Semigroups with fixed multiplicity and embedding dimension[*]

Juan Ignacio García-García [†]

Dpto. de Matemáticas/INDESS (Instituto Universitario para el Desarrollo Social Sostenible). Universidad de Cádiz, E-11510, Puerto Real (Cádiz, Spain)

Daniel Marín-Aragón [†]

Dpto. de Matemáticas. Universidad de Cádiz, E-11510, Puerto Real (Cádiz, Spain)

María Ángeles Moreno-Frías [‡]

Dpto. Matemáticas, Universidad de Cádiz, E-11510, Puerto Real (Cádiz, Spain)

José Carlos Rosales [§]

Dpto. de Álgebra, Universidad de Granada, E-18071, Granada (Spain)

Alberto Vigneron-Tenorio [¶]

Dpto. de Matemáticas/INDESS (Instituto Universitario para el Desarrollo Social Sostenible). Universidad de Cádiz, E-11406 Jerez de la Frontera (Cádiz, Spain)

Abstract

Given $m \in \mathbb{N}$, a numerical semigroup with multiplicity m is called a packed numerical semigroup if its minimal generating set is included in $\{m, m+1, \ldots, 2m-1\}$. In this work, packed numerical semigroups are used to build the set of numerical semigroups with a given multiplicity and embedding dimension, and to create a partition of this set. Wilf's conjecture is verified in the tree associated to some packed numerical semigroups. Furthermore, given two positive integers m and e, some algorithms for computing the minimal Frobenius number and minimal genus of the set of numerical semigroups with

---

multiplicity m and embedding dimension e are provided. We also compute the semigroups where these minimal values are achieved.

Keywords: Embedding dimension, Frobenius number, genus, multiplicity, numerical semigroup.

Math. Subj. Class.: 20M14, 20M05

1   Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. A numerical semigroup is a subset S of $\mathbb{N}$ which is closed under addition, such that $0 \in S$ and $\mathbb{N} \setminus S$ is finite. If S is a numerical semigroup, we define the multiplicity of S, denoted by $m(S)$, to be the least positive integer in S, the Frobenius number ($F(S)$) to be the greatest integer that is not in S, and the genus, $g(S)$, to be the cardinality of $\mathbb{N} \setminus S$.

Given a non-empty subset A of $\mathbb{N}$ we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}, \ a_1, \ldots, a_n \in A, \ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

It is well known (for example, see Lemma 2.1 from [11]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. If S is a numerical semigroup and $S = \langle A \rangle$, we say that A is a system of generators of S. Moreover, A is a minimal system of generators of S if $S \neq \langle B \rangle$ for every $B \subsetneq A$. In Theorem 2.7 from [11] it is shown that every numerical semigroup has a unique minimal system of generator and this system is finite. We denote by $\mathrm{msg}(S)$ and $e(S)$ the minimal system of generators of S and its cardinality, also called the embedding dimension of S.

If m and e are positive integers we use the following notation:

$$\mathcal{L}(m, e) = \{S \mid S \text{ is a numerical semigroup, } m(S) = m, \ e(S) = e\}.$$

In this work, one of our aims is to present a procedure that allows us to recursively construct the set $\mathcal{L}(m, e)$.

We say that a numerical semigroup S is a packed numerical semigroup if

$$\mathrm{msg}(S) \subseteq \{m(S), m(S) + 1, \ldots, 2m(S) - 1\}.$$

The set of all packed numerical semigroups with multiplicity m and embedding dimension e is denoted by $\mathcal{C}(m, e)$.

In Section 2, an equivalence relation $\mathcal{R}$ in the set $\mathcal{L}(m, e)$ is defined. For each $S \in \mathcal{L}(m, e)$ we denote by $[S]$, the equivalence class of S. We show that if $S \in \mathcal{L}(m, e)$ then $[S] \cap \mathcal{C}(m, e)$ has cardinality 1, so $\{[S] \mid S \in \mathcal{C}(m, e)\}$ is a partition of $\mathcal{L}(m, e)$. Hence, for computing all the elements of the set $\mathcal{L}(m, e)$ it is only necessary to perform the following steps:

1. Compute $\mathcal{C}(m, e)$.
2. For every $S \in \mathcal{C}(m, e)$ compute $[S]$.

E-mail addresses: ignacio.garcia@uca.es (Juan Ignacio García-García), daniel.marin@uca.es (Daniel Marín-Aragón), mariangeles.moreno@uca.es (María Ángeles Moreno-Frías), jrosales@ugr.es (José Carlos Rosales), alberto.vigneron@uca.es (Alberto Vigneron-Tenorio)

We see that it is possible to compute $\mathcal{C}(m, e)$, since this problem is equivalent to computing all the subsets A of $\{1, 2, \ldots, m-1\}$ such that A has cardinality $e-1$ and $\gcd(A \cup \{m\}) = 1$. For computing $[S]$, we order its elements by making a tree whose root is S, and describing the children of each of the vertices. In this way, we can recursively build the elements of $[S]$ by adding in at each step the children of the vertices that were obtained in the previous step. This procedure is not algorithmic because $[S]$ is infinite so we can not build it in a finite number of steps.

The Frobenius number and genus have been widely studied (see [7]) and they, together with the embedding dimension, are the background of one the most important problems in this theory: Wilf's conjecture which asserts that if S is a numerical semigroup then $e(S)g(S) \leq (e(S) - 1)(F(S) + 1)$ (see [15]). At present, it is still open.

In this work, we show that if we go along through a branch of the tree associated to $[S]$, the numerical semigroups have a greater Frobenius number and genus. These facts enable us to give an algorithm for building all the elements of $\mathcal{L}(m, e)$ with a fixed Frobenius number and/or genus. Finally, in order to compute the Frobenius number and the genus of the numerical semigroups of $[S]$, we give an algorithm based on [3]. We would like to note that these new algorithms enable us to study the tree of numerical semigroups with a given multiplicity and embedding dimension and with Frobenius number and/or genus up to any given bound. In some works, there appear algorithms for the computation of the tree of numerical semigroups up to a certain genus (see for example [4]). In our work, there can also be a bound on the Frobenius number. In addition, the computation of the complete tree of numerical semigroups up to a certain genus is not a practical method to obtain the numerical semigroups with a fixed multiplicity and embedding dimension, since it performs unnecessary calculations and does not obtain as large a genus as we can get with our algorithms. We are also interested in giving algorithms for computing

$$g(m, e) = \min\{g(S) \mid S \in \mathcal{L}(m, e)\},$$
$$F(m, e) = \min\{F(S) \mid S \in \mathcal{L}(m, e)\},$$
$$\{S \in \mathcal{L}(m, e) \mid g(S) = g(m, e)\},$$

and

$$\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}.$$

These methods are illustrated with several examples. To accomplish this, we have used the library `FrobeniusNumberAndGenus` developed by the authors in Mathematica ([16]). This library is freely available online at [5].

The content of this work is organized as follows. In Section 2, a partition of the set $\mathcal{L}(m, e)$ is studied and we construct a map $\phi \colon \mathcal{L}(m, e) \to \mathcal{C}(m, e)$ such that $[S] \cap \mathcal{C}(m, e)$ is equal to $\{\phi(S)\}$ for every $S \in \mathcal{L}(m, e)$. Theorem 3.3, in Section 3, is used to recursively compute the elements of $[S]$. In Section 4, we give some algorithms for computing the elements of $[S]$ with Frobenius number and/or genus less than fixed integer bounds. In Section 5, we show how the Apéry set of the elements of $[S]$ is used to compute their Frobenius number and genus. We also check that Wilf's conjecture is satisfied for some elements of $[S]$. Section 6 illustrates the preceding section and Section 7 contains some known results on Frobenius pseudo-varieties which allow us to construct the tree of all numerical semigroups with any given multiplicity. In Section 8 and Section 9, the minimal genus and minimal Frobenius number of the set of numerical semigroups with fixed multiplicity and embedding dimension are studied, giving some algorithms for computing them and obtaining the semigroups with these minimal values.

2 A partition of $\mathcal{L}(m, e)$

If A and B are subsets of $\mathbb{N}$ we denote by $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. It is well known (for example see Proposition 2.10 from [11]) that if S is a numerical semigroup then $e(S) \le m(S)$. Note that if $e(S) = 1$ then $S = \mathbb{N}$. Therefore, in the sequel, we assume that e and m are integers such that $2 \le e \le m$.

Given $S \in \mathcal{L}(m, e)$ we denote by $\phi(S)$ the numerical semigroup generated by $\{m\} + \{x \bmod m \mid x \in \mathrm{msg}(S)\}$. Clearly, $\phi(S)$ is a packed numerical semigroup and therefore we have the following result.

Lemma 2.1. *With the previous assumptions, ϕ defines a surjective map from $\mathcal{L}(m, e)$ to $\mathcal{C}(m, e)$.*

We define in $\mathcal{L}(m, e)$ the following equivalence relation: $S \mathcal{R} T$ if $\phi(S) = \phi(T)$. Given $S \in \mathcal{L}(m, e)$, $[S]$ denotes the set $\{T \in \mathcal{L}(m, e) \mid S \mathcal{R} T\}$. Therefore, the quotient set $\mathcal{L}(m, e)/\mathcal{R} = \{[S] \mid S \in \mathcal{L}(m, e)\}$ is a partition of $\mathcal{L}(m, e)$.

Lemma 2.2. *If $S \in \mathcal{L}(m, e)$, then $[S] \cap \mathcal{C}(m, e) = \{\phi(S)\}$.*

Proof. By Lemma 2.1, we know that $\phi(S) \in \mathcal{C}(m, e)$. Moreover, it is clear that $\phi(\phi(S)) = \phi(S)$. Therefore, $S \mathcal{R} \phi(S)$ and $\phi(S) \in [S] \cap \mathcal{C}(m, e)$.

