On graphs with exactly two positive eigenvalues[*]

Fang Duan [†]

College of Mathematics and Systems Science, Xinjiang University, Urumqi, P. R. China
School of Mathematics Science, Xinjiang Normal University, Urumqi, P. R. China

Qiongxiang Huang [‡]

College of Mathematics and Systems Science, Xinjiang University, Urumqi, P. R. China

Xueyi Huang [§]

School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, P. R. China

Abstract

The inertia of a graph G is defined to be the triplet $\text{In}(G) = (p(G), n(G), \eta(G))$, where $p(G)$, $n(G)$ and $\eta(G)$ are the numbers of positive, negative and zero eigenvalues (including multiplicities) of the adjacency matrix $A(G)$, respectively. Traditionally $p(G)$ (resp. $n(G)$) is called the positive (resp. negative) inertia index of G. In this paper, we introduce three types of congruent transformations for graphs that keep the positive inertia index and negative inertia index. By using these congruent transformations, we determine all graphs with exactly two positive eigenvalues and one zero eigenvalue.

Keywords: Congruent transformation, positive (negative) inertia index, nullity.

Math. Subj. Class.: 05C50

---

1   Introduction

All graphs considered here are undirected and simple. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The order of G is the number of vertices of G, denoted by $|G|$. For $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ the *neighborhood* of v, $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v and $d(v) = |N_G(v)|$ the degree of v. A vertex of G is said to be *pendant* if it has degree 1. By $\delta(G)$ we mean the minimum degree of vertices of G. As usual, we denote by $G + H$ the disjoint union of two graphs G and H, $K_{n_1,\dots,n_l}$ the complete multipartite graph with l parts of sizes $n_1, \dots, n_l$, and K_n, C_n, P_n the complete graph, cycle, path on n vertices, respectively.

The *adjacency matrix* of G, denoted by $A(G) = (a_{ij})$, is the square matrix with $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Clearly, $A(G)$ is a symmetric matrix with zeros on the diagonal, and thus all the eigenvalues of $A(G)$ are real, which are defined to be the *eigenvalues* of G. The multiset consisting of eigenvalues along with their multiplicities is called the *spectrum* of G denoted by $\mathrm{Spec}(G)$. To characterize graphs in terms of their eigenvalues has always been of the great interests for researchers, for instance to see [2, 4, 5, 8, 9] and references therein.

The *inertia* of a graph G is defined as the triplet $\mathrm{In}(G) = (p(G), n(G), \eta(G))$, where $p(G), n(G)$ and $\eta(G)$ are the numbers of positive, negative and zero eigenvalues (including multiplicities) of G, respectively. Traditionally $p(G)$ (resp. $n(G)$) is called the *positive* (resp. *negative*) *inertia index* of G and $\eta(G)$ is called the *nullity* of G. Obviously, $p(G) + n(G) = r(G) = n - \eta(G)$ if G has n vertices, where $r(G)$ is the rank of $A(G)$. Let B and D be two real symmetric matrices of order n. Then D is called *congruent* to B if there is an real invertible matrix C such that $D = C^T B C$. Traditionally we say that D is obtained from B by congruent transformation. The famous Sylvester's law of inertia states that the inertia of two matrices is unchanged by congruent transformation.

Since the adjacency matrix $A(G)$ of G has zero diagonal, we have $p(G) \geq 1$ if G has at least one edge. One of the attractive problems is to characterize those graphs with a few positive eigenvalues. In [9] Smith characterized all graphs with exactly one positive eigenvalue. Recently, Oboudi [6] completely determined the graphs with exactly two nonnegative eigenvalues, i.e., those graphs satisfying $p(G) = 1$ and $\eta(G) = 1$ or $p(G) = 2$ and $\eta(G) = 0$.

In this paper, we introduce three types congruent transformations for graphs. By using these congruent transformations and Oboudi's results in [6], we completely characterize the graphs satisfying $p(G) = 2$ and $\eta(G) = 1$.

2   Preliminaries

In this section, we will introduce some notions and lemmas for the latter use.

