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Vertex-quasiprimitive 2-arc-transitive digraphs*

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Abstract

We study vertex-quasiprimitive 2-arc-transitive digraphs, and reduce the problem of vertex-primitive 2-arc-transitive digraphs to almost simple groups. This includes a complete classification of vertex-quasiprimitive 2-arc-transitive digraphs where the action on vertices has O'Nan-Scott type SD or CD.

Keywords: Digraphs, vertex-quasiprimitive, 2-arc-transitive.

Math. Subj. Class.: 05C20, 05C25

1 Introduction

A *digraph* Γ is a pair $(V, \to)$ with a set V (of vertices) and an antisymmetric irreflexive binary relation $\to$ on V. All digraphs considered in this paper will be finite. For a non-negative integer s, an *s-arc* of Γ is a sequence $v_0, v_1, \ldots, v_s$ of vertices with $v_i \to v_{i+1}$ for each $i = 0, \ldots, s-1$. A 1-arc is also simply called an *arc*. We say Γ is *s-arc-transitive* if the group of all automorphisms of Γ (that is, all permutations of V that preserve the relation $\to$) acts transitively on the set of s-arcs. More generally, for a group G of automorphisms of Γ, we say Γ is (G, s)*-arc-transitive* if G acts transitively on the set of s-arcs of Γ.

A transitive permutation group G on a set Ω is said to be *primitive* if G does not preserve any nontrivial partition of Ω, and is said to be *quasiprimitive* if each nontrivial normal subgroup of G is transitive. It is easy to see that primitive permutation groups are necessarily quasiprimitive, but there are quasiprimitive permutation groups that are not primitive. We say a digraph is *vertex-primitive* if its automorphism group is primitive on the vertex set. The aim of this paper is to investigate finite vertex-primitive s-arc transitive digraphs with $s \geqslant 2$. However, we will often work in the more general quasiprimitive setting.

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There are many s-arc-transitive digraphs, see for example [2, 6, 7, 8]. In particular, for every integer $k \geqslant 2$ and every integer $s \geqslant 1$ there are infinitely many k-regular (G, s)-arc-transitive digraphs with G quasiprimitive on the vertex set (see the proof of Theorem 1 of [2]). On the other hand, the first known family of vertex-primitive 2-arc-transitive digraphs besides directed cycles was only recently discovered in [3]. The digraphs in this family are not 3-arc-transitive, which prompted the following question:

Question 1.1. Is there an upper bound on s for vertex-primitive s-arc-transitive digraphs that are not directed cycles?

The O'Nan-Scott Theorem divides the finite primitive groups into eight types and there is a similar theorem for finite quasiprimitive groups, see [9, Section 5]). For four of the eight types, a quasiprimitive group of that type has a normal regular subgroup. Praeger [8, Theorem 3.1] showed that if Γ is a $(G, 2)$-arc-transitive digraph and G has a normal subgroup that acts regularly on V, then Γ is a directed cycle. Thus to investigate vertex-primitive and vertex-quasiprimitive 2-arc-transitive digraphs, we only need to consider the four remaining types. One of these types is where G is an almost simple group, that is, where G has a unique minimal normal subgroup T, and T is a nonabelian simple group. The examples of primitive 2-arc-transitive digraphs constructed in [3] are of this type. This paper examines the remaining three types, which are labelled SD, CD and PA, and reduces Question 1.1 to almost simple vertex-primitive groups (Corollary 1.6). We now define these three types and state our results.

We say that a quasiprimitive group G on a set Ω is of type SD if G has a unique minimal normal subgroup N, there exists a nonabelian simple group T and positive integer $k \geqslant 2$ such that $N \cong T^k$, and for $\omega \in \Omega$, N_ω is a full diagonal subgroup of N (that is, $N_\omega \cong T$ and projects onto T in each of the k simple direct factors of N). It is incorrectly claimed in [8, Lemma 4.1] that there is no 2-arc-transitive digraph with a vertex-primitive group of automorphisms of type SD. However, there is an error in the proof which occurs when concluding "σx also fixes $D\mathtt{t}^{-1}$". Indeed, given a nonabelian simple group T, our Construction 3.1 yields a $(G, 2)$-arc-transitive digraph $\Gamma(T)$ with G primitive of type SD. These turn out to be the only examples.

Theorem 1.2. *Let Γ be a connected $(G, 2)$-arc-transitive digraph such that G is quasiprimitive of type* SD *on the set of vertices. Then there exists a nonabelian simple group T such that $\Gamma \cong \Gamma(T)$, as obtained from* Construction 3.1. *Moreover,* $\mathrm{Aut}(\Gamma)$ *is vertex-primitive of type SD and Γ is not 3-arc-transitive.*

The remaining two quasiprimitive types, CD and PA, both arise from product actions. For any digraph Σ and positive integer m, Σ^m denotes the direct product of m copies of Σ as in Notation 2.6. The wreath product $\mathrm{Sym}(\Delta) \wr \mathrm{S}_m = \mathrm{Sym}(\Delta)^m \rtimes \mathrm{S}_m$ acts naturally on the set Δ^m with product action. Let G_1 be the stabiliser in G of the first coordinate and let H be the projection of G_1 onto $\mathrm{Sym}(\Delta)$. If G projects onto a transitive subgroup of S_m, then a result of Kovács [4, (2.2)] asserts that up to conjugacy in $\mathrm{Sym}(\Delta)^m$ we may assume that $G \leqslant H \wr \mathrm{S}_m$. A reduction for 2-arc-transitive digraphs was sought in [8, Remark 4.3] but only partial results were obtained. Our next result yields the desired reduction.

Theorem 1.3. *Let $H \leqslant \mathrm{Sym}(\Delta)$ with transitive normal subgroup N and let $G \leqslant H \wr \mathrm{S}_m$ acting on $V = \Delta^m$ with product action such that G projects to a transitive subgroup of* S_m *and G has component H. Moreover, assume that $N^m \leqslant G$. If Γ is a (G, s)-arc-transitive* *digraph with vertex set V such that $s \geqslant 2$, then $\Gamma \cong \Sigma^m$ for some (H, s)-arc-transitive digraph Σ with vertex set Δ.*

A quasiprimitive group of type CD on a set Ω is one that has a product action on Ω and the component is quasiprimitive of type SD, while a quasiprimitive group of type PA on a set Ω is one that acts faithfully on some partition $\mathcal{P}$ of Ω and G has a product action on $\mathcal{P}$ such that the component H is an almost simple group. When G is primitive of type PA, H is primitive and the partition $\mathcal{P}$ is the partition into singletons, that is, G has a product action on Ω. As a consequence, we have the following corollaries.

Corollary 1.4. *Suppose Γ is a connected $(G, 2)$-arc-transitive digraph such that G is vertex-quasiprimitive of type* CD. *Then there exists a nonabelian simple group T and positive integer $m \geqslant 2$ such that $\Gamma \cong \Gamma(T)^m$, where $\Gamma(T)$ is as obtained from* Construction 3.1. *Moreover, Γ is not* 3*-arc-transitive.*

Corollary 1.5. *Suppose Γ is a (G, s)-arc-transitive digraph such that G is vertex-primitive of type* PA. *Then $\Gamma \cong \Sigma^m$ for some (H, s)-arc-transitive digraph Σ and integer $m \geqslant 2$ for some almost simple primitive permutation group $H \leqslant \mathrm{Aut}(\Sigma)$.*

We give an example in Section 2.3 of an infinite family of $(G, 2)$-arc-transitive digraphs Γ with G vertex-quasiprimitive of PA type such that Γ is not a direct power of a digraph Σ (indeed the number of vertices of Γ is not a proper power). We leave the investigation of such digraphs open.

We note that Theorem 1.2 and Corollaries 1.4 and 1.5, reduce Question 1.1 to studying almost simple primitive groups.

Corollary 1.6. *There exists an absolute upper bound C such that every vertex-primitive s-arc-transitive digraph that is not a directed cycle satisfies $s \leqslant C$, if and only if for every (G, s)-arc-transitive digraph with G a primitive almost simple group we have $s \leqslant C$.*

Theorem 1.2 follows immediately from a more general theorem (Theorem 3.15) given at the end of Section 3. Then in Section 4, we prove Theorem 1.3 as well as Corollaries 1.4–1.5 after establishing some general results for normal subgroups of s-arc-transitive groups.