If $T \in [S] \cap \mathcal{C}(m, e)$, then $\phi(T) = \phi(S)$ and $\phi(T) = T$, so $T = \phi(S)$.   □

The following result is a consequence of the previous lemmas.

Theorem 2.3. *Let m and e be integers such that $2 \le e \le m$. Then $\{[S] \mid S \in \mathcal{C}(m, e)\}$ is a partition of $\mathcal{L}(m, e)$. Moreover, if $\{S, T\} \subseteq \mathcal{C}(m, e)$ and $S \neq T$ then $[S] \cap [T] = \emptyset$.*

Therefore, as a consequence of Theorem 2.3, for computing all the elements of the set $\mathcal{L}(m, e)$ it is only necessary to do the following steps:

1. Compute $\mathcal{C}(m, e)$.

2. For every $S \in \mathcal{C}(m, e)$ compute $[S]$.

$\mathcal{C}(m, e)$ is easy to compute using the following result.

Proposition 2.4. *Let m and e be integers such that $2 \le e \le m$, and let A be a subset of $\{1, \ldots, m - 1\}$ with cardinality $e - 1$ such that $\gcd(A \cup \{m\}) = 1$. Then*

$$S = \langle \{m\} + (A \cup \{0\}) \rangle \in \mathcal{C}(m, e).$$

Moreover, every element of $\mathcal{C}(m, e)$ has this form.

Proof. The set S is a numerical semigroup because

$$\gcd(\{m\} + (A \cup \{0\})) = \gcd(A \cup \{m\}) = 1.$$

It is straightforward to prove that $\mathrm{msg}(S) = \{m\} + (A \cup \{0\})$, so $S \in \mathcal{C}(m, e)$.

If $S \in \mathcal{C}(m, e)$ then $\mathrm{msg}(S) = \{m, m + r_1, \ldots, m + r_{e-1}\}$ with $\{r_1, \ldots, r_{e-1}\} \subseteq \{1 \ldots, m - 1\}$. Moreover, since $\gcd\{m, m + r_1, \ldots, m + r_{e-1}\} = 1$,

$$\gcd\{m, r_1, \ldots, r_{e-1}\} = 1.$$ □

We illustrate the content of the previous proposition with an example.

Example 2.5. We are going to compute the set $\mathcal{C}(6,3)$ formed by all the packed numerical semigroups of multiplicity 6 and embedding dimension 3. For this purpose, and using Proposition 2.4, it is enough computing the subsets A of $\{1,2,3,4,5\}$ of cardinality 2 such that $\gcd(A \cup \{6\}) = 1$. This set is equal to

$$\{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}.$$

Therefore,

$$\mathcal{C}(6,3) = \{\langle 6,7,8\rangle, \langle 6,7,9\rangle, \langle 6,7,10\rangle, \langle 6,7,11\rangle,$$
$$\langle 6,8,9\rangle, \langle 6,8,11\rangle, \langle 6,9,10\rangle, \langle 6,9,11\rangle, \langle 6,10,11\rangle\}.$$

Note that if m is a prime number then every subset A of $\{1,\ldots,m-1\}$ with cardinality $e-1$ verifies that $\gcd(A \cup \{m\}) = 1$. Therefore, we have the following result.

Proposition 2.6. *If m is a prime number and e is an integer number such that $2 \le e \le m$ then $\mathcal{C}(m,e)$ has cardinality $\binom{m-1}{e-1}$.*

Our next goal in this work is to show a recursive procedure to compute $[S]$ for every $S \in \mathcal{C}(m,e)$. In order to achieve it, in the next section, we set the elements of $[S]$ in a tree.

3 The tree associated to $[S]$

A graph G is pair (V,E) where V is a set (with elements called vertices) and E is a subset of $\{(v,w) \in V \times V \mid v \ne w\}$ (with elements called edges). A path which connects the vertices x and y of G is a sequence of different edges of the form (v_0,v_1), $(v_1,v_2),\ldots,(v_{n-1},v_n)$ such that $v_0 = x$ and $v_n = y$. A graph G is a tree if there exists a vertex r (known as the root of G) such that for any other vertex x of G there exists a unique path connecting x and r. If (x,y) is an edge of a tree, we say that x is a child of y.

Lemma 3.1. *If $\{n_1 < n_2 < \cdots < n_e\}$ is a minimal system of generators of a numerical semigroup and $n_e - n_1 > n_1$ then $\{n_1,\ldots,n_{e-1}, n_e - n_1\}$ is also a minimal system of generators of a numerical semigroup.*

Proof. In other case, there exists $k \in \{1,\ldots,e-1\}$ such that

$$n_k \in \{n_e - n_1\} + \langle n_1,\ldots,n_{k-1},n_{k+1},\ldots,,n_{e-1},n_e - n_1\rangle.$$

But it is not possible because $n_e - n_1 + n_1 = n_e > n_k$. $\qquad\square$

Let S be a numerical semigroup. We denote by $M(S)$ the maximum of $\mathrm{msg}(S)$. If $S \in \mathcal{L}(m,e)$, we define the following sequence of elements of $\mathcal{L}(m,e)$:

- $S_0 = S$,
- $S_{n+1} = \langle(\mathrm{msg}(S_n) \setminus \{M(S_n)\}) \cup \{M(S_n) - m\}\rangle$ if $M(S_n) - m > m$.

Because of Lemma 3.1, there exists a sequence:

$$S = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_k = \phi(S) \in \mathcal{C}(m,e).$$

Example 3.2. Let $S \in \mathcal{L}(5,3)$ be the semigroup minimally generated by $\{5, 13, 21\}$. Then, we have the following sequence of elements of $\mathcal{L}(5,3)$:

$$S_0 = \langle 5, 13, 21 \rangle \subsetneq S_1 = \langle 5, 13, 16 \rangle \subsetneq S_2 = \langle 5, 11, 13 \rangle$$
$$\subsetneq S_3 = \langle 5, 8, 11 \rangle \subsetneq S_4 = \langle 5, 6, 8 \rangle = \phi(S) \in \mathcal{C}(5,3).$$

Let S be in $\mathcal{C}(m, e)$. We define the graph $G([S])$ as follows: $[S]$ is the set of vertices and $(A, B) \in [S] \times [S]$ is an edge if $\mathrm{msg}(B) = (\mathrm{msg}(A) \setminus \{M(A)\}) \cup \{M(A) - m\}$.

Theorem 3.3. *If $S \in \mathcal{C}(m, e)$ then $G([S])$ is a tree with root S. Moreover, if $P \in [S]$ and $\mathrm{msg}(P) = \{n_1 < n_2 < \cdots < n_e\}$ then the children of P in $G([S])$ are the numerical semigroups of the form $\langle (\{n_1, \ldots, n_e\} \setminus \{n_k\}) \cup \{n_k + n_1\} \rangle$ such that $k \in \{2, \ldots, e\}$, $n_k + n_1 > n_e$ and $n_k + n_1 \notin \langle \{n_1, \ldots, n_e\} \setminus \{n_k\} \rangle$.*

Proof. From the definition and the comment after Lemma 3.1, we have that $G([S])$ is a tree with root S.

Let k be in $\{2, \ldots, e\}$ such that $n_k + n_1 > n_e$ and $n_k + n_1 \notin \langle \{n_1, \ldots, n_e\} \setminus \{n_k\} \rangle$. If $H = \langle (\{n_1, \ldots, n_e\} \setminus \{n_k\}) \cup \{n_k + n_1\} \rangle$ is clear that

$$\mathrm{msg}(H) = (\{n_1, \ldots, n_e\} \setminus \{n_k\}) \cup \{n_k + n_1\} \quad \text{and}$$
$$\mathrm{msg}(P) = (\mathrm{msg}(H) \setminus \{M(H)\}) \cup \{M(H) - m\}.$$

Therefore H is a child of P.

Conversely, if H is a child of P then (H, P) is an edge of $G([S])$ and we obtain that H is as the theorem describes. $\qquad\square$

The previous theorem provides us with a method to recursively build the elements of $[S]$ as it is shown in the next example.

Example 3.4. Figure 1 shows some levels of the tree $G([\langle 5, 6, 8 \rangle])$.

Note that the cardinality of $[S]$ is infinity, so it is impossible to compute all the elements of $[S]$. However, in the next section, we show that it is possible to compute all the elements of $[S]$ with a fixed Frobenius number or genus.

4   Frobenius number and genus

Let P be a numerical semigroup with minimal generating set $\{n_1 < n_2 < \cdots < n_e\}$, $k \in \{2, \ldots, e\}$ and H be the numerical semigroup generated by $(\{n_1, \ldots, n_e\} \setminus \{n_k\}) \cup \{n_k + n_1\}$. Then $H \subset P$, $F(P) \leq F(H)$ and $g(P) < g(H)$. We can formulate the following result.

