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On the rank two geometries of the groups $\mathrm{PSL}(2, q)$: part II*

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Abstract

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Appendix

Proof of Lemma 9

Proof. In order to determine all subgroups H of $\mathrm{PSL}(2, q)$ such that (H, D_{10}) is a two-transitive pair we scan the list of maximal subgroups of $\mathrm{PSL}(2, q)$. For each maximal subgroup we analyse its subgroup lattice. There are six cases to consider.

1. The group $E_q : \frac{q-1}{(2,q-1)}$ contains a subgroup $D_{10} \cong E_5 : 2$ if $5|q$. In this situation and in view of (1) in Proposition 7, $H \cong E_5 : 4$ which is not a subgroup of $\mathrm{PSL}(2, q)$, under the given conditions.
2. Take D_{2d} with $d \mid \frac{q\pm1}{2}$. In view of (16)-(18) in Proposition 7, D_{2d} acts two-transitively on the cosets of D_{10} if and only if the index of D_{10} in D_{2d} equals 2 or 3 ($d = 10$ or 15). Therefore (D_{20}, D_{10}) and (D_{30}, D_{10}) are two-transitive pairs.
3. A_4 and S_4 do not contain any subgroup of order 10.
4. In view of (6) in Proposition 7 (A_5, D_{10}) is a two-transitive pair.
5. In view of (6), (7), (8) and (10) in Proposition 7, $\mathrm{PSL}(2, q')$ acts two-transitively on the cosets of $D_{10} \cong E_{q'} : \frac{q'-1}{2}$ only if $q' = 5$, therefore $q = 5^r$ for r an odd prime. $(\mathrm{PSL}(2, 5), D_{10})$ is a two-transitive pair.

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6. In view of (12) in Proposition 7, $\mathrm{PSL}(2, q')$ acts two-transitively on the cosets of $D_{10} \cong E_{q'} : \frac{q'-1}{2}$ only if $q' = 5$ and $q' - 1 = 2$, which leads to a contradiction.

□

Proof of Lemma 10

Proof. In order to determine all subgroups H of $\mathrm{PSL}(2, q)$ such that (H, A_4) is a two-transitive pair we scan the list of maximal subgroups of $\mathrm{PSL}(2, q)$. For each maximal subgroup we analyse its subgroup lattice. There are six cases to consider.

1. If $q = 5^r$, the group $E_q : \frac{q-1}{2}$ does not contain any subgroup isomorphic to $A_4 \cong E_4 : 3$ because $4 \mid q$ is in contradiction with the condition $q = 5^r$.
   If $q = p = \pm 1(5)$, the group $E_q : \frac{q-1}{2}$ does not contain any subgroup isomorphic to $E_4 : 3$ because $4 \mid p$ implies that $4 = p$, which is in contradiction with p an odd prime, the same argument holds for $q = p^2 = -1(5)$.
   If $q = 4^r$ with r prime, the $(2T)_1$ condition, the maximality and the conditions given on q imply that the only candidate of the form $E_q : \frac{q-1}{2}$ is $E_{16} : 3$. Now $(E_{16} : 3, E_4 : 3)$ is a two-transitive pair.
2. Take D_{2d} with $d \mid \frac{q \pm 1}{(2, q-1)}$. We know that dihedral groups only contain cyclic groups and dihedral groups, they do not contain an A_4.
3. If $q = 4^r$ with r prime, the group $\mathrm{PSL}(2, q)$ does not contain a subgroup isomorphic to S_4, because this is in contradiction with $q = \pm 1(8)$. The same argument holds for $q = 5^r$ with r an odd prime.
   If $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$, in view of (11) in Proposition 7 (S_4, A_4) is a two-transitive pair provided $q = \pm 1(8)$.
4. In view of (6) in Proposition 7 (A_5, A_4) is a two-transitive pair.
5. If $q = p = \pm 1(5)$, the group $\mathrm{PSL}(2, q)$ cannot contain any $\mathrm{PSL}(2, q')$ with $q'^m = q$, m an odd prime, the same argument holds for $q = p^2 = -1(5)$.
   If $q = 5^r$ with r an odd prime; orif $q = 4^r$ with r prime, the only candidates q' for $\mathrm{PSL}(2, q')$ are 4 and 5. In this situation we have $\mathrm{PSL}(2, q') \cong \mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5) \cong A_5$. This situation has been treated in (4).
6. If $q = p = \pm 1(5)$; or $q = 5^r$ with r an odd prime, the group $\mathrm{PSL}(2, q)$ cannot contain any $\mathrm{PGL}(2, q')$ with $q'^2 = q$.
   If $q = p^2 = -1(5)$ in view of (12) in Proposition 7 $\mathrm{PGL}(2, q')$ with $q'^2 = q$ acts two-transitively on the cosets of A_4 if $q' = 4$. In this situation we have $\mathrm{PSL}(2, q') \cong \mathrm{PGL}(2, 4) \cong A_5$. This situation has been treated in (4).
   If $q = 4^r$ with r prime, the group $\mathrm{PSL}(2, q)$ contains $\mathrm{PGL}(2, q')$ if $q'^2 = q$ which implies that $q' = 2^r$. In view of (12) in Proposition 7, $(\mathrm{PGL}(2, 4), E_4 : 3)$ is a two-transitive pair provided $q' = 2^r$ with $r = 2$.

□

Part of proof of Proposition 13.

Proof. <u>Subcase 1:</u> $G_{01} = G_0 \cap G_1 \cong D_{10}$.
By Lemma 9 the four possibilities for G_1 are D_{20} provided $10 \mid \frac{q \pm 1}{(2, q-1)}$, D_{30} provided

$15 \mid \frac{q\pm1}{(2,q-1)}$, $\mathrm{PSL}(2,5) \cong A_5$ provided $q = 5^r$ and A_5.

1.1 We consider the case where $G_1 \cong D_{20}$, provided $10 \mid \frac{q\pm1}{(2,q-1)}$.
The given conditions imply that either $q = p = \pm1(20)$ or $q = p^2 = -1(20)$. In both situations there are two conjugacy classes of A_5 in $\mathrm{PSL}(2,q)$. Since $\frac{q\pm1}{10}$ is even there are two conjugacy classes of D_{10} in $\mathrm{PSL}(2,q)$. The index of D_{10} in D_{20} equals two, therefore the D_{10} in a D_{20} are not all conjugate. The number of conjugacy classes of D_{20} depends on whether $\frac{q\pm1}{20}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases where $\frac{q\pm1}{20}$ is even or odd.

• $\frac{q\pm1}{20}$ is even. This implies that $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{20}$ and $N_{\mathrm{PSL}(2,q)}(D_{20}) = D_{40}$, with two conjugacy classes of D_{20}. Therefore the number of D_{20} containing a given D_{10} is one.

There are two classes of A_5 and D_{10} and the latter is contained in one D_{20}; therefore there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); A_5, D_{20}, D_{10})$ up to conjugacy, provided $\frac{q\pm1}{20}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of D_{10}, D_{20} and A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); A_5, D_{20}, D_{10})$ up to isomorphism provided $\frac{q\pm1}{20}$ is even.

• $\frac{q\pm1}{20}$ is odd. In this situation there is one conjugacy class of D_{20} in $\mathrm{PSL}(2,q)$. The condition on q implies that $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{20}$ and $N_{\mathrm{PSL}(2,q)}(D_{20}) = D_{20}$. Therefore the number of D_{20} containing a given D_{10} is one.

Up to conjugacy, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); A_5, D_{20}, D_{10})$ provided $\frac{q\pm1}{20}$ is odd.

Let us deal with the fusion of non-conjugate classes. Up to isomorphism there is exactly one such geometry, since following Lemma 8 the two classes of D_{10} and A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$.

To summarize, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_5 = \Gamma(\mathrm{PSL}(2,q); A_5, D_{20}, D_{10})$ provided $q = p = \pm1(20)$. Up to isomorphism there exists exactly one such geometry. Also, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{12} = \Gamma(\mathrm{PSL}(2,q); A_5, D_{20}, D_{10})$ provided $q = p^2 = -1(20)$. Up to isomorphism there exists exactly one such geometry.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 19, 41, 61$. For $q = 19$, it is also confirmed by [20].

1.2. We consider the case where $G_1 \cong D_{30}$, provided $15 \mid \frac{q\pm1}{(2,q-1)}$.
The condition $15 \mid \frac{q\pm1}{(2,q-1)}$ implies that either $q = 4^r$ with r prime, $q = p = \pm1(5)$ or $q = p^2 = -1(5)$. Hence, there are three cases namely $q = 4^r = \pm1(15)$ with r prime; $q = p = \pm1(30)$; or $q = p^2 = -1(30)$. We distinguish the first case from the other two.

• Let us first assume that $q = 4^r = \pm1(15)$ with r prime. In this situation there is only one conjugacy class of A_5. The number of classes of D_{30} and D_{10} in $\mathrm{PSL}(2,q)$ depends on whether $\frac{q\pm1}{15}$ is even or odd. The even case cannot occur because of the condition $q = 4^r$ given on q. If $\frac{q\pm1}{15}$ is odd there is only one conjugacy class of D_{30} and also one of D_{10} in $\mathrm{PSL}(2,q)$. Then the index $\frac{|D_{30}|}{|D_{10}|} \neq 2$, and therefore all D_{10} in D_{30} are conjugate. And A_5 contains one D_{10} up to conjugacy. The odd condition on $\frac{q\pm1}{15}$ implies that $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{10}$ and $N_{\mathrm{PSL}(2,q)}(D_{30}) = D_{30}$. Therefore the number of D_{30} containing a given D_{10} is one.

To summarize, up to conjugacy there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_1 = \Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ and thus also exactly one up to isomorphism provided either $q = 4^r$ with r prime; or $\frac{q\pm1}{15}$ odd. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 16$ and is also confirmed by [20].

