addition, $n \neq 8$ and the Sylow* 3-*subgroup of A has order* 3.

Not much is known about which of the groups under consideration in this paper are CI-groups. Let p be a prime. Babai [1, Theorem 4.4] showed D_{2p} is a CI-group. The first author [4, Theorem 22] extended this to some special values of square-free integers. With Joy Morris, the first and third authors [8] showed that D_{6p} is a CI-group, $p \geq 5$. Also, Li, Lu, and Pálfy showed $E(p,4)$ and $E(p,8)$ are CI-groups.

We have one other result of interest, for which we will need an additional definition.

Definition 1.5. Let G be a group, and $S \subseteq G$. A *Haar graph* of G with connection set S has vertex set $G \times \mathbb{Z}_2$ and edge set $\{\{(g,0),(sg,1)\} : g \in G \text{ and } s \in S\}$.

So a Haar graph is a bipartite analogue of a Cayley graph. There is a corresponding isomorphism problem for Haar graphs, and if the group A is abelian, it is equivalent to the isomorphism problem for Cayley graphs of generalised dihedral groups $\mathrm{Dih}(A)$ that are bipartite (for nonabelian groups the problems are not equivalent, as for non-abelian groups Haar graphs need not be transitive), see [17, Lemma 2.2]. If isomorphic bipartite Cayley graphs of $\mathrm{Dih}(A)$ are isomorphic by group automorphisms of A, we say A is a *BCI-group*. We will also show that $\mathbb{Z}_3^k$ is not a BCI-group for every $k \geq 3$, while it is known that $\mathbb{Z}_3^k$ is a CI-group for every $1 \leq k \leq 5$ [32].

1.1 Some notation

Babai [1, Lemma 3.1] has proved a very useful criterion for determining when a finite group is a DCI-group and, more generally, when $\mathsf{Cay}(R,S)$ is a DCI-graph.

Lemma 1.6. *Let R be a finite group, and let S be a subset of R. Then,* $\mathsf{Cay}(R,S)$ *is a* DCI-*graph if and only if* $\mathrm{Aut}(\mathsf{Cay}(R,S))$ *contains a unique conjugacy class of regular subgroups isomorphic to R.*

Let Ω be a finite set and let G be a permutation group on Ω. An *orbital graph* of G is a digraph with vertex set Ω and with arc set a G-orbit $(\alpha,\beta)^G = \{(\alpha^g,\beta^g) \mid g \in G\}$, where $(\alpha,\beta) \in \Omega \times \Omega$. In particular, each orbital graph has for its arcs one orbit on the ordered pairs of elements of Ω, under the action of G. Moreover, we say that the orbital graphs $(\alpha,\beta)^G$ and $(\beta,\alpha)^G$ are *paired*. When $(\alpha,\beta)^G = (\beta,\alpha)^G$, we say that the orbital graph is *self-paired*.

When G is transitive and $\omega_0 \in \Omega$, there exists a natural one-to-one correspondence between the orbits of G on $\Omega \times \Omega$ (a.k.a. orbitals or 2-orbits of G) and the orbits of the stabiliser G_{ω_0} on Ω (a.k.a. *suborbits* of G). Therefore, under this correspondence, we may naturally define paired and self-paired suborbits.

Two subgroups of the symmetric group $\mathsf{Sym}(\Omega)$ are called 2-*equivalent* if they have the same orbitals. A subgroup of $\mathsf{Sym}(\Omega)$ generated by all subgroups 2-equivalent to a given $G \leq \mathsf{Sym}(\Omega)$ is called the 2-*closure* of G, denoted $G^{(2)}$.

The group G is said to be 2-*closed* if $G = G^{(2)}$. It is easy to verify that $G^{(2)}$ is a subgroup of $\mathrm{Sym}(\Omega)$ containing G and, in fact, $G^{(2)}$ is the largest (with respect to inclusion) subgroup of $\mathrm{Sym}(\Omega)$ preserving every orbital of G.

2 Construction and basic results

Let q be a power of an odd prime and let $\mathbb{F}$ be a field of cardinality q. We let

$$G := \left\{ \begin{pmatrix} a & x & z \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \mid x, y, z \in \mathbb{F}, a, b, c \in \{-1, 1\}, abc = 1 \right\},$$
$$D := \left\{ \begin{pmatrix} a & ax & ax^2/2 \\ 0 & 1 & x \\ 0 & 0 & a \end{pmatrix} \mid x \in \mathbb{F}, a \in \{-1, 1\} \right\},$$
$$H := \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{F}, a \in \{-1, 1\} \right\},$$
$$K := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid x, y \in \mathbb{F}, a \in \{-1, 1\} \right\}.$$

It is elementary to verify that G, D, H and K are subgroups of the special linear group $\mathrm{SL}_3(\mathbb{F})$. Moreover, D, H and K are subgroups of G, $|G| = 4q^3$, $|D| = 2q$ and $|H| = |K| = 2q^2$. We summarise in Proposition 2.1 some more facts.

Proposition 2.1. *The group D is generalised dihedral over the abelian group $(\mathbb{F}, +)$ and, H and K are generalised dihedral over the abelian group $(\mathbb{F} \oplus \mathbb{F}, +)$. The core of D in G is 1. Moreover,*

$$DK = DH = G = HD = KD \text{ and } D \cap H = 1 = D \cap K.$$

Proof. The first two assertions follow with easy matrix computations. Let

$$g := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in G$$

and observe that

$$g^{-1} \begin{pmatrix} a & ax & ax^2/2 \\ 0 & 1 & x \\ 0 & 0 & a \end{pmatrix} g = \begin{pmatrix} a & -ax & -ax^2/2 \\ 0 & 1 & x \\ 0 & 0 & a \end{pmatrix}.$$

As the characteristic of $\mathbb{F}$ is odd, from this it follows that

$$D \cap D^g = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

It is now easy to see that D is core-free in G.

It is readily seen from the definitions that $D \cap H = 1 = D \cap K$. Therefore, $|DH| = |D||H| = 4q^3$ and $|DK| = |D||K| = 4q^3$. As DH and DK are subsets of G and $|G| = 4q^3$, we deduce $DH = G = DK$ and hence also $HD = G = KD$. $\square$

We let $D\backslash G := \{Dg \mid g \in G\}$ be the set of right cosets of D in G. In view of Proposition 2.1, G acts faithfully by right multiplication on $D\backslash G$ and H and K act regularly by right multiplication on $D\backslash G$.

Proposition 2.2. *The subgroups H and K are normal in G and, therefore, are in distinct G-conjugacy classes.*

Proof. The normality of H and K in G can be checked by direct computations. □

2.1 Schur notation

Since $G = DH$ and $D \cap H = 1$, for every $g \in G$, there exists a unique $h \in H$ with $Dg = Dh$. In this way, we obtain a bijection $\theta : D\backslash G \to H$, where $\theta(Dg) = h \in H$ satisfies $Dg = Dh$.

Using the method of Schur (see [33]), we may identify via θ the G-set $D\backslash G$ with H. Moreover, we may define an action of G on H via the following rule: for every $g \in G$ and for every $h \in H$,

$$h^g = h' \text{ if and only if } Dhg = Dh'.$$

A classic observation of Schur yields that the action of G on $D\backslash G$ is permutation isomorphic to the action of G on H. In the rest of the paper, we use both points of view.

In the action of G on H, D is a stabiliser of the identity $e \in H$, i.e. $G_e = D$, and H acts on itself via its right regular representation. Since H is normal in G, the action of the point stabiliser G_e on H is permutation equivalent to the action of G_e via conjugation on H (Proposition 20.2 [33]). More precisely, $h^g = g^{-1}hg$ for any $g \in G_e$ and $h \in H$.

In what follows, we represent the elements of H and D as pairs $[a, x]$ and $[a, \vec{w}]$, where $x \in \mathbb{F}$, $\vec{w} \in \mathbb{F}^2$ and $a \in \{\pm 1\}$. In particular, $[a, x]$ represents the matrix

$$\begin{pmatrix} a & ax & ax^2/2 \\ 0 & 1 & x \\ 0 & 0 & a \end{pmatrix}$$

of D and, if $\vec{w} = (x, y)$, then $[a, \vec{w}]$ represents the matrix

$$\begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix}$$

of H. Under this identification, the product in D and H greatly simplifies. Indeed, for every $[a, x], [b, y] \in D$ and for every $[a, \vec{v}], [b, \vec{w}] \in H$, we have

$$\begin{aligned} [a, x][b, y] &= [ab, bx + y], \\ [a, \vec{v}][b, \vec{w}] &= [ab, b\vec{v} + \vec{w}]. \end{aligned} \tag{2.1}$$

Using this identification, the action of D on H also becomes slightly easier. Indeed, for every $[a, \vec{v}] \in H$ (with $\vec{v} = (x, y)$) and for every $[b, z] \in D$, we have

$$[a, (x, y)]^{[b,z]} = [a, \big((1-a)z^2/2 - byz + x, (-1+a)z + by\big)]. \tag{2.2}$$

This equality can be verified observing that

$$\begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b & bz & bz^2/2 \\ 0 & 1 & z \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} b & bz & bz^2/2 \\ 0 & 1 & z \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} a & 0 & (1-a)z^2/2 - byz + x \\ 0 & a & (-1+a)z + by \\ 0 & 0 & 1 \end{pmatrix}.$$

2.2 One special case

Let $A := \langle e_1, e_2, e_3\rangle$, where $e_1 := (1\,2\,3)$, $e_2 := (4\,5\,6)$, $e_1 := (7\,8\,9)$, let $x := (1\,2)(4\,5)(7\,8)$ and let $R := \langle A, x\rangle$. Then R is a generalised dihedral group over the elementary abelian 3-group A of order $3^3 = 27$. Let

$$S := \{x,\, e_1x,\, e_2x,\, e_3x,\, e_1e_2x,\, e_1^2e_2^2x,\, e_2e_3x,\, e_2^2e_3^2x,\, e_1^2e_2^2e_3^2x\}$$

and define

$$\Gamma := \mathsf{Cay}(R, S).$$

It can be verified with the computer algebra system Magma that $\mathrm{Aut}(\Gamma)$ has order $46656 = 2^6 \cdot 3^6$, acts transitively on the arcs of Γ and (most importantly) contains two conjugacy classes of regular subgroups isomorphic to R and hence, via Babai's lemma, R is not a CI-group.

This example has another interesting property from the isomorphism problem point of view. Observe that each element of S is an involution contained in $R \setminus A$. This implies that Γ is a bipartite graph, in which case Γ is isomorphic to a Haar graph, also called a bi-coset graph. In our example above, as every element of the connection set is an involution, it is a Haar graph of $\mathbb{Z}_3^3$ but as it is not a CI-graph of $\mathrm{Dih}(\mathbb{Z}_3^3)$, $\mathbb{Z}_3^3$ is not a BCI-group. This is the first example the authors are aware of where a group is an abelian DCI-group but not a BCI-group, as $\mathbb{Z}_p^3$ is a DCI-group [3]. Our next result shows $\mathbb{Z}_3^k$ is not a BCI-group for any $k \geq 3$.

Lemma 2.3. *Let R be an abelian group and let $H \leq R$. If R is BCI-group, then R/H is BCI-group.*

Proof. For this result, it is most convenient to have the vertex sets of Haar graphs and Cayley graphs of dihedral groups be the same. So, for an abelian group R, we will have $\mathrm{Dih}(R)$ permuting the set $R \times \mathbb{Z}_2$ (the vertex set of a Haar graph of R), where an element $s \in R$ is identified with the map $s_t \colon R \times \mathbb{Z}_2 \to R \times \mathbb{Z}_2$ given by $s_t(r, i) \mapsto (r + s, i)$. Define $\iota \colon R \times \mathbb{Z}_2 \to R \times \mathbb{Z}_2$ by $\iota(r, i) = (-r, i + 1)$. Then $\mathrm{Dih}(R)$ is canonically isomorphic to $G = \langle \iota, s_t : s \in R\rangle$. It is straightforward to show that $\iota \in \mathsf{Aut}(\mathrm{Haar}(R, S))$, and so we have $G \leq \mathsf{Aut}(\mathrm{Haar}(R, S))$ for every $S \subseteq R$. By [28, Theorem 2], we have $\mathrm{Haar}(R, S) \cong \mathrm{Cay}(\mathrm{Dih}(R), T)$, for some $T \subseteq G$, by the map ϕ which identifies (r, i) with the unique element of G which maps $(0, 0)$ to (r, i), $r \in R$, $i \in \mathbb{Z}_2$. Hence $\phi(r, i) = r_t\iota^i$, and $T = \{s\iota : s \in S\} = S \cdot \iota$.

If R is a BCI-group, then $\mathrm{Haar}(R, S)$ is a BCI graph. Let $\mathcal{C} = \{R \times \{0\}, R \times \{1\}\}$, $\mathcal{B}$ be the set of right cosets of H in $\mathrm{Dih}(R)$, and $U = \{sH : s \in S\}$. Then, as partitions of $R \times \mathbb{Z}_2$, $\mathcal{B}$ refines $\mathcal{C}$. As $\mathcal{C}$ is a bipartition of $\mathrm{Cay}(\mathrm{Dih}(R), S \cdot \iota)$, $\mathrm{Cay}(\mathrm{Dih}(R/H), U \cdot \iota)$ is bipartite with bipartition $\{\{(rH, i) : r \in R\} : i \in \mathbb{Z}_2\}$ and so $\mathrm{Cay}(\mathrm{Dih}(R/H), U \cdot \iota) = \mathrm{Haar}(R/H, U)$.

As $\mathrm{Cay}(\mathrm{Dih}(R), S \cdot \iota)$ is a CI-graph of $\mathrm{Dih}(R)$, by the proof of [6, Theorem 8], we see $\mathrm{Cay}(\mathrm{Dih}(R/H), U \cdot \iota)$ is a CI-graph of $\mathrm{Dih}(R/H)$ and any Cayley graph of $\mathrm{Dih}(R/H)$ isomorphic to $\mathrm{Cay}(\mathrm{Dih}(R/H), U \cdot \iota)$ is isomorphic by a group automorphism of $\mathrm{Dih}(R/H)$. But this means any two Haar graphs of R/H are isomorphic by a group automorphism of $\mathrm{Dih}(R/H)$, and so R/H is a BCI-group. □

Finally, Γ, as well as the graphs constructed in the next section, have the property that the Sylow p-subgroups of their automorphism groups are not isomorphic to Sylow p-subgroups of any 2-closed group of degree 3^3 or p^2 (in the next section). For the example above, the Sylow p-subgroups of the automorphism groups of Cayley digraphs of $\mathbb{Z}_p^3$ can be obtained from [5, Theorem 1.1], and none have order 3^6 as a Sylow p-subgroup of $\mathrm{AGL}(3,3)$ is not 2-closed (for p^2 in the next section, the Sylow p-subgroup has order p^3, but Sylow p-subgroups of the automorphism groups of Cayley digraphs of $\mathbb{Z}_p^2$ have order p^2 or p^{p+1} [10, Theorem 14]).

3 The permutation group G is 2-closed

In this section we prove the following.

Proposition 3.1. *The group G in its action on H is 2-closed.*

We start with some preliminary observations.

Lemma 3.2. *The orbits of G_e on H have one of the following forms:*

(1) $S_t := \{[1,(t,0)]\}$, *for every* $t \in \mathbb{F}$;

(2) $C_t \cup C_{-t}$, *where* $C_t := \{[1,(z,t)] \mid z \in \mathbb{F}\}$ *and* $t \in \mathbb{F} \setminus \{0\}$;

(3) $P_t := \big\{[-1,(t+z^2,2z)] \mid z \in \mathbb{F}\big\}$ *with* $t \in \mathbb{F}$.

Proof. Let $g := [a,(x,y)] \in H$. If $a = 1$ and $y = 0$, then (2.2) yields

$$g^{[b,z]} = [1,(x,0)] = g$$

and hence the G_e-orbit containing g is simply $\{g\}$. Therefore we obtain the orbits in Case (1).

Suppose then $a = 1$ and $y \neq 0$. Now, 2.2 yields

$$g^{[1,z]} = [1,(-yz+x,y)],$$
$$g^{[-1,z]} = [1,(yz+x,-y)].$$

In particular, $C_y = \{g^{[1,z]} \mid z \in \mathbb{F}\}$ and $C_{-y} = \{g^{[-1,z]} \mid z \in \mathbb{F}\}$ and we obtain the orbits in Case (2).

Finally suppose $a = -1$. Now, (2.2) yields

$$g^{[b,z]} = [1,(z^2 - byz + x, -2z + by)].$$

In particular, if we choose $z := by/2$ and $t = -y^2/4 + x$, then g and $[-1,(t,0)]$ are in the same G_e-orbit. Therefore $[-1,(x,y)]^{G_e} = [-1,(t,0)]^{G_e}$. Using again (2.2), we get

$$[-1,(t,0)]^{[b,-z]} = [-1,(t+z^2,2z)].$$

In particular, $P_t = \{g^{[b,z]} \mid [b,z] \in G_e\}$ and we obtain the orbits in Case (3). □

We call the G_e-orbits in (1) *singleton orbits*, the G_e-orbits in (2) *coset orbits* and the G_e-orbits in (3) *parabolic orbits*. Clearly, singleton orbits have cardinality 1, coset orbits have cardinality $2q$ and parabolic orbits have cardinality q. Also, it follows from Lemma 3.2 that there are q singleton orbits, $\frac{q-1}{2}$ coset orbits and q parabolic orbits. Indeed,

$$q \cdot 1 + \frac{q-1}{2} \cdot 2q + q \cdot q = 2q^2 = |H|.$$

It is also clear from Lemma 3.2 that all non-singleton orbits are self-paired and the only self-paired singleton orbit is S_0.

Before continuing, we recall [14, Definitions 2.5.3 and 2.5.4] tailored to our needs.

Definition 3.3. We say that $h \in H$ ***separates*** the pair $(h_1, h_2) \in H \times H$, if (h, h_1) and (h, h_2) belong to distinct G-orbitals, that is, hh_1^{-1} and hh_2^{-1} are in distinct G_e-orbits.

We also say that a subset $S \subseteq H$ ***separates*** G-orbitals if, for any two distinct elements $h_1, h_2 \in H \setminus S$, there exists $s \in S$ separating the pair (h_1, h_2).

Proposition 3.4. *If $q \geq 5$, then $\{e\} \cup P_0$ separates G-orbitals.*

Proof. Set $S := \{e\} \cup P_0$. Let $h_1, h_2 \in H \setminus S$ be two distinct elements. If h_1 and h_2 belong to distinct G_e-orbits, then $e \in S$ separates (h_1, h_2). Therefore, we assume that h_1 and h_2 belong to the same G_e-orbit, say, O. Since $h_1 \neq h_2$, O is not a singleton orbit and hence O is either a coset or a parabolic orbit.

Assume first that O is a parabolic orbit, that is, $O = P_t$, for some $t \in \mathbb{F}$. By Lemma 3.2, for each $i \in \{1, 2\}$, there exists $x_i \in \mathbb{F}$ with $h_i = [-1, (t + x_i^2, 2x_i)]$. As $q = |\mathbb{F}| \geq 5$, it is easy to verify that there exists $x \in \mathbb{F}$ with $x \notin \{x_1, x_2\}$ and with $x - x_1 \neq -(x - x_2)$. Now, let $s := [-1, (x^2, 2x)] \in P_0 \subseteq S$. From (2.1), we deduce

$$sh_i^{-1} = [1, (t + x_i^2 - x^2, 2x_i - 2x)].$$

As $2x_i - 2x \neq 0$, from Lemma 3.2, we obtain $sh_i^{-1} \in C_{2(x-x_i)} \cup C_{-2(x-x_i)}$. As $x - x_1 \neq -(x - x_2)$, we deduce that sh_1^{-1} and sh_2^{-1} are in distinct G_e-orbits and hence s separates (h_1, h_2).

Assume now that O is a coset orbit, that is, $O = C_t \cup C_{-t}$, for some $t \in \mathbb{F} \setminus \{0\}$. In this case, for each $i \in \{1, 2\}$, there exist $x_i \in \mathbb{F}$ and $a_i \in \{\pm 1\}$ with $h_i = [1, (x_i, a_i t)]$. Let $x \in \mathbb{F}$ with

$$xt(a_2 - a_1) \neq x_2 - x_1.$$

(The existence of x is clear when $a_1 \neq a_2$ and it follows from the fact that $h_1 \neq h_2$ when $a_1 = a_2$.) Set $s := [-1, (x^2, 2x)] \in P_0 \subseteq S$. From (2.1), we have

$$sh_i^{-1} \in [-1, (x^2 - x_i, 2x - a_i t)].$$

In particular, from Lemma 3.2, we have $sh_i^{-1} \in P_{t_i}$, for some $t_i \in \mathbb{F}$. Thus, $(x^2 - x_i, 2x - a_i t) = (t_i + y^2, 2y)$, for some $y \in \mathbb{F}$. From this it follows that

$$t_i = x^2 - x_i - \frac{(2x - a_i t)^2}{4}.$$

As $xt(a_2 - a_1) \neq x_2 - x_1$, a simple computation yields $t_1 \neq t_2$ and hence sh_1^{-1} and sh_2^{-1} are in distinct G_e-orbits. Therefore, s separates (h_1, h_2). $\square$

Proof of Proposition 3.1. When $q = 3$, the proof follows with a computation with the computer algebra system Magma. Therefore, for the rest of the proof we suppose $q \geq 5$. Let T be the 2-closure of G. As $\{e\} \cup P_0$ separates the G-orbitals, it follows from [14, Theorem 2.5.7] that the action of T_e on P_0 is faithful, and hence so is the action of G_e on P_0. We denote by $G_e^{P_0}$ (respectively, $T_e^{P_0}$) the permutation group induced by G_e (respectively, T_e) on P_0. In particular, $G_e \cong G_e^{P_0}$ and $T_e \cong T_e^{P_0}$.

We claim that

$$(T_e)^{P_0} = (G_e)^{P_0}. \tag{3.1}$$

Observe that from (3.1) the proof of Proposition 3.1 immediately follows. Indeed, $T_e \cong T_e^{P_0} = G_e^{P_0} \cong G_e$ and hence $T_e = G_e$. As H is a transitive subgroup of G, we deduce that

$G = G_eH = T_eH = T$ and hence G is 2-closed. Therefore, to complete the proof, we need only establish (3.1).

From Lemma 3.2, $|P_0| = q$. Hence $(G_e)^{P_0}$ is a dihedral group of order $2q$ in its natural action on q points.

For each $t \in \mathbb{F}^*$ let Φ_t be the subgraph of $\mathsf{Cay}(H, C_t \cup C_{-t})$ induced by P_0. Let (h_1, h_2) be an arc of Φ_t. As $h_1, h_2 \in P_0$, there exist $x_1, x_2 \in \mathbb{F}$ with $h_1 = [-1, (x_1^2, 2x_1)]$ and $h_2 = [-1, (x_2^2, 2x_2)]$. Moreover, $h_2h_1^{-1} \in C_t \cup C_{-t}$ and hence, by (2.1), we obtain

$$h_2h_1^{-1} = [1, (x_2^2 - x_1^2, 2x_2 - 2x_1)] \in C_t \cup C_{-t},$$

that is, $2x_2 - 2x_1 \in \{-t, t\}$. This shows that the mapping

$$\begin{aligned} P_0 &\to \mathbb{F}^+ \\ (x^2, 2x) &\mapsto 2x \end{aligned}$$

is an isomorphism between the graphs Φ_t and $\mathsf{Cay}(\mathbb{F}^+, \{-t, t\})$. Therefore

$$(G_e)^{P_0} \le (T_e)^{P_0} \le \bigcap_{t\in\mathbb{F}^*} \mathrm{Aut}(\Phi_t) \cong \bigcap_{t\in\mathbb{F}^*} \mathrm{Aut}(\mathsf{Cay}(\mathbb{F}^+, \{-t, t\})) \cong \mathrm{Dih}(\mathbb{F}^+).$$

Since $(G_e)^{P_0}$ and $\mathrm{Dih}(\mathbb{F}^+)$ are dihedral groups of order $2q$, we conclude that $(G_e)^{P_0} = (T_e)^{P_0} = \bigcap_{t\in\mathbb{F}^*} \mathrm{Aut}(\Phi_t)$, proving 3.1. □

4 Generating graph

Combining Proposition 3.1, Proposition 2.2, and Lemma 1.6, we have proven that $\mathrm{Dih}(\mathbb{Z}_p^2)$ is not a CI-group with respect to colour Cayley digraphs for odd primes p. In this section we strengthen that result to Cayley graphs.

4.1 Schur rings

Let R be a finite group with identity element e. We denote the group algebra of R over the field $\mathbb{Q}$ by $\mathbb{Q}R$. For $Y \subseteq R$, we define

$$\underline{Y} := \sum_{y\in Y} y \in \mathbb{Q}R.$$

Elements of $\mathbb{Q}R$ of this form will be called *simple quantities*, see [33]. A subalgebra $\mathcal{A}$ of the group algebra $\mathbb{Q}R$ is called a *Schur ring* over R if the following conditions are satisfied:

(1) there exists a basis of $\mathcal{A}$ as a $\mathbb{Q}$-vector space consisting of simple quantities $\underline{T}_0, \dots, \underline{T}_r$;

(2) $T_0 = \{e\}$, $R = \bigcup_{i=0}^r T_i$ and, for every $i, j \in \{0, \dots, r\}$ with $i \ne j$, $T_i \cap T_j = \emptyset$;

(3) for each $i \in \{0, \dots, r\}$, there exists i' such that $T_{i'} = \{t^{-1} \mid t \in T_i\}$.

Now, $\underline{T}_0, \dots, \underline{T}_r$ are called the *basic quantities* of $\mathcal{A}$. A subset S of R is said to be an $\mathcal{A}$- *subset* if $\underline{S} \in \mathcal{A}$, which is equivalent to $S = \bigcup_{j\in J} T_j$, for some $J \subseteq \{0, \dots, r\}$.

Given two elements $a := \sum_{x\in R} a_x x$ and $b := \sum_{y\in R} b_y y$ in $\mathbb{Q}R$, the *Schur-Hadamard* product $a \circ b$ is defined by

$$a \circ b := \sum_{z\in R} a_z b_z z.$$

It is an elementary exercise to observe that, if $\mathcal{A}$ is a Schur ring over R, then $\mathcal{A}$ is closed by the Schur-Hadamard product.

The following statement is known as the *Schur-Wielandt principle*, see [33, Proposition 22.1].

Proposition 4.1. *Let $\mathcal{A}$ be a Schur ring over R, let $q \in \mathbb{Q}$ and let $x := \sum_{r\in R} a_r r \in \mathcal{A}$. Then*

$$x_q := \sum_{\substack{r\in R\\ a_r=q}} r \in \mathcal{A}.$$

Let X be a permutation group containing a regular subgroup R. As in Section 2.1, we may identify the domain of X with R. Let $T_0, \dots, T_r$ be the orbits of X_e with $T_0 = \{e\}$. A fundamental result of Schur [33, Theorem 24.1] shows that the $\mathbb{Q}$-vector space spanned by $\underline{T}_0, \underline{T}_1, \dots, \underline{T}_r$ in $\mathbb{Q}R$ is a Schur ring over R, which is called the *transitivity module* of the permutation group X and is usually denoted by $V(R, G_e)$. In particular, the $V(R, G_e)$-subsets of the Schur ring $V(R, G_e)$ are unions of G_e-orbits.

Let $\mathcal{A} := \langle \underline{T}_0, \dots, \underline{T}_r \rangle$ be a Schur ring over R (where $T_0, \dots, T_r$ are the basic quantities spanning $\mathcal{A}$). The *automorphism group* of $\mathcal{A}$ is defined by

$$\mathrm{Aut}(\mathcal{A}) := \bigcap_{i=0}^{r} \mathrm{Aut}(\mathsf{Cay}(R, T_i)). \tag{4.1}$$

Given a subset S of R, we denote by

$$\langle\langle \underline{S} \rangle\rangle,$$

the smallest (with respect to inclusion) Schur ring containing $\underline{S}$. Now, $\langle\langle \underline{S} \rangle\rangle$ is called the *Schur ring generated* by $\underline{S}$.

We conclude this brief introduction to Schur rings recalling [25, Theorem 2.4].

Proposition 4.2. *Let S be a subset of R. Then* $\mathrm{Aut}(\langle\langle \underline{S} \rangle\rangle) = \mathrm{Aut}(\mathsf{Cay}(R, S))$.

4.2 The group G is the automorphism group of a single (di)graph

It was shown above that the group G is 2-closed, i.e. it is the automorphism of a coloured digraph. In this section we give a Cayley digraph $\mathsf{Cay}(H, T)$ having automorphism group G. To build such a digraph it is sufficient to find a subset $T \subseteq H$ such that $\langle\langle \underline{T} \rangle\rangle = V(H, G_e)$ (Proposition 4.2). Such a set is constructed in Proposition 4.3. Note that T is symmetric for $q \geq 7$, so the digraph $\mathsf{Cay}(H, T)$ is undirected. The cases of $q = 3, 5$ are exceptional, because in those cases no inverse-closed subset of H has the required property.

Proposition 4.3. *Let q be prime, and*

$$T := \begin{cases} P_0 \cup P_1 \cup P_x \cup C_1 \cup C_{-1} & \textit{where } x \in \mathbb{F} \textit{ with } x \notin \{0, \pm 1, \pm 2, \frac{1}{2}\} \textit{ and } x^6 \neq 1, \\ & \textit{when } q > 7, \\ P_0 \cup P_1 \cup P_3 \cup C_1 \cup C_{-1} & \textit{when } q = 7, \\ S_1 \cup P_0 & \textit{when } q = 5, \\ S_1 \cup P_0 & \textit{when } q = 3. \end{cases}$$

Then $\langle\langle \underline{T} \rangle\rangle = V(H, G_e)$. In particular, T is not a (D)CI-subset of H.

Proof. When $q \leq 7$, the result follows by computations with the computer algebra system Magma. Therefore for the rest of the proof we suppose $q > 7$.

According to Proposition 3.2 the basic sets of $V(H, G_e)$ are of three types: $S_a, C_b \cup C_{-b}, P_c$ with $a, b, c \in \mathbb{F}$ and $b \neq 0$. Thus we have three types of basic quantities $\underline{S_a}$, $\underline{C_b} + \underline{C_{-b}}, \underline{P_c}$ and

$$V(H, G_e) = \langle \underline{S_a}, \underline{C_b} + \underline{C_{-b}}, \underline{P_c} \big| a, b, c \in \mathbb{F}, b \neq 0 \rangle.$$

Set

$$H_1 := \{[1, \vec{v}] \mid \vec{v} \in \mathbb{F}^2\},$$
$$H_2 := \{[1, (t, 0)] \mid t \in \mathbb{F}\}.$$

By (2.1), H_1 and H_2 are subgroups of H with $|H_2| = q$, $|H_1| = q^2$ and, by Lemma 3.2, $H_2 = \cup_{t \in \mathbb{F}} S_t$. In Table 4.2 we have reported the multiplication table among the basic quantities of $V(H, G_e)$: this will serve us well.

