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Perfect matchings, Hamiltonian cycles and edge-colourings in a class of cubic graphs

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Abstract

A graph G has the Perfect-Matching-Hamiltonian property (PMH-property) if for each one of its perfect matchings, there is another perfect matching of G such that the union of the two perfect matchings yields a Hamiltonian cycle of G. The study of graphs that have the PMH-property, initiated in the 1970s by Las Vergnas and Häggkvist, combines three well-studied properties of graphs, namely matchings, Hamiltonicity and edge-colourings. In this work, we study these concepts for cubic graphs in an attempt to characterise those cubic graphs for which every perfect matching corresponds to one of the colours of a proper 3-edge-colouring of the graph. We discuss that this is equivalent to saying that such graphs are even-2-factorable (E2F), that is, all 2-factors of the graph contain only even cycles. The case for bipartite cubic graphs is trivial, since if G is bipartite then it is E2F. Thus, we restrict our attention to non-bipartite cubic graphs. A sufficient, but not necessary, condition for a cubic graph to be E2F is that it has the PMH-property. The aim of this work

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is to introduce an infinite family of E2F non-bipartite cubic graphs on two parameters, which we coin *papillon graphs*, and determine the values of the respective parameters for which these graphs have the PMH-property or are just E2F. We also show that no two papillon graphs with different parameters are isomorphic.

Keywords: Cubic graph, perfect matching, Hamiltonian cycle, 3-edge-colouring.

Math. Subj. Class. (2020): 05C15, 05C45, 05C70

1 Introduction

Let G be a simple connected graph of even order with vertex set $V(G)$ and edge set $E(G)$. A k-*factor* of G is a k-regular spanning subgraph of G (not necessarily connected). Two very well-studied concepts in graph theory are *perfect matchings* and *Hamiltonian cycles*, where the former is the edge set of a 1-factor and the latter is a connected 2-factor of a graph. For $t \geq 3$, a *cycle* of length t (or a t-cycle), denoted by $C_t = (v_1, \ldots, v_t)$, is a sequence of mutually distinct vertices $v_1, v_2, \ldots, v_t$ with corresponding edge set $\{v_1v_2, \ldots, v_{t-1}v_t, v_tv_1\}$. For definitions not explicitly stated here we refer the reader to [4]. A graph G admitting a perfect matching is said to have the *Perfect-Matching-Hamiltonian property* (for short the *PMH-property*) if for every perfect matching M of G there exists another perfect matching N of G such that the edges of $M \cup N$ induce a Hamiltonian cycle of G. For simplicity, a graph admitting this property is said to be PMH. This property was first studied in the 1970s by Las Vergnas [15] and Häggkvist [9], and for more recent results about the PMH-property we suggest the reader to [2, 1, 3, 7, 8]. In [3], a property stronger than the PMH-property is studied: the Pairing-Hamiltonian property, for short the PH-property. Before proceeding to the definition of this property, we first define what a pairing is. For any graph G, K_G denotes the complete graph on the same vertex set $V(G)$ of G. A perfect matching of K_G is said to be a *pairing* of G, and a graph G is said to have the *Pairing-Hamiltonian property* if every pairing M of G can be extended to a Hamiltonian cycle H of K_G such that $E(H) - M \subseteq E(G)$. Clearly, a graph having the PH-property is also PMH, although the converse is not necessarily true. Amongst other results, the authors of [3] show that the only cubic graphs admitting the PH-property are the complete graph K_4, the complete bipartite graph $K_{3,3}$, and the cube $\mathcal{Q}_3$. However, this does not mean that these are the only three cubic graphs admitting the PMH-property. For instance, all cubic 2-factor Hamiltonian graphs (all 2-factors of such a graph form a Hamiltonian cycle) are PMH (see for example [5, 6, 11, 12, 13]).

If a cubic graph G is PMH, then every perfect matching of G corresponds to one of the colours of a (proper) 3-edge-colouring of the graph, and we say that every perfect matching can be extended to a 3-edge-colouring. This is achieved by alternately colouring the edges of the Hamiltonian cycle containing a predetermined perfect matching using two colours, and then colouring the edges not belonging to the Hamiltonian cycle using a third colour. However, there are cubic graphs which are not PMH but have every one of their perfect matchings that can be extended to a 3-edge-colouring (see for example Figure 1). The following proposition characterises all cubic graphs for which every one of their perfect matchings can be extended to a 3-edge-colouring of the graph.

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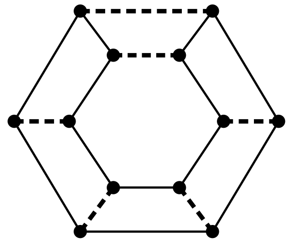

Figure 1: The bold dashed edges can be extended to a proper 3-edge-colouring but not to a Hamiltonian cycle.

Proposition 1.1. *Let G be a cubic graph admitting a perfect matching. Every perfect matching of G can be extended to a 3-edge-colouring of G if and only if all 2-factors of G contain only even cycles.*

Proof. Let F be a 2-factor of G, and let M be the perfect matching $E(G) - E(F)$. Since M can be extended to a 3-edge-colouring of G, F can be 2-edge-coloured, and hence F does not contain any odd cycles. Conversely, let M' be a perfect matching of G, and let F' be its complementary 2-factor, that is, $E(F') = E(G) - M'$. Since F' contains only even cycles, M' can be extended to a 3-edge-colouring, by assigning a first colour to all of its edges and then alternately colouring the edges of the 2-factor F' using another two colours. □

We shall call graphs in which all 2-factors consist only of even cycles as *even*-2-*factorable* graphs, denoted by E2F for short. In particular, from Proposition 1.1, if a cubic graph G has the PMH-property, then it is also E2F. As in the proof of Proposition 1.1, in the sequel, given a perfect matching M of a cubic graph G, the 2-factor obtained after deleting the edges of M from G is referred to as the *complementary* 2-*factor* of M.

If a cubic graph is bipartite, then trivially, each of its perfect matchings can be extended to a 3-edge-colouring, since it is E2F. But what about non-bipartite cubic graphs? In Table 1, we give the number of non-isomorphic non-bipartite 3-connected cubic graphs (having girth at least 4) such that each one of their perfect matchings can be extended to a 3-edge-colouring. As is the case of *snarks* (bridgeless cubic graphs which are not 3-edge-colourable), these seem to be difficult to find, as one can notice after comparing these numbers to the total number of non-isomorphic 3-edge-colourable (Class I) non-bipartite 3-connected cubic graphs having girth at least 4, also given in Table 1. The numbers shown in this table were obtained thanks to a computer check done by Jan Goedgebeur, and the data is sorted according to the cyclic connectivity of the graphs considered. We remark that E2F cubic graphs having girth 3 can be obtained by applying a star product between an E2F cubic graph of smaller order and the complete graph K_4—this has been investigated further by the last two authors in [14]. This is the reason why only graphs having girth at least 4 are considered in this work. More results on star products (also known in the literature as 3-cut connections) in cubic graphs can be found in [5, 6, 10, 11, 12, 13].

A complete characterisation of which cubic graphs are PMH is still elusive, so considering the Class I non-bipartite cubic graphs having the property that each one of their perfect matchings can be extended to a 3-edge-colouring may look presumptuous. As far as we

	Cyclic connectivity				Total no. of graphs		
Number of vertices	3	4	5	6	E2F	Class I	ratio E2F : Class I
8	/	1	/	/	1	1	100%
10	/	/	/	/	0	3	0%
12	2	5	2	/	9	17	52.94%
14	2	2	2	/	6	92	6.52%
16	35	56	4	/	95	716	13.27%
18	84	21	9	/	114	7343	1.55%
20	926	655	15	2	1598	93946	1.70%
22	2978	331	17	6	3332	1400203	0.24%

Table 1: The number of non-isomorphic non-bipartite 3-connected cubic graphs with girth at least 4 which are E2F and Class I.

know this property and the corresponding characterisation problem were never considered before and tackling the following problem seems a reasonable step to take.

Problem 1.2. Characterise the Class I non-bipartite cubic graphs for which each one of their perfect matchings can be extended to a 3-edge-colouring, that is, are E2F.