2 Preliminaries

We say that a digraph Γ is *k-regular* if both the set $\Gamma^-(v) = \{u \in V \mid u \to v\}$ of in-neighbours of v and the set $\Gamma^+(v) = \{w \in V \mid v \to w\}$ of out-neighbours of v have size k for all $v \in V$, and we say that Γ is *regular* if it is k-regular for some positive integer k. Note that any vertex-transitive digraph is regular. Moreover, if Γ is regular and (G, s)-arc-transitive with $s \geqslant 2$ then it is also $(G, s-1)$-arc-transitive.

Recall that a digraph is said to be connected if and only if its underlying graph is connected. A vertex-primitive digraph is necessarily connected, for otherwise its connected components would form a partition of the vertex set that is invariant under digraph automorphisms.

2.1 Group factorizations

All the groups we consider in this paper are assumed to be finite. An expression of a group G as the product of two subgroups H and K of G is called a *factorization* of G. The following lemma lists several equivalent conditions for a group factorization, whose proof is fairly easy and so is omitted.

Lemma 2.1. *Let H and K be subgroups of G. Then the following are equivalent.*

(a) $G = HK$.

(b) $G = KH$.

(c) $G = (x^{-1}Hx)(y^{-1}Ky)$ *for any* $x, y \in G$.

(d) $|H \cap K||G| = |H||K|$.

(e) *H acts transitively on the set of right cosets of K in G by right multiplication.*

(f) *K acts transitively on the set of right cosets of H in G by right multiplication.*

The s-arc-transitivity of digraphs can be characterized by group factorizations as follows:

Lemma 2.2. *Let Γ be a G-arc-transitive digraph, $s \geqslant 2$ be an integer, and $v_0 \to v_1 \to \cdots \to v_{s-1} \to v_s$ be an s-arc of Γ. Then Γ is (G, s)-arc-transitive if and only if $G_{v_1 \dots v_i} = G_{v_0 v_1 \dots v_i} G_{v_1 \dots v_i v_{i+1}}$ for each i in $\{1, \dots, s-1\}$.*

Proof. For any i such that $1 \leqslant i \leqslant s-1$, the group $G_{v_1 \dots v_i}$ acts on the set $\Gamma^+(v_i)$ of out-neighbours of v_i. Since $v_{i+1} \in \Gamma^+(v_i)$ and $G_{v_1 \dots v_i v_{i+1}}$ is the stabilizer in $G_{v_1 \dots v_i}$ of v_{i+1}, by Frattini's argument, the subgroup $G_{v_0 v_1 \dots v_i}$ of $G_{v_1 \dots v_i}$ is transitive on $\Gamma_+(v_i)$ if and only if $G_{v_1 \dots v_i} = G_{v_0 v_1 \dots v_i} G_{v_1 \dots v_i v_{i+1}}$. Note that Γ is (G, s)-arc-transitive if and only if Γ is $(G, s-1)$-arc-transitive and $G_{v_0 v_1 \dots v_i}$ is transitive on $\Gamma_+(v_i)$. One then deduces inductively that Γ is (G, s)-arc-transitive if and only if $G_{v_1 \dots v_i} = G_{v_0 v_1 \dots v_i} G_{v_1 \dots v_i v_{i+1}}$ for each i in $\{1, \dots, s-1\}$. ☐

If Γ is a G-arc-transitive digraph and $u \to v$ is an arc of Γ, then since G is vertex-transitive we can write $v = u^g$ for some $g \in G$ and it follows that

$$v^{g^{-1}} \to v \to \cdots \to v^{g^{s-2}} \to v^{g^{s-1}} \tag{2.1}$$

is an s-arc of Γ. In this setting, Lemma 2.2 is reformulated as follows.

Lemma 2.3. *Let Γ be a G-arc-transitive digraph, $s \geqslant 2$ be an integer, v be a vertex of Γ, and $g \in G$ such that $v \to v^g$. Then Γ is (G, s)-arc-transitive if and only if*

$$\bigcap_{j=0}^{i-1} g^{-j} G_v g^j = \left(\bigcap_{j=0}^{i} g^{-(j-1)} G_v g^{j-1} \right) \left(\bigcap_{j=0}^{i} g^{-j} G_v g^j \right)$$

for each i in $\{1, \dots, s-1\}$.

Proof. Let $v_j = v^{g^{j-1}}$ for any integer j such that $0 \leqslant j \leqslant s-1$. Then the s-arc (2.1) of Γ turns out to be $v_0 \to v_1 \to \cdots \to v_{s-1} \to v_s$, and for any i in $\{1, \dots, s\}$ we have

$$G_{v_1 \dots v_i} = \bigcap_{j=1}^{i} G_{v_j} = \bigcap_{j=1}^{i} g^{-(j-1)} G_v g^{j-1} = \bigcap_{j=0}^{i-1} g^{-j} G_v g^j$$

and

$$G_{v_0 v_1 \dots v_i} = \bigcap_{j=0}^{i} G_{v_j} = \bigcap_{j=0}^{i} g^{-(j-1)} G_v g^{j-1}.$$

Hence the conclusion of the lemma follows from Lemma 2.2. ☐

2.2 Constructions of s-arc-transitive digraphs

Let G be a group, H be a subgroup of G, V be the set of right cosets of H in G and g be an element of $G \setminus H$ such that $g^{-1} \notin HgH$. Define a binary relation $\to$ on V by letting $Hx \to Hy$ if and only if $yx^{-1} \in HgH$ for any $x, y \in G$. Then $(V, \to)$ is a digraph, denoted by $\mathrm{Cos}(G, H, g)$. Right multiplication gives an action R_H of G on V that preserves the relation $\to$, so that $R_H(G)$ is a group of automorphisms of $\mathrm{Cos}(G, H, g)$.

Lemma 2.4. *In the above notation, the following hold.*

(a) $\mathrm{Cos}(G, H, g)$ *is* $|H{:}H \cap g^{-1}Hg|$*-regular.*

(b) $\mathrm{Cos}(G, H, g)$ *is* $R_H(G)$*-arc-transitive.*

(c) $\mathrm{Cos}(G, H, g)$ *is connected if and only if* $\langle H, g\rangle = G$*.*

(d) $\mathrm{Cos}(G, H, g)$ *is* $R_H(G)$*-vertex-primitive if and only if* H *is maximal in* G*.*

(e) Let $s \geqslant 2$ *be an integer. Then* $\mathrm{Cos}(G, H, g)$ *is* $(R_H(G), s)$*-arc-transitive if and only if for each* i *in* $\{1, \ldots, s-1\}$*,*

$$\bigcap_{j=0}^{i-1} g^{-j}Hg^{j} = \left(\bigcap_{j=0}^{i} g^{-(j-1)}Hg^{j-1}\right)\left(\bigcap_{j=0}^{i} g^{-j}Hg^{j}\right).$$

Proof. Parts (a)–(d) are folklore (see for example [2]), and part (e) is derived in light of Lemma 2.3. □

Remark 2.5. Lemma 2.4 establishes a group theoretic approach to constructing s-arc-transitive digraphs. In particular, $\mathrm{Cos}(G, H, g)$ is $(R_H(G), 2)$-arc-transitive if and only if $H = (gHg^{-1} \cap H)(H \cap g^{-1}Hg)$.

Next we show how to construct s-arc-transitive digraphs from existing ones. Let Γ be a digraph with vertex set U and Σ be a digraph with vertex set V. The *direct product* of Γ and Σ, denoted $\Gamma \times \Sigma$, is the digraph (it is easy to verify that this is indeed a digraph) with vertex set $U \times V$ and $(u_1, v_1) \to (u_2, v_2)$ if and only if $u_1 \to u_2$ and $v_1 \to v_2$, where $u_i \in U$ and $v_i \in V$ for $i = 1, 2$.

Notation 2.6. For any digraph Σ and positive integer m, denote by Σ^m the direct product of m copies of Σ.