• The cases $q = p = \pm1(30)$ and $q = p^2 = -1(30)$ with p an odd prime can be treated together. In this situation there are two conjugacy classes of A_5, but the number of conjugacy classes of D_{30} and D_{10} depends on whether $\frac{q\pm1}{30}$ is even or odd.

Assume $\frac{q\pm1}{30}$ is even. This implies that $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{20}$ and $N_{\mathrm{PSL}(2,q)}(D_{30}) = D_{60}$, with two conjugacy classes of D_{10} and also two of D_{30}. The number of subgroups D_{30} containing a given subgroup D_{10} in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ D_{60}\ |}\cdot\frac{|\ D_{30}\ |}{|\ D_{10}\ |}\cdot\frac{|\ D_{20}\ |}{|\ \mathrm{PSL}(2,q)\ |} = 1.$$

Up to conjugacy, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ provided $\frac{q\pm1}{30}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of D_{10}, D_{30} and A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ up to isomorphism provided $\frac{q\pm1}{30}$ is even.

Assume $\frac{q\pm1}{30}$ is odd. This implies that $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{10}$ and $N_{\mathrm{PSL}(2,q)}(D_{30}) = D_{30}$, with one conjugacy class of D_{10} and also one of D_{30}. The number of subgroups D_{30} containing a given subgroup D_{10} in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ D_{30}\ |}\cdot\frac{|\ D_{30}\ |}{|\ D_{10}\ |}\cdot\frac{|\ D_{10}\ |}{|\ \mathrm{PSL}(2,q)\ |} = 1.$$

Up to conjugacy, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ provided $\frac{q\pm1}{30}$ is odd.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ up to isomorphism provided $\frac{q\pm1}{30}$ is odd.

To summarize, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_6 = \Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ up to conjugacy and exactly one up to isomorphism provided $q = p = \pm1(30)$. Also, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{13} = \Gamma(\mathrm{PSL}(2,q); A_5, D_{30}, D_{10})$ and exactly one up to isomorphism provided $q = p^2 = -1(30)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 29, 31, 61$.

1.3. Consider the case $G_0 \cong G_1 \cong A_5$.

There are four situations, which are $q = 5^r$ with r odd prime, $q = p = \pm1(5)$, $q = p^2 = -1(5)$ with p an odd prime and $q = 4^r$ with r prime. Cases 2 and 3 can be treated together. We distinguish them from the others in our discussion.

• Assume $q = 5^r$ with r an odd prime. Using that $\mathrm{PSL}(2,5) \cong A_5$, there is only one conjugacy class of $\mathrm{PSL}(2,5)$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to A_5 in $\mathrm{PSL}(2,5^r)$ that have a subgroup D_{10} in common. There is only one conjugacy class of $E_5:2$. Since $\mathrm{PSL}(2,5^r)$ is simple and A_5 maximal, A_5 is self-normalized. Also, since $\mathrm{PSL}(2,5)$ is simple and $E_5:2$ maximal, $E_5:2$ is self-normalized in $\mathrm{PSL}(2,5)$ and also in $\mathrm{PSL}(2,5^r)$. Therefore the number of subgroups $\mathrm{PSL}(2,5)$ containing a given subgroup $E_5:2$ in $\mathrm{PSL}(2,5^r)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,5^r)\mid}{\mid \mathrm{PSL}(2,5)\mid}\cdot\frac{\mid \mathrm{PSL}(2,5)\mid}{\mid E_5:2\mid}\cdot\frac{\mid E_5:2\mid}{\mid \mathrm{PSL}(2,5^r)\mid}=1$$

which implies that the geometry does not exist.

• Assume that either $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$ with p an odd prime. There are two conjugacy classes of A_5. The number of conjugacy classes of D_{10} depends on whether $\frac{q\pm 1}{10}$ is even or odd.
If $\frac{q\pm 1}{10}$ is even there are two conjugacy classes of D_{10}. Notice that all D_{10} in an A_5 are conjugate and $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{20}$. The number of subgroups A_5 containing a given subgroup D_{10} in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q)\mid}{\mid A_5\mid}\cdot\frac{\mid A_5\mid}{\mid D_{10}\mid}\cdot\frac{\mid D_{20}\mid}{\mid \mathrm{PSL}(2,q)\mid}=2.$$

Therefore there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_7 = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, D_{10})$ up to conjugacy, provided $\frac{q\pm 1}{10}$ is even with q an odd prime and also exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{14} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, D_{10})$ up to conjugacy, provided $\frac{q+1}{10}$ is even with $q = p^2$; one geometry for each class of A_5.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of A_5 and D_{10} are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_7 = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, D_{10})$ up to isomorphism provided $\frac{q\pm 1}{10}$ is even with q an odd prime and also exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{14} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, D_{10})$ up to isomorphism provided $\frac{q+1}{10}$ is even with $q = p^2$.

Assume that $\frac{q\pm 1}{10}$ is odd. There is only one conjugacy class of D_{10} and $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{10}$. The number of subgroups A_5 containing a given subgroup D_{10} in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q)\mid}{\mid A_5\mid}\cdot\frac{\mid A_5\mid}{\mid D_{10}\mid}\cdot\frac{\mid D_{10}\mid}{\mid \mathrm{PSL}(2,q)\mid}=1.$$

Since there are two conjugacy classes of A_5 there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_8 = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, D_{10})$ up to conjugacy and thus also exactly one up to isomorphism provided $\frac{q\pm 1}{10}$ is odd with q an odd prime. Also, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{15} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, D_{10})$ up to conjugacy and thus also exactly one up to isomorphism provided $\frac{q+1}{10}$ is odd with $q = p^2$.
This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 11, 19, 29, 31, 41, 49$. For $q = 9$, it is also confirmed by

[3] and for $q = 11, 19$ by [20].
• If $q = 4^r$ with r prime. We know that there is only one conjugacy class of A_5. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to A_5 in $\mathrm{PSL}(2, 4^r)$ that have a subgroup D_{10} in common. The condition given on q implies that $\frac{q\pm1}{5}$ is odd, therefore there is only one class of D_{10} and $N_{\mathrm{PSL}(2,q)}(D_{10}) = D_{10}$. The number of subgroups A_5 containing a given subgroup D_{10} in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid A_5 \mid} \cdot \frac{\mid A_5 \mid}{\mid D_{10} \mid} \cdot \frac{\mid D_{10} \mid}{\mid \mathrm{PSL}(2,q) \mid} = 1.$$

In this situation there is only one conjugacy class of A_5, therefore we may conclude that there exists no such geometry.

□

Proof of Proposition 14

Proof. Let $G_0 \cong A_4$ with q prime, $q > 3$ and either $q = 3, 13, 27, 37(40)$ or $q = 5$.
In view of (5) in Proposition 7 the only possibility for G_{01} is the cyclic subgroup of order 3.
If H is a subgroup of G such that $(H, 3)$ is a two-transitive pair then one of the following holds: $H \cong Z_6$ provided $6 \mid \frac{q\pm1}{2}$, $H \cong D_6$ and $H \cong A_4$. They are the three only G_1-candidates.
Notice that q prime, $q > 3$ and so 3 divides either $\frac{q+1}{2}$ or $\frac{q-1}{2}$.
We review all possibilities for G_1 as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism).

1. Consider the case where $G_1 \cong Z_6$, provided $6 \mid \frac{q\pm1}{2}$.
The conditions on q prime are that $q = \pm1(12)$ and $q = 3, 13, 27, 37(40)$. This implies that $q = 13, 37, 83, 107(120)$ with q prime. The group A_4 contains one cyclic group of order 3 up to conjugacy. The cyclic group of order 3 is contained in exactly one Z_6 and all Z_6 in $\mathrm{PSL}(2, q)$ are conjugate. Since $\mathrm{PSL}(2, q)$ is simple and A_4 maximal, A_4 is self-normalized. It is also the case for the cyclic subgroups of order 3 in A_4. Now $N_{Z_6}(3) = Z_6$ and $N_{\mathrm{PSL}(2,q)}(3) = N_{\mathrm{PSL}(2,q)}(Z_6) = D_{q+1}$ provided $6 \mid \frac{q+1}{2}$ and D_{q-1} provided $6 \mid \frac{q+1}{2}$.
The number of subgroups Z_6 containing a given cyclic subgroup of order 3 in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid q\pm1 \mid} \cdot 1 \cdot \frac{\mid q\pm1 \mid}{\mid \mathrm{PSL}(2,q) \mid} = 1.$$

Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_1 = \Gamma(\mathrm{PSL}(2, q); A_4, Z_6, 3)$ up to conjugacy, and also exactly one up to isomorphism, provided $q = 13, 37, 83, 107(120)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 13, 37, 83$.

2. Consider the case where $G_1 \cong D_6$.
All cyclic subgroups of order 3 are conjugate in $\mathrm{PSL}(2, q)$. The number of conjugacy classes of D_6 depends on whether $\frac{q\pm1}{6}$ is odd or even. We distinguish the cases $\frac{q\pm1}{6}$ odd or even.

The group A_4 contains one cyclic group of order 3 up to conjugacy. We know that the normalizer of D_6 in $\mathrm{PSL}(2, q)$ is D_6 provided $\frac{q\pm1}{6}$ is odd, and that it is D_{12} provided

$\frac{q\pm1}{6}$ is even. The normalizer of the cyclic group of order 3 in D_6 is D_6 and its normalizer in $\mathrm{PSL}(2,q)$ is a dihedral group of order $q\pm1$. Therefore the number of subgroups D_6 containing a given cyclic subgroup of order 3 in $\mathrm{PSL}(2,q)$ is equal to

$$\begin{cases} \frac{|\mathrm{PSL}(2,q)|}{|D_6|}\cdot 1\cdot \frac{|q\pm1|}{|\mathrm{PSL}(2,q)|} = \frac{q\pm1}{6} & \text{if} \quad \frac{q\pm1}{6} \text{ odd} \\ \frac{|\mathrm{PSL}(2,q)|}{|D_{12}|}\cdot 1\cdot \frac{|q\pm1|}{|\mathrm{PSL}(2,q)|} = \frac{q\pm1}{12} & \text{if} \quad \frac{q\pm1}{6} \text{ even.} \end{cases}$$

To get the number of geometries up to conjugacy we need to know whether the subgroup A_4 normalizes each of the D_6, which is the case because

$$|N_{\mathrm{PSL}(2,q)}(3)\cap N_{\mathrm{PSL}(2,q)}(A_4)| = 3.$$

In order to determine the number of classes of geometries up to conjugacy we distinguish the cases $\frac{q\pm1}{6}$ odd or even.