We remark that although the PMH-property is an appealing property in its own right, Problem 1.2 continues to justify its study in relation to cubic graphs. Observe that in the family of cubic graphs, whilst snarks are not 3-edge-colourable, even-2-factorable graphs are quite the opposite being "very much 3-edge-colourable", since the latter can be 3-edge-coloured by assigning a colour to one of its perfect matchings, and then alternately colour the edges of the complementary 2-factor.

1.1 Cycle permutation graphs

Consider two disjoint cycles each of length t, referred to as the first and second t-cycles and denoted by $(x_1, \ldots, x_t)$ and $(y_1, \ldots, y_t)$, respectively. Let σ be a permutation of the symmetric group $\mathcal{S}_t$ on the t symbols $\{1, \ldots, t\}$. The *cycle permutation graph corresponding to* σ is the cubic graph obtained by considering the first and second t-cycles in which x_i is adjacent to $y_{\sigma(i)}$, where $\sigma(i)$ is the image of i under the permutation σ.

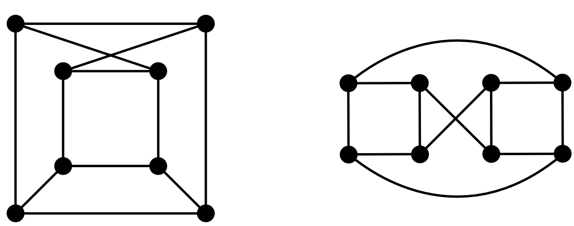

Figure 2: Two different drawings of the smallest non-bipartite E2F cubic graph.

The smallest non-bipartite cubic graph (from Table 1) which is E2F is in fact a cycle permutation graph corresponding to $\sigma = (1\ 2) \in \mathcal{S}_4$, where $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3$, and $\sigma(4) = 4$ (see Figure 2). This shows that the edges between the vertices of the first and second 4-cycles of the cycle permutation graph are $x_1y_2, x_2y_1, x_3y_3, x_4y_4$. In what follows we shall denote permutations in cycle notation and, for simplicity, fixed points shall be suppressed. With the help of Wolfram Mathematica, in Table 2 we provide the number of non-isomorphic non-bipartite cycle permutation graphs up to 20 vertices which are PMH or just E2F. Recall that PMH cubic graphs are also E2F, and so, PMH cycle permutation graphs should be searched for from amongst the cycle permutation graphs which are E2F. We also remark that, in the sequel, cycle permutation graphs with total number of vertices equal to twice an odd number are not considered because, in this case, the first and second cycles form a 2-factor consisting of two odd cycles, and so they are trivially not E2F.

No. of vertices	E2F	PMH
8	1	0
12	5	1
16	28	2
20	175	0

Table 2: The number of non-isomorphic non-bipartite cycle permutation graphs with girth at least 4 which are E2F and PMH.

This work is a first structured attempt at tackling Problem 1.2. We give an infinite family of non-bipartite cycle permutation graphs which admit the PMH-property or are just E2F. In Section 2, we generalise the smallest cubic graph which is E2F into a family of non-bipartite cycle permutation graphs, namely papillon graphs $\mathcal{P}_{r,\ell}$ (for $r, \ell \in \mathbb{N}$), whose smallest member $\mathcal{P}_{1,1}$ is, in fact, the graph in Figure 2. We show that papillon graphs are E2F for all values of r and ℓ (Theorem 2.3) and PMH if and only if both r and ℓ are even (Theorem 3.8 and Theorem 3.9).

2 Papillon graphs

Let $[n] = \{1, \ldots, n\}$, for some positive integer n.

Definition 2.1. Let r and ℓ be two positive integers. The *papillon graph* $\mathcal{P}_{r,\ell}$ is the graph on $4r + 4\ell$ vertices such that $V(\mathcal{P}_{r,\ell}) = \{u_i, v_i : i \in [2r + 2\ell]\}$, where:

(i) $(u_1, u_2, \ldots, u_{2r+2\ell})$ is a cycle of length $2r + 2\ell$;

(ii) u_i is adjacent to v_i, for each $i \in [2r + 2\ell]$; and

(iii) the adjacencies between the vertices v_i, for $i \in [2r + 2\ell]$, form a cycle of length $2r + 2\ell$ given by the edge set

$$\{v_{2i-1}v_{2i} : i \in [r + \ell]\} \cup \{v_{2i-1}v_{2i+2} : i \in [r + \ell - 1] \setminus \{s\}\} \\ \cup \{v_2v_{2s+2}, v_{2s-1}v_{2r+2\ell-1}\},$$

where $s = \min\{r, \ell\}$.

Clearly, the two papillon graphs $\mathcal{P}_{r,\ell}$ and $\mathcal{P}_{\ell,r}$ are isomorphic, and henceforth, without loss of generality, we shall tacitly assume that $r \leq \ell$. The papillon graph $\mathcal{P}_{r,\ell}$ for $r \geq 2$ is depicted in Figure 3. When r and ℓ are equal, say $r = \ell = n$, the papillon graph $\mathcal{P}_{r,\ell}$ is said to be *balanced*, and simply denoted by $\mathcal{P}_n$ (see, for example, Figure 4). Otherwise, $\mathcal{P}_{r,\ell}$ is said to be *unbalanced* (see, for example, Figure 11). The $(2r + 2\ell)$-cycle induced by the sets of vertices $\{u_i : i \in [2r + 2\ell]\}$ is referred to as the *outer-cycle*, whilst the $(2r + 2\ell)$-cycle induced by the vertices $\{v_i : i \in [2r + 2\ell]\}$ is referred to as the *inner-cycle*. The edges on these two $(2r + 2\ell)$-cycles are said to be the *outer-edges* and *inner-edges* accordingly, whilst the edges u_iv_i are referred to as *spokes*. The edges $u_1u_{2r+2\ell}$, $v_{2r-1}v_{2r+2\ell-1}$, v_2v_{2r+2}, $u_{2r}u_{2r+1}$ are denoted by a, b, c, d, respectively, and we shall also denote the set $\{a, b, c, d\}$ by $\mathcal{X}$. The set $\mathcal{X}$ is referred to as the *principal* 4*-edge-cut* of $\mathcal{P}_{r,\ell}$.

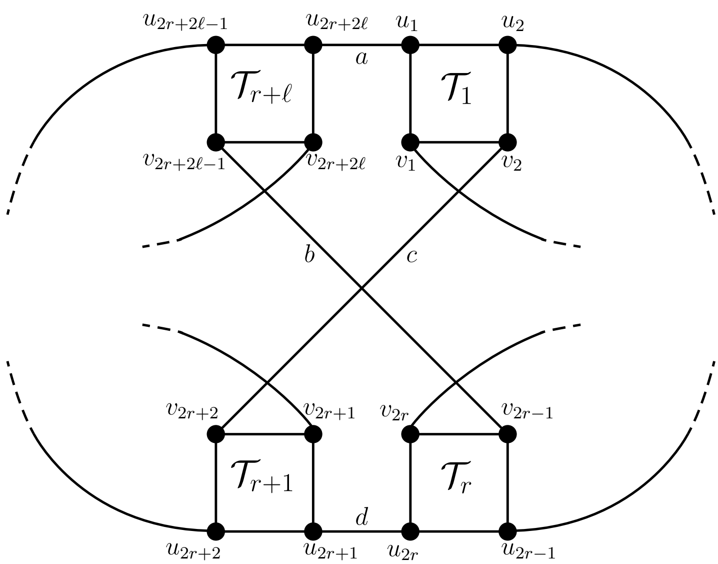

Figure 3: The papillon graph $\mathcal{P}_{r,\ell}$, for $\ell \geq r \geq 2$, and the 4-pole $\mathcal{T}_j$, for $j \in [r + \ell]$.