Lemma 2.7. *Let* s *be a positive integer,* Γ *be a* (G, s)*-arc-transitive digraph and* Σ *be a* (H, s)*-arc-transitive digraph. Then* $\Gamma \times \Sigma$ *is a* $(G \times H, s)$*-arc-transitive digraph, where* $G \times H$ *acts on the vertex set of* $\Gamma \times \Sigma$ *by product action.*

Proof. Let $(u_0, v_0) \to (u_1, v_1) \to \cdots \to (u_s, v_s)$ and $(u'_0, v'_0) \to (u'_1, v'_1) \to \cdots \to (u'_s, v'_s)$ be any two s-arcs of $\Gamma \times \Sigma$. Then $u_0 \to u_1 \to \cdots \to u_s$ and $u'_0 \to u'_1 \to \cdots \to u'_s$ are s-arcs of Γ while $v_0 \to v_1 \to \cdots \to v_s$ and $v'_0 \to v'_1 \to \cdots \to v'_s$ are s-arcs of Σ. Since Γ is (G, s)-arc-transitive, there exists $g \in G$ such that $u_i^g = u'_i$ for each i with $0 \leqslant i \leqslant s$. Similarly, there exists $h \in H$ such that $v_i^h = v'_i$ for each i with $0 \leqslant i \leqslant s$. It follows that $(u_i, v_i)^{(g,h)} = (u'_i, v'_i)$ for each i with $0 \leqslant i \leqslant s$. This means that $\Gamma \times \Sigma$ is a $(G \times H, s)$-arc-transitive. □

2.3 Example

In this subsection we give an example of an infinite family of $(G, 2)$-arc-transitive digraphs Γ with G vertex-quasiprimitive of PA type such that Γ is not a direct power of a digraph Σ. In fact, we prove in Lemma 2.9 that the number of vertices of Γ is not a proper power.

Let $n \geqslant 5$ be odd, $G_1 = \mathrm{Alt}(\{1, 2, \ldots, n\})$ and $G_2 = \mathrm{Alt}(\{n+1, n+2, \ldots, 2n\})$. Take permutations

$$a = (1, n+1)(2, n+2)\cdots(n, 2n), \quad b = (1, 2)(3, 4)(n+1, n+2)(n+3, n+4)$$

and

$$\begin{aligned} g = (&1, n+2, 2, n+3, 5, n+6, 7, n+8, \ldots, 2i-1, n+2i, \ldots, n-2, 2n-1, n, \\ &n+1, 3, n+4, 4, n+5, 6, n+7, \ldots, 2j, n+2j+1, \ldots, n-1, 2n). \end{aligned}$$

In fact, $g = ac$ with

$$c = (1, 3, 5, 6, 7, \ldots, n)(n+1, n+2, \ldots, 2n).$$

Let $G = (G_1 \times G_2) \rtimes \langle a \rangle$, and note that $g \in G$ as $c \in G_1 \times G_2$. Let $H = \langle a, b \rangle = \langle a \rangle \times \langle b \rangle$ and $\Gamma_n = \mathrm{Cos}(G, H, g)$.

Lemma 2.8. *For all odd $n \geqslant 5$, Γ_n is a connected $(G, 2)$-arc-transitive digraph with G quasiprimitive of PA type on the vertex set.*

Proof. As $(G_1 \times G_2) \cap H = \langle b \rangle$ we see that G is quasiprimitive of PA type on the vertex set. To show that Γ_n is connected, we shall show $\langle H, g \rangle = G$ in light of Lemma 2.4(c). Let $M = \langle H, g \rangle \cap (G_1 \times G_2)$. Then we only need to show $M = G_1 \times G_2$ since $a \in \langle H, g \rangle$.

Denote the projections of $G_1 \times G_2$ onto G_1 and G_2, respectively, by π_1 and π_2. Note that g^2 fixes both $\{1, \ldots, n\}$ and $\{n+1, \ldots, 2n\}$ setwise with

$$\pi_1(g^2) = (1, 2, 5, 7, \ldots, 2i-1, \ldots, n, 3, 4, 6, \ldots, 2j, \ldots, n-1)$$

and

$$\pi_1(g^{n+1}) = (1, 3, 2, 4, 5, \ldots, n).$$

We have $g^2 \in M$ and

$$\pi_1(g^{-(n+1)} b g^{n+1} b) = \pi_1(g^{-(n+1)} b g^{n+1}) \pi_1(b) = (3, 4)(2, 5)(1, 2)(3, 4) = (1, 2, 5),$$

which implies

$$\pi_1(M) \geqslant \pi_1(\langle g^2, b \rangle) \geqslant \pi_1(\langle g^2, g^{-(n+1)} b g^{n+1} b \rangle) = \langle \pi_1(g^2), \pi_1(g^{-(n+1)} b g^{n+1} b) \rangle = G_1$$

using the fact that the permutation group generated by a 3-cycle (α, β, γ) and an n-cycle with first 3-entries α, β, γ is A_n. It follows that

$$\pi_2(M) = \pi_2(M^a) = (\pi_2(M))^a = G_1^a = G_2,$$

and so M is either $G_1 \times G_2$ or a full diagonal subgroup of $G_1 \times G_2$. However, $c = ag \in M$ while $\pi_1(c)$ and $\pi_2(c)$ have different cycle types. We conclude that M is not a diagonal subgroup of $G_1 \times G_2$, and so $M = G_1 \times G_2$ as desired.

Now we prove that Γ_n is $(G, 2)$-arc-transitive, which is equivalent to proving that $H = (gHg^{-1} \cap H)(H \cap g^{-1}Hg)$ according to Lemma 2.4(e). In view of

$$(ab)^g = (ab)^{ac} = (ab)^c = a \tag{2.2}$$

we deduce that $a \in H \cap H^g$. Since H is not normal in $G = \langle H, g\rangle$, we have $H^g \neq H$. Consequently, $H \cap H^g = \langle a\rangle$. Then again by (2.2) we deduce that

$$H \cap H^{g^{-1}} = (H \cap H^g)^{g^{-1}} = \langle a\rangle^{g^{-1}} = \langle a^{g^{-1}}\rangle = \langle ab\rangle.$$

This yields

$$(gHg^{-1} \cap H)(H \cap g^{-1}Hg) = \langle a\rangle\langle ab\rangle = H. \tag{2.3}$$

Finally, the condition $g^{-1} \notin HgH$ holds as a consequence (see [3, Lemma 2.3]) of (2.3) and the conclusion $H^g \neq H$. This completes the proof. □

Lemma 2.9. *The number of vertices of Γ_n is not a proper power for any odd $n \geqslant 5$.*

Proof. Suppose that the number of vertices of Γ_n is m^k for some $m \geqslant 2$ and $k \geqslant 2$. Then we have

$$m^k = \frac{|G|}{|H|} = \frac{2(n!/2)^2}{4} = \frac{(n!)^2}{8} \tag{2.4}$$

If $k = 2$, then (2.4) gives $(n!)^2 = 2(2m)^2$, which is not possible. Hence $k \geqslant 3$. By Bertrand's Postulate, there exists a prime number p such that $n/2 < p < n$. Thus, the largest p-power dividing $n!$ is p, and so the largest p-power dividing the right hand side of (2.4) is p^2. However, this implies that the largest p-power dividing m^k is p^2, contradicting the conclusion $k \geqslant 3$. □

2.4 Normal subgroups

Lemma 2.10. *Let Γ be a (G, s)-arc-transitive digraph with $s \geqslant 2$, M be a vertex-transitive normal subgroup of G, and $v_1 \to \cdots \to v_s$ be an $(s-1)$-arc of Γ. Then $G = MG_{v_1 \dots v_i}$ for each i in $\{1, \dots, s\}$.*

Proof. Since M is transitive on the vertex set of Γ, there exists $m \in M$ such that $v_1^m = v_2$. Denote $u_i = v_1^{m^{i-1}}$ for each i such that $0 \leqslant i \leqslant s$. Then $G_{u_0 u_1 \dots u_i} = mG_{u_1 \dots u_i u_{i+1}} m^{-1}$ for each i such that $0 \leqslant i \leqslant s-1$, and $u_0 \to u_1 \to \cdots \to u_s$ is an s-arc of Γ since $v_1 \to v_2$ and m is an automorphism of Γ. For each i in $\{1, \dots, s-1\}$, we deduce from Lemma 2.2 that

$$G_{u_1 \dots u_i} = G_{u_0 u_1 \dots u_i} G_{u_1 \dots u_i u_{i+1}} = (mG_{u_1 \dots u_i u_{i+1}} m^{-1}) G_{u_1 \dots u_i u_{i+1}}.$$