Proposition 4.1. *If $S \in \mathcal{C}(m, e)$, $P \in [S]$ and (H, P) is an edge of $G([S])$ then $F(P) \leq F(H)$ and $g(P) < g(H)$.*

As a consequence of the previous proposition, we have that if we go along through the branches of the tree $G([S])$, the numerical semigroups that we are finding have greater or equal Frobenius number, and also a greater genus. This fact enables us to formulate the Algorithms 1 and 3 in order to compute all the elements in $[S]$ with Frobenius number less than or equal to a given integer, and genus less than or equal to another given integer, respectively.
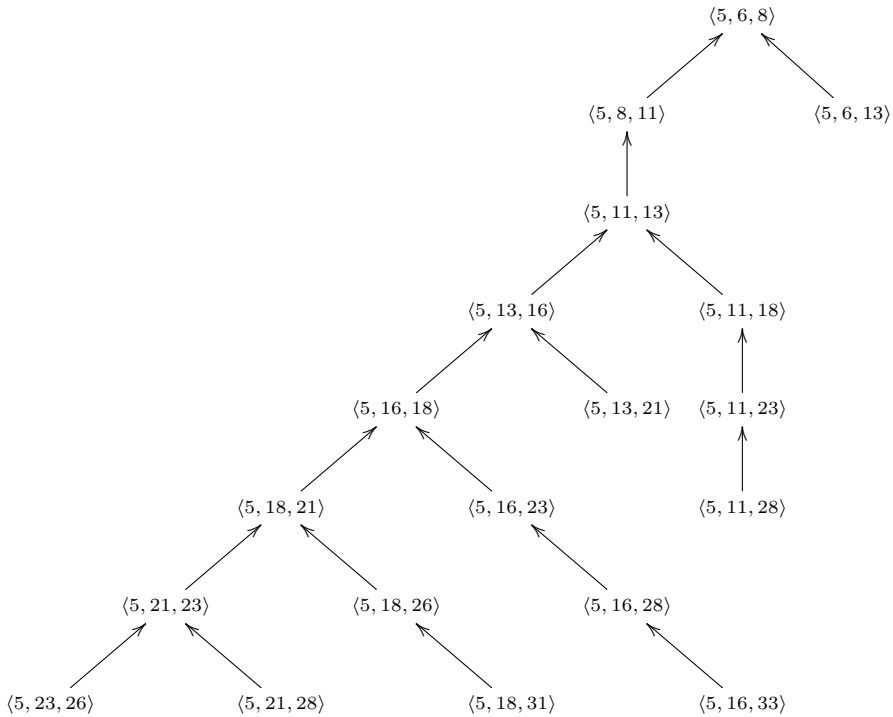
Figure 1: Seven levels of the tree of the packed numerical semigroup $\langle 5, 6, 8 \rangle$.

Algorithm 1 An algorithm to determinate the elements $T \in [S]$ such that $F(T) \leq F$ for a fixed integer F.

INPUT: (S, F) where S is a packed numerical semigroup and F is a positive integer.
OUTPUT: $\{T \in [S] \mid F(T) \leq F\}$.

1: **if** $F(S) > F$ **then**
2:     **return** $\emptyset$
3: **while** true **do**
4:     $A = \{S\}$ and $B = \{S\}$.
5:     $C = \{H \mid H$ is a child of an element of $B, \; F(H) \leq F\}$.
6:     **if** $C = \emptyset$ **then**
7:         **return** A
8:     $A = A \cup C, B = C$.

The following example illustrates how the previous algorithm works.

Example 4.2. We compute all the elements of $[\langle 5, 6, 8 \rangle]$ with Frobenius number less than or equal to 25.

- $A = \{\langle 5, 6, 8 \rangle\}$, $B = \{\langle 5, 6, 8 \rangle\}$ and $C = \{\langle 5, 8, 11 \rangle, \langle 5, 6, 13 \rangle\}$.
- $A = \{\langle 5, 6, 8 \rangle, \langle 5, 8, 11 \rangle, \langle 5, 6, 13 \rangle\}$, $B = \{\langle 5, 8, 11 \rangle, \langle 5, 6, 13 \rangle\}$ and $C = \{\langle 5, 11, 13 \rangle\}$.

- $A = \{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle, \langle 5, 11, 13\rangle\}$, $B = \{\langle 5, 11, 13\rangle\}$ and $C = \{\langle 5, 11, 18\rangle\}$.

- $A = \{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle, \langle 5, 11, 13\rangle, \langle 5, 11, 18\rangle\}$, $B = \{\langle 5, 11, 18\rangle\}$ and $C = \emptyset$.

Therefore, the set $\{T \in [\langle 5, 6, 8\rangle] \mid F(T) \leq 25\}$ is equal to $\{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle, \langle 5, 11, 13\rangle, \langle 5, 11, 18\rangle\}$.

The next algorithm allows us to compute all the numerical semigroups with multiplicity m, embedding dimension e and Frobenius number less than or equal to F. Note that if S is a numerical semigroup, such that $S \neq \mathbb{N}$ then $m(S) - 1 \notin S$ and then $m(S) - 1 \leq F(S)$.

Algorithm 2 An algorithm to determinate the numerical semigroups with a fixed embedding dimension and multiplicity, and bounded Frobenius number.

INPUT: m, e, and F positive integers such that $2 \leq e \leq m \leq F + 1$.
OUTPUT: $\{S \mid S$ numerical semigroup, $m(S) = m$, $e(S) = e$ and $F(S) \leq F\}$.

1: compute $\mathcal{C}(m, e)$, using Proposition 2.4.
2: **for all** $S \in \mathcal{C}(m, e)$ **do**
3:   compute $A(S) = \{T \in [S] \mid F(T) \leq F\}$, using Algorithm 1.
4: **return** $\cup_{S \in \mathcal{C}(m,e)} A(S)$.

Now, we change Frobenius number by the genus in Algorithm 1 and Algorithm 2.

Algorithm 3 An algorithm to determinate the elements $T \in [S]$ such that $g(T) \leq g$ for a fixed integer g.

INPUT: (S, g) where S is a packed numerical semigroup and g is a positive integer.
OUTPUT: $\{T \in [S] \mid g(T) \leq g\}$.

1: **if** $g(S) > g$ **then**
2:   **return** $\emptyset$
3: $A = \{S\}$ and $B = \{S\}$.
4: **while** true **do**
5:   $C = \{H \mid H$ is a child of an element of B, $g(H) \leq g\}$.
6:   **if** $C = \emptyset$ **then**
7:     **return** A
8:   $A = A \cup C, B = C$.

We illustrate now the above algorithm.

Example 4.3. We compute all the elements of $[\langle 5, 6, 8\rangle]$ with genus less than or equal to 15.

- $A = \{\langle 5, 6, 8\rangle\}$, $B = \{\langle 5, 6, 8\rangle\}$ and $C = \{\langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle\}$.

- $A = \{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle\}$, $B = \{\langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle\}$ and $C = \{\langle 5, 11, 13\rangle\}$.

- $A = \{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle, \langle 5, 11, 13\rangle\}$, $B = \{\langle 5, 11, 13\rangle\}$ and $C = \{\langle 5, 11, 18\rangle\}$.

- $A = \{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle, \langle 5, 11, 13\rangle, \langle 5, 11, 18\rangle\}$, $B = \{\langle 5, 11, 18\rangle\}$ and $C = \emptyset$.

Algorithm 3 returns $\{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle, \langle 5, 11, 13\rangle, \langle 5, 11, 18\rangle\}$.

Note that if S is a numerical semigroup such that $S \neq \mathbb{N}$ then $\{1, \ldots, m(S) - 1\} \subseteq \mathbb{N} \setminus S$ and then $m(S) - 1 \leq g(S)$.

Combining the above results, we obtain Algorithm 4.

Algorithm 4 An algorithm to compute numerical semigroups with fixed multiplicity, embedding dimension and bounded genus.

INPUT: m, e, and g positive integers such that $2 \leq e \leq m \leq g + 1$.
OUTPUT: $\{S \mid S$ numerical semigroup, $m(S) = m$, $e(S) = e$ and $g(S) \leq g\}$.

1: compute $\mathcal{C}(m, e)$, using Proposition 2.4.
2: **for all** $S \in \mathcal{C}(m, e)$ **do**
3:     compute $A(S) = \{T \in [S] \mid g(T) \leq g\}$, using Algorithm 3.
4: **return** $\cup_{S \in \mathcal{C}(m,e)} A(S)$.

Note that applying Algorithms 1 and 2 we have to compute the Frobenius number and the genus, respectively, of the numerical semigroups we recursively obtain when we build $[S]$. Results of the next section enable us to easily compute the Frobenius number and the genus of every semigroup of $[S]$.

5 The Apéry set of the elements of $[S]$

Let S be a numerical semigroup and $n \in S \setminus \{0\}$. The Apéry set (named by [1]) of n in S is $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. The next result is a consequence of Lemma 2.4 from [11].

Lemma 5.1. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then $\mathrm{Ap}(S, n)$ has cardinality n. Moreover, $\mathrm{Ap}(S, n) = \{w(0) = 0, w(1), \ldots, w(n - 1)\}$ where $w(i)$ is the least element in S congruent with i modulo n.*

The set $\mathrm{Ap}(S, n)$ gives us a lot of information of S. The following result is found in [13].

Lemma 5.2. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then:*

- $F(S) = \max(\mathrm{Ap}(S, n)) - n$.
- $g(S) = \frac{1}{n}(\sum_{w \in \mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$.

The following result is a consequence of Lemma 5.1.

Lemma 5.3. *Let S be a numerical semigroup with minimal system of generators $\{n_1, n_2, \ldots, n_e\}$ and $\mathrm{Ap}(S, n_1) = \{0, w(1), \ldots, w(n_1 - 1)\}$. Then*

$$w(i) = \min\{a_2 n_2 + \cdots + a_e n_e \mid (a_2, \ldots, a_e) \in \mathbb{N}^{e-1} \text{ and}$$
$$a_2 n_2 + \cdots + a_e n_e \equiv i \pmod{n_1}\}.$$

Note that the set $\{(a_2, \ldots, a_e) \in \mathbb{N}^{e-1} \mid a_2n_2 + \cdots + a_en_e \equiv i \pmod{n_1}\}$ has a finite number of minimal elements (using the usual ordering in $\mathbb{N}^{e-1}$) by Dickson's Lemma (Theorem 5.1 from [10]). We denote the set of these minimal elements by $\mathcal{M}((n_1, \ldots, n_e), i)$. The following result is obtained from Lemma 5.3.