Theorem 2.1 (Interlacing theorem [1])**.** *Let G be a graph of order n and H be an induced subgraph of G with order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Lemma 2.2 ([1])**.** *Let H be an induced subgraph of graph G. Then $p(H) \leq p(G)$.*

Lemma 2.3 ([3]). *Let G be a graph containing a pendant vertex, and let H be the induced subgraph of G obtained by deleting the pendant vertex together with the vertex adjacent to it. Then $p(G) = p(H) + 1$, $n(G) = n(H) + 1$ and $\eta(G) = \eta(H)$.*

Lemma 2.4 (Sylvester's law of inertia). *If two real symmetric matrices A and B are congruent, then they have the same positive (resp., negative) inertia index, the same nullity.*

Theorem 2.5 ([9]). *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Let G_1 be a graph containing a vertex u and G_2 be a graph of order n that is disjoint from G_1. For $1 \le k \le n$, the *k-joining* graph of G_1 and G_2 with respect to u, denoted by $G_1(u) \odot^k G_2$, is a graph obtained from $G_1 \cup G_2$ by joining u to arbitrary k vertices of G_2. By using the notion of k-joining graph, Yu et al. [11] completely determined the connected graphs with at least one pendant vertex that have positive inertia index 2.

Theorem 2.6 ([11]). *Let G be a connected graph with pendant vertices. Then $p(G) = 2$ if and only if $G \cong K_{1,r}(u) \odot^k K_{n_1,\dots,n_l}$, where u is the center of $K_{1,r}$ and $1 \le k \le n_1 + \cdots + n_l$.*

Theorem 2.7 ([6]). *Let G be a graph of order $n \ge 2$ with eigenvalues $\lambda_1(G) \ge \cdots \ge \lambda_n(G)$. Assume that $\lambda_3(G) < 0$, then the following hold:*

(1) If $\lambda_1(G) > 0$ and $\lambda_2(G) = 0$, then $G \cong K_1 + K_{n-1}$ or $G \cong K_n \setminus e$ for $e \in E(K_n)$;

(2) If $\lambda_1(G) > 0$ and $\lambda_2(G) < 0$, then $G \cong K_n$.

Let $\mathcal{H}$ be set of all graphs satisfying $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ (in other words, $p(G) = 2$ and $\eta(G) = 0$). Oboudi [6] determined all the graphs of $\mathcal{H}$. To give a clear description of this characterization, we introduce the class of graphs G_n defined in [6].

For every integer $n \ge 2$, let $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ be two disjoint complete graphs with vertex set $V = \{v_1, \dots, v_{\lceil \frac{n}{2} \rceil}\}$ and $W = \{w_1, \dots, w_{\lfloor \frac{n}{2} \rfloor}\}$. G_n is defined to be the graph obtained from $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ by adding some edges distinguishing whether n is even or not below:

(1) If n is even, then add some new edges to $K_{\frac{n}{2}} + K_{\frac{n}{2}}$ satisfying

$$\emptyset = N_W(v_1) \subset N_W(v_2) = \{w_{\frac{n}{2}}\} \subset N_W(v_3) = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}\} \subset \cdots \subset$$
$$N_W(v_{\frac{n}{2}-1}) = \{w_{\frac{n}{2}}, \dots, w_3\} \subset N_W(v_{\frac{n}{2}}) = \{w_{\frac{n}{2}}, \dots, w_2\}.$$

(2) If n is odd, then add some new edges to $K_{\frac{n+1}{2}} + K_{\frac{n-1}{2}}$ satisfying

$$\emptyset = N_W(v_1) \subset N_W(v_2) = \{w_{\frac{n-1}{2}}\} \subset N_W(v_3) = \{w_{\frac{n-1}{2}}, w_{\frac{n-1}{2}-1}\} \subset \cdots \subset$$
$$N_W(v_{\frac{n+1}{2}-1}) = \{w_{\frac{n-1}{2}}, \dots, w_2\} \subset N_W(v_{\frac{n+1}{2}}) = \{w_{\frac{n-1}{2}}, \dots, w_1\}.$$

By deleting the maximum (resp. minimum) degree vertex from G_{n+1} if n is an even (resp. odd), we obtain G_n. It follows the result below.

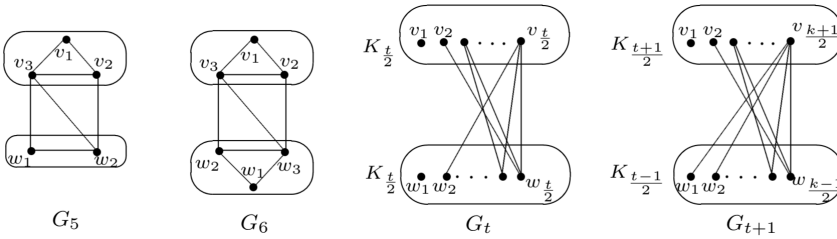**Remark 2.8** (See [6]). *G_n is an induced subgraph of G_{n+1} for every $n \ge 2$.*

Figure 1: G_5, G_6, G_t and G_{t+1}.

