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On the Hamilton-Waterloo problem: the case of two cycles sizes of different parity*

Melissa S. Keranen
Michigan Technological University, Department of Mathematical Sciences, Houghton, MI 49931, USA

Adrián Pastine
Instituto de Matemática Aplicada San Luis (IMASL), Universidad Nacional de San Luis, CONICET, Ejército de los Andes 950 (D5700HHW), San Luis, Argentina

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Abstract

The Hamilton-Waterloo problem asks for a decomposition of the complete graph of order v into r copies of a 2-factor F_1 and s copies of a 2-factor F_2 such that $r+s = \left\lfloor \frac{v-1}{2} \right\rfloor$. If F_1 consists of m-cycles and F_2 consists of n cycles, we say that a solution to (m,n)-$\mathrm{HWP}(v;r,s)$ exists. The goal is to find a decomposition for every possible pair (r,s). In this paper, we show that for odd x and y, there is a solution to $(2^k x, y)$-$\mathrm{HWP}(vm;r,s)$ if $\gcd(x,y) \geq 3$, $m \geq 3$, and both x and y divide v, except possibly when $1 \in \{r,s\}$.

Keywords: 2*-factorizations, Hamilton-Waterloo problem, Oberwolfach problem, cycle decomposition, resolvable decompositions.*

Math. Subj. Class.: 05C51, 05C70

1 Introduction

The Oberwolfach problem asks for a decomposition of the complete graph K_v into $\frac{v-1}{2}$ copies of a 2-factor F. To achieve this decomposition, v needs to be odd, because the vertices must have even degree. The problem with v even asks for a decomposition of K_v into $\frac{v-2}{2}$ copies of a 2-factor F, and one copy of a 1-factor. The uniform Oberwolfach problem (all cycles of the 2-factor have the same size) has been completely solved by

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E-mail addresses: msjukuri@mtu.edu (Melissa S. Keranen), adrian.pastine.tag@gmail.com (Adrián Pastine)

Alspach and Haggkvist [1] and Alspach, Schellenberg, Stinson and Wagner [2]. The non-uniform Oberwolfach problem has been studied as well, and a survey of results up to 2006 can be found in [8]. Furthermore, one can refer to [6, 7, 9, 23, 24] for more recent results.

In [19] Liu first worked on the generalization of the Oberwolfach problem to equipartite graphs. He was seeking to decompose the complete equipartite graph $K_{(m:n)}$ with n partite sets of size m each into $\frac{(n-1)m}{2}$ copies of a 2-factor F. For such a decomposition to exist $(n-1)m$ has to be even. In [14] Hoffman and Holliday worked on the equipartite generalization of the Oberwolfach problem when $(n-1)m$ is odd, decomposing into $\frac{(n-1)m-1}{2}$ copies of a 2-factor F, and one copy of a 1-factor. The uniform Oberwolfach problem over equipartite graphs has since been completely solved by Liu [20] and Hoffman and Holliday [14]. For the non-uniform case, Bryant, Danziger and Pettersson [7] completely solved the case when the 2-factor is bipartite. In particular, Liu showed the following.

Theorem 1.1 ([20]). *For $m \geq 3$ and $u \geq 2$, $K_{(h:u)}$ has a resolvable C_m-factorization if and only if hu is divisible by m, $h(u-1)$ is even, m is even if $u = 2$, and $(h, u, m) \notin \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$.*