	$\underline{S_r}$	$\underline{C_s}$	$\underline{P_t}$
$\underline{S_a}$	$\underline{S_{a+r}}$	$\underline{C_s}$	$\underline{P_{t-a}}$
$\underline{C_b}$	$\underline{C_b}$	$\begin{cases} q\underline{C_{b+s}} & \text{if } b+s \neq 0 \\ q\underline{H_2} & \text{if } b+s = 0 \end{cases}$	$\underline{H \setminus H_1}$
$\underline{P_c}$	$\underline{P_{c+r}}$	$\underline{H \setminus H_1}$	$q\underline{S_{-c+t}} + \underline{H_1 \setminus H_2}$

Table 1: Multiplication table for the basic quantities of $V(H, G_e)$.

Fix $a, b, c \in \mathbb{F}$ with $b, c \neq 0$ and let $\mathcal{A}$ be the smallest Schur ring of the group algebra $\mathbb{Q}H$ containing $\underline{P_a}, \underline{C_b} + \underline{C_{-b}}, \underline{S_c}$. We claim that

$$\mathcal{A} = V(H, G_e). \tag{4.2}$$

Clearly, $\mathcal{A} \leq V(H, G_e)$. From Table 4.2, for every $k \in \{0, \ldots, q-1\}$, we have $\underline{S_c}^k = \underline{S_{ck}}$ and hence $\underline{S_{ck}} \in \mathcal{A}$. As $c \neq 0$, $\underline{S_i} \in \mathcal{A}$, for each $i \in \{0, \ldots, q-1\}$. Now, as $\underline{P_a} \in \mathcal{A}$, from Table 4.2, we have $\underline{P_a} \cdot \underline{S_i} = \underline{P_{a+i}} \in \mathcal{A}$ for any $i \in \{0, \ldots, q-1\}$. The equality $(\underline{C_b} + \underline{C_{-b}})^2 = 2q\underline{H_2} + q\underline{C_{2b}} + q\underline{C_{-2b}}$ implies $\underline{C_{2b}} + \underline{C_{-2b}} \in \mathcal{A}$. Now arguing inductively we deduce $\underline{C_k} + \underline{C_{-k}} \in \mathcal{A}$, for all $k \in \{1, \ldots, q-1\}$. Thus (4.2) follows.

Let $x \in \mathbb{F}$ with $x \notin \{0, \pm 1, \pm 2, \frac{1}{2}\}$ and $x^6 \neq 1$, let $T := P_0 \cup P_1 \cup P_x \cup C_1 \cup C_{-1}$ and let $\mathcal{T} := \langle\langle \underline{T} \rangle\rangle$ (the existence of x is guaranteed by the fact that $q > 7$). We claim that

$$\underline{H_2},\ \underline{H_1},\ \underline{C_2} + \underline{C_{-2}},\ \underline{S_1} + \underline{S_{-1}} + \underline{S_x} + \underline{S_{-x}} + \underline{S_{1-x}} + \underline{S_{x-1}} \in \mathcal{T}. \tag{4.3}$$

Using Table 4.2 for squaring $\underline{T}$, we obtain (after rearranging the terms):

$$\begin{aligned} \underline{T}^2 =& 3q\underline{S_0} + q\underline{S_1} + q\underline{S_{-1}} + q\underline{S_x} + q\underline{S_{-x}} + q\underline{S_{1-x}} + q\underline{S_{x-1}} \\ &+ 9\underline{H_1 \setminus H_2} + 12\underline{H \setminus H_1} + q\underline{C_2} + q\underline{C_{-2}} + 2q\underline{H_2}. \end{aligned}$$

From the assumptions on x, the elements $-1, 1, -x, x, -(x-1), x-1$ are pairwise distinct. Therefore

$$\underline{T}^2 \circ \underline{S_b} = \begin{cases} 5q\underline{S_0}, & b = 0, \\ 3q\underline{S_b}, & \text{if } b \in \{\pm 1, \pm x, \pm(x-1)\}, \\ 2q\underline{S_b}, & \text{if } b \notin \{0, \pm 1, \pm x, \pm(x-1)\}, \end{cases}$$
$$\underline{T}^2 \circ \underline{C_b} = \begin{cases} (q+9)\underline{C_b}, & \text{if } b \in \{\pm 2\}, \\ 9\underline{C_b}, & \text{if } b \notin \{0, \pm 2\}, \end{cases}$$
$$\underline{T}^2 \circ \underline{P_b} = 12\underline{P_b}, \quad \text{if } b \in \mathbb{F}.$$

Since the numbers $6, 9, q+9, 2q, 3q, 5q$ are also pairwise distinct (because $q \neq 3$), an application of the Schur-Wielandt principle yields

$$\begin{aligned} (\underline{T}^2)_{3q} &= \underline{S_1} + \underline{S_{-1}} + \underline{S_x} + \underline{S_{-x}} + \underline{S_{1-x}} + \underline{S_{x-1}} \in \mathcal{T}, \\ (\underline{T}^2)_{12} &= \underline{H \setminus H_1} \in \mathcal{T}, \\ (\underline{T}^2)_{2q} &= \underline{H_2} - (\underline{S_0} + \underline{S_1} + \underline{S_{-1}} + \underline{S_x} + \underline{S_{-x}} + \underline{S_{1-x}} + \underline{S_{x-1}}) \in \mathcal{T}, \\ (\underline{T}^2)_{q+9} &= \underline{C_2} + \underline{C_{-2}} \in \mathcal{T}. \end{aligned}$$

From this, (4.3) immediately follows.

We claim that

$$\underline{S_1} + \underline{S_{-1}} \in \mathcal{T}. \tag{4.4}$$

Let

$$\mathcal{T}_{H_2} := \mathcal{T} \cap \mathbb{Q}H_2$$

and observe that $\mathcal{T}_{H_2}$ is a Schur ring over the cyclic group $H_2 \cong \mathbb{Z}_q$ of prime order q. It is well known that every Schur ring over $\mathbb{Z}_q$ is determined by a subgroup $M \leq \mathrm{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_q^*$ such that, every basic set of the corresponding Schur ring is an M-orbit. Let M be such a subgroup for $\mathcal{T}_{H_2}$. From (4.3), the simple quantity $\underline{S_1} + \underline{S_{-1}} + \underline{S_x} + \underline{S_{-x}} + \underline{S_{1-x}} + \underline{S_{x-1}}$ belongs to $\mathcal{T}_{H_2}$ and hence $\{\pm 1, \pm x, \pm(1-x)\}$ is a $\mathcal{T}_{H_2}$-subset of cardinality 6. It follows that $|M|$ divides six and $M \subseteq \{\pm 1, \pm x, \pm(1-x)\}$. If $|M| \in \{3, 6\}$, then $\{\pm 1, \pm x, \pm(1-x)\}$ is a subgroup of $\mathbb{Z}_q^*$, contrary to the assumption $x^6 \neq 1$. Therefore

$$\text{either } M = \{1\} \text{ or } |M| = \{\pm 1\}. \tag{4.5}$$

In both cases, $\{-1, 1\}$ is a union of M-orbits. Therefore, $\underline{S_1} + \underline{S_{-1}} \in \mathcal{T}_{H_2}$. From this, (4.4) follows immediately.

We are now ready to conclude the proof. Clearly, $\underline{T} \in V(H, G_e)$ and hence $\mathcal{T} \subseteq V(H, G_e)$. From (4.3), $\underline{H_1} \in \mathcal{T}$ and, from (4.4), $\underline{S_1} + \underline{S_{-1}} \in \mathcal{T}$. Therefore $\underline{H_1} \circ \underline{T} = \underline{C_1} + \underline{C_{-1}} \in \mathcal{T}$ and $(\underline{T} - \underline{H_1}) \circ \underline{T} = \underline{P_0} + \underline{P_1} + \underline{P_x} \in \mathcal{T}$. Therefore

$$\Big((\underline{P_0} + \underline{P_1} + \underline{P_x})(\underline{S_1} + \underline{S_{-1}})\Big) \circ (\underline{P_0} + \underline{P_1} + \underline{P_x}) \in \mathcal{T}.$$

As $(\underline{P_0} + \underline{P_1} + \underline{P_x})(\underline{S_1} + \underline{S_{-1}}) = \underline{P_1} + \underline{P_2} + \underline{P_{x+1}} + \underline{P_{-1}} + \underline{P_0} + \underline{P_{x-1}}$, we deduce

$$\Big((\underline{P_0} + \underline{P_1} + \underline{P_x})(\underline{S_1} + \underline{S_{-1}})\Big) \circ (\underline{P_0} + \underline{P_1} + \underline{P_x}) = \underline{P_0} + \underline{P_1}$$

and hence $\underline{P_0} + \underline{P_1} \in \mathcal{T}$. Therefore, $\underline{P_x} = (\underline{P_0} + \underline{P_1} + \underline{P_x}) - (\underline{P_0} + \underline{P_1}) \in \mathcal{T}$. As

$$(\underline{P_0} + \underline{P_1})\underline{P_x} = q\underline{S_x} + q\underline{S_{x-1}} + 2(\underline{H \setminus H_1}),$$

from the Schur-Wielandt principle, we obtain $\underline{S_x} + \underline{S_{x-1}} \in \mathcal{T}$. Therefore $\underline{S_x} + \underline{S_{x-1}} \in \mathcal{T}_{H_2}$ and hence $\{x, x-1\}$ is a $\mathcal{T}_{H_2}$-subset. Thus $\{x, x-1\}$ is an M-orbit. Recall (4.5). If $M = \{-1, 1\}$, then $x - 1 = -1 \cdot x = -x$, contrary to the assumption $x \neq 1/2$. Therefore $M = \{1\}$ and $\mathcal{T}_{H_2} = \mathbb{Q}H_2$. Thus $\underline{S_i} \in \mathcal{T}$, for each $i \in \mathbb{Z}_q$. Thus $\underline{S_1}, \underline{P_x}, \underline{C_1} + \underline{C_{-1}} \in \mathcal{T}$ and (4.2) implies $V(H, G_e) \subseteq \mathcal{T}$. □

5 Proof of Theorem 1.2

Proof of Theorem 1.2. The list of candidate CI-groups is on page 323 in [20]. From here, we see that, if R is in this list and if $R = \mathrm{Dih}(A)$ is generalised dihedral, then for every odd prime p the Sylow p-subgroup of R is either elementary abelian or cyclic of order 9.

Assume that the Sylow p-subgroup (p is an odd prime) of A is elementary abelian of rank at least 2. Let $P \leq A$ be a subgroup isomorphic to $\mathbb{Z}_p^2$ and let $x \in R \setminus A$. Then $\langle P, x\rangle \cong \mathrm{Dih}(\mathbb{Z}_p^2)$. By Proposition 4.3, $\mathrm{Dih}(\mathbb{Z}_p^2)$ contains a non-DCI subset. Therefore $\mathrm{Dih}(\mathbb{Z}_p^2)$ is a non-DCI-group. Since subgroups of a (D)CI-group are also (D)CI, we conclude that R is a not a DCI-group as well. The non-DCI set T constructed in Proposition 4.3 is symmetric for $p \geq 7$. Hence $\mathrm{Dih}(\mathbb{Z}_p^2)$ and, therefore, R are non-CI groups when $p \geq 7$. If $p = 5$, then the group $\mathrm{Dih}(\mathbb{Z}_p^2)$ contains a non-CI subset, namely: $P_0 \cup S_1 \cup S_{-1}$ (this was checked by Magma[1]). Combining these arguments we conclude that if $\mathrm{Dih}(A)$ is a CI-group, then its Sylow p-subgroup is cyclic if $p \geq 5$. If $p = 3$, then the Sylow 3-subgroup is either cyclic of order 9 or elementary abelian. The example in Section 2.2 shows that the rank of an elementary abelian group is bounded by 2. □

We now give the updated list of CI-groups. It is a combination of the list in [20], together with our results here and [12, Corollary 13] (note [12, Corollary 13] contains an error, and should list Q_8 on line (1c), not on line (1b)). We need to define one more group:

Definition 5.1. Let M be a group of order relatively prime to 3, and $\exp(M)$ be the largest order of any element of M. Set $E(M,3) = M \rtimes_\phi \mathbb{Z}_3$, where $\phi(g) = g^\ell$, and ℓ is an integer satisfying $\ell^3 \equiv 1 \pmod{\exp(M)}$ and $\gcd(\ell(\ell-1), \exp(M)) = 1$.

Theorem 5.2. *Let G, M, and K be* CI-*groups with respect to graphs such that M and K are abelian, all Sylow subgroups of M are elementary abelian, and all Sylow subgroups of K are elementary abelian of order* 9 *or cyclic of prime order.*

(1) *If G does not contain elements of order* 8 *or* 9*, then $G = H_1 \times H_2 \times H_3$, where the orders of H_1, H_2, and H_3 are pairwise relatively prime, and*

  (a) *H_1 is an abelian group, and each Sylow p-subgroup of H_1 is isomorphic to $\mathbb{Z}_p^k$ for $k < 2p + 3$ or $\mathbb{Z}_4$;*

  (b) *H_2 is isomorphic to one of the groups $E(K,2)$, $E(M,3)$, $E(K,4)$, A_4, or* 1*;*

  (c) *H_3 is isomorphic to one of the groups D_{10}, Q_8, or* 1.

[1] The automorphism group of the corresponding Cayley graph is 4 times bigger than G but the subgroups H and K are non-conjugate inside it.

(2) *If* G *contains elements of order* 8*, then* $G \cong E(K, 8)$ *or* $\mathbb{Z}_8$*.*

(3) *If* G *contains elements of order* 9*, then* G *is one of the groups* $\mathbb{Z}_9 \rtimes \mathbb{Z}_2$*,* $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$*,* $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$*, or* $\mathbb{Z}_2^n \times \mathbb{Z}_9$*, with* $n \leq 5$*.*

Remark 5.3. The rank bound of an elementary abelian group used in part (1)(a) is due to [29].

Other than positive results already mentioned, the abelian groups known to be CI-groups are $\mathbb{Z}_{2n}$ [22], $\mathbb{Z}_{4n}$ [23] with n an odd square-free integer, $\mathbb{Z}_q \times \mathbb{Z}_p^2$ [18], $\mathbb{Z}_q \times \mathbb{Z}_p^3$ [31], and $\mathbb{Z}_q \times \mathbb{Z}_p^4$ [19] with q and p and distinct primes, and $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ [9]. Additional results are given in [4, Theorem 16] and [11] with technical restrictions on the orders of the groups. A similar result with technical restrictions on M is given in [4, Theorem 22] for some $E(M, 3)$. Also, $E(\mathbb{Z}_p, 4)$ and $E(\mathbb{Z}_p, 8)$ were shown to be CI-groups in [21], and $Q_8 \times \mathbb{Z}_p$ in [30]. Finally, Holt and Royle have determined all CI-groups of order at most 47 [16]. Applying Theorem 5.2 to determine possible CI-groups, and then checking the positive results above to see that all possible CI-groups are known to be CI-groups, we extend the census of CI-groups up to groups of order at most 59. The isomorphism problem for circulant digraphs was independently solved in [13] and [26] (in both cases a polynomial time algorithm for solving the isomorphism problem was given). A polynomial time algorithm for finding the automorphism group of circulant digraph was provided in [27]. Finally, we remark that the groups $E(M, 3)$ and $E(M, 8)$ are *not* DCI-groups.

Appendix A An alternative approach

In this section we give an alternative approach to the proof of Theorem 1.2. We do not give all of the details - just the basic idea. In principle, this section is independent from the previous sections, but for convenience we deduce the main result from our previous work.

For each $g \in \mathsf{GL}_3(\mathbb{F})$, let $g^\top$ denote the transpose of the matrix g and let $g^\iota := (g^{-1})^\top$. It is easy to verify that $\iota : \mathsf{GL}_3(\mathbb{F}) \to \mathsf{GL}_3(\mathbb{F})$ is an automorphism. Let

$$s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and let α be the automorphism of $\mathsf{GL}_3(\mathbb{F})$ defined by

$$g^\alpha := s^{-1} g^\iota s = s^{-1} (g^{-1})^\top s, \tag{A.1}$$

for every $g \in \mathsf{GL}_3(\mathbb{F})$.

We now define $\hat{\alpha} \in \mathrm{Sym}(H)$ by

$$[a, (x, y)]^{\hat{\alpha}} = [a, (y^2/2 - x, ay)], \tag{A.2}$$

for every $[a, (x, y)] \in H$.

Lemma A.1. *Let* α *and* $\hat{\alpha}$ *be as in* (A.1) *and* (A.2)*. We have*

(1) $G^\alpha = G$ *and* $D^\alpha = D$*;*

(2) $K = H^\alpha$ *and* $H = K^\alpha$*;*

(3) *for every* $h \in H$, $(Dh)^\alpha = Dh^{\hat{\alpha}}$*;*

(4) *for every* $x \in \mathbb{F}$ *and for every* $t \in \mathbb{F}^*$, $S_x^{\hat{\alpha}} = S_{-x}, C_t^{\hat{\alpha}} = C_t, P_x^{\hat{\alpha}} = P_{-x}$.

Proof. The proof follows from straightforward computations. For every $a \in \{-1, 1\}$ and $x \in \mathbb{F}$, we have

$$\begin{aligned}
\begin{pmatrix} a & ax & ax^2/2 \\ 0 & 1 & x \\ 0 & 0 & a \end{pmatrix}^\alpha &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} a & ax & ax^2/2 \\ 0 & 1 & x \\ 0 & 0 & a \end{pmatrix}^{-1} \right)^\top \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & -x & a(-x)^2/2 \\ 0 & 1 & a(-x) \\ 0 & 0 & a \end{pmatrix}^\top \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ -x & 1 & 0 \\ a(-x)^2/2 & a(-x) & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a & a(-x) & a(-x)^2/2 \\ 0 & 1 & -x \\ 0 & 0 & a \end{pmatrix} \in D.
\end{aligned}$$

This shows $D^\alpha = D$. The computations for proving $G = G^\alpha$, $K = H^\alpha$ and $H = K^\alpha$ are similar.

Let $h := [a, (x, y)] \in H$. A direct computation shows that

$$h^\alpha = \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix}^\alpha = \begin{pmatrix} 1 & -ay & -ax \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

and hence

$$\begin{aligned}
h^\alpha (h^{\hat{\alpha}})^{-1} &= \begin{pmatrix} 1 & -ay & -ax \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \left(\begin{pmatrix} a & 0 & y^2/2 - x \\ 0 & a & ay \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} 1 & -ay & -ax \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} a & 0 & -ay^2/2 + ax \\ 0 & a & -y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a & -y & ay^2/2 \\ 0 & 1 & -ay \\ 0 & 0 & a \end{pmatrix} \in D.
\end{aligned}$$

Therefore

$$(Dh)^\alpha = D^\alpha h^\alpha = Dh^\alpha = Dh^{\hat{\alpha}}$$

and part (3) follows. Now, part (4) follows immediately from Lemma 3.2 and part (3). □

Lemma A.2. *Let* $x \in \mathbb{F}$ *with* $x \notin \{0, \pm 1, \pm 2, \frac{1}{2}\}$ *and* $x^6 \neq 1$*, and let*

$$\begin{aligned}
T &:= P_0 \cup P_1 \cup P_x \cup C_1 \cup C_{-1}, \\
T' &:= P_0 \cup P_{-1} \cup P_{-x} \cup C_1 \cup C_{-1}.
\end{aligned}$$

Then $\mathsf{Cay}(H, T)$ *and* $\mathsf{Cay}(H, T')$ *are isomorphic but not Cayley isomorphic. In particular,* H *is not a* CI-*group.*

Proof. We view G as a permutation group on $D\backslash G$, which we may identify with H via the Schur notation.

It follows from Lemma A.1(1) and (3) that $\hat{\alpha}$ normalizes G. Therefore, $\hat{\alpha}$ permutes the orbitals of G. Since $\hat{\alpha}$ fixes $e = [1, (0, 0)]$, $\hat{\alpha}$ permutes the suborbits of G and, from Lemma A.1(4), we have $\mathsf{Cay}(H, T^{\hat{\alpha}}) = \mathsf{Cay}(H, T')$. Hence $\mathsf{Cay}(H, T)^{\hat{\alpha}} = \mathsf{Cay}(H, T')$ and $\mathsf{Cay}(H, T) \cong \mathsf{Cay}(H, T')$.

Assume that there exists $\beta \in \mathsf{Aut}(H)$ with $\mathsf{Cay}(H, T)^{\beta} = \mathsf{Cay}(H, T')$. Then $\hat{\alpha}\beta^{-1}$ is an automorphism of $\mathsf{Cay}(H, T)$. It follows from Propositions 4.2 and 4.3 that $\hat{\alpha}\beta^{-1} \in \mathrm{Aut}(\mathsf{Cay}(H, T)) = G$. Therefore $\hat{\alpha} \in G\beta$. Since G and β normalize H, so does α. However, this contradicts Lemma A.1(2). $\square$

On the previous proof, one could prove directly that there exists no automorphism β of H with $T^{\beta} = T'$; however, this requires some detailed computations, in the same spirit as the computations in Section 4.2.

ORCID iDs

Mikhail Muzychuk https://orcid.org/0000-0002-6346-8976
Pablo Spiga https://orcid.org/0000-0002-0157-7405
Ted Dobson https://orcid.org/0000-0003-2013-4594

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The antiprism of an abstract polytope*

Ian Gleason
Univeristy of California, Berkeley, United States

Isabel Hubard †
Instituto de Matemáticas, Universidad Nacional Autonoma de México, Circuito Exterior, C.U. Coyoacán 04510, México D.F.

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Abstract

Antiprisms of polygons are classical convex vertex-transitive polyhedra. In this paper, for any given (abstract) polytope, we define its *antiprism*. We then find the automorphism group of the antiprism of $\mathcal{P}$ in terms of the extended group of $\mathcal{P}$ (the groups of automorphisms and dualities) as well as some transitivity properties. We also give a relation between some products of abstract polytopes and their antiprisms.

Keywords: Antiprism, abstact polytopes.

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The antiprism is a classical convex polyhedron. The antiprism of a polygon can be constructed by taking, in Euclidian 3-space, two identical copies of a regular n-gon in parallel planes, in such a way that the vertices of one of the polygons are "aligned" with the mid points of the edges of the other. By taking the convex hull of all the vertices, we obtain the antiprism over an n-gon (see Figure 1).

For higher dimensions, the concept of a convex antiprism is not always defined (see [1, 2] and [3] for further discussion of the subject). In this paper we define the antiprism of any abstract polytope and show that it is indeed again an abstract polytope. The given definition generalizes the antiprism of a polygon and satisfies that the antiprism of a polytope and its dual is the same.

The paper uses some of the ideas and notation of the products of polytopes described in [4]. Moreover, we give relations between some products and their antiprisms. We then use such relations to compute the automorphism group of an antiprism. These results are summarized in the following theorem.

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†Corresponding author.
E-mail addresses: ianandreigf@gmail.com (Ian Gleason), isahubard@im.unam.mx (Isabel Hubard)

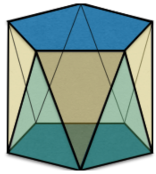

Figure 1: Antiprism over a pentagon.

Theorem A. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two abstract polytopes and $Ant(\mathcal{P}), Ant(\mathcal{Q})$ be their antiprisms.*

(a) *If $\bowtie$ and $\oplus$ denote the join product and the direct sum of abstract polytopes, respectively, then $Ant(\mathcal{P} \bowtie \mathcal{Q}) \cong Ant(\mathcal{P}) \oplus Ant(\mathcal{Q})$.*

(b) *If $\hat{\Gamma}$ denotes the extended group of $\mathcal{P}$ and $\mathcal{P} = \mathcal{Q}_1^{m_1} \bowtie \mathcal{Q}_2^{m_2} \bowtie \cdots \bowtie \mathcal{Q}_r^{m_r}$, where each $\mathcal{Q}_i$ is a prime polytope with respect to the join product, then*

$$\Gamma(Ant(\mathcal{P})) = \Pi_{i=1}^{r}\big((\hat{\Gamma}(\mathcal{Q}_i))^{m_i} \rtimes S_{m_i}\big).$$

In particular, if $\mathcal{P}$ is prime with respect to the join product, then $\Gamma(Ant(\mathcal{P})) \cong \Gamma(\mathcal{P})$ whenever $\mathcal{P}$ is a not a self-dual polytope, while if $\mathcal{P}$ is self-dual, then $\Gamma(\mathcal{P})$ has index 2 in $\Gamma(Ant(\mathcal{P})) = \hat{\Gamma}(\mathcal{P})$.

(c) *Let $\mathcal{P}$ be a prime polytope with respect to the join product. If $\mathcal{P}$ is a k-orbit polytope of rank n, then $Ant(\mathcal{P})$ is either a $2^n k$-orbit polytope (if $\mathcal{P}$ is self-dual) or a $2^{n+1}k$-orbit polytope (if $\mathcal{P}$ is not self-dual).*

The paper is organized as follows. Section 1 reviews the basic notions about abstract polytopes and their products. In Section 2 we define the antiprism and show that is always an abstract polytope and analyse the flags of the antiprism in terms of the flags of the polytope. Sections 3 and 4 deal with the interaction between some products and the antiprism, and with the study of the automorphism group of an antiprism, respectively.

1 Abstract polytopes, their join product and direct sum

Abstract polytopes are combinatorial generalizations of the face lattice of convex polytopes. In this section we give the basic definitions from the theory of abstract polytopes, as well as two of their products. For details on these subjects we refer the reader to [5] and [4], respectively.

An *(abstract) polytope* is a partially ordered set (poset) $\mathcal{P}$, whose elements are called *faces*, such that it has a minimal and a maximal element and is ranked: all its maximal chains, called *flags*, have the same number of elements. This endows the poset with a rank function r satisfying that if $F, G \in \mathcal{P}$ with $F \leq G$, then $r(F) \leq r(G)$, and if $r(F) = r(G)$, then F and G are either equal or they are not incident in $\mathcal{P}$. We say that the minimal face has rank -1, and if the range of the rank function is $\{-1, 0, \ldots, n\}$, then we say that $\mathcal{P}$ has rank n or is a n-polytope. A face of rank i is said to be an *i-face* and the 0-, 1- and $n-1$-faces are the vertices, edges and facets of $\mathcal{P}$, respectively. The minimal and maximal faces are the *improper* faces of $\mathcal{P}$, and all other faces are *proper*. We also require that $\mathcal{P}$ satisfies the *diamond condition*, meaning that whenever $F, G \in \mathcal{P}$ are two incident faces such that their ranks differ by 2, then there are exactly two faces H, H' of $\mathcal{P}$ satisfying that $F < H, H' < G$. Finally, we ask that $\mathcal{P}$ be *strongly connected* in the sense that the poset is connected and each of its open intervals with more than two elements is connected as well.

A *section* of $\mathcal{P}$ is a closed interval of $\mathcal{P}$. Every section of $\mathcal{P}$ is a polytope in its own right. The diamond condition is equivalent to saying that all sections of rank 1 have exactly 4 faces. This condition also implies that for each $i \in \{0, 1, \dots, n-1\}$ and every flag Φ, there is a unique i*-adjacent* flag to Φ that differs from Φ only in the element of rank i. We shall denote the set of all flags of $\mathcal{P}$ by $\mathcal{F}(\mathcal{P})$, and the i-adjacent flag of Φ by Φ^i.

The *dual* of a polytope $\mathcal{P}$ is the poset that has the same elements as $\mathcal{P}$, but with the reverse order. If a polytope is isomorphic to its dual, it is said to be *self-dual*.

An *automorphism* of $\mathcal{P}$ is an order preserving bijection. The group of all automorphisms of $\mathcal{P}$ is its *automorphism group* and it shall be denoted by $\Gamma(\mathcal{P})$. A *duality* of a self-dual polytope is an order reversing bijection. The composition of two dualities of a self-dual polytope is not a duality, but an automorphim. Thus, the *extended group* of $\mathcal{P}$ is the group that contains all automorphisms and dualities of $\mathcal{P}$ and it will be denoted by $\hat{\Gamma}(\mathcal{P})$. Note that $\hat{\Gamma}(\mathcal{P})$ has $\Gamma(\mathcal{P})$ as a subgroup of index at most 2; the groups coincide whenever $\mathcal{P}$ is not self-dual.

Given two polytopes $\mathcal{P}$ and $\mathcal{Q}$, their join product, $\mathcal{P} \bowtie \mathcal{Q}$, is the polytope whose elements are the pairs (F, G), with $F \in \mathcal{P}$ and $G \in \mathcal{Q}$. Two elements (F, G) and (F', G') are incident in $\mathcal{P} \bowtie \mathcal{Q}$ if and only if $F \leq_{\mathcal{P}} F'$ and $G \leq_{\mathcal{Q}} G'$. The rank of (F, G) is $\mathrm{rank}_{\mathcal{P}}(F) + \mathrm{rank}_{\mathcal{Q}}(G) + 1$. A polytope $\mathcal{P}$ is said to be *prime with respect to the join product* if it cannot be decomposed as the join product of two polytopes of ranks at least 0.

The *direct sum* of the polytopes $\mathcal{P}$ and $\mathcal{Q}$, with maximum elements F_n and G_m, respectively is $\mathcal{P} \oplus \mathcal{Q} = \{(F, G) \in \mathcal{P} \bowtie \mathcal{Q} \mid F \neq F_n, G \neq G_m\} \cup \{(F_n, G_m)\}$. The order of the direct sum is given by $(F, G) \leq_{\mathcal{P} \oplus \mathcal{Q}} (F', G')$ if and only if $F \leq_{\mathcal{P}} F'$ and $G \leq_{\mathcal{Q}} G'$, and the rank of the face (F, G) is $\mathrm{rank}_{\mathcal{P}}(F) + \mathrm{rank}_{\mathcal{Q}}(G)$, which implies that the rank of $\mathcal{P} \oplus \mathcal{Q}$ is $n + m$. A polytope $\mathcal{P}$ is said to be *prime with respect to the direct sum* if it cannot be decomposed as the join product of two polytopes of ranks at least 1. The following lemma falls straightforward from the definitions.

Lemma 1.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes, and let (F, G) be a proper face of $\mathcal{P} \oplus \mathcal{Q}$. Then the section $\{(H, K) \in \mathcal{P} \oplus \mathcal{Q} \mid (H, K) \leq_{\mathcal{P} \oplus \mathcal{Q}} (F, G)\}$ is isomorphic to the join product of the sections $\{H \in \mathcal{P} \mid H \leq_{\mathcal{P}} F\}$ and $\{K \in \mathcal{Q} \mid K \leq_{\mathcal{Q}} G\}$.*

In [4] the authors study the automorphism group of a product in terms of the automorphisms groups of the factors. In particular we have the following result.

Theorem 1.2 ([4]). *Let $\mathcal{P} = \mathcal{Q}_1^{m_1} \oplus \mathcal{Q}_2^{m_2} \oplus \cdots \oplus \mathcal{Q}_r^{m_r}$, where each $\mathcal{Q}_i$ is a prime polytope with respect to the direct sum. Then*

$$\Gamma(\mathcal{P}) = \Pi_{i=1}^{r} \big((\Gamma(\mathcal{Q}_i))^{m_i} \rtimes S_{m_i} \big).$$

2 The antiprism

The antiprism of a polygon is a convex polyhedron in ordinary 3-space. Its faces are two regular n-gons and $2n$ equilateral triangles. When $n = 3$, we obtain the regular octahedron. Otherwise, the antiprism over an n-gon is an Archemidian solid, as their faces are all regular polygons and its group of symmetries acts transitively on the vertices. In this section, for each polytope $\mathcal{P}$, we give a construction of a new polytope $Ant(\mathcal{P})$ which generalizes the construction of the antiprism of a polygon.