Let φ be the projection from G to G/M. It follows that

$$\begin{aligned}
\varphi(G_{u_1 \dots u_i}) &= \varphi(m)\varphi(G_{u_1 \dots u_i u_{i+1}})\varphi(m)^{-1}\varphi(G_{u_1 \dots u_i u_{i+1}}) \\
&= \varphi(G_{u_1 \dots u_i u_{i+1}})\varphi(G_{u_1 \dots u_i u_{i+1}}) \\
&= \varphi(G_{u_1 \dots u_i u_{i+1}})
\end{aligned}$$

and so $G_{u_1 \dots u_i} M = G_{u_1 \dots u_i u_{i+1}} M$ for each i in $\{1, \dots, s-1\}$. Again as M is transitive on the vertex set of Γ, we have $G = MG_{u_1}$. Hence

$$G = MG_{u_1} = MG_{u_1 u_2} = \cdots = MG_{u_1 \dots u_i} = \cdots = MG_{u_1 \dots u_s}.$$

Now for each i in $\{1, \ldots, s\}$, the digraph Γ is $(G, i-1)$-arc-transitive, so there exists $g \in G$ such that $(v_1^g, \ldots, v_i^g) = (u_1, \ldots, u_i)$. Hence

$$G = MG_{u_1 \ldots u_i} = M(g^{-1} G_{v_1 \ldots v_i} g) = MG_{v_1 \ldots v_i}$$

by Lemma 2.1(c). □

By Frattini's argument, we have the following consequence of Lemma 2.10:

Corollary 2.11. *Let Γ be a (G, s)-arc-transitive digraph with $s \geqslant 2$, and M be a vertex-transitive normal subgroup of G. Then Γ is $(M, s-1)$-arc-transitive.*

To close this subsection, we give a short proof of the following result of Praeger [8, Theorem 3.1] using Lemma 2.10.

Proposition 2.12. *Let Γ be a $(G, 2)$-arc-transitive digraph. If G has a vertex-regular normal subgroup, then Γ is a directed cycle.*

Proof. Let N be a vertex-regular normal subgroup of G, and $u \to v$ be an arc of Γ. Then $|G|/|N| = |G_v|$, and $G = G_{uv}N$ by Lemma 2.10. Hence by Lemma 2.1(d), $|G_{uv}| \geqslant |G|/|N| = |G_v|$ and so $|G_{uv}| = |G_v|$. Consequently, Γ is 1-regular, which means that Γ is a directed cycle. □

2.5 Two technical lemmas

Lemma 2.13. *Let A be an almost simple group with socle T and L be a nonabelian simple group. Suppose $L^n \leqslant A$ and $|T| \leqslant |L^n|$ for some positive integer n. Then $n = 1$ and $L = T$.*

Proof. Note that $L^n/(L^n \cap T) \cong (L^n T)/T \leqslant A/T$, which is solvable by the Schreier conjecture. If $L^n \cap T \neq L^n$, then $L^n/(L^n \cap T) \cong L^m$ for some positive integer m, a contradiction. Hence $L^n \cap T = L^n$, which means $L^n \leqslant T$. This together with the condition that $|T| \leqslant |L^n|$ implies $L^n = T$. Hence $n = 1$ and $L = T$, as the lemma asserts. □

Lemma 2.14. *Let A be an almost simple group with socle T and S be a primitive permutation group on $|T|$ points. Then S is not isomorphic to any subgroup of A.*

Proof. Suppose for a contradiction that $S \lesssim A$. Regard S as a subgroup of A, and write $\mathrm{Soc}(S) = L^n$ for some simple group L and positive integer n. Since S is primitive on $|T|$ points, $\mathrm{Soc}(S)$ is transitive on $|T|$ points, and so $|T|$ divides $|\mathrm{Soc}(S)| = |L|^n$. Consequently, L is nonabelian for otherwise T would be solvable. Then by Lemma 2.13 we have $\mathrm{Soc}(S) = L = T$. It follows that S is an almost simple primitive permutation group with $\mathrm{Soc}(S)$ regular, contradicting [5]. □

3 Vertex-quasiprimitive of type SD

3.1 Constructing the graph $\Gamma(T)$

Construction 3.1. *Let T be a nonabelian simple group of order k with $T = \{t_1, \ldots, t_k\}$. Let $D = \{(t, \ldots, t) \mid t \in T\}$ be a full diagonal subgroup of T^k and let $g = (t_1, \ldots, t_k)$. Define $\Gamma = \mathrm{Cos}(T^k, D, g)$ and let V be the set of right cosets of D in T^k, i.e. the vertex set of $\Gamma(T)$.*

Lemma 3.2. *$\Gamma(T)$ is a $|T|$-regular digraph.*

Proof. Suppose that $D\cap g^{-1}Dg\neq 1$. Then there exist $s,t\in T\setminus\{1\}$ such that $(s,\dots,s)=(t_1^{-1}tt_1,\dots,t_k^{-1}tt_k)$. Thus $s=t_i^{-1}tt_i$ for each i such that $1\leqslant i\leqslant k$. Since $\{t_i\mid 1\leqslant i\leqslant k\}=T$, we have $t_j=1$ for some $1\leqslant j\leqslant k$. It then follows from the equality $s=t_j^{-1}tt_j$ that $s=t$. Thus $t=t_i^{-1}tt_i$ for each i such that $1\leqslant i\leqslant k$. Hence t lies in the center of T, which implies $t=1$ as T is nonabelian simple, a contradiction. Consequently, $D\cap g^{-1}Dg=1$, and so $\mathrm{Cos}(T^k,D,g)$ is $|T|$-regular as $|D|/|D\cap g^{-1}Dg|=|D|=|T|$.

Suppose that $g^{-1}\in DgD$. Then there exist $s,t\in T$ such that $(t_1^{-1},\dots,t_k^{-1})=(st_1t,\dots,st_kt)$. It follows that $t_i^{-1}=st_it$ for each i such that $1\leqslant i\leqslant k$. Since $\{t_i\mid 1\leqslant i\leqslant k\}=T$, we have $t_j=1$ for some $1\leqslant j\leqslant k$. Then the equality $t_j^{-1}=st_jt$ leads to $s=t^{-1}$. Thus $t_i^{-1}=t^{-1}t_it$ for each i such that $1\leqslant i\leqslant k$. This implies that the inverse map is an automorphism of T and so T is abelian, a contradiction. Hence $g^{-1}\notin DgD$, from which we deduce that $\mathrm{Cos}(T^k,D,g)$ is a digraph, completing the proof. □

Next we show that up to isomorphism, the definition of $\Gamma(T)$ does not depend on the order of $t_1,t_2,\dots,t_k$.

Lemma 3.3. *Let $g'=(t_1',\dots,t_k')$ such that $T=\{t_1',\dots,t_k'\}$. Then $\mathrm{Cos}(T^k,D,g)\cong\mathrm{Cos}(T^k,D,g')$.*

Proof. Since $\{t_1',\dots,t_k'\}=\{t_1,\dots,t_k\}$, there exists $x\in\mathrm{S}_k$ such that $t_{i^x}=t_i'$ for each i with $1\leqslant i\leqslant k$. Define an automorphism λ of T^k by $(g_1,\dots,g_k)^\lambda=(g_{1^x},\dots,g_{k^x})$ for all $(g_1,\dots,g_k)\in T^k$. Then λ normalizes D and $\lambda^{-1}g\lambda=g'$. Hence the map $Dh\mapsto Dh^\lambda$ gives an isomorphism from $\mathrm{Cos}(T^k,D,g)$ to $\mathrm{Cos}(T^k,D,g')$. □

For any $t\in T$, let $x(t)$ and $y(t)$ be the elements of S_k such that $t_{i^{x(t)}}=tt_i$ and $t_{i^{y(t)}}=t_it^{-1}$ for any $1\leqslant i\leqslant k$, and define permutations $\lambda(t)$ and $\rho(t)$ of V by letting

$$D(g_1,\dots,g_k)^{\lambda(t)}=D(g_{1^{x(t)}},\dots,g_{k^{x(t)}})$$

and

$$D(g_1,\dots,g_k)^{\rho(t)}=D(g_{1^{y(t)}},\dots,g_{k^{y(t)}})$$

for any $(g_1,\dots,g_k)\in T^k$. For any $\varphi\in\mathrm{Aut}(T)$, let $z(\varphi)\in\mathrm{S}_k$ such that $t_{i^{z(\varphi)}}=t_i^\varphi$ for any $1\leqslant i\leqslant k$, and define $\delta(\varphi)\in\mathrm{Sym}(V)$ by letting

$$D(g_1,\dots,g_k)^{\delta(\varphi)}=D((g_{1^{z(\varphi^{-1})}})^\varphi,\dots,(g_{k^{z(\varphi^{-1})}})^\varphi)$$

for any $(g_1,\dots,g_k)\in T^k$. In particular, $\delta(\varphi)$ both permutes the coordinates and acts on each entry.