The Hamilton-Waterloo problem is a variation of the Oberwolfach problem, in which we consider two 2-factors, F_1 and F_2. It asks for a factorization of K_v when v is odd or $K_v - I$ (I is a 1-factor) when v is even into r copies of F_1 and s copies of F_2 such that $r+s = \left\lfloor \frac{v-1}{2} \right\rfloor$, where F_1 and F_2 are two 2-regular graphs on v vertices. Most of the results for the Hamilton-Waterloo problem are uniform, meaning F_1 consists of cycles of size m (C_m-factors), and F_2 consists of cycles of size n (C_n-factors). If there is a decomposition of K_v into r C_m-factors and s C_n-factors we say that a solution to (m, n)-HWP$(v; r, s)$ exists. The case where both m and n are odd positive integers and v is odd is almost completely solved by [11, 12]; and if m and n are both even, then the problem again is almost completely solved (see [5, 6]). However, if m and n are of differing parities, then we only have partial results. Most of the work has been done in the case where one of the cycle sizes is constant. The case of $(m, n) = (3, 4)$ is solved in [4, 13, 21, 25]. Other cases which have been studied include $(m, n) = (3, v)$ [18], $(m, n) = (3, 3x)$ [3], and $(m, n) = (4, n)$ [16, 21] .

In this paper, we consider the case of m and n being of different parity. This case has gained attention recently, where it has been shown that the necessary conditions are sufficient for a solution to (m, n)-HWP$(v; r, s)$ to exist whenever $m \mid n$, $v > 6n > 36m$, and $s \geq 3$ [10]. We provide a complementary result to this in our main theorem, which covers cases in which $m \nmid n$ and solves a major portion of the problem.

Theorem 1.2. *Let x, y, v, k and m be positive integers such that:*

(i) $v, m \geq 3$,

(ii) x, y *are odd,*

(iii) $\gcd(x, y) \geq 3$,

(iv) x *and* y *divide* v,

(v) 4^k *divides* v.

Then there exists a solution to $(2^k x, y)$-HWP$(vm; r, s)$ for every pair r, s with $r + s = \lfloor (vm-1)/2 \rfloor$, $r, s \neq 1$.

2 Preliminaries

The *complete cyclic multipartite graph* $C_{(x:n)}$ is the graph with n partite sets of size x, where two vertices (g,i) and (h,j) are neighbors if and only if $i-j=\pm 1 \pmod n$, with subtraction being done modulo n. The *directed complete cyclic multipartite graph* $\overrightarrow{C}_{(x:n)}$ is the graph with n parts of size x, with arcs of the form $\big((g,i),(h,i+1)\big)$ for every $0 \le g,h \le x-1$, $0 \le i \le n-1$.

One of the main tools in [17] is a Lemma that combines decompositions of $C_{(x:k)}$ to obtain decompositions of $K_{(v:m)}$. We present a version of the Lemma for uniform decompositions, as those are the focus of this manuscript.

Lemma 2.1 ([17])**.** *Let m, x, y, and v be positive integers. Let $s_1,\dots,s_{\frac{m-1}{2}}$ be non-negative integers. Suppose the following conditions are satisfied:*

- *There exists a decomposition of K_m into C_n-factors.*
- *For every $1 \le t \le \frac{m-1}{2}$ there exists a decomposition of $C_{(v:n)}$ into s_t C_{xn}-factors and r_t C_{yn}-factors.*

Let

$$s=\sum_{t=1}^{\frac{(m-1)}{2}} s_t \qquad \text{and} \qquad r=\sum_{t=1}^{\frac{(m-1)}{2}} r_t.$$

Then there exists a decomposition of $K_{(v:m)}$ into s C_{xn}-factors and r C_{yn}-factors.

In order to decompose $\overrightarrow{C}_{(x:n)}$, x and n odd, into C_n-factors and C_{xn}-factors, the authors of [17] labeled the vertices by $\mathbb{Z}_x \times \{0,\dots,n-1\}$. They build a 2-factor F by providing n permutations of G. The ith permutation is used to connect vertices in column $i-1$ to vertices in column i, in particular the n-th permutation is used to connect vertices in column $n-1$ to vertices in column 0. It must be said that these permutations were used implicitly in [17], as no permutation language was used for this part of the construction.

