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Geometrijska optimizacija pri uklonu palice

Optimizing the Geometry for the Buckling of a Bar

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Pojav uklona vitke elastične palice spremenljivega prereza je obravnavan po teoriji majhnih premikov (teorija II. reda po Chwalli [2]) in je predstavljen z ustreznim robnim problemom. Na temelju matematičnega modela uklona je z uporabo variacijskega računa izvedena geometrijska optimizacija palice ob predpisanih geometrijskih in robnih pogojih. Prikazana je splošna uporabnost metode optimizacije z reševanjem variacijskega problema ter primerjava med lastnostmi palice z optimalno geometrijsko obliko in referenčne palice nespremenljivega prereza. Glavna lastnost palice z optimalno geometrijsko obliko je nespremenljiva največja upogibna napetost vzdolž palice v mejnem stanju – gradivo je stabilnostno v celoti izkoriščeno.

(Ključne besede: konstrukcija, nosilci, uklon, oblike optimalne)

Using the small-displacement theory (a theory of the second order, according to Chwalla [2]), the buckling process for a slender, elastic bar with a changeable cross-sectional area is considered and represented with a corresponding boundary problem. Based on a mathematical model of buckling, which considers the geometric and boundary conditions, an optimum geometry is obtained using the calculus of variation. By comparing the properties of a bar with optimum geometry to those of a reference bar with a constant cross-section, the paper shows that the presented optimization method is generally applicable. The main feature of a bar with optimum geometry is a constant maximum bending stress along the whole length of the bar in its deflected form, which means that in terms of stability the material is completely exploited.

(Keywords: design, beams, buckling, optimal shape design)

0 UVOD

Pri problemu uklona se tlačno obremenjena palica v mejnem stanju, pod vplivom poljubno majhne motnje, ukloni – pojavi se upogibna obremenitev. Ker ta prehod povzroči dodatne obremenitve palice, je treba pri postopku dimenzioniranja zagotoviti, da obremenitev ne doseže kritične vrednosti. Zato je v primerih vitkih elementov, pri katerih smo omejeni z mejo stabilnosti, nosilnost gradiva slabo izkoriščena. Eden od načinov zvečanja meje stabilnosti in s tem izkoriščenosti nosilnosti gradiva elementa je geometrijska optimizacija.

Namen prispevka je predstaviti analitično metodo geometrijske optimizacije, ki je splošno uporabna pri problemih uklona palice v elastičnem območju za različna vpetja in obremenitve. Optimizacija je izvedena na podlagi matematičnega modela – robnega problema, ki popiše mejno stanje pri pojavu uklona palice po teoriji II. reda. To pomeni, da so ravnotežne enačbe zapisane za

0 INTRODUCTION

The buckling of a compressed bar is a stability problem where a small lateral disturbance in an unstable equilibrium state produces a deflection of the bar, and as a result a bending load appears. This transition causes an additional load on the bar, so in the design process it is important to ensure that the load does not exceed its critical value. That is the reason why, in cases of slender elements where the stability limit is the main criterion, the load-carrying capacity of the material is poorly exploited. One possible way to increase the stability limit and exploit the load-carrying capacity of the element is to optimize its geometry.

The purpose of this paper is to present an analytical method of geometry optimization that can be generally applied to the problems of the buckling of an elastic bar for various conditions and loads. The optimization is based on a boundary-condition mathematical model that describes the unstable state of the buckling process of a bar using second-order theory. This means that equilibrium equations are deformiran element, pri čemer so upoštevani majhni premiki v mejnem stanju. Za preverjanje so uporabljeni rezultati optimizacije za dva posebna primera vpetja palice, ki so podani v literaturi [1], kjer je uporabljena druga analitična metoda reševanja.

written for the deflected element by considering small displacements in the unstable state. To verify our results we will refer to the results of an optimization for two particular cases of boundary conditions that were obtained by [1], where a different analytical solving method was used.

1 TEORETIČNE OSNOVE

1 THEORY

Postopek geometrijske optimizacije temelji na diferencialni enačbi, ki je povzeta iz teorije elastične stabilnosti [2]:

The procedure of optimizing the geometry of a bar is based on the differential equation from the theory of elastic stability [2]:

$$\left[v''(x)EI(x)\right]'' + Fl^2 v''(x) = 0 \quad ; \quad 0 \le x \le 1 \qquad (1).$$

Z enačbo (1) so opisane razmere v mejnem stanju pri uklonu vitke elastične palice (sl. 1) spremenljivega prereza, ki je obremenjena s kritično uklonsko silo F. Parameter E pomeni elastični modul gradiva palice, $I(x)$ pa je najmanjši vztrajnostni moment prereza v točki x. Enačba (1) je zapisana v normirani obliki – spremenljivka x in funkcija prečnega premika $v(x)$ sta izraženi na enoto dolžine palice l.

Za uporabo pri zapisu robnega problema definiramo funkcijo brezrazsežnega upogibnega momenta $m(x)$:

Equation (1) describes the buckling process of a slender elastic bar with a variable cross-sectional area (Fig. 1) that is loaded with a critical buckling load F. The parameter E is the Young's modulus of elasticity of a bar, $I(x)$ is the minimum moment of inertia of the cross section at point x. Equation (1) is written in a normalized form – the variable x and the transverse displacement function $v(x)$ are given per unit of bar length l.

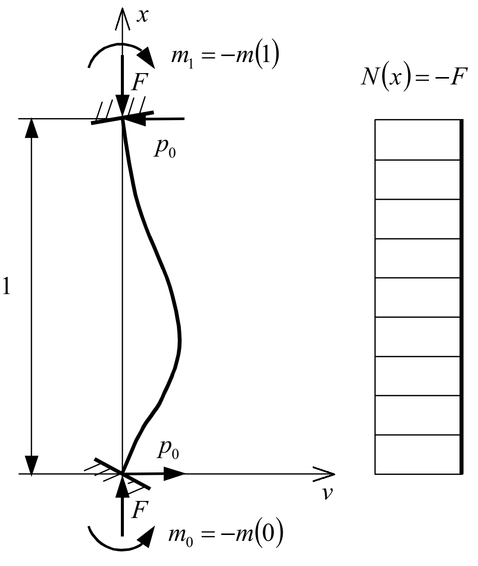
To express the boundary problem a nondimensional bending-moment function $m(x)$ is defined:

$$m(x) = -\frac{v''(x)EI(x)}{Fl^2} \qquad (2).$$

Na sliki 1 je ponazorjena elastično vpeta palica v deformiranem stanju in potek notranje osne sile $N(x)$ v nedeformiranem stanju za dano obremenitev. Na podlagi pogoja za ravnotežje momentov velja za krajiščne vrednosti naslednja zveza (predpostavljene usmeritve obravnavanih veličin so prikazane na sliki 1):

