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Reading Mathematical Texts as a Problem-Solving Activity: The Case of the Principle of Mathematical Induction

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☞ Reading mathematical texts is closely related to the effort of the reader to understand its content; therefore, it is reasonable to consider such reading as a problem-solving activity. In this paper, the Principle of Mathematical Induction was given to secondary education students, and their effort to comprehend the text was examined in order to identify whether significant elements of problem solving are involved. The findings give evidence that while negotiating the content of the text, the students went through Polya's four phases of problem solving. Moreover, this approach of reading the Principle of Mathematical Induction in the sense of a problem that must be solved seems a promising idea for the conceptual understanding of the notion of mathematical induction

Keywords: mathematical induction, reading mathematical text, problem solving, secondary education students

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Branje matematičnih besedil kot dejavnost reševanja problemov: primer principa matematične indukcije

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≈ Branje matematičnih besedil je tesno povezano s prizadevanjem bralca, da razume njegovo vsebino, zato je smiselno takšno branje obravnavati kot dejavnost reševanja problemov. V tem prispevku je obravnavan primer principa matematične indukcije, ki je bil posredovan v branje dijakom srednješolskega izobraževanja. S preučevanjem njihovega prizadevanja po razumevanju besedila smo želeli ugotoviti, ali ta dejavnost vsebuje ključne elemente reševanja problemov. Ugotovitve pokažejo, da so dijaki med razpravljanjem o vsebini besedila prešli skozi štiri faze reševanja problemov po Polyi. Poleg tega menimo, da je pristop branja korakov matematične indukcije na način reševanja problema obetaven za pridobitev konceptualnega razumevanja matematične indukcije.

Ključne besede: matematična indukcija, branje matematičnega besedila, reševanje problemov, dijaki

Introduction

Mamona-Downs and Downs (2005), considering the ‘identity’ of problem solving, raised a series of issues including, among others, the reading of mathematical texts and considered whether this could involve significant elements of problem solving. More precisely, they acknowledge that ‘reading mathematical text often needs an effort from the reader to understand and assimilate its content’ (p. 386), and this prompted the question of whether it is reasonable to consider such reading as a problem-solving activity.

However, not all kinds of mathematical text can be appropriate. It seems that the kind that is the most suitable is proof. As Mamona-Downs and Downs (2005) explain, there are three aspects in examining a proof relevant to the issue. The first concerns ‘the locating and examining of the implications occurring in the argument’ (p. 397). The students can check whether the implications are logically sound and ensure that the necessary conditions are properly accounted for. The second concerns the understanding of how the overall reasoning is structured. Finally, the third concerns the extracting of meaning from the exposition in the sense that ‘the reader creates concept images in order to relate the material to intuitively understood schemas’ (p. 397).

Moreover, reading a proof seems difficult simply because of the style of exposition. The technical complexity in the use of notation, the leaping from one statement to another without justifying the leap, thus assuming too much, obscures much of the real argument that takes place in the proof. The readers must independently ‘retrieve’ their own version of how to interpret, evaluate and assimilate the material in front of them. A typical proof cannot be too detailed and excessively broken up into little lemmas, because one can become lost in the details. Thus, reading proofs is not easy if the student attempts to read them like a novel, in a comfortable way with little concentration.

This perspective opens a new area in the research agenda of problem solving; we are not aware of a study that examines this perspective. The few studies that exist in the relevant literature approach this issue from the authors’ point of view, focusing on how mathematical texts should be written to make their text as comprehensible as possible (Konior, 1993; Morgan, 1996).

As mentioned above, reading mathematical text has been seen mainly from the authors’ point of view. Morgan (1996) acknowledges that the language used in a textbook does not transparently transmit the authors’ intentions; different readers may construct different meanings from the same text. This is in alignment with Freudenthal’s (1983) view that the way each one of us is thinking is not always possible to be transferred satisfactorily to other people, especially

if we differ in background and experiences.