Let $\mathcal{P}$ be an n-polytope. To formally define the antiprism of $\mathcal{P}$, $Ant(\mathcal{P})$, we let P be a symbol, and define

$$Ant(\mathcal{P}) := \{(F, G) \mid F, G \in \mathcal{P}, F \leq_{\mathcal{P}} G\} \cup \{P\},$$

where the order is given by

$$(F, G) \leq (H, K) \text{ if and only if } F \leq_{\mathcal{P}} H \leq_{\mathcal{P}} K \leq_{\mathcal{P}} G; \tag{2.1}$$

$$(F, G) \leq P \qquad \text{for every } F, G \in \mathcal{P}. \tag{2.2}$$

Throughout this section, when we say that an ordered pair of elements of $\mathcal{P}$ is an element of $Ant(\mathcal{P})$, we shall be referring to an element of $Ant(\mathcal{P})$ different than P.

Note then that P is the maximum element of $Ant(\mathcal{P})$ and that, if F_{-1} and F_n denote the minimum and maximum elements of $\mathcal{P}$, respectively, then (F_{-1}, F_n) is in fact the minimum element of $Ant(\mathcal{P})$. Moreover, for $H, F, G \in \mathcal{P}$, with $H \leq_{\mathcal{P}} F \leq_{\mathcal{P}} G$, we have that $(H, F) \leq (F, F)$ and $(F, G) \leq (F, F)$, but the only element of $Ant(\mathcal{P})$ greater than (F, F) is P.

Suppose that $\mathcal{P}$ has rank n and its rank function is $\operatorname{rank}_{\mathcal{P}}$. Define

$$\operatorname{rank}_{Ant(\mathcal{P})}(F, G) := \operatorname{rank}(F, G) = n + \operatorname{rank}_{\mathcal{P}}(F) - \operatorname{rank}_{\mathcal{P}}(G), \tag{2.3}$$

$$\operatorname{rank}_{Ant(\mathcal{P})}(P) := \operatorname{rank}(P) = n + 1.$$

Note that for every $(F, G) \in Ant(\mathcal{P})$, we have that $0 \leq \operatorname{rank}_{\mathcal{P}}(G) - \operatorname{rank}_{\mathcal{P}}(F) \leq n + 1$, implying that $\operatorname{rank}(F, G) \in \{-1, \dots, n\}$ and therefore $\operatorname{rank}\colon Ant(\mathcal{P}) \to \{-1, \dots, n + 1\}$.

Moreover, if $\operatorname{rank}(F, G) = -1$, then

$$n + \operatorname{rank}_{\mathcal{P}}(F) - \operatorname{rank}_{\mathcal{P}}(G) = -1.$$

This is equivalent to have that

$$\operatorname{rank}_{\mathcal{P}}(G) = n + 1 + \operatorname{rank}_{\mathcal{P}}(F).$$

But $\operatorname{rank}_{\mathcal{P}}(G) \leq n$ which implies that F should have rank -1, and thus G has rank n; in other words, $\operatorname{rank}(F, G) = -1$ if and only if $F = F_{-1}$ and $G = F_n$. We can further see that $\operatorname{rank}(F, G) = n$ if and only if $F = G$. Hence, the facets of $Ant(\mathcal{P})$ are the elements (F, F), with $F \in \mathcal{P}$.

There are other faces of $Ant(\mathcal{P})$ that are easy to identify. For example, if (F, G) is a vertex, it should satisfy that $\operatorname{rank}(F, G) = n + \operatorname{rank}_{\mathcal{P}}(F) - \operatorname{rank}_{\mathcal{P}}(G) = 0$. That is,

$$\operatorname{rank}_{\mathcal{P}}(G) = n + \operatorname{rank}_{\mathcal{P}}(F).$$

Again, since $\operatorname{rank}_{\mathcal{P}}(G) \leq n$ we have two options: either $\operatorname{rank}_{\mathcal{P}}(G) = n$ and $\operatorname{rank}_{\mathcal{P}}(F) = 0$, or $\operatorname{rank}_{\mathcal{P}}(G) = n - 1$ and $\operatorname{rank}_{\mathcal{P}}(F) = F_{-1}$. This implies that the vertices of $Ant(\mathcal{P})$ are either of the form (v, F_n), where v is a vertex of $\mathcal{P}$, or of the form (F_{-1}, f), where f is a facet of $\mathcal{P}$.

Before showing that $Ant(\mathcal{P})$ is a polytope, let us analyze the case when $\mathcal{P}$ is 2-polytope. Let $\mathcal{P}$ be a 2-polytope with vertices $\{v_1, \dots, v_p\}$ and edges $\{e_1, \dots e_p\}$ in such a way that for every $i = 1, \dots, p - 1$, $v_i, v_{i+1} \leq e_i$, and $v_1, v_p \leq e_p$. Let m and M be the minimum and maximum elements of $\mathcal{P}$, respectively. We already know that $Ant(\mathcal{P})$ has a unique minimum (m, M) and a unique maximum P, that there are $2p$ vertices, namely:

$$(v_1, M), \ldots, (v_p, M), (m, e_1), \ldots, (m, e_p),$$

and that the facets, 2-faces in this case, are of the form (F, F), where F is any element of $\mathcal{P}$. Thus there are $2p + 2$ facets. Finally, the 1-faces are the elements

$$\begin{aligned} &(e_1, M), \ldots, (e_p, M), \\ &(m, v_1), \ldots, (m, v_p), \\ (v_1, e_p), (v_1, e_1), &(v_2, e_1), \ldots, (v_p, e_p), \end{aligned}$$

and there are $4p$ of them.

We note that the facets (m, m) and (M, M) are p-gons, as their vertices are of the form (m, e_i) and (v_i, M), respectively. In contrast, the facets of type (v_i, v_i) and (e_i, e_i) are triangles, as their only vertices are either of the form $(v_i, M), (m, e_{i-1}), (m, e_i)$ or of the form $(v_i, M), (v_{i+1}, M), (m, e_i)$. It is not too difficult now to see that $Ant(\mathcal{P})$ is in fact isomorphic to the classical antiprism.

Given an abstract polytope $\mathcal{P}$, we should say that $Ant(\mathcal{P})$ is the *antiprism of* $\mathcal{P}$.

In order to show that the antiprism of any polytope is again a polytope, we shall start by analyzing the sections of $Ant(\mathcal{P})$. As we noted before, the only elements of rank n are of the type (F, F), where $F \in \mathcal{P}$. Let us take a look into the sections $\mathcal{Q}_F := (F, F)/(F_{-1}, F_n)$ where, as before, F_{-1}, F_n are the minimum and maximum faces of the n-polytope $\mathcal{P}$, respectively.

Let us fix a face F of $\mathcal{P}$. If $(H, G) \in \mathcal{Q}_F$, then $(H, G) \leq (F, F)$, which implies that $H \leq F \leq F \leq G$. In other words, F is a face of the section G/H of $\mathcal{P}$. On the other hand, if $H, G \in \mathcal{P}$ are such that $H \leq F$ and $F \leq G$, then $(H, G) \in \mathcal{Q}_F$. That means that the faces of the section $\mathcal{Q}_F$ are in one to one correspondence with the order pairs (H, G) of elements of $\mathcal{P}$ such that $H \leq F \leq G$.

Since $\mathcal{P}$ is a polytope, then $\mathcal{P}_F^- := F/F_{-1}$ and $\mathcal{P}_F^+ := F_n/F$ are also polytopes. Let $\delta\colon \mathcal{P}_F^+ \to (\mathcal{P}_F^+)^*$ be a duality mapping $\mathcal{P}_F^+$ to its dual. Now, $H \in \mathcal{P}_F^-$ if and only if $H \leq F$; on the other hand, $F \leq G$ if and only if $G\delta \in (\mathcal{P}_F^+)^*$. Consider now the join product of $\mathcal{P}_F^-$ with $(\mathcal{P}_F^+)^*$. We have that

$$\begin{aligned} \psi\colon \mathcal{P}_F^- \bowtie (\mathcal{P}_F^+)^* &\to \mathcal{Q}_F \\ (H, G\delta) &\mapsto (H, G) \end{aligned} \tag{2.4}$$

is a well-defined bijection between $\mathcal{P}_F^- \bowtie (\mathcal{P}_F^+)^*$ and $\mathcal{Q}_F$. Furthermore, note that $(H, G\delta) \leq_{\bowtie} (H', G'\delta)$ if and only if $H \leq H'$ and $G\delta \leq G'\delta$, which is equivalent to have $H \leq H'$ and $G' \leq G$. That is, $(H, G\delta) \leq_{\bowtie} (H', G'\delta)$ if and only if $H \leq H' \leq F \leq G' \leq G$, which is equivalent to have that $(H, G) \leq_{\mathcal{Q}_F} (H', G')$. Thus, ψ is an isomorphism between $\mathcal{P}_F^- \bowtie (\mathcal{P}_F^+)^*$ and $\mathcal{Q}_F$. This implies that all the facets of $Ant(\mathcal{P})$ are abstract polytopes. In particular we note that $\mathcal{Q}_{F_{-1}} \cong \mathcal{P}^*$, while $\mathcal{Q}_{F_n} \cong \mathcal{P}$.

We turn now our attention to the co-faces $P/(F, G)$ of $Ant(\mathcal{P})$. We observe that

$$\begin{aligned} P/(F, G) &= \{(H, K) \in Ant(\mathcal{P}) \mid (F, G) \leq_{Ant(\mathcal{P})} (H, K)\} \cup \{P\} \\ &\cong \{(H, K) \in \mathcal{P} * \mathcal{P} \mid F \leq_{\mathcal{P}} H \leq_{\mathcal{P}} K \leq_{\mathcal{P}} G)\} \cup \{P\} \\ &\cong \{(H, K) \in \mathcal{P} * \mathcal{P} \mid H, K \in G/F, H \leq_{G/F} K\} \cup \{P\} \\ &\cong Ant(G/F). \end{aligned}$$

This says that all the co-faces of $Ant(\mathcal{P})$ are antiprisms of polytopes of smaller rank than that of $\mathcal{P}$.

Theorem 2.1. *Let $\mathcal{P}$ be an n-polytope, then $Ant(\mathcal{P})$ is an $n+1$ polytope.*

Proof. The function given in (2.3) is the desired rank function, with range $\{-1, \dots, n+1\}$, and it is clear from the definition that $Ant(\mathcal{P})$ has a minimum and a maximum face.

We now proceed by induction over n.

Let $\mathcal{P} = \{F_{-1}, F_0\}$ be a 0-polytope. Then $Ant(\mathcal{P}) = \{(F_{-1}, F_0), (F_{-1}, F_{-1}), (F_0, F_0), P\}$, where $(F_{-1}, F_0) \leq (F_{-1}, F_{-1}), (F_0, F_0) \leq P$. Hence, $Ant(\mathcal{P})$ is isomorphic to an edge, that is, $Ant(\mathcal{P})$ is a 1-polytope.

Suppose now that the antiprism of any polytope of rank $(n-1)$ is a polytope and let $\mathcal{P}$ be an n-polytope. Since the facets of $Ant(\mathcal{P})$ are a join product of polytopes, then they are polytopes. In particular, every flag of $Ant(\mathcal{P})$, when taking away the maximum face, can be seen to be contained flag of a facet of $Ant(\mathcal{P})$. Since the flags of the facets have all $n+2$ elements, every flag of $Ant(\mathcal{P})$ has exactly $n+3$ elements.

The diamond condition is satisfied and all the proper sections of $Ant(\mathcal{P})$ are connected: this is straightforward to see as a proper section of $Ant(\mathcal{P})$ is contained either in a facet or in a vertex figure of $Ant(\mathcal{P})$. The facets of $Ant(\mathcal{P})$ are joins of polytopes (hence polytopes) and the vertex figures are anitprisms over proper sections of $\mathcal{P}$, which by hypothesis of induction are also polytopes.

We only have to see that $Ant(\mathcal{P})$ itself is connected. Let $(F, G), (H, K)$ be two proper faces of $Ant(\mathcal{P})$. We divide the analysis in several cases, depending on whether or not F, G, H and K are proper or improper faces of $\mathcal{P}$. Note that F and G (resp. H and K) cannot be improper face of $\mathcal{P}$ simultaneously, unless they are equal. Without loss of generality, we may assume that $rank_{\mathcal{P}}(G) \leq rank_{\mathcal{P}}(K)$.

- If $G, K \neq F_n$, then (F, G), (F_{-1}, G), (F_{-1}, F_{-1}), (F_{-1}, K), (H, K) is a sequence of incident proper faces of $Ant(\mathcal{P})$.
- If $F, H \neq F_{-1}$, then (F, G), (F, F_n), (F_n, F_n), (H, F_n), (H, K) is a sequence of incident proper faces of $Ant(\mathcal{P})$.
- If $K = F_n$, $F = F_{-1}$ and H is a proper face of $\mathcal{P}$, then $H \neq F_{-1}, F_n$ and $G \neq F_n$. Since $\mathcal{P}$ is connected, then there exists a sequence $G = G_1, G_2, \dots, G_h = H$ of incident faces of $\mathcal{P}$ all of which, except perhaps for G, are proper faces. Then $(F, G) = (F_{-1}, G_1)$, $(F_{-1}, G_2), \dots, (F_{-1}, G_h) = (F_{-1}, H)$, (H, H), $(H, F_n) = (H, K)$ is a sequence of incident proper faces of $Ant(\mathcal{P})$.
- If $K = H = F_n$, $F = F_{-1}$ and G is a proper face of $\mathcal{P}$, then (F_{-1}, G), (G, G), (G, F_n), (F_n, F_n) is a sequence of incident proper faces of $Ant(\mathcal{P})$.
- If $K = H = F_n$, $F = G = F_{-1}$, then let $J \in \mathcal{P}$ be any proper face of $\mathcal{P}$ (exists as we are assuming $n > 0$). Hence (F_{-1}, F_{-1}), (F_{-1}, J), (J, J), (J, F_n), (F_n, F_n) is a sequence of incident proper faces of $Ant(\mathcal{P})$.

Hence $Ant(\mathcal{P})$ is connected and therefore it is a polytope. □

Note that given a polytope $\mathcal{P}$ and its dual $\mathcal{P}^*$, there is a duality $\delta\colon \mathcal{P} \to \mathcal{P}^*$. We know that δ is a bijection that reverses the order, and hence every element of $\mathcal{P}^*$ can be written as $F\delta$, where F is a face of $\mathcal{P}$. Hence, there is a natural bijection between the faces of the antirpism of $\mathcal{P}$ and the faces of the antiprism of $\mathcal{P}^*$. In fact, we have the following proposition.

Proposition 2.2. *For any polytope $\mathcal{P}$, $Ant(\mathcal{P}) \cong Ant(\mathcal{P}^*)$, where $\mathcal{P}^*$ denotes the dual of $\mathcal{P}$.*

Proof. Let P and P^* be the maximum elements of $Ant(\mathcal{P})$ and $Ant(\mathcal{P}^*)$, respectively, and let $\delta\colon \mathcal{P} \to \mathcal{P}^*$ be a duality. Let $\psi\colon\ Ant(\mathcal{P}) \to Ant(\mathcal{P}^*)$ be given by:

$$\begin{array}{rcl} (F,G) & \mapsto & (G\delta, F\delta), \\ P & \mapsto & P^*. \end{array}$$

Then clearly ψ is a well-defined bijection between $Ant(\mathcal{P})$ and $Ant(\mathcal{P}^*)$. Furthermore $(F,G) \leq_{Ant(\mathcal{P})} (H,K)$ if and only if $F \leq_{\mathcal{P}} H \leq_{\mathcal{P}} K \leq_{\mathcal{P}} G$ if and only if $G\delta \leq_{\mathcal{P}^*} K\delta \leq_{\mathcal{P}^*} H\delta \leq_{\mathcal{P}^*} F\delta$ if and only if $(G\delta, F\delta) \leq_{\mathcal{P}^*} (K\delta, H\delta)$ which is equivalent to $(F,G)\psi \leq_{\mathcal{P}^*} (H,K)\psi$. Since it is now straightforward to see that δ^{-1} also induces a bijection that preserves the order and is the inverse of ψ. This settles the proposition. $\square$

2.1 The flags of a polytope and the flags of its antiprism

In this section we study the relation between the flags of $Ant(\mathcal{P})$ and the flags of $\mathcal{P}$.

Let $\mathcal{P}$ be an n-polytope (with maximum element F_n and minimum element F_{-1}) and consider V to be the set of all ordered $(n+1)$-tuples with entries 0 and 1. We are going to see that there is a bijection between $\mathcal{F}(Ant(\mathcal{P}))$ and $\mathcal{F}(\mathcal{P}) \times V$. For this, consider a flag of $Ant(\mathcal{P})$, $\{A_{-1}, A_0, \dots A_{n+1}\}$, where $\operatorname{rank} A_i = i$. Then $A_{n+1} = P$ and for each $i = -1, 0 \dots n$, there exist $F^i, G^i \in \mathcal{P}$ such that $A_i = (F^i, G^i)$. It is straightforward to see that $F^{-1} = F_{-1}$, $G^{-1} = F_n$ and $F^n = G^n := F$, for some $F \in \mathcal{P}$. Furthermore, observe that

$$F_{-1} \leq F^0 \leq F^1 \leq \dots \leq F^n = F = G^n \leq G^{n-1} \leq \dots \leq G^0 \leq G^{-1} = F_n, \quad (2.5)$$

is a sequence of faces of $\mathcal{P}$ in which, of course, many of the elements might repeat. For example, the sequence could be such that $F^0 = F^1 = F^2 = \dots = F^n = F_{-1}$.

On one hand, note that for a given $i \in \{0, \dots n\}$, we have that either

$$\operatorname{rank}(F^i) = \operatorname{rank}(F^{i+1}) \text{ and } \operatorname{rank}(G^i) = \operatorname{rank}(G^{i+1}) + 1$$

or

$$\operatorname{rank}(F^i) + 1 = \operatorname{rank}(F^{i+1}) \text{ and } \operatorname{rank}(G^i) = \operatorname{rank}(G^{i+1}).$$

In particular, either $\operatorname{rank}(F^0) = -1$ and $\operatorname{rank}(G^0) = n-1$ or $\operatorname{rank}(F^0) = 0$ and $\operatorname{rank}(G^0) = n$. Hence, we can regard the sequence in (2.5) as a sequence of incident faces of $\mathcal{P}$ that has exactly one element of each rank. That is, a flag of $\mathcal{P}$. In other words, each flag of $Ant(\mathcal{P})$ induces a flag of $\mathcal{P}$ in a natural way.

On the other hand, the sequence in (2.5) also defines an element of V in the following way. For each $i \in \{0, \dots, n\}$, let $a_i = 0$ if $\operatorname{rank}(F^{i-1}) = \operatorname{rank}(F^i)$ and $a_i = 1$ otherwise. It should be clear that $(a_0, \dots a_n)$ is an element of V.

The above assignment is a bijection. To see this, take $\Phi \in \mathcal{F}(\mathcal{P})$ and $v \in V$. Denote by Φ_i the i-face of Φ and by v_i the i-th element of v, i.e. $v = (v_0, v_1, \dots, v_n)$. We define the flag $\{A_{-1}, A_0, \dots A_{n+1}\}$ of $Ant(\mathcal{P})$, where each $A_i = (F^i, G^i)$, in the following way.

First, $A_{n+1} = P$ and $A_{-1} = (F_{-1}, F_n) = (\Phi_{-1}, \Phi_n)$. Now, we define inductively the elements F^i and G^i. Suppose F^{i-1} is defined as the j-face Φ_j, then we define $F^i := \Phi_{j+v_i}$. (For example, if $v_0 = 0$, then $F^0 = \Phi_{-1+v_0} = \Phi_{-1} = F_{-1}$, and if $v_0 = 1$, then $F^0 = \Phi_{-1+v_0} = \Phi_0$.) Similarly, we first define $G^n := F^n$ and, inductively, suppose that G^{i+1} is defined as the k-face Φ_k, then we define $G^i := \Phi_{k+1-v_{i+1}}$. Thus, $| \{v_j \in v \mid v_j = 1\} |= m + 1$, for some $-1 \leq m \leq n$ and hence $G^n = F^n = \Phi_m$ and $| \{v_j \in v \mid v_j = 0\} |= n - m$. This implies that

$$G^{-1} = \Phi_{m+(1-v_n)+(1-v_{n-1}+\cdots+(1-v_0))} = \Phi_{m+(n-m)} = \Phi_n.$$

It should not be difficult to see that this assignment of a flag of $Ant(\mathcal{P})$, given a pair $(\Phi, v) \in \mathcal{F}(\mathcal{P}) \times V$ is inverse to the above description, where each flag of $Ant(\mathcal{P})$ induces a flag of $\mathcal{P}$ and an element of V. We have therefore established that

Lemma 2.3. *Let $\mathcal{P}$ be an n-polytope. Then the flags of $Ant(\mathcal{P})$ are in one-to-one correspondence with the set $\mathcal{F}(\mathcal{P}) \times V$, where $\mathcal{F}(\mathcal{P})$ denotes the set of flags of $\mathcal{P}$ and V the set of all ordered $(n+1)$-tuples with entries* 0 *and* 1.

3 Products and the antiprism

In the next section we will study the automorphism group of an antiprism. We shall see that computing it for polytopes that are prime with respect to the join product is straighforward. To completely determine the automorphism group of any antiprism, we need some of the results given in this section.

All our results here deal with the interaction of the join product and the direct sum with the anitprism.

Proposition 3.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes. Then $Ant(\mathcal{P} \bowtie \mathcal{Q}) \cong Ant(\mathcal{P}) \oplus Ant(\mathcal{Q})$.*

Proof. Let $\psi\colon\ Ant(\mathcal{P} \bowtie \mathcal{Q}) \to Ant(\mathcal{P}) \oplus Ant(\mathcal{Q})$ be such that

$$\big((F, G), (H, K)\big)\psi = \big((F, H), (G, K)\big)$$

and, if $P, P_{\mathcal{P}}$ and $P_{\mathcal{Q}}$ are the maximum elements of $Ant(\mathcal{P} \bowtie \mathcal{Q}), Ant(\mathcal{P})$ and $Ant(\mathcal{Q})$, respectively, then $P\psi = (P_{\mathcal{P}}, P_{\mathcal{Q}})$. We shall show that ψ is an isomorphism.

First note that $\big((F, G), (H, K)\big) \in Ant(\mathcal{P} \bowtie \mathcal{Q})$ implies that $(F, G), (H, K) \in \mathcal{P} \bowtie \mathcal{Q}$ and that $(F, G) \leq_{\mathcal{P}\bowtie\mathcal{Q}} (H, K)$. Hence, we have that $F, H \in \mathcal{P}$ with $F \leq_{\mathcal{P}} H$, and that $G, K \in \mathcal{Q}$ with $G \leq_{\mathcal{Q}} K$; that is, $(F, H) \in Ant(\mathcal{P})$ and $(G, K) \in Ant(\mathcal{Q})$. Moreover, (F, G) is not the maximum element of $Ant(\mathcal{P})$, and (H, K) is not the maximum element of $Ant(\mathcal{Q})$, which implies that $\big((F, H), (G, K)\big) \in Ant(\mathcal{P}) \oplus Ant(\mathcal{Q})$. Furthermore, observe that different elements of $Ant(\mathcal{P} \bowtie \mathcal{Q})$ go to different elements of $Ant(\mathcal{P}) \oplus Ant(\mathcal{Q})$ under ψ and therefore ψ is a well-defined function from $Ant(\mathcal{P} \bowtie \mathcal{Q})$ to $Ant(\mathcal{P}) \oplus Ant(\mathcal{Q})$.

Similarly, let $\phi\colon\ Ant(\mathcal{P}) \oplus Ant(\mathcal{Q}) \to Ant(\mathcal{P} \bowtie \mathcal{Q})$ be such that

$$\big((F, H), (G, K)\big) \mapsto \big((F, G), (H, K)\big).$$

A similar argument as the one above shows that ϕ is also a well-defined function. Note that both $\psi\phi$ and $\phi\psi$ are the identity map, which implies that both functions are bijections, and one is the inverse of the other.

We need to show that these two functions preserve the orders. Let $\big((F_0,G_0),(H_0,K_0)\big)$, $\big((F_1,G_1),(H_1,K_1)\big) \in Ant(\mathcal{P} \bowtie \mathcal{Q})$, then

$$\begin{aligned}
&\big((F_0,G_0),(H_0,K_0)\big) \leq_{Ant(\mathcal{P}\bowtie\mathcal{Q})} \big((F_1,G_1),(H_1,K_1)\big)\\
&\Leftrightarrow (F_0,G_0) \leq_{\mathcal{P}\bowtie\mathcal{Q}} (F_1,G_1) \leq_{\mathcal{P}\bowtie\mathcal{Q}} (H_1,K_1) \leq_{\mathcal{P}\bowtie\mathcal{Q}} (H_0,K_0)\\
&\Leftrightarrow F_0 \leq_{\mathcal{P}} F_1 \leq_{\mathcal{P}} H_1 \leq_{\mathcal{P}} H_0 \text{ and } G_0 \leq_{\mathcal{Q}} G_1 \leq_{\mathcal{Q}} K_1 \leq_{\mathcal{Q}} K_0\\
&\Leftrightarrow (F_0,H_0) \leq_{Ant(\mathcal{P})} (F_1,H_1) \text{ and } (G_0,K_0) \leq_{Ant(\mathcal{Q})} (G_1,K_1)\\
&\Leftrightarrow \big((F_0,H_0),(G_0,K_0)\big) \leq_{Ant(\mathcal{P})\oplus Ant(\mathcal{Q})} \big((F_1,H_1),(G_1,K_1)\big).
\end{aligned}$$

Therefore both ψ and ϕ preserve the orders and hence ψ is an isomorphism. □

Lemma 3.2. *If $\mathcal{P}$ is a prime polytope with respect to the join product, then $Ant(\mathcal{P})$ is a prime polytope with respect to the direct sum.*

Proof. Suppose otherwise. Then there exists a polytope $\mathcal{P}$ that is prime with respect to the join product, but such that $Ant(\mathcal{P})$ is not prime with respect to the direct sum. Let $Ant(\mathcal{P}) = \mathcal{Q} \oplus \mathcal{K}$, where $\mathcal{Q}$ and $\mathcal{K}$ are polytopes of rank at least 1.

Note that $Ant(\mathcal{P})$ contains a facet that is isomorphic to $\mathcal{P}$. In fact, if F_n denotes the maximum element of $\mathcal{P}$, then $(F_n,F_n) \in Ant(\mathcal{P})$ has rank n and if $(F,G) \leq (F_n,F_n)$, then $G = F_n$ (since $F \leq_{\mathcal{P}} F_n \leq_{\mathcal{P}} F_n \leq_{\mathcal{P}} G$). That means that the section $(F_n,F_n)/(F_{-1},F_n)$ of $Ant(\mathcal{P})$ is isomorphic to $\mathcal{P}$.

But by Lemma 1.1, a facet of the direct product $\mathcal{Q} \oplus \mathcal{K}$ is isomorphic to a non-trivial join product. Hence $\mathcal{P}$ is not prime with respect to the join product, which contradicts our hypothesis. □

4 Automorphism groups

We now turn our attention to the study of the automorphism group of the antiprism of $\mathcal{P}$.

It is not difficult to see that every automorphism of $\mathcal{P}$ induces an automorphism of $Ant(\mathcal{P})$. In fact, given $\gamma \in \Gamma(\mathcal{P})$ the mapping $\hat{\gamma}\colon\ Ant(\mathcal{P}) \to Ant(\mathcal{P})$ given by $(F,G)\hat{\gamma} := (F\gamma, G\gamma)$, for $(F,G) \in Ant(\mathcal{P})$, and $P\hat{\gamma} := P$ is clearly an automorphism of $Ant(\mathcal{P})$. Similarly, if $\mathcal{P}$ is a self-dual polytope and δ is a duality of $\mathcal{P}$, then $\hat{\delta}\colon (F,G) \mapsto (G\delta, F\delta)$ (and $P\hat{\delta} = P$) is also an automorphism of $Ant(\mathcal{P})$. In other words, we have the following lemma. Keep in mind that we have defined the extended group of a non-self-dual polytope simply as its automorphism group.

Lemma 4.1. *Let $\mathcal{P}$ be a polytope and let $\hat{\Gamma}(\mathcal{P})$ denote its extended group. Then, $\hat{\Gamma}(\mathcal{P})$ is (isomorphic to) a subgroup of $G(Ant(\mathcal{P}))$.*

It is not difficult to see that if $\psi : \mathcal{F}(Ant(\mathcal{P})) \to \mathcal{F}(\mathcal{P}) \times V$ is the bijection from Lemma 2.3, $\gamma \in \widehat{\Gamma}(\mathcal{P})$ and $\tilde{\gamma}$ is the automorphism of $Ant(\mathcal{P})$ induced by γ, then for every flag $\Phi \in \mathcal{F}(\mathcal{P})$ and every $(n+1)$-tuple $v \in V$, we have that $(\Phi,v)\psi^{-1}\tilde{\gamma}\psi = (\Phi\gamma, v)$. This implies that if $\mathcal{P}$ is a self-dual polytope, dualities of $\mathcal{P}$ induce automorphisms of $Ant(\mathcal{P})$.

The above observation, together with Lemmas 4.1 and 2.3 imply the following result.

Proposition 4.2. *Let $\mathcal{P}$ an n-polytope and let $Ant(\mathcal{P})$ be its antiprism. If $\mathcal{P}$ is a k-orbit polytope, and $Ant(\mathcal{P})$ is an m-orbit polytope, then:*

- *if $\mathcal{P}$ is self-dual, then $m \leq k \cdot 2^n$,*

- *if $\mathcal{P}$ is not self-dual, then $m \leq k \cdot 2^{n+1}$.*

Observe that, by the isomorphism given in (2.4), the facets of $Ant(\mathcal{P})$ can be seen as the join product of sections of $\mathcal{P}$. This means that, maybe with the exception of the facets $(F_{-1}, F_{-1}) \cong \mathcal{P}^*$ and $(F_n, F_n) \cong \mathcal{P}$, the facets of $Ant(\mathcal{P})$ are not prime with respect to the join product. Whenever $\mathcal{P}$ is a prime polytope with respect to the join product, we can obtain a lot of information about $\Gamma(Ant(\mathcal{P})$.

Proposition 4.3. *Let $\mathcal{P}$ be a prime polytope with respect to the join product. Then, $\Gamma(Ant(\mathcal{P})) \cong \hat{\Gamma}(\mathcal{P})$.*

Proof. By Lemma 4.1 we only need to show that any automorphism of $Ant(\mathcal{P})$ is in fact induced by either an automorphism or a duality (if $\mathcal{P}$ is self-dual) of $\mathcal{P}$. In this proof we abuse notation and refer to a polytope that is prime with respect to the join product simply as a prime polytope.

As pointed out above, when $\mathcal{P}$ is a prime polytope, the only two facets of $Ant(\mathcal{P})$ that are also prime are (F_{-1}, F_{-1}) and (F_n, F_n). This means that any automorphism $\alpha \in \Gamma(Ant(\mathcal{P}))$ either fixes both such faces or interchanges them (as they cannot be permuted with any other, or they would not be prime). It is then easy to see that if α fixes them, then it induces an automorphism of $\mathcal{P}$ and that if interchanges them, then $\mathcal{P}$ is self-dual and α induces a duality. $\square$

Propositions 4.2 and 4.3 immediately imply the following result.