• Assume $\frac{q\pm1}{6}$ is odd. There is only one class of D_6 and every given cyclic subgroup of order 3 in $\mathrm{PSL}(2,q)$ is contained in exactly $\frac{q\pm1}{6}$ dihedral groups D_6. Up to conjugacy there exist exactly $\frac{q\pm1}{6}$ geometries.

• Assume $\frac{q\pm1}{6}$ is even. There are two classes of D_6 and every given cyclic subgroup of order 3 in $\mathrm{PSL}(2,q)$ is contained in exactly $\frac{q\pm1}{12}$ dihedral groups D_6. Up to conjugacy there exist exactly $\frac{q\pm1}{6}$ geometries.

To summarize, up to conjugacy there exist exactly $\frac{q-1}{6}$ RWPRI and $(2T)_1$ geometries $\Gamma_3 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ provided $\frac{q-1}{6}$ is odd and exactly $\frac{q-1}{6}$ RWPRI and $(2T)_1$ geometries $\Gamma_5 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to conjugacy, provided $\frac{q-1}{6}$ is even. Also, there exist exactly $\frac{q+1}{6}$ RWPRI and $(2T)_1$ geometries $\Gamma_2 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to conjugacy, provided $\frac{q+1}{6}$ is odd and exactly $\frac{q+1}{6}$ RWPRI and $(2T)_1$ geometries $\Gamma_4 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to conjugacy, provided $\frac{q+1}{6}$ is even.

Let us deal with the fusion of non-conjugate classes. We remember that q is prime and thus $P\Gamma L(2,q)\cong \mathrm{PGL}(2,q)$. We also find that $N_{\mathrm{PGL}(2,q)}(A_4) = S_4$, $N_{\mathrm{PGL}(2,q)}(3) = D_{2(q\pm1)}$ and $N_{\mathrm{PGL}(2,q)}(D_6) = D_{12}$. In order to determine the number of classes of geometries up to isomorphism we distinguish the cases $\frac{q\pm1}{6}$ odd or even.

• Assume $\frac{q\pm1}{6}$ odd. There is only one conjugacy class of D_6. If we fix A_4 and the cyclic group of order 3, there is one D_6 which is fixed and the others are exchanged two by two, because D_6 in $\mathrm{PSL}(2,q)$ is its own normalizer. They merge two by two under the action of $P\Gamma L(2,q)$. Therefore, the number of RWPRI and $(2T)_1$ geometries $\Gamma_2 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to isomorphism, provided $\frac{q+1}{6}$ odd, is exactly $\frac{\frac{q+1}{6}-1}{2}+1$, and the number of RWPRI and $(2T)_1$ geometries $\Gamma_3 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to isomorphism, provided $\frac{q-1}{6}$ odd, is exactly $\frac{\frac{q-1}{6}-1}{2}+1$.

• Assume $\frac{q\pm1}{6}$ is even. There are two conjugacy classes of D_6. They both merge under the action of $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$ (see Lemma 11). If we fix A_4 and the cyclic group of order 3, we fix two D_6, one of each conjugacy class and all others are exchanged two by two. They merge two by two under the action of $P\Gamma L(2,q)$. Therefore, the number of RWPRI and $(2T)_1$ geometries $\Gamma_4 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to isomorphism, provided $\frac{q+1}{6}$ even, is exactly $\left(\frac{\frac{q+1}{6}-2}{2}+1\right) = \frac{q+1}{12}$ and the number of RWPRI and $(2T)_1$ geometries $\Gamma_5 = \Gamma\left(\mathrm{PSL}(2,q); A_4, D_6, 3\right)$ up to isomorphism, provided $\frac{q-1}{6}$ even, is exactly $\left(\frac{\frac{q-1}{6}-2}{2}+1\right)=\frac{q-1}{12}$.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 5, 13, 37, 43, 53, 67$. For $q = 5$, it is also confirmed by [3] and for $q = 13$ by [20].

3. Consider the case where $G_0 \cong G_1 \cong A_4$.
We must check whether this geometry exists or not, that is whether there are two subgroups isomorphic to A_4 in $\mathrm{PSL}(2,q)$ that have a cyclic subgroup of order 3 in common. We know that $N_{\mathrm{PSL}(2,q)}(A_4) = A_4$ and that $N_{A_4}(3) = 3$. Moreover, the group A_4 contains 4 maximal cyclic subgroups of order 3, all conjugate. The normalizer of 3 in $\mathrm{PSL}(2,q)$ is D_{q-1} if $3 \mid q-1$ and D_{q+1} if $3 \mid q+1$. Therefore the number of subgroups A_4 containing a given cyclic subgroup of order 3 in $\mathrm{PSL}(2,q)$ is equal to

$$\begin{cases} \frac{|\mathrm{PSL}(2,q)|}{|A_4|}\cdot 4\cdot\frac{q-1}{|\mathrm{PSL}(2,q)|}=\frac{q-1}{3} & \text{if} \quad 3 \mid q-1 \\ \frac{|\mathrm{PSL}(2,q)|}{|A_4|}\cdot 4\cdot\frac{q+1}{|\mathrm{PSL}(2,q)|}=\frac{q+1}{3} & \text{if} \quad 3 \mid q+1. \end{cases}$$

Knowing that there exists only one conjugacy class of A_4 and using the conditions on q we know that this geometry exists. There exist exactly, up to conjugacy, $\frac{q-1}{3}-1$ RWPRI and $(2T)_1$ geometries $\Gamma_7 = \Gamma\left(\mathrm{PSL}(2,q); A_4, A_4, 3\right)$, provided $3 \mid q-1$ and exactly $\frac{q+1}{3}-1$ RWPRI and $(2T)_1$ geometries $\Gamma_6 = \Gamma\left(\mathrm{PSL}(2,q); A_4, A_4, 3\right)$ up to conjugacy, provided $3 \mid q+1$.

Let us deal with the fusion of non-conjugate classes. We remember that q is prime and thus $\mathrm{PGL}(2,q) \cong P\Gamma L(2,q)$. We find that $N_{\mathrm{PGL}(2,q)}(A_4) = S_4$ and $N_{\mathrm{PGL}(2,q)}(3) = D_{2(q\pm1)}$. Therefore the number of subgroups A_4 containing a given cyclic subgroup of order 3 in $\mathrm{PGL}(2,q)$ is equal to $\frac{q\pm1}{3}$. To count the geometries up to isomorphism we need to know the action of $\mathrm{PGL}(2,q)$ on subgroups A_4 containing a given cyclic subgroup of order 3. If we fix $A_4 \cong G_0$ and the cyclic subgroup of order 3 we know that $|N_{\mathrm{PGL}(2,q)}(3) \cap N_{\mathrm{PGL}(2,q)}(A_4)| = |D_6| = 2|3|$. This D_6 is contained in two S_4 in $\mathrm{PGL}(2,q)$, which implies that there is one other A_4 fixed and all others are exchanged two by two. Thus they merge under the action of $\mathrm{PGL}(2,q)$. Hence, there exist exactly $\frac{(\frac{q-1}{3}-2)}{2}+1$ RWPRI and $(2T)_1$ geometries
$\Gamma_7 = \Gamma\left(\mathrm{PSL}(2,q); A_4, A_4, 3\right)$ up to isomorphism, provided $3 \mid q-1$ and exactly $\frac{(\frac{q+1}{3}-2)}{2}+1$ RWPRI and $(2T)_1$ geometries $\Gamma_6 = \Gamma\left(\mathrm{PSL}(2,q); A_4, A_4, 3\right)$ up to isomorphism, provided $3 \mid q+1$.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 5, 13, 37, 43, 53, 67$. For $q = 13$, it is also confirmed by [20]. □

Proof of Proposition 18

Proof. Let $G_0 \cong S_4$.
We subdivide our discussion in three cases, namely the three G_{01}-candidates given by (11), (12) and (13) in Proposition 7 which are: D_6, D_8 and A_4. In each of these three cases we review all possibilities for G_1 given in the previous Lemmas as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism).

Subcase 1: $G_{01} = G_0 \cap G_1 \cong D_6$.
By Lemma 15 the three possibilities for G_1 are D_{12} provided $6 \mid \frac{q\pm1}{2}$, D_{18} provided $9 \mid \frac{q\pm1}{2}$ and S_4.

The number of conjugacy classes of D_6 depends on whether $\frac{q\pm1}{6}$ is odd or even. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q\pm1}{6}$ odd or even.

Recall that when $q > 2$ is a prime and $q = \pm1(8)$ there are two conjugacy classes of S_4 in $\mathrm{PSL}(2, q)$.