Lemma 3.4. *λ and ρ are monomorphisms from T to $\mathrm{Sym}(V)$, and δ is a monomorphism from $\mathrm{Aut}(T)$ to $\mathrm{Sym}(V)$.*

Proof. For any $s,t\in T$, noting that $x(t)x(s)=x(st)$, we have

$$\begin{aligned}D(g_1,\dots,g_k)^{\lambda(s)\lambda(t)} &= D(g_{1^{x(s)}},\dots,g_{k^{x(s)}})^{\lambda(t)}\\ &= D(g_{1^{x(t)x(s)}},\dots,g_{k^{x(t)x(s)}})\\ &= D(g_{1^{x(st)}},\dots,g_{k^{x(st)}})\\ &= D(g_1,\dots,g_k)^{\lambda(st)}\end{aligned}$$

for each $(g_1,\dots,g_k) \in T^k$, and so $\lambda(st) = \lambda(s)\lambda(t)$. This means that λ is a homomorphism from T to $\mathrm{Sym}(V)$. Moreover, since $\lambda(t)$ acts on V as the permutation $x(t)$ on the entries, $\lambda(t) = 1$ if and only if $x(t) = 1$, which is equivalent to $t = 1$. Hence λ is a monomorphism from T to $\mathrm{Sym}(V)$. Similarly, ρ is a monomorphism from T to $\mathrm{Sym}(V)$.

For any $\varphi,\psi \in \mathrm{Aut}(T)$, since $z(\psi^{-1})z(\varphi^{-1}) = z(\psi^{-1}\varphi^{-1}) = z((\varphi\psi)^{-1})$, we have

$$\begin{aligned} D(g_1,\dots,g_k)^{\delta(\varphi)\delta(\psi)} &= D((g_{1^{z(\varphi^{-1})}})^{\varphi},\dots,(g_{k^{z(\varphi^{-1})}})^{\varphi})^{\delta(\psi)} \\ &= D((g_{1^{z(\psi^{-1})z(\varphi^{-1})}})^{\varphi\psi},\dots,(g_{k^{z(\psi^{-1})z(\varphi^{-1})}})^{\varphi\psi}) \\ &= D(g_1,\dots,g_k)^{\delta(\varphi\psi)} \end{aligned}$$

for all $(g_1,\dots,g_k) \in T^k$. This means that δ is a homomorphism from $\mathrm{Aut}(T)$ to $\mathrm{Sym}(V)$. Next we prove that δ is a monomorphism. Let $\varphi \in \mathrm{Aut}(T)$ such that

$$D((g_{1^{z(\varphi^{-1})}})^{\varphi},\dots,(g_{k^{z(\varphi^{-1})}})^{\varphi}) = D(g_1,\dots,g_k)^{\delta(\varphi)} = D(g_1,\dots,g_k) \tag{3.1}$$

for each $(g_1,\dots,g_k) \in T^k$. Take any $i \in \{1,\dots,k\}$ and $(g_1,\dots,g_k) \in T^k$ such that $g_j = 1$ for all $j \neq i$ and $g_i \neq 1$. By (3.1), there exists $t \in T$ such that $(g_{j^{z(\varphi^{-1})}})^{\varphi} = tg_j$ for each $j \in \{1,\dots,k\}$. As a consequence, we obtain $t = 1$ by taking any $j \in \{1,\dots,k\}\setminus\{i\}$ such that $j^{z(\varphi^{-1})} \neq i$. Also, for $j \in \{1,\dots,k\}$, $(g_{j^{z(\varphi^{-1})}})^{\varphi} \neq t$ if and only if $j = i$. It follows that $i^{z(\varphi^{-1})} = i$. As i is arbitrary, this implies that $z(\varphi^{-1}) = 1$, and so $\varphi = 1$. This shows that δ is a monomorphism from $\mathrm{Aut}(T)$ to $\mathrm{Sym}(V)$. □

Let M be the permutation group on V induced by the right multiplication action of T^k. For any group X, the *holomorph* of X, denoted by $\mathrm{Hol}(X)$, is the normalizer of the right regular representation of X in $\mathrm{Sym}(X)$. Note that $\langle x(T), y(T), z(\mathrm{Aut}(T))\rangle = x(T) \rtimes z(\mathrm{Aut}(T)) = y(T) \rtimes z(\mathrm{Aut}(T))$ is primitive on $\{1,\dots,k\}$ and permutation isomorphic to $\mathrm{Hol}(T)$. Thus,

$$X := \langle M, \lambda(T), \rho(T), \delta(\mathrm{Aut}(T))\rangle \tag{3.2}$$

is a primitive permutation group on V of type SD with socle M, and the conjugation action of X on the set of k factors of $M \cong T^k$ is permutation isomorphic to $\mathrm{Hol}(T)$. Let $v = D \in V$, a vertex of $\Gamma(T)$. For any $t \in T$ let $\sigma(t) \in M$ be the permutation of V induced by right multiplication by $(t,\dots,t)$. Then

$$X_v/\sigma(T) = X_v/(X_v \cap M) \cong X_vM/M = X/M \cong \mathrm{Hol}(T)$$

since M acts transitively on V, and therefore

$$|X_v| = |\sigma(T)||\mathrm{Hol}(T)| = |T|^3|\mathrm{Out}(T)|. \tag{3.3}$$

Lemma 3.5. $X \leqslant \mathrm{Aut}(\Gamma(T))$.

Proof. Clearly $M \leqslant \mathrm{Aut}(\Gamma(T))$, so it remains to verify that $\lambda(T)$, $\rho(T)$ and $\delta(\mathrm{Aut}(T))$ are subgroups of $\mathrm{Aut}(\Gamma(T))$. Let $D(g_1,\dots,g_k) \in V$ and $D(g_1',\dots,g_k') \in V$. Then we have $D(g_1,\dots,g_k) \to D(g_1',\dots,g_k')$ in $\Gamma(T)$ if and only if

$$(g_1'g_1^{-1},\dots,g_k'g_k^{-1}) \in D(t_1,\dots,t_k)D. \tag{3.4}$$

Let $t \in T$. Since (3.4) holds if and only if

$$\begin{aligned}(g'_{1^{x(t)}}g^{-1}_{1^{x(t)}}, \dots, g'_{k^{x(t)}}g^{-1}_{k^{x(t)}}) &\in D(t_{1^{x(t)}}, \dots, t_{k^{x(t)}})D \\ &= D(tt_1, \dots, tt_k)D \\ &= D(t_1, \dots, t_k)D,\end{aligned}$$

we conclude that $D(g_1, \dots, g_k) \to D(g'_1, \dots, g'_k)$ if and only if $D(g_1, \dots, g_k)^{\lambda(t)} \to D(g'_1, \dots, g'_k)^{\lambda(t)}$. This shows $\lambda(t) \in \mathrm{Aut}(\Gamma(T))$ for any $t \in T$. Similarly, we have $\rho(t) \in \mathrm{Aut}(\Gamma(T))$ for any $t \in T$. Let $\varphi \in \mathrm{Aut}(T)$. Then (3.4) holds if and only if

$$\begin{aligned}((g'_{1^{z(\varphi^{-1})}}g^{-1}_{1^{z(\varphi^{-1})}})^{\varphi}, \dots, (g'_{k^{z(\varphi^{-1})}}g^{-1}_{k^{z(\varphi^{-1})}})^{\varphi}) &\in D((t_{1^{z(\varphi^{-1})}})^{\varphi}, \dots, (t_{k^{z(\varphi^{-1})}})^{\varphi})D \\ &= D((t_1^{\varphi^{-1}})^{\varphi}, \dots, (t_k^{\varphi^{-1}})^{\varphi})D \\ &= D(t_1, \dots, t_k)D.\end{aligned}$$

It follows that $D(g_1, \dots, g_k) \to D(g'_1, \dots, g'_k)$ if and only if

$$D(g_1, \dots, g_k)^{\delta(\varphi)} \to D(g'_1, \dots, g'_k)^{\delta(\varphi)},$$

and so $\delta(\varphi) \in \mathrm{Aut}(\Gamma(T))$ for any $\varphi \in \mathrm{Aut}(T)$. This completes the proof. □

Denote $H = \langle M, \lambda(T)\rangle = M \rtimes \lambda(T) \leqslant X$.