Corollary 4.4. *Let $\mathcal{P}$ an n-polytope that is prime with respect to the join product and let $Ant(\mathcal{P})$ be its antiprism. If $\mathcal{P}$ is a k-orbit polytope, and $Ant(\mathcal{P})$ is an m-orbit polytope, then:*

- *if $\mathcal{P}$ is self-dual, then $m = k \cdot 2^n$,*
- *if $\mathcal{P}$ is not self-dual, then $m = k \cdot 2^{n+1}$.*

Lemma 3.2, together with Propositions 3.1 and 4.3 and Theorem 1.2, give us all the necessary tools to compute the automorphism of the antiprism of any polytope.

Theorem 4.5. *Let $\mathcal{P} = \mathcal{Q}_1^{m_1} \bowtie \mathcal{Q}_2^{m_2} \bowtie \cdots \bowtie \mathcal{Q}_r^{m_r}$, where each $\mathcal{Q}_i$ is a prime polytope with respect to the join product. Then*

$$\Gamma(Ant(\mathcal{P})) = \Pi_{i=1}^{r}\big((\hat{\Gamma}(\mathcal{Q}_i))^{m_i} \rtimes S_{m_i}\big).$$

ORCID iDs

Isabel Hubard https://orcid.org/0000-0002-0960-3671

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Linkedness of Cartesian products of complete graphs*

Leif K. Jørgensen

Department of Mathematical Sciences, Aalborg University, Denmark

Guillermo Pineda-Villavicencio †, Julien Ugon ‡

Federation University, Ballarat, Australia and
School of Information Technology, Deakin University, Geelong, Australia

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Abstract

This paper is concerned with the linkedness of Cartesian products of complete graphs. A graph with at least $2k$ vertices is k*-linked* if, for every set of $2k$ distinct vertices organised in arbitrary k pairs of vertices, there are k vertex-disjoint paths joining the vertices in the pairs.

We show that the Cartesian product $K^{d_1+1} \times K^{d_2+1}$ of complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1+d_2)/2 \rfloor$-linked for $d_1, d_2 \geq 2$, and this is best possible.

This result is connected to graphs of simple polytopes. The Cartesian product $K^{d_1+1} \times K^{d_2+1}$ is the graph of the Cartesian product $T(d_1) \times T(d_2)$ of a d_1-dimensional simplex $T(d_1)$ and a d_2-dimensional simplex $T(d_2)$. And the polytope $T(d_1) \times T(d_2)$ is a *simple polytope*, a (d_1+d_2)-dimensional polytope in which every vertex is incident to exactly d_1+d_2 edges.

While not every d-polytope is $\lfloor d/2 \rfloor$-linked, it may be conjectured that every simple d-polytope is. Our result implies the veracity of the revised conjecture for Cartesian products of two simplices.

Keywords: k-linked, cyclic polytope, connectivity, dual polytope, linkedness, Cartesian product.

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E-mail addresses: leifkjorgensen@gmail.com (Leif K. Jørgensen), work@guillermo.com.au (Guillermo Pineda-Villavicencio), julien.ugon@deakin.edu.au (Julien Ugon)

1 Introduction

Denote by $V(X)$ the vertex set of a graph. Given sets A, B of vertices in a graph, a path from A to B, called an $A-B$ *path*, is a (vertex-edge) path $L := u_0 \dots u_n$ in the graph such that $V(L) \cap A = \{u_0\}$ and $V(L) \cap B = \{u_n\}$. We write $a - B$ path instead of $\{a\} - B$ path, and likewise, write $A - b$ path instead of $A - \{b\}$.

Let G be a graph and X a subset of $2k$ distinct vertices of G. The elements of X are called *terminals*. Let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be an arbitrary labelling and (unordered) pairing of all the vertices in X. We say that Y is *linked* in G if we can find disjoint $s_i - t_i$ paths for $i \in [1, k]$, the interval $1, \dots, k$. The set X is *linked* in G if every such pairing of its vertices is linked in G. Throughout this paper, by a set of disjoint paths, we mean a set of vertex-disjoint paths. If G has at least $2k$ vertices and every set of exactly $2k$ vertices is linked in G, we say that G is *k-linked*.

This paper studies the linkedness of Cartesian products of complete graphs. Linkedness of Cartesian products has been studied in the past [4]. The *Cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph defined on the pairs (v_1, v_2) with $v_i \in G_i$ and with two pairs (u_1, u_2) and (v_1, v_2) being adjacent if, for some $\ell \in \{1, 2\}$, $u_\ell v_\ell \in E(G_\ell)$ and $u_i = v_i$ for $i \neq \ell$. We prove that the Cartesian product $K^{d_1+1} \times K^{d_2+1}$ of complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1 + d_2)/2 \rfloor$-linked for $d_1, d_2 \geq 0$, and that there are products that are not $\lfloor (d_1 + d_2 + 1)/2 \rfloor$-linked; hence this result is best possible. Here K^t denotes the complete graph on t vertices.

Our result is connected to questions on the linkedness of a polytope. A (convex) polytope is the convex hull of a finite set X of points in $\mathbb{R}^d$; the *convex hull* of X is the smallest convex set containing X. The *dimension* of a polytope in $\mathbb{R}^d$ is one less than the maximum number of affinely independent points in the polytope; a set of points $\vec{p}_1, \dots, \vec{p}_k$ in $\mathbb{R}^d$ is *affinely independent* if the $k-1$ vectors $\vec{p}_1 - \vec{p}_k, \dots, \vec{p}_{k-1} - \vec{p}_k$ are linearly independent. A polytope of dimension d is referred to as a *d-polytope*.

The *Cartesian product* $P \times P'$ of a d-polytope $P \subset \mathbb{R}^d$ and a d'-polytope $P' \subset \mathbb{R}^{d'}$ is the Cartesian product of the sets P and P':

$$P \times P' = \left\{ \begin{pmatrix} p \\ p' \end{pmatrix} \in \mathbb{R}^{d+d'} \,\middle|\, p \in P,\, p' \in P \right\}.$$

The resulting polytope is $(d + d')$-dimensional. The *graph* $G(P)$ of a polytope P is the undirected graph formed by the vertices and edges of the polytope. It follows that the graph $G(P \times P')$ of the Cartesian product $P \times P'$ is the Cartesian product $G(P) \times G(P')$ of the graphs $G(P)$ and $G(P')$.

A d-simplex $T(d)$ is the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^d$. The graph of $T(d)$ is the complete graph K^{d+1}. As a consequence, our result implies that the graph of the Cartesian product $T(d_1) \times T(d_2)$ is $\lfloor (d_1 + d_2)/2 \rfloor$-linked for $d_1, d_2 \geq 0$. Henceforth, if the graph of a polytope is k-linked we say that the polytope is also *k-linked*.

The first edition of the Handbook of Discrete and Computational Geometry [3, Problem 17.2.6] posed the question of whether or not every d-polytope is $\lfloor d/2 \rfloor$-linked. This question was answered in the negative by [2]. None of the known counterexamples are *simple d-polytopes*, d-polytopes in which every vertex is incident to exactly d edges. Hence, it may be hypothesised that the conjecture holds for such polytopes.

Conjecture 1.1. *Every simple d-polytope is $\lfloor d/2 \rfloor$-linked for $d \geq 2$.*

Cartesian products of simplices are simple polytopes, and so our result supports this revised conjecture. Furthermore, Cartesian products of simplices and duals of cyclic polytopes are related; the dual of a cyclic d-polytope with $d+2$ vertices is the Cartesian product of a $\lfloor d/2 \rfloor$-simplex and a $\lceil d/2 \rceil$-simplex [6, Example 0.6]. Hence we obtain that the dual of a cyclic d-polytope on $d+2$ vertices is also $\lfloor d/2 \rfloor$-linked for $d \geq 2$.

Unless otherwise stated, the graph theoretical notation and terminology follows from [1] and the polytope theoretical notation and terminology from [6]. Moreover, when referring to graph-theoretical properties of a polytope such as linkedness and connectivity, we mean properties of its graph.

2 Linkedness of Cartesian products of complex graphs

The contribution of this section is a sharp theorem (Theorem 2.1) that tells the story of the linkedness of Cartesian product of two complete graphs.

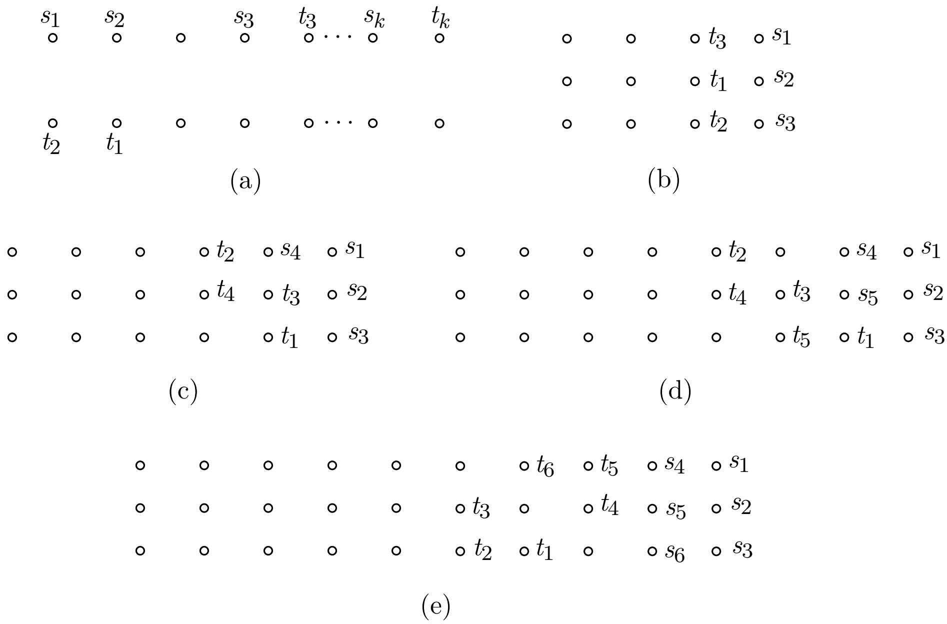

Figure 1: No feasible linkage problems for $K^{d_1+1} \times K^{d_2+1}$, $k = \lfloor (d_1 + d_2 + 1)/2 \rfloor$, $d_1 \leq 2$ and $d_2 > d_1$. (a) The case $d_1 = 1$ and even d_2 with $d_2 > d_1$. (b) The case $d_1 = 2$ and $d_2 = 3$. (c) The case $d_1 = 2$ and $d_2 = 5$. (d) The case $d_1 = 2$ and $d_2 = 7$. (e) The case $d_1 = 2$ and $d_2 = 9$. Each row of each part (a)-(e) is a complete graph whose edges have not been drawn.

Theorem 2.1. *The Cartesian product of two complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1 + d_2)/2 \rfloor$-linked for every $d_1, d_2 \geq 0$.*

Remark 2.2. Theorem 2.1 is best possible. There are products $K^{d_1+1} \times K^{d_2+1}$ that are not $\lfloor (d_1 + d_2 + 1)/2 \rfloor$-linked:

1. $K^2 \times K^{d_2+1}$ for even $d_2 \geq 1$, and

2. $K^3 \times K^{d_2+1}$ for $d_2 = 1, 3, 5, 7, 9$.

For each of these cases, Figure 1 provides a pairing of terminals that cannot be $\lfloor (d_1 + d_2 + 1)/2 \rfloor$-linked. We conjecture these are the only such cases.

An immediate corollary of Theorem 2.1 is the following.

Corollary 2.3. *The Cartesian product of two simplices $T(d_1)$ and $T(d_2)$ is $\lfloor (d_1+d_2)/2 \rfloor$-linked for every $d_1, d_2 \geq 0$.*

The notions of linkage, linkage problem, and valid path will simplify our arguments. A *linkage* in a graph is a subgraph in which every component is a path. Let X be a set of vertices in a graph and let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a pairing of all the vertices of X. A Y*-linkage* $\{L_1, \ldots, L_k\}$ is a set of disjoint paths with the path L_i joining the pair $\{s_i, t_i\}$ for $i = 1, \ldots, k$. We may also say that Y represents our *linkage problem*, and if Y is linked in G then our linkage problem is *feasible* and *infeasible* otherwise. A path in the graph is called X*-valid* if no inner vertex of the path is in X. Let X be a set of vertices in a graph G. Denote by $G[X]$ the subgraph of G induced by X, the subgraph of G that contains all the edges of G with vertices in X. Write $G - X$ for $G[V(G) \setminus X]$.

Consider a linkage problem $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ on a set X of $2k$ vertices in a graph G. Consider a linkage $\mathcal{L}$ from a subset Z of X to some set Z' disjoint from X and label the vertices of Z' such that the path in $\mathcal{L}$ with end $z_i \in Z$ has its other end $z_i' \in Z'$. Then the linkage $\mathcal{L}$ in G *induces* a linkage problem Y' in $(G - V(\mathcal{L})) \cup Z'$ where the vertices of $X \setminus Z$ remain and the vertices of Z have been replaced by the vertices of Z'. Slightly abusing terminology, we also call terminals the vertices of Z'. If the problem Y' is feasible in $(G - V(\mathcal{L})) \cup Z'$, so is the problem Y in G.

Since we make heavy use of Menger's theorem [1, Theorem. 3.3.1], we next remind the reader of one of its consequences.

Theorem 2.4 (Menger's theorem)**.** *Let G be a k-connected graph, and let A and B be two subsets of its vertices, each of cardinality at least k. Then there are k disjoint $A - B$ paths in G.*

We fix some notation and terminology for the remaining of the section. Let G denote the graph $K^{d_1+1} \times K^{d_2+1}$. We think of $G = K^{d_1+1} \times K^{d_2+1}$ as a grid with $d_1 + 1$ rows and $d_2 + 1$ columns. In this way, the entry in Row i and Column j can be referred to as $G[i, j]$.

When we write about a row r of subgraph G' of G, we think of r as a subgraph of G' and as the number r so that we can write about the rth row of G' or G; this ambiguity should cause no confusion. An entry in the grid $K^{d_1+1} \times K^{d_2+1}$ with no terminal is said to be *free*, as is a row or a column of a subgraph of G with no terminal. A row or a column of a subgraph of G with every entry being occupied by a terminal is said to be *full*.

We need the following induced subgraphs of G:

$C_{ab\ldots z}$, the subgraph formed by the union of Columns $a, b, \ldots, z$;

$\bar{C}_{ab\ldots z}$, the subgraph obtained by removing Columns $a, b, \ldots, z$;

$R_{ab\ldots z}$, the subgraph formed by the union of Rows $a, b, \ldots, z$;

$\bar{R}_{ab\ldots z}$, the subgraph obtained by removing Rows $a, b, \ldots, z$;

A_α, the induced subgraph of $\bar{C}_{12}$ obtained by removing its first α rows; and

B_α, the subgraph of C_{12} obtained by removing its first α rows.

For instance, $\bar{C}_1$ denotes the subgraph of G obtained by removing the first column, C_{12} the subgraph formed by the first two columns of G, and $\bar{C}_{12}$ denotes the subgraph obtained by removing the first two columns of G; observe $\bar{C}_{12}$ is isomorphic to $K^{d_1+1} \times K^{d_2-1}$. Figure 2 depicts some of the aforementioned subgraphs of $K^{d_1+1} \times K^{d_2+1}$.

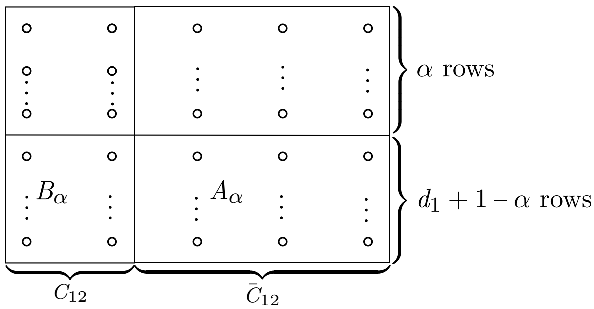

Figure 2: Depiction of the subgraphs B_α, A_α, C_{12}, and $\bar{C}_{12}$ of $K^{d_1+1} \times K^{d_2+1}$.

The connectivity of $K^{d_1+1} \times K^{d_2+1}$ is stated below.

Lemma 2.5 (Špacapan [5, Theorem 1]). *The (vertex)connectivity of $K^{d_1+1} \times K^{d_2+1}$ is precisely $d_1 + d_2$.*

We continue fixing further notation. Henceforth let $k := \lfloor (d_1 + d_2)/2 \rfloor$. And let X be a subset of $2k$ vertices of G and let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be a pairing of all the vertices in X.

We first settle the simple cases of $(0, d_2)$ and $(1, d_2)$ for $d_2 \geq 0$.

Proposition 2.6 (Base cases). *For $d_2 \geq 0$ the Cartesian products $K^1 \times K^{d_2+1}$ and $K^2 \times K^{d_2+1}$ are both $\lfloor (1 + d_2)/2 \rfloor$-linked. This statement is best possible.*

Proof. The lemma is true for the pair $(0, d_2)$ for each $d_2 \geq 0$, since $K^1 \times K^{d_2+1} = K^{d_2+1}$ and K^{d_2+1} is $\lfloor (1 + d_2)/2 \rfloor$-linked. This is best possible.

The graph $K^2 \times K^{d_2+1}$ is $(1 + d_2)$-connected by Lemma 2.5. Use Menger's theorem (Theorem 2.4) to bring the $1 + d_2$ terminals to the subgraph $\bar{R}_1$ through a linkage $\{S_1, \dots, S_k, T_1, \dots, T_k\}$ with $S_i := s_i - \bar{R}_1$ and $T_i := t_i - \bar{R}_1$ for $i \in [1, k]$. Letting $\{\bar{s}_i\} := V(S_i) \cap V(\bar{R}_1)$ and $\{\bar{t}_i\} := V(T_i) \cap V(\bar{R}_1)$, we produce a new linkage problem $Y' := \{\{\bar{s}_1, \bar{t}_1\}, \dots, \{\bar{s}_k, \bar{t}_k\}\}$ in $\bar{R}_1$ whose feasibility implies that of Y in G. To solve Y' link the pairs of Y' in the subgraph $\bar{R}_1$, which is isomorphic to K^{d_2+1}, using the $\lfloor (1 + d_2)/2 \rfloor$-linkedness of K^{d_2+1}. For even even d_2, Figure 1(a) shows an infeasible linkage problem with $\lfloor (2 + d_2)/2 \rfloor$ pairs in the graph $K^2 \times K^{d_2+1}$. $\square$

In what follows we aim to find a Y-linkage $\{L_1, \dots, L_k\}$ in G with L_i joining the pair $\{s_i, t_i\}$ of Y for $i \in [1, k]$. Our proof is by induction on (d_1, d_2) with the base cases settled in Proposition 2.6. If there is a pair of Y, say $\{s_1, t_1\}$, lying in some column or row of G, say in Column 1, we send every terminal $s_i \in C_1$ that is different from s_1 and t_1 and that is not adjacent to t_i to the subgraph $\bar{C}_1$, and apply the induction hypothesis on $\bar{C}_1$. Otherwise, we may assume every pair of Y lies in two distinct columns or rows, say the pair $\{s_1, t_1\}$ lies in C_{12}; then we send every terminal $s_i \in C_{12}$ that is different from s_1 and

t_1 and that is not adjacent to t_i to the subgraph $\bar{C}_{12}$, and apply the induction hypothesis to $\bar{C}_{12}$. We develop these ideas below.

The definition of k-linkedness gives the following lemma at once; we will use it implicitly hereafter.

Lemma 2.7. *Let $\ell \leq k$. Let X be a set of 2ℓ distinct vertices of a k-linked graph K, let Y be a labelling and pairing of the vertices in X, and let Z be a set of $2k - 2\ell$ vertices in K such that $X \cap Z = \emptyset$. Then there exists a Y-linkage in K that avoids every vertex in Z.*

Besides, basic algebraic manipulation yields the following inequality.

Lemma 2.8. *If $x \geq 2$ and $y \geq 2$ then $x(y-1) > x + y - 3$.*

Proof. The inequality simplifies to $(x-1)(y-2) > -1$. □

We are now ready to put together all the elements of the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $k := \lfloor (d_1 + d_2)/2 \rfloor$. Then $d_1 + d_2 \geq 2k$.

Proposition 2.6 gives the result for the pairs $(d_1, 0)$, $(0, d_2)$, $(d_1, 1)$, and $(1, d_2)$ for each $d_1, d_2 \geq 0$. Hence, our bidimensional induction on (d_1, d_2) can start with the assumption of $d_1, d_2 \geq 2$.

We first deal with the case where a pair in Y, say $\{s_1, t_1\}$, lies in some column or some row of G, say in Column 1.

Case 1. A pair in Y, say $\{s_1, t_1\}$, lies in Column 1.

The induction hypothesis ensures that the subgraph $\bar{C}_1$ is $(k-1)$-linked. Hence it suffices to show that all the terminals in C_1 other than s_1, t_1 can be moved to $\bar{C}_1$ via a linkage; Menger's theorem (Theorem 2.4) guarantees this.

Let U be the set of terminals in C_1 other than s_1 and t_1, and let W be the set of terminals in $\bar{C}_1$. Then $|U|+|W| \leq d_1+d_2-2$, as $|U|+|W| = 2k-2$ and $2k \leq d_1+d_2$. Besides, the subgraph $G - (W \cup \{s_1, t_1\})$ is $|U|$-connected, as G is $(d_1 + d_2)$-connected (Lemma 2.5). In the case of $d_1, d_2 \geq 2$, Lemma 2.8 yields that $\bar{C}_1$ has more than $|U \cup W|$ vertices:

$$|\bar{C}_1| = (d_1 + 1)d_2 > d_1 + 1 + d_2 + 1 - 3 > d_1 + d_2 - 2 = |U| + |W|.$$

Use Menger's theorem (Theorem 2.4) to bring the $|U|$ terminals in C_1 to the subgraph $\bar{C}_1$ through a linkage Y_U. For every path L in Y_U, if $s_i \in L$, let $\{\bar{s}_i\} := V(L) \cap V(\bar{C}_1)$ and if $t_i \in L$ let $\{\bar{t}_i\} := V(L) \cap V(\bar{C}_1)$. For $s_i \in W$ (respectively $t_i \in W$) let $\bar{s}_i = s_i$ (respectively $\bar{t}_i = t_i$). This produces a new linkage problem $Y' := \{\{\bar{s}_2, \bar{t}_2\}, \ldots, \{\bar{s}_k, \bar{t}_k\}\}$ in $\bar{C}_1$ whose feasibility implies that of Y in G, since s_1 and t_1 are adjacent in C_1. The $(k-1)$-linkedness of $\bar{C}_1$ now settles the case.

By symmetry, we can assume that every pair $\{s_i, t_i\}$ in Y lies in two different columns or rows and that s_i, t_i are not adjacent. Without loss of generality, assume that

$$s_1 \text{ is in Column 1 and } t_1 \text{ is in Column 2 of } C_{12}. \tag{$*$}$$

The induction hypothesis also ensures that both $\bar{C}_{12}$ and $\bar{R}_{12}$ are $(k-1)$-linked. We consider two further cases based on the number of terminals in C_{12} or R_{12}.

Case 2. The subgraph C_{12} contains precisely $d_1 + 2 - \alpha$ terminals, including $\{s_1, t_1\}$, where $0 \leq \alpha \leq d_1$.

Excluding $\{s_1, t_1\}$, there are at most d_1 terminals in C_{12}, and there are d_1+1 internally-disjoint $s_1 - t_1$ paths in C_{12} of length at most three: two length-two paths and $d_1 - 1$ length-three paths. One of these $s_1 - t_1$ paths, say L_1, avoids every other terminal in C_{12}.

Without loss of generality, assume that Row 1 in C_{12} is part of the path L_1; that is,

$$\{G[1,1], G[1,2]\} \subseteq V(L_1). \tag{$**$}$$

In the subcase $\alpha = d_1$, every pair in $Y \setminus \{s_1, t_1\}$ is in $\bar{C}_{12}$, and the induction hypothesis on $\bar{C}_{12}$ settles the subcase.

Suppose that $\alpha = d_1 - 1$, say C_{12} contains $\{s_1, t_1, s_2\}$. Then $s_2 \in B_1$ and $t_2 \in \bar{C}_{12}$. We may assume s_1, s_2 are in Column 1 and t_1 is in Column 2. We show there is an X-valid $s_2 - A_1$ path L_2' such that the vertex $\bar{s_2} \in V(L_2') \cap V(A_1)$ is either t_2 or a nonterminal.

Through each entry of Column 1 of B_1, there are $d_2 - 1$ paths form s_2 to A_1 of length at most two (one for each column in A_1). Moreover, there are at least $d_1 - 1$ free entries in Column 1 of B_1. Therefore, to ensure the existence of L_2', we need to show that at least one of these $(d_1 - 1)(d_2 - 1)$ paths from s_2 to A_1 either contains t_2 or a nonterminal in A_1. Indeed, according to Lemma 2.8, the inequality

$$(d_1 - 1)(d_2 - 1) > d_1 - 1 + d_2 - 3 \geq |X \setminus \{s_1, t_1, s_2, t_2\}|$$

holds for $d_1, d_2 \geq 2$. Hence we get the existence of L_2'. As a result, the solution of the new problem $Y' := \{\{\bar{s_2}, t_2\}, \{s_3, t_3\}, \ldots, \{s_k, t_k\}\}$ in $\bar{C}_{12}$ induces a solution of the problem Y in G. And the solution of Y' follows from the $(k-1)$-linkedness of $\bar{C}_{12}$.

Henceforth assume that $\alpha \leq d_1 - 2$. To finalise Case 2, we require a couple of claims.

Claim 2.9. *Suppose that there are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$. Then there is an injection from the set of rows of $B_{\alpha+1}$ that contain two terminals x_1, x_2 such that $\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ to the set of rows of $B_{\alpha+1}$ that contain no terminal other than possibly s_1 and t_1.*

Proof. This follows from a simple counting argument. The number of rows in $B_{\alpha+1}$ is $d_1 - \alpha$. Let m denote the number of rows of $B_{\alpha+1}$ that contain two terminals x_1, x_2 such that $\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ and let $n := |(X \cap V(B_{\alpha+1})) \setminus \{s_1, t_1\}|$; that is, n counts the total number of terminals in $B_{\alpha+1}$ other than s_1 and t_1. It follows that the number of rows of $B_{\alpha+1}$ that contain precisely one terminal $x \notin \{s_1, t_1\}$ is $n - 2m$; either s_1 or t_1 may be in these rows. As a result, the number of rows of $B_{\alpha+1}$ that contain no terminal other than $\{s_1, t_1\}$ is $d_1 - \alpha - m - (n - 2m)$. Combining $n \leq d_1 - \alpha$ with all these numbers, we get that

$$d_1 - \alpha - m - (n - 2m) = d_1 - \alpha - n + m \geq d_1 - \alpha - (d_1 - \alpha) + m = m.$$

The claim is proved. $\square$

Claim 2.10. *Suppose that there are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$. If every row in the subgraph $A_{\alpha+1} = K^{d_1-\alpha} \times K^{d_2-1}$ of $\bar{C}_{12}$ has a free entry, then, for every terminal $x \notin \{s_1, t_1\}$ in $B_{\alpha+1}$, there is an X-valid $x - A_{\alpha+1}$ path L to a free entry in $A_{\alpha+1}$; and all these X-valid paths are disjoint.*

Proof. If a row of $B_{\alpha+1}$ contains exactly one terminal $x \notin \{s_1, t_1\}$, then send x to a free entry in the same row of $A_{\alpha+1}$. Let x_1 and x_2 be two terminals in $B_{\alpha+1}$ that satisfy

$\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ and occupy a row r_f of $B_{\alpha+1}$. From Claim 2.9 ensues the existence of a row r_e of $B_{\alpha+1}$ that contain no terminal other than possibly s_1 and t_1; in short, there is at least a free entry in r_e.

Consider a pair (r_f, r_e) of rows granted by Claim 2.9. Send either x_1 or x_2, say x_1, to the free entry in the row r_e of $A_{\alpha+1}$ passing through the corresponding free entry in the row r_e of $B_{\alpha+1}$, and send x_2 to a free entry in the row r_f of $A_{\alpha+1}$. The proof of the claim is now complete. □

Now suppose that $\alpha = 0$ or $2 \le \alpha \le d_1 - 2$. In this subcase, the subgraph $\bar{C}_{12}$ contains at most α full rows: if $\alpha + 1$ rows were full in $\bar{C}_{12}$ then there would be at least $(\alpha+1)(d_2-1)$ terminals in $\bar{C}_{12}$ but $(\alpha+1)(d_2-1) > d_2 - 2 + \alpha$ (Lemma 2.8). Even when the path L_1 uses the first row of C_{12} by $(**)$, there is no loss of generality by assuming that the full rows of $\bar{C}_{12}$ are among the *first* $\alpha + 1$ *rows* of $\bar{C}_{12}$. It follows that every row of $A_{\alpha+1}$ has a free entry.

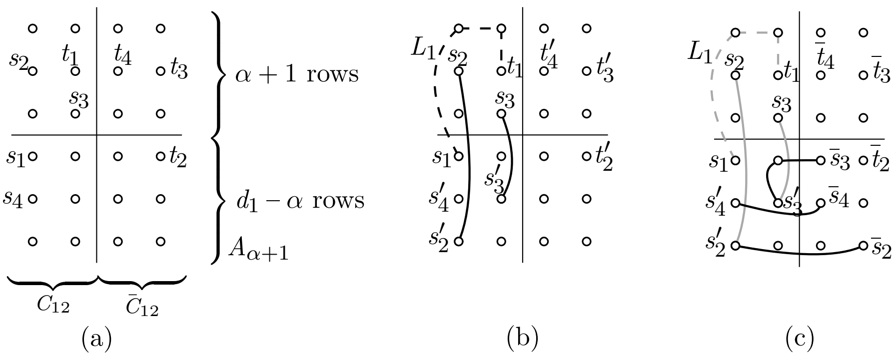

Figure 3: Auxiliary figure for Case 2 (a) This shows a scenario where $d_1 = 5$, $d_2 = 3$, and $\alpha = 2$. (b) The path $L_1 = s_1 - t_1$ in dashed line, the paths that send the terminals in $B_1 \setminus B_3$ other than s_1 and t_1 to B_3, and the resulting new linkage $Y' = \{\{s'_2, t'_2\}, \{s'_3, t'_3\}, \{s'_4, t'_4\}\}$ in $\bar{C}_{12} \cup B_{\alpha+1}$. (c) The paths that send the terminals in B_3 to A_3, and the resulting new linkage $Y'' = \{\{\bar{s}_2, \bar{t}_2\}, \{\bar{s}_3, \bar{t}_3\}, \{\bar{s}_4, \bar{t}_4\}\}$ in $\bar{C}_{12}$.