The graph in Figure 2 is actually the smallest (balanced) papillon graph $\mathcal{P}_1$. In general, since $\{u_i : i \in [2r + 2\ell]\}$ and $\{v_i : i \in [2r + 2\ell]\}$ induce two disjoint $(2r + 2\ell)$-cycles in $\mathcal{P}_{r,\ell}$, and since every vertex belonging to the outer-cycle is adjacent to exactly one vertex on the inner-cycle, there exists an isomorphism π between the papillon graph $\mathcal{P}_{r,\ell}$ and a cycle permutation graph corresponding to some $\sigma \in \mathcal{S}_{2r+2\ell}$ satisfying $\pi(x_i) = u_i$ and $\pi(y_i) = v_{\sigma^{-1}(i)}$, for each $i \in [2r + 2\ell]$. In fact, the papillon graph $\mathcal{P}_{r,\ell}$ is the cycle permutation graph, with $(u_1, \dots, u_{2r+2\ell})$ as the first cycle, corresponding to the permutation:

- $\sigma_{1,\ell} := (3\ \ 4) \dots (2\ell + 1\ \ 2\ell + 2)$, with fixed points 1 and 2, when $\ell \geq 1$;
- $\sigma_{2,2} := (1\ \ 2)(3\ \ 4)(5\ \ 7)(6\ \ 8)$;
- $\sigma_{r,3} := (1\ \ 2) \dots (2r - 1\ \ 2r)(2r + 1\ \ 2r + 5)(2r + 2\ \ 2r + 6)$, with fixed points $2r + 3$ and $2r + 4$, when $r \in \{2, 3\}$; and

- $\sigma_{r,\ell} := (1\ \ 2)\dots(2r-1\ \ 2r)(2r+1\ \ 2r+2\ell-1)(2r+2\ \ 2r+2\ell)(2r+3\ \ 2r+2\ell-3)(2r+4\ \ 2r+2\ell-2)\dots(\alpha\ \ \beta)$, when $\ell \geq r \geq 4$, where $(\alpha\ \ \beta) = (2r+\ell\ \ 2r+\ell+2)$ if ℓ is even, and $(\alpha\ \ \beta) = (2r+\ell-1\ \ 2r+\ell+3)$ if ℓ is odd.

We remark that when $r > 1$, the above permutations has no fixed points when ℓ is even, but, when ℓ is odd, $2r+\ell$ and $2r+\ell+1$ are fixed points, and thus, in this case, $x_{2r+\ell}$ is adjacent to $y_{2r+\ell}$, and $x_{2r+\ell+1}$ is adjacent to $y_{2r+\ell+1}$ in $\mathcal{P}_{r,\ell}$. Note that since $\sigma_{r,\ell}$ is an involution for all positive integers r and ℓ, the isomorphism π mentioned above can be rewritten as follows: $\pi(x_i) = u_i$ and $\pi(y_i) = v_{\sigma(i)}$, for each $i \in [2r+2\ell]$. The papillon graph $\mathcal{P}_{r,\ell}$ admits a natural automorphism ψ which exchanges the two cycles, given by $\psi(u_i) = v_{\sigma_{r,\ell}(i)}$ and $\psi(v_i) = u_{\sigma_{r,\ell}(i)}$, for each $i \in [2r+2\ell]$. In fact, the function ψ is clearly bijective. Moreover, it maps edges of the outer-cycle to edges of the inner-cycle (and vice-versa), and maps spokes to spokes, since the edges u_iv_i are mapped to $u_{\sigma_{r,\ell}(i)}v_{\sigma_{r,\ell}(i)}$.

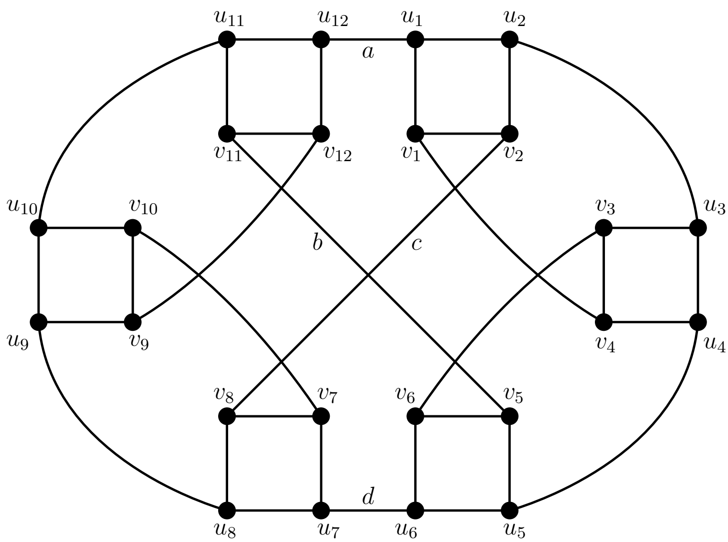

Figure 4: The balanced papillon graph $\mathcal{P}_3$ on 24 vertices.

Before proceeding, we introduce multipoles which generalise the notion of graphs. This will become useful when describing papillon graphs. A *multipole* $\mathcal{Z}$ consists of a set of vertices $V(\mathcal{Z})$ and a set of generalised edges such that each generalised edge is either an edge in the usual sense (that is, it has two endvertices) or a semiedge. A *semiedge* is a generalised edge having exactly one endvertex. The set of semiedges of $\mathcal{Z}$ is denoted by $\partial\mathcal{Z}$ whilst the set of edges of $\mathcal{Z}$ having two endvertices is denoted by $E(\mathcal{Z})$. Two semiedges are *joined* if they are both deleted and their endvertices are made adjacent. A *k-pole* is a multipole with k semiedges. A perfect matching M of a k-pole $\mathcal{Z}$ is a subset of generalised edges of $\mathcal{Z}$ such that every vertex of $\mathcal{Z}$ is incident with exactly one generalised edge of M. In what follows, we shall construct papillon graphs by joining together semiedges of a number of multipoles. In this sense, given a perfect matching M of a graph G, and a multipole $\mathcal{Z}$ used as a building block to construct G, we shall say that M contains a semiedge e of the multipole $\mathcal{Z}$, if M contains the edge in G obtained by joining e to another semiedge in the process of constructing G.

The 4-pole $\mathcal{Z}$ with vertex set $\{z_1, z_2, z_3, z_4\}$, such that $E(\mathcal{Z})$ induces the 4-cycle (z_1, z_2, z_3, z_4) and with exactly one semiedge incident to each of its vertices is referred to as a C_4*-pole* (see Figure 5). For each $i \in [4]$, let the semiedge incident to z_i be denoted by f_i. The semiedges f_1 and f_2 are referred to as the *upper left semiedge* and the *upper right semiedge* of $\mathcal{Z}$, respectively. On the other hand, the semiedges f_3 and f_4 are referred to as the *lower left semiedge* and the *lower right semiedge* of $\mathcal{Z}$, respectively (see Figure 5).

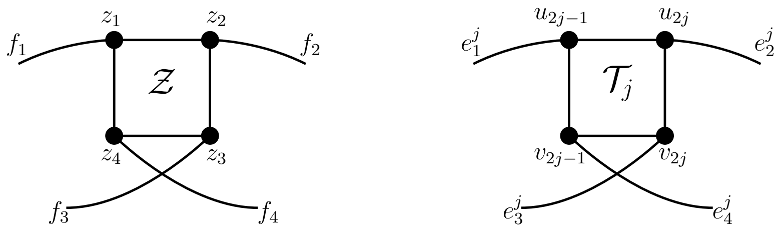

Figure 5: A C_4-pole $\mathcal{Z}$ and the 4-pole $\mathcal{T}_j$ in $\mathcal{P}_{r,\ell}$.

For some integer $n \geq 1$, let $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ be n copies of the above C_4-pole $\mathcal{Z}$. For each $j \in [n]$, let $V(\mathcal{Z}_j) = \{z_1^j, z_2^j, z_3^j, z_4^j\}$, and let $f_1^j, f_2^j, f_3^j, f_4^j$ be the semiedges of $\mathcal{Z}_j$ respectively incident to $z_1^j, z_2^j, z_3^j, z_4^j$ such that f_1^j and f_2^j are the upper left and upper right semiedges of $\mathcal{Z}_j$, whilst f_3^j and f_4^j are the lower left and lower right semiedges of $\mathcal{Z}_j$. A *chain of* C_4*-poles* of length $n \geq 2$, is the 4-pole obtained by respectively joining f_2^j and f_4^j (upper and lower right semiedges of $\mathcal{Z}_j$) to f_1^{j+1} and f_3^{j+1} (upper and lower left semiedges of $\mathcal{Z}_{j+1}$), for every $j \in [n-1]$. When $n = 1$, a chain of C_4-poles of length 1 is just a C_4-pole. For simplicity, we shall refer to a chain of C_4-poles of length n, as a n*-chain of* C_4*-poles*, or simply a n*-chain*. The semiedges f_1^1 and f_3^1 (similarly, f_2^n and f_4^n) are referred to as the upper left and lower left (respectively, upper right and lower right) semiedges of the n-chain. A chain of C_4-poles of any length has exactly four semiedges. For simplicity, when we say that e_1, e_2, e_3, e_4 are the four semiedges of a chain $\mathcal{Z}'$ of C_4-poles (possibly of length 1), we mean that e_1 and e_2 are respectively the upper left and upper right semiedges of $\mathcal{Z}'$, whilst e_3 and e_4 are respectively the lower left and lower right semiedges of the same chain $\mathcal{Z}'$ (see Figure 6). The semiedges e_1 and e_2 (similarly, e_3 and e_4) are referred to collectively as the *upper semiedges* (respectively, *lower semiedges*) of $\mathcal{Z}'$. In a similar way, the semiedges e_1 and e_3 (similarly, e_2 and e_4) are referred to collectively as the *left semiedges* (respectively, *right semiedges*) of $\mathcal{Z}'$.