1.1. Consider the case where $G_1 \cong D_{12}$, provided $6 \mid \frac{q\pm1}{2}$.
Since $\frac{q\pm1}{6}$ is even, following Lemma 4 there are two conjugacy classes of D_6 in $\mathrm{PSL}(2, q)$. The number of conjugacy classes of D_{12} depends on whether $\frac{q\pm1}{12}$ is even or odd. The conditions on q are that $q = \pm1(8)$ and $q = \pm1(12)$. Which implies that $\frac{q\pm1}{12}$ even. In this situation there are two classes of D_{12} in $\mathrm{PSL}(2, q)$. Now the index of $\frac{|D_{12}|}{|D_6|} = 2$, therefore the D_6 in a D_{12} are not all conjugate. Also, every D_{12} contains two D_6 which are not conjugate. And S_4 contains one D_6 up to conjugacy. Since $\frac{q\pm1}{12}$ is even we have $N_{\mathrm{PSL}(2,q)}(D_6) = D_{12} = N_{D_{12}}(D_6)$ and $N_{\mathrm{PSL}(2,q)}(D_{12}) = D_{24}$. Therefore the number of D_{12} containing a given D_6 is one. Since there are two classes of S_4, D_6 and D_{12}, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_1 = \Gamma\left(\mathrm{PSL}(2, q); S_4, D_{12}, D_6\right)$ up to conjugacy when $\frac{q\pm1}{12}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of D_6, D_{12} and S_4 are fused under the action of $\mathrm{PGL}(2, q)$ and thus also under the action of $P\Gamma L(2, q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_1 = \Gamma\left(\mathrm{PSL}(2, q); S_4, D_{12}, D_6\right)$ up to isomorphism, provided $\frac{q\pm1}{12}$ is even.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 23$.

1.2. Consider the case where $G_1 \cong D_{18}$, provided $9 \mid \frac{q\pm1}{2}$.
The number of conjugacy classes of D_{18} and D_6 depends on whether $\frac{q\pm1}{18}$ is even or odd. The conditions on q are that $q = \pm1(8)$ and $q = \pm1(18)$. Which implies that $\frac{q\pm1}{18}$.

Now the index $\frac{|D_{18}|}{|D_6|} \neq 2$, therefore all D_6 in a D_{18} are conjugate. And S_4 contains one D_6 up to conjugacy.

• Assume $\frac{q\pm1}{18}$ is even. This implies that $N_{\mathrm{PSL}(2,q)}(D_6) = D_{12}$ and $N_{\mathrm{PSL}(2,q)}(D_{18}) = D_{36}$. In this situation there are two conjugacy classes of D_6 and also two of D_{18}. The number of subgroups D_{18} containing a given subgroup D_6 in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2, q) \mid}{\mid D_{36} \mid} \cdot \frac{\mid D_{18} \mid}{\mid D_6 \mid} \cdot \frac{\mid D_{12} \mid}{\mid \mathrm{PSL}(2, q) \mid} = 1.$$

Since there are two conjugacy classes of S_4 there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2, q); S_4, D_{18}, D_6)$ up to conjugacy, provided $\frac{q\pm1}{18}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of D_6, D_{18} and S_4 are fused under the action of $\mathrm{PGL}(2, q)$ and thus also under the action of $P\Gamma L(2, q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2, q); S_4, D_{18}, D_6)$ up to isomorphism, provided $\frac{q\pm1}{18}$ is even.

• Assume $\frac{q\pm1}{18}$ is odd. This implies that $N_{\mathrm{PSL}(2,q)}(D_6) = D_6$ and $N_{\mathrm{PSL}(2,q)}(D_{18}) = D_{18}$. In this situation there is one conjugacy class of D_6 and also one of D_{18}. The number of subgroups D_{18} containing a given subgroup D_6 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ D_{18}\ |}\cdot\frac{|\ D_{18}\ |}{|\ D_6\ |}\cdot\frac{|\ D_6\ |}{|\ \mathrm{PSL}(2,q)\ |}=1.$$

Since there are two conjugacy classes of S_4 there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q);S_4,D_{18},D_6)$ up to conjugacy, provided $\frac{q\pm1}{18}$ is odd.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q);$ $S_4,D_{18},D_6)$ up to isomorphism, provided $\frac{q\pm1}{18}$ is odd.

To summarize, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_2=\Gamma\,(\mathrm{PSL}(2,q);S_4,D_{18},D_6)$ up to conjugacy and one up to isomorphism, provided $q=\pm1(72)$ or $q=\pm17(72)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q=17$ and is also confirmed by [20].

1.3. Finally we consider the case where $G_0\cong G_1\cong S_4$.

• Assume $\frac{q\pm1}{6}$ is even. There are two conjugacy classes of D_6. Now all the D_6 are contained in a S_4 and all D_6 in a S_4 are conjugate. The normalizer of D_6 in $\mathrm{PSL}(2,q)$ is D_{12}. The number of subgroups S_4 containing a given subgroup D_6 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ S_4\ |}\cdot\frac{|\ S_4\ |}{|\ D_6\ |}\cdot\frac{|\ D_{12}\ |}{|\ \mathrm{PSL}(2,q)\ |}=2.$$

Therefore, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_3=\Gamma(\mathrm{PSL}(2,q);S_4,$ $S_4,D_6)$ up to conjugacy, provided $\frac{q\pm1}{6}$ is even, one for each class of S_4.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_3=\Gamma\,(\mathrm{PSL}(2,q);S_4,S_4,D_6)$ up to isomorphism, provided $\frac{q\pm1}{6}$ is even.

• Assume $\frac{q\pm1}{6}$ is odd. There is one conjugacy class of D_6. This implies that normalizer $N_{\mathrm{PSL}(2,q)}(D_6)=D_6$. The number of subgroups S_4 containing a given subgroup D_6 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ S_4\ |}\cdot\frac{|\ S_4\ |}{|\ D_6\ |}\cdot\frac{|\ D_6\ |}{|\ \mathrm{PSL}(2,q)\ |}=1.$$

Since there are two conjugacy classes of S_4, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_4=\Gamma\,(\mathrm{PSL}(2,q);S_4,S_4,D_6)$ up to conjugacy and thus also one up to isomorphism, provided $\frac{q\pm1}{6}$ is odd.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q=7,17,23,31,41$. For $q=17$, it is also confirmed by [20].

Subcase 2: $G_{01}=G_0\cap G_1\cong D_8$.

By Lemma 16 the three possibilities for G_1 are D_{16} provided $8\mid\frac{q\pm1}{2}$, D_{24} provided $12\mid\frac{q\pm1}{2}$ and S_4. Observe that under the hypothesis there are two conjugacy classes of S_4 in $\mathrm{PSL}(2,q)$.

2.1. Consider the case where $G_1 \cong D_{16}$, provided $8 \mid \frac{q\pm1}{2}$.
Since $\frac{q\pm1}{8}$ is even there are two conjugacy classes of D_8. The conditions on q are that $q \pm 1(8)$ and $q \pm 1(16)$. Which implies that $q = \pm1(16)$. The index of D_8 in D_{16} equals two, therefore the D_8 in a D_{16} are not all conjugate. And also, every D_{16} contains two D_8 which are not conjugate. Moreover S_4 contains one D_8 up to conjugacy. The number of conjugacy classes of D_{16} depends on whether $\frac{q\pm1}{16}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q\pm1}{16}$ odd or even.

• Assume $\frac{q\pm1}{16}$ is even. This implies that $N_{\mathrm{PSL}(2,q)}(D_8) = D_{16} = N_{D_{16}}(D_8)$ and $N_{\mathrm{PSL}(2,q)}(D_{16}) = D_{32}$, with two conjugacy classes of D_{16}. Therefore the number of D_{16} containing a given D_8 is one.

Since there are two classes of S_4, D_8 and D_{16}, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); S_4, D_{16}, D_8)$ up to conjugacy, provided $\frac{q\pm1}{16}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of D_8, D_{16} and S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); S_4, D_{16}, D_8)$ up to isomorphism provided $\frac{q\pm1}{16}$ is even.

• Assume $\frac{q\pm1}{16}$ is odd. This implies that $N_{\mathrm{PSL}(2,q)}(D_8) = D_{16}$ and $N_{\mathrm{PSL}(2,q)}(D_{16}) = D_{16}$, with one conjugacy class of D_{16}. Therefore the number of D_{16} containing a given D_8 is one.

Hence, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); S_4, D_{16}, D_8)$ up to conjugacy, provided $\frac{q\pm1}{16}$ is odd.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of D_8 and S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); S_4, D_{16}, D_8)$ up to isomorphism, provided $\frac{q\pm1}{16}$ is odd.

To summarize, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_5 = \Gamma\,(\mathrm{PSL}(2,q); S_4, D_{16}, D_8)$ up to conjugacy and exactly one up to isomorphism, provided $q = \pm1(16)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 17, 31$. For $q = 17$, it is also confirmed by [20].

2.2. We now consider the case $G_1 \cong D_{24}$, provided $12 \mid \frac{q\pm1}{2}$.
The index $\frac{|D_{24}|}{|D_8|} \neq 2$, therefore all D_8 in a D_{24} are conjugate. And S_4 contains one D_8 up to conjugacy. The number of conjugacy classes of D_8 and D_{24} depends on whether $\frac{q\pm1}{24}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q\pm1}{24}$ odd or even.

• Assume $\frac{q\pm1}{24}$ is even. This implies that $N_{\mathrm{PSL}(2,q)}(D_8) = D_{16}$ and $N_{\mathrm{PSL}(2,q)}(D_{24}) = D_{48}$. In this situation there are two conjugacy classes of D_8 and also two of D_{24}. The number of subgroups D_{24} containing a given subgroup D_8 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid D_{48} \mid} \cdot \frac{\mid D_{24} \mid}{\mid D_8 \mid} \cdot \frac{\mid D_{16} \mid}{\mid \mathrm{PSL}(2,q) \mid} = 1.$$

Therefore, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); S_4, D_{24}, D_8)$ up to conjugacy, provided $\frac{q\pm1}{24}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of D_8, D_{24} and S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); S_4, D_{24}, D_8)$ up to isomorphism provided $\frac{q\pm1}{24}$ is even.

• Assume $\frac{q\pm1}{24}$ is odd. This implies that $N_{\mathrm{PSL}(2,q)}(D_8) = D_8$ and $N_{\mathrm{PSL}(2,q)}(D_{24}) = D_{24}$. In this situation there is one conjugacy class of D_8 and also one of D_{24}. The number of subgroups D_{18} containing a given subgroup D_8 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid D_{24} \mid} \cdot \frac{\mid D_{24} \mid}{\mid D_8 \mid} \cdot \frac{\mid D_8 \mid}{\mid \mathrm{PSL}(2,q) \mid} = 1.$$

To summarize, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2,q); S_4, D_{24}, D_8)$ up to conjugacy.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2,q); S_4, D_{24}, D_8)$ up to isomorphism, provided $\frac{q\pm1}{24}$ is odd.