As Konior (1993) mentions, reading such mathematical texts demands certain techniques that can be taught but are not given to the students together with the alphabet. He emphasises the importance of the authors structuring their text in a certain way to determine the process of its reading. This is related to the fact that mathematical texts are mainly conceived as ‘highly compact, precise, complex, and containing technical vocabulary’ (Österholm & Bergqvist, 2013, p. 751), and it is not clear how, or even if, mathematical texts, in general, can be described in common linguistic terms. Mathematical reading involves both linguistic comprehension skills and knowledge of the ‘language of mathematics’ (Adams, 2003). Moreover, Österholm (2006) found that comprehending a mathematical text (concerning basic concepts of group theory) becomes even more difficult when the text includes symbols. In particular, he found that if the texts include symbols, they require a special type of skill for reading comprehension, while if the texts are written in natural language, they then merely need a more general reading ability.

If, however, we are restricted to the mathematical texts of proofs, it seems that when students read and reflect on them, they tend to focus on the superficial features of the proofs’ arguments, and their ability to determine whether these arguments are proven is very limited (Selden & Selden, 2003). A proof-text can be read in two different ways. One is to validate the proof, meaning to determine whether or not it is valid (Selden & Selden, 1995). The other is reading for comprehension. In the latter case, the validity of the proof is assumed by virtue of its author or source, and the goal of the reader is to understand the proof, not to check its validity (Mejía-Ramos & Inglis, 2009). Furthermore, the students’ skills in reading comprehension, in general, are closely linked to their reading and understanding of mathematical texts. Vilenius-Tuohimaa, Aunola, and Nurmi (2008), working with Grade 4 students (9–10 years old), found that their reading comprehension was strongly related to their performance in mathematical word problems.

For securing the validity of proof, the step-by-step presentation of the mathematical proof moving from hypothesis to conclusion is considered suitable. In the case, however, of comprehension and therefore mathematical communication, this linear way does not work. Instead, Leron (1983) proposes the ‘structural method’ whose basic idea is to divide the proof into levels proceeding from the top down. These levels can be considered short autonomous ‘modules’, each embodying one major idea of the proof. In a very general (but precise) manner, the top level provides the main line of the proof. The next level proceeds to elaborate on these generalities of the top level. Proofs for

unsubstantiated statements, more details for general descriptions, and similarly, are provided at this level. If a sub-procedure is somehow complicated, then a 'top level' description is given in this second level, and details are pushed further down to lower levels. The hierarchy continues similarly. Leron (1983) claims that this method could increase the comprehensibility of these ideas retaining at the same time their rigour. This process is often supported by what Raman (2003) calls a 'heuristic idea', which is an idea based on informal understanding, which gives a sense of understanding but not a conviction. It is more like a sense that something ought to be true. Another method, suggested by Grugnetti and Jaque (2005), is to ask students to look for a mistake in arguments or to examine the validity of their peers' evidence (see also Selden & Selden, 2003).

Mamona-Downs and Downs (2005) claim that reading a mathematical text can be a real problem-solving activity and that understanding a mathematical text can be just as challenging as developing a strategy for solving it. In this sense, this endeavour is connected to the theory of problem-solving (Polya, 1957; Schoenfeld, 1985, 2013). A recent study in this spirit is by Papadopoulos and Iatridou (2010), who presented the Pick's theorem in the form of an open problem to Grade 11 students and recorded the different ways the students approached this problem. The main feature of this approach is that the responsibility of understanding the mathematical text is transferred to the students themselves and not to the teachers or the textbook authors, who by default are considered responsible for making mathematical reading as clear and easy as possible.