Proposition 5.4. *Let S be a numerical semigroup with minimal system of generators $\{n_1, n_2, \ldots, n_e\}$ and $\mathrm{Ap}(S, n_1) = \{0, w(1), \ldots, w(n_1 - 1)\}$. Then*

$$w(i) = \min\{a_2n_2 + \cdots + a_en_e \mid (a_2, \ldots, a_e) \in \mathcal{M}((n_1, \ldots, n_e), i)\}.$$

We illustrate the above proposition with an example.

Example 5.5. In this example we try to compute the Apéry set of the numerical semigroups of $[\langle 5, 6, 8 \rangle]$ that we obtained in Example 3.4.

For every $i \in \{1, 2, 3, 4\}$ let $A(i)$ be the set

$$\{(a_2, a_3) \in \mathbb{N}^2 \mid a_2 \cdot 1 + a_3 \cdot 3 \equiv i \pmod 5\},$$

and let $\mathcal{M}(i)$ be the set of the minimal elements of $A(i)$. Then,

$$\begin{aligned}
\mathcal{M}(1) &= \{(1, 0), (0, 2)\}, \\
\mathcal{M}(2) &= \{(2, 0), (0, 4), (1, 2)\}, \\
\mathcal{M}(3) &= \{(3, 0), (0, 1)\} \quad \text{and} \\
\mathcal{M}(4) &= \{(4, 0), (0, 3), (1, 1)\}.
\end{aligned}$$

Now, if we take an element from $[\langle 5, 6, 8 \rangle]$, for example $S = \langle 5, 21, 13 \rangle$, and we want to compute $\mathrm{Ap}(S, 5) = \{0, w(1), w(2), w(3), w(4)\}$, by applying Proposition 5.4 we have that $w(1) = \min\{21, 26\} = 21$, $w(2) = \min\{42, 52, 47\} = 42$, $w(3) = \min\{63, 13\} = 13$ and $w(4) = \min\{84, 39, 34\} = 34$.

Note that in the previous example it was easy to compute $\mathcal{M}(i)$ for every $i \in \{1, 2, 3, 4\}$. Now, our next goal is to give an algorithm for computing $\mathcal{M}((n_1, \ldots, n_e), i)$. In order to do it, we introduce the following sets:

$$\begin{aligned}
C(1) &= \{(x_2, \ldots, x_e) \in \mathbb{N}^{e-1} \mid n_2x_2 + \cdots + n_ex_e \equiv i \pmod{n_1}\}, \\
C(2) &= \{(x_1, x_2, \ldots, x_e) \in \mathbb{N}^e \mid (-n_1)x_1 + n_2x_2 + \cdots + n_ex_e = i\}, \\
C(3) &= \{(x_1, x_2, \ldots, x_e, x_{e+1}) \in \mathbb{N}^{e+1} \mid \\
&\qquad\qquad (-n_1)x_1 + n_2x_2 + \cdots + n_ex_e + (-i)x_{e+1} = 0\}.
\end{aligned}$$

Lemma 5.6. *If $(a_2, \ldots, a_e) \in C(1)$ then there exists $a_1 \in \mathbb{N}$ such that $(a_1, a_2, \ldots, a_e) \in C(2)$.*

Proof. It is enough to note that if $n_2a_2 + \cdots + n_ea_e \equiv i \pmod{n_1}$ then, there exist $a_1 \in \mathbb{N}$ such that $n_2a_2 + \cdots + n_ea_e = i + a_1n_1$. $\qquad\square$

Thanks to [12] we know that $C(3)$ is a finitely generated submonoid of $\mathbb{N}^{e+1}$. The next result can be deduced from Lemma 2 of [12].

Lemma 5.7. *Let A be the set $\{\alpha_1, \ldots, \alpha_t\}$ with $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ie}, \alpha_{i\,e+1})$ a system of generators of $C(3)$. If we suppose that $\alpha_1, \ldots, \alpha_d$ are the elements in A with the last coordinate equal to zero and $\alpha_{d+1}, \ldots, \alpha_q$ are the elements of S with the last coordinate equal to 1, then $C(2) = \{\bar{\alpha}_{d+1}, \ldots \bar{\alpha}_q\} + \langle \bar{\alpha}_1, \ldots, \bar{\alpha}_d \rangle$ where $\bar{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ie})$.*

Note that $\mathcal{M}((n_1, \ldots, n_e), i)$ are the minimal elements of $C(1)$. Hence, the following result allows us to compute it.

Proposition 5.8. *The minimal elements of $C(1)$ are the same that the minimal elements of the set* $\{(\alpha_{d+1,2}, \ldots, \alpha_{d+1,e}), \ldots, (\alpha_{q2}, \ldots, \alpha_{qe})\}$.

Proof. Let k be in $\{d+1, \ldots, q\}$. We check if $(\alpha_{k2}, \ldots, \alpha_{ke}) \in C(1)$. Since $(\alpha_{k1}, \ldots, \alpha_{ke}, 1) \in C(3)$, then $(-n_1)\alpha_{k1} + n_2\alpha_{k2} + \cdots + n_e\alpha_{ke} - i = 0$. Therefore $n_2\alpha_{k2} + \cdots + n_e\alpha_{ke} \equiv i \pmod{n_1}$ so $(\alpha_{k2}, \ldots, \alpha_{ke}) \in C(1)$.

We finish the proof checking that if $(a_2, \ldots, a_e) \in C(1)$ then there exists $k \in \{d+1, \ldots, q\}$ such that $(\alpha_{k2}, \ldots, \alpha_{ke}) \leq (a_2, \ldots, a_e)$. By Lemma 5.6, there exists $a_1 \in \mathbb{N}$ such that $(a_1, a_2, \ldots, a_e) \in C(2)$. Hence by Lemma 5.7, there exists $k \in \{d+1, \ldots, q\}$ such that $(\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{ke}) \leq (a_1, a_2, \ldots, a_e)$. Therefore, we have that $(\alpha_{k2}, \ldots, \alpha_{ke}) \leq (a_2, \ldots, a_e)$. □

An efficient algorithm for computing a finite system of generators of $C(3)$ is given in [3]. So, applying the previous result we have an algorithm which allows us to compute the minimal elements of $C(1)$. Therefore, using Proposition 5.4 and the idea exposed in Example 5.5, we have an algorithm for computing easily $\mathrm{Ap}(T, m)$ for every $T \in [S]$. Finally, thanks to Lemma 5.2 we can compute $F(T)$ and $g(T)$ for every $T \in [S]$.

6   Examples

We devote this section to illustrate the previous results with several examples. They show all the semigroups with a fixed multiplicity, embedding dimension, and Frobenius number or genus. Besides, we check Wilf's conjecture for many semigroups in the tree associated to $[S]$ for several packed numerical semigroups. The computations have been done in an Intel i7 with 32 GB of RAM, and using `Mathematica` ([16]).

Example 6.1. In this example we compute all the numerical semigroups with multiplicity 6, embedding dimension 3, and Frobenius number equal to 23.

With these fixed conditions, the set $\mathcal{C}(m, e)$ is

$$\{\langle 6, 7, 8\rangle, \langle 6, 7, 9\rangle, \langle 6, 7, 10\rangle, \langle 6, 7, 11\rangle, \langle 6, 8, 9\rangle,$$
$$\langle 6, 8, 11\rangle, \langle 6, 9, 10\rangle, \langle 6, 9, 11\rangle, \langle 6, 10, 11\rangle\}.$$

The Frobenius number of these semigroups are $17, 17, 15, 16, 19, 21, 23, 25$ and 25, respectively. So, by Proposition 4.1, for computing the semigroups with Frobenius number 23, we only consider the packed numerical semigroups $L = \{\langle 6, 7, 8\rangle, \langle 6, 7, 9\rangle, \langle 6, 7, 10\rangle, \langle 6, 7, 11\rangle, \langle 6, 8, 9\rangle, \langle 6, 8, 11\rangle, \langle 6, 9, 10\rangle\}$. Applying Algorithm 2, we compute the elements in $G([S])$ with the fixed Frobenius number. For example, from the first packed numerical semigroups in L only one numerical semigroup with Frobenius number equal to 23 is obtained (see Figure 2), but there is no numerical semigroups with Frobenius number equal to 23 in $G[\langle 6, 8, 9\rangle]$ (see Figure 3). Hence, the set of numerical semigroups with multiplicity 6, embedding dimension 3, and Frobenius number equal to 23 is

$$\{\langle 6, 8, 13\rangle, \langle 6, 7, 15\rangle, \langle 6, 7, 22\rangle, \langle 6, 7, 29\rangle, \langle 6, 9, 10\rangle\}.$$
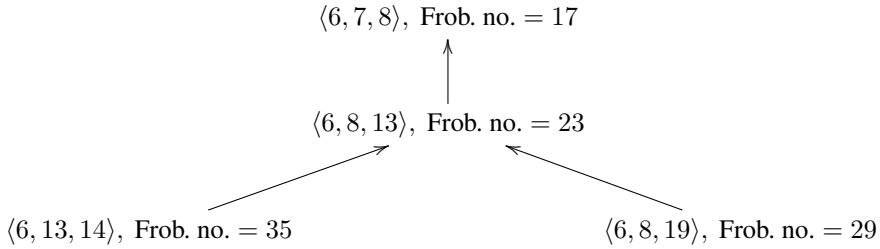
$\langle 6, 7, 8 \rangle$, Frob. no. $= 17$

$\uparrow$

$\langle 6, 8, 13 \rangle$, Frob. no. $= 23$

$\langle 6, 13, 14 \rangle$, Frob. no. $= 35$ $\langle 6, 8, 19 \rangle$, Frob. no. $= 29$

Figure 2: Two levels of the tree associated to the semigroup $\langle 6, 7, 8 \rangle$.