Next we show how to send to $B_{\alpha+1}$ the terminals other than s_1 and t_1 that are in the rows 2 to $\alpha + 1$ of C_{12}; that is, the terminals other than s_1 and t_1 that are in $B_1 \setminus B_{\alpha+1}$. For $\alpha = 0$, $B_1 \setminus B_{\alpha+1} = \emptyset$ and there is nothing to do. We now focus on the subcase $2 \le \alpha \le d_1 - 2$. Let n_1 and n_2 denote the number of terminals in $B_1 \setminus B_{\alpha+1}$ and $B_{\alpha+1}$, respectively. Then the following inequalities hold

$$n_1 + n_2 \le d_1 + 2 - \alpha \le d_1 \quad (\text{since } 2 \le \alpha),$$
$$n_1 + n_2 \le d_1 + 2 - \alpha \le 2d_1 - 2\alpha = |V(B_{\alpha+1})| \quad (\text{since } \alpha \le d_1 - 2).$$

From the second inequality, it follows that there are at least n_1 free vertices in $B_{\alpha+1}$. Since B_1 is d_1-connected by Lemma 2.5, Menger's theorem gives n_1 disjoint paths in B_1 from the terminals in $B_1 \setminus B_{\alpha+1}$ to n_1 free entries in $B_{\alpha+1}$, avoiding the n_2 terminals in $B_{\alpha+1}$. For a terminal s_i in $B_1 \setminus B_{\alpha+1}$, let L'_i be the path from s_i to $B_{\alpha+1}$ and let $s'_i := V(L'_i) \cap B_{\alpha+1}$. Define t'_i similarly for a terminal t_i in $B_1 \setminus B_{\alpha+1}$. Furthermore, for s_i (respectively, t_i) in

$B_{\alpha+1} \cup \bar{C}_{12}$, let $s_i' := s_i$ (respectively, $t_i' := t_i$). This produces a new linkage problem $Y' := \{\{s_2', t_2'\}, \ldots, \{s_k', t_k'\}\}$ in $\bar{C}_{12} \cup B_{\alpha+1}$. See Figure 3(b).

There are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$, and every row in $A_{\alpha+1} = K^{d_1-\alpha} \times K^{d_2-1}$ has a free entry. Hence, Claim 2.10 applies, and there is a linkage formed by X-valid paths from the terminals in $B_{\alpha+1}$, other than s_1 and t_1, to free entries in $A_{\alpha+1}$. For every such path L_i'', if $s_i' \in V(L_i'') \cap V(B_{\alpha+1})$, let $\{\bar{s}_i\} := V(L_i'') \cap V(A_{\alpha+1})$, and if $t_i' \in V(L_i'') \cap V(B_{\alpha+1})$, let $\{\bar{t}_i\} := V(L_i'') \cap V(A_{\alpha+1})$. Besides, for $s_i' \in \bar{C}_{12}$ (respectively $t_i' \in \bar{C}_{12}$), let $\bar{s}_i = s_i'$ (respectively, $\bar{t}_i = t_i'$). This produces a new linkage problem $Y'' := \{\{\bar{s}_2, \bar{t}_2\}, \ldots, \{\bar{s}_k, \bar{t}_k\}\}$ in $\bar{C}_{12}$ whose feasibility implies that of Y', and therefore that of Y in G, by completing each linkage problem with the path L_1. See Figure 3(c).

Now we have a new linkage problem Y'' in $\bar{C}_{12}$ with $(k-1)$ pairs. The solution of Y'' in $\bar{C}_{12}$ implies a solution of the linkage problem Y in G. To link the pairs of Y'' use the $(k-1)$-linkedness of $\bar{C}_{12}$.

Finally assume that $\alpha = 1$. Then there are exactly $d_1 + 1$ terminals in C_{12} and at most $d_2 - 1$ terminals in $\bar{C}_{12}$. In a first scenario suppose that either both entries in $B_1 \setminus B_2$ are nonterminals or each terminal other than s_1 and t_1 in $B_1 \setminus B_2$ is adjacent to a nonterminal in B_2. Then we can send these terminals in $B_1 \setminus B_2$ to B_2. In the second scenario, suppose that there is a terminal s_i ($i \neq 1$) in $B_1 \setminus B_2$ whose neighbours in B_2 are all terminals. Then the column of s_i in B_1 would contain exactly d_1 terminals, including s_i. We send s_i to a free entry in A_1, in the same row as s_i (the first row of A_1): if this free entry didn't exist, then s_i would be adjacent to the $d_2 - 1$ terminals in A_1 and the $d_1 - 1$ terminals in B_2. Since there are $d_1 + d_2$ terminals in total, it would follow that s_i is adjacent to t_i. This contradiction shows that we can send s_i to a free entry in A_1.

In both scenarios, it remains to send the terminals other than s_1 and t_1 in $B_2 = K^{d_1-1} \times K^2$ to $A_2 = K^{d_1-1} \times K^{d_2-1}$. To do so, we reason as in the subcase $2 \leq \alpha \leq d_1 - 2$. It follows that there are at most $d_1 + 2 - 1$ terminals in B_2, and that every row in A_2 has a free entry. Claim 2.10 applies again and gives a linkage formed by X-valid paths from the terminals in B_2, other than s_1, t_1, to free entries in A_2.

With all the terminals other than s_1 and t_1 in $\bar{C}_{12}$, therein we have a new linkage problem Y' with $k-1$ pairs whose solution in $\bar{C}_{12}$ implies a solution of the linkage problem Y in G. To solve Y' in $\bar{C}_{12}$ use the $(k-1)$-linkedness of $\bar{C}_{12}$.

By symmetry, we also have the result if there are at most $d_2 + 2$ terminals in R_{12}, including $\{s_1, t_1\}$.

Case 1. The subgraph C_{12} contains at least $d_1 + 3$ terminals, including $\{s_1, t_1\}$.

This case reduces to the previous case. If C_{12} contains at least $d_1 + 3$ terminals then R_{12} contains at most $d_2 - 3 + 4 = d_2 + 1$ terminals, since there are four entries shared by C_{12} and R_{12}. Because we make no distinction between columns and rows, this case is already covered. This completes the proof of the theorem. □

3 Duals of cyclic polytopes

There is a close connection between duals of cyclic d-polytopes with $d + 2$ vertices and Cartesian products of complete graphs.

The *moment curve* in $\mathbb{R}^d$ is defined by $x(t) := (t, t^2, \ldots, t^d)$ for $t \in \mathbb{R}$, and the convex hull of any $n > d$ points on it gives a *cyclic polytope* $C(n, d)$. The *combinatorics* of a cyclic polytope, the face lattice of the polytope faces partially ordered by inclusion, is independent of the points chosen on the moment curve. Hence we talk of the cyclic d-polytope on n vertices [6, Example 0.6].

For a polytope P that contains the origin in its interior, the *dual polytope* P^* is defined as

$$P^* = \{y \in \mathbb{R}^d \mid x \cdot y \leq 1 \text{ for all } x \text{ in } P\}.$$

If P does not contain the origin, we translate the polytope so that it does. Translating the polytope P changes the geometry of P^* but not its face lattice. The face lattice of P^* is the inclusion reversed face lattice of P. In particular, the vertices of P^* correspond to the facets of P, and the edges of P^* correspond to the $(d-2)$-faces of P. The *dual graph* of a polytope P is the graph of the dual polytope, or equivalently, the graph on the set of facets of P where two facets are adjacent in the dual graph if they share a $(d-2)$-face.

Duals of cyclic d-polytopes are simple d-polytopes. It is also the case that the dual of a cyclic d-polytope with $d+2$ vertices can be expressed as $T(\lfloor d/2 \rfloor) \times T(\lceil d/2 \rceil)$ ([6, Example 0.6]). From this observation and Theorem 2.1 the next corollary follows at once.

Corollary 3.1. *Duals of cyclic polytopes with $d+2$ vertices are $\lfloor d/2 \rfloor$-linked for every $d \geq 2$.*

ORCID iDs

Leif K. Jørgensen https://orcid.org/0000-0003-4922-3937
Guillermo Pineda-Villavicencio https://orcid.org/0000-0002-2904-6657
Julien Ugon https://orcid.org/0000-0001-5290-8051

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Characterization of a family of rotationally symmetric spherical quadrangulations

Lowell Abrams
The George Washington University, Washington DC 20052, USA

Daniel Slilaty *
Wright State University, Dayton OH 45435, USA

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Abstract

A *spherical quadrangulation* is an embedding of a graph G in the sphere in which each facial boundary walk has length four. Vertices that are not of degree four in G are called *curvature vertices*. In this paper we classify all spherical quadrangulations with n-fold rotational symmetry ($n \geq 3$) that have minimum degree 3 and the least possible number of curvature vertices, and describe all such spherical quadrangulations in terms of nets of quadrilaterals. The description reveals that such rotationally symmetric quadrangulations necessarily also have a pole-exchanging symmetry.

Keywords: Quadrangulation, spherical quadrangulation, rotational symmetry.

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1 Introduction

If S is a closed surface, a graph G embedded in S in which all facial boundary walks have length four is called a *quadrangulation* of S. When S is the sphere, the graph G is necessarily bipartite. Considering quadrilateral faces to be geometrically flat squares, vertices of degree 4 extend this flatness to neighboring faces. Thus, when "most" of the vertices of a spherical quadrangulation are of degree four, large areas will appear as a portion of the geometrically-flat, infinite $\{4, 4\}$-planar lattice. The curvature is therefore localized at vertices of degree other than 4.

In spherical triangulations where each face is considered to be a flat equilateral triangle, vertices of degree 6 play a similar role in extending flatness, and curvature is thus localized

*Corresponding author.
E-mail addresses: labrams@gwu.edu (Lowell Abrams), daniel.slilaty@wright.edu (Daniel Slilaty)

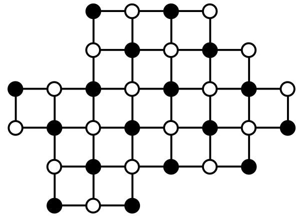

Figure 1: A flat region with all internal vertices of degree 4.

at vertices of degree other than 6. Such spherical triangulations link to several well-studied structures; for example, those for which all of the curvature is localized at 12 vertices of degree 5 were popularized by the *geodesic domes* of Buckminster Fuller. Moreover, these triangulations were later noted to be the topological dual graphs of what are now called *fullerene* graphs. ([4] is a good starting source for fullerenes and other chemical graphs.)

As such, in a quadrangulation of S, a vertex of degree other than 4 is called a *curvature vertex*. In this paper we investigate spherical quadrangulations with n-fold rotational symmetry ($n \geq 3$) that have minimum degree 3 and the least possible number of curvature vertices, which is 8 when $n = 4$ and $2n + 2$ otherwise. (See Proposition 2.1 and Corollary 2.2.) The fact that curvature is localized at a relatively small number of vertices suggests that G may have a description in terms of a geometric net of polygons. For example, in Figure 2 we have a net of six congruent quadrilaterals which closes up to yield a spherical quadrangulation with 3-fold rotational symmetry about poles p and q. Note that there are $2n + 2 = 8$ curvature vertices of degree 3 each.

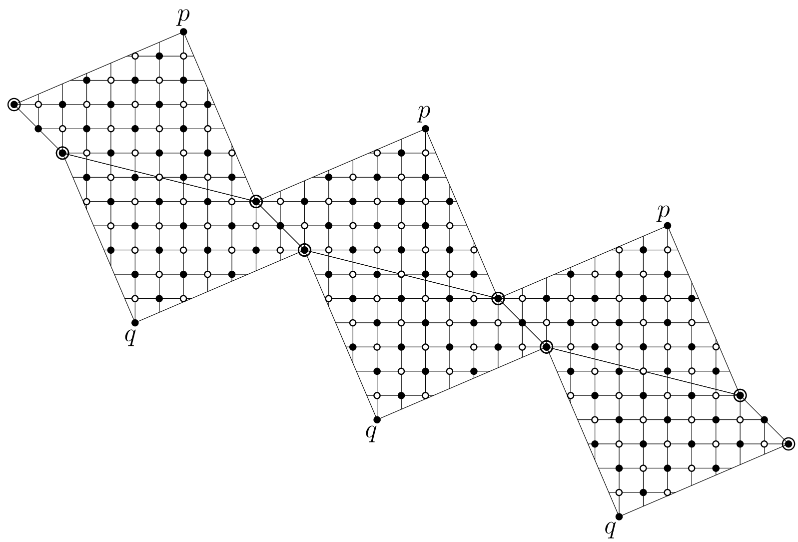

Figure 2: A quadrilateral net which describes a spherical quadrangulation with 3-fold rotational symmetry.

Our main result (Theorem 1.1) is that every such spherical quadrangulation has a similar description as a very simple net of $2n$ congruent quadrilaterals. The Three- and Four-Parameter Constructions mentioned in Theorem 1.1 are stated in full detail in Section 3. We also identify which of these nets produce quadrangulations that are overlay graphs of self-dual planar embeddings.

Theorem 1.1. *Let G be a spherical quadrangulation with minimum degree 3, having n-fold rotational symmetry ($n \geq 3$), and having the least possible number of curvature vertices. If*

(1) *all white vertices have degree 4 and*

(2) *the poles of the rotational symmetry are at two black vertices,*

then G can be obtained from either the Three- or Four-Parameter Construction of Section 3.

Remark 1.2. The reader may note that not all spherical quadrangulations with n-fold rotational symmetry and the least possible number of curvature vertices satisfy Conditions (1) and (2) of Theorem 1.1. Nevertheless, any such spherical quadrangulation can still be obtained using Theorem 1.1 by making use of Proposition 1.3. In short, even if G does not satisfy both (1) and (2), it is necessarily the case that G overlayed with its topological dual graph will be a spherical quadrangulation which does satisfy both (1) and (2). This will be described in more detail in Section 1.1 in the paragraph after Proposition 1.3.

The reader may note that the quadrangulation of Figure 2, and indeed all of the quadrangulations constructed in Section 3, not only possesses n-fold rotational symmetry at poles p and q but also has a 2-fold symmetry which exchanges p and q. This is interesting in that this 2-fold symmetry is not *a priori* implied by our hypotheses. Hence our spherical quadrangulations possess, at the very least, an order-$2n$ symmetry group. Any such quadrangulations possessing additional symmetries will, of course, also be included in our constructions.

1.1 Overlay graphs and other background on quadrangulations

An important fact about quadrangulations is that a spherical quadrangulation is always bipartite while quadrangulations of other surfaces need not be. Given any graph H that is cellularly embedded in a closed surface S, two bipartite quadrangulations that are naturally associated with the embedding of H and its topological dual graph H^* are the *overlay graph* and the *radial graph*. For the sphere, in fact, any quadrangulation G is a radial graph for some embedding H and its dual H^*. Applications of radial graphs, overlay graphs, and the closely associated *medial graph* can be found in [1, 2, 3, 5, 7, 8, 9].

Consider a graph H cellularly embedded in a closed surface S and also consider its topological dual graph H^*. Say that all of H (both vertices and edges) is colored "red" and all of H^* is colored "blue". Embed H and H^* simultaneously in S and at each edge/dual-edge crossing point create a new vertex of degree four (which now has alternating red and blue edges in rotation around it) and say that this new vertex is "white". The graph obtained is called the *overlay graph* $\mathcal{O}(H, H^*)$. Note that H is self dual if and only if $\mathcal{O}(H, H^*)$ has a cellular automorphism which leaves white invariant and switches red and blue colors.

The overlay graph $\mathcal{O}(H, H^*)$ was used by Servatius and Servatius [8] to classify self-dual embeddings in the sphere along with the pairing of their groups of color-preserving cellular automorphisms of $\mathcal{O}(H, H^*)$ as index-2 subgroups of the groups of the white-preserving cellular automorphisms. Graver and Hartung [5] do the same in the special case of self-dual embeddings of graphs having four trivalent vertices and the remaining vertices all of degree four. Their results, however, are much more explicit than those in [8].

The overlay graph $\mathcal{O}(H, H^*)$ is a bipartite quadrangulation of a closed surface S with partite sets $\mathsf{Red} \cup \mathsf{Blue}$ and White. The *radial graph* of H and H^* (denote it by $\mathcal{R}(H, H^*)$) can be constructed from $\mathcal{O}(H, H^*)$ by placing a diagonal edge in each face of $\mathcal{O}(H, H^*)$ which connects the red and blue vertices on that face and then erasing all of the edges and white vertices of $\mathcal{O}(H, H^*)$. Thus $\mathcal{R}(H, H^*)$ is also a bipartite quadrangulation of S. Conversely, if G is a bipartite quadrangulation of S with bipartition $\mathsf{Red} \cup \mathsf{Blue}$, then $G = \mathcal{R}(H, H^*)$ for some H and H^* as follows. In each face of G, place a red edge connecting the two red vertices and a blue edge connecting the two blue vertices. The resulting red and blue graphs are H and H^*.

Unlike the radial graph, even if G is a bipartite quadrangulation of S in which all white vertices have degree four it is not necessarily true that G is of the form $\mathcal{O}(H, H^*)$ for some H. An additional condition that does ensure that G has the form $\mathcal{O}(H, H^*)$ is given in Proposition 1.3. In Proposition 1.3, $\mathcal{D}(G)$ is the graph obtained from quadrangulation G by placing a diagonal edge connecting the black corners of each face and then deleting the white vertices of G. As mentioned in the previous paragraph, $\mathcal{D}(G)$ is the radial graph for some K and K^* in S when $\mathcal{D}(G)$ is bipartite.

Proposition 1.3. *If G is a quadrangulation of closed surface S, then $G = \mathcal{O}(H, H^*)$ for some H if and only if every white vertex of G has degree 4 and both G and $\mathcal{D}(G)$ are bipartite. In the particular case that S is the sphere, $G = \mathcal{O}(H, H^*)$ for some H if and only if every white vertex of G has degree 4.*

Proof. First, let S be any closed surface. The one direction of the equivalence statement is trivial. For the other direction, the fact that $\mathcal{D}(G)$ is bipartite allows us to properly 2-color (red and blue) the vertices of $\mathcal{D}(G)$, which shows G is of the form $\mathcal{O}(H, H^*)$, as required.

The statement for the sphere follows from the first statement and the fact that any spherical quadrangulation is automatically bipartite. □

Now say H is a spherical quadrangulation with n-fold rotational symmetry and the minimum number of curvature vertices, but does not satisfy the other two conditions in Theorem 1.1. In this case $\mathcal{O}(H, H^*)$ inherits the n-fold rotational symmetry of H and does satisfy the two additional conditions.

Quadrangulations have also been studied for other surfaces. Thomassen [10] and also Márquez, de Mier, Noy, Revuelta [6] give explicit constructions for all 4-regular quadrangulations of the torus and Klein bottle. If G is a quadrangulation of S having no curvature vertices, then in fact S must be the torus or Klein bottle. (See Proposition 2.1.)

2 Basic properties of spherical quadrangulations

Proposition 2.1 gives an arithmetic constraint on the quantities and degrees of curvature vertices in quadrangulations, and Corollary 2.2 is an immediate consequence. We use $\chi(S)$ to denote the Euler Characteristic of the surface S.

Proposition 2.1. *If G is a quadrangulation of closed surface S with minimum degree 3 and v_i vertices of degree i then,*

$$v_3 = 4\chi(S) + \sum_{i \geq 5} (i-4)v_i.$$

Furthermore, if $\chi(S) \neq 0$, then there are curvature vertices.

Proof. If G has f faces and e edges, then $\sum_i iv_i = 2e$. Also, $4f = 2e$ and $\left(\sum_i v_i\right) - e + f = \chi(S)$ which when combined together yield $4(\sum_i v_i) = 4\chi(S) + 2e$. Now subtracting we obtain $\sum_i (4-i)v_i = 4\chi(S)$ which yields our desired results. $\square$

Corollary 2.2. *If G is a spherical quadrangulation with n-fold rotational symmetry ($n \geq 3$), minimum degree 3, and having the least possible number of curvature vertices, then*

- *if $n = 3$, then G has eight vertices of degree 3 of which two are poles of the rotational symmetry;*
- *if $n = 4$, then G has 8 vertices of degree 3 and the two poles of the rotational symmetry are either both at vertices of degree 4 or both at the centers of faces; and*
- *if $n > 4$, then G has two vertices of degree n and $2n$ vertices of degree 3 where the two vertices of degree n are the two poles of the rotational symmetry.*

In [5], Graver and Hartung give a complete construction of spherical quadrangulations of the form $G = \mathcal{O}(H, H^*)$ where $H \cong H^*$ is a planar graph with four vertices of degree 3 and all other vertices of degree 4. (They do not assume any rotational symmetry). Here, for $n = 3$, we assume only that G is a spherical quadrangulation with 3-fold rotational symmetry and discover structures not found in [5].

Given a vertex v in a graph G and a subgraph H of G, the difference $d_G(v) - d_H(v)$ (that is, the degree of v in G minus the degree of v in H) is called the *codegree* of v with respect to H and G. We say that v is *saturated* by a subgraph H when v has codegree zero with respect to H and G.

Consider a spherical quadrangulation G with topological dual graph G^*. A collection X of faces of G corresponds to a collection of vertices X^* of G^*. If the induced subgraph of G^* on vertex set X^* is connected, then the union of the faces in X along their incident edges and vertices is called a *face-connected subsurface*. Let F be the face-connected subsurface corresponding to X. The *boundary* of F, call it ∂F, is the collection of edges (and their endpoints) incident to exactly one face in X. Topologically speaking, a face-connected subsurface F is a sphere with holes (including the possibility of no holes) where, of course, if there is exactly one hole, then F is topologically a disk. The boundary ∂F is now an edge-disjoint union of cycles in G bounding the holes of F. The total length of the union of boundary cycles is called the *circumference* or *total circumference* of F. An *interior vertex* of F is a vertex not on ∂F while a *boundary vertex* is a vertex on ∂F.

The *distance* between two vertices u and v in a graph G is the length (edge length) of a shortest uv-path in G. Denote this distance by $\mathsf{d}_G(u,v)$. Of course, $\mathsf{d}_G(u,v)$ is even iff u and v are either both black or both white. Given a vertex v in a spherical quadrangulation G, consider a face F with white vertices w_1 and w_2 and black vertices b_1 and b_2. For any vertex v in G, evidently, $|\mathsf{d}_G(v,w_i) - \mathsf{d}_G(v,b_j)| = 1$ for each i and $j \in \{1,2\}$. Additionally, the following three possibilities may occur for F with respect to v: $\mathsf{d}_G(v,w_1) = \mathsf{d}_G(v,w_2)$ and $|\mathsf{d}_G(v,b_1) - \mathsf{d}_G(v,b_2)| = 2$; $|\mathsf{d}_G(v,w_1) - \mathsf{d}_G(v,w_2)| = 2$ and $\mathsf{d}_G(v,b_1) = \mathsf{d}_G(v,b_2)$; and $\mathsf{d}_G(v,w_1) = \mathsf{d}_G(v,w_2)$ and $\mathsf{d}_G(v,b_1) = \mathsf{d}_G(v,b_2)$.

Given a vertex v and a face f in a spherical quadrangulation G, let u be a vertex on f of smallest distance from v, say distance t. The vertices in cyclic ordering around f now have distances $t, t+1, t+d, t+1$ from v where $d \in \{0, 2\}$. Given integer $k \geq 0$, define $X_k(v)$ to be the set of faces f for which $t+1 \leq k$. The face-connected subsurface of G given by $X_k(v)$ in Proposition 2.3 is called the *k-ball* centered at v and is denoted $\mathsf{B}_k(v)$.

Proposition 2.3. *The faces in $X_k(v)$ define a face-connected subsurface of G.*

Proof. Certainly $X_1(v)$ is a face-connected subsurface. Using induction, assume that $X_{k-1}(v)$ is a face-connected subsurface and consider a face f in $X_k(v)$ that is not in $X_{k-1}(v)$. We will complete the proof by showing that there is a facial path (that is, a path in G^*) from f to a face in $X_{k-1}(v)$. Let u be a vertex on f whose distance from v is the smallest and let x_1 and x_2 be the neighbors of u on f. If we write $\mathsf{d}_G(v, u) = t$, then $\mathsf{d}_G(v, x_1) = \mathsf{d}_G(v, x_2) = t+1$. It must be that $t+1 = k$ or else f would be in $X_{k-1}(v)$.

Now consider the neighbors of u in G, say $x_1, x_2, \ldots, x_m$. Since $\mathsf{d}_G(v, u) = k-1$, we get that $\mathsf{d}_G(v, x_i) \in \{k-2, k\}$ for each i which implies that all faces of G incident to u are in $X_k(v)$. Now there must be some i for which $\mathsf{d}_G(v, x_i) = k-2$ and so there is some face f' incident to u which is in $X_{k-1}(v)$. The rotation of faces in G around u contains a path of adjacent faces from f to f'. □

The reader can easily verify that Proposition 2.4 follows directly from definitions.

Proposition 2.4. *If v is a vertex in a spherical quadrangulation G and $k \geq 1$, then the following hold.*

(1) *If e is an edge of G whose vertices have distances t and $t+1$ from v with $t+1 \leq k$, then both faces of G incident to e are in $\mathsf{B}_k(v)$.*

(2) *If u is a vertex of G having distance $t \leq k$ from v, then u is in $\mathsf{B}_k(v)$.*

(3) *If f is a face of $\mathsf{B}_k(v)$ sharing an edge with $\partial\mathsf{B}_k(v)$, then the vertices of f have distances $k-1, k, k+1, k$ from v; furthermore, an edge of f on $\partial\mathsf{B}_k(v)$ has endpoints with distances k and $k+1$ from v.*

(4) *If f' is a face that is not in $\mathsf{B}_k(v)$ and which shares an edge with $\partial\mathsf{B}_k(v)$, then the vertices of f' have distances $k, k+1, k+d, k+1$ from v where $d \in \{0, 2\}$.*

(5) *The vertices of a cycle C on $\partial\mathsf{B}_k(v)$ have distance from v alternating k and $k+1$.*

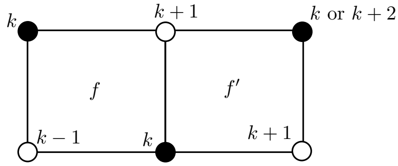

Figure 3: Proposition 2.4(4).

A *standard k-disk* with n-fold rotational symmetry around a fixed black vertex is constructed as follows. Consider the standard $\{4, 4\}$ planar quadrangulation and designate one black vertex as an origin, and then label perpendicular x- and y-axes. Consider the part of the quadrangulation in the first quadrant, with coordinate axes included, consisting of the union of the closed faces whose interiors are either beneath or intersecting the line

$x + y = k$ (see Figure 4). Call this planar graph a *k-wedge*. Taking n copies of a k-wedge, $W_1, \ldots, W_n$, identify the x-axis of W_i with the y-axis of W_{i+1} (subscript addition taken modulo n) with origin vertex identified to origin vertex to obtain the standard k-disk. In Figure 5 we show the standard 3- and 4-disks with a central vertex of degree $n = 5$.

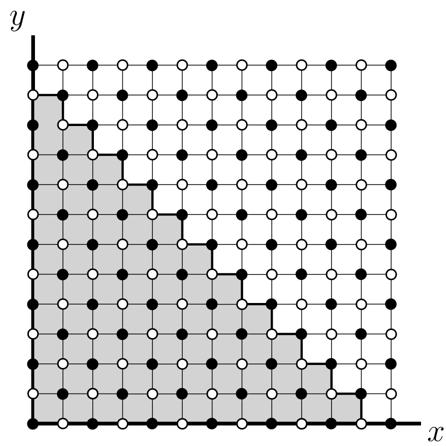

Figure 4: The 11-wedge.

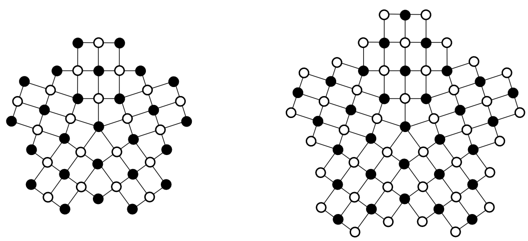

Figure 5: Standard 3- and 4-disks.

Consider a degree-4 vertex in G with incident edges e_1, e_2, e_3, e_4 in rotational order. We call each of the pairs e_1, e_3 and e_2, e_4 *transverse*. A path P in a planar graph G is said to be a *transverse* if each of its interior vertices has degree four in G with consecutive edges on P forming a transverse pair. In a standard k-disk, there are n distinct transverse paths of length k emanating from the origin (call it p) which we call the *central rays*. Note that every vertex of distance k from p in a standard k-disk K has degree 4 in K aside from the n endpoints of the central rays which have degree 3 in K. The vertices of distance $k + 1$ from p in K have degree 2 in K.

Proposition 2.5. *The circumference of the standard k-disk is $2nk$ and the number of faces in the standard k-disk is $n\binom{k+1}{2}$.*

Proof. The number of faces in the first quadrant of the coordinatized $\{4, 4\}$ planar quadrangulation that intersect the line $x + y = k$ is k. Therefore the number of boundary edges of the k-wedge is $2k$ and the number of faces in the k-wedge is a triangular sum. The result follows. □

Proposition 2.6. *If G is a spherical quadrangulation and $\mathsf{B}_k(p)$ is a standard k-disk in G, then every vertex of distance k from p is on $\partial\mathsf{B}_k(p)$. Furthermore, any vertex of distance $k+1$ from p is either on $\partial\mathsf{B}_k(p)$ or is a neighbor of some vertex of distance k from p on $\partial\mathsf{B}_k(p)$ that is not saturated by $\mathsf{B}_k(p)$.*

Proof. Here $\partial\mathsf{B}_k(p)$ is a single cycle separating the sphere into regions R_1 and R_2 where without loss of generality p is in R_1. Since $\mathsf{B}_k(p)$ is a standard k-disk, we know that every interior vertex of $\mathsf{B}_k(p)$ has distance less than k from p. Given a vertex v in R_2, a shortest pv-path in G must pass through $\partial\mathsf{B}_k(p)$ and so v has distance strictly larger than k. This implies our desired result. □

Proposition 2.7. *If D is a disk in a spherical quadrangulation G defined by a set of faces X with $|X| \geq 2$, then there are two distinct faces $f_1, f_2 \in X$ such that each $X - f_i$ defines a disk D' for which the intersection of the boundary of face f_i with $\partial D'$ is a path.*

Proof. Consider a face $f \in X$ whose boundary shares an edge with ∂D, call this a *boundary face*. Using the fact that D is a disk, the reader can confirm that the following are equivalent. (Here $G^*[Y^*]$ is the induced subgraph of G^* on vertices Y^* where Y is a collection of faces in G.)

- The faces $X - f$ define a disk.
- The faces $X - f$ define a face-connected subsurface of G, that is, $G^*[X^* - f^*]$ is connected.
- The intersection of f with ∂D is a single path.

Assume by way of contradiction that $G^*[X^* - f^*]$ is disconnected for each boundary face $f \in X$. By disconnectedness, the degree of f^* in $G^*[X^*]$ is not 1; furthermore, since f is a boundary face, the degree of f^* in $G^*[X^*]$ is either 2 or 3. The number of connected components of $G^*[X^* - f^*]$ is two when the degree is 2 and is two or three when the degree is 3. We get 3 connected components precisely when f intersects ∂D in three paths of lengths $1, 0$, and 0. Each connected component of $G^*[X^* - f^*]$ must contain a vertex corresponding to a boundary face of D.

Let $f \in X$ be a boundary face for which the induced subgraph of $G^*[X^* - f^*]$ contains a connected component on vertex set $C \subseteq X^* - f^*$ with $|C|$ as small as possible. Pick $f_0^* \in C$ that is a boundary face of D. By assumption, $G^*[X^* - f_0^*]$ is disconnected; however, by planarity, one of its connected components has vertex set which is a proper subset of C, a contradiction of minimality.