To summarize, there exist exactly two RWPRI and $(2T)_1$ geometries
$\Gamma_6 = \Gamma\,(\mathrm{PSL}(2,q); S_4, D_{24}, D_8)$ up to conjugacy and exactly one up to isomorphism provided $q = \pm1(24)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 23$.

2.3. At last, consider the case $G_0 \cong G_1 \cong S_4$.
The number of conjugacy classes of D_8 depends on whether $\frac{q\pm1}{8}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q\pm1}{8}$ odd or even.

• Assume $\frac{q\pm1}{8}$ is even. There are two conjugacy classes of D_8. In S_4 all D_8 are conjugate and the normalizer of D_8 in $\mathrm{PSL}(2,q)$ is D_{16}. The number of subgroups S_4 containing a given subgroup D_8 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid S_4 \mid} \cdot \frac{\mid S_4 \mid}{\mid D_8 \mid} \cdot \frac{\mid D_{16} \mid}{\mid \mathrm{PSL}(2,q) \mid} = 2.$$

Therefore, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries
$\Gamma_7 = \Gamma\,(\mathrm{PSL}(2,q); S_4, S_4, D_8)$ provided $\frac{q\pm1}{8}$ is even, one for each class of S_4.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of S_4 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_7 = \Gamma\,(\mathrm{PSL}(2,q); S_4, S_4, D_8)$ up to isomorphism, provided $\frac{q\pm1}{8}$ is even.

• Assume $\frac{q\pm1}{8}$ is odd. There is one conjugacy class of D_8. This implies that normalizer $N_{\mathrm{PSL}(2,q)}(D_8) = D_8$. The number of subgroups S_4 containing a given subgroup D_8 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q) \mid}{\mid S_4 \mid} \cdot \frac{\mid S_4 \mid}{\mid D_8 \mid} \cdot \frac{\mid D_8 \mid}{\mid \mathrm{PSL}(2,q) \mid} = 1.$$

Since there are two conjugacy classes of S_4, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_8 = \Gamma\,(\mathrm{PSL}(2,q); S_4, S_4, D_8)$ up to conjugacy and thus also exactly one up to isomorphism provided $\frac{q\pm1}{8}$ is odd.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 7, 17, 23, 31, 41$. For $q = 7$, it is also confirmed by [3] and for $q = 17$ by [20].

Subcase 3: $G_{01} = G_0 \cap G_1 \cong A_4$.
By Lemma 17 the possibilities for G_1 are S_4 and A_5 provided $q = \pm 1(5)$. In the latter situation there are two conjugacy classes of A_5.

3.1. Consider the case where $G_0 \cong G_1 \cong S_4$.
We have $q = \pm 1(8)$ which implies that there are two conjugacy classes of S_4 and also two of A_4. Now all A_4 in a S_4 are conjugate and every given A_4 is contained in exactly one S_4, which implies that there exists no geometry in this situation.

3.2. Consider the case where $G_1 \cong A_5$.
If $q = p = \pm 1(5) = \pm 1(8)$ with p prime, this case has already been dealt with in Proposition 13. Therefore, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_9 = \Gamma\left(\mathrm{PSL}(2,q); S_4, A_5, A_4\right)$ up to conjugacy and exactly one up to isomorphism for $q = p = \pm 1(40)$ and for $q = p = \pm 9(40)$ with p an odd prime. □

Proof of Proposition 20

Proof. Let $G_0 \cong \mathrm{PSL}(2, 2^n)$.
We subdivide our discussion in three cases according to the three G_{01}-candidates given by (3), (4), (6) and (10) in Proposition 7 namely: the case of the cyclic subgroup of order 3 provided $q' = 2$; the case of D_{10} provided $q' = 4$ and the case of $E_{2^n} : (2^n - 1)$.

In each of these three cases we review all possibilities for G_1 given in the previous Lemmas as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism). In order to determine all geometries under the given conditions we subdivide our discussion in a particular case and a general one depending on whether $n = 1$ or not.

Particular case: $n = 1$ and $m = 2$.
In this situation $q' = 2$ and $q = 4$. In view of (3) and (4) in Proposition 7 there are two cases to consider: the cyclic group of order 3 and the cyclic group of order 2.

Subcase 1: $G_{01} = G_0 \cap G_1 \cong 2$.
Since $G \cong \mathrm{PSL}(2,4)$, $(\mathrm{PSL}(2,2), 2)$ and $(2^2, 2)$ are the only two-transitive pairs. We obtain the following geometries

$$\Gamma_2 = \Gamma\left(\mathrm{PSL}(2,4); \mathrm{PSL}(2,2), \mathrm{PSL}(2,2), 2\right) \text{ and } \Gamma_3 = \Gamma\left(\mathrm{PSL}(2,4); \mathrm{PSL}(2,2), 2^2, 2\right).$$

They are indeed RWPRI and $(2T)_1$ geometries as we need because we already met them in [5], Proposition 15. Since $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,2) \cong S_3$, these are the RWPRI and $(2T)_1$ geometries corresponding to the Petersen graph and the Desargues' configuration.

Subcase 2: $G_{01} = G_0 \cap G_1 \cong 3$.
Since $G \cong \mathrm{PSL}(2,4) \cong A_5$, $(\mathrm{PSL}(2,2), 3)$ and $(A_4, 3)$ are the only two-transitive pairs.

The geometry $\Gamma\left(\mathrm{PSL}(2,4); \mathrm{PSL}(2,2), \mathrm{PSL}(2,2), 3\right)$ has been treated in [5] Proposition 15 since $\mathrm{PSL}(2,2) \cong D_6$ and it does not exist. We obtain the following geometry

$\Gamma_4 = \Gamma\left(\mathrm{PSL}(2,4); \mathrm{PSL}(2,2), A_4, 3\right)$, which is indeed a RWPRI and $(2T)_1$ geometry as we need because we already met it in Proposition 14 since $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$.

General case: $n \neq 1$ and m is a prime.
In view of (10) in Proposition 7 there are two cases to consider: $E_{2^n}:(2^n-1)$ and D_{10} provided $q' = 4$.

Subcase 1: $G_{01} = G_0 \cap G_1 \cong D_{10}$, provided $q' = 4$.
This situation has been treated in Proposition 13, Subcase 1. We obtained the following RWPRI and $(2T)_1$ geometry $\Gamma_5 = \Gamma\,(\mathrm{PSL}(2,4^m); \mathrm{PSL}(2,4), D_{30}, D_{10})$, provided $\frac{q\pm1}{15}$ is odd.

Subcase 2: $G_{01} = G_0 \cap G_1 \cong E_{2^n}:(2^n-1)$.
By Lemma 19 the possibilities for G_1 are $E_{2^{2n}}:(2^n-1)$ provided $m=2$, and $\mathrm{PSL}(2,2^n)$.

Notice that if $n = 2$, $\mathrm{PSL}(2,2^n) \cong A_5$.

2.1. Consider the case where $G_1 \cong E_{2^{2n}}:(2^n-1)$ provided $m = 2$.
In this situation there is only one conjugacy class of $\mathrm{PSL}(2,2^n)$ and also one of $E_{2^{2n}}:(2^n-1)$ in $\mathrm{PSL}(2,2^{2n})$. There is one conjugacy class of $E_{2^n}:(2^n-1)$ in $\mathrm{PSL}(2,2^n)$ and also one in $\mathrm{PSL}(2,2^{2n})$. Notice that there are 2^n+1 conjugacy classes of $E_{2^n}:(2^n-1)$ in $E_{2^{2n}}:(2^n-1)$. Since $\mathrm{PSL}(2,2^{2n})$ is simple and both $\mathrm{PSL}(2,2^n)$ and $E_{2^n}:(2^n-1)$ are maximal, $\mathrm{PSL}(2,2^n)$ and $E_{2^n}:(2^n-1)$ are self-normalized. Moreover the normalizer of $E_{2^n}:(2^n-1)$ in $\mathrm{PSL}(2,2^{2n})$ is itself. We also find that $N_{\mathrm{PSL}(2,2^{2n})}(E_{2^{2n}}:(2^n-1)) = E_{2^{2n}}:(2^{2n}-1)$. Therefore the number of subgroups $E_{2^{2n}}:(2^n-1)$ containing a given subgroup $E_{2^n}:(2^n-1)$ in $\mathrm{PSL}(2,2^{2n})$ is equal to

$$\frac{\mid \mathrm{PSL}(2,2^{2n}) \mid}{\mid E_{2^{2n}}:(2^{2n}-1) \mid} \cdot \frac{\mid E_{2^{2n}}:(2^n-1) \mid}{\mid E_{2^n}:(2^n-1) \mid} \cdot (2^n+1) \cdot \frac{\mid E_{2^n}:(2^n-1) \mid}{\mid \mathrm{PSL}(2,2^{2n}) \mid} = 1.$$

Hence, the RWPRI and $(2T)_1$ geometry $\Gamma_1 = \Gamma(\mathrm{PSL}(2,2^{2n}); \mathrm{PSL}(2,2^n), E_{2^{2n}}:(2^n-1), E_{2^n}:(2^n-1))$ provided $n \neq 1$ exists and is unique up to conjugacy and also up to isomorphism.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 16, 64$. For $q = 16$, it is also confirmed by [20].

The particular situation where $n = 2$, has also been dealt with in Proposition 13, which showed that $\Gamma(\mathrm{PSL}(2,4^2); A_5, E_{16}:3, A_4)$ exists and is unique up to conjugacy, and also up to isomorphism.