Lemma 3.6. $\Gamma(T)$ *is* $(H, 2)$*-arc-transitive.*

Proof. It is readily seen that $H_v = \sigma(T) \times \lambda(T) \cong T^2$. Let $K = \{\sigma(t)\lambda(t) \mid t \in T\}$. For any $t \in T$ and any $(g_1, \dots, g_k) \in T^k$ we have

$$\begin{aligned}D(g_1, \dots, g_k)^{g^{-1}\sigma(t)\lambda(t)g} &= D(g_1t_1^{-1}t, \dots, g_kt_k^{-1}t)^{\lambda(t)g} \\ &= D(g_{1^{x(t)}}t^{-1}_{1^{x(t)}}t, \dots, g_{k^{x(t)}}t^{-1}_{k^{x(t)}}t)^g \\ &= D(g_{1^{x(t)}}(tt_1)^{-1}t, \dots, g_{k^{x(t)}}(tt_k)^{-1}t)^g \\ &= D(g_{1^{x(t)}}t_1^{-1}, \dots, g_{k^{x(t)}}t_k^{-1})^g \\ &= D(g_{1^{x(t)}}, \dots, g_{k^{x(t)}}) \\ &= D(g_1, \dots, g_k)^{\lambda(t)}.\end{aligned}$$

Hence $g^{-1}\sigma(t)\lambda(t)g = \lambda(t)$ for all $t \in T$. Consequently, $g^{-1}Kg = \lambda(T) < H_v$ and so $K \leqslant H_v \cap gH_vg^{-1}$. Now for any elements s and t of T,

$$\sigma(s)\lambda(t) = (\sigma(s)\lambda(s))\lambda(s^{-1}t) \in K\lambda(T) = K(g^{-1}Kg).$$

It follows that

$$H_v \leqslant K(g^{-1}Kg) \leqslant (H_v \cap gH_vg^{-1})(H_v \cap g^{-1}H_vg),$$

so $H_v = (H_v \cap gH_vg^{-1})(H_v \cap g^{-1}H_vg)$. Thus by Remark 2.5, $\Gamma(T)$ is $(H, 2)$-arc-transitive, as the lemma asserts. □

An immediate consequence of Lemma 3.6 is that $\Gamma(T)$ is $(X, 2)$-arc-transitive. However, X is not transitive on the set of 3-arcs of $\Gamma(T)$, as we shall see in the next lemma.

Lemma 3.7. $\Gamma(T)$ *is not* $(X,3)$*-arc-transitive.*

Proof. Suppose that $\Gamma(T)$ is $(X,3)$-arc-transitive. Then since M is a vertex-transitive normal subgroup of X, Corollary 2.11 asserts that $\Gamma(T)$ is $(M,2)$-arc-transitive. As a consequence, M_v is transitive on $A_2(v) := \{(v_1,v_2) \in V^2 \mid v \to v_1 \to v_2\}$, the set of 2-arcs starting from v. However, $|M_v| = |T|$ while $|A_2(v)| = |T|^2$ as $\Gamma(T)$ is $|T|$-regular. This is not possible. □

3.2 Classification

Throughout this subsection, let T be a nonabelian simple group, $k \geqslant 2$ be an interger, $D = \{(t,\dots,t) \mid t \in T\}$ be a full diagonal subgroup of T^k, V be the set of right cosets of D in T^k, and M be the permutation group induced by the right multiplication action of T^k on V. Suppose that G is a permutation group on V with $M \leqslant G \leqslant M.(\mathrm{Out}(T)\times \mathrm{S}_k)$, and Γ is a connected $(G,2)$-arc-transitive digraph. Let $v = D \in V$ and w be an out-neighbour of v. Then $w = D(t_1,\dots,t_k) \in V$ for some elements $t_1,\dots,t_k$ of T which are not all equal. Without loss of generality, we assume $t_k = 1$. Let $u = D(t_1^{-1},\dots,t_k^{-1}) \in V$ and $g \in M$ be the permutation of V induced by right multiplication by $(t_1,\dots,t_k) \in T^k$. Moreover, define $\{\Omega_1,\dots,\Omega_n\}$ to be the partition of $\{1,\dots,k\}$ such that $t_i = t_j$ if and only if i and j are in the same part of $\{\Omega_1,\dots,\Omega_n\}$. Note that $G_v \leqslant \mathrm{Aut}(T)\times \mathrm{S}_k$. Let α be the projection of G_v into $\mathrm{Aut}(T)$ and β be the projection of G_v into S_k. Let $A = \alpha(G_v)$ and $S = \beta(G_v)$, so that $G_v \leqslant A \times S$, where each element σ of A is induced by an automorphism of T acting on V as

$$D(g_1,\dots,g_k)^\sigma = D(g_1^\sigma,\dots,g_k^\sigma)$$

and each element x of S is induced by a permutation on $\{1,\dots,k\}$ acting on V as

$$D(g_1,\dots,g_k)^x = D(g_{1^{x^{-1}}},\dots,g_{k^{x^{-1}}}).$$

As $G \geqslant M$ we have $\mathrm{Inn}(T) \leqslant A \leqslant \mathrm{Aut}(T)$. Moreover, since G is 2-arc-transitive, Lemma 2.2 implies that $G_v = G_{uv}G_{vw}$. Let R be the stabilizer in S of k in the set $\{1,\dots,k\}$.

Take any $\sigma \in A$ and $x \in S$. Then $\sigma x \in G_u$ if and only if $x^{-1}\sigma^{-1}$ fixes u, that is

$$D((t_{1^x}^{-1})^{\sigma^{-1}},\dots,(t_{(k-1)^x}^{-1})^{\sigma^{-1}},(t_{k^x}^{-1})^{\sigma^{-1}}) = D(t_1^{-1},\dots,t_{k-1}^{-1},1),$$

or equivalently,

$$D(t_{k^x}t_{1^x}^{-1},\dots,t_{k^x}t_{(k-1)^x}^{-1},1) = D((t_1^{-1})^\sigma,\dots,(t_{k-1}^{-1})^\sigma,1). \tag{3.5}$$

Similarly, $\sigma x \in G_w$ if and only if $x^{-1}\sigma^{-1}$ fixes w, which is equivalent to

$$D(t_{k^x}^{-1}t_{1^x},\dots,t_{k^x}^{-1}t_{(k-1)^x},1) = D(t_1^\sigma,\dots,t_{k-1}^\sigma,1). \tag{3.6}$$

Lemma 3.8. $\langle t_1,\dots,t_k\rangle = T$.

Proof. For all $\sigma \in \alpha(G_{uv})$, there exists $x \in S$ such that $\sigma x \in G_u$. Then (3.5) implies that $t_{k^x}t_{i^x}^{-1} = (t_i^{-1})^\sigma$ and thus $t_i^\sigma = t_{i^x}t_{k^x}^{-1}$ for all i such that $1 \leqslant i \leqslant k$. This shows that $\alpha(G_{uv})$ stabilizes $\langle t_1,\dots,t_k\rangle$. Similarly, for all $\sigma \in \alpha(G_{vw})$, there exists

$x \in S$ such that $\sigma x \in G_w$. Then (3.6) implies that $t_i^\sigma = t_{k^x}^{-1} t_{i^x}$ for all i such that $1 \leqslant i \leqslant k$. Accordingly, $\alpha(G_{vw})$ also stabilizes $\langle t_1, \dots, t_k \rangle$. It follows that $A = \alpha(G_v) = \alpha(G_{uv}G_{vw}) = \alpha(G_{uv})\alpha(G_{vw})$ stabilizes $\langle t_1, \dots, t_k \rangle$. Hence $\langle t_1, \dots, t_k \rangle = T$ since $\mathrm{Inn}(T) \leqslant A \leqslant \mathrm{Aut}(T)$. □

Lemma 3.9. $G_{uv} \cap (A \times R) = G_{vw} \cap (A \times R)$.