Figure 1 shows a buckled bar with elastically supported ends and an internal axial load $N(x)$ in the nondeformed state for a given load. Based on the equilibrium of moments, the following relation holds for the end values (the assumed directions of the discussed quantities are shown in Figure 1):

$$m_1 = 1 \cdot p_0 + m_0 \qquad (3),$$



Sl. 1. *Uklon elastično vpete palice*

Fig. 1. *Buckling of a bar with elastically supported ends*

kjer je p_0 brezrazsežna strižna sila. Na mestih vpetja palice predpišemo robne pogoje za funkcijo $v(x)$:

where p_0 is the nondimensional shear force. The required boundary conditions for the function $v(x)$ are:

$$v(0)=0 \quad , \quad v(1)=0 \tag{4}$$

in definiramo zvezi med upogibnima momentoma in zasukoma palice pri elastičnem vpetju:

Now we define the relation between the bending moments and the twisting motion of the bar with elastically supported ends:

$$m_0 = c_0 v'(0) \quad , \quad m_1 = -c_1 v'(1) \tag{5},$$

kjer sta c_0 in c_1 brezrazsežna parametra togosti v podporah.

Diferencialno enačbo (1) dvakrat integriramo in jo ob upoštevanju robnih pogojev zapišemo v končni obliki:

where c_0 and c_1 are the nondimensional rigidity parameters of the supports.

After integrating the differential equation (1) twice and considering the boundary conditions, the following expression is obtained:

$$\eta''(x)EI(x)+Fl^2\eta(x)=0 \tag{6},$$

kjer je $\eta(x)$ nova funkcija:

where $\eta(x)$ represents a new function:

$$\eta(x)=v(x)-p_0x-m_0 \tag{7}.$$

Z uporabo zvez (3), (4) in (5) zapišemo robni problem pri uklonu elastično vpete palice:

Using relations (3), (4) and (5), the final form of the boundary problem for the buckling of the bar with elastically supported ends can be written:

$$\begin{aligned} &c_0 \neq 0: \quad \eta'(0) = -\left(1+\frac{1}{c_0}\right)\eta(0)+\eta(1) \quad ; \quad c_0 = 0: \quad \eta(0)=0 \\ &c_1 \neq 0: \quad \eta'(1) = -\eta(0)+\left(1+\frac{1}{c_1}\right)\eta(1) \quad ; \quad c_1 = 0: \quad \eta(1)=0 \end{aligned} \tag{8}.$$

2 GEOMETRIJSKA OPTIMIZACIJA

Definicija problema geometrijske optimizacije: za elastično vpeto palico pravokotnega prečnega prereza in določene dolžine iščemo takšno obliko po dolžini, pri kateri je kritična uklonska sila največja ob geometrijskem pogoju, da je prostornina optimirane palice enaka prostornini referenčne palice nespremenljivega prečnega prereza.

Rezultate uklona palice optimalne geometrijske oblike želimo primerjati z rezultati uklona referenčne palice nespremenljivega prereza. Zato definiramo ustrezne brezrazsežne parametre, s katerimi bo pojav uklona obravnavan relativno glede na lastnosti referenčne palice.

– Relativna višina prereza:

2 GEOMETRY OPTIMIZATION

The definition of the geometry optimization problem is as follows: for an elastically supported bar with a rectangular cross-sectional area and fixed length we are trying to find an appropriate longitudinal shape of the bar that would give the maximum critical buckling load assuming that the volume of the optimized bar is equal to the volume of the reference bar with a constant cross-sectional area.

The results of the buckling of the bar with optimum geometry will be compared with the results of the buckling of the reference bar with a constant cross-sectional area. For this purpose we define appropriate nondimensional parameters that will be used for the analysis of the buckling process with respect to the properties of the reference bar.

– Relative height of the cross section

$$\bar{h}(x)=h(x)/h_0 \tag{9},$$

kjer sta $h(x)$ višina prereza v točki x in h_0 višina prereza referenčne palice.

– Relativna debelina prereza:

where $h(x)$ is the height of the cross section at point x, and h_0 is the height of the cross section of the reference bar.

– Relative thickness of the cross section:

$$\bar{t}(x)=t(x)/t_0 \tag{10},$$

kjer sta $t(x)$ debelina prereza v točki x in t_0 debelina prereza referenčne palice.

– Na podlagi predpostavke:

where $t(x)$ is the thickness of the cross section at point x, and t_0 is the thickness of the cross section of the reference bar.

– Based on the assumption:

$$t(x) \le h(x) \tag{11}$$

zapišemo relativni vztrajnostni moment pravokotnega prereza:

the relative moment of inertia of the rectangular cross section is written as:

$$\bar{I}(x) = I(x)/I_0 = \bar{h}(x)\bar{t}^3(x) \tag{12},$$

kjer sta $I(x)=h(x)t^3(x)/12$ najmanjši vztrajnostni moment prereza v točki x in $I_0= h_0t_0^3/12$ najmanjši vztrajnostni moment prereza referenčne palice.

– Relativna kritična uklonska sila:

where $I(x)=h(x)t^3(x)/12$ is the minimum moment of inertia of the cross section at point x and $I_0= h_0t_0^3/12$ is the minimum moment of inertia of the cross section of the reference bar.

– Relative critical buckling load:

$$f = F/F_0 \tag{13},$$

kjer sta F kritična uklonska sila optimirane palice in F_0 kritična uklonska sila referenčne palice. Kritično silo F_0 določimo z uporabo nastavka rešitve diferencialne enačbe (6) za referenčno palico:

where F is the critical buckling load of the optimized bar and F_0 is the critical buckling load of the reference bar. The critical load F_0 is determined by solving the differential equation (6) for the reference bar:

$$\eta_{ref} = A\cos\omega_0 x + B\sin\omega_0 x \tag{14}.$$

Dobimo izraz:

We obtain the following expression:

$$F_0 = \frac{\omega_0^2 EI_0}{l^2} \tag{15}.$$

kjer je ω_0 lastna vrednost robnega problema (8) za referenčno palico in je določena s transcendentno enačbo:

where ω_0 is the eigenvalue of the boundary problem (8) for the reference bar, which is determined with a transcendental equation:

$$2c_0c_1\omega_0 - (c_0 + c_1 + 2c_0c_1)\omega_0\cos\omega_0 + (1 + c_0 + c_1 - c_0c_1\omega_0^2)\sin\omega_0 = 0 \tag{16}.$$

2.1 Variacijski problem

Z uporabo brezrazsežnih veličin (12) in (13) ter izraza (15) zapišemo diferencialno enačbo (6) v normirani obliki:

2.1 Variational problem

Using nondimensional quantities (12) and (13) and expression (15), differential equation (6) can be written in the normal form:

$$\eta''(x)\bar{I}(x) + \omega_0^2 f\ \eta(x) = 0 \tag{17}.$$

Pri geometrijski optimizaciji izhajamo iz geometrijskega pogoja o nespremenljivi prostornini palice. Definiramo funkcijo relativnega prečnega prereza $a(x)$:

The geometry optimization is conditioned by the requirement that the volume of the bar is a constant. Now we define the function of the relative cross-sectional area $a(x)$:

$$a(x) = \bar{h}(x)\bar{t}(x) \tag{18}$$

in zapišemo pogoj za relativno prostornino palice $\bar{V}$:

and express the condition for the relative volume of a bar $\bar{V}$:

$$\bar{V} = \int_0^1 a(x)\,dx = 1 \tag{19}.$$

V nadaljevanju je prikazan postopek geometrijske

Below is the procedure for optimizing the geometry of a optimizacije palice pravokotnega prečnega prereza za primere, v katerih velja naslednja zveza:

bar with a rectangular cross-sectional area for particular examples in which the following relation is valid:

$$\bar{I}(x) = a^k(x) \tag{20}$$

Dodatni geometrijski pogoji in pripadajoče vrednosti eksponenta k za posamezne primere so podani v preglednici 1.

Some additional geometrical conditions and corresponding values of exponent k in particular cases are given in Table 1.

Preglednica 1. *Dodatni geometrijski pogoji*
Table 1. *Additional geometrical conditions*

Geometrijski pogoj Geometrical condition		k
Nespremenljiva debelina prereza Constant thickness of the cross section	$\bar{t}(x) \equiv 1$	1
Nespremenljivo razmerje med višino in debelino prereza Constant height-to-thickness ratio of the cross section	$\bar{h}(x)/\bar{t}(x) \equiv 1$	2
Nespremenljiva višina prereza Constant height of the cross section	$\bar{h}(x) \equiv 1$	3

Iz diferencialne enačbe (17) z uporabo zveze (20) izrazimo funkcijo $a(x)$:

From differential equation (17) using relation (20), we obtain the function $a(x)$:

$$a(x) = \left[\omega_0^2 f\right]^{\frac{1}{k}} \left[\frac{-\eta(x)}{\eta''(x)}\right]^{\frac{1}{k}} \tag{21}$$

Izraz vstavimo v pogoj (19) in zapišemo relativno kritično silo v naslednji obliki:

This expression is now introduced into condition (19), so the relative critical load can be written in the following form:

$$f = \frac{1}{\omega_0^2 \left[J_k(\eta)\right]^k} \tag{22}$$

kjer je $J_k(\eta)$ funkcional:

where $J_k(\eta)$ represents a functional:

$$J_k(\eta) = \int_0^1 \left[\frac{-\eta(y)}{\eta''(y)}\right]^{\frac{1}{k}} dy \tag{23}$$

Izraz (22) vstavimo v zvezo (21) in dobimo končno obliko zapisa za rešitev optimizacijskega problema:

Expression (22) is further used in relation (21), from which we get the final form of the solution of the optimization problem:

$$a(x) = \frac{\left[\frac{-\eta(x)}{\eta''(x)}\right]^{\frac{1}{k}}}{J_k(\eta)} \tag{24}$$

Relativna kritična sila f bo največja v primeru, ko bo vrednost funkcionala $J_k(\eta)$ najmanjša. Zato definiramo variacijski problem: med vsemi, na območju [0,1] odsekoma zveznimi in dvakrat zvezno odvedljivimi funkcijami $\eta(x)$, ki rešijo dani robni problem, je treba določiti tisto, pri kateri ima funkcional $J_k(\eta)$ najmanjšo vrednost.

Pri obravnavanem problemu geometrijske optimizacije iščemo obliko palice, pri kateri je kritična uklonska sila največja ob pogoju, da je prostornina palice nespremenljiva. Optimizacijski problem pa bi lahko definirali tudi drugače: iščemo obliko palice , pri kateri je prostornina najmanjša ob pogoju, da je kritična uklonska sila nespremenljiva. Izkaže se, da obema definicijama pripada isti variacijski problem in s tem ista rešitev $\eta(x)$, zveza med največjo relativno

The relative critical load will be a maximum if the value of the functional $J_k(\eta)$ is a minimum. So we define the variational problem as follows: among all the functions $\eta(x)$ that are in the interval [0,1], intermittently continuous and twice continuously differentiable, we are looking for the one that would give a minimum value to the functional $J_k(\eta)$.

In the discussed geometry-optimization problem we are looking for the shape of the bar that will withstand the maximum critical buckling load, with the condition that the volume of the bar is constant. In other words, we are looking for the shape of the bar that has the minimum volume for the condition that the critical buckling load is constant. It turns out that both definitions correspond to the same variational problem and consequently the same kritično silo f_{max} in najmanjšo relativno prostornino palice $\bar{V}_{min}$ pa je naslednja:

solution $\eta(x)$. Given this, the relation between the maximum relative critical load f_{max} and the minimum relative volume of the bar $\bar{V}_{min}$ can be expressed by:

$$\bar{V}_{min} = \left(f_{max}\right)^{-\frac{1}{k}} \quad (25).$$

2.2 Osnovni robni problem in lastnosti rešitve

2.2 The fundamental boundary problem and the properties of the solution

Osnovni robni problem predstavlja primer, ko je palica členkasto vpeta: c_0=0, c_1=0. Na podlagi zapisa splošnega robnega problema (8) lahko za rešitev osnovnega robnega problema $\eta_0(x)$ predpišemo robne pogoje:

The fundamental boundary problem is represented by the case of a bar with simply supported ends: c_0=0, c_1=0. Based on the general boundary problem (8) we can write the boundary conditions for the solution of the fundamental boundary problem $\eta_0(x)$:

$$\eta_0(0) = 0 \quad , \quad \eta_0(1) = 0 \quad (26).$$

Obremenitev palice in robni pogoji so v primeru osnovnega robnega problema simetrični, zato je simetrična funkcija pomika $v(x)$. V primeru osnovnega robnega problema velja $\eta_0(x)=v(x)$, zato je simetrična tudi rešitev $\eta_0(x)$:

In the case of the fundamental boundary problem, the load of the bar and the boundary conditions are symmetrical. As a result the displacement function $v(x)$ is symmetrical too. For the fundamental boundary problem it holds that $\eta_0(x)=v(x)$, so the solution $\eta_0(x)$ is also symmetrical:

$$\eta_0(x) = \eta_0(1-x) \quad (27).$$

Rešitev osnovnega robnega problema je enoparametrična družina krivulj, zato funkcijo $\eta_0(x)$ normiramo z dodatnim pogojem: $\eta_0'(0)=1$ (pogoj ne vpliva na lastno vrednost problema). Variacijski problem je rešen z uporabo nastavka za funkcijo $\eta_0(x)$, ki ustreza simetriji (27) in izpolnjuje predpisane pogoje:

The solution of the fundamental boundary problem is a one-parametric family of curves, so we normalize the function $\eta_0(x)$ with an additional condition: $\eta_0'(0)=1$ (this condition does not affect the eigenvalue of the problem). The variational problem is solved using the expression for the function $\eta_0(x)$ in the form of a series that corresponds to symmetry (27) and fulfils the prescribed conditions:

$$\eta_0 = x(1-x) + \sum_{i=1}^{i=\nu} \alpha_i x^{i+1} (1-x)^{i+1} \quad (28).$$

Z uporabo nastavka pretvorimo funkcional $J_k(\eta_0)$ v funkcijo ν spremenljivk: $J_k(\eta_0)=g_k(\alpha_1, \alpha_2,... \alpha_\nu)$. Potrebni pogoj za najmanjšo vrednost funkcije ν realnih spremenljivk je sistem ν nelinearnih enačb:

Using this expression the functional $J_k(\eta_0)$ is transformed into a function of ν variables: $J_k(\eta_0)=g_k(\alpha_1, \alpha_2,... \alpha_\nu)$. The necessary condition for the minimum of the function of ν real variables is represented by the system of ν nonlinear equations:

$$\frac{\partial g_k}{\partial \alpha_i} = 0 \quad , \quad i = 1, 2, \ldots, \nu \quad (29).$$

Nelinearni sistem enačb je rešen numerično [5]:
- približek integralov po Simpsonovi formuli,
- reševanje nelinearnega sistema enačb po Newtonovi metodi.

Število parametrov ν povečujemo, dokler ni izpolnjen pogoj:

The nonlinear system of equations is solved numerically using [5]:
- the approximation of the integrals with Simpson's formula,
- solving a nonlinear system of equations with Newton's method.

The number of parameters ν is increased until the next condition is fulfilled:

$$\left| g_k^{(\nu+1)} - g_k^{(\nu)} \right| < \varepsilon \quad , \quad \nu = 1, 2, 3, \ldots \quad (30),$$

kjer je ε izbrano pozitivno realno število.

Rešitev osnovnega robnega problema $\eta_0(x)$ mora ustrezati Eulerjevi diferencialni enačbi [3], ki

where ε is the chosen positive real number.

The solution of the fundamental boundary problem $\eta_0(x)$ is also the solution of Euler's differential pomeni potrebni pogoj za rešitev variacijskega problema (23):

equation [3], which represents the necessary condition for the solution of the variational problem (23):

$$\frac{\partial}{\partial \eta_0}\left[\frac{-\eta_0(x)}{\eta_0''(x)}\right]^{\frac{1}{k}} + \frac{d^2}{dx^2}\left[\frac{\partial}{\partial \eta_0''}\left[\frac{-\eta_0(x)}{\eta_0''(x)}\right]^{\frac{1}{k}}\right] = 0 \qquad (31).$$

Z integracijo Eulerjeve diferencialne enačbe in ustreznim preoblikovanjem rezultata dobimo naslednjo lastnost rešitve osnovnega robnega problema:

With the integration of Euler's differential equation and the appropriate transformation of the result, the following property of the solution of the fundamental boundary problem is obtained:

$$\frac{\eta_0(x)}{\bar{h}(x)\bar{t}^2(x)} \equiv C \qquad (32),$$

kjer je C nespremenljiva vrednost.

where C is a constant value.

Za členkasto vpeto palico v mejnem stanju zapišemo izraz za potek največje upogibne napetosti prereza σ_u vzdolž palice v brezrazsežni obliki:

The expression for the maximum nondimensional bending stress of cross section σ_u along the bar with simply supported ends in an unstable state is the following:

$$\bar{\sigma}_u(x) = \frac{\sigma_u(x)\cdot\left(\frac{l}{t_0}\right)}{E} = \frac{\pi^2 f}{2}\cdot\frac{\eta_0(x)}{\bar{h}(x)\bar{t}^2(x)} \qquad (33).$$

Ob primerjavi izrazov (32) in (33) ugotovimo, da je v mejnem stanju največja upogibna napetost prereza, ki ustreza rešitvi osnovnega robnega problema, vzdolž palice nespremenljiva – torej je nosilnost pri palici z optimalno geometrijsko obliko glede na predpisane pogoje v celoti izkoriščena.

A comparison between expressions (32) and (33) shows that the maximum bending stress of the cross section in the unstable state, which corresponds to the solution of the fundamental boundary problem, is constant along the bar. Thus, it can be stated that the load-carrying capacity of the bar with optimal geometry, considering the prescribed conditions, is completely exploited.

2.3 Splošni robni problem

2.3 General boundary problem

Rešitev osnovnega robnega problema $\eta_0(x)$ razširimo prek krajišč definicijskega območja [0,1]:

The solution of the fundamental boundary problem $\eta_0(x)$ is expanded beyond the end points of the definition area [0,1]:

$$\eta_0(-x) = -\eta_0(x) \quad , \quad \eta_0(1+x) = -\eta_0(1-x) \quad ; \quad 0 \le x \le 1 \qquad (34).$$

Tako definirana funkcija $\eta_0(x)$ reši Eulerjevo diferencialno enačbo osnovnega robnega problema (31) v vsaki točki razširjenega definicijskega območja [-1,2].

The expanded function $\eta_0(x)$ solves Euler's differential equation of the fundamental boundary problem (31) at every point of the definition area [-1,2].

Rešitev splošnega robnega problema $\eta(x)$ izrazimo z uporabo nove funkcije μ ter parametrov x_1 in x_2:

The solution of the general boundary problem $\eta(x)$ is written with the use of a new function μ and parameters x_1 and x_2:

$$\eta(x) = (x_2 - x_1)\cdot\mu\left(\frac{x - x_1}{x_2 - x_1}\right) \quad ; \quad x_2 > x_1 \quad ; \quad 0 \le x \le 1 \qquad (35).$$

V izraz (35) uvedemo spremenljivko y:

The variable y is introduced into expression (35):

$$y = \frac{x - x_1}{x_2 - x_1} \qquad (36)$$

in zapišemo funkcional (23):

and functional (23) is written in the form:

$$J_k(\mu) = (x_2 - x_1)^{\frac{k+2}{k}} \cdot \int_{\frac{-x_1}{x_2 - x_1}}^{\frac{1-x_1}{x_2 - x_1}} \left[\frac{-\mu(y)}{\mu''(y)}\right]^{\frac{1}{k}} dy \qquad (37).$$