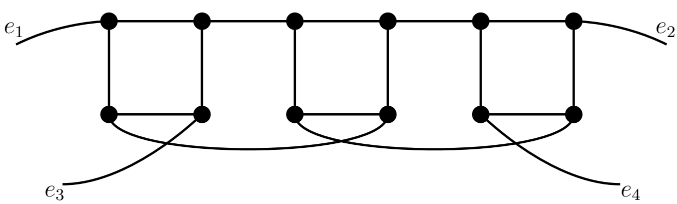

Figure 6: A chain of C_4-poles of length 3 having semiedges e_1, e_2, e_3, e_4.

In order to construct the papillon graph $\mathcal{P}_{r,\ell}$ using C_4-poles as building blocks, for each $j \in [r+\ell]$, we consider the 4-pole $\mathcal{T}_j$ arising from the cycle $(u_{2j-1}, u_{2j}, v_{2j}, v_{2j-1})$ of $\mathcal{P}_{r,\ell}$, whose semiedges are $e_1^j, e_2^j, e_3^j, e_4^j$ as in Figure 5. The r-chain and ℓ-chain giving rise to $\mathcal{P}_{r,\ell}$ consist of $\mathcal{T}_1, \ldots, \mathcal{T}_r$ (referred to as the *right r-chain* of $\mathcal{P}_{r,\ell}$), and $\mathcal{T}_{r+1}, \ldots, \mathcal{T}_{r+\ell}$ (referred to as the *left ℓ-chain* of $\mathcal{P}_{r,\ell}$), which have semiedges $e_1^1, e_2^r, e_3^1, e_4^r$, and $e_1^{r+1}, e_2^{r+\ell}, e_3^{r+1}, e_4^{r+\ell}$, respectively. The papillon graph $\mathcal{P}_{r,\ell}$ is then obtained by joining the semiedges in pairs as follows: e_1^1 to $e_2^{r+\ell}$, e_2^r to e_1^{r+1}, e_3^1 to e_3^{r+1}, and e_4^r to $e_4^{r+\ell}$.

2.1 Odd cycles and isomorphisms in the class of papillon graphs

In this section we shall discuss the presence and behaviour of odd cycles in papillon graphs. Consider the balanced papillon graph $\mathcal{P}_n$ and let C be an odd cycle in $\mathcal{P}_n$. Since cycles intersect C_4-poles in 2, 3 or 4 vertices, there must exist some $t_1 \in [2n]$, such that $|V(C) \cap V(\mathcal{T}_{t_1})| = 3$. Without loss of generality, assume that $t_1 \in [n]$, that is, $\mathcal{T}_{t_1}$ belongs to the right n-chain of $\mathcal{P}_n$. If $t_1 \notin \{1, n\}$, we must have exactly one of the following:

- $|V(C) \cap V(\mathcal{T}_i)| = 4$, for all $i \in \{1, \ldots, t_1 - 1\}$; or
- $|V(C) \cap V(\mathcal{T}_i)| = 4$, for all $i \in \{t_1 + 1, \ldots, n\}$.

Without loss of generality, assume that we either have $t_1 = 1$, or $|V(C) \cap V(\mathcal{T}_i)| = 4$, for all $i \in \{1, \ldots, t_1 - 1\}$. This implies that the number of vertices in C belonging to $\cup_{i=1}^{t_1} V(\mathcal{T}_i)$ is odd and at least 3. Moreover, the edges a and c must belong to C. We claim that $b \notin E(C)$. For, suppose that $b \in E(C)$. Since $\mathcal{X}$ is a 4-edge-cut, $d \in E(C)$ as well. This implies that $n > 1$ and there exist:

- $t_2 \in \{t_1 + 1, \ldots, n\}$, such that $|V(C) \cap V(\mathcal{T}_{t_2})| = 3$;
- $s_1 \in \{n + 1, \ldots, 2n - 1\}$, such that $|V(C) \cap V(\mathcal{T}_{s_1})| = 3$; and
- $s_2 \in \{s_1 + 1, \ldots, 2n\}$, such that $|V(C) \cap V(\mathcal{T}_{s_2})| = 3$.

Let $\Omega = \{1, \ldots, t_1\} \cup \{t_2, \ldots, n, n+1, \ldots, s_1\} \cup \{s_2, \ldots, 2n\}$. If $\Omega \setminus \{t_1, t_2, s_1, s_2\} \neq \emptyset$, then for any $j \in \Omega \setminus \{t_1, t_2, s_1, s_2\}$, $|V(C) \cap V(\mathcal{T}_j)| = 4$. Additionally, for any $k \in [2n] \setminus \Omega$, $|V(C) \cap V(\mathcal{T}_k)| = 0$. However, this means that C has even length, a contradiction. Thus, $\{b, d\} \cap E(C) = \emptyset$. As a result, C intersects none of the C_4-poles $\mathcal{T}_{t_1+1}, \ldots, \mathcal{T}_n$, but intersects each of the C_4-poles $\mathcal{T}_{n+1}, \ldots, \mathcal{T}_n$ in exactly 2 or 4 vertices. Hence, the length of C is at least $2n + 3$. When $n = 1$, $(u_1, u_2, v_2, v_4, u_4)$ is a 5-cycle, and when $n > 1$, $(u_1, u_2, v_2, v_{2n+2}, u_{2n+2}, u_{2n+3}, u_{2n+4}, \ldots, u_{4n})$ is an odd cycle of length exactly $2n+3$. Therefore, a shortest odd cycle in $\mathcal{P}_n$ has length $2n + 3$. By using similar arguments, a shortest odd cycle in $\mathcal{P}_{r,\ell}$ has length $2r + 3$.

Remark 2.2. The papillon graph $\mathcal{P}_{r,\ell}$ is not bipartite and has a shortest odd cycle of length $2r + 3$.

Consequently, we can show that any two distinct papillon graphs $\mathcal{P}_{r_1,\ell_1}$ and $\mathcal{P}_{r_2,\ell_2}$ are not isomorphic, where by distinct we mean that $(r_1, \ell_1) \neq (r_2, \ell_2)$. Suppose not, for contradiction. Since $\mathcal{P}_{r_1,\ell_1} \simeq \mathcal{P}_{r_2,\ell_2}$, we must have $r_1 + \ell_1 = r_2 + \ell_2$, and so if $r_1 = r_2$, then this implies that $\ell_1 = \ell_2$, and conversely. Hence, $r_1 \neq r_2$ and $\ell_1 \neq \ell_2$. Thus, without loss of generality, we can assume that $r_1 < r_2$. However, this means that a shortest odd cycle in $\mathcal{P}_{r_1,\ell_1}$ (of length $2r_1 + 3$), is shorter than a shortest odd cycle in $\mathcal{P}_{r_2,\ell_2}$ (of length $2r_2 + 3$), a contradiction.

We are now in a position to give our first result.