2.2. Consider the case where $G_0 \cong G_1 \cong \mathrm{PSL}(2,2^n)$.
In this situation there is only one conjugacy class of $\mathrm{PSL}(2,2^n)$ in $\mathrm{PSL}(2,2^{nm})$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PSL}(2,2^n)$ in $\mathrm{PSL}(2,2^{nm})$ that have the subgroup $E_{2^n}:(2^n-1)$ in common. Since $\mathrm{PSL}(2,2^{nm})$ is simple and $\mathrm{PSL}(2,2^n)$ maximal, $\mathrm{PSL}(2,2^n)$ is self-normalized. Moreover, the group $\mathrm{PSL}(2,2^n)$ contains 2^n+1 maximal subgroups $E_{2^n}:(2^n-1)$ all conjugate. The normalizer of $E_{2^n}:(2^n-1)$ in $\mathrm{PSL}(2,q)$ is the group itself. Therefore the number of subgroups $\mathrm{PSL}(2,2^n)$ containing a given subgroup $E_{2^n}:(2^n-1)$ in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,2^{mn}) \mid}{\mid \mathrm{PSL}(2,2^n) \mid} \cdot \frac{\mid \mathrm{PSL}(2,2^n) \mid}{\mid E_{2^n}:(2^n-1) \mid} \cdot \frac{\mid E_{2^n}:(2^n-1) \mid}{\mid \mathrm{PSL}(2,2^{mn}) \mid} = 1$$

which implies that the geometry does not exist.

The particular situation where $n = 2$, has also been treated in Proposition 13 since $\mathrm{PSL}(2,4) \cong A_5$, which showed that $\Gamma(\mathrm{PSL}(2,4^m); \mathrm{PSL}(2,4), \mathrm{PSL}(2,4), A_4)$ does not exist. □

Proof of Proposition 24

Proof. Let $G_0 \cong \mathrm{PSL}(2,p^n)$.
We subdivide our discussion in three cases according to the four G_{01}-candidates given by (5)-(10) in Proposition 7 namely: A_4 provided $q' = 5$, S_4 provided $q' = 7$, A_5 provided $q' = 9, 11$ and $E_{q'} : \frac{q'-1}{2}$.

In each of these four cases we review all possibilities for G_1 given in the previous Lemmas as well as the number of classes of such geometries with respect to conjugacy (resp. isomorphism). In order to determine all geometries under the given conditions we subdivide our discussion in a particular case and a general one depending on whether $n = 1$ or not.

Particular case: $n = 1$.
In this situation $q' = p$. The candidates for G_{01} are $E_p : \frac{p-1}{2}$, A_4 provided $q' = 5$, S_4 provided $q' = 7$, A_5 provided $q' = 11$.

Subcase 1: $G_{01} = G_0 \cap G_1 \cong E_p : \frac{p-1}{2}$.
By Lemma 21 the only possibility for G_1 is $\mathrm{PSL}(2,p)$. We distinguish two particular situations, namely $\mathrm{PSL}(2,3) \cong A_4$ (provided $p = 3$) and $\mathrm{PSL}(2,5) \cong A_5$ (provided $p = 5$). All other situations will be treated in the general case, where n can take any value.

1.1 Consider the case where $G_0 \cong \mathrm{PSL}(2,3) \cong A_4 \cong G_1$.
In this situation G_{01} is the cyclic group of order 3. There is only one conjugacy class of A_4 in $\mathrm{PSL}(2,3^m)$. We must check whether this geometry exists, that is whether there exist two subgroups isomorphic to A_4 in $\mathrm{PSL}(2,3^m)$ that have the cyclic subgroup of order 3 in common. Since $\mathrm{PSL}(2,3^m)$ is simple and A_4 maximal, A_4 is self-normalized. The cyclic subgroup of order 3 is self-normalized in A_4. Moreover A_4 contains four cyclic subgroups of order 3 which are all conjugate. The normalizer of 3 in $\mathrm{PSL}(2,3^m)$ is an elementary abelian subgroup of order 3^m. Therefore the number of subgroups A_4 containing a given subgroup 3 in $\mathrm{PSL}(2,3^m)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,3^m) \mid}{\mid A_4 \mid} . \frac{\mid A_4 \mid}{\mid 3 \mid} . \frac{\mid 3^m \mid}{\mid \mathrm{PSL}(2,3^m) \mid} = 3^{m-1}$$

and thus the geometry exists. There exist exactly $3^{m-1} - 1$ RWPRI and $(2T)_1$ geometries $\Gamma_1 = \Gamma(\mathrm{PSL}(2,3^m); A_4, A_4, 3)$ up to conjugacy when $m \neq 3$. There exist exactly 8 RWPRI and $(2T)_1$ geometries $\Gamma_2 = \Gamma\left(\mathrm{PSL}(2,3^3); A_4, A_4, 3\right)$ up to conjugacy when $m = 3$.

Let us deal with the fusion of non-conjugate classes. We find that $N_{P\Gamma L(2,q)}(A_4) = (S_4 : C_m)$ and $N_{P\Gamma L(2,q)}(3) = (3^m.2 : C_m)$. Therefore the number of subgroups A_4 containing a given cyclic subgroup of order 3 in $P\Gamma L(2,3^m)$ is equal to

$$\frac{\mid P\Gamma L(2,3^m) \mid}{\mid S_4.m \mid} . \frac{\mid A_4 \mid}{\mid 3 \mid} . \frac{\mid 3^m.2.m \mid}{\mid P\Gamma L(2,3^m) \mid} = 3^{m-1}.$$

To count the geometries up to isomorphism we need to know the action of $P\Gamma L(2,3^m)$ on the subgroups A_4 containing a given cyclic subgroup of order 3. If we fix $A_4 \cong G_0$ and the cyclic subgroup of order 3 we know that $|N_{P\Gamma L(2,3^m)}(A_4) \cap N_{P\Gamma L(2,3^m)}(3)| = |D_6|.|C_m|$.

We distinguish the cases $m = 3$ and $m \neq 3$:

• Let us first assume that $m = 3$. In this situation there are three subgroups A_4 fixed and the others are exchanged 6 by 6. Thus they merge under the action of $P\Gamma L(2,3^m)$. Therefore, there exist exactly $\frac{3^{3-1}-3}{6} + 1 = 2$ RWPRI and $(2T)_1$ geometries $\Gamma_2 = \Gamma\left(\mathrm{PSL}(2,3^3); A_4, A_4, 3\right)$ up to isomorphism for $m = 3$.

• Now we assume $m \neq 3$. Using Fermat's Last Theorem for m an odd prime we know that $m \mid 3^{m-1} - 1$. In this situation there is only one $A_4 \cong G_0$ fixed. All others are exchanged $2m$ by $2m$. Therefore, there exist exactly $\frac{3^{m-1}-1}{2m}$ RWPRI and $(2T)_1$ geometries $\Gamma_1 = \Gamma\left(\mathrm{PSL}(2,3^m); A_4, A_4, 3\right)$ up to isomorphism, provided $m \neq 3$ is an odd prime.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 27$.

1.2 Consider the case where $G_0 \cong \mathrm{PSL}(2,5) \cong A_5 \cong G_1$.
This RWPRI and $(2T)_1$ geometry $\Gamma\left(\mathrm{PSL}(2,5^m), \mathrm{PSL}(2,5), A_5, E_5{:}2\right)$ has already been dealt with in Proposition 13 and it does not exist.

<u>Subcase 2:</u> $G_{01} = G_0 \cap G_1 \cong A_4$, provided $q = 5^m$ with m an odd prime.
This RWPRI and $(2T)_1$ geometry $\Gamma\left(\mathrm{PSL}(2,5^m), \mathrm{PSL}(2,5), A_5, A_4\right)$ has already been dealt with in Proposition 13, Subcase 2.3 and it does not exist.

<u>Subcase 3:</u> $G_{01} = G_0 \cap G_1 \cong S_4$, when $q = 7^m$ with m an odd prime.
By Lemma 23 the possibility for $G_1 \cong \mathrm{PSL}(2,7) \cong G_0$. In this situation there is only one conjugacy class of $\mathrm{PSL}(2,7)$ in $\mathrm{PSL}(2,11^m)$ and two conjugacy classes of S_4. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PSL}(2,7)$ in $\mathrm{PSL}(2,7^m)$ which have the subgroup S_4 in common. Since $\mathrm{PSL}(2,7^m)$ is simple and $\mathrm{PSL}(2,7)$ maximal, $\mathrm{PSL}(2,7)$ is self-normalized. The normalizer of S_4 in $\mathrm{PSL}(2,7^m)$ and in $\mathrm{PSL}(2,7)$ is the group S_4 itself. Therefore the number of subgroups $\mathrm{PSL}(2,7)$ containing a given subgroup S_4 in $\mathrm{PSL}(2,7^m)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,7^m)\ |}{|\ \mathrm{PSL}(2,7)\ |} \cdot \frac{|\ \mathrm{PSL}(2,7)\ |}{|\ S_4\ |} \cdot \frac{|\ S_4\ |}{|\ \mathrm{PSL}(2,7^m)\ |} = 1$$

which implies that the geometry does not exist.

<u>Subcase 4:</u> $G_{01} = G_0 \cap G_1 \cong A_5$, when $q = 11^m$ with m an odd prime.
By Lemma 25 the possibility for $G_1 \cong \mathrm{PSL}(2,11) \cong G_0$. In this situation there is only one conjugacy class of $\mathrm{PSL}(2,11)$ in $\mathrm{PSL}(2,11^m)$ and two conjugacy classes of A_5. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PSL}(2,11)$ in $\mathrm{PSL}(2,11^m)$ which have the subgroup A_5 in common. Since $\mathrm{PSL}(2,11^m)$ is simple and $\mathrm{PSL}(2,11)$ maximal, $\mathrm{PSL}(2,11)$ is self-normalized. The normalizer of A_5 in $\mathrm{PSL}(2,11^m)$ and in $\mathrm{PSL}(2,11)$ is the group A_5 itself. Therefore the number of subgroups $\mathrm{PSL}(2,11)$ containing a given subgroup A_5 in $\mathrm{PSL}(2,11^m)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,11^m)\ |}{|\ \mathrm{PSL}(2,11)\ |} \cdot \frac{|\ \mathrm{PSL}(2,11)\ |}{|\ A_5\ |} \cdot \frac{|\ A_5\ |}{|\ \mathrm{PSL}(2,11^m)\ |} = 1$$

which implies that the geometry does not exist.