Given a face $f_1 \in X$ such that $G^*[X^* - f_1^*]$ is connected, we will now find a face $f_2 \neq f_1$ such that $G^*[X^* - f_2^*]$ is connected. Since $|X| \geq 2$, there must be boundary faces in D other than f_1. By way of contradiction, assume that for every boundary face $f_2 \neq f_1$ we have that $G^*[X^* - f_2^*]$ is disconnected. Pick f_2 such that $G^*[X^* - f_2^*]$ has a connected component on vertices C with $f_1^* \notin C$ and $|C|$ as small as possible. Following the same argument as above, we will contradict the minimality of C. □

Proposition 2.8 is surely an expected outcome; however, there are subtleties that require verification. Proposition 2.8 provides a standard model for iteratively building up disks in G.

Proposition 2.8. *Let D be a disk in a spherical quadrangulation G such that D contains a black vertex p having degree $n \geq 3$ in G and all other vertices of D have degree 4 in G. Then for k large enough, D is isomorphic to a face-connected subsurface of the standard k-disk with n-fold rotational symmetry around a black central vertex p_0. Furthermore, p is identified with p_0.*

Proof. Let X be the collection of faces defining D. If $|X| = 1$, then the result is clearly true. If $|X| \geq 2$, then by Proposition 2.7, there is an ordering $f_1, \ldots, f_m$ of the elements of X, such that p is on f_1 and $X_i = \{f_1, \ldots, f_i\}$ defines a disk D_i such that f_{i+1} intersects ∂D_i in a path. Inductively D_i is a face-connected subsurface of a standard k-disk, S_k, for some large enough value of k. If ∂D_i intersects ∂S_k, then increase k by 2 so that ∂D_i no longer intersects the boundary of the k-disk.

Let P_i be the path of intersection of f_{i+1} with ∂D_i (the length of P_i being $1, 2$, or 3). Every internal vertex of P_i must be saturated by D_i as a subgraph of G and each endpoint of P_i is not saturated. Additionally, every vertex of D other than p has degree 4 in G, so the codegree of any vertex of D_i is the same with respect to being a subgraph of G or S_k. We now have that every internal vertex of P_i is saturated by D_i as a subgraph of S_k and each endpoint of P_i is not saturated. Thus P_i is incident to a unique face f' of S_k that is not in D_i. The face f' may now be identified with f_{i+1} and we have D_{i+1} as a subgraph of the standard k-disk S_k. □

3 The two constructions

We will define two families of spherical quadrangulations, one defined with three independent parameters and the other with four. Each spherical quadrangulation is described as a net of $2n$ congruent convex quadrilaterals with vertices on the 2-dimensional integer lattice in which two sides of the quadrilateral are perpendicular and of the same length. For lack of a more specific term, we will call such a quadrilateral a *special integer quadrilateral*.

3.1 Three Parameters

Choose positive even integer a, non-negative integer s, and $l \in \{0, \ldots, a-1\}$. Consider the special integer quadrilateral of Figure 6. By reflecting along the line $y = x$ we may assume that $l \in \{0, \ldots, \frac{a}{2}\}$

We assemble a net of $2n$ such quadrilaterals as indicated in Figure 7 to obtain a spherical quadrangulation with n-fold rotational symmetry with poles at black vertices p and q and with all white vertices of degree 4. Note that the arrangement of the quadrilaterals in this construction will always yield a quadrangulation with an order-2 rotational symmetry which exchanges p and q.

In Proposition 3.2 we characterize when the Three-Parameter Construction yields a spherical grid which is the overlay graph of a self-dual embedding. Proposition 3.1 gives us a necessary and sufficient condition for making this characterization.

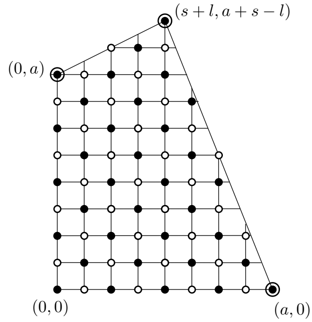

Figure 6: The Three-Parameter Construction with $(a, s, l) = (8, 3, 1)$.

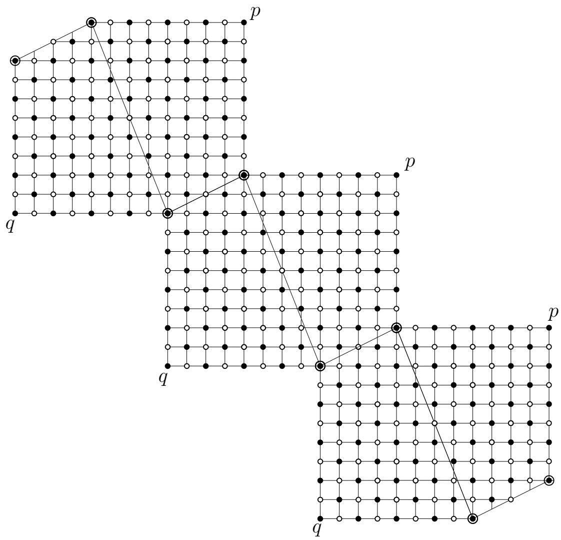

Figure 7: A net constructed from 6 copies of the quadrilateral in Figure 6.

Proposition 3.1. *If $\mathcal{O}(H, H^*)$ has n-fold rotational symmetry with poles at black vertices, the minimum possible number of curvature vertices, and an additional symmetry that exchanges the poles, then $H \cong H^*$ if and only if one pole is in H and the other in H^*.*

Proof. Let p and q be the poles of the n-fold rotational symmetry of $\mathcal{O}(H, H^*)$. If one of p and q is in H and the other in H^*, then because $\mathcal{R}(H, H^*)$ is bipartite, the symmetry which exchanges poles must exchange all of H and H^* and so is an isomorphism between H and H^*.

Conversely, assume that $H \cong H^*$ and say that H is red and H^* is blue. As such, the rotational symmetry, which fixes p and q, preserves the red and blue colors whereas the symmetry which exchanges p and q exchanges red and blue colors.

If $n = 3$, then four of the degree-3 vertices are red and four are blue and p and q both have degree 3. Furthermore, the six degree-3 vertices other than p and q are therefore divided into two orbits under the rotational symmetry of three vertices each, one being red and the other blue. Therefore the fourth red degree-3 vertex is one of p and q and the fourth blue degree-3 vertex is the other.

If $n \geq 4$, then the rotational symmetry divides the $2n$ degree-3 vertices into two orbits of n vertices each; one must be red and the other blue. Call these O_r and O_b. Without loss of generality, say that p is red. Thus the distance in $\mathcal{R}(H, H^*)$ from p to the vertices in O_r is even while the distance to the vertices in O_b is odd. Now the symmetry which exchanges p and q must therefore exchange O_r and O_b, more generally, exchange red and blue colors. Thus $p \in H$ and $q \in H^*$. $\square$

Proposition 3.2. *A spherical quadrangulation constructed from the Three-Parameter Construction is the overlay graph of a self-dual graph if and only if s and l have different parities.*

Proof. Let G be a quadrangulation constructed using the Three-Parameter Construction and let p and q be the poles of the rotational symmetry. By Proposition 1.3, $G \cong \mathcal{O}(H, H^*)$ for some H. By Proposition 3.1, a necessary and sufficient condition for $H \cong H^*$ would without loss of generality be that $p \in H$ and $q \in H^*$. This is true if and only if the distance from p to q in $\mathcal{D}(G)$ is odd.

Consider one quadrilateral of the construction. In this quadrilateral, there is a path in the graph $\mathcal{D}(G)$, from $(0, 0)$ to $(0, a)$ of length a. From $(0, a)$ to $(s + l, a + s - l)$, there is a path in $\mathcal{D}(G)$ of length $(s + l) + (s - l) = 2s$ when $s \equiv l \mod 2$ and of length $(s + l - 1) + (s - l - 1) + 1 = 2s - 1$ when $s \not\equiv l \mod 2$. Thus there is a path from p to q in $\mathcal{D}(G)$ of length $2a + 2s$ when $s \equiv l \mod 2$ and of length $2a + 2s - 1$ when $s \not\equiv l \mod 2$, as required. $\square$

3.2 Four parameters

Choose positive integers a and b of the same parity. Assume that $a \geq b$. Choose non-negative integers h and w of the same parity, not both zero, and which satisfy

$$-\frac{h}{w} \leq \frac{b - a}{b + a} \qquad \text{and} \qquad -\frac{a}{b}(a - w) \leq b + h.$$

(In the case that $w = 0$ say that $-\frac{h}{w} = -\infty$.) Consider the special integer quadrilateral of Figure 8.

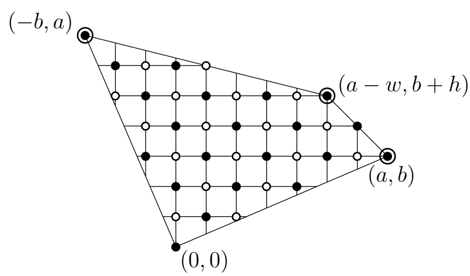

Figure 8: A special integer quadrilateral with $(a, b, h, w) = (7, 3, 2, 2)$.

Given a choice of a and b, the constraints placed on non-negative integers h and w guarantee that the quadrilateral defined will indeed be convex. The first inequality guarantees that the point $(a-w, b+h)$ lies above the line containing $(-b, a)$ and (a, b). The second inequality guarantees that at $x = a - w$, the y-coordinate $b + h$ is greater than the y-coordinate of the line of slope $-\frac{b}{a}$. (See Figure 9.)

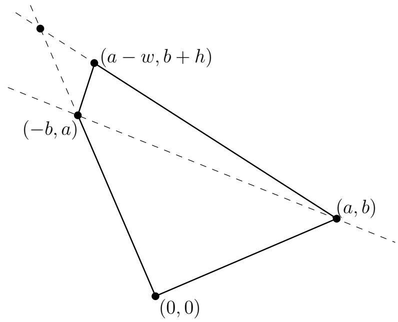

Figure 9: Constraints on (a, b, h, w) guarantee convexity.

We assemble a net of $2n$ such quadrilaterals as indicated in Figure 10 to obtain a spherical quadrangulation with n-fold rotational symmetry with poles at black vertices p and q and with every white vertex of degree 4. As with the three-parameter construction, the four-parameter construction always yields a quadrangulation with an order-2 rotational symmetry which exchanges p and q.

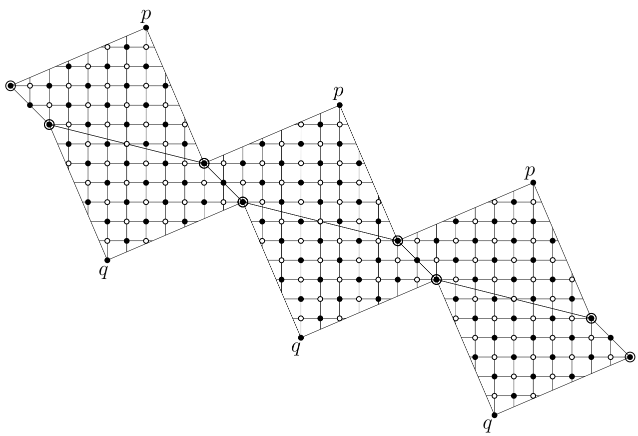

Figure 10: A net constructed from 6 copies of the quadrilateral in Figure 8.

Proposition 3.3. *If G is constructed from the four-parameter construction with parameters (a, b, h, w), then G is the overlay graph of a self-dual graph if and only if h and w are both odd.*

Proof. Say G is constructed using the Four-Parameter Construction and let p and q be the poles of the rotational symmetry. As in the proof of Proposition 3.2, $G \cong \mathcal{O}(H, H^*)$ for some H (say H is red and H^* is blue) and a necessary and sufficient condition for $H \cong H^*$ would be that the distance from p to q in $\mathcal{D}(G)$ is odd.

Consider one quadrilateral of the construction with p at $(0,0)$. There is a path in $\mathcal{D}(G)$ from p to the vertex at (a,b) of length $a+b$ when a and b are both even and $a+b-1$ when a and b are both odd. Call this length x. There is a path in $\mathcal{D}(G)$ from (a,b) to $(a-w,b+h)$ of length $w+h$ when w and h are both even and of length $w+h-1$ when h and w are both odd. Call this length y. Thus there is a pq-path in $\mathcal{D}(G)$ of length $2x+y$ which is even when h and w are both even and is odd when h and w are both odd, as required. $\square$

4 The two constructions are sufficient

In this section we prove Theorem 1.1. So, throughout this section, let $n \geq 3$ be a fixed integer and G a spherical quadrangulation satisfying the hypothesis of Theorem 1.1. Recall that making use of Proposition 1.3 allows this theorem to cover all spherical quadrangulations with minimum degree three and the minimum number of curvature vertices. Our first step is to prove Proposition 4.2, which separates the remainder of this proof into two distinct cases. In Section 4.3 we find that all graphs in the first case are given by the Three-Parameter Construction. In Section 4.4 we find that all graphs in the second case are given by the Four-Parameter Construction.

4.1 Initial k-balls are standard disks

Proposition 4.1. *If every vertex of* $\mathsf{B}_k(p)$ *aside from* p *has degree 4 in* G*, then* $\mathsf{B}_{k+1}(p)$ *is a standard* $(k+1)$*-disk.*

Proof of Proposition 4.1. First we prove that $\mathsf{B}_1(p)$ is a standard 1-disk. The n white neighbors of p are all distinct because G is simple. Now the only way in which $\mathsf{B}_1(p)$ is not a disk would be if the black vertices along the boundary walk are not all distinct. If, by way of contradiction, we assume that these black vertices are not all distinct, then rotational symmetry implies that there is only one black vertex on $\partial\mathsf{B}_1(p)$. This black vertex must therefore have degree $2n$, a contradiction.

Inductively assume that the result holds up to some $k-1 \geq 0$ in G. Assume, by way of contradiction, that every vertex $v \neq p$ of $\mathsf{B}_k(p)$ has degree 4 in G and yet $\mathsf{B}_{k+1}(p)$ is not a standard $(k+1)$-disk. Since every vertex $v \neq p$ of $\mathsf{B}_k(p)$ has degree 4 in G, we get that every vertex $v \neq p$ of $\mathsf{B}_{k-1}(p)$ has degree 4 in G and so inductively $\mathsf{B}_k(p)$ is a standard k-disk.

By Proposition 2.4, every face f that is not in $\mathsf{B}_k(p)$ but shares an edge e with $\partial\mathsf{B}_k(p)$ is in $\mathsf{B}_{k+1}(p)$. Each vertex of distance k from p is in $\mathsf{B}_k(p)$ (again by Proposition 2.4) and so has degree 4 in G. Therefore each edge e on $\partial\mathsf{B}_k(p)$ satisfies the following: if e is incident to a central ray then the two edges of f incident to e are not on $\partial\mathsf{B}_k(p)$ and if e is not incident to a central ray, then there are two consecutive edges of f on $\partial\mathsf{B}_k(p)$ (see Figure 11).

The former type of face we will call a “radial” face and the latter a “notch” face. For a notch face there cannot be 3 edges of f that are on $\partial\mathsf{B}_k(v)$ because this would either force a vertex on $\partial\mathsf{B}_k(p)$ to have degree less that 3 in G, a contradiction, or force two vertices on $\partial\mathsf{B}_k(p)$ to be identified, a contradiction because $\mathsf{B}_k(p)$ is a standard k-disk. In Case 1 assume that there is a radial face f in $\mathsf{B}_{k+1}(p)$ having opposing edges that are both on $\partial\mathsf{B}_k(p)$. In Case 2, every radial face has exactly one of its edges on $\partial\mathsf{B}_k(p)$.

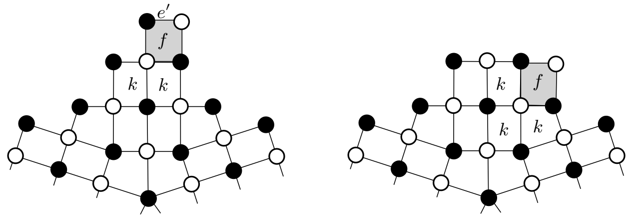

Figure 11: A radial face and a notch face.

Case 1: Let e and e' be opposing edges of radial face f that are both on $\partial\mathsf{B}_k(p)$. Since e and e' are nonconsecutive on f, they must also be non-consecutive along $\partial\mathsf{B}_k(p)$ (see the left in Figure 11). Since e and e' are the only edges of f on $\partial\mathsf{B}_k(p)$, it must also be the case that e' is incident to a central ray of $\mathsf{B}_k(p)$ as well (which cannot be the same ray). Let O be the orbit of e under the n-fold rotational symmetry. (Note that $|O| = n$.) The black and white coloring of the vertices, along with orientability, forces edges e and e' to both point in the same direction along the disk from their central rays, which implies that $e' \in O$. Therefore the orbit of f under the n-fold rotational symmetry does not have order n and so f contains a pseudofixed point in its interior, a contradiction of the fact that p and q are the only pseudofixed points of the rotational symmetry of the sphere.

Case 2: Let B denote the face-connected subsurface consisting of $\mathsf{B}_k(p)$ along with the faces not in $\mathsf{B}_k(p)$ that share an edge with $\partial\mathsf{B}_k(p)$. Note that $B \subseteq \mathsf{B}_{k+1}(p)$. Take two faces f and f' of B that are not in $\mathsf{B}_k(p)$ which are consecutive with respect to their edges on $\partial\mathsf{B}_k(p)$. Because no vertex $v \neq p$ of $B \subseteq \mathsf{B}_{k+1}(p)$ has degree other than 4, if one of f and f' is a notch face then f and f' share no edges in common, and if f and f' are both radial faces then they must share one edge in common. Thus B is obtained from a standard $(k+1)$-disk, call it K, after perhaps making identifications along ∂K. (See Figure 12.)

$$\begin{tikzcd}
B_k(p) \arrow[r, hook] \arrow[d, hook] & B \arrow[r, hook] & B_{k+1}(p) \\
K \arrow[ur, squiggly] & &
\end{tikzcd}$$

Figure 12: The relationships between B, K, $B_k(p)$ and $B_{k+1}(p)$.

Note that there can be no facial identifications in obtaining B from K because that forces the identified face to have two opposite edges on $\partial\mathsf{B}_k(p)$ which would put us back into Case 1. In Case 2.1 there are no identifications, in Case 2.2 there is an edge identification, and in Case 2.3 there are no edge identifications but there are vertex identifications.

Case 2.1: Here we have that $B = K$ is a standard $(k+1)$-disk. In this case, the vertices on the cycle ∂B in G must have distances from p in G alternating between $k+1$ and $k+2$. If $\mathsf{B}_{k+1}(p) = B$, then we are done. If not, then there is a face f in $\mathsf{B}_{k+1}(p)$ that is not in B and shares an edge with ∂B. Because f is in $\mathsf{B}_{k+1}(p)$, the distances of the vertices on f from p must be $k+1, k+2, k+1, k$; however, ∂B separates p from f in the sphere which means that no vertex on f can have distance less than $k+1$ from p.

Case 2.2: Let e and e' be two edges on ∂K that are identified in obtaining B from K. If e and e' are consecutive along ∂K with e having endpoints u and v and e' having endpoints v and w, then the distances in K from p to u, v, and w are either $k+1$, $k+2$, $k+1$ or $k+2$, $k+1$, and $k+2$. In the first case, the identification of e and e' would create a vertex in G of degree 1, a contradiction. In the second case, the identification of e and e' would create either a vertex of degree 2 in G (again a contradiction) or a vertex of degree 3 in G that is in $B_k(p)$, a contradiction of our inductive hypothesis.

If e and e' are in the same orbit under the rotational symmetry of K, then the interior of e would contain a pseudofixed point of the rotational symmetry; however, p and q are the only pseudofixed points in G of the rotational symmetry, a contradiction.

Now, given that e and e' are not consecutive along ∂K and not in the same orbit, rotational symmetry yields an orbit of n distinct edges $e_1, \ldots, e_n$ identified to an orbit of n distinct edges $e'_1, \ldots, e'_n$. Planarity of G forces the cyclic ordering of these edges along ∂K to be $e_1, e'_1, \ldots, e_n, e'_n$.

If either e_1 or e'_1 (say e_1) is not incident to a central ray of K, then e has an endpoint v of degree 4 in K. Thus v has distance $k+1$ from p in K and say without loss of generality that v is black. The identification of e_1 and e'_1 would force v to have degree at least five in B because the black endpoint of e'_1 has degree at least 3 in K. This is impossible unless the vertex resulting from the identification of e_1 and e'_1 is the other pole q; however, this implies that the corresponding endpoints of $e_2, \ldots, e_n$ and $e'_2, \ldots, e'_n$ are also the pole q. This is impossible because q has degree n in G and such identifications would force q to have degree at least $2n$, which contradicts the fact that G has maximum degree n.

If both e_1 and e'_1 are incident to central rays, then these two orbits of edges account for all of the $2n$ edges on ∂K that are incident to the central rays. In Figure 13 the edges with the same numbers are identified and therefore, by the rotational symmetry, the endpoints of the central rays must be identified to the other pole of the rotational symmetry, call it q. Making these edge identifications results in a surface K' that is topologically a sphere with n holes; that is, $\partial K'$ consists of vertex-disjoint cycles $C_1, \ldots, C_n$. As discussed above, there can be no further edge identifications in going from K' to B. If there are vertex identifications in going from K' to B, then each identification is between two white degree-2 vertices on $\partial K'$. These white vertices must be on the same cycle C_i. The reason for this is as follows. If vertex x on C_i is identified to vertex y on C_j, then let Q be a simple xy-path in K' whose interior avoids $\partial K'$ (not necessarily a path in the graph). Now any cycle on K' which avoids its boundary (again, not necessarily a cycle in the graph itself) and separates C_i from C_j must transversely intersect Q an odd number of times. Thus the spherical embedding G would have two cycles drawn on it which intersect transversely an odd number of times, a contradiction.

Now say that two white vertices on C_i (call them x and y) are identified in going from K' to B. Note that the black vertices on C_i all have degree 4 in K' and the white vertices on C_i all have degree 2 in K' save for one which has degree 3. Let P_i be the xy-path on C_i which avoids the degree-3 white vertex. After identifying x and y, facial boundaries of length four and saturated black vertices forces the identification of the adjacent pair of white vertices on P_i, and so on. These identifications will eventually result either in a white vertex being forced to have degree 2 in G (a contradiction) or a face in G being forced to have length 2 (again a contradiction).

Lastly, assuming there are no further vertex or edge identifications, we have $K' = B$. Let D_i be the disk in G bounded by C_i whose faces are not in B. The black vertices on

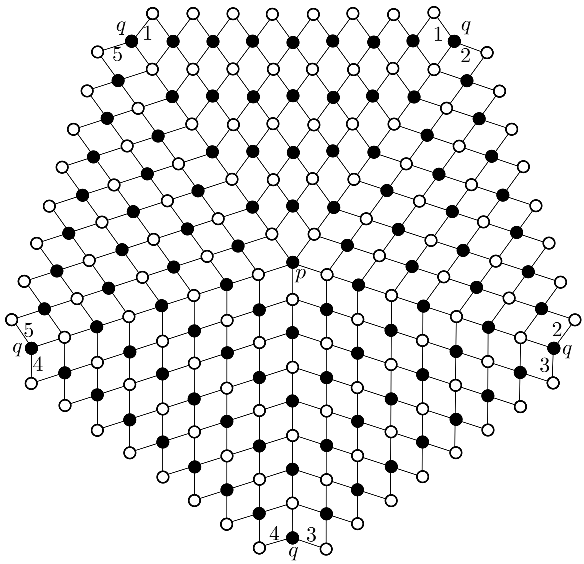

Figure 13: Edges with the same numbers are identified which then implies that the vertex q is the other pole.

C_i have degree 2 in D_i and the white vertices on C_i have degree 4 in D_i save for one white vertex which has degree 3 in D_i. By rotational symmetry, there must be two black degree-3 vertices in the interior of D_i, with the remaining interior vertices having degree 4. This forces D_i to have exactly three vertices of odd degree, a contradiction.

Case 2.3: If two identified vertices on ∂K are in the same orbit under the rotational symmetry, then the resulting vertex will be pseudofixed and so the identified vertex is the pole q. However, q will now be forced to have degree at least $2n$, a contradiction. So now take an orbit of n distinct vertices $v_1, \ldots, v_n$ on ∂K that are pairwise identified to the n distinct vertices $v'_1, \ldots, v'_n$ in going from K to B. There can be no additional identifications among these vertices. Planarity now forces these $2n$ vertices to have cyclic ordering $v_1, v'_1, \ldots, v_n, v'_n$ along ∂K. Thus the vertex $v_1 = v'_1$ in G has degree 4 in G. So now if K' is obtained from K by making these n identifications only, then K' is obtained as shown in Figure 14 (for $n = 5$).

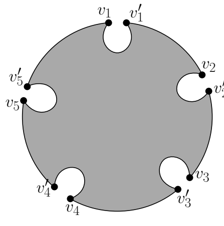

Figure 14: The surface K' from Case 2.3.

Rechoose the identified vertex pairs $v_1, v_1', \ldots, v_n, v_n'$ so that v_i and v_i' are as close together along ∂K as possible. Thus the orbit of n holes in K' have boundaries that are cycles in G, say $C_1, \ldots, C_n$. Now either the endpoint of a central ray of K' is on C_1 and is adjacent to $v_1 = v_1'$ or not. Let these be Cases 2.3.1 and 2.3.2.

Case 2.3.1: Consider the disk H in G with $\partial H = C_1$ whose faces are not in K. By rotational symmetry either $0, 1$, or 2 of the degree-3 vertices of G appear in H and the pole q does not appear in H. Assume for the moment that 0 of the degree-3 vertices of G appear in H. Say that C_1 has length $2m$. The degrees in H of the vertices on C_1 are therefore $2, 3, 4, 2, 4, 2, \ldots, 4, 2$ in which the first degree-2 vertex is $v_1 = v_1'$ and the degree-3 vertex is the endpoint of the central ray of K'. The remaining vertices of H all have degree 4. Thus the sum of the degrees of the vertices in H is

$$4i + 3 + 4(m-1) + 2m = 4i + 6m - 1$$

where i is the number of interior vertices. So now if ϵ is the number of degree-3 vertices of G appearing in H, then the sum of the degrees of the vertices in H is $4i + 6m - 1 - \epsilon$. Now if e is the number of edges in H, we obtain

$$4i + 6m - 1 - \epsilon = 2e.$$

If f is the number of quadrilateral faces in H, then

$$4f + 2m = 2e.$$

Now Euler's Formula implies that

$$1 = i + 2m - e + f = \frac{1}{4}(2e + \epsilon + 1 - 6m) + 2m - e + \frac{1}{4}(2e - 2m) = \frac{1}{4}(1 + \epsilon) \leq \frac{3}{4},$$

which is a contradiction.

Case 2.3.2: Let u_1 and w_1 be the neighbors of $v_1 = v_1'$ on C_1. Note that these three vertices all have degree 4 in K' and so have no edges extending into the interior of H. This forces these three vertices to be on the same quadrilateral face of G and this face is in H. Because v_1 and v_1' are chosen to be as close together as possible along ∂K, it must be that v_1 and v_1' are at distance 4 apart along ∂K. Hence C_1 has length four and H has only one face. Let x be the fourth vertex of C_1. Since u_1 and w_1 both have degree 4 in K' and K, it must be that x has degree 2 in K' which implies that x also has degree 2 in G, a contradiction. □

Proposition 4.2. *If every vertex $v \neq p$ of $\mathsf{B}_{k-1}(p)$ has degree 4 in G but there are curvature vertices of G in $\mathsf{B}_k(p) - p$, then $\mathsf{B}_k(p)$ is a standard k-disk and the following hold.*

(1) *If k is even, then the n endpoints of the central rays of $\mathsf{B}_k(p)$ have degree 3 in G and all other vertices of $\mathsf{B}_k(p) - p$ have degree 4 in G.*

(2) *If k is odd, then there are either n or $2n$ degree-3 vertices of G which have distance $k+1$ from p on $\partial\mathsf{B}_k(p)$ and all other vertices of $\mathsf{B}_k(p) - p$ have degree 4 in G, including the endpoints of the central rays.*

Proof. By Proposition 4.1, $\mathsf{B}_k(v)$ is a standard k-disk. By Proposition 2.4, the vertices of $\mathsf{B}_k(p)$ that are not in $\mathsf{B}_{k-1}(p)$ come in two types: those on $\partial\mathsf{B}_k(p)$ having distance $k+1$

from p and the endpoints of the central rays of $\mathsf{B}_k(p)$, which have distance k from p. If k is odd, then because the curvature vertices of G are black, $\mathsf{B}_k(p)$ is of Type (2). If k is even, then because the curvature vertices of G are black, $\mathsf{B}_k(p)$ is of Type (1) and there are either n or $2n$ such curvature vertices. $\square$

Proposition 4.2 gives us two cases for G. In Section 4.3 we will discuss the case for G in Part (1) of Proposition 4.2 and in Section 4.4 the case for Part (2) of Proposition 4.2.

4.2 Necklaces

Take quadrilaterals $q_1, \ldots, q_n$ whose vertices are properly colored black and white. Label the black vertices of q_i with $b_{i,1}$ and $b_{i,2}$. A *diamond necklace* of length n with a *black diagonal* is the graph obtained from $q_1, \ldots, q_n$ by identifying $b_{i,2}$ with $b_{i+1,1}$ for each $i \in \{1, \ldots, n\}$ where addition in subscripts is taken modulo n so as to obtain a cyclic arrangement of these quadrilaterals. The top of Figure 15 shows a diamond necklace with a black diagonal. A *diamond necklace* of length n with a *white diagonal* is defined similarly. When t diamond necklaces of the same length with diagonals of alternating colors are stacked together as on the bottom of Figure 15, we obtain a *straight thorax* of *thickness* t.

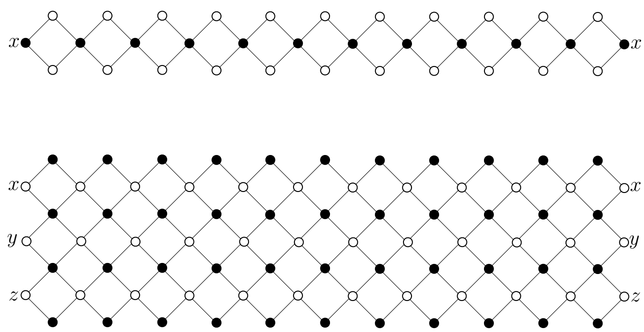

Figure 15: A diamond necklace with a black diagonal and a straight thorax of thickness 5.

Consider a diamond necklace of length $n(k+1)$ for some $k \geq 1$ with black-diagonal vertices $b_0, \ldots, b_{n(k+1)-1}$ and choose a positive integer $1 \leq l \leq k$. If we embed our necklace in the plane with $b_0, \ldots, b_{n(k+1)-1}$ oriented in the clockwise direction, then there are well-defined inner and outer boundary cycles of length $2n(k+1)$ each. For each vertex $i(k+1) \in \{0, k+1, \ldots, (n-1)(k+1)\}$ identify the two inner-boundary edges incident to $b_{i(k+1)}$ and identify the two outer-boundary edges incident to $b_{i(k+1)+l}$. The resulting graph with n-fold rotational symmetry is called a (k,l)-*zig-zag necklace* with a *black diagonal*. Note that the lengths of each of the two boundary cycles of the (k,l)-*zig-zag necklace* is $2nk$. A (k,l)-*zig-zag necklace* with a *white diagonal* is defined similarly. The graph in Figure 16 shows a portion of a $(k,6)$-zig-zag necklace.

Any number of (k,l)-zig-zag necklaces with diagonals of alternating colors may be stacked as shown in Figure 17.