This process of comprehending a mathematical text involves aspects of executive control, but, as Schoenfeld (1985a) indicates, the students' metacognitive skills, in general, are remarkably poor. The explanation is that standard instruction focuses on the mastery of facts and procedures and does not deal with metacognition. This is why Schoenfeld (1985) suggests some approaches that could be adapted to support students' needs while reading a mathematical text (Mamona-Downs & Downs, 2005). Yang (2012), working with Grade 9 students, found that good comprehenders tend to use more metacognitive reading strategies for planning and monitoring comprehension compared with moderate and poor comprehenders. In contrast, Weber et al. (2008), working with advanced undergraduate students, found that they use sophisticated comprehension-fostering and monitoring strategies in comprehending texts in the 'definition-theorem-proof' format.

In our study, the focus is on the students' responsibility to be engaged in comprehending the given mathematical text. The text chosen for the purpose

of the study is negotiating the notion of Mathematical Induction, taken from a mathematics textbook that is no longer used in regular mathematics teaching and, therefore, is completely unknown for the students.

Mathematical induction and students' difficulties with mathematical induction

The Principle of Mathematical Induction (PMI) is usually (on the grounds of simplicity) expressed in the terms of the properties of natural numbers and with two options about the first number (since 0 and 1 are commonly used). Some authors, however, use a non-negative integer a_0 .

The PMI can be stated as: *If 1 has a property P, and if any n having the property P implies that n+1 also has the property P, then every n has the property P.* In a more formal way, this can be expressed as: *If P(1) and if for all n, P(n) implies P(n+1), then for all n, P(n).* A typical proof by induction, therefore, must follow the steps below (Ernest, 1984):

Theorem: $\forall n \in \mathbb{N}, P(n)$.

Proof: By mathematical induction.

Basis: Prove that P(1) is true.

Inductive hypothesis: Assume P(n) is true.

Induction step: Prove that P(n+1) is true from the inductive hypothesis.

End of proof: Hence, from the PMI, P(n) is true for all natural numbers n.

The research shows that mathematical induction is a very difficult concept for secondary education students to learn, as well as for undergraduate students (Dubinsky, 1989). Induction presents specific cognitive obstacles (some of them will be presented below); therefore, students still fail with this given that the teaching methods do not pay attention to these difficulties. Bell (1920) very early expressed concerns about mathematical induction in secondary education: "[...] mathematical induction has no place in elementary teaching" (p. 413). Baker (1996) examined thoroughly the difficulties students faced when dealing with mathematical induction and identified nine such difficulties. Among others, he found that (i) students have difficulty with the mathematical content of induction, especially with being unable to operate with symbols, (ii) they rely exclusively on procedures lacking thus conceptual understanding, (iii) they are mainly reliant on examples to recognise that something is proven, and (iv) they exhibit poor metacognitive control abilities. To a certain extent, these can be attributed to the way the proof, in general, is negotiated in classrooms. In most textbooks, the relevant proof activities (not only for mathematical induction)

begin with phrases such as: ‘show that ...’ or ‘prove that...’. The theorem is given, and the students have to accept its truth and to prove it. This approach leaves behind how the theorem emerged (Avital & Hansen, 1976; Papadopoulos & Iatridou, 2010). Ernest (1984) relates these students’ difficulties to the fact that the solvers assume what they have to prove, and then they prove it. He also finds it reasonable that the students ‘wonder why this rather complex and seemingly arbitrary principle is adopted’ (p. 183). The consequence is that the students are able to deal with mathematical induction tasks and feel comfortable with them, but it is questionable whether they really learned induction and are able to provide a coherent explanation of the induction steps (Allen, 2001).

Despite all these difficulties that the students face with PMI, there are also some findings showing that in some cases students are able to identify the critical properties that justify why mathematical induction works (Palla et al., 2012).

The fact is, however, that no matter the students’ difficulties, the significance of mathematical induction cannot be ignored. NCTM (2000), in the section about reasoning and proof in Grades 9 through 12, acknowledges that ‘students should learn that certain types of results are proved using the technique of mathematical induction’ (p. 345).