Notice that in general, if the columns are labeled by an abelian group G, f is the ith permutation and $g \in G$, in the 2-factor F, vertex $(g,i-1)$ is connected to vertex $(f(g),i)$. Let $\mathcal{F}$ be the composition of all n permutations of the 2-factor F, such that $(\mathcal{F}(g),0)$ is the vertex at which we finish if we start at vertex $(g,0)$ and move through F until we reach column 0 again. In the constructions in [17], G is abelian, and $g-\mathcal{F}(g)$ depends only on $\mathcal{F}$ and not on g. If this is the case, the length of the cycles of F is n times the order of the element $g-\mathcal{F}(g)$.

Lemma 2.2. *Assume F is a 2-factor built with the permutation $\mathcal{F}$, and $g-\mathcal{F}(g)$ depends only on $\mathcal{F}$. If q is the order of $g-\mathcal{F}(g)$, then F is a $\overrightarrow{C}_{qn}$-factor of $\overrightarrow{C}_{(xy4^k:n)}$.*

As we will need to use the permutations of $\mathbb{Z}_x$, we will introduce them. For $\alpha \in \mathbb{Z}_x$, let f_α be the permutation that adds α to every element of $\mathbb{Z}_x$, i.e. $f_\alpha(g)=g+\alpha$. Let $H(\alpha,\beta)$ be the 2-factor made with the following permutations:

- f_α from column $i-1$ to column i if $1 \le i \le n-3/2$;
- $f_{-\alpha}$ from column $i-1$ to column i if $n-3/2+1 \le i \le n-3$;
- f_α from column $n-3$ to column $n-2$;

- $f_{-2\alpha}$ from column $n-2$ to column $n-1$ (this is a permutation because x is odd);
- f_β from column $n-1$ to column 0.

In [17] the first $n-3$ permutations were different, but the end result was the same. Notice that $\mathcal{F}(g) = g - \alpha + \beta$. For every $r \in \{0, 1, 2, \ldots, x-3, x-2, x\}$, the authors of [17] gave permutations ϕ of $\mathbb{Z}_x$ that satisfied:

(a) $\phi(\alpha) = \alpha$ for r elements of $\mathbb{Z}_x$;

(b) $\gcd(\alpha - \phi(\alpha), x) = 1$ for the remaining $x - r$ elements of $\mathbb{Z}_x$.

Then, the decomposition of $\overrightarrow{C}_{(x:n)}$ was given by the 2-factors $H(\alpha, \phi(\alpha))$, $\alpha \in \mathbb{Z}_x$.

In order for such a decomposition to work, for every $\alpha, \beta \in \mathbb{Z}_x$ the permutations f_α, f_β needed to satisfy $f_\alpha = f_\beta$ if and only if $\alpha = \beta$, as otherwise some arcs would be repeated in the factor $H(\alpha, \phi(\alpha))$ and the factor $H(\beta, \phi(\beta))$.

Then, in [17], decompositions of $\overrightarrow{C}_{(x:n)}$, $\overrightarrow{C}_{(y:n)}$, and $\overrightarrow{C}_{(4:n)}$ were combined using a graph product and permutations of $\mathbb{Z}_x \times \mathbb{Z}_y \times \mathbb{Z}_4$ to decompose $\overrightarrow{C}_{(4xy:n)}$. Instead of doing so, we will use group products to label the vertices of $\overrightarrow{C}_{(4^k xy:n)}$, although we will make use of permutations of the group product.

In Section 3, we give permutations of $\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$, and show that they satisfy the necessary conditions to be used for decompositions. In Section 4, we use multivariate bijections to give decompositions of $\overrightarrow{C}_{(4^k xy:n)}$ into $\overrightarrow{C}_{2^k xk}$-factors and $\overrightarrow{C}_{yk}$-factors. Finally, in Section 5, we use these decompositions to prove our main results.

3 The permutation $f_\alpha(a, b)$ of $\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$

Consider the group $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$, an element $\alpha = (\alpha_1, \alpha_2)$ and the function $f_\alpha(a, b) = (-b + \alpha_1, a - b + \alpha_2)$.