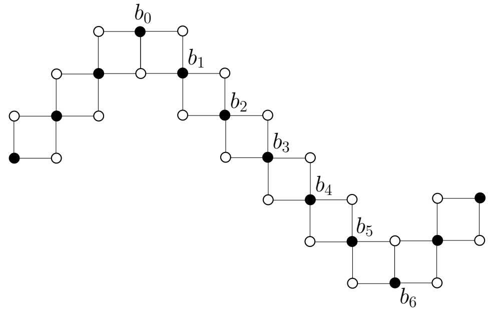

Figure 16: A portion of a $(k, 6)$-zig-zag necklace.

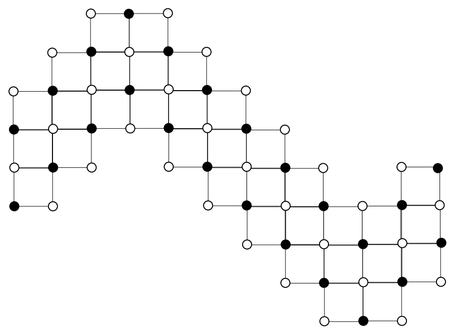

Figure 17: Stacking zig-zag necklaces.

4.3 Curvature vertices on the ends of central rays

Proposition 4.3. *Let* $\mathsf{B}_k(p)$ *be a standard* k*-disk in* G *as described in* Proposition 4.2(1). *If the only curvature vertices of* $\mathsf{B}_{k+t}(p)$ *are* $p, v_1, \dots, v_n$*, then* $\mathsf{B}_{k+t+1}(p)$ *is a disk obtained from the standard disk* $\mathsf{B}_k(p)$ *by adding a straight thorax with thickness* $t+1$*. Furthermore, the circumferences of* $\mathsf{B}_k(p)$ *and* $\mathsf{B}_{k+t}(p)$ *are the same.*

Proof. First we observe that the statement about the circumference of $\mathsf{B}_{k+t}(p)$ is evident by the structure of necklaces. We now proceed with the rest of the proof. Certainly for $t = 0$, $\mathsf{B}_{k+t}(p)$ is a disk obtained from the standard disk $\mathsf{B}_k(p)$ by adding a straight thorax with thickness $t = 0$. So now assume that this same statement holds for some $t \geq 0$ and the only curvature vertices of $\mathsf{B}_{k+t}(p)$ are $p, v_1, \dots, v_n$. Also as part of the induction hypothesis we include that $\partial\mathsf{B}_{k+t}(p)$ has vertices in one color class (those of distance $k+t$ from p) saturated by $\mathsf{B}_{k+t}(p)$ and vertices in the other color class (those of distance $k+t+1$ from p) having degree 2 in $\mathsf{B}_{k+t}(p)$.

Now consider $\mathsf{B}_{k+t+1}(p)$. Every face f of $\mathsf{B}_{k+t+1}(p)$ that is not in $\mathsf{B}_{k+t}(p)$ yet shares an edge with $\partial\mathsf{B}_{k+t}(p)$ must share two consecutive edges with $\partial\mathsf{B}_{k+t}(p)$ because the vertices in one color class of $\partial\mathsf{B}_{k+t}(p)$ are saturated by $\mathsf{B}_{k+t}(p)$. Furthermore, since $\mathsf{B}_{k+t}(p)$ is a disk, f cannot share three edges with $\partial\mathsf{B}_{k+t}(p)$ because if it did, then f would have two vertices that are saturated by $\mathsf{B}_{k+t}(p)$ which would force f to share all four of its edges with $\partial\mathsf{B}_{k+t}(p)$. Since $\partial\mathsf{B}_{k+t}(p)$ is a cycle, this would imply that the length of $\partial\mathsf{B}_{k+t}(p)$ is four; however, it must have length at least $2n \geq 6$.

Let B be the face-connected subsurface obtained from $\mathsf{B}_{k+t}(p)$ by adding to it the faces of $\mathsf{B}_{k+t+1}(p)$ that share an edge with $\partial\mathsf{B}_{k+t}(p)$. We claim that B is a disk. If B is not a disk, then B is obtained as follows. Let B' be the disk obtained from $\mathsf{B}_{k+t}(p)$ by adding to it a diamond necklace of faces. Now B is obtained from B' by identifying edges or vertices on $\partial B'$. Note that it is not possible to identify faces because $\partial\mathsf{B}_{k+t}(p)$ is a cycle.

It is not possible to identify two non-consecutive edges of $\partial B'$ because any two such edges on $\partial B'$ contain endpoints that are on $\partial\mathsf{B}_{k+t}(p)$, which is a disk, and therefore are distinct. Now suppose that e_1 and e_2 are consecutive edges on $\partial B'$ whose common endpoint is v. The degree of v in B' is either 2 or 4. It is not possible that v has degree 2 in B' because identifying e_1 and e_2 would then yield a vertex of degree 1 in G. It is not possible that v has degree 4 in B' because identifying e_1 and e_2 would create a vertex of degree 3 in $\mathsf{B}_{k+t}(p)$ that is not among $p, v_1, \dots, v_n$.

So it must be that B is obtained from B' by identifications of vertices along $\partial B'$. Note that the degree in B' of the vertices of $\partial B'$ alternate between 2 and 4 where the degree-2 vertices have distance $k+t+2$ from p and the degree-4 vertices have distance $k+t+1$ from p. Thus identification of any two degree-2 vertices on $\partial B'$ will then force the identification of another pair of degree-2 vertices on $\partial B'$. These identifications will continue until we force G to contain either a facial cycle of length 2 (a contradiction) or a vertex of degree 2 (again a contradiction).

Thus $B' = B$ is a disk and the vertices of ∂B alternate with distances $k+t+1$ and $k+t+2$ from p. Thus $B = \mathsf{B}_{k+t+1}(p)$ because if there is a face f in $\mathsf{B}_{k+t+1}(p)$ but not in B, its closest vertex to p has distance at least $k+t+1$ and so this face is not in $\mathsf{B}_{k+t+1}(p)$, a contradiction. □

Proposition 4.4. *If* $\mathsf{B}_{k+t}(p)$ *is a disk in a spherical quadrangulation* G *as given in* Proposition 4.3*,* $\mathsf{B}_{k+t}(p)$ *contains no curvature vertices of* G *other than* $p, v_1, \dots, v_n$*, but* $\mathsf{B}_{k+t+1}(p)$ *contains an additional curvature vertex of* G*, then* $\mathsf{B}_{k+t+1}(p)$ *is also a disk as given in* Proposition 4.3*, contains curvature vertices* $u_1, \dots, u_n$ *on its boundary cycle, and each* u_i *has distance* $k+t+2$ *from* p*.*

Proof. This follows from Proposition 4.3 and the fact that the only vertices in $\mathsf{B}_{k+t+1}(p)$ that are not in $\mathsf{B}_{k+t}(p)$ are the outer vertices of the new diamond-necklace layer of the thorax. □

Proposition 4.5. *If* $\mathsf{B}_{k+t}(p)$ *is a disk as given in* Proposition 4.4 *which contains curvature vertices* $p, v_1, \dots, v_n$ *in its interior and curvature vertices* $u_1, \dots u_n$ *on its boundary, then* G *is obtained from* $\mathsf{B}_{k+t}(p)$ *by identifying* $\partial\mathsf{B}_{k+t}(p)$ *with the boundary of a standard k-disk* D *such that* $u_1, \dots, u_n$ *are identified with the endpoints of the central rays of* D*.*

Proof. By Proposition 4.4, each u_i has distance $k+t+1$ from p. This implies that $k+t+1$ is even, and since k is even, we must have that t is odd.

Say that l is the smallest distance in G from the pole q to any vertex on $\partial B_{k+t}(p)$ and let u be such a vertex. It must be that $\mathsf{d}(u,p) = k+t+1$ rather than $k+t$, because the vertices of $\partial B_{k+t}(p)$ of distance $k+t$ from p are saturated by $B_{k+t}(p)$ and so any path from q to one of these vertices of distance $k+t$ from p must go through the vertices of distance $k+t+1$ from p. Therefore u is black and has degree 2 in $B_{k+t}(p)$. Also, since u is black, l must be even.

First suppose that u can be chosen to be in $\mathsf{B}_{l-1}(q)$. The intersection of $\mathsf{B}_{l-1}(q)$ and $\mathsf{B}_{k+t}(p)$ may only consist of a collection of black vertices because the white vertices of

$\mathsf{B}_{l-1}(q)$ have distance at most $l-1$ from q. Therefore $\mathsf{B}_{l-2}(q)$ contains no curvature vertices aside from q and Proposition 4.1 implies that $\mathsf{B}_{l-1}(q)$ is a standard $(l-1)$-disk. Thus u has degree 2 in $\mathsf{B}_{l-1}(q)$ and must have degree 4 in G. The neighbor w_p of u on $\partial\mathsf{B}_{k+t}(p)$ in the direction of rotation is saturated by $\mathsf{B}_{k+t}(p)$ and the neighbor w_q of u on $\partial\mathsf{B}_{l-1}(q)$ in the rotational direction is either saturated or, alternatively, is the endpoint of a central ray and, because it is white, has codegree 1 with respect to $\mathsf{B}_{l-1}(q)$. Let u_p be the next vertex in rotational order along $\partial\mathsf{B}_{k+t}(p)$ and let u_q be the next vertex in rotational order along $\partial\mathsf{B}_{l-1}(q)$. If w_q is saturated by $\mathsf{B}_{l-1}(q)$, then because every face of G has length four we must have that $u_p = u_q$ and this vertex has degree 4 in G. (See the left configuration in Figure 18). If w_q is the end of a central ray of $\mathsf{B}_{l-1}(q)$ (which has codegree 1), then again, the fact all faces have length four implies that u_p is adjacent to w_q and so u_p is a curvature vertex of G and $u_q = u'_p$ where u'_p is the next black vertex in the rotational direction on $\partial\mathsf{B}_{k+t}(p)$. (See the right configuration in Figure 18).

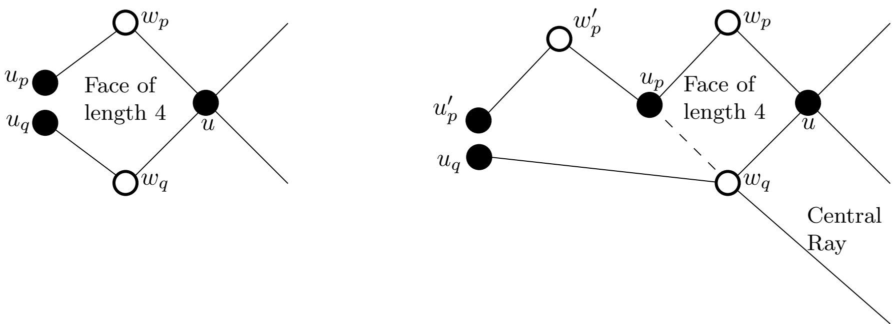

Figure 18: Forced vertex identifications on the boundaries of $\mathsf{B}_{k+1}(p)$ and a standard k-disk.

This process of identifying vertices and adjacencies continues all the way around $\partial\mathsf{B}_{k+t}(p)$ and $\partial\mathsf{B}_{l-1}(q)$ so that the black vertices on $\partial\mathsf{B}_{k+t}(p)$ correspond to the black vertices and endpoints of the central rays of $\mathsf{B}_{l-1}(q)$. Therefore $l = k$ and G is obtained as stated in the proposition.

Next suppose that u cannot be chosen to be in $\mathsf{B}_{l-1}(q)$. Proposition 2.6 implies that u is the endpoint of a central ray of $B_l(q)$. Let w be this endpoint of the central ray of $\mathsf{B}_{l-1}(q)$ that is adjacent to u. Thus u has codegree 1 or 2 with respect to $B_{k+t}(p)$. In either case there is a face f incident to the uw-edge that contains an edge wb_1 of $\partial\mathsf{B}_{l-1}(q)$ and an edge uw_1 of $\partial\mathsf{B}_{k+t}(p)$. So now a fourth edge for f would be w_1b_1; however, w_1 is saturated by $\mathsf{B}_{k+t}(p)$ and $b_1 \notin \mathsf{B}_{k+t}(p)$, a contradiction (see Figure 19). □

Theorem 4.6. *The graph described in* Proposition 4.5 *is given by the Three-Parameter Construction.*

Proof. Consider the part of $\mathsf{B}_k(p)$ between two consecutive central rays, call it W_k. Let o_1 and o_2 be the curvature vertices on the central rays of W_k which have distance k from p. Consider the black diagonal line D in W_k from o_1 to o_2. Now let W be the portion of $\mathsf{B}_{k+t}(p)$ consisting of W_k along with the faces between the black diagonals emanating from o_1 and o_2 which are perpendicular to D. Let o be the curvature vertex in W which

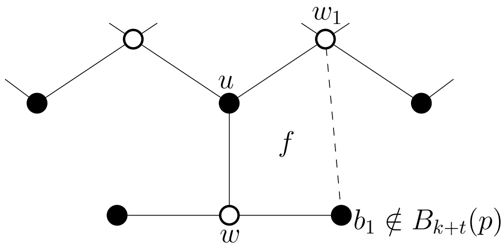

Figure 19: Final contradiction in the proof of Proposition 4.5.

has distance $k + t$ from p. As shown in Figure 20, there is a special integer quadrilateral for the Three-Parameter Construction contained within W.

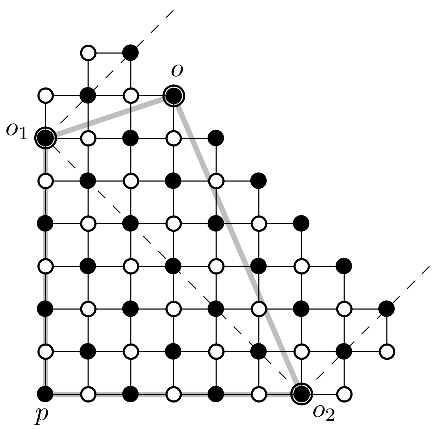

Figure 20: A special integer quadrilateral within a single wedge.

By inspection, the spherical quadrangulation constructed by the special integer quadrilateral from Figure 20 contains $B_{k+t}(p)$ with curvature vertices positioned as shown. By Proposition 4.5, there is only one spherical quadrangulation which contains $\mathsf{B}_{k+t}(p)$ with curvature vertices in a given position. Thus G is given by the Three-Parameter Construction. □

4.4 Curvature vertices off the central rays

Let $\mathsf{B}_k(p)$ be a standard k-disk in G as described in Proposition 4.2(2). The disk $\mathsf{B}_k(p)$ contains at least one orbit of n degree-3 curvature vertices. Let $t \geq 0$ be the smallest integer for which $\mathsf{B}_{k+t}(p)$ contains both orbits of n degree-3 curvature vertices. In Proposition 4.8 we describe three possible structures for $\mathsf{B}_{k+t}(p)$. Finally, we show that each structure is given by the Four-Parameter Construction.

4.4.1 Two new types of disk

Note that the length of $\partial\mathsf{B}_k(p)$ is $2nk$. The black vertices along $\partial\mathsf{B}_k(p)$ have degree 2 in $\mathsf{B}_k(p)$ and the white vertices along $\partial\mathsf{B}_k(p)$ have degree 4 in $\mathsf{B}_k(p)$ save for the endpoints of the central rays which have degree 3 in $\mathsf{B}_k(p)$. Consider vertices $w_1, v_1, \ldots, w_n, v_n$ in clockwise order around $\partial\mathsf{B}_k(p)$ in which $w_1, \ldots, w_n$ are the ends of the central rays and $v_1, \ldots, v_n$ is one orbit of black vertices on $\partial\mathsf{B}_k(p)$. Say that the distance from w_i to v_i along $\partial\mathsf{B}_k(p)$ is $2l-1$. Let T be a t-layered stack of (k,l)-zig-zag necklaces. Note that ∂T consists of two cycles. Let $\partial_{\text{in}}T$ be the inner cycle of ∂T and say that the necklace along $\partial_{\text{in}}T$ has a black diagonal and label these black vertices as $b_0, \ldots, b_{n(k+1)-1}$. Note that b_j for j not divisible by $k+1$ appears on $\partial_{\text{in}}T$ and has degree four in T except when $j = i(k+1)+l$, in which case b_j has degree 3 in T. Also, the white vertices on $\partial_{\text{in}}T$ all have degree 2 in T save for the white vertices on $\partial_{\text{in}}T$ adjacent to $b_{i(k+1)}$'s, which have degree 3 in T. Thus we can identify $\partial\mathsf{B}_k(p)$ with $\partial_{\text{in}}T$ so that v_i is identified with $b_{i(k+1)+l}$ and w_i is adjacent to $b_{i(k+1)}$. We call the resulting disk $\mathsf{Z}_{k,l,t}(p)$. Our discussion assumes that $t \geq 1$, but as a convention we can define $\mathsf{Z}_{k,l,0}(p)$ to be the standard k-disk with l defined by either one of the two orbits of degree-3 curvature vertices on $\partial\mathsf{B}_k(p)$. Note that $\mathsf{Z}_{k,l,t}(p)$ is a $(k+t)$-ball centered at p and every vertex $v \neq p$ in the interior of $\mathsf{Z}_{k,l,t}(p)$ has degree 4 in $\mathsf{Z}_{k,l,t}(p)$ save for $v_1, \ldots, v_n$ which all have degree 3. Figure 21 depicts $\mathsf{Z}_{5,2,3}(p)$ for $n = 5$ (ignore the shading in the outer faces for the moment).

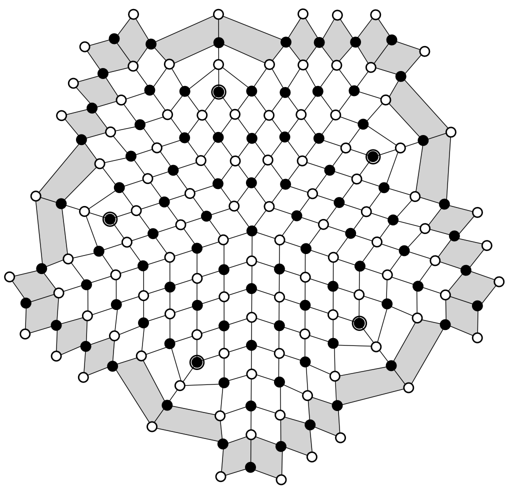

Figure 21: The disk $\mathsf{Z}_{5,2,3}(p)$. If the shaded faces are removed, then the remaining faces define $\mathsf{Z}_{5,2,2}(p)$.

Now for $t \geq 2$ that is even we define a disk $\widehat{\mathsf{Z}}_{k,l,t}(p)$ from $\mathsf{Z}_{k,l,t}(p)$. Say that v_i' is the black vertex on $\partial\mathsf{Z}_{k,l,t}(p)$ that is on the transverse path from v_i emanating outwards from $\mathsf{B}_k(p)$. (Call this transverse path from v_i a *curvature ray*.) Also, say that w_i' is the endpoint of the central ray of $\mathsf{Z}_{k,l,t}(p)$ that contains w_i. Now since t is even, the black vertices on $\partial\mathsf{Z}_{k,l,t}(p)$ all have degree 2 except for $v_1', \ldots, v_n'$, which have degree 3 in $\mathsf{Z}_{k,l,t}(p)$. Let $l' = \min\{l-1, k-l\}$. Label the l' black vertices on $\partial\mathsf{Z}_{k,l,t}(p)$ in the clockwise direction from v_i' with $1, 2, \ldots, l'$ and do the same for the l' black vertices on $\partial\mathsf{Z}_{k,l,t}(p)$

in the counter-clockwise direction from v_i' (see the left-hand configuration in Figure 22 in which $k = 7$, $l = 3$, $t = 2$, and $l' = l - 1 = 2$).

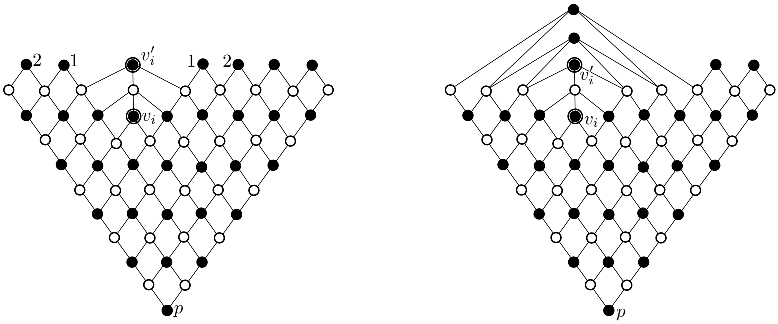

Figure 22: Constructing $\widehat{\mathsf{Z}}_{k,l,t}(p)$ with $k = 7$, $l = 3$, $t = 2$, and $l' = l - 1 = 2$.

Now identify the black vertices having the same labels as shown on the right in Figure 22. Repeat these identifications for each i. The resulting disk is $\widehat{\mathsf{Z}}_{k,l,t}(p)$. Note that $\widehat{\mathsf{Z}}_{k,l,t}(p)$ is a $(k+t)$-ball centered at p and that every vertex $v \neq p$ in the interior of $\widehat{\mathsf{Z}}_{k,l,t}(p)$ has degree 4 in $\widehat{\mathsf{Z}}_{k,l,t}(p)$ save for $v_1, \dots, v_n, v_1', \dots, v_n'$ which all have degree 3.

4.4.2 The three structures

Proposition 4.7. *Let $\mathsf{B}_k(p)$ be a standard k-disk in G with k odd and with exactly n degree-3 vertices $v_1, \dots, v_n$ of G appearing on $\partial\mathsf{B}_k(p)$. For each $t \geq 0$, if the only curvature vertices in $\mathsf{B}_{k+t-1}(p)$ are among $p, v_1, \dots, v_n$, then $\mathsf{B}_{k+t}(p)$ is either $\mathsf{Z}_{k,l,t}(p)$ or $\widehat{\mathsf{Z}}_{k,l,t}(p)$ where l is specified by the position of $v_1, \dots, v_n$ on $\partial\mathsf{B}_k(p)$.*

Proof. The proof will be by induction on t where the case for $t = 0$ is given by Proposition 4.1. Assuming for some $t \geq 1$ that the only curvature vertices in $\mathsf{B}_{k+t-1}(p)$ are among $p, v_1, \dots, v_n$ we now consider $\mathsf{B}_{k+t}(p)$. For $t = 1$, we already know that $\mathsf{B}_{k+t-1}(p) = \mathsf{B}_k(p)$ is a standard k-disk which is also $\mathsf{Z}_{k,l,0}(p)$. For $t \geq 2$, the induction hypothesis assumes that $\mathsf{B}_{k+t-1}(p)$ is either $\mathsf{Z}_{k,l,t-1}(p)$ or $\widehat{\mathsf{Z}}_{k,l,t-1}(p)$. However, while the only curvature vertices of $\mathsf{B}_{k+t-1}(p)$ are among $p, v_1, \dots, v_n$, in fact, $\widehat{\mathsf{Z}}_{k,l,t-1}(p)$ contains more curvature vertices than this. Hence $\mathsf{B}_{k+t-1}(p) = \mathsf{Z}_{k,l,t-1}(p)$.

By Proposition 2.4 every face of G not in $\mathsf{B}_{k+t-1}(p)$ but sharing an edge with $\partial\mathsf{B}_{k+t-1}(p)$ is in $\mathsf{B}_{k+t}(p)$. Consider the face-connected subsurface $B \subseteq \mathsf{B}_{k+t}(p)$ consisting of $\mathsf{B}_{k+t-1}(p)$ along with the faces not in $\mathsf{B}_{k+t-1}(p)$ but sharing an edge with $\partial\mathsf{B}_{k+t-1}(p)$. We will show that $B = \mathsf{Z}_{k,l,t}(p)$ or $B = \widehat{\mathsf{Z}}_{k,l,t}(p)$ and that $B = \mathsf{B}_{k+t}(p)$.

Given an edge e of $\partial\mathsf{B}_{k+t-1}(p) = \partial\mathsf{Z}_{k,l,t-1}(p)$, let f_e be the face of B that is not in $\mathsf{B}_{k+t-1}(p)$ and is incident to e. For comparison as a standard model, consider the disk $\mathsf{Z}_{k,l,t}(p)$ (separate from G) whose subdisk $\mathsf{Z}_{k,l,t-1}(p)$ is identified with $\mathsf{B}_{k+t-1}(p)$ in G. Let f_e' be the face of $\mathsf{Z}_{k,l,t}(p)$ that is not in $\mathsf{B}_{k+t-1}(p) = \mathsf{Z}_{k,l,t-1}(p)$ and is incident to e. If f_e (or f_e') is incident to a central ray, then call f_e (or f_e') a *radial face*; otherwise, call f_e (or f_e') a *notch face*. Note that $f_{e_1}' = f_{e_2}'$ if and only if f_{e_1}' is a notch face with e_1 and e_2 both incident to a common vertex that is saturated with respect to $\mathsf{B}_{k+t-1}(p)$ and G (see

Figure 21). We must show that the corresponding necessary and sufficient condition holds for $f_{e_1} = f_{e_2}$.

If f_e is a notch face, then f_e shares two consecutive edges (say e_1 and e_2) with $\partial \mathsf{B}_{k+t-1}(p)$ where the common endpoint of e_1 and e_2, call it v_{12}, is a vertex saturated by $\mathsf{B}_{k+t-1}(p)$ in G. It cannot be that a third edge of f_e is on $\partial \mathsf{B}_{k+t-1}(p)$ because such an edge would have to be consecutive with e_1 or e_2 on the cycle $\partial \mathsf{B}_{k+t-1}(p)$, whereas the two endpoints of e_1 and e_2 other than v_{12} have codegree 1 or 2 with respect to $\mathsf{B}_{k+t-1}(p)$.

If f_e is a radial face of B, then denote the edges of f_e by e_1, e_2, e_3, e_4 in rotational order along f_e. Assuming that e_1 is on $\partial \mathsf{B}_{k+t-1}(p)$ and is incident to a central ray of $\mathsf{B}_{k+t-1}(p)$ (call it r) we get that each endpoint of e_1 has positive codegree with respect to $\mathsf{B}_{k+t-1}(p)$. Thus e_2 and e_4 are not on the cycle $\partial \mathsf{B}_{k+t-1}(p)$. Without loss of generality assume that edge e_2 is the transverse continuation of r. Assume by way of contradiction that e_3 is on $\partial \mathsf{B}_{k+t-1}(p)$. Since e_2 and e_4 are not on $\partial \mathsf{B}_{k+t-1}(p)$, we again must have that e_3 is also incident to a central ray of $\mathsf{B}_{k+t-1}(p)$, call it r'. Note that $r \neq r'$ because $r = r'$ would imply that G is not simple, a contradiction. Now either e_1 and e_3 are in the same orbit under the rotational symmetry or not. If so, then the orbit of f_e under the rotational symmetry consists of $n/2$ faces and so there is a pseudofixed point in the interior of f_e, a contradiction. If not, then when adding f_e to $\mathsf{B}_{k+t-1}(p)$, the black and white bipartition forces there to be a half twist which creates a Möbius band in G, a contradiction.

The previous two paragraphs show that $f_e \mapsto f'_e$ is a one to one correspondence between the faces of B that are not in $\mathsf{B}_{k+t-1}(p)$ and the faces of $\mathsf{Z}_{k,l,t}(p)$ that are not in $\mathsf{B}_{k+t-1}(p)$. Furthermore, $f_e \mapsto f'_e$ takes notch faces to notch faces and radial faces to radial faces; also, if f_1 and f_2 are two consecutive faces of B, then their common vertex along $\partial \mathsf{B}_{k+t-1}(p)$, call it v, has codegree one or two and this determines whether or not f_1 and f_2 share an edge incident to v. Therefore B is obtained from $\mathsf{Z}_{k,l,t}(p)$ by making zero or more identifications along $\partial \mathsf{Z}_{k,l,t}(p)$. If there are no identifications, then we have that $B = \mathsf{Z}_{k,l,t}(p)$. We also get that $B = B_{k+t}(p)$ because no face outside of B can have a vertex of distance $k + t - 1$ from p and so we are done. So now, in Case 1 say that there are edges on $\partial \mathsf{Z}_{k,l,t}(p)$ that are identified and in Case 2 say that no edges along $\partial \mathsf{Z}_{k,l,t}(p)$ are identified but that there are vertices that are identified.

Case 1: Assume that e_1 and e_2 are on $\partial \mathsf{Z}_{k,l,t}(p)$ and are identified in going from $\mathsf{Z}_{k,l,t}(p)$ to B. If e_i is not incident to a central ray of $\mathsf{Z}_{k,l,t}(p)$, then e_i has one endpoint that is on $B_{k+t-1}(p) = \mathsf{Z}_{k,l,t-1}(p)$, is not a curvature vertex, and has degree 4 in $\mathsf{Z}_{k,l,t}(p)$. Any identification with another vertex of the same color would yield a vertex of degree more than four, a contradiction. Thus e_1 and e_2 are both incident to central rays of $\mathsf{Z}_{k,l,t}(p)$. Because the rotational symmetry has only two fixed points (i.e., the poles), the n endpoints of the central rays of $\mathsf{Z}_{k,l,t}(p)$ must either correspond to n distinct vertices in B or one vertex in B that is fixed under the n-fold rotational symmetry (that is, the other pole of the rotational symmetry, call it q). We assume the latter is true as this is the only way in which edges of $\partial \mathsf{Z}_{k,l,t}(p)$ may be identified. Now the $2n$ edges of $\partial \mathsf{Z}_{k,l,t}(p)$ incident to the central rays are identified as per the numbering in Figure 23 to obtain Z'.

There are two subcases to consider here: in Case 1.1 say that $2 \leq l \leq k-1$ and in Case 1.2 say that $l \in \{1, k\}$.

Case 1.1: Now $\partial Z'$ consists of n vertex-disjoint cycles. Since q is black, the black vertices on $\partial Z'$ all have degree 4 in Z' and the white vertices on $\partial Z'$ all have degree 2 in Z' except

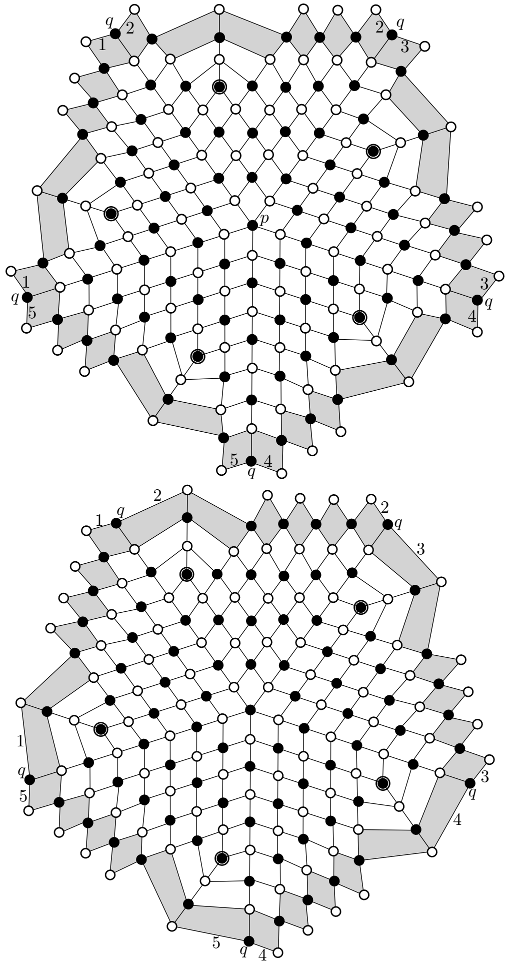

Figure 23: The disk Z' for $l = 2$ and $l = 1$, respectively.

for exactly $2n$ white vertices on $\partial Z'$ having degree 3 in Z': the n endpoints of the curvature rays along with the n neighbors of q. As shown at the beginning of Case 1, no two edges of $\partial Z'$ may be identified and so B is obtained from Z' by the identification of zero or more pairs of white vertices having degree 2 in Z' to obtain white vertices of degree 4 in G. We cannot, of course, identify white vertices from two distinct cycles of $\partial Z'$ because this would create a non-separating cycle in the embedding of G in the sphere, a contradiction.