Funkcionalu (37) pripada Eulerjeva

Euler's differential equation of the fundamental boundary problem (31) corresponds to diferencialna enačba osnovnega robnega problema (31) za funkcijo μ. To pomeni, da razširjena rešitev osnovnega robnega problema reši splošni robni problem in velja zveza:

functional (37) for the function μ. This means that the expanded solution of the fundamental boundary problem solves the general boundary problem and so the next relation is valid:

$$\mu(y)=\eta_0(y) \quad ; \quad -1\le y\le 2 \tag{38}$$

Glede na robne pogoje (26) ter izraza (35) in (38) se izkaže, da parametra x_1 in x_2 predstavljata ničli funkcije $\eta(x)$. Rešitev $\eta(x)$ mora izpolnjevati splošne robne pogoje (8), ki jih z uporabo zvez (35) in (38) zapišemo:

Considering the boundary conditions (26) and expressions (35) and (38) it turns out that parameters x_1 and x_2 represent roots of the function $\eta(x)$. The solution $\eta(x)$ should fulfil the general boundary conditions (8), which are written using relations (35) and (38):

$$\begin{aligned}\frac{1}{x_2-x_1}\cdot\eta_0'\left(\frac{-x_1}{x_2-x_1}\right)&=-\left(1+\frac{1}{c_0}\right)\cdot\eta_0\left(\frac{-x_1}{x_2-x_1}\right)+\eta_0\left(\frac{1-x_1}{x_2-x_1}\right)\\ \frac{1}{x_2-x_1}\cdot\eta_0'\left(\frac{1-x_1}{x_2-x_1}\right)&=-\eta_0\left(\frac{-x_1}{x_2-x_1}\right)+\left(1+\frac{1}{c_1}\right)\cdot\eta_0\left(\frac{1-x_1}{x_2-x_1}\right)\end{aligned} \tag{39}$$

Zapis (39) je sistem dveh nelinearnih enačb z neznankama x_1 in x_2. Torej so ob ustreznih vrednostih x_1 in x_2 splošni robni pogoji izpolnjeni in funkcija $\eta(x)$, izražena v obliki (35) z uporabo zveze (38), pomeni rešitev splošnega robnega problema. Izkaže se, da imata ničli funkcije $\eta(x)$ x_1 in x_2 naslednje lastnosti: $x_1,x_2\ge 0$, $x_1<(x_2-x_1)$, $(1-x_2)<(x_2-x_1)$. To pomeni, da spremenljivka y (36) v vseh primerih vpetja leži znotraj območja [-1,2] in je rešitev $\eta(x)$ definirana v vseh točkah $x\in[0,1]$.

Sistem nelinearnih enačb (39) je rešen numerično po Newtonovi metodi.

Expression (39) represents a system of two nonlinear equations with unknowns x_1 and x_2. So at appropriate values of x_1 and x_2, the general boundary conditions are fulfilled and the function $\eta(x)$, expressed in form (35) using relation (38), represents the solution of the general boundary problem. It turns out that the roots of the function $\eta(x)$ x_1 and x_2 have the following properties: $x_1,x_2\ge 0$, $x_1<(x_2-x_1)$, $(1-x_2)<(x_2-x_1)$. This means that in all cases of the boundary conditions, the variable y (36) lies inside the interval [-1,2] and the solution $\eta(x)$ is defined at every point $x\in[0,1]$.

The system of nonlinear equations (39) is solved numerically with Newton's method.

3 REZULTATI IN RAZPRAVA

Matematični model pojava uklona vitke elastične palice, na katerem temelji postopek geometrijske optimizacije, je predstavljen v brezrazsežni obliki. Zato so rezultati optimizacije, ki so podani relativno glede na lastnosti referenčne palice nespremenljivega prereza, ob izpolnjevanju predpisanih pogojev splošno veljavni. Rezultati geometrijske optimizacije za kombinacije mejnih primerov elastičnega vpetja (c=0 oziroma $c\to\infty$) so prikazani v preglednici 2 ter na slikah 2, 3, 4. Največje vrednosti relativne kritične sile dobimo v primeru nespremenljive višine prereza (k=3), najmanjše vrednosti relativne prostornine palice pa v primeru nespremenljive debeline prereza (k=1).

Pri uporabi rešitev optimizacije moramo biti pozorni na geometrijski pogoj (11). Ta pogoj je izpolnjen, če velja za geometrijsko obliko referenčne palice naslednja lastnost:

3 RESULTS AND DISCUSSION

A mathematical model of the buckling process of a slender elastic bar, which is used in a geometry optimization procedure, is represented in a nondimensional form. Therefore the results of the optimization, which are defined relative to the properties of the reference bar with a constant cross-sectional area, fulfil the prescribed conditions and have a general validity. These results are shown for combinations of the limit cases of the boundary conditions (c=0 or $c\to\infty$) in Table 2 and in Figures 2, 3, and 4. The highest values of the maximum relative critical load are obtained in the case of a constant height of the cross section (k=3), and the lowest values of the minimum relative volume of the bar are obtained in the case of a constant thickness of the cross section (k=1).

When using the results of the optimization we should pay attention to the geometry condition (11). This condition is fulfilled if the following property of the geometry of the reference bar is valid:

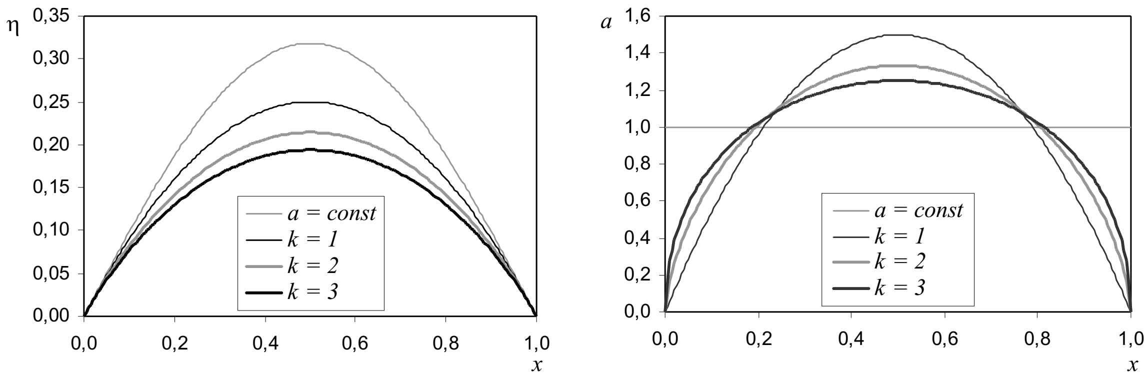
$$t_0/h_0\le\bar{h}(x)/\bar{t}(x) \quad ; \quad 0\le x\le 1 \tag{40}$$

V točkah ničel rešitve variacijskega problema x_1 in x_2 se pojavijo singularnosti – površina prereza je enaka nič. To pomeni, da tlačna