$\langle 6, 8, 9 \rangle$, Frob. no. $= 19$

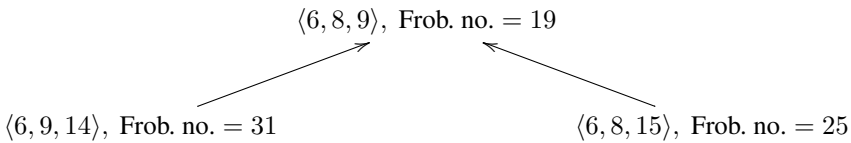$\langle 6, 9, 14 \rangle$, Frob. no. $= 31$ $\langle 6, 8, 15 \rangle$, Frob. no. $= 25$

Figure 3: One level of the tree associated to the semigroup $\langle 6, 8, 9 \rangle$.

Example 6.2. In this example, all the numerical semigroups with multiplicity 6, embedding dimension 3, and genus equal to 16 are computed. From Example 6.1, the set $\mathcal{C}(6, 3)$ is

$$\{\langle 6, 7, 8 \rangle, \langle 6, 7, 9 \rangle, \langle 6, 7, 10 \rangle, \langle 6, 7, 11 \rangle,$$
$$\langle 6, 8, 9 \rangle, \langle 6, 8, 11 \rangle, \langle 6, 9, 10 \rangle, \langle 6, 9, 11 \rangle, \langle 6, 10, 11 \rangle\}.$$

The genus of these semigroups are $9, 9, 9, 10, 10, 11, 12, 13$ and 13, respectively. So, by Proposition 4.1, for computing the semigroups with genus 16, we apply Algorithm 3 to all elements in $\mathcal{C}(6, 3)$. For example, for the semigroups $\langle 6, 7, 8 \rangle$ and $\langle 6, 8, 9 \rangle$ we obtain the trees showed in Figures 4 and 5, respectively. Thus, the set of numerical semigroups with multiplicity 6, embedding dimension 3, and genus 16 is

$$\{\langle 6, 14, 9 \rangle, \langle 6, 8, 21 \rangle, \langle 6, 15, 11 \rangle, \langle 6, 10, 17 \rangle\}.$$
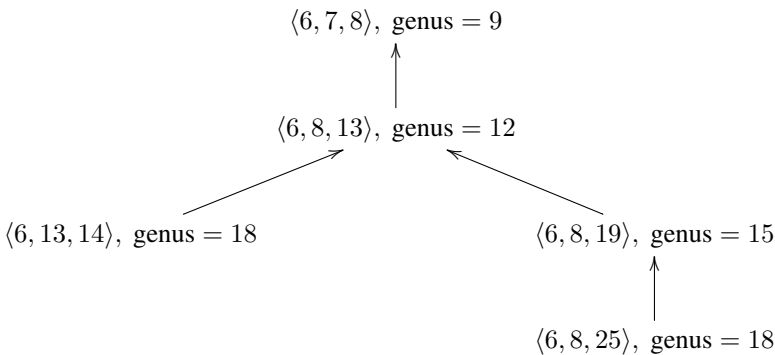
$\langle 6, 7, 8 \rangle$, genus $= 9$

$\uparrow$

$\langle 6, 8, 13 \rangle$, genus $= 12$

$\langle 6, 13, 14 \rangle$, genus $= 18$ $\langle 6, 8, 19 \rangle$, genus $= 15$

$\uparrow$

$\langle 6, 8, 25 \rangle$, genus $= 18$

Figure 4: Three levels of the tree associated to the semigroup $\langle 6, 7, 8 \rangle$.

$\langle 6, 8, 9 \rangle$, genus = 10

$\langle 6, 9, 14 \rangle$, genus = 16 $\langle 6, 8, 15 \rangle$, genus = 13
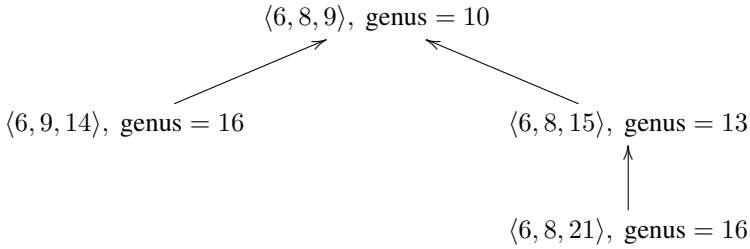
$\langle 6, 8, 21 \rangle$, genus = 16

Figure 5: Two levels of the tree associated to the semigroup $\langle 6, 8, 9 \rangle$.

Example 6.3. Now, we check Wilf's conjecture for several elements in the tree associated to some packed numerical semigroups. In this example, the elements are showed as a set with three entries $\{A, f, g\}$ where A is the minimal generating set of a numerical semigroup, and f and g are its Frobenius number and genus, respectively. Figure 6 illustrates two levels of the tree associated to the semigroup $S = \langle 110, 216, 217, 218, 219 \rangle$. Note that for all its elements the inequality $\frac{e(S)}{e(S)-1} = \frac{5}{4} \le \frac{F(S)+1}{g(S)}$ is held, and therefore they all satisfy Wilf's conjecture.

$\{\{110, 216, 217, 218, 219\}, 5941, 2971\}$

  $\{\{110, 217, 218, 219, 326\}, 7811, 3925\}$

    $\{\{110, 218, 219, 326, 327\}, 8911, 4456\}$

    $\{\{110, 217, 219, 326, 328\}, 7811, 3927\}$

    $\{\{110, 217, 218, 326, 329\}, 7811, 3926\}$

  $\{\{110, 216, 218, 219, 327\}, 5941, 2972\}$

    $\{\{110, 216, 219, 327, 328\}, 5941, 3025\}$

    $\{\{110, 216, 218, 327, 329\}, 6051, 3026\}$

  $\{\{110, 216, 218, 219, 327\}, 5941, 2972\}$

    $\{\{110, 216, 217, 328, 329\}, 5941, 2974\}$

  $\{\{110, 216, 218, 219, 327\}, 5941, 2972\}$
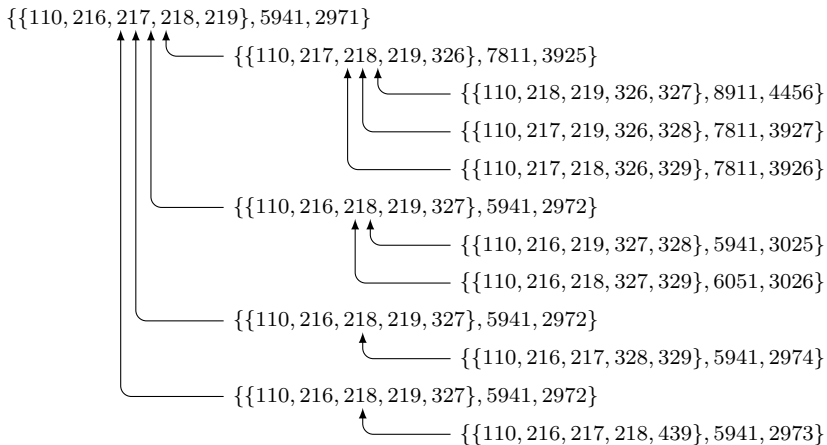
    $\{\{110, 216, 217, 218, 439\}, 5941, 2973\}$

Figure 6: Tree for checking Wilf's conjecture.

In Table 1 we show some packed numerical semigroups and the minimum and maximum of the quotients $(F(T) + 1)/g(T)$ of the semigroups T in their associated trees until a fixed level. Note that all tested semigroups (more than 71 000) satisfy Wilf's conjecture.

7 Frobenius pseudo-variety of numerical semigroups with a fixed multiplicity

According to the notation of [8], a Frobenius pseudo-variety is a non-empty family $\mathcal{P}$ of numerical semigroups which verifies the following conditions:

1. $\mathcal{P}$ has a maximum (according to the inclusion order).

Table 1: Checking Wilf's conjecture (up to level 15).

Semigroup	Number	min $\{\frac{F(\bullet)+1}{g(\bullet)}\}$	max $\{\frac{F(\bullet)+1}{g(\bullet)}\}$
$\{\{97, 111, 142, 159, 171\}, 958, 525\}$	3694	1496/981	2705/1357
$\{\{110, 216, 217, 218, 219\}, 5941, 2971\}$	425	2055/1081	2
$\{\{115, 151, 172, 189, 201\}, 1282, 724\}$	2656	1937/1224	670/339
$\{\{111, 115, 122, 171, 181, 200, 201\}, 702, 445\}$	35735	1488/1027	2012/1041
$\{\{117, 125, 142, 173, 191, 203, 213\}, 794, 476\}$	28688	382/261	899/458

2. If $\{S, T\} \subseteq \mathcal{P}$ then $S \cap T \in \mathcal{P}$.

3. If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$ then $S \cup \{F(S)\} \in \mathcal{P}$.

If $\mathcal{P}$ is a Frobenius pseudo-variety we define the graph $G(\mathcal{P})$ as follows: $\mathcal{P}$ is its set of vertices and $(S, T) \in \mathcal{P} \times \mathcal{P}$ is an edge if $T = S \cup \{F(S)\}$.

The following result is a direct consequence from Lemma 12 and Theorem 3 of [8].