Lemma 3.1. *f_α is a permutation of G.*

Proof. As $|G|$ is finite, it is enough to prove that f_α is an injective function.

Assume $f_\alpha(a, b) = f_\alpha(c, d)$. Then

$$(-b + \alpha_1, a - b + \alpha_2) = (-d + \alpha_1, c - d + \alpha_2).$$

The equality $-b + \alpha_1 = -d + \alpha_1$ implies $b = d$. Using $b = d$, the equality $a - b + \alpha_2 = c - d + \alpha_2$ implies $a = c$. Therefore, f_α is a permutation of G. □

Lemma 3.2. $f_\beta(f_\alpha^2(a, b)) = (a, b) - \alpha + \beta$.

Proof. We will prove this lemma by computing $f_\beta(f_\alpha^2(a, b))$.

$$\begin{aligned}
f_\alpha(a, b) &= (-b + \alpha_1, a - b + \alpha_2) \\
f_\alpha^2(a, b) &= f(-b + \alpha_1, a - b + \alpha_2) \\
&= (-a + b - \alpha_2 + \alpha_1, -b + \alpha_1 - a + b - \alpha_2 + \alpha_2) \\
&= (-a + b - \alpha_2 + \alpha_1, \alpha_1 - a)
\end{aligned}$$

$$\begin{aligned} f_\beta(f_\alpha^2(a,b)) &= f_\beta(-a+b-\alpha_2+\alpha_1, \alpha_1-a) \\ &= (a-\alpha_1+\beta_1, -a+b-\alpha_2+\alpha_1-\alpha_1+a+\beta_2) \\ &= (a-\alpha_1+\beta_1, b-\alpha_2+\beta_2) \\ &= (a,b)-\alpha+\beta. \end{aligned}$$

□

Letting $\beta=\alpha$ in Lemma 3.2 yields $f_\alpha^3(a,b)=(a,b)$.

Corollary 3.3. $f_\alpha^3(a,b)=(a,b)$.

As it was mentioned in Section 2, we need to show that if $\alpha \neq \beta$, then $f_\alpha(a,b) \neq f_\beta(a,b)$ for every $(a,b) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$; so that each arc is used exactly once. The statement of the following lemma is an equivalent claim.

Lemma 3.4. $f_\alpha(a,b)=f_\beta(a,b)$ *for some* $(a,b) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ *if and only if* $\alpha=\beta$.

Proof. Assume $f_\alpha(a,b)=f_\beta(a,b)$. Then

$$(-b+\alpha_1, a-b+\alpha_2) = (-b+\beta_1, a-b+\beta_2).$$

Hence, $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$. Therefore $\alpha=\beta$. □

4 Decomposing $\overrightarrow{C}_{(4^kxy:n)}$ into $\overrightarrow{C}_{yn}$-factors and $\overrightarrow{C}_{x2^kn}$-factors

Let $G=\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ and label each column of $\overrightarrow{C}_{(4^kxy:n)}$ with the elements of the group $G \times \mathbb{Z}_x \times \mathbb{Z}_y$.

Let $R = G \times \mathbb{Z}_x \times \mathbb{Z}_y$. For every $\lambda \in R$, let $\lambda=(\alpha,\beta,\gamma)$, with $\alpha \in G$, $\beta \in \mathbb{Z}_x$ and $\gamma \in \mathbb{Z}_y$. For $\alpha \in G$, let f_α be defined as in Section 3. For $\beta \in \mathbb{Z}_x$ let f_β be the permutation of $\mathbb{Z}_x$ defined by $f_\beta(a)=a+\beta$. Similarly, for $\gamma \in \mathbb{Z}_y$ let f_γ be the permutation of $\mathbb{Z}_y$ defined by $f_\gamma(a)=a+\gamma$. Finally, for $\lambda=(\alpha,\beta,\gamma) \in R$ let f_λ be the permutation of R defined by $f_\lambda(a,b,c)=(f_\alpha(a), f_\beta(b), f_\gamma(c))$.