Consider a cycle C in $\partial Z'$ with vertices $w_1, b_1, w_2, b_2, \ldots, w_m, b_m$ and say by way of contradiction that w_i and w_j are identified in going from Z' to B. Call the resulting face-connected subsurface after this identification Z''. Now w_i, b_i, b_{j-1} (and also w_i, b_{i-1}, b_j) all have degree four in Z''. Because $2 \leq l \leq k-1$, we now get that these three vertices are on a common face of G and so w_{i+1} and w_{j-1} (and also w_{i-1} and w_{j+1}) must be identified in going from Z'' to B. This identification process will continue and eventually yield a contradiction by either: trying to identify a white vertex of degree 3 with a white vertex of degree 2 or 3, by creating a face of length four having a white vertex of positive codegree on its boundary, or by creating a cycle of length 2.

Thus we must have that $Z' = B$; however, this is not possible for the following reason. Consider a cycle C in ∂B and let P be the disk of G with $\partial P = C$. The black vertices on C have degree 2 in P, the white vertices on C have degree 4 in P save for two of them which have degree 3 in P. By the rotational symmetry, the interior vertices of P include exactly one black degree-3 vertex and all other interior vertices have degree 4 in P. Thus P has an odd number of vertices of odd degree, a contradiction.

Case 1.2: As in Case 1.1, $\partial Z'$ consists of n vertex-disjoint cycles and the black vertices on $\partial Z'$ all have degree 4 in Z'. However, in this case, the white vertices on $\partial Z'$ all have degree 2 in Z' except for the n white vertices adjacent to q which have degree 4 in Z'. Let C be one cycle in $\partial Z'$. As in Case 1.1, no two edge of C may be identified in going from Z' to B. Label the vertices in rotational order along C with $w_1, b_1, \ldots, w_m, b_m$ where w_1 is the white vertex of C having degree 4 in Z'. Since b_m, w_1, b_1 all have degree 4, they must be on the same face of G and so we must have that $w_m = w_2$. This in turn implies that $w_{m-1} = w_3$, etc. These identifications are not possible, however, because $m = 2k-2$ and k is even and thus these identifications would create a face of length 2, a contradiction.

Case 2: In this case, the only possible identifications along $\partial \mathsf{Z}_{k,l,t}(p)$ in going from $\mathsf{Z}_{k,l,t}(p)$ to B are pairs of vertices which have degree 2 in $\mathsf{Z}_{k,l,t}(p)$. Say that the vertices on $\partial \mathsf{Z}_{k,l,t}(p)$ of distance $k+t+1$ from p have color κ and the vertices of distance $k+t$ have color λ. Hence $\{\kappa, \lambda\} = \{\text{black}, \text{white}\}$ (e.g., in Figure 21 $\kappa = $ white). The vertices of color κ on $\partial \mathsf{Z}_{k,l,t}(p)$ have degree 2 in $\mathsf{Z}_{k,l,t}(p)$ save for the n endpoints of the curvature rays (which have degree 3 in $\mathsf{Z}_{k,l,t}(p)$) and the vertices of color λ on $\partial \mathsf{Z}_{k,l,t}(p)$ have degree 4 in $\mathsf{Z}_{k,l,t}(p)$ save for the n endpoints of the central rays (which have degree 3 in $\mathsf{Z}_{k,l,t}(p)$).

Label the vertices along $\partial \mathsf{Z}_{k,l,t}(p)$ in rotational order with $\lambda_1, \kappa_1, \ldots, \lambda_m, \kappa_m$ in which λ_1 is the endpoint of a central ray. Say that κ_i and κ_j are identified in B, and say that $v_1, \ldots, v_n$ is the orbit of κ_i under the rotational symmetry and $u_1, \ldots, u_n$ the orbit of κ_j. Then $|\{u_1, \ldots, u_n, v_1, \ldots, v_n\}| = 2n$ in $\mathsf{Z}_{k,l,t}(p)$ and $|\{u_1, \ldots, u_n, v_1, \ldots, v_n\}| = n$ or 1 in G. It cannot be that $|\{u_1, \ldots, u_n, v_1, \ldots, v_n\}| = 1$ in G because then these $2n$ degree-2 vertices in $\mathsf{Z}_{k,l,t}(p)$ would then identify to one vertex of degree $4n$ in G, a contradiction. Thus $|\{u_1, \ldots, u_n, v_1, \ldots, v_n\}| = n$ in G. Because of the rotational symmetry these two orbits of vertices must alternate along the cycle $\partial \mathsf{Z}_{k,l,t}(p)$ and because G is spherical, identified pairs of vertices (e.g., κ_i and κ_j) must appear consecutively along $\partial \mathsf{Z}_{k,l,t}(p)$.

Let γ_{ij} be the $\kappa_i\kappa_j$-path along $\partial \mathsf{Z}_{k,l,t}(p)$ which contains no other vertices from $v_1, \ldots, v_n, u_1, \ldots, u_n$. Again, because of the rotational symmetry, at most one endpoint of a curvature ray and at most one endpoint of a central ray occurs on γ_{ij}. Suppose that λ_{i+1} and λ_j are the neighbors of $\kappa_i = \kappa_j$ on γ_{ij}. If λ_{i+1} and λ_j both have degree 4 in $\mathsf{Z}_{k,l,t}(p)$, then κ_i, λ_{i+1}, and λ_j must all be on the same face of G and so we must have that $\kappa_{i+1} = \kappa_{j-1}$ in B. Similarly if λ_i and λ_{j+1} both have degree 4 in $\mathsf{Z}_{k,l,t}(p)$, then $\kappa_{i-1} = \kappa_{j+1}$ in B. These identifications of degree-2, κ-colored vertices must continue in each direction along $\partial \mathsf{Z}_{k,l,t}(p)$ until either we reach the endpoint of a curvature ray or central ray. Thus γ_{ij} contains either the endpoint of a curvature ray or the endpoint of a central ray, but not both.

We claim that γ_{ij} contains the endpoint of a curvature ray and not the endpoint of a central ray. This is because if the latter were true, then, because the endpoint of a central ray is of color λ, this identification of κ-colored vertices along γ_{ij} would end with either a face of length two (a contradiction) or a face of length four containing a κ-colored vertex of degree 2 (again a contradiction).

Now since the endpoint of a curvature ray is contained in γ_{ij}, the identification of κ-colored vertices along γ_{ij} ends with the identification of some κ_{a-1} and κ_{a+1} where κ_a is the endpoint of the curvature ray and so has degree 3 in G. Also note that this implies that κ_a is the midpoint of γ_{ij}. Identifications of κ-colored vertices from $\kappa_i = \kappa_j$ that are off of γ_{ij} must then stop at the endpoint of the central rays. These identifications of vertices in $\mathsf{Z}_{k,l,t}(p)$ result in the disk $\widehat{\mathsf{Z}}_{k,l,t}(p)$. These are the only vertex identifications that can happen in going from $\mathsf{Z}_{k,l,t}(p)$ to B because we started with an arbitrary vertex identification. Thus $B = \widehat{\mathsf{Z}}_{k,l,t}(p)$ and we must also have that $B = \widehat{\mathsf{Z}}_{k,l,t}(p) = \mathsf{B}_{k+t}(p)$ because any face of G outside of $\widehat{\mathsf{Z}}_{k,l,t}(p)$ cannot contain a vertex of distance $k+t-1$ from p. □

Proposition 4.8 gives us the three possible structures for $B_{k+t}(p)$. The proof of Proposition 4.8 is similar to the proof of Proposition 4.2 using Proposition 4.7 in the place of Proposition 4.1.

Proposition 4.8. *Let $k \geq 1$ be odd and let O_1 be one orbit of n degree-3 curvature vertices contained in $\partial \mathsf{B}_k(p)$. Let O_2 be the second orbit of n degree-3 curvature vertices of G. Let $t \geq 0$ be such that $B_{k+t-1}(p)$ contains no curvature vertices aside from $O_1 \cup \{p\}$ and $\mathsf{B}_{k+t}(p)$ contains O_2. One of the following holds.*

(1) *If t is odd (that is, $k+t$ is even), then $\mathsf{B}_{k+t}(p) = \mathsf{Z}_{k,l,t}(p)$ and the vertices of O_2 are the endpoints of the central rays of B_{k+t}.*

(2) *If t is even (that is, $k+t$ is odd), then either*

   (a) *$\mathsf{B}_{k+t}(p) = \widehat{\mathsf{Z}}_{k,l,t}(p)$ and the vertices of O_2 are the endpoints of the curvature rays and appear in the interior of $\mathsf{B}_{k+t}(p)$, or*

   (b) *$\mathsf{B}_{k+t}(p) = \mathsf{Z}_{k,l,t}(p)$ and the vertices of O_2 are on $\partial\mathsf{B}_{k+t}(p)$ but not the endpoints of the central rays or curvature rays.*

Proposition 4.9. *If $\mathsf{B}_{k+t}(p) = \mathsf{Z}_{k,l,t}(p)$ is as given in* Proposition 4.8(2)(b), *then G is given by the Four-Parameter Construction with uniquely determined parameters.*

Proof. Consider two consecutive central rays of $\mathsf{Z}_{k,l,t}(p)$ along with the vertices, edges and faces of $\mathsf{Z}_{k,l,t}(p)$ between these central rays, that is, the closure of a fundamental region of the rotational symmetry, call it F. A rendering of F is shown on the left in Figure 24 in which the dashed lines have length t (with $t = 0$ a possibility) and are identified.
Let $o_1 \in O_1$ and $o_2 \in O_2$ be the vertices of $O_1 \cup O_2$ in F. Since o_2 is not on the endpoint of the central ray or curvature ray, o_2 appears on $\partial\mathsf{Z}_{k,l,t}(p)$ in one of the two circled areas shown on the left of the figure; take the right of Figure 24 as an illustration. We may assume without loss of generality that o_2 is in the upper region because if o_2 is in the lower circled region on the left of Figure 24, then we may reflect $\mathsf{Z}_{k,l,t}(p)$ to get $\mathsf{Z}_{k,k-l+1,t}(p)$ and then have o_2 in the upper region.

Now take the fundamental region F' adjacent to and in the counterclockwise direction from F. The rendering of $F \cup F'$ shown in Figure 25 is geometrically flat and so we coordinatize in the obvious way with p at $(0,0)$. As a result, the grey lines shown in Figure 25

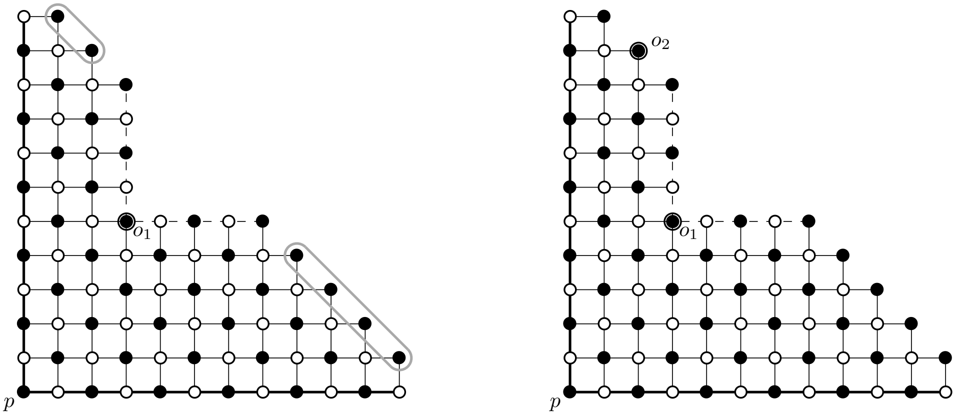

Figure 24: The fundamental region F.

form a quadrilateral with a right angle at the origin. Evidently the quadrilateral is also convex because each of the interior angles is less than $180°$. Hence the grey quadrilateral is of the type used in the Four-Parameter Construction; furthermore, the parameters defining this quadrilateral are uniquely determined by the positions of o_1 and o_2 within F. Hence $\mathsf{Z}_{k,l,t}(p)$ contains n of these special quadrilaterals at p.

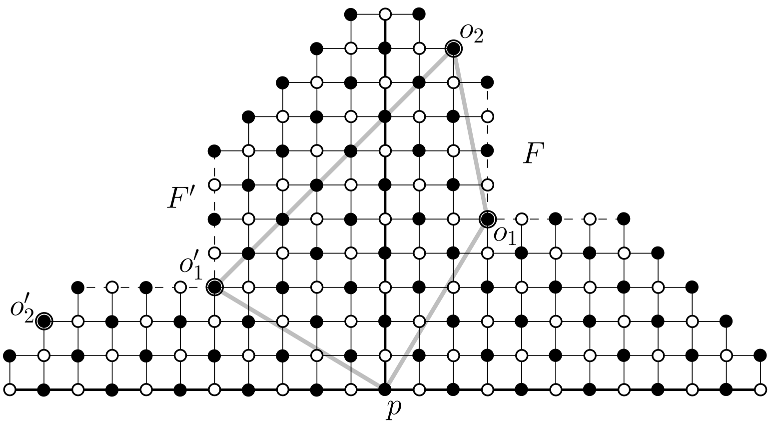

Figure 25: Two fundamental regions F and F' along with a four-parameter special integer quadrilateral.

Conversely, by Proposition 4.7, the entire Four-Parameter Construction using this quadrilateral must contain $\mathsf{Z}_{k,l,t}(p)$ because the positions of the curvature vertices in the Four-Parameter Construction come from the corners of the quadrilaterals. So one possibility for G is given by the Four-parameter construction. Now, we will show that there is only one spherical quadrangulation which contains $\mathsf{Z}_{k,l,t}(p)$ and has O_2 in this position on $\partial\mathsf{Z}_{k,l,t}(p)$, which will complete our proof.

Consider the vertices on the boundary $\partial\mathsf{Z}_{k,l,t}(p)$. The black vertices on this boundary have degree 2 in $\mathsf{Z}_{k,l,t}(p)$ save for the n endpoints of the curvature rays which have degree 3. The white vertices on this boundary have degree 4 in $\mathsf{Z}_{k,l,t}(p)$ save for the n endpoints of the central rays which have degree 3. Let $w_1, b_1, c_1, \dots, w_n, b_n, c_n$ be vertices in clockwise rotational order on $\partial\mathsf{Z}_{k,l,t}(p)$ in which the w_i's are the endpoints of the central rays, b_i's the vertices of O_2, and c_i's the endpoints of the curvature rays.

Now let D be the disk of G constructed from the faces of G not in $\mathsf{Z}_{k,l,t}(p)$. Thus D contains the pole $q \neq p$ and $\partial D = \partial\mathsf{Z}_{k,l,t}(p)$; furthermore, by rotational symmetry the pole q is in the interior of D. Now,

- the w_i's, b_i's, and c_i's all have degree 3 in D,
- the remaining black vertices on ∂D have degree 4 in D, and
- the remaining white vertices on ∂D have degree 2 in D.

By Proposition 2.8, we may consider D as a subgraph of a standard r-disk S_r with a black central vertex, call it q_0, for some large enough value of r. Of course the embedding of D must have q corresponding to q_0 and, since q is in the interior of D, the central rays of D must lie on the central rays of S_r. Thus the embedding of D in S_r is unique up to dihedral symmetry. For the uniqueness of D as a completion of G, we need to show that there is no other disk D' in S_r having n-fold rotational symmetry around q with a bijection between the vertices of $\partial D'$ and ∂D which respects degrees and cyclic ordering.

Consider the black vertices of S_r and connect pairs of black vertices on the same face with an edge (say it is also black). This black graph is a quadrangulation with every internal vertex of degree 4 aside from q which has degree n. Call any transverse path in the black graph a *black diagonal path* of S_r or D. Call the n black diagonal paths of S_r or D that originate from q the *diagonal rays* of S_r or D.

Now consider the boundary faces of D and the black diagonal edges in each. These black edges form a cycle, call it C, in the black-diagonal graph and C is contained entirely inside the disk D. Note that the cycle C consists of black diagonal paths whose endpoints are the c_i's, b_i's, and w_i''s where w_i' is the black neighbor of w_i that is not in $\mathsf{Z}_{k,l,t}(p)$. Traversing C in S_r with q to our right, the c_i's and b_i's represent a right turn rather than a transverse path and the w_i''s represent a left turn. We will now show that C (and hence the boundary faces of D) is uniquely determined by the positions of c_i's, b_i's, and w_i''s on ∂D. This will imply the uniqueness of D.

Now let V be the region of S_r between and including two consecutive diagonal rays, call them Y_1 and Y_2 in the clockwise direction. The intersection of C with Y_1 has one or more connected components, each of which is either an isolated vertex or a path of positive length. If there is no path of positive length, then let y_1 be some vertex of C on Y_1. If there is a path of positive length in the intersection, then let y_1 be the last vertex of some intersection path when traversing C in the clockwise direction. Let y_2 be the corresponding vertex on Y_2 under the rotational symmetry in the clockwise direction, and let P be the y_1y_2-path in C in the clockwise direction. Consider the square Q in V given by the black diagonals of S_r shown in Figure 26.
In the clockwise traversal of C, P contains two right turns and one left turn and at the rest of the vertices of P, a transverse crossing. The sequence of turns is either left-right-right, right-left-right, or right-right-left; however, if necessary we can reflect R around the axis Y_1 and reverse the traversal of C so that the first turn is right. In Figure 26, V is rendered as part of the standard 4×4 grid in the xy-plane between the perpendicular lines $y = x$

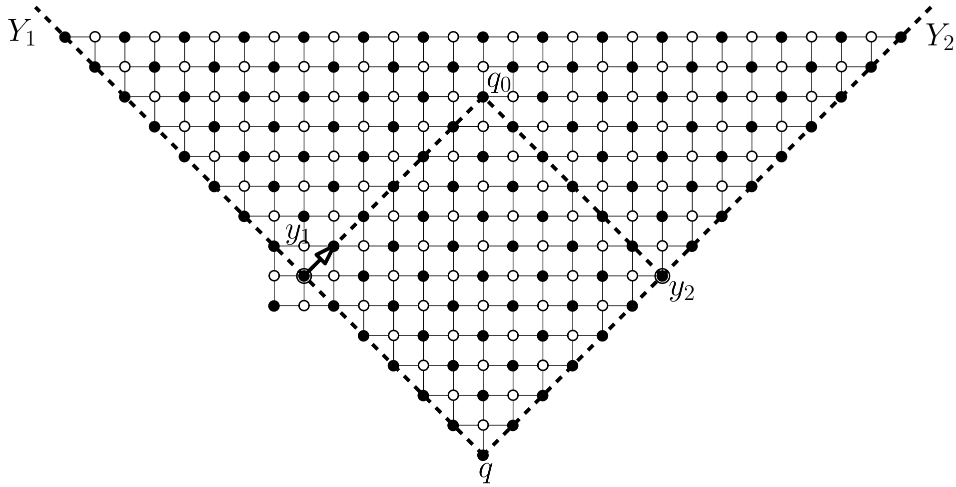

Figure 26: The square defined by Y_1, y_1, Y_2, and y_2 in V.

and $y = -x$. In the traversal of P, two right turns and a left turn yield a net change of $90°$ degrees in the clockwise direction. Because of the way in which y_1 is chosen, the first edge of P is in the direction of the arrow shown in Figure 26. Now the path P is in V and is completely determined by the placement of the two right turns in the 3-turn sequence. In Figure 27 we have three examples of P. Since these turns are determined by the placements of the c_i's, b_i's, and w_i's on ∂D, there is only one possibility for P and so for C and hence for D. □

Proposition 4.10. *If* $\mathsf{B}_{k+t}(p) = \widehat{\mathsf{Z}}_{k,l,t}(p)$ *is as given in* Proposition 4.8(2)(a)*, then* G *is given by the Four-Parameter Construction with uniquely determined parameters.*

Proof. As in the proof of Proposition 4.9, consider two adjacent fundamental regions F and F' of $\widehat{\mathsf{Z}}_{k,l,t}(p)$ between three consecutive central rays. These may be rendered in a geometrically flat fashion as in Figure 28 with identically labeled vertices being identified in G and appropriate identifications of dashed edges. Note that the path of dashed edges has positive length. Since the rendering is flat we have a special integer quadrilateral as used in the Four-Parameter Construction with parameters uniquely determined by k, l, and t as shown in the figure. Therefore the Four-Parameter Construction yields one possibility for G. In order to show that this is the only possibility for G, we will show that there is only one possibility for the disk in G around q sharing its boundary with $\widehat{\mathsf{Z}}_{k,l,t}(p)$.

Consider the dashed edge shown in Figure 29 along with its orbit of n edges under the rotational symmetry. Let $\widetilde{Z}$ be the disk around p consisting of $\widehat{\mathsf{Z}}_{k,l,t}(p)$ along with the n faces bounded by these n edges and $\widehat{\mathsf{Z}}_{k,l,t}(p)$.

Let D be the disk defined by the faces of G not contained in $\widetilde{Z}$. Note that D contains q in its interior and $\partial D = \partial\widetilde{Z}$. All of the white vertices of $\partial D = \partial\widetilde{Z}$ have degree 4 in $\widetilde{Z}$ and degree 2 in D. Among the black vertices of $\partial D = \partial\widetilde{Z}$, n have degree 3 in both $\widetilde{Z}$ and D and the rest have degree 2 in $\widetilde{Z}$ and degree 4 in D.

Say that l is the smallest distance in G from the pole q to any vertex on $\partial D = \partial\widetilde{Z}$. Let u be one such vertex on the common boundary. It must be that $\mathsf{d}(u,p) = k + t + 1$ rather

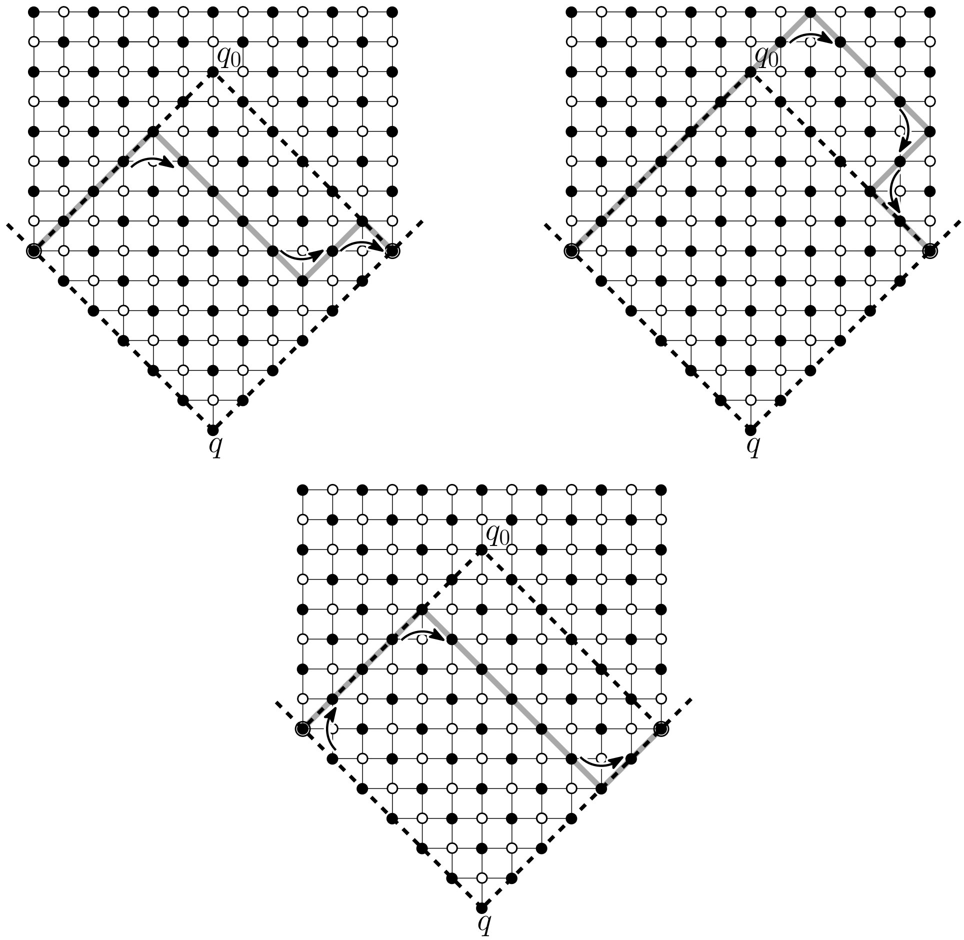

Figure 27: Three examples of the path P. In the third example, B intersects Y_1 in a path.

than $k+t$. This is because the vertices of $\partial B_{k+t}(p)$ of distance $k+t$ from p are all white and are saturated by $\widetilde{Z}$ and so any path from q to one of these vertices of distance $k+t$ from p must go through the vertices of distance $k+t+1$ from p. Therefore u is black and has degree 2 or 3 in $\widetilde{Z}$.

Since u is black, l must be even. First suppose that u can be chosen to be in $\mathsf{B}_{l-1}(q)$ (which by Proposition 4.1 is a standard $(l-1)$-disk); that is, u is a vertex of degree 2 on the boundary of $\mathsf{B}_{l-1}(q)$. Since the white vertices of $\partial\mathsf{B}_{l-1}(q)$ have distance $l-1$ from q and l is the smallest distance of a vertex from q to $\widetilde{Z}$, these white vertices on $\partial\mathsf{B}_{l-1}(q)$ are not in $\widetilde{Z}$. So any black vertex on $\partial\mathsf{B}_{l-1}(q)$ which is identified to a black vertex on $\partial\widetilde{Z}$ forces another identification of two black boundary vertices. Eventually these identifications will run to the degree-3 vertices of $\widetilde{Z}$ on $\partial\widetilde{Z}$. But this forces these black vertices to have degree 5 in G, a contradiction.

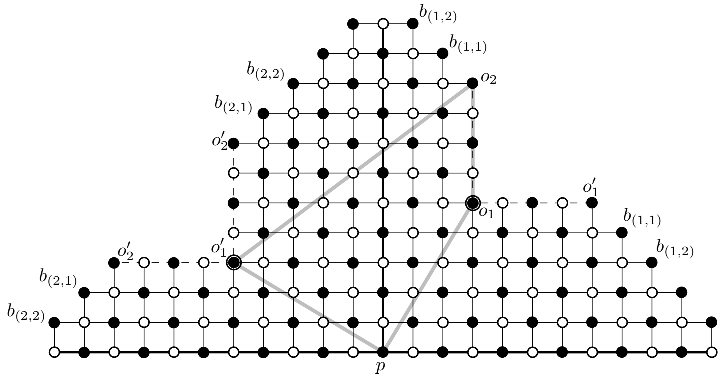

Figure 28: A flat rendering of two fundamental regions for $\widehat{\mathsf{Z}}_{k,l,t}(p)$.

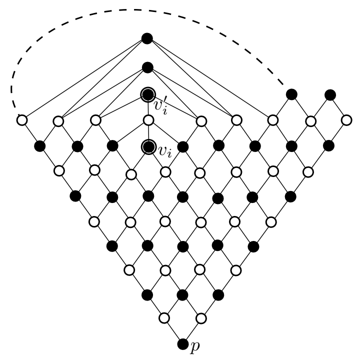

Figure 29: There are n edges in G with endpoints in $\widehat{\mathsf{Z}}_{k,l,t}(p)$. One such edge is shown as a dashed curve.

Thus u is a vertex of degree 3 on the boundary of $\mathsf{B}_l(q)$; that is, u is the endpoint of a central ray of $\mathsf{B}_l(q)$ which by Proposition 4.1 is a standard l-disk. Degree considerations now force u and the vertices in its orbit to be identified with the degree-3 vertices of $\widetilde{Z}$ on $\partial\widetilde{Z}$. From here $\partial\mathsf{B}_l(q)$ must then be identified with $\partial\widetilde{Z}$. □

Proposition 4.11. *If* $\mathsf{B}_{k+t}(p) = \mathsf{Z}_{k,l,t}(p)$ *is as given in* Proposition 4.8(1)*, then* G *is given by the Four-Parameter Construction with uniquely determined parameters.*

Proof. Again, as in the proof of Proposition 4.9, consider two adjacent fundamental regions F and F' of $\mathsf{Z}_{k,l,t}(p)$ between three consecutive central rays rendered in a geometrically flat fashion as in Figure 30. Again we have a special integer quadrilateral contained in $F \cup F'$ as shown in the figure with uniquely determined parameters. Therefore the Four-Parameter Construction yields one possibility for G, and we will now show that there is only one possibility for the disk in G around q sharing its boundary with $\mathsf{Z}_{k,l,t}(p)$.

Let m be the shortest distance from q to a vertex u on $\partial\mathsf{Z}_{k,l,t}(p)$. Since all of the black vertices on $\partial\mathsf{Z}_{k,l,t}(p)$ are saturated by $\mathsf{Z}_{k,l,t}(p)$, it must be that u is white and hence m is odd. By Proposition 4.1 and the definition of m, $\mathsf{B}_m(q)$ is a standard m-disk. If u is not the endpoint of a central ray of $\mathsf{B}_m(q)$, then u is saturated by $\mathsf{B}_m(q)$. Since u has degree

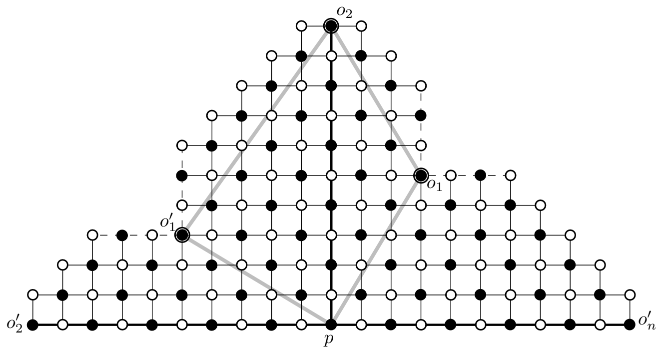

Figure 30: A flat rendering of two fundamental regions for $\mathsf{Z}_{k,l,t}(p)$.

4 in G, it must be that u has degree 2 in $\partial\mathsf{Z}_{k,l,t}(p)$ and that the boundary edges incident to u in $\mathsf{B}_m(q)$ are identified to the boundary edges incident to u in $\partial\mathsf{Z}_{k,l,t}(p)$. The only boundary edges of both disks that are left are those incident to the central rays of $\mathsf{B}_m(q)$ and the curvature rays of $\partial\mathsf{Z}_{k,l,t}(p)$. Degree considerations and the fact that all faces must have length 4 now force all edges of $\partial\mathsf{B}_m(q)$ to be identified to all edges of $\partial\mathsf{Z}_{k,l,t}(p)$. $\square$

ORCID iDs

Lowell Abrams https://orcid.org/0000-0002-8174-5957
Daniel Slilaty https://orcid.org/0000-0002-7918-3641

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