Proposition 7.1. *If $\mathcal{P}$ is a Frobenius pseudo-variety, then $G(\mathcal{P})$ is a tree with root $\max(\mathcal{P})$. Moreover, the set of children of a vertex $S \in \mathcal{P}$ is*

$$\{S \setminus \{x\} \in \mathcal{P} \mid x \in \mathrm{msg}(S), \ x > F(S)\}.$$

Let m be a positive integer. We denote by $\mathcal{L}(m)$ the set

$$\{S \mid S \text{ is a numerical semigroup with } m(S) = m\}.$$

Clearly $\mathcal{L}(m)$ is a Frobenius pseudo-variety and $\max(\mathcal{L}(m)) = \{0, m, \rightarrow\} = \langle m, m+1, \ldots, 2m-1 \rangle$. So, as a consequence of Proposition 7.1, we have the following result which is fundamental in this work.

Theorem 7.2. *The graph $G(\mathcal{L}(m))$ is a tree rooted in $\langle m, m+1, \ldots, 2m-1 \rangle$. Moreover, the set formed by the children of a vertex $S \in \mathcal{L}(m)$ is*

$$\{S \setminus \{x\} \mid x \in \mathrm{msg}(S), \ x > F(S) \text{ and } x \neq m\}.$$

The previous theorem allows us to recursively construct $\mathcal{L}(m)$ from its root by recursively adding its children to the computed vertices. We illustrate this with an example.

Example 7.3. We show some levels of the tree $G(\mathcal{L}(4))$ giving its vertices and edges, and the minimal removed generators for obtaining the children.
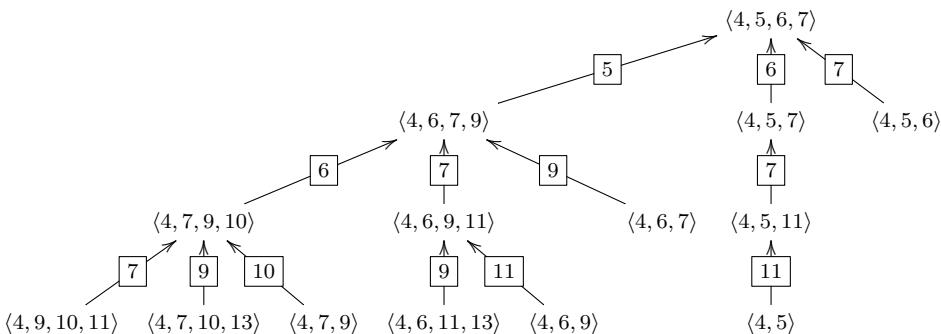
If G is a tree with root r, the level of a vertex x is the length of the only path which connect x and r. The height of a tree is the value of its maximum level. If $k \in \mathbb{N}$, we denote by $N(k, G) = \{v \in G \mid v \text{ has level } k\}$. So in Example 7.3 we have:

$N(0, \mathcal{L}(4)) = \{\langle 4, 5, 6, 7 \rangle\},$

$N(1, \mathcal{L}(4)) = \{\langle 4, 6, 7, 9 \rangle, \langle 4, 5, 7 \rangle, \langle 4, 5, 6 \rangle\},$

$N(2, \mathcal{L}(4)) = \{\langle 4, 7, 9, 10 \rangle, \langle 4, 6, 9, 11 \rangle, \langle 4, 6, 7 \rangle, \langle 4, 5, 11 \rangle\},$

$N(3, \mathcal{L}(4)) = \{\langle 4, 9, 10, 11 \rangle, \langle 4, 7, 10, 13 \rangle, \langle 4, 7, 9 \rangle, \langle 4, 6, 11, 13 \rangle, \langle 4, 6, 9 \rangle, \langle 4, 5 \rangle\}.$

8   Elements of $\mathcal{L}(m, e)$ with minimum genus

Our aim in this section is to give an algorithm that allows us to compute $g(m, e)$ and $\{S \mid S \in \mathcal{L}(m, e) \text{ and } g(S) = g(m, e)\}$. The following result is a consequence of Theorem 7.2.

Proposition 8.1. *If m is a positive integer and (S, T) an edge of $G(\mathcal{L}(m))$, then*

$$g(S) = g(T) + 1.$$

As a direct consequence of the previous proposition we have the following result.

Corollary 8.2. *Let us fix $m, e \in \mathbb{N}$. If*

$$P = \min\{k \in \mathbb{N} \mid N(k, G(\mathcal{L}(m))) \cap \mathcal{L}(m, e) \neq \emptyset\}$$

then

$$\{S \in \mathcal{L}(m, e) \mid g(S) = g(m, e)\} = N(P, G(\mathcal{L}(m))) \cap \mathcal{L}(m, e).$$

Moreover, $g(m, e) = m - 1 + P$.

It is clear that if $m \geq e \geq 2$ then $\langle m, m + 1, \ldots, m + e - 1 \rangle \in \mathcal{L}(m, e)$. In this way, we have the following result.

Proposition 8.3. *Let m and e be positive integers.*

1. *If $m < e$ then $\mathcal{L}(m, e) = \emptyset$.*
2. *If $e = 1$ and $\mathcal{L}(m, e) \neq \emptyset$ then $m = 1$ and $\mathcal{L}(m, e) = \{\mathbb{N}\}$.*
3. *If $m \geq e \geq 2$ then $\mathcal{L}(m, e) \neq \emptyset$.*

We now give an algorithm to compute $g(m, e)$ and $\{S \in \mathcal{L}(m, e) \mid g(S) = g(m, e)\}$.

Algorithm 5 An algorithm to compute $g(m, e)$ and the set of semigroups with a fixed multiplicity and embedding dimension such that its genus is $g(m, e)$.

INPUT: m and e positive integers such that $m \geq e \geq 2$.
OUTPUT: $g(m, e)$ and $\{S \mid S \in \mathcal{L}(m, e) \text{ and } g(S) = g(m, e)\}$.

1: Set $k = 0$ and $A = \{\langle m, m + 1, \ldots, 2m - 1 \rangle\}$.
2: **while** True **do**
3:      **if** $A \cap \mathcal{L}(m, e) \neq \emptyset$ **then**
4:          **return** $m - 1 + k$ and $A \cap \mathcal{L}(m, e)$
5:      **for** $S \in A$ **do**
6:          $C(S) = \{T \mid T \text{ is a child of } S\}$.
7:      $A = \bigcup_{S \in A} C(S)$, $k = k + 1$.

We illustrate the above algorithm in the following example.

Example 8.4. We compute $g(5, 3)$ and $\{S \in \mathcal{L}(5, 3) \mid g(S) = g(5, 3)\}$ using Algorithm 5.

- $k = 0$ and $A = \{\langle 5, 6, 7, 8, 9 \rangle\}$.
- $k = 1$ and $A = \{\langle 5, 7, 8, 9, 11 \rangle, \langle 5, 6, 8, 9 \rangle, \langle 5, 6, 7, 9 \rangle, \langle 5, 6, 7, 8 \rangle\}$.
- $k = 2$ and

$$A = \{\langle 5, 8, 9, 11, 12 \rangle, \langle 5, 7, 9, 11, 13 \rangle, \langle 5, 7, 8, 11 \rangle,$$
$$\langle 5, 7, 8, 9 \rangle, \langle 5, 6, 9, 13 \rangle, \langle 5, 6, 8 \rangle, \langle 5, 6, 7 \rangle\}.$$

It returns $g(5, 3) = 6$ and $\{S \in \mathcal{L}(5, 3) \mid g(S) = 6\} = \{\langle 5, 6, 8 \rangle, \langle 5, 6, 7 \rangle\}$.

In the package `FrobeniusNumberAndGenus` ([5]), we can run the command `ComputeMinimumGenusLme[5,3]` to obtain this result.

If S is a numerical semigroup, $n \in S \setminus \{0\}$ and $\mathrm{Ap}(S, n) = \{w(0) = 0, w(1), \ldots, w(n-1)\}$ (see [11, Lemma 2.4, Lemma 2.6]), then $w(i) = k_i n + i$ for some $k_i \in \mathbb{N}$ and $kn + i \in S$ if and only if $k \geq k_i$. Therefore, using Lemma 5.2, we have the following (see the proof of [11, Proposition 2.12]).

Lemma 8.5. *Let S be a numerical semigroup, $n \in S \setminus \{0\}$ and $\mathrm{Ap}(S, n) = \{0, k_1 n + 1, \ldots, k_{n-1} n + n - 1\}$. Then $g(S) = k_1 + \cdots + k_{n-1}$.*

The next result is easily deduced from Corollary 4 of [6].

Lemma 8.6. *Let m, e, q, r be integers such that $m \geq e \geq 2$, $S = \langle m, m+1, \ldots, m+e-1 \rangle$, and $m - 1 = q(e-1) + r$, with $q, r \in \mathbb{N}$ and $r \leq e - 2$. Then*

$$\mathrm{Ap}(S, m) = \{0, m+1, \ldots, m+e-1, 2m+(e-1)+1, \ldots,$$
$$2m + 2(e-1), \ldots, qm + (q-1)(e-1) + 1, \ldots, qm + q(e-1),$$
$$(q+1)m + q(e-1) + 1, \ldots, (q+1)m + q(e-1) + r\}.$$

If $a, b \in \mathbb{N}$ and $b \neq 0$ we denote by $a \bmod b$ the remainder of dividing a by b. If q is a rational number we denote by $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$. Note that $a = \lfloor \frac{a}{b} \rfloor b + (a \bmod b)$. From Lemma 8.5 and Lemma 8.6 we have the following result.