Therefore, in this context, our research questions are shaped as follows:

1. Is it plausible to consider the reading of a mathematical text as a problem-solving activity? To what extent can elements of problem-solving be identified in the students’ work while they try to comprehend a mathematical text (in our case, the Theorem of the Principle of Mathematical Induction)?
2. Does this approach contribute and/or facilitate its conceptual understanding?

Method

Sample of participants

The participants were 15 students from Grades 10, 11, and 12 (15 to 17 years old) in a public school in northern Greece who participated voluntarily. The students worked in pairs or small groups. For the purposes of this work, we follow one group of five students of Grade 11 since their work provided us with a representative complete set of instances found across the whole sample. The students’ performance in mathematics was average and, according to the curriculum, they should have developed some of the mathematics skills required

for learning mathematical induction, such as being (i) familiar with the reasoning and proof process, (ii) able to eliminate parentheses and reduce similar terms through the distributive property, (iii) able to use basic identities, and (iv) able to factorise polynomials.

Even though mathematical induction is included in the students' textbooks, the Number Theory chapter that negotiates this topic has been excluded from teaching for the last five years. Therefore, the participants had fluency in the above-mentioned algebraic acts, but they were not familiar with proof by mathematical induction.

Presentation of the task

In the context of the present study, the mathematical text given to students was the Principle of Mathematical Induction. It was an extract of an older textbook (Ntzioras, 1979) that was addressed to high school students (Figure 1). Therefore, despite the content being completely unknown to them, it was written for students of their age and, therefore, it was considered proper to use it in our study.

Figure 1

The PMI in the form given to the students

✓ § 19. Θεώρημα (πρώτη μορφή της τέλειας επαγωγής).—'Αν $p(v)$ είναι ένας προτασιακός τύπος με σύνολο αναφοράς τό σύνολο \mathbb{N} τών φυσικῶν ἀριθμῶν, τέτοιος ὡστε :

α) νά εἶναι ἀληθής ἡ πρόταση $p(1)$, καί

β) νά εἶναι ἀληθής ἡ πρόταση : $\forall k \in \mathbb{N}, p(k) \Rightarrow p(k + 1)$,

τότε (δηλ. ὅταν συμβαίνουν τά α) καί β)) ὁ προτασιακός τύπος $p(v)$ εἶναι ἀληθής (ισχύει) γιά κάθε $v \in \mathbb{N}$.

Note. Adapted from Ntzioras, 1979.

Its translation is as follows:

If $P(n)$ is a mathematical statement in the set \mathbb{N} of natural numbers, and:

(a) $P(1)$ is true, and

(b) $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$, is also true,

Then the $P(n)$ statement is also true $\forall n \in \mathbb{N}$

At the same time, the students were given an application of the theorem taken from the same textbook (Figure 2)

Figure 2

The application of the PMI given to the students

ΕΦΑΡΜΟΓΕΣ

1η: Να αποδείξετε ότι για κάθε φυσικό αριθμό n ισχύει ο τύπος:

$$p(n): \quad 1+2+3+\dots+n = \frac{n(n+1)}{2} \quad (1)$$

Απόδειξη α) Για $n = 1$ ή (1) γίνεται: $1 = \frac{1(1+1)}{2}$, αληθής.

β) *Ας δεχθούμε ότι η (1) ισχύει για $n = k$ ($k \in \mathbb{N}$), δηλ. ότι:

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad (2)$$

Θά αποδείξουμε τώρα ότι η (1) ισχύει και για $n = k + 1$, δηλ. ότι:

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{(k+1)(k+2)}{2} \quad (3)$$

Πράγματι, αν προσθέσουμε και στα δύο μέλη της (2) τό $(k + 1)$ έχουμε:

$$(1 + 2 + 3 + \dots + k) + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2}, \text{ δηλαδή η}$$

(3) ισχύει.