Theorem 2.3. *Every papillon graph $\mathcal{P}_{r,\ell}$ is E2F.*

Proof. Let $\mathcal{P}_{r,\ell}$ be a counterexample to the above statement, and let M be a perfect matching of $\mathcal{P}_{r,\ell}$ whose complementary 2-factor contains an odd cycle C. As previously discussed, C must intersect some $\mathcal{T}_j$, for some $j \in [r+\ell]$, in exactly 3 (consecutive) vertices. Without loss of generality, assume that these 3 vertices are u_{2j-1}, u_{2j}, v_{2j}. This means that both the left semiedges (e_1^j and e_3^j) of $\mathcal{T}_j$ belong to this odd cycle. However, since C is in the complementary 2-factor of M, the two edges $u_{2j-1}v_{2j-1}$ and $v_{2j-1}v_{2j}$ (which do not belong to $E(C)$) must both belong to M, a contradiction. $\square$

3 The PMH-property in papillon graphs

3.1 The balanced case $r = \ell$

Let M be a perfect matching of the balanced papillon graph $\mathcal{P}_n$. Since $\mathcal{X} = \{a, b, c, d\}$ is a 4-edge-cut of $\mathcal{P}_n$, $|M \cap \mathcal{X}| \equiv 0 \pmod 2$, that is, $|M \cap \mathcal{X}|$ is $0, 2$ or 4. The following is a useful lemma which shall be used frequently in the results that follow.

Lemma 3.1. *Let M be a perfect matching of the balanced papillon graph $\mathcal{P}_n$ and let $\mathcal{X}$ be its principal 4-edge-cut. If $|M \cap \mathcal{X}| = k$, then $|M \cap \partial\mathcal{T}_j| = k$, for each $j \in [2n]$.*

Proof. Let M be a perfect matching of $\mathcal{P}_n$. We first note that the left semiedges of a C_4-pole are contained in a perfect matching if and only if the right semiedges of the C_4-pole are contained in the same perfect matching. The lemma is proved by considering three cases depending on the possible values of k, that is, $0, 2$ or 4. When $n = 1$, the result clearly follows since $\mathcal{X}$ is made up by joining $\partial\mathcal{T}_1$ and $\partial\mathcal{T}_2$ accordingly. So assume $n \geq 2$.

Case I: $k = 0$.
Since a and c do not belong to M, the left semiedges of $\mathcal{T}_1$ are not contained in M, and so M cannot contain its right semiedges. Therefore, $|M \cap \partial\mathcal{T}_1| = 0$. Consequently, the left semiedges of $\mathcal{T}_2$ are not contained in M implying again that $|M \cap \partial\mathcal{T}_2| = 0$. By repeating the same argument up till the n^{th} C_4-pole, we have that $|M \cap \partial\mathcal{T}_j| = 0$, for every $j \in [n]$. By noting that c and d do not belong to M and repeating a similar argument to the 4-poles in the left n-chain, we can deduce that $|M \cap \partial\mathcal{T}_j| = 0$ for every $j \in [2n]$.

Case II: $k = 4$.
Since a and c belong to M, the left semiedges of $\mathcal{T}_1$ are contained in M, and so M contains its right semiedges as well. Therefore, $|M \cap \partial\mathcal{T}_1| = 4$. Consequently, the left semiedges of $\mathcal{T}_2$ are contained in M implying again that $|M \cap \partial\mathcal{T}_2| = 4$. As in Case I, by noting that both c and d belong to M and repeating a similar argument to the 4-poles in the left n-chain, we can deduce that $|M \cap \partial\mathcal{T}_j| = 4$ for every $j \in [2n]$.

Case III: $k = 2$.
We first claim that when $k = 2$, $M \cap \mathcal{X}$ must be equal to $\{a, d\}$ or $\{b, c\}$. For, suppose that $M \cap \mathcal{X} = \{a, c\}$, without loss of generality. This means that the right semiedges of $\mathcal{T}_1$ are also contained in M, implying that $|M \cap \partial\mathcal{T}_1| = 4$. This implies that the left semiedges of $\mathcal{T}_2$ are contained in M, which forces $|M \cap \partial\mathcal{T}_j|$ to be equal to 4, for every $j \in [2n]$. In particular, $|M \cap \partial\mathcal{T}_n| = 4$, implying that the edges b and d belong to M, a contradiction since $M \cap \mathcal{X} = \{a, c\}$. This proves our claim. Since the natural automorphism ψ of $\mathcal{P}_n$,

which exchanges the outer- and inner-cycles, exchanges also $\{a, d\}$ with $\{b, c\}$, without loss of generality, we may assume that $M \cap \mathcal{X} = \{a, d\}$. Since $c \notin M$, $1 \leq |M \cap \partial \mathcal{T}_1| < 4$. But, $\partial \mathcal{T}_1$ corresponds to a 4-edge-cut in $\mathcal{P}_n$, and so, by using a parity argument, $|M \cap \partial \mathcal{T}_1|$ must be equal to 2, implying that exactly one of the right semiedges of $\mathcal{T}_1$ is contained in M. This means that exactly one left semiedge of $\mathcal{T}_2$ is contained in M, and consequently, by a similar argument now applied to $\mathcal{T}_2$, we obtain $|M \cap \partial \mathcal{T}_2| = 2$. By repeating the same argument and noting that $\mathcal{T}_{n+1}$ has exactly one left semiedge (corresponding to the edge d) contained in M, one can deduce that $|M \cap \partial \mathcal{T}_j| = 2$ for every $j \in [2n]$. □

The following two results are two consequences of the above lemma and they both follow directly from the proof of Case III. In a few words, if a perfect matching M of $\mathcal{P}_n$ intersects its principal 4-edge-cut in exactly two of its edges, then these two edges are either the pair $\{a, d\}$ or the pair $\{b, c\}$, and, for every $j \in [2n]$, M contains only one pair of semiedges of $\mathcal{T}_j$ which does not consist of the pair of left semiedges of $\mathcal{T}_j$ nor the pair of right semiedges of $\mathcal{T}_j$.

Corollary 3.2. *Let M be a perfect matching of $\mathcal{P}_n$ and let $\mathcal{X}$ be its principal 4-edge-cut. If $|M \cap \mathcal{X}| = 2$, then $M \cap \mathcal{X}$ is equal to $\{a, d\}$ or $\{b, c\}$.*

Corollary 3.3. *Let M be a perfect matching of $\mathcal{P}_n$ and let $\mathcal{X}$ be its principal 4-edge-cut such that $|M \cap \mathcal{X}| = 2$. For each $j \in [2n]$, M contains exactly one of the following sets of semiedges: $\{e_1^j, e_2^j\}, \{e_3^j, e_4^j\}, \{e_1^j, e_4^j\}, \{e_2^j, e_3^j\}$, that is, of all possible pairs of semiedges of $\mathcal{T}_j$, $\{e_1^j, e_3^j\}$ and $\{e_2^j, e_4^j\}$ cannot be contained in M.*

In the sequel, the process of traversing one path after another shall be called *concatenation of paths*. If two paths P and Q have endvertices x, y and y, z, respectively, we write PQ to denote the path starting at x and ending at z obtained by traversing P and then Q. Without loss of generality, if x is adjacent to y, that is, P is a path on two vertices, we may write xyQ instead of PQ.

Lemma 3.4. *Let M_1 be a perfect matching of $\mathcal{P}_n$ such that $|M_1 \cap \mathcal{X}| = 2$.*

(i) *There exists a perfect matching M_2 of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$.*

(ii) *The complementary 2-factors of M_1 and M_2 are both Hamiltonian cycles.*

Proof. (i) Since $|M_1 \cap \mathcal{X}| = 2$, by Lemma 3.1 we get that $|M_1 \cap \partial \mathcal{T}_j| = 2$ for every $j \in [2n]$. For each j, let $P^{(j)}$ be the subgraph of $\mathcal{P}_n$ which is induced by $E(\mathcal{T}_j) - M_1$. Note that $\cup_{j=1}^{2n} V(P^{(j)}) = V(\mathcal{P}_n)$. By Corollary 3.3, each $P^{(j)}$ is a path of length 3. Letting N be the unique perfect matching of $\mathcal{P}_n$ which intersects each $E(P^{(j)})$ in exactly two edges, we note that $M_1 \cap N = \emptyset$. Let $M_2 = E(\mathcal{P}_n) - (M_1 \cup N)$. Since M_1 and N are two disjoint perfect matchings, M_2 is also a perfect matching of $\mathcal{P}_n$ and, in particular, M_2 contains $\mathcal{X} - (M_1 \cap \mathcal{X})$. Thus, $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$, proving part (i).