Proposition 8.7. *Let m and e be integers such that $m \geq e \geq 2$ and $S = \langle m, m+1, \ldots, m+e-1 \rangle$. Then,*

$$g(S) = \left(\left\lfloor \frac{m-1}{e-1} \right\rfloor + 1 \right) \left(\frac{\left\lfloor \frac{m-1}{e-1} \right\rfloor (e-1)}{2} + (m-1) \bmod (e-1) \right).$$

Clearly $\langle m, m+1, \ldots, m+e-1 \rangle \in \mathcal{L}(m, e)$ and therefore we have the following result.

Corollary 8.8. *If m and e are integers such that $m \geq e \geq 2$ then*

$$g(m, e) \leq \left(\left\lfloor \frac{m-1}{e-1} \right\rfloor + 1 \right) \left(\frac{\left\lfloor \frac{m-1}{e-1} \right\rfloor (e-1)}{2} + (m-1) \bmod (e-1) \right).$$

For many examples the equality holds. However, there are some cases where the semi-group $\langle m, m+1, \ldots, m+e-1 \rangle$ does not have minimum genus in the set $\mathcal{L}(m, e)$ as we show in the next example.

Example 8.9. $S = \langle 8, 9, 10 \rangle$ is a numerical semigroup and $g(S) = 16$. $\bar{S} = \langle 8, 9, 11 \rangle$ is a numerical semigroup and $g(\bar{S}) = 14$. Therefore, in this case $g(\langle 8, 9, 10 \rangle) \neq g(8, 3)$.

The following result is a consequence from Proposition 2.4.

Proposition 8.10. *If $S \in \mathcal{L}(m, e)$ then*

$$\bar{S} = \langle \{m\} + \{x \bmod m \mid x \in \mathrm{msg}(S)\} \rangle \in \mathcal{C}(m, e) \quad and \quad g(\bar{S}) \leq g(S).$$

Moreover, if $S \notin \mathcal{C}(m, e)$ then $g(\bar{S}) < g(S)$.

We illustrate the previous proposition with an example.

Example 8.11. If $S = \langle 5, 11, 17 \rangle \in \mathcal{L}(5, 3)$ then $\bar{S} = \langle \{5\} + \{0, 1, 2\} \rangle = \langle 5, 6, 7 \rangle \in \mathcal{C}(5, 3)$. Therefore, $g(\bar{S}) \leq g(S)$. Moreover, $S \notin \mathcal{C}(5, 3)$, so $g(\bar{S}) < g(S)$.

The next result is a consequence of Proposition 8.10.

Corollary 8.12. *Let m and e be integers such that $m \geq e \geq 2$. Then*

1. $g(m, e) = \min\{g(S) \mid S \in \mathcal{C}(m, e)\}$.
2. $\{S \in \mathcal{L}(m, e) \mid g(S) = g(m, e)\} = \{S \in \mathcal{C}(m, e) \mid g(S) = g(m, e)\}$.

Note that $\mathcal{C}(m, e)$ is finite and therefore the previous corollary gives us another algorithm for computing $g(m, e)$ and $\{S \in \mathcal{L} \mid g(S) = g(m, e)\}$. We give more details about this method using Proposition 2.4 and the calculations which appear in Example 2.5.

Example 8.13. From the calculations of Example 2.5, we have

$$\mathcal{C}(6, 3) = \{\langle 6, 7, 8 \rangle, \langle 6, 7, 9 \rangle, \langle 6, 7, 10 \rangle, \langle 6, 7, 11 \rangle,$$
$$\langle 6, 8, 9 \rangle, \langle 6, 8, 11 \rangle, \langle 6, 9, 10 \rangle, \langle 6, 9, 11 \rangle, \langle 6, 10, 11 \rangle\}.$$

A simple computation shows us

$$\begin{array}{lll}
g(\langle 6, 7, 8 \rangle) = 9, & g(\langle 6, 7, 9 \rangle) = 9, & g(\langle 6, 7, 10 \rangle) = 9, \\
g(\langle 6, 7, 11 \rangle) = 10, & g(\langle 6, 8, 9 \rangle) = 10, & g(\langle 6, 8, 11 \rangle) = 11, \\
g(\langle 6, 9, 10 \rangle) = 12, & g(\langle 6, 9, 11 \rangle) = 13 \quad \text{and} & g(\langle 6, 10, 11 \rangle) = 13.
\end{array}$$

Therefore, $g(6, 3) = 9$ and the set $\{S \in \mathcal{L}(6, 3) \mid g(S) = 9\}$ is equal to $\{\langle 6, 7, 8 \rangle, \langle 6, 7, 9 \rangle, \langle 6, 7, 10 \rangle\}$.

9 Elements of $\mathcal{L}(m, e)$ with minimum Frobenius number

Our aim in this section is to obtain algorithmic methods for computing $F(m, e)$ and $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}$. The next result is a consequence of Theorem 7.2.

Proposition 9.1. *If m is a positive integer and (S, T) is an edge of $G(\mathcal{L}(m))$, then $F(T) < F(S)$.*

The following result can be deduced from [9].

Proposition 9.2. *If m is a positive integer and (S, T) is an edge of $G(\mathcal{L}(m))$, then $e(S) \leq e(T)$.*

Clearly $F(m, m) = m - 1$ and

$$\{S \in \mathcal{L}(m, m) \mid F(S) = m - 1\} = \{\langle m, m + 1, \ldots, 2m - 1 \rangle\}.$$

It is well known (see [14] for example) that if $S = \langle n_1, n_2 \rangle$ is a numerical semigroup, then $F(S) = n_1 n_2 - n_1 - n_2$. Therefore, we obtain the following result.

Proposition 9.3. *Let m be an integer such that $m \geq 2$.*

1. *$F(m, m) = m - 1$ and $\{S \in \mathcal{L}(m, m) \mid F(S) = m - 1\} = \{\langle m, m + 1, \ldots, 2m - 1 \rangle\}$.*

2. *$F(m, 2) = m^2 - m - 1$ and $\{S \in \mathcal{L}(m, 2) \mid F(S) = m^2 - m - 1\} = \{\langle m, m + 1 \rangle\}$.*

If q is a rational number we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$. The next result is deduced from Corollary 5 of [6].

Proposition 9.4. *If m and e are integers such that $m \geq e \geq 2$, then*

$$F(\langle m, m + 1, \ldots, m + e - 1 \rangle) = \left\lceil \frac{m - 1}{e - 1} \right\rceil m - 1.$$

As a consequence of the previous proposition we get the following result.

Corollary 9.5. *If m and e are integers such that $m \geq e \geq 2$, then*

$$F(m, e) \leq \left\lceil \frac{m - 1}{e - 1} \right\rceil m - 1.$$

In the previous corollary, equality often holds, but in some cases

$$F(\langle m, m + 1, \ldots, m + e - 1 \rangle) \neq \min\{F(S) \mid S \in \mathcal{L}(m, e)\}.$$

For example, $F(\langle 4, 5, 6 \rangle) = 7$ and $F(\langle 4, 5, 7 \rangle) = 6$.

From the above results, we obtain the following algorithm where the projections from the cartesian product $\mathcal{L}(m) \times \mathbb{N}$ are denoted by π_1 and π_2.

Algorithm 6 An algorithm to compute $F(m, e)$ and the set of semigroups with a fixed multiplicity and embedding dimension such that its Frobenius number is $F(m, e)$.

INPUT: m and e integers such that $m \geq e \geq 2$.
OUTPUT: $F(m, e)$ and $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}$.

1: $A = \{\langle m, m + 1, \ldots, 2m - 1 \rangle\}$, $I = \emptyset$ and $\alpha = \lceil \frac{m-1}{e-1} \rceil m - 1$.
2: **while** True **do**
3:     $C = \{(S, F(S)) \mid S$ is child of some element of A and $F(S) \leq \alpha\}$.
4:     $K = \{S \in \pi_1(C) \mid e(S) \geq e\}$.
5:     **if** $K = \emptyset$ **then**
6:         **return** $F(m, e) = \pi_2(I)$ and $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\} = \pi_1(I)$
7:     $A = K$, $B = \{(S, F(S)) \mid S \in K$ and $e(S) = e\}$.
8:     $\alpha = \min(\pi_2(B) \cup \{\alpha\})$, $I = \{(S, F(S)) \in I \cup B \mid F(S) = \alpha\}$.

We illustrate how this algorithm works with an example.

Example 9.6. We compute $F(4, 3)$ and $\{S \in \mathcal{L}(4, 3) \mid F(S) = F(4, 3)\}$ using Algorithm 6.

- $A = \{\langle 4, 5, 6, 7 \rangle\}$, $I = \emptyset$ and $\alpha = \lceil \frac{3}{2} \rceil 4 - 1 = 7$.
- $C = \{(\langle 4, 6, 7, 9 \rangle, 5), (\langle 4, 5, 7 \rangle, 6), (\langle 4, 5, 6 \rangle, 7)\}$ and
  $K = \{\langle 4, 6, 7, 9 \rangle, \langle 4, 5, 7 \rangle, \langle 4, 5, 6 \rangle\}$.
- $A = \{\langle 4, 6, 7, 9 \rangle, \langle 4, 5, 7 \rangle, \langle 4, 5, 6 \rangle\}$, $B = \{(\langle 4, 5, 7 \rangle, 6), (\langle 4, 5, 6 \rangle, 7)\}$,
  $\alpha = \min\{6, 7\} = 6$ and $I = \{(\langle 4, 5, 7 \rangle, 6)\}$.
- $C = \{(\langle 4, 7, 9, 10 \rangle, 6)\}$ and $K = \{\langle 4, 7, 9, 10 \rangle\}$.
- $A = \{\langle 4, 7, 9, 10 \rangle\}$, $B = \emptyset$, $\alpha = 6$ and $I = \{(\langle 4, 5, 7 \rangle, 6)\}$.
- $C = \emptyset$ and $K = \emptyset$.