γ) **Συμπέρασμα:** Στο α) αποδείξαμε ότι $p(1)$ είναι αληθής. Στο β) αποδείξαμε ότι: αν $p(k)$ είναι αληθής, τότε και η πρόταση $p(k + 1)$ είναι αληθής*. Συνεπώς, σύμφωνα με τό θεώρημα της τέλειας επαγωγής, η (1) ισχύει για κάθε $n \in \mathbb{N}$.

Note. Adapted from Ntzioras, 1979.

Its translation is as follows:

Statement: Prove that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}. \quad (1)$$

Proof: (a) if $n = 1$, then $1 = \frac{1(1+1)}{2}$, true.

(b) Let us assume that (1) is true for $n = k$ ($k \in \mathbb{N}$), that is

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}, \quad (2)$$

We will prove that (1) is true for $n = k + 1$, e.g.,

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{(k+1)(k+2)}{2}, \quad (3)$$

Indeed, if we add $(k + 1)$ at both sides of (2) we take

$$(1 + 2 + 3 + \dots + k) + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2},$$

which means that (3) is true.

(c) Conclusion: we proved $P(1)$ is true (step a). We proved that if $P(k)$ is true, then $P(k+1)$ is also true (step b). Therefore, according to the PMI, (1) is true $\forall n \in \mathbb{N}$.

The participants were asked to read the two texts (Figures 1 and 2) and then answer the questions:

- Do you understand the Theorem of Mathematical Induction?
- Can you identify the series of steps in a proof by Mathematical Induction?

- *Are you able to explain why this series of steps constitute proof?*
- *How do these steps convince you about the validity of the statement to be proved?*

The specific choice ensured that the students would be involved in a completely unknown mathematical text but, given that it was addressed to high school students, the suitability of its wording and use of symbolism is also ensured.

The students had almost one and a half hours to complete the task. They were asked to vocalise and discuss their thoughts with each other. The only interventions aimed to answer questions related to terminology and symbols in the text.

Collecting and analysing data

The session was audio-recorded, and the students were constantly asked to ‘think aloud’. The students’ worksheets collected at the end of the session, combined with the transcribed protocols, constituted the data of this study. These data were examined and analysed in the context of qualitative content analysis (Mayring, 2014) at two different levels: first, detecting instances related to Polya’s problem-solving steps (*Getting familiar with the problem, Devise a plan, Carrying out the plan, Looking back*); second, seeking evidence of the students’ conceptual understanding of mathematical induction. In the context of this study, by the conceptual understanding of mathematical induction, we mean (i) the students’ convincing explanation of why the steps of induction constitute evidence for the general truth of the statement, (ii) how these steps relate to each other, (iii) why the case of $n = 1$ is used as a first step, and (iv) why these series of steps constitute a proof.

The data were initially examined independently by the two authors. The coding results were compared, codes were clarified, and some data were re-coded until agreement.

Results

In this section, the students’ interaction while reading the task will be presented. The presentation will be based on the use of certain extracts from the transcribed protocols. All the excerpts will be accompanied by an alphanumeric string on the left indicating the group (e.g., B), the number of the task (e.g., 2), and the lines of the protocol (e.g., 22–24). Students’ discussion and actions will be commented on.

The students initially spend some time individually to think about the task before starting any discussion. After that, their first effort was to connect this task with concepts already known to them, such as the concept of sequence:

[B2.22–24]: *The application is about a sequence that increases by 1 each time. Is this theorem true for every sequence that increases by 1?*

[B2.25–26]: *I thought it was an arithmetic progression.*

They verify the statement for different values of n :

[B2.29–33]: *If we put $n = 2$...*

[B2.34]: *The statement is true for $n = 3$.*

They understand that according to the theorem, a statement is true if the two conditions (a) and (b) of the theorem are satisfied (see above), and they describe the problem in their own words:

[B2.41–43]: *Any statement referring to natural numbers and satisfies these two conditions (a) and (b) is true for every natural number.*

They match without difficulty the first step, $P(1)$, of the theorem with the first step of its implementation, and they observe that the proof includes the generalisation element:

[B2.67–68]: *It starts with $n = 1$, which is a natural number to show that the statement is true for natural numbers in order to proceed then to the case of the statement being true for $n = k$. Actually, the aim is to generalise.*

They understand the algorithmic part of the inductive step, and they suspect that the inductive step is somehow related to the generalisation of the statement:

[B2. 71–74]: *...Then it adds $k + 1$ on both sides... to show that we can take any natural number.*

They understand that the second step of the proof serves the generality of the statement:

[B2.104]: *I think that the 2nd step just generalises.*

They wonder whether the information given in the statement of the task is unnecessary:

[B2.107–109]: *The third step seems useless.*

This part of their negotiation brings them quite close to the question of why the given content is a proof:

[B2.113–114]: *Because $k + 1$ is again a natural number. That is, from the moment we found that the statement is true for $n=k$ that belongs to N , it means it will be true also for $n = k + 1$.*

They start to notice that the idea of successive numbers might be a key element in the proof by mathematical induction:

[B2.115–116]: *Let us say... If we put $k = 2$... we could also put $k = 3$.*

[B2.117–118]: *It is not actually two random numbers. $n = k$ is a random number, and $n = k + 1$ is a bigger random number. Why was $k + 1$ chosen?*

They regularly come back to the wording of the task and analyse more closely words and data:

[B2.121]: *The step for $n = k$ is not proof. It says, 'assuming that the statement is true for $n = k$ '.*

This part of the induction makes them feel confused, and they spent time moving back and forth to understand the situation. In particular, they focus on the issue of successive numbers and on assuming what they have to prove before proving it.

[B2.185–186]: *2 is a random number, n and $k+1$ are random too.*

[B2.200–201]: *It assumes that the statement is true for $n = k$... How is it possible to assume that? It is supposed that we want to prove it and we did not it yet.*

They gradually realise that k and $k+1$ are not specific numbers. They can be any natural numbers, but they always remain successive.

[B2.211–213]: *We cannot determine the size of this sequence. The numbers can be until 100, until 1000, and so on...*

[B2. 230–231]: *Also, they are random... k is random. And $k + 1$ is the next one...*

However, they remain unable to describe fully why these steps are proof and re-match the theorem's steps with the implementation steps. They mistakenly believe that the steps of the theorem are themselves the proof.

[B2.254–255]: *If we can show that it is possible to go from $P(k)$ to $P(k + 1)$ then I think this is enough to convince us.*

Therefore, at this point they think they have solved the problem:
 [B2.269]: *We cannot find any arithmetical mistakes in the process. Therefore, the question has been answered.*

They were satisfied with their thinking that if the statement has been verified for a known specific natural number and assuming it is true for a random k , then it is not necessary to go further.

The setting changes from the moment one of the members observed that both 1 and k are values that are assigned to n .

[B2.350]: *Actually, when we say $n = 1$ or $n = k$, it is the same action. We have the variable n , and we substitute n with 1 or k .*

Finally, they approach the essence of the theorem.

[B2.353–355, 359]: *Oh, I think I understood. In the beginning, we show it is true for $n = 1$. Then we assume that it is true for $n = k$, and then we show that this will be true **for every next number**. So, it is true if we put 1 in the place of k . Then, it is true for its next number 2 and the same for the next 3 and so on...*

They end with an implicit reference to the general validity of the theorem for all the natural numbers.