(ii) Let M_2 be as in part (i), that is, $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$. When $n = 1$, the result clearly follows. So assume $n \geq 2$. For distinct i and j in $[2n]$, let $Q^{(i,j)}$ be the subgraph of $\mathcal{P}_n$ which is induced by $M_2 \cap \{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_i), y \in V(\mathcal{T}_j)\}$, that is, $E(Q^{(i,j)})$ is either empty or consists of exactly one edge, that is, $Q^{(i,j)}$ is a path of length 1. When $M_1 \cap \mathcal{X} = \{a, d\}$, we can form a Hamiltonian cycle of $\mathcal{P}_n$ (not containing M_1) by considering the following concatenation of paths:

$$P^{(1)}Q^{(1,2)} \ldots Q^{(n-1,n)}P^{(n)}Q^{(n,2n)}P^{(2n)}Q^{(2n,2n-1)} \ldots P^{(n+1)}Q^{(n+1,1)},$$

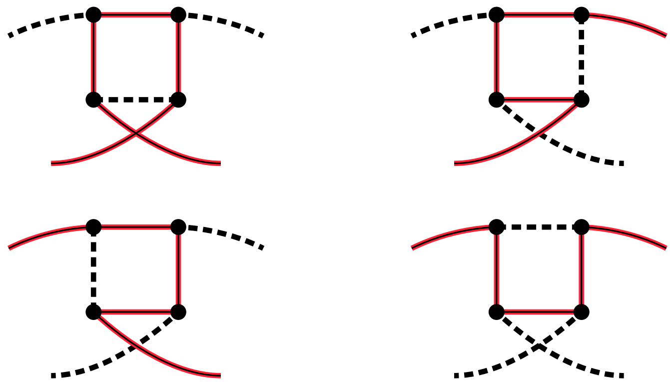

Figure 7: Perfect matching M_1 (bold dashed edges) with $|M_1 \cap \mathcal{X}| = 2$ and its complementary 2-factor (highlighted edges).

where $Q^{(1,2)}$ and $Q^{(2n,2n-1)}$ are respectively followed by $P^{(2)}$ and $P^{(2n-1)}$, and, $Q^{(n,2n)}$ and $Q^{(n+1,1)}$ consist of the edges b and c, respectively. On the other hand, when $M_1 \cap \mathcal{X} = \{b, c\}$, we can form a Hamiltonian cycle of $\mathcal{P}_n$ (not containing M_1) by considering the following concatenation of paths:

$$P^{(1)}Q^{(1,2)} \dots Q^{(n-1,n)}P^{(n)}Q^{(n,n+1)}P^{(n+1)}Q^{(n+1,n+2)} \dots P^{(2n)}Q^{(2n,1)},$$

where $Q^{(1,2)}$ and $Q^{(n+1,n+2)}$ are respectively followed by $P^{(2)}$ and $P^{(n+2)}$, and, $Q^{(n,n+1)}$ and $Q^{(2n,1)}$ consist of the edges d and a, respectively. Thus, the complementary 2-factor of M_1 is a Hamiltonian cycle. This is depicted in Figure 7. The proof that the complementary 2-factor of M_2 is a Hamiltonian cycle follows analogously. □

Proposition 3.5. *Let n be a positive odd integer. Then, the balanced papillon graph $\mathcal{P}_n$ is not PMH.*

Proof. Consider the following perfect matching of the balanced papillon graph $\mathcal{P}_n$:

$$M = \cup_{i=1}^{2n}\{u_{2i-1}u_{2i}, v_{2i-1}v_{2i}\}.$$

It is clear that when $n = 1$, the perfect matching M cannot be extended to a Hamiltonian cycle of the balanced papillon graph $\mathcal{P}_1$. So assume that $n \geq 3$. We claim that M cannot be extended to a Hamiltonian cycle of $\mathcal{P}_n$. For, let F be a 2-factor of $\mathcal{P}_n$ containing M. Since $u_1u_2 \in M$ and $\mathcal{P}_n$ is cubic, F contains exactly one of the following two edges: u_1u_{4n} or u_1v_1. In the former case, if $u_1u_{4n} \in E(F)$, then, $u_{2n}u_{2n+1}$ and all the edges of the outer- and inner-cycle will belong to F (at the same time, the choice of u_1u_{4n} forbids all the spokes of $\mathcal{P}_n$ to belong to F), yielding two disjoint cycles each of length $4n$. In the latter case, if $u_1v_1 \in E(F)$, then F must also contain all spokes u_iv_i, for $1 < i \leq 4n$. In fact, the subgraph induced by the set of spokes is exactly the complement of the 2-factor obtained in the former case. Consequently, F will consist of $2n$ disjoint 4-cycles. □

Consider $\mathcal{P}_n$, with $n \geq 2$, and let M be a perfect matching of $\mathcal{P}_n$ with $M \cap \mathcal{X} = 0$, which by Lemma 3.1 implies that $|M \cap \partial\mathcal{T}_j| = 0$ for all $j \in [2n]$. Now consider $j \in [2n] \setminus \{n, 2n\}$ and let $\mathcal{T}_{(j,j+1)}$ denote a 2-chain composed of $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$. We say that $\mathcal{T}_{(j,j+1)}$ is *symmetric with respect to M* if exactly one of the following occurs:

(i) $\{u_{2j-1}v_{2j-1}, u_{2j}v_{2j}, u_{2j+1}v_{2j+1}, u_{2j+2}v_{2j+2}\} \subset M$; or

(ii) $\{u_{2j-1}u_{2j}, v_{2j-1}v_{2j}, u_{2j+1}u_{2j+2}, v_{2j+1}v_{2j+2}\} \subset M$.

If neither (i) nor (ii) occur, $\mathcal{T}_{(j,j+1)}$ is said to be *asymmetric with respect to* M. This is shown in Figure 8.

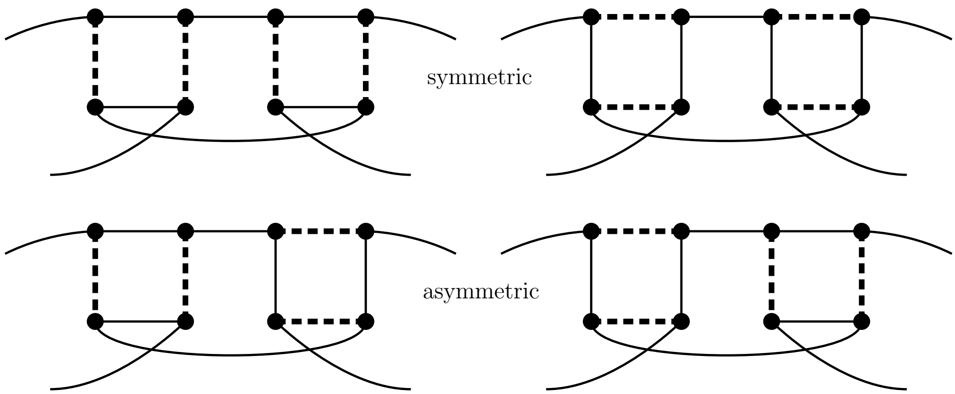

Figure 8: Symmetric and asymmetric 2-chains with the bold dashed edges belonging to M.

Remark 3.6. Let $n \geq 2$. Consider a perfect matching M_1 of $\mathcal{P}_n$ such that M_1 does not intersect the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, that is, $M_1 \cap \mathcal{X} = \emptyset$, and consider a 2-chain of $\mathcal{P}_n$, say $\mathcal{T}_{(j,j+1)}$ with $j \in [2n] \setminus \{n, 2n\}$, having semiedges e_1, e_2, e_3, e_4, where $e_1 = e_1^j, e_2 = e_2^{j+1}, e_3 = e_3^j$ and $e_4 = e_4^{j+1}$. Assume there exists a perfect matching M_2 of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 2$ and $M_1 \cap M_2 = \emptyset$ (see Figure 9). If $\mathcal{T}_{(j,j+1)}$ is symmetric with respect to M_1, then we have exactly one of the following instances:

$$M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_1, e_2\} \text{ (upper);} \quad \text{or} \quad M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_3, e_4\} \text{ (lower).}$$

Otherwise, if $\mathcal{T}_{(j,j+1)}$ is asymmetric with respect to M_1, then exactly one of the following must occur:

$$M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_1, e_4\} \text{ (upper left, lower right); or}$$
$$M_2 \cap \partial\mathcal{T}_{(j,j+1)} = \{e_2, e_3\} \text{ (upper right, lower left).}$$

Notwithstanding whether $\mathcal{T}_{(j,j+1)}$ is symmetric or asymmetric with respect to M_1, $(M_1 \cup M_2) \cap E(\mathcal{T}_{(j,j+1)})$ induces a path (see Figure 9) which contains all the vertices of $V(\mathcal{T}_{(j,j+1)})$, and whose endvertices are the endvertices of the semiedges in $M_2 \cap \partial\mathcal{T}_{(j,j+1)}$.