Proof. Let $\sigma \in A$ and $x \in R$. Then $t_{k^x} = t_k = 1$, and thus (3.6) shows that $\sigma x \in G_w$ if and only if $t_{i^x} = t_i^\sigma$ for all i such that $1 \leqslant i \leqslant k$. Similarly, (3.5) shows that $\sigma x \in G_u$ if and only if $t_{i^x}^{-1} = (t_i^{-1})^\sigma$ for all i such that $1 \leqslant i \leqslant k$. Since this is equivalent to $t_{i^x} = t_i^\sigma$ for all i, we conclude that $\sigma x \in G_w$ if and only if $\sigma x \in G_u$. As a consequence, $G_{uv} \cap (A \times R) = G_{vw} \cap (A \times R)$. □

Lemma 3.10. $G_{uv} \cap A = G_{vw} \cap A = 1$.

Proof. In view of Lemma 3.9 we only need to prove that $G_{vw} \cap A = 1$. For any $\sigma \in G_{vw} \cap A$, (3.6) shows that $D(t_1, \dots, t_{k-1}, 1) = D(t_1^\sigma, \dots, t_{k-1}^\sigma, 1)$, and so $t_i^\sigma = t_i$ for all i such that $1 \leqslant i \leqslant k$. By Lemma 3.8, this implies that $\sigma = 1$ and so $G_{vw} \cap A = 1$, as desired. □

Lemma 3.11. *Both $\beta(G_{uv})$ and $\beta(G_{vw})$ preserve the partition $\{\Omega_1, \dots, \Omega_n\}$.*

Proof. Let $x \in \beta(G_{uv})$. Then there exists $\sigma \in A$ such that $\sigma x \in G_u$, and so (3.5) gives

$$t_{k^x} t_{i^x}^{-1} = (t_i^{-1})^\sigma \tag{3.7}$$

for all i such that $1 \leqslant i \leqslant k$. For any $i, j \in \{1, \dots, k\}$, if i and j are in the same part of $\{\Omega_1, \dots, \Omega_n\}$, then $t_i = t_j$ and so $(t_i^{-1})^\sigma = (t_j^{-1})^\sigma$, which leads to $t_{i^x} = t_{j^x}$ by (3.7). Since $t_{i^x} = t_{j^x}$ if and only if i^x and j^x are in the same part of $\{\Omega_1, \dots, \Omega_n\}$, this shows that x, hence $\beta(G_{uv})$, preserves the partition $\{\Omega_1, \dots, \Omega_n\}$. The proof for $\beta(G_{vw})$ is similar. □

Lemma 3.12. $t_1, \dots, t_k$ *are pairwise distinct.*

Proof. Let U be the subset of V consisting of the elements $D(g_1, \dots, g_k)$ with $g_i = g_j$ whenever i and j are in the same part of $\{\Omega_1, \dots, \Omega_n\}$. By Lemma 3.11, both $\beta(G_{uv})$ and $\beta(G_{vw})$ preserve the partition $\{\Omega_1, \dots, \Omega_n\}$. Then since $S = \beta(G_v) = \beta(G_{uv}G_{vw}) = \beta(G_{uv})\beta(G_{vw})$, we derive that S preserves the partition $\{\Omega_1, \dots, \Omega_n\}$. As a consequence, S stabilizes U setwise. Meanwhile, A and g stabilize U setwise. Hence $G = \langle G_v, g \rangle \leqslant \langle A \times S, g \rangle$ stabilizes U setwise, which implies $U = V$. Thus each Ω_i has size 1 and so $t_1, \dots, t_k$ are pairwise distinct. □

Lemma 3.13. $G_{uv} \cap R = G_{vw} \cap R = 1$.

Proof. In view of Lemma 3.9 we only need to prove that $G_{vw} \cap R = 1$. Let $x \in G_{vw} \cap R$. Then $t_{k^x} = t_k = 1$, and so (3.6) shows that $t_{i^x} = t_i$ for all i such that $1 \leqslant i \leqslant k$. Note that $t_1, \dots, t_k$ are pairwise distinct by Lemma 3.12. We conclude that $x = 1$ and so $G_{vw} \cap R = 1$, as desired. □

Lemma 3.14. $k = |T|$, $\{t_1, \dots, t_k\} = T$ *and* $\Gamma \cong \Gamma(T)$ *as given in* Construction 3.1. *Moreover, if G is vertex-primitive, then the induced permutation group of G on the k copies of T is a subgroup of* $\mathrm{Hol}(T)$ *containing* $\mathrm{Soc}(\mathrm{Hol}(T))$.

Proof. It follows from Lemma 3.9 that $G_{uvw} \cap (A \times R) = G_{uv} \cap (A \times R)$. Then as G is 2-arc-transitive on Γ, we have

$$\begin{aligned} \frac{|G_v|}{|G_{uv}|} = \frac{|G_{uv}|}{|G_{uvw}|} &\leqslant \frac{|G_{uv}|}{|G_{uvw} \cap (A \times R)|} \\ &= \frac{|G_{uv}|}{|G_{uv} \cap (A \times R)|} = \frac{|G_{uv}(A \times R)|}{|A \times R|} \leqslant \frac{|A \times S|}{|A \times R|} = k. \end{aligned} \tag{3.8}$$

We thus obtain $|G_v| \leqslant k|G_{uv}| = k|G_{vw}|$. From Lemma 3.10 we deduce $\beta(G_{uv}) \cong G_{uv}$ and $\beta(G_{vw}) \cong G_{vw}$. Moreover, $t_1, \dots, t_k$ are pairwise distinct by Lemma 3.12, which implies $|T| \geqslant k$. Therefore,

$$k|S| \leqslant |T||S| \leqslant |G_v \cap A||S| = |G_v| \leqslant k|G_{uv}| = k|\beta(G_{uv})| \leqslant k|S|$$

and

$$k|S| \leqslant |T||S| \leqslant |G_v \cap A||S| = |G_v| \leqslant k|G_{vw}| = k|\beta(G_{vw})| \leqslant k|S|.$$

Hence $|G_v \cap A| = |T| = k$, $|G_v| = k|G_{uv}| = k|G_{vw}|$ and $\beta(G_{uv}) = \beta(G_{vw}) = S$. As a consequence, $T = \{t_1, \dots, t_k\}$ by Lemma 3.12, and so $\Gamma \cong \mathrm{Cos}(T^k, D, g) \cong \Gamma(T)$. Also, (3.8) implies that $G_{uvw} = G_{uvw} \cap (A \times R)$. If $G_{uv} \cap S = 1$ or $G_{vw} \cap S = 1$, then Lemma 3.10 implies $S = \beta(G_{uv}) \cong G_{uv} \lesssim A$ or $S = \beta(G_{vw}) \cong G_{vw} \lesssim A$, contradicting Lemma 2.14. Thus $G_{uv} \cap S$ and $G_{vw} \cap S$ are both nontrivial normal subgroups of $\beta(G_{uv}) = \beta(G_{vw}) = S$.

From now on suppose that G is primitive and so S is a primitive subgroup of S_k. By Lemma 3.13, $G_{uv} \cap R = G_{vw} \cap R = 1$, so we derive that $G_{uv} \cap S$ and $G_{vw} \cap S$ are both regular normal subgroups of S. Moreover, $G_{uv} \cap S \neq G_{vw} \cap S$ for otherwise $G_{uvw} \cap S = G_{uv} \cap S$ would be a regular subgroup of S, contrary to the condition $G_{uvw} = G_{uvw} \cap (A \times R) \leqslant A \times R$. This indicates that S has at least two regular normal subgroups, and so $\mathrm{Soc}(S) = N^{2n}$ for some nonabelian simple group N and positive integer n such that $k = |N|^n$ and $S/(G_{uv} \cap S)$ has a normal subgroup isomorphic to N^n. It follows that

$$N^n \lesssim S/(G_{uv} \cap S) = \beta(G_{uv})/(G_{uv} \cap S) \cong \alpha(G_{uv})/(G_{uv} \cap A) \cong \alpha(G_{uv}) \leqslant A,$$

and then Lemma 2.13 implies that $n = 1$ and $N \cong T$. Thus, $\mathrm{Soc}(S) \cong T^2$, and so $\mathrm{Soc}(\mathrm{Hol}(T)) \leqslant S \leqslant \mathrm{Hol}(T)$. □

We are now ready to give the main theorem of this section. Recall X defined in (3.2).