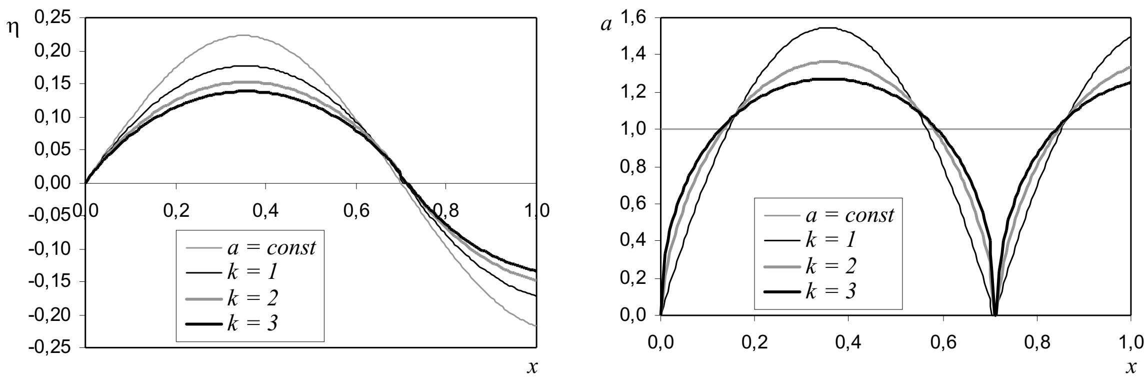
In the roots of the solution of the variational problem x_1 and x_2 singularities appear, the cross-sectional area is equal to zero. This means that the

Preglednica 2. *Rezultati geometrijske optimizacije*
Table 2. *The results of the geometry optimization*

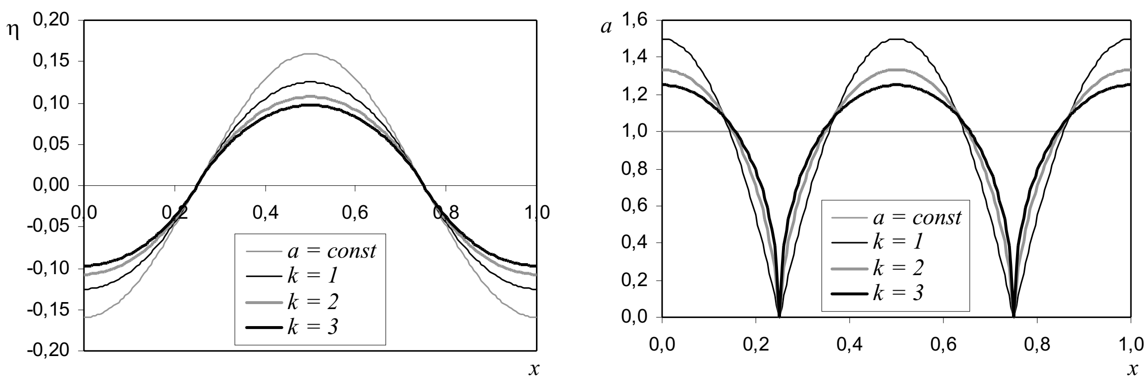
Ciljna funkcija Goal function	Pogoj optimizacije Condition of optimization	k	$c_0 = 0, c_1 = 0$	$c_0 = 0, c_1 \to \infty$	$c_0 \to \infty, c_1 \to \infty$
$f_{\max}$	$\overline{V} = 1$	1	1,216	1,225	1,216
		2	1,334	1,348	1,334
		3	1,408	1,425	1,408
$\overline{V}_{\min}$	$f = 1$	1	0,822	0,816	0,822
		2	0,866	0,861	0,866
		3	0,892	0,889	0,892



Sl. 2. *Potek rešitve variacijskega problema η(x) in funkcije relativnega prereza a(x) za primer $c_0=0$, $c_1=0$*
Fig. 2. *Solution of variational problem η(x) and the relative cross-sectional area a(x) in the case $c_0=0$, $c_1=0$*



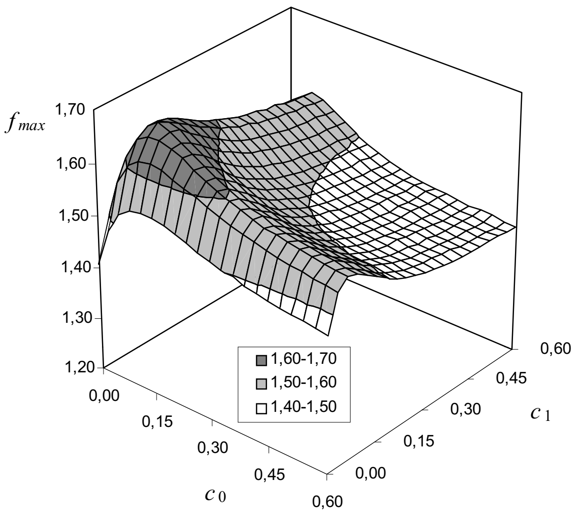
Sl. 3. *Potek rešitve variacijskega problema η(x) in funkcije relativnega prereza a(x) za primer $c_0=0$, $c_1\to\infty$*
Fig. 3. *Solution of the variational problem η(x) and the relative cross-sectional area a(x) in the case $c_0=0$, $c_1\to\infty$*



Sl. 4. *Potek rešitve variacijskega problema η(x) in funkcije relativnega prereza a(x) za primer $c_0\to\infty$, $c_1\to\infty$*
Fig. 4. *Solution of variational problem η(x) and the relative cross-sectional area a(x) in the case $c_0\to\infty$, $c_1\to\infty$*

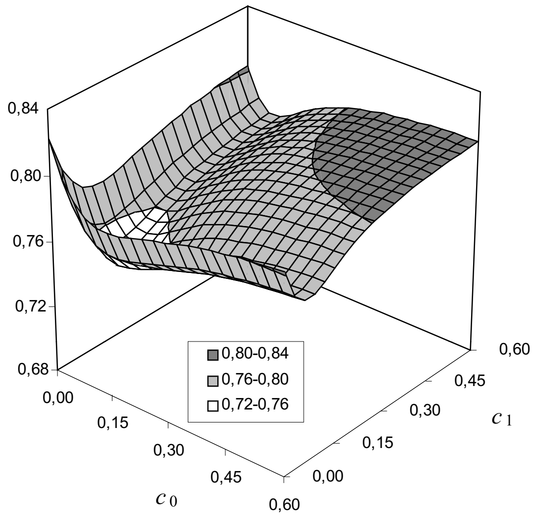
napetost, ki je opazna pred uklonom palice, v singularnih točkah ni omejena. Poleg tega je v primeru $k = 1$ v singularnih točkah kršen pogoj (40). Za praktično uporabnost rezultatov bi bilo treba v postopek optimizacije vključiti omejitev napetosti zaradi prvotne tlačne obremenitve ter z ustrezno izbiro geometrijskih parametrov in gradiva referenčne palice zagotoviti, da so napetosti v mejnem stanju v elastičnem področju ter izpolnitev pogoja (40). Omejitev napetosti bi imela znaten vpliv na rešitev le v okolici singularnosti, pri čemer bi bila relativna kritična uklonska sila manjša od največje, saj je pri prikazani rešitvi brez omejitev gradivo stabilnostno popolnoma izkoriščeno.