Remark 3.7. Let $n \geq 2$. Consider a perfect matching M_1 of $\mathcal{P}_n$ such that M_1 does not intersect the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, that is, $M_1 \cap \mathcal{X} = \emptyset$, and consider a 2-chain of $\mathcal{P}_n$, say $\mathcal{T}_{(j,j+1)}$ with $j \in [2n] \setminus \{n, 2n\}$. Let M_2 be the perfect matching of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 4$. Clearly $M_1 \cap M_2 = \emptyset$. Notwithstanding whether $\mathcal{T}_{(j,j+1)}$ is symmetric or asymmetric with respect to M_1, we have that $(M_1 \cup M_2) \cap E(\mathcal{T}_{(j,j+1)})$ induces two disjoint paths of equal length (see Figure 10) whose union contains all the vertices of $\mathcal{T}_j$

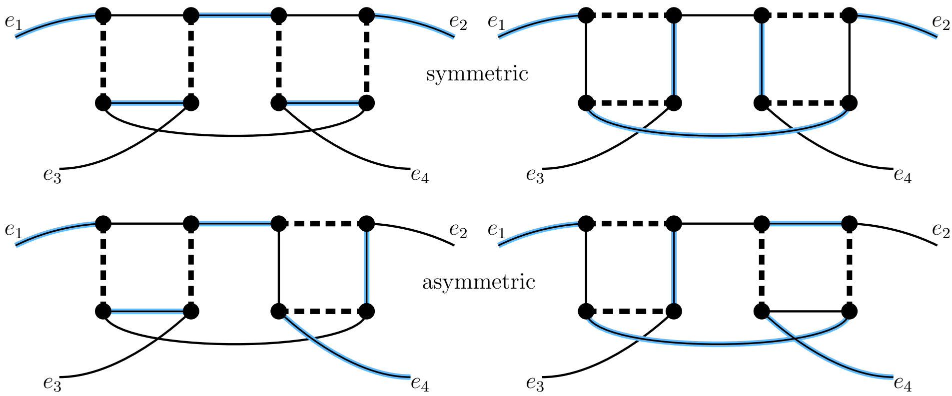

Figure 9: 2-chains when $M_1 \cap \mathcal{X} = \emptyset$ and $|M_2 \cap \mathcal{X}| = 2$ (bold dashed edges belong to M_1 and highlighted edges to M_2).

and $\mathcal{T}_{j+1}$. Let Q be one of these paths. We first note that Q contains exactly one vertex from $\{u_j, v_{j+1}\}$ and exactly one vertex from $\{u_{j+3}, v_{j+2}\}$. If $\mathcal{T}_{(j,j+1)}$ is symmetric with respect to M_1, then Q contains u_j if and only if Q contains u_{j+3}. Otherwise, if $\mathcal{T}_{(j,j+1)}$ is asymmetric with respect to M_1, then Q contains u_j if and only if Q contains v_{j+2}.

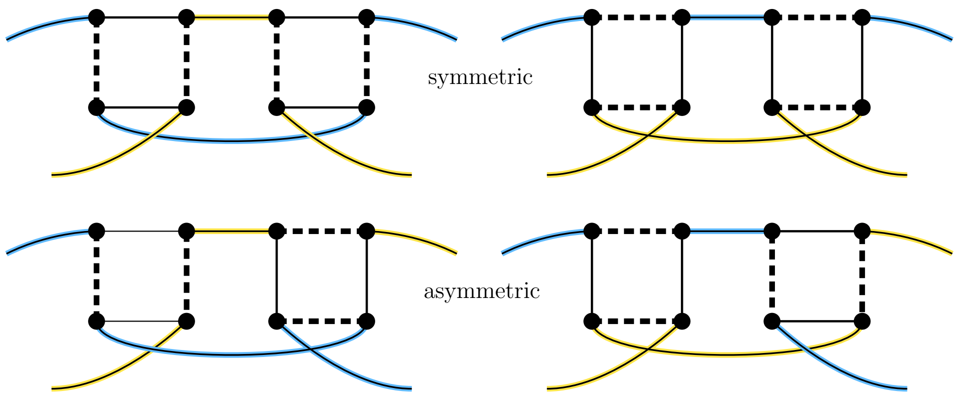

Figure 10: 2-chains when $M_1 \cap \mathcal{X} = \emptyset$ and $|M_2 \cap \mathcal{X}| = 4$ (bold dashed edges belong to M_1 and highlighted edges to M_2).

Theorem 3.8. *Let n be a positive even integer. Then, the balanced papillon graph $\mathcal{P}_n$ is PMH.*

Proof. Let M_1 be a perfect matching of $\mathcal{P}_n$. We need to show that there exists a perfect matching M_2 of $\mathcal{P}_n$ such that $M_1 \cup M_2$ induces a Hamiltonian cycle of $\mathcal{P}_n$. Three cases, depending on the intersection of M_1 with the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, are considered. If $|M_1 \cap \mathcal{X}| = 2$, then, by Lemma 3.4, there exists a perfect matching N of $\mathcal{P}_n$ such that $|N \cap \mathcal{X}| = 2$ and $M_1 \cap N = \emptyset$. Moreover, the complementary 2-factor of N is a Hamiltonian cycle. Since M_1 is contained in the mentioned 2-factor, the result follows. When $|M_1 \cap \mathcal{X}| = 4$, we can define M_2 to be the following perfect matching:

$$M_2 = \{u_1v_1, u_2v_2\} \bigcup \cup_{j=2}^{2n} \{u_{2j-1}u_{2j}, v_{2j-1}v_{2j}\}.$$

In fact, $M_1 \cup M_2$ induces the following Hamiltonian cycle: $(u_1, v_1, v_4, \ldots, v_{2n}, v_{2n-1}, v_{4n-1}, v_{4n}, v_{4n-3}, \ldots, v_{2n+1}, v_{2n+2}, v_2, u_2, u_3, u_4, \ldots, u_{4n})$, where v_4 and v_{4n-3} are respectively followed by v_3 and v_{4n-2}.

What remains to be considered is the case when $|M_1 \cap \mathcal{X}| = 0$. Clearly, $|M_2 \cap \mathcal{X}|$ cannot be zero, because, if so, choosing M_2 to be disjoint from M_1, $M_1 \cup M_2$ induces $2n$ disjoint 4-cycles. Therefore, $|M_2 \cap \mathcal{X}|$ must be equal to 2 or 4. Let $\mathcal{R} = \{\mathcal{T}_{(1,2)}, \ldots, \mathcal{T}_{(n-1,n)}\}$ and $\mathcal{L} = \{\mathcal{T}_{(n+1,n+2)}, \ldots, \mathcal{T}_{(2n-1,2n)}\}$ be the sets of 2-chains within the left and right n-chains of $\mathcal{P}_n$—namely the right and left n-chains each split into $\frac{n}{2}$ 2-chains. We consider two cases depending on the parity of the number of 2-chains in $\mathcal{L}$ and $\mathcal{R}$ which are asymmetric with respect to M_1. Let the function $\Phi \colon \mathcal{R} \cup \mathcal{L} \to \{-1, +1\}$ be defined on the 2-chains $\mathcal{T} \in \mathcal{R} \cup \mathcal{L}$ such that:

$$\Phi(\mathcal{T}) = \begin{cases} +1 & \text{if } \mathcal{T} \text{ is symmetric with respect to } M_1, \\ -1 & \text{otherwise.} \end{cases}$$

Case 1: $\mathcal{L}$ and $\mathcal{R}$ each have an even number (possibly zero) of asymmetric 2-chains with respect to M_1.