Theorem 3.15. *Let T be a nonabelian simple group, $k \geqslant 2$ be an interger, and $T^k \leqslant G \leqslant T^k.(\mathrm{Out}(T) \times \mathrm{S}_k)$ with diagonal action on the set V of right cosets of $\{(t, \dots, t) \mid t \in T\}$ in T^k. Suppose Γ is a connected $(G, 2)$-arc-transitive digraph with vertex set V. Then $k = |T|$, $\Gamma \cong \Gamma(T)$, $\mathrm{Aut}(\Gamma) = X$ is vertex-primitive of type SD with socle T^k and the conjugation action on the k copies of T permutation isomorphic to $\mathrm{Hol}(T)$, and Γ is not 3-arc-transitive.*

Proof. We have by Lemma 3.14 that $k = |T|$, $\{t_1, \dots, t_k\} = T$ and $\Gamma \cong \Gamma(T)$. In the following, we identify Γ with $\Gamma(T)$. Let X be as in (3.2) and $Y = \mathrm{Aut}(\Gamma(T))$. Then X is vertex-primitive of type SD with socle $T^{|T|}$, and the conjugation action of X on the $|T|$ copies of T is permutation isomorphic to $\mathrm{Hol}(T)$. Also, $X \leqslant Y$ by Lemma 3.5. It follows from [1, Theorem 1.2] that Y is vertex-primitive of type SD with the same socle of X. Then again by Lemma 3.14 we have $Y_v \leqslant \mathrm{Aut}(T) \times \mathrm{Hol}(T)$. Thus by (3.3) $Y_v = X_x$. Since X is vertex-transitive, it follows that $Y = XY_v = X$, and so Γ is not 3-arc-transitive by Lemma 3.7. □

4 Product action on the vertex set

In this section, we study (G, s)-arc-transitive digraphs with vertex set Δ^m such that G acts on Δ^m by product action. We first prove Theorem 1.3.

Proof of Theorem 1.3. Let G_1 be the stabiliser in G of the first coordinate and π_1 be the projection of G_1 into $\mathrm{Sym}(\Delta)$. Then $\pi_1(G_1) = H$. Since N is normal in H and transitive on Δ, N^m is normal in G and transitive on $\Delta^m = V$. Hence Corollary 2.11 implies that Γ is $(N^m, s-1)$-arc-transitive. In particular, since $s \geqslant 2$, N^m is transitive on the set of arcs of Γ, and so Γ has arc set $A = \{u^n \to v^n \mid n \in N^m\}$ for any arc $u \to v$ of Γ.

Let $\alpha \in \Delta$, $u = (\alpha, \dots, \alpha) \in V$ and $v = (\beta_1, \dots, \beta_m)$ be an out-neighbour of u in Γ. By Lemma 2.10 we have $G = N^m G_{uv}$. Let φ be the projection of G to S_m, and we regard $\varphi(G)$ as a subgroup of $\mathrm{Sym}(V)$. Then

$$\varphi(G) \leqslant H^m G = H^m(N^m G_{uv}) = H^m G_{uv}.$$

Take any i in $\{1, \dots, m\}$. Since $\varphi(G)$ is transitive on $\{1, \dots, m\}$, there exists $x \in \varphi(G)$ such that $1^x = i$ and $x = yz$ with $y = (y_1, \dots, y_m) \in H^m$ and $z \in G_{uv}$. Note that $z \in G_{uv}$ and $x \in \mathrm{S}_m$ both fix u. We conclude that y fixes u and hence $y_j \in H_\alpha$ for each j in $\{1, \dots, m\}$. Also, $y^{-1}x = z \in G_{uv} \leqslant G_v$ implies $\beta_1^{y_1^{-1}} = \beta_i$. It follows that for each i in $\{1, \dots, m\}$ there exists $h_i \in H_\alpha$ with $\beta_i^{h_i} = \beta_1$. Let $w = (\beta_1, \dots, \beta_1) \in V$, $h = (h_1, \dots, h_m) \in (H_\alpha)^m$ and Γ^h be the digraph with vertex set V and arc set $A^h := \{u^{nh} \to v^{nh} \mid n \in N^m\}$. It is evident that $u^h = u$, $v^h = w$, and h gives an isomorphism from Γ to Γ^h. Let Σ be the digraph with vertex set Δ and arc set $I := \{\alpha^n \to \beta_1^n \mid n \in N\}$. Then $N \leqslant \mathrm{Aut}(\Sigma)$, and viewing $N^m h = hN^m$ we have

$$\begin{aligned} A^h &= \{u^{hn} \to v^{hn} \mid n \in N^m\} = \{u^n \to w^n \mid n \in N^m\} \\ &= \{(\alpha^{n_1}, \dots, \alpha^{n_m}) \to (\beta_1^{n_1}, \dots, \beta_1^{n_m}) \mid n_1, \dots, n_m \in N\}. \end{aligned}$$

This implies that $\Gamma^h = \Sigma^m$. Consequently, $\Gamma \cong \Sigma^m$.

For any $\beta \in \Delta$, denote by $\delta(\beta)$ the point in $V = \Delta^m$ with all coordinates equal to β. Then $\delta(\alpha) \to \delta(\beta_1)$ in Γ^h since $\alpha \to \beta_1$ in Σ. Let x be any element of H. Then since

$$H = h_1^{-1} H h_1 = h_1^{-1} \pi_1(G_1) h_1 = \pi_1(h)^{-1} \pi_1(G_1) \pi_1(h) = \pi_1(h^{-1} G_1 h),$$

there exists $g \in h^{-1} G_1 h$ such that $x = \pi_1(g)$. As g is an automorphism of Γ^h and $\delta(\alpha) \to \delta(\beta_1)$ in Γ^h, we have $\delta(\alpha)^g \to \delta(\beta_1)^g$ in Γ^h. Comparing first coordinates, this implies that $\alpha^{\pi_1(g)} \to \beta_1^{\pi_1(g)}$ in Σ, which turns out to be $\alpha^x \to \beta_1^x$ in Σ. In other words, $\alpha^x \to \beta_1^x$ is in I. It follows that

$$I = \{\alpha^{xn} \to \beta_1^{xn} \mid n \in N\} = \{\alpha^{nx} \to \beta_1^{nx} \mid n \in N\}$$

as $xN = Nx$. Hence x preserves I, and so $H \leqslant \mathrm{Aut}(\Sigma)$.

Let $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_s$ be an s-arc of Σ. Since Γ is $(N^m, s-1)$-arc-transitive and $N^m = h^{-1} N^m h$, it follows that Γ^h is $(N^m, s-1)$-arc-transitive. Then for any $(s-1)$-arc $\alpha_1' \to \cdots \to \alpha_s'$ of Σ, since $\delta(\alpha_1) \to \cdots \to \delta(\alpha_s)$ and $\delta(\alpha_1') \to \cdots \to \delta(\alpha_s')$ are both $(s-1)$-arcs of Γ^h, there exists $(n_1, \dots, n_m) \in N^m$ such that $\delta(\alpha_i)^{(n_1, \dots, n_m)} = \delta(\alpha_i')$ for each i with $1 \leqslant i \leqslant s$. Hence $\alpha_i^{n_1} = \alpha_i'$ for each i with $1 \leqslant i \leqslant s$. Therefore, Σ is $(N, s-1)$-arc-transitive. Let $\Sigma^+(\alpha_{s-1})$ be the set of out-neighbours of α_{s-1} in

Σ. Take any $\beta \in \Sigma^+(\alpha_{s-1})$. As $\delta(\alpha_s)$ and $\delta(\beta)$ are both out-neighbours of $\delta(\alpha_{s-1})$ in Γ^h and Γ^h is $(h^{-1}Gh, s)$-arc-transitive, there exists $g \in h^{-1}Gh \leqslant H \wr \mathrm{S}_m$ such that g fixes $\delta(\alpha_0), \delta(\alpha_1), \ldots, \delta(\alpha_{s-1})$ and maps $\delta(\alpha_s)$ to $\delta(\beta)$. Write $g = (x_1, \ldots, x_m)\sigma$ with $(x_1, \ldots, x_m) \in H^m$ and $\sigma \in \mathrm{S}_m$. Then x_1 fixes $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ and maps α_s to β. This shows that Σ is (H, s)-arc-transitive, completing the proof. □

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