Let φ be a permutation of R, and for each $\lambda \in G$ let $H_{4^kxy}(\lambda,\varphi(\lambda))$ be the 2-factor formed with the following permutations:

1. f_λ from column i to $i+1$ if $1 \leq i \leq n-3/2$;
2. f_λ^{-1} from column i to $i+1$ if $n-3/2+1 \leq i \leq n-3$;
3. f_λ from column $n-2$ to column $n-1$;
4. $f(\alpha,-2\beta,-2\gamma)$ from column $n-1$ to column n;
5. $f_{\varphi(\alpha)}$ from column n to column 1.

Notice that if you start in column 1 at vertex (a,b,c) the first time you reach column 1 again you reach vertex

$$(a,b,c)-(\alpha,\beta,\gamma)+\varphi(\alpha,\beta,\gamma) = (a,b,c)-\lambda+\varphi(\lambda).$$

Hence, we can apply Lemma 2.2 to obtain the length of the cycles in the 2-factor.

Let $\lambda \in R$. If $\lambda-\varphi(\lambda)=(a,b,0)$ with $a \in \pm\{(1,0),(0,1),(1,1)\}$ and $\gcd(b,x)=1$, then by Lemma 2.2 the 2-factor $H_{4^k}(\lambda,\varphi(\lambda))$ is a $\overrightarrow{C}_{2^kxn}$-factor. If $\lambda-\varphi(\lambda)=(0,0,c)$ with $\gcd(c,y)=1$, Lemma 2.2 implies that $H_{4^k}(\lambda,\varphi(\lambda))$ is a $\overrightarrow{C}_{yn}$-factor.

Therefore, to obtain a decomposition of $\overrightarrow{C}_{(4^kxy:n)}$ into r $\overrightarrow{C}_{2^kxn}$-factors and s $\overrightarrow{C}_{yn}$-factors, we need a permutation φ satisfying

(A) $\lambda - \varphi(\lambda) = (a, b, 0)$ with $a \in \pm\{(1,0),(0,1),(1,1)\}$ and $\gcd(b, x) = 1$ for r elements $\lambda \in R$;

(B) $\lambda - \varphi(\lambda) = (0, 0, c)$ with $\gcd(c, y) = 1$ for $s = 4^k xy - r$ elements $\lambda \in R$.

In order to obtain the permutation φ, consider the subgroup $2G$ of G of index 4, and let

$$K = \{(0,0),(1,0),(0,1),(1,1)\}.$$

Notice that K is a set of representatives of the cosets of $2G$ in G. Let $\epsilon \in 2G$, and let ϕ be a permutation of G. If $g, \phi(g) \in \epsilon + K$, then either $g = \phi(g)$ or $|g - \phi(g)| = 2^k$ because $g - \phi(g) \in \pm K$. Hence, we can obtain φ by providing 4^{k-1} permutations ρ_ϵ of $K \times \mathbb{Z}_x \times \mathbb{Z}_y$ satisfying

(A′) $\lambda - \rho_\epsilon(\lambda) = (a, b, 0)$ with $a \in \pm\{(1,0),(0,1),(1,1)\}$ and $\gcd(b, x) = 1$ for r_ϵ elements $\lambda \in K \times \mathbb{Z}_x \times \mathbb{Z}_y$;

(B′) $\lambda - \rho_\epsilon(\lambda) = (0, 0, c)$ with $\gcd(c, y) = 1$ for $s_\epsilon = 4xy - r_\epsilon$ elements $\lambda \in K \times \mathbb{Z}_x \times \mathbb{Z}_y$;

having $r = \sum_{\epsilon \in 2G} r_\epsilon$, and having φ act in each $(\epsilon + K) \times \mathbb{Z}_x \times \mathbb{Z}_y$ as ρ_ϵ, i.e. if $g = (\epsilon, c, d) + (\mu, 0, 0)$, with $\mu \in K$, $\varphi(g) = (\epsilon, c, d) + \rho_\epsilon(\mu, c, d)$. Notice that if $a \in K$, $a \in \pm\{(1,0),(0,1),(1,1)\}$ if and only if $a \neq (0,0)$.