compressive stress, which is present before the bar buckles, is not limited at the points of singularities. Beside this, in the case $k = 1$ condition (40) is violated at the singular points. For the practical use of the results a compressive stress constraint should be included in the optimization procedure and appropriate values of the geometry and the material parameters of the reference bar would ensure that the stress in the unstable state lies in the elastic region of the material and that condition (40) is fulfilled. The stress constraint would have a considerable impact on the solution only around the singularities, and the relative critical buckling load would be lower than the maximum, since in the shown solution with no constraints, in terms of stability, the material is completely exploited.



Sl. 5. *Največja relativna kritična uklonska sila za primer k = 3*

Fig. 5. *Maximum relative critical buckling load in the case k = 3*



Sl. 6. *Najmanjša relativna prostornina palice za primer k = 1*

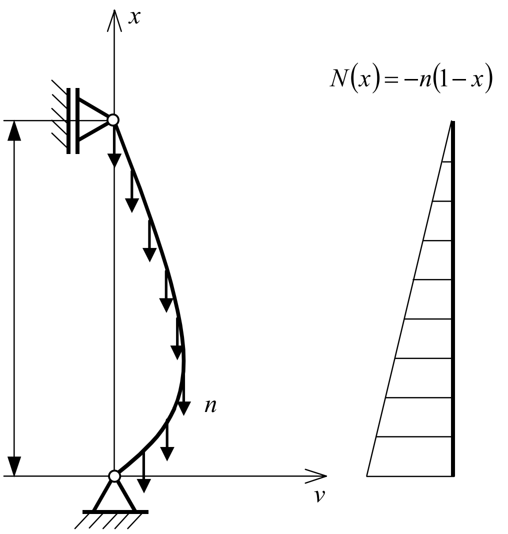
Fig. 6. *Minimal relative volume of the bar in the case k = 1*

Na slikah 5 in 6 je prikazan vpliv parametrov togosti v podporah na velikost največje relativne kritične uklonske sile oziroma najmanjše relativne prostornine palice. Geometrijska optimizacija je najbolj učinkovita pri vrednostih parametrov togosti $c \approx 0,1$.

V literaturi [1] je obravnavan problem optimizacije pri uklonu palice za primera vpetja $c_0=0$, $c_1=0$ ter $c_0=0$, $c_1 \to \infty$. Rezultati optimizacije za ta dva primera vpetja, ki so prikazani zgoraj, se glede na upoštevano natančnost ujemajo s tistimi v literaturi [1], kjer ni uporabljena metoda reševanja pripadajočega variacijskega problema, ampak so rezultati dobljeni z rešitvijo ustrezne diferencialne enačbe. Za reševanje optimizacijskih problemov v bolj splošnih primerih obremenitev in vpetja palice je uporabnejša prikazana metoda pripadajočega variacijskega problema, saj je reševanje nelinearnih diferencialnih enačb zelo zahtevno že za preproste primere. Kot primer geometrijske optimizacije pri bolj splošnih obremenitvah palice bo prikazan postopek optimizacije pri členkasto vpeti palici, obremenjeni s trikotno osno obremenitvijo $N(x)=-n(1-x)$ (sl. 7).

Figures 5 and 6 show the impact of rigidity parameters on the maximum relative critical buckling load and the minimum relative volume of the bar. The geometry optimization is most effective at the rigidity parameter value $c \approx 0.1$.

Troickij et al. [1] discuss the optimization problem for the buckling of a bar in cases of boundary conditions $c_0=0$, $c_1=0$ and $c_0=0$, $c_1 \to \infty$. For the two cases shown above and considering the degree of accuracy, our results for the optimization are in good agreement with their results. In [1] the method of solving the corresponding variational problem was not used, they obtained the results by solving the corresponding differential equation. For solving optimization problems in more general cases of loadings and boundary conditions the represented method of the corresponding variational problem is more useful because solving nonlinear differential equations is a very complex task, even for simple cases. As an example of geometry optimization for more general loads we will show the optimization procedure for a bar with simply supported ends, loaded with a triangular axial load $N(x)=-n(1-x)$



Sl. 7. *Uklon členkasto vpete palice, obremenjene s trikotno osno obremenitvijo N(x)=-n(1-x)*
Fig. 7. *Buckling of a bar with simply supported ends, loaded with a triangular axial load N(x)=-n(1-x)*

Razmere v mejnem stanju pri uklonu tako obremenjene palice so opisane z diferencialno enačbo [2], za katero veljajo robni pogoji (4) in jo zapišemo v brezrazsežni obliki:

(Fig. 7). The conditions in the unstable state for the given load of a bar are described with a differential equation [2]. Considering boundary conditions (4) this equation, written in nondimensional form, is:

$$v''(x)\bar{I}(x)+\frac{nl^3}{EI_0}\left[(1-x)\int_0^x(1-y)v'(y)\,dy-x\int_x^1(1-y)v'(y)\,dy\right]=0 \qquad (41).$$

Problemu geometrijske optimizacije priredimo po predhodno opisanem postopku variacijski problem, pri katerem iščemo najmanjšo vrednost funkcionala:

For the problem of geometry optimization we define the variational problem by the previously discussed procedure, where we are searching for the minimum of the functional:

$$J_k(v)=\int_0^1\left[\frac{-(1-x)\int_0^x(1-y)v'(y)\,dy+x\int_x^1(1-y)v'(y)\,dy}{v''(x)}\right]^{\frac{1}{k}}dx \qquad (42).$$

Pri rešitvi za linijsko obremenjeno palico simetrija ne velja več, zato pri reševanju variacijskega problema uporabimo naslednji nastavek za funkcijo $v(x)$:

In solving the problem for a linearly loaded bar, the symmetry no longer holds, so the variational problem is solved using the next expression for the function $v(x)$:

$$v=x(1-x)+\sum_{i=1}^{i=\nu}\alpha_i\left(x^{i+1}-x^{i+2}\right) \qquad (43).$$

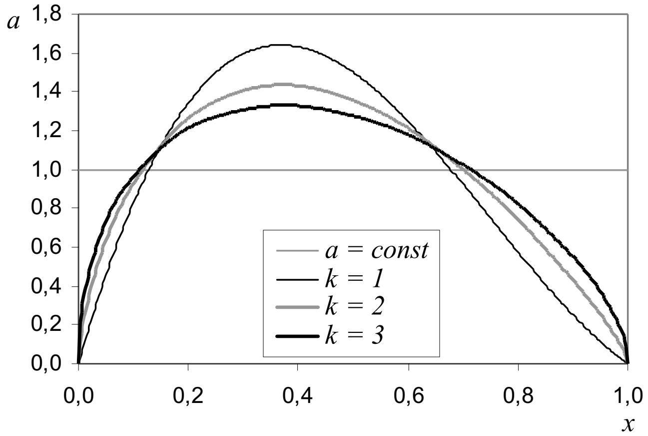
Z uporabo nastavka pretvorimo funkcional (42) v funkcijo ν spremenljivk, vrednosti spremenljivk pa določimo z numerično rešitvijo sistema ν nelinearnih enačb (29). Rešitev optimizacijskega problema $a(x)$ izrazimo z rešitvijo variacijskega problema $v(x)$:

Using this expression, functional (42) is transformed into a function of ν variables. The values of the variables are obtained with a numerical solution of the system of ν nonlinear equations (29). The solution of the optimization problem $a(x)$ is expressed with the solution of the variational problem $v(x)$:

$$a(x)=\frac{\left[\dfrac{-(1-x)\int_0^x(1-y)v'(y)\,dy+x\int_x^1(1-y)v'(y)\,dy}{v''(x)}\right]^{\frac{1}{k}}}{J_k(v)} \qquad (44).$$

Iz opisanega postopka optimizacije je razvidna splošnost metode reševanja pripadajočega variacijskega problema, pri kateri hkrati določimo funkcijo prečnega premika $v(x)$ in iskani potek relativnega prečnega prereza palice

From the presented optimization procedure it is evident that the method of solving the corresponding variational problem is more generally applicable. It allows the simultaneous determination of the transverse displacement function $v(x)$ and the relative cross-



Sl. 8. *Potek funkcije relativnega prereza pri členkasto vpeti palici, obremenjeni s trikotno osno obremenitvijo* $N(x)=-n(1-x)$

Fig. 8. *Relative cross-sectional area function for a bar with simply supported ends, loaded with a triangular axial load* $N(x)=-n(1-x)$

$a(x)$ (sl. 8), ne da bi poznali vrednost relativne kritične uklonske sile oziroma lastnosti referenčne palice.

4 SKLEPI

V prispevku je predstavljen analitični postopek problema geometrijske optimizacije pri uklonu elastično vpete vitke palice. Rešitev optimizacijskega problema je ob izpolnjevanju predpisanih pogojev splošno veljavna. Metoda reševanja pripadajočega variacijskega problema optimizacije, ki je obravnavana v prispevku, se izkaže za splošno uporabno, saj je postopek mogoče prenesti na različne primere vpetja in obremenitev palice.

Povzetek rezultatov geometrijske optimizacije pri uklonu palice:

– Pri palici z optimalno geometrijsko obliko je največja upogibna napetost prereza v mejnem stanju vzdolž palice nespremenljiva, nosilnost gradiva je stabilnostno v celoti izkoriščena.
– Kritična uklonska sila doseže največje vrednosti v primeru nespremenljive višine pravokotnega prereza, prihranek materiala pa je največji v primeru nespremenljive debeline pravokotnega prereza.
– Geometrijska optimizacija je v primerjavi z lastnostmi referenčne palice najbolj učinkovita pri majhnih vrednostih brezrazsežnih parametrov togosti elastičnega vpetja - $c\approx0.1$.
– V rešitvah se pojavljajo singularne točke, ki bi jih odpravili z definicijo in rešitvijo optimizacijskega problema z omejitvijo tlačnih napetosti. Ob tem je z ustrezno izbiro lastnosti referenčne palice treba zagotoviti, da so napetosti v palici v elastičnem področju in da so izpolnjeni predpisani geometrijski pogoji.
– V literaturi [1] je predstavljen postopek optimizacije z analizo ustreznih diferencialnih enačb za primera vpetja $c_0=0$, $c_1=0$ ter $c_0=0$, $c_1\rightarrow\infty$. Rezultati za ta

sectional area function $a(x)$ (Fig. 8), without the need to know the value of the relative critical buckling load or the properties of the reference bar.

4 CONCLUSIONS

This paper describes an analytical approach to the geometry optimization of a slender bar with elastically supported ends. The solution of the optimization problem is valid in general, considering certain conditions. The disscused method of solving the variational problem of optimization has a general validity, since it can be applied to different cases of boundary conditions and loads of a bar.

Summary of the results of the geometry optimization for the buckling process of a bar:

– The maximum bending stress of the cross section in an unstable state is constant along the bar with optimum geometry, in terms of stability the load-carrying capacity of the material is completely exploited.
– The critical buckling load is a maximum in the case of a constant height of the cross section, and material savings are a maximum in the case of a constant thickness of the cross section.
– The geometry optimization, compared to the properties of the reference bar, is the most effective if the values of the nondimensional rigidity parameters of the ends are small - $c\approx0.1$.
– Points of singularity appear in the solution, which could be eliminated by defining and solving the optimization problem with a compressive stress constraint. In addition, with an appropriate selection of the properties of the reference bar it could be ensured that the stress in a bar lies in the elastic region and the prescribed geometry conditions are fulfilled.
– The results of our method have been compared with those from the available literature. The same two cases of boundary conditions $c_0=0$, $c_1=0$ and

dva primera vpetja, ki so prikazani v prispevku, se glede na upoštevano natančnost, kljub različni metodi reševanja, ujemajo s tistimi v literaturi [1].

c_0=0, $c_1 \to \infty$ as presented in [1] were solved, and despite a different method, the results, considering the degree of accuracy, are in good agreement.

5 LITERATURA
5 REFERENCES

[1] Troickij, V. A., L. V. Petuhov (1982) Optimizacija formy uprugih tel. *Nauka*, Moskva. Prečrkovano iz cir./ Transliterated from Cyrillic.
[2] Timoshenko,S.P., J.M. Gere (1961) Theory of elastic stability. *McGraw-Hill*, New York.
[3] Vidav, I. (1991) Variacijski račun. *Društvo matematikov, fizikov in astronomov Slovenije.*
[4] Križanič, F. (1991) Navadne diferencialne enačbe. *Društvo matematikov, fizikov in astronomov Slovenije.*
[5] Hoffman, J.D. (1992) Numerical methods for engineers and scientists. *McGraw-Hill*, New York.
[6] Arora, J.S. (1989) Introduction to optimum design. *McGraw-Hill*, New York.

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