[B2.370]: *We assumed it is true for a random number, and we proved it for its successor. So, the statement applies to all the natural numbers.*

[B2.371–372]: *Because it applies for $n = 1$ and ... it also applies to $n = 1 + 1 = 2$. In the same way, it applies for $n = 3$...*

Discussion

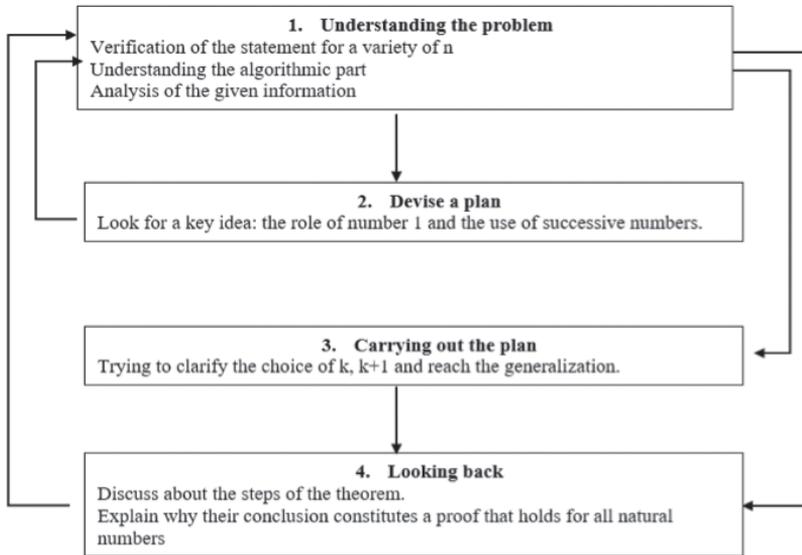
Before discussing the two research questions, it might be said that we do not intend to overgeneralise our findings. This study might better be considered as a case study that gives evidence for thinking on our research questions and support our better understanding of the situation. The next subsections try to shed light on these questions building upon the research findings presented above.

Reading a mathematical text as a problem-solving activity.

The analysis of the findings gave evidence on whether it is plausible to consider the reading of mathematical texts as a problem-solving activity (first research question) since Polya's problem-solving steps can be easily detected in the students' work, as presented in Figure 3.

Figure 3

Schematic representation of the problem-solving process of the group while reading the mathematical text



In ‘Getting familiar with the problem’, the students took the initiative to ensure they understand the problem correctly. Thus, they verified the statement for different values of n (B2.29–33), they invested time to analyse the given (B2.67–68, 71–74, 75–78, 104) and examine whether the included information was sufficient, insufficient, or redundant (B2.107–109).

Then they started wondering whether they had seen something familiar before, which is in accordance with Polya’s 2nd step of ‘Devising a plan’. They recalled sequences and arithmetic progression (B2.22–26). Polya, in this step, also invites the solvers to pose to themselves the question ‘Could you restate the problem’, an action taken by the students at B2.41–43.

They turned towards looking for a key idea, which was used later to solve the problem (‘Carrying out the plan’). This idea was evolved around the questions: Why do we start with number 1? What is the role of the use of the consecutive numbers k and $k + 1$? (B2. 117–118). This leads to the solution (B2.353–355, 359).

Finally, actions that could be linked to Polya’s fourth step of problem-solving (Looking back) are the efforts made by the students to explain the generality of their arguments for all the natural numbers, starting from the smallest one and its successor (numbers 1 and 2) and then expanding this to the

numbers 2 and 3, and so on (B2.370–372).

Therefore, it can be said that the students' effort to deal with this task can be seen as a problem-solving process, and the students exhibited instances of all of Polya's four steps, although in a non-linear way. There was a continuous interplay between the 'Getting familiar' and 'Devising a plan' phases. However, it was 'Getting familiar with the problem' that dominated the students' problem-solving process. The group came back several times to this step after devising or carrying out a plan (see relevant arrows in Fig. 2). This was the reaction of the participants every time they did not know how to move on or when a plan did not seem promising.

Reading mathematical texts and conceptual understanding.

The second research question concerned whether this approach of reading the particular mathematical text facilitated or contributed to the conceptual understanding of mathematical induction in the sense of whether the students have come to substantiate why the sequence of steps in the theorem of mathematical induction is a proof.