In [17], for every $r \in \{0, 2, 3, \dots, 4xy - 3, 4xy - 2, 4xy\}$, permutations ϕ of $\mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$ were given satisfying:

(A″) $\lambda - \phi(\lambda) = (a, b, 0)$, with $a \neq 0$ and $\gcd(b, x) = 1$ for r elements $\lambda \in \mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$;

(B″) $\lambda - \phi(\lambda) = (0, 0, c)$, with $\gcd(c, y) = 1$ for the remaining $4xy - r$ elements $\lambda \in \mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$.

Let $\pi\colon K \to \mathbb{Z}_4$ be a bijection such that $\pi(0,0) = 0$, and let $\psi\colon K \times \mathbb{Z}_x \times \mathbb{Z}_y \to \mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$ be the bijection that fixes the coordinates of $\mathbb{Z}_x$ and $\mathbb{Z}_y$, and that behaves like π in the coordinate of K. Then if ϕ_ϵ is a permutation of $\mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$, satisfying Conditions (A″) and (B″) with r_ϵ and s_ϵ, $\rho_\epsilon = \psi^{-1}\phi_\epsilon\psi$ is a permutation of $K \times \mathbb{Z}_x \times \mathbb{Z}_y$ satisfying Conditions (A′) and (B′) with r_ϵ and s_ϵ.

If we wanted either $x = 1$ or $y = 1$, we would need to change Conditions (A) and (B), but it is easy to see that the necessary permutations to decompose $\vec{C}_{(4^k xy:n)}$ exist.

Therefore we have the following.

Lemma 4.1. *Let $r \notin \{1, 4^k xy - 1\}$, then there is a decomposition of $\vec{C}_{(4^k xy:n)}$ into r $\vec{C}_{2^k xn}$-factors and $s = 4^k xy - r$ C_{yn}-factors.*

5 Main results

The complete solution to the uniform case of the Oberwolfach problem will be vital to the proof of our main result.

Theorem 5.1 ([1, 2, 15, 22]). *K_v can be decomposed into C_m-factors (and a 1-factor if v is even) if and only if $v \equiv 0 \pmod{m}$, $(v, m) \neq (6, 3)$ and $(v, m) \neq (12, 3)$.*

We now apply the results from Section 4 to produce the following important result for the uniform equipartite version of the Hamilton-Waterloo problem where the two factor types consist of cycle sizes of distinct parities.

Theorem 5.2. *Let x, y, z, v, m, k be positive integers $v, m, k \geq 3$ satisfying the following:*

(i) $v, m \geq 3$,

(ii) $k \geq 2$,

(iii) x, y, z *odd,*

(iv) $z \geq 3$,

(v) $\gcd(x, y) = 1$,

(vi) $vm \equiv 0 \pmod{4^k xyz}$, $v \equiv 0 \pmod{4^k xy}$,

(vii) $\frac{v(m-1)}{4^k xy}$ *is even,*

(viii) $\left(\frac{v}{4^k xy}, m, z\right) \notin \{(2,3,3), (6,3,3), (2,6,3), (6,2,6)\}$,

then there is a decomposition of $K_{(v:m)}$ into r $C_{2^k xz}$-factors and s C_{yz}-factors, for any $s, r \neq 1$.