We claim that there exists a perfect matching such that its union with M_1 gives a Hamiltonian cycle of $\mathcal{P}_n$. Since the number of asymmetric 2-chains in $\mathcal{R}$ is even, $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = +1$, and consequently, by appropriately concatenating paths as in Remark 3.6, there exists a path R with endvertices u_1 and u_{2n} whose vertex set is $\cup_{i=1}^{2n} \{u_i, v_i\}$ such that it contains all the edges in $M_1 \cap (\cup_{i=1}^{n} E(\mathcal{T}_i))$. We remark that this path intersects exactly one edge of $\{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_j), y \in V(\mathcal{T}_{j+1})\}$, for each $j \in [n-1]$. By a similar reasoning, since $\prod_{\mathcal{T} \in \mathcal{L}} \Phi(\mathcal{T}) = +1$, there exists a path L with endvertices u_{2n+1} and u_{4n} whose vertex set is $\cup_{i=2n+1}^{4n} \{u_i, v_i\}$, such that it contains all the edges in $M_1 \cap (\cup_{i=n+1}^{2n} E(\mathcal{T}_i))$. Once again, this path intersects exactly one edge of $\{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_j), y \in V(\mathcal{T}_{j+1})\}$, for each $j \in \{n+1, \ldots, 2n-1\}$. These two paths, together with the edges a and d form the required Hamiltonian cycle of $\mathcal{P}_n$ containing M_1, proving our claim. We remark that this shows that there exists a perfect matching M_2 of $\mathcal{P}_n$ such that $M_2 \cap \mathcal{X} = \{a, d\}$, $M_1 \cap M_2 = \emptyset$ and with $M_1 \cup M_2$ inducing a Hamiltonian cycle of $\mathcal{P}_n$. One can similarly show that there exists a perfect matching M_2' of $\mathcal{P}_n$ such that $M_2' \cap \mathcal{X} = \{b, c\}$, $M_1 \cap M_2' = \emptyset$ and with $M_1 \cup M_2'$ inducing a Hamiltonian cycle of $\mathcal{P}_n$.

Case 2: One of $\mathcal{L}$ and $\mathcal{R}$ has an odd number of asymmetric 2-chains with respect to M_1.

Without loss of generality, assume that $\mathcal{R}$ has an odd number of asymmetric 2-chains with respect to M_1, that is, $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = -1$. Let M_2 be the perfect matching of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 4$. We claim that $M_1 \cup M_2$ induces a Hamiltonian cycle of $\mathcal{P}_n$. Since $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = -1$, by appropriately concatenating paths as in Remark 3.7 we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths R_1 and R_2, such that:

(i) $|V(R_1)| = |V(R_2)| = 2n$;

(ii) $V(R_1) \cup V(R_2) = \cup_{i=1}^{2n} \{u_i, v_i\}$;

(iii) the endvertices of R_1 are u_1 and v_{2n-1}; and

(iv) the endvertices of R_2 are v_2 and u_{2n}.

Next, we consider two subcases depending on the value of $\prod_{\mathcal{T}\in\mathcal{L}}\Phi(\mathcal{T})$. We shall be using the fact that $\{u_1u_{4n}, v_{2n-1}v_{4n-1}, v_2v_{2n+2}, u_{2n}u_{2n+1}\} = \{a, b, c, d\} = \mathcal{X} \subset M_2$.

Case 2a: $\prod_{\mathcal{T}\in\mathcal{L}}\Phi(\mathcal{T}) = -1$

As above, by Remark 3.7, we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths L_1 and L_2, such that:

(i) $|V(L_1)| = |V(L_2)| = 2n$;

(ii) $V(L_1) \cup V(L_2) = \cup_{i=2n+1}^{4n}\{u_i, v_i\}$;

(iii) the endvertices of L_1 are u_{2n+1} and v_{4n-1}; and

(iv) the endvertices of L_2 are v_{2n+2} and u_{4n}.

The concatenation of the following paths and edges gives a Hamiltonian cycle of $\mathcal{P}_n$ containing M_1:

$$R_1v_{2n-1}v_{4n-1}L_1u_{2n+1}u_{2n}R_2v_2v_{2n+2}L_2u_{4n}u_1.$$

Case 2b: $\prod_{\mathcal{T}\in\mathcal{L}}\Phi(\mathcal{T}) = +1$.

Once again, by Remark 3.7 we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths L_1 and L_2, such that:

(i) $|V(L_1)| = |V(L_2)| = 2n$;

(ii) $V(L_1) \cup V(L_2) = \cup_{i=2n+1}^{4n}\{u_i, v_i\}$;

(iii) the endvertices of L_1 are u_{2n+1} and u_{4n}; and

(iv) the endvertices of L_2 are v_{2n+2} and v_{4n-1}.

The concatenation of the following paths and edges gives a Hamiltonian cycle of $\mathcal{P}_n$ containing M_1:

$$R_1v_{2n-1}v_{4n-1}L_2v_{2n+2}v_2R_2u_{2n}u_{2n+1}L_1u_{4n}u_1.$$

This completes the proof. □

3.2 The unbalanced case $r < \ell$ and final remarks

By following the proofs in Section 2, the results obtained for balanced papillon graphs are now extended to unbalanced papillon graphs.

Theorem 3.9. *The unbalanced papillon graph $\mathcal{P}_{r,\ell}$ is PMH if and only if r and ℓ are both even.*

Proof. This is an immediate consequence of Proposition 3.5 and Theorem 3.8. In particular, when at least one of r and ℓ is odd, $\mathcal{P}_{r,\ell}$ is not PMH because the perfect matching $\cup_{i=1}^{r+\ell}\{u_{2i-1}u_{2i}, v_{2i-1}v_{2i}\}$ of $\mathcal{P}_{r,\ell}$ (illustrated in Figure 11) cannot be extended to a Hamiltonian cycle. □

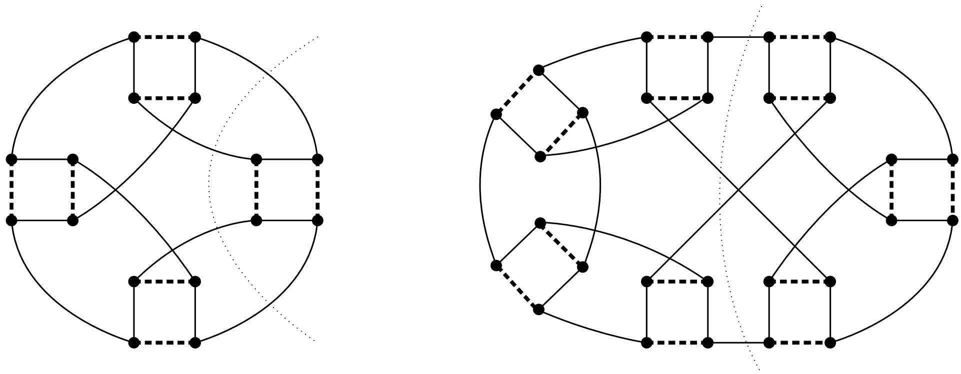

Figure 11: $\mathcal{P}_{1,3}$ and $\mathcal{P}_{3,4}$: unbalanced papillon graphs are not always PMH. The above perfect matchings do not extend to a Hamiltonian cycle.

Corollary 3.10. *The papillon graph $\mathcal{P}_{r,\ell}$ is PMH if and only if r and ℓ are both even.*

Finally, we remark that since $\mathcal{P}_n$ is PMH for every even $n \in \mathbb{N}$, balanced papillon graphs provide us with examples of non-bipartite PMH cubic graphs which are cyclically 4-edge-connected and have girth 4 such that their order is a multiple of 16. Additionally, by considering unbalanced papillon graphs, say $\mathcal{P}_{2,\ell}$, for some even $\ell > 2$, we can obtain non-bipartite PMH cubic graphs having the above characteristics (that is, cyclically 4-edge-connected and having girth 4) such that their order is 8ν, for odd $\nu \geq 3$.

It would also be very compelling to see whether there exist other 4-poles instead of the C_4-poles that can be used as building blocks when constructing papillon graphs and which yield non-bipartite PMH or just E2F cubic graphs.

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