It can be said that there was a certain path leading this group towards conceptual understanding. They made choices; these choices were negotiated and revised, were abandoned or improved, and the participants gradually started exhibiting bits of understanding about the cognitively demanding mathematical induction. Based on the analysis of our collected data, we can distinguish five steps in the process of conceptualisation followed by the participants in this study.

Step One: The first step is connected with the design principle of the study to transfer responsibility from authors and/or teachers to students. This takes place through the experience of reading a mathematical text as a potential problem that has to be solved. In our case, the 'problem' we gave was an original mathematical text, the Principle of Mathematical Induction. Its solution is multistep. The students had to identify the series of necessary steps and then explain why these steps constitute proof.

Step Two: The students considered the algorithmic part as the actual proof (B2.254–255). Baker (1996) characterises this as a 'difficulty with proof by mathematical induction predicted in conceptual understanding'.

Step Three: The students gradually started being aware that if the statement is true for a random number then it is also true for its successor. This is a unique property of natural numbers. The set of all-natural numbers forms a (well-) ordered sequence. So, if the initial number (one) has a property and if

it is passed along the ordered sequence from any natural number to its successor, then the property will hold for all natural numbers since they all occur in the sequence (all of them can be generated from a single initial number, e.g., number one) (Ernest, 1984).

Step Four: The students realised that the concept of successive numbers was of critical importance. As Movshovitz-Hadar (1993) explains, this step, if completed successfully, makes it possible to deduce the truth of “For all $n \in \mathbb{N}$, $P(n)$ ” and presents it as an infinite chain of applications of the basic law of inference.

Step Five: The students became finally able to make the connection between the steps (B2.353-372): Given the awareness of Step-3, if one starts with the smallest natural number 1 then the statement is true for its successor 2, and then for the next successor 3, and so on. This substantiates the validity of the statement for the whole set of natural numbers.

It seems, therefore, that this series of steps worked as a scaffolding that initially facilitated students’ understanding of the process of proof by mathematical induction. At the same time, there was evidence that the students finally became able to appreciate why this process warrants the truth of the given statement.

Conclusions

This paper has attempted to show that reading a mathematical text could be considered a problem-solving activity. Students are engaged in reading mathematical texts with content completely unknown to them. This signals a shift. Traditionally, the teacher was responsible for the transmission of knowledge and the clarification or explanation of concepts. Now, this responsibility is transferred to the students. The students’ effort to comprehend the mathematical meaning of the text became a problem that required a solution. In the students’ effort to solve this ‘problem’, it was feasible to identify all the four problem-solving steps of Polya with the step of ‘Getting familiar with the task’ to dominate the students’ work which confirms that reading of mathematical text might be considered as a problem-solving activity.

Moreover, it seems that this approach contributed to the conceptual understanding of mathematical induction. According to the literature, students do not conceptually understand mathematical induction regardless of whether they are able to apply it and prove statements (Allen, 2001; Baker, 1996). This is why alternative methods of teaching proof beyond the traditional model have been suggested. In our study, however, we introduce the students to the

proof by mathematical induction as something that is progressively revealed to them. Therefore, our approach provides them with the mathematical text, and they are required to interpret it. We believe that the findings of this study gave evidence that this process leads to conceptual understanding. The participants were able to grasp the reason the steps of the theorem of mathematical induction constitute proof.

The difficulty to linguistically comprehend the text (Adams, 2003), especially when it includes symbols (Österholm, 2006), became obvious in the students' effort. In the end, however, we can say that this endeavour was successful, and this success seems to have its origin in the combination of three elements: (i) responsibility for understanding the mathematical content was transferred to students, (ii) proof had been selected as the most suitable kind of text, and (iii) the collaborative nature of the problem-solving process.

Given that this study might better be considered a case study, it is obvious that we cannot overgeneralise its results. However, they are promising for planning a future study since some questions arise. Is it possible to obtain similar results when the selected text is not relevant to proof? What is the role of the metacognitive skills the participants already possess? What is the role of the teacher? What aspects of social metacognitive control emerge while students attempt to cope with the task?

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