Proof. Let $v_1 = v/4^k xy$. Consider $K_{(v_1:m)}$. Item (vi) ensures that z divides $v_1 m$; and items (vii), (i), and (viii) give us $v_1(m-1)$ is even, $m \neq 2$, and

$$\left(\frac{v}{4^k xy}, m, z\right) \notin \{(2,3,3), (6,3,3), (2,6,3), (6,2,6)\}.$$

Thus by Theorem 1.1 there is a decomposition of $K_{(v_1:m)}$ into C_z-factors.

Replace each vertex in $a \in K_{(v_1:m)}$ by $4^k xy$ vertices (a, b), with $0 \leq b \leq 4^k xy - 1$, having an edge between (a_1, b_1) and (a_2, b_2) if and only if there was an edge between a_1 and a_2. This yields $K_{(v:m)}$. Even more, each C_z-factor becomes a copy of $\frac{v_1 m}{z} C_{(4^k xy:z)}$. By Lemma 4.1, we have that each $\frac{v_1 m}{z} C_{(4^k xy:z)}$ can be decomposed into r_p $C_{2^k xz}$-factors and s_p C_{yz}-factors as long as $r_p, s_p \neq 1$. Choosing s_p such that $\sum_p s_p = s$ and $s_p, r_p \neq 1$, provides a decomposition of $K_{(v:m)}$ into r $C_{2^k xz}$-factors and s C_{yz}-factors by Lemma 2.1. □

The next lemma, given in [17] shows how to find solutions to the Hamilton-Waterloo problems by combining solutions for the problem on complete graphs and solutions for the problem on equipartite graphs.

Lemma 5.3 ([17])**.** *Let m and v be positive integers. Let F_1 and F_2 be two 2-factors on vm vertices. Suppose the following conditions are satisfied:*

- *There exists a decomposition of $K_{(v:m)}$ into s_α copies of F_1 and r_α copies of F_2.*
- *There exists a decomposition of mK_v into s_β copies of F_1 and r_β copies of F_2.*

Then there exists a decomposition of K_{vm} into $s = s_\alpha + s_\beta$ copies of F_1 and $r = r_\alpha + r_\beta$ copies of F_2.

We are now in a position to provide a proof of the main theorem.

Theorem 5.4. *Let x, y, v, k and m be positive integers such that:*

(i) $v, m \geq 3$,

(ii) x, y *are odd,*

(iii) $\gcd(x,y) \geq 3$,

(iv) x *and* y *divide* v,

(v) 4^k *divides* v.

Then there exists a solution to $(2^k x, y)$-HWP$(vm; r, s)$ *for every pair* r, s *with* $r + s = \lfloor (vm-1)/2 \rfloor$, $r, s \neq 1$.

Proof. Let r and s be positive integers with $r + s = \lfloor (vm-1)/2 \rfloor$ and $r, s \neq 1$. Write $r = r_\alpha + r_\beta$ and $s = s_\alpha + s_\beta$, where $r_\alpha, r_\beta, s_\alpha, s_\beta$ are positive integers that satisfy $r_\alpha, s_\alpha \neq 1$, $r_\alpha + s_\alpha = v(m-1)/2$, $r_\beta + s_\beta = \lfloor (v-1)/2 \rfloor$, and $r_\beta, s_\beta \in \{0, \lfloor (v-1)/2 \rfloor\}$.

Start by decomposing K_{vm} into $K_{(v:m)} \oplus mK_v$. Let $z = \gcd(x,y)$, $x_1 = x/z$, $y_1 = y/z$. By Theorem 5.2 there is a decomposition of $K_{(v:m)}$ into r_α $C_{2^k x_1 z}$-factors and s_α $C_{y_1 z}$-factors. This is a decomposition of $K_{(v:m)}$ into r_α $C_{2^k x}$-factors and s_α C_y-factors. By Theorem 5.1 there is a decomposition of mK_v into r_β $C_{2^k x}$-factors and s_β C_y-factors. Lemma 5.3 shows that all of this together yields a decomposition of K_{vm} into r C_x-factors and s C_y-factors. □

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