General case:
Let us now discuss the general case, where n can take any value and p^n is different from 3 and 5 because these two cases have been discussed in the particular case. The two candidates for G_{01} are $E_{q'} : \frac{q'-1}{2}$ and A_5 provided $q' = 3^2$.

Subcase 1: $G_{01} = G_0 \cap G_1 \cong E_{p^n} : \frac{p^n-1}{2}$.
By Lemma 21 the only possibility for G_1 is $\mathrm{PSL}(2,p^n) \cong G_0$. In this situation there is only one conjugacy class of $\mathrm{PSL}(2,p^n)$ in $\mathrm{PSL}(2,p^{nm})$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PSL}(2,p^n)$ in $\mathrm{PSL}(2,p^{nm})$ that have the subgroup $E_{p^n} : \frac{p^n-1}{2}$ in common. Since $\mathrm{PSL}(2,q)$ is simple and $\mathrm{PSL}(2,p^n)$ maximal, $\mathrm{PSL}(2,p^n)$ is self-normalized. Moreover, the group $\mathrm{PSL}(2,p^n)$ contains 2^n+1 maximal subgroups $E_{p^n} : \left(\frac{p^n-1}{2}\right)$ all conjugate. There is only one conjugacy class of $E_{p^n} : \left(\frac{p^n-1}{2}\right)$ in $\mathrm{PSL}(2,p^{mn})$. The normalizer of $E_{p^n} : \frac{p^n-1}{2}$ in $\mathrm{PSL}(2,q)$ is the group itself. Therefore the number of subgroups $\mathrm{PSL}(2,p^n)$ containing a given subgroup $E_{p^n} : \left(\frac{p^n-1}{2}\right)$ in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\mathrm{PSL}(2,p^{mn})|}{|\mathrm{PSL}(2,p^n)|} \cdot \frac{|\mathrm{PSL}(2,p^n)|}{|E_{p^n} : \frac{p^n-1}{2}|} \cdot \frac{|E_{p^n} : \frac{p^n-1}{2}|}{|\mathrm{PSL}(2,p^{mn})|} = 1$$

which implies that the geometry does not exist.

Subcase 2: $G_{01} = G_0 \cap G_1 \cong A_5$, when $q = 9^m$ with m an odd prime.
By Lemma 22 the possibility for $G_1 \cong \mathrm{PSL}(2,9) \cong G_0$. In this situation there is only one conjugacy class of $\mathrm{PSL}(2,9)$ in $\mathrm{PSL}(2,9^m)$ and two conjugacy classes of A_5. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PSL}(2,9)$ in $\mathrm{PSL}(2,9^m)$ which have the subgroup A_5 in common. Since $\mathrm{PSL}(2,9^m)$ is simple and $\mathrm{PSL}(2,9)$ maximal, $\mathrm{PSL}(2,9)$ is self-normalized. The normalizer of A_5 in $\mathrm{PSL}(2,9^m)$ and in $\mathrm{PSL}(2,9)$ is the group A_5 itself. Therefore the number of subgroups $\mathrm{PSL}(2,9)$ containing a given subgroup A_5 in $\mathrm{PSL}(2,9^m)$ is equal to

$$\frac{|\mathrm{PSL}(2,9^m)|}{|\mathrm{PSL}(2,9)|} \cdot \frac{|\mathrm{PSL}(2,9)|}{|A_5|} \cdot \frac{|A_5|}{|\mathrm{PSL}(2,9^m)|} = 1$$

which implies that the geometry does not exist. □

Proof of Proposition 29

Proof. Let $G_0 \cong \mathrm{PGL}(2,p^n)$.
We subdivide our discussion in four cases, namely the four G_{01}-candidates given by (11), (12), (13) and (20) in Proposition 7 namely: $E_{p^n} : (p^n-1)$, $\mathrm{PSL}(2,p^n)$, D_8 for $p^n = 3$ and the case of S_4 provided $q = 5^2$. In each of these four cases we review all possibilities for G_1 given in the previous Lemmas as well as the number of classes of such geometries with respect to conjugacy (resp. isomorphism).

Subcase 1: $G_{01} = G_0 \cap G_1 \cong D_8$, provided $q = 9$.

By Lemma 25 the only case to consider is $G_0 \cong G_1 \cong \mathrm{PGL}(2,3)$.

Since $q = 9$, there is only one conjugacy class of D_8 and D_8 is self-normalized in $\mathrm{PSL}(2,9)$. Therefore the number of subgroups $\mathrm{PGL}(2,3)$ containing a given subgroup D_8 in $\mathrm{PSL}(2,9)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,9) \mid}{\mid \mathrm{PGL}(2,3) \mid} \cdot \frac{\mid \mathrm{PGL}(2,3) \mid}{\mid D_8 \mid} \cdot \frac{\mid D_8 \mid}{\mid \mathrm{PSL}(2,9) \mid} = 1.$$

There are 2 conjugacy classes of $\mathrm{PGL}(2,3)$ in $\mathrm{PSL}(2,9)$. Hence, up to conjugacy and also up to isomorphism there exists exactly one RWPRI and $(2T)_1$ geometry
$\Gamma_3 = \Gamma\,(\mathrm{PSL}(2,9); \mathrm{PGL}(2,3); \mathrm{PGL}(2,3); D_8)$. This is confirmed by [3].

Subcase 2: $G_{01} = G_0 \cap G_1 \cong E_{p^n} : (p^n - 1)$.
By Lemma 26 the possibilities for G_1 are $E_{p^{2n}} : (p^n - 1)$ and $\mathrm{PGL}(2, p^n)$. Notice that S_4 is a particular case of $\mathrm{PGL}(2, p^n)$ provided $p^n = 3$.

2.1. Consider the case where $G_1 \cong E_{p^{2n}} : (p^n - 1)$.
In this situation there is only one conjugacy class of $E_{p^{2n}} : (p^n - 1)$ and two conjugacy classes of $\mathrm{PGL}(2, p^n)$ in $\mathrm{PSL}(2, p^{2n})$. Each $\mathrm{PGL}(2, p^n)$ contains one conjugacy class of $E_{p^n} : (p^n - 1)$ and there are two conjugacy classes of $E_{p^n} : (p^n - 1)$ in $\mathrm{PSL}(2, p^{2n})$. Notice that there are $p^n + 1$ conjugacy classes of $E_{p^n} : (p^n - 1)$ in $E_{p^{2n}} : (p^n - 1)$. Since $\mathrm{PSL}(2, p^{2n})$ is simple and both $\mathrm{PGL}(2, p^n)$ and $E_{p^n} : (p^n - 1)$ maximal, $\mathrm{PGL}(2, p^n)$ and $E_{p^n} : (p^n - 1)$ are self-normalized. Moreover the normalizer of $E_{p^n} : (p^n - 1)$ in $\mathrm{PSL}(2, p^{2n})$ is itself. We also find that $N_{\mathrm{PSL}(2,p^{2n})}(E_{p^{2n}} : (p^n - 1)) = E_{p^{2n}} : \frac{p^{2n}-1}{2}$. Therefore the number of subgroups $\mathrm{PGL}(2, p^n)$ containing a given subgroup $E_{p^n} : (p^n - 1)$ in $\mathrm{PSL}(2, p^{2n})$ is equal to

$$\frac{\mid \mathrm{PSL}(2,p^{2n}) \mid}{\mid \mathrm{PGL}(2,p^n) \mid} \cdot \frac{\mid \mathrm{PGL}(2,p^n) \mid}{\mid E_{p^n} : (p^n - 1) \mid} \cdot \frac{\mid E_{p^n} : (p^n - 1) \mid}{\mid \mathrm{PSL}(2,2^{2n}) \mid} = 1.$$

Therefore, up to conjugacy, there exist exactly two RWPRI and $(2T)_1$ geometries
$\Gamma_1 = \Gamma\left(\mathrm{PSL}(2, p^{2n}); \mathrm{PGL}(2, p^n); E_{p^{2n}} : (p^n - 1); E_{p^n} : (p^n - 1)\right)$, corresponding to the two conjugacy classes of subgroups isomorphic to $E_{p^n} : (p^n - 1)$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $\mathrm{PGL}(2, p^n)$ are fused under the action of $\mathrm{PGL}(2, p^{2n})$ and thus also under the action of $P\Gamma L(2, p^{2n})$. This is also the case for the two classes of $E_{p^n} : (p^n - 1)$. Therefore, up to isomorphism there exists exactly one RWPRI and $(2T)_1$ geometry
$\Gamma_1 = \Gamma\left(\mathrm{PSL}(2, p^{2n}); \mathrm{PGL}(2, p^n); E_{p^{2n}} : (p^n - 1); E_{p^n} : (p^n - 1)\right)$.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 25, 49$.