Therefore, $F(4, 3) = 6$ and $\{S \in \mathcal{L}(4, 3) \mid F(S) = 6\} = \{\langle 4, 5, 7 \rangle\}$. Using the Mathematica package [5], we obtain 6 and $\langle 4, 5, 7 \rangle$, running the commands `MinFrob[4,3]` and `FrobeniusEmbeddingDimensionMultiplicity[6,3,4]`, respectively.

Our next goal is to give an alternative algorithm for computing $F(m, e)$ and $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}$. The next result is deduced from Proposition 2.4.

Proposition 9.7. *If $S \in \mathcal{L}(m, e)$ then $\bar{S} = \langle \{m\} + \{x \bmod m \mid x \in \mathrm{msg}(S)\} \rangle \in \mathcal{C}(m, e)$ and $F(\bar{S}) \leq F(S)$.*

As a consequence of the previous proposition we get the following result.

Corollary 9.8. *If m and e are integers such that $m \geq e \geq 2$, then $F(m, e) = \min\{F(S) \mid S \in \mathcal{C}(m, e)\}$.*

The set $\mathcal{C}(m, e)$ is finite, so previous corollary give us an algorithmic method for computing $F(m, e)$.

Example 9.9. We compute $F(6, 5)$. First, we calculate $\mathcal{C}(6, 5)$ by using Proposition 2.4 and then we apply Corollary 9.8. So,

$$\mathcal{C}(6, 5) = \{\langle 6, 7, 8, 9, 10 \rangle, \langle 6, 7, 8, 9, 11 \rangle,$$
$$\langle 6, 7, 8, 10, 11 \rangle, \langle 6, 7, 9, 10, 11 \rangle, \langle 6, 8, 9, 10, 11 \rangle\}$$

and therefore

$$F(6, 5) = \min\{F(\langle 6, 7, 8, 9, 10 \rangle) = 11, \qquad F(\langle 6, 7, 8, 9, 11 \rangle) = 10,$$
$$F(\langle 6, 7, 8, 10, 11 \rangle) = 9, \qquad F(\langle 6, 7, 9, 10, 11 \rangle) = 8,$$
$$F(\langle 6, 8, 9, 10, 11 \rangle) = 13\} = 8.$$

We are now interested in giving a method for computing $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}$. The next example shows us that there exist semigroups $S \in \mathcal{L}(m, e)$ such that $S \notin \mathcal{C}(m, e)$ and $F(S) = F(m, e)$.

Example 9.10. The numerical semigroups $S_1 = \langle 7, 9, 10, 15 \rangle$ and $S_2 = \langle 7, 8, 10, 19 \rangle$ verify that $S_1, S_2 \in \mathcal{L}(7, 4) \setminus \mathcal{C}(7, 4)$ and $F(S_1) = F(S_2) = 13 = F(7, 4)$.

If $S \in \mathcal{L}(m, e)$ we denote by $\theta(S)$ the numerical semigroup generated by $\{m\} + \{x \bmod m \mid x \in \mathrm{msg}(S)\}$. Clearly, $\theta(S) \in \mathcal{C}(m, e)$.

Using the partition given in Section 2 and Theorem 2.3, the following two steps are sufficient for computing $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}$.

1. Compute $A = \{S \in \mathcal{C}(m, e) \mid F(S) = F(m, e)\}$.

2. For every $S \in A$, compute $\{T \in [S] \mid F(T) = F(S)\}$.

We already know how to compute 1. We now focus on giving an algorithm that allows us to compute 2.

Using Algorithm 1, for $S \in \mathcal{C}(m, e)$ and $F \in \mathbb{N}$ we get the set $\{T \in [S] \mid F(T) \leq F\}$. Clearly if $S \in \mathcal{C}(m, e)$ then $\{T \in [S] \mid F(T) = F(S)\} = \{T \in [S] \mid F(T) \leq F(S)\}$. We are going to adapt Algorithm 1 to our needs for computing 2.

We now recall some definitions of Section 3. If S is a numerical semigroup, $M(S) = \max(\mathrm{msg}(S))$. If $S \in \mathcal{C}(m, e)$ the graph $G([S])$ was defined as follows: $[S]$ is its set of vertices and $(A, B) \in [S] \times [S]$ is an edge if

$$\mathrm{msg}(B) = (\mathrm{msg}(A) \setminus \{M(A)\}) \cup \{M(A) - m\}.$$

Now, using the Theorem 3.3, we give an algorithm which for a semigroup $S \in \mathcal{C}(m, e)$ computes the set $\{T \in [S] \mid F(T) = F(S)\}$.

Algorithm 7 An algorithm to compute the semigroups of each equivalence class such that their Frobenius number is minimum.

INPUT: $S \in \mathcal{C}(m, e)$.
OUTPUT: $\{T \in [S] \mid F(T) = F(S)\}$.
1: $A = \{S\}$ and $B = \{S\}$.
2: **while** True **do**
3:     $C = \{H \mid H \text{ is child of an element of } B \text{ and } F(H) = F(S)\}$.
4:     **if** $C = \emptyset$ **then**
5:         **return** A
6:     $A = A \cup C, B = C$.

We finish this section with an example to illustrate the above algorithm.

Example 9.11. We use now Algorithm 7 for computing $\{T \in [S] \mid F(T) = F(S) = 10\}$ where $S = \langle 6, 7, 8, 9, 11 \rangle \in \mathcal{C}(6, 5)$.

- $A = \{\langle 6, 7, 8, 9, 11 \rangle\}$ and $B = \{\langle 6, 7, 8, 9, 11 \rangle\}$.

- $C = \{\langle 6, 8, 9, 11, 13 \rangle, \langle 6, 8, 11, 13, 15 \rangle\}$.

- $A = \{\langle 6, 7, 8, 9, 11 \rangle, \langle 6, 8, 9, 11, 13 \rangle, \langle 6, 8, 11, 13, 15 \rangle\}$ and
  $B = \{\langle 6, 8, 9, 11, 13 \rangle, \langle 6, 8, 11, 13, 15 \rangle\}$.

- $C = \emptyset$.

Thus, $\{T \in [S] \mid F(T) = 10\} = \{\langle 6, 7, 8, 9, 11 \rangle, \langle 6, 8, 9, 11, 13 \rangle, \langle 6, 8, 11, 13, 15 \rangle\}$. This result is also obtained with the package `FrobeniusNumberAndGenus` ([5]) by executing the command `ComputeEquivalenceClass[{6,7,8,9,11}]`.

References

[1] R. Apéry, Sur les branches superlinéaires des courbes algébriques, *C. R. Acad. Sci. Paris* **222** (1946), 1198–1200.

[2] V. Barucci, D. E. Dobbs and M. Fontana, *Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains*, Memoirs of the American Mathematical Society, American Mathematical Society, Providence, RI, 1997, doi:10.1090/memo/0598.

[3] E. Contejean and H. Devie, An efficient incremental algorithm for solving systems of linear Diophantine equations, *Inform. and Comput.* **113** (1994), 143–172, doi:10.1006/inco.1994.1067.

[4] J. Fromentin and F. Hivert, Exploring the tree of numerical semigroups, *Math. Comp.* **85** (2016), 2553–2568, doi:10.1090/mcom/3075.

[5] J. I. García-García, D. Marín-Aragón and A. Vigneron-Tenorio, FrobeniusNumberAndGenus, a Mathematica package for computations in numerical semigroups, 2019, `http://hdl.handle.net/10498/21658`.

[6] P. A. García-Sánchez and J. C. Rosales, Numerical semigroups generated by intervals, *Pacific J. Math.* **191** (1999), 75–83, doi:10.2140/pjm.1999.191.75.

[7] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, volume 30 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2005, doi:10.1093/acprof:oso/9780198568209.001.0001.

[8] A. M. Robles-Pérez and J. C. Rosales, Frobenius pseudo-varieties in numerical semigroups, *Ann. Mat. Pura Appl.* **194** (2015), 275–287, doi:10.1007/s10231-013-0375-1.

[9] J. C. Rosales, On numerical semigroups, *Semigroup Forum* **52** (1996), 307–318, doi:10.1007/bf02574106.

[10] J. C. Rosales and P. A. García-Sánchez, *Finitely Generated Commutative Monoids*, Nova Science Publishers, Commack, New York, 1999, `https://books.google.com/books?id=LQsH6m-x8ysC`.

[11] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, volume 20 of *Developments in Mathematics*, Springer, New York, 2009, doi:10.1007/978-1-4419-0160-6.

[12] J. C. Rosales, P. A. García-Sánchez, J. I. García-García and M. B. Branco, Systems of inequalities and numerical semigroups, *J. London Math. Soc.* **65** (2002), 611–623, doi:10.1112/s0024610701003052.

[13] E. S. Selmer, On the linear Diophantine problem of Frobenius, *J. Reine Angew. Math.* **1977** (1977), 1–17, doi:10.1515/crll.1977.293-294.1.

[14] J. J. Sylvester, Problem 7382, *Educat. Times J. College Preceptors New Ser.* **36** (1883), 177.

[15] H. S. Wilf, A circle-of-lights algorithm for the "money-changing problem", *Amer. Math. Monthly* **85** (1978), 562–565, doi:10.2307/2320864.

[16] Wolfram Research, Inc., Mathematica, Version 11.2, Champaign, Illinois, 2017, `https://www.wolfram.com/mathematica`.