2.2 Let us now consider the case where $G_1 \cong G_0 \cong \mathrm{PGL}(2, p^n)$.
In this situation there are two conjugacy classes of $\mathrm{PGL}(2, p^n)$ and also two conjugacy classes of $E_{p^n} : (p^n - 1)$ in $\mathrm{PSL}(2, p^{2n})$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PGL}(2, p^n)$ in $\mathrm{PSL}(2, p^{2n})$ that have the subgroup $E_{p^n} : (p^n - 1)$ in common. Since $\mathrm{PSL}(2, p^{2n})$ is simple and $\mathrm{PGL}(2, p^n)$ is maximal, $\mathrm{PGL}(2, p^n)$ is self-normalized. The subgroup $E_{p^n} : (p^n - 1)$ is also its own normalizer

in $\mathrm{PGL}(2,p^n)$ and in $\mathrm{PSL}(2,p^{2n})$. Therefore the number of subgroups $\mathrm{PGL}(2,p^n)$ containing a given subgroup $E_{p^n}:(p^n-1)$ in $\mathrm{PSL}(2,p^{2n})$ is equal to

$$\frac{\mid \mathrm{PSL}(2,p^{2n})\mid}{\mid \mathrm{PGL}(2,p^n)\mid}\cdot\frac{\mid \mathrm{PGL}(2,p^n)\mid}{\mid E_{p^n}:(p^n-1)\mid}\cdot\frac{\mid E_{p^n}:(p^n-1)\mid}{\mid \mathrm{PSL}(2,2^{2n})\mid}=1.$$

Now all $E_{p^n}:(p^n-1)$ in $\mathrm{PGL}(2,p^n)$ are conjugate. This implies that the RWPRI and $(2T)_1$ geometry $\Gamma\left(\mathrm{PSL}(2,p^{2n});\mathrm{PGL}(2,p^n);\mathrm{PGL}(2,p^n);E_{p^n}:(p^n-1)\right)$ does not exist.

Notice that in the particular case where $p^n=3$ and thus $G_1\cong S_4\cong\mathrm{PGL}(2,3)$ the geometry does not exist.

<u>Subcase 3:</u> $G_{01}=G_0\cap G_1\cong\mathrm{PSL}(2,p^n)$.

By Lemma 27 the possibilities for G_1 are A_5 provided $p^n=3$, $\mathrm{PGL}(2,p^n)$. Notice that S_4 is a particular case of $\mathrm{PGL}(2,p^n)$ provided $p^n=3$.

3.1. Consider the case where $G_1\cong A_5$ when $p^n=3$.
There are two conjugacy classes of $\mathrm{PGL}(2,3)\cong S_4$, of A_4 and of A_5 in $\mathrm{PSL}(2,9)$. All A_4 in A_5 are conjugate, it is also the case for all A_4 in S_4. Since $\mathrm{PSL}(2,9)$ is simple and both S_4 and A_5 are maximal, S_4 and A_5 are self-normalized. The normalizer of A_4 in $\mathrm{PSL}(2,9)$ and in S_4 is S_4. A_4 is self-normalized in A_5. The number of subgroups A_5 containing a given subgroup A_4 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{\mid \mathrm{PSL}(2,q)\mid}{\mid A_5\mid}\cdot\frac{\mid A_5\mid}{\mid A_4\mid}\cdot\frac{\mid S_4\mid}{\mid \mathrm{PSL}(2,q)\mid}=2.$$

To count the geometries up to conjugacy we need to know if the S_4 normalizes each of the A_5 which is not the case because $|N_{\mathrm{PSL}(2,q)}(A_4)\cap N_{\mathrm{PSL}(2,q)}(S_4)|=|S_4|=2|A_4|$. Therefore, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_2=\Gamma\left(\mathrm{PSL}(2,9);S_4,A_5,A_4\right)$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of A_4, S_4 and A_5 are fused under the action of $\mathrm{PGL}(2,9)$ and thus also under the action of $P\Gamma L(2,9)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_2=\Gamma\left(\mathrm{PSL}(2,9);S_4,A_5,A_4\right)$ up to isomorphism. This is confirmed by [3].

3.2 Consider the case where $G_1\cong G_0\cong\mathrm{PGL}(2,p^n)$.
In this situation there are two conjugacy classes of $\mathrm{PGL}(2,p^n)$ and also two conjugacy classes of $\mathrm{PSL}(2,p^n)$ in $\mathrm{PSL}(2,p^{2n})$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\mathrm{PGL}(2,p^n)$ in $\mathrm{PSL}(2,p^{2n})$ that have the subgroup $\mathrm{PSL}(2,p^n)$ in common. Since $\mathrm{PSL}(2,p^{2n})$ is simple and $\mathrm{PGL}(2,p^n)$ maximal, $\mathrm{PGL}(2,p^n)$ is self-normalized. The normalizer of the subgroup $\mathrm{PSL}(2,p^n)$ in $\mathrm{PGL}(2,p^n)$ and in $\mathrm{PSL}(2,p^{2n})$ is $\mathrm{PGL}(2,p^n)$. Therefore the number of $\mathrm{PGL}(2,p^n)$ containing a given $\mathrm{PSL}(2,p^n)$ is one.

Now all $\mathrm{PSL}(2,p^n)$ in $\mathrm{PGL}(2,p^n)$ are conjugate, which implies that the RWPRI and $(2T)_1$ geometry $\Gamma\left(\mathrm{PSL}(2,p^{2n});\mathrm{PGL}(2,p^n);\mathrm{PGL}(2,p^n);\mathrm{PSL}(2,p^n)\right)$ does not exist.

Notice that in the particular case where $p^n=3$ we get $G_1\cong S_4\cong\mathrm{PGL}(2,3)$.

<u>Subcase 4:</u> $G_{01}=G_0\cap G_1\cong S_4$, provided $q=5^2$.
By Lemma 28 the only case to consider is $G_0\cong G_1\cong\mathrm{PGL}(2,5)$.

In this situation where $q = 25$, there are two conjugacy classes of $\mathrm{PGL}(2,5)$ and also two conjugacy classes of S_4 in $\mathrm{PSL}(2,5^2)$. Since $\mathrm{PSL}(2,5^2)$ is simple and $\mathrm{PGL}(2,5)$ is maximal, $\mathrm{PGL}(2,5)$ is self-normalized and S_4 is self-normalized in $\mathrm{PGL}(2,5)$ and also in $\mathrm{PSL}(2,5^2)$. Therefore the number of $\mathrm{PGL}(2,5)$ containing a given S_4 is one. Now all S_4 in $\mathrm{PGL}(2,5)$ are conjugate, which implies that the RWPRI and $(2T)_1$ geometry $\Gamma\left(\mathrm{PSL}(2,5^2); \mathrm{PGL}(2,5); \mathrm{PGL}(2,5); S_4\right)$ does not exist. □

Case of Table 3, geometry $\mathbf{\Gamma_2}$

We know that $s \geq 2$. Consider a path (a, b, c) as in the preceding case. Here, $G_{abc} = Z_3$. This acts on the three 1-elements d_1, d_2, d_3 other than b in $c^\perp$. The action is transitive since otherwise Z_3 would be in the kernel of the action of G_c on $c^\perp$. This kernel for the action of S_4 on the cosets of D_6 is reduced to the identity, a contradiction. This provides $s \geq 3$ for paths starting at a $0 - element$.

Next consider a path (h, i, j) as in the preceding case. Here, $G_{hij} = Z_2$. This acts on the two 0-elements k_1, k_2 other than i in $j^\perp$. The action is transitive since otherwise Z_2 would be in the kernel of the action of G_j on $j^\perp$. This kernel for the action D_{18} on the cosets of D_6 is a group Z_3, a contradiction. Hence $s \geq 3$.

Applying Leemans' method we get $s = 2$ or 3. Thus $s = 3$.

Case of Table 3, geometry $\mathbf{\Gamma_5}$

This geometry $\Gamma\left(\mathrm{PSL}(2,q); D_{16}, S_4, D_8\right)$ is known as a locally 7-arc-transitive graph due to Wong [22], hence $s = 7$.

Case of Table 3, geometry $\mathbf{\Gamma_7}$ and $\mathbf{\Gamma_8}$.

This geometry $\Gamma\left(\mathrm{PSL}(2,q); S_4, S_4, D_8\right)$ is known as a locally 4-arc-transitive graph due to Biggs-Hoare [1], hence $s = 4$ in this case.

Case of Table 4, geometry $\mathbf{\Gamma_1}$

We know that $s \geq 2$. Consider a path (a, b, c) as in the preceding case. Here, $G_{abc} = 2^n$. This acts on the 2^n elements of type 1, $d_1, ..., d_{2^n}$ other than b in $c^\perp$. The action is transitive since otherwise a subgroup of order 2 would be in the kernel of the action of G_c on $c^\perp$. This kernel for the action of $PSL(2, 2^n)$ on the cosets of $2^n : (2^n - 1)$ is reduced to the identity, a contradiction. This provides $s \geq 3$ for paths starting at a $0 - element$.

Next consider a path (h, i, j) as in the preceding case. Here, $G_{hij} = Z_{2^n-1}$. This acts on the $2^n - 1$ elements of type 0, k_1, k_{2^n-1} other than i in $j^\perp$. The action is transitive since otherwise Z_t with t prime and dividing $2^n - 1$ would be in the kernel of the action of G_j on $j^\perp$. This kernel for the action of $2^{2n} : (2^n - 1)$ on the cosets of $2^n : (2^n - 1)$ is not determined but its order divides 2^n, a contradiction. Hence $s \geq 3$.

Applying Leemans' method we get $s = 2$ or 3. Thus $s = 3$.

Case of Table 6, geometry Γ_1

We know that $s \geq 2$. Consider a path (a, b, c) as in the preceding case. Here, $G_{abc} = p^n$. This acts on the p^n elements of type 1, $d_1, ..., d_{p^n}$ other than b in $c^\perp$. The action is transitive since otherwise a subgroup of order p would be in the kernel of the action of G_c on $c^\perp$. This kernel for the action of $PGL(2, 2^n)$ on the cosets of $p^n : (p^n - 1)$ is reduced to the identity, a contradiction. This provides $s \geq 3$ for paths starting at a $0 - element$.

Next consider a path (h, i, j) as in the preceding case. Here, $G_{hij} = Z_{p^n-1}$. This acts on the $p^n - 1$ elements of type 0, k_1, k_{p^n-1} other than i in $j^\perp$. The action is transitive since otherwise Z_t with t prime and dividing $p^n - 1$ would be in the kernel of the action of G_j on $j^\perp$. This kernel for the action of $p^{2n} : (p^n - 1)$ on the cosets of $p^n : (p^n - 1)$ is not determined but its order divides p^n, a contradiction. Hence $s \geq 3$.

Applying Leemans' method we get $s = 2$ or 3. Thus $s = 3$.