For example, $G_2 \cong 2K_1$, $G_3 \cong P_3$ and $G_4 \cong P_4$. The graphs G_5 and G_6 are shown in Figure 1. In general, G_t and G_{t+1} are also shown in Figure 1 for an even number t.

Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. By $G[K_{t_1}, \ldots, K_{t_n}]$ we mean the *generalized lexicographic product* of G (by $K_{t_1}, K_{t_2}, \ldots, K_{t_n}$), which is the graph obtained from G by replacing the vertex v_j with K_{t_j} and connecting each vertex of K_{t_i} to each vertex of K_{t_j} if v_i is adjacent to v_j in G.

Theorem 2.9 ([6]). *Let $G \in \mathcal{H}$ of order $n \geq 4$ with eigenvalues $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$.*

 (1) If G is disconnected, then $G \cong K_p + K_q$ for some integers $p, q \geq 2$;

 (2) If G is connected, there exist some positive integers s and $t_1, \ldots, t_s$ such that $G \cong G_s[K_{t_1}, \ldots, K_{t_s}]$ where $3 \leq s \leq 12$ and $t_1 + \cdots + t_s = n$.

Furthermore, Oboudi gave all the positive integers $t_1, \ldots, t_s$ such that $G_s[K_{t_1}, \ldots, K_{t_s}] \in \mathcal{H}$ in Theorems $3.4 - 3.14$ of [6].

Let $\mathcal{G}$ be the set of all graphs with positive inertia index $p(G) = 2$ and nullity $\eta(G) = 1$. In next section, we introduce some new congruent transformations for graph that keep to the positive inertia index. By using such congruent transformations we characterize those graphs in $\mathcal{G}$ based on $\mathcal{H}$.

3   Three congruent transformations of graphs

In this section, we introduce three types of congruent transformations for graphs.

Lemma 3.1 ([10]). *Let u, v be two non-adjacent vertices of a graph G. If u and v have the same neighborhood, then $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$.*

Remark 3.2. Two non-adjacent vertices u and v are said to be *congruent vertices of I-type* if they have the same neighbors. Lemma 3.1 implies that if one of congruent vertices of I-type is deleted from a graph then the positive and negative inertia indices left unchanged, but the nullity reduces just one. Conversely, if we add a new vertex that joins all the neighbors of some vertex in a graph (briefly we refer to add a vertex of I-type in what follows) then the positive and negative inertia indices left unchanged, but the nullity adds just one. The graph transformation of deleting or adding vertices of I-type is called the (graph) *transformation of I-type*.

Since $\mathrm{Spec}(K_s) = [(s-1)^1, (-1)^{s-1}]$. By applying the transformation of I-type, we can simply find the inertia of $K_{n_1, n_2, \ldots, n_s}$.

Corollary 3.3. *Let $G = K_{n_1, n_2, \ldots, n_s}$ be a multi-complete graph where $n_1 \geq n_2 \geq \cdots \geq n_s$ and $i_0 = \min\{1 \leq i \leq s \mid n_i \geq 2\}$. Then G has the inertia index:* $\text{In}(G) = (p(G), \eta(G), n(G)) = (1, n_{i_0} + n_{i_0+1} + \cdots + n_s - s + i_0 - 1, s - 1)$.

The following transformation was mentioned in [4], but the author didn't prove the result. For the completeness we give a proof below.

Lemma 3.4. *Let $\{u, v, w\}$ be an independent set of a graph G. If $N(u)$ is a disjoint union of $N(v)$ and $N(w)$, then $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$.*

Proof. Since u, v, w are not adjacent to each other, we may assume that $(0, 0, 0, \alpha^T)$, $(0, 0, 0, \beta^T)$ and $(0, 0, 0, \gamma^T)$ are the row vectors of $A(G)$ corresponding to the vertices u, v, w, respectively. Thus $A(G)$ can be written as

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & \alpha^T \\ 0 & 0 & 0 & \beta^T \\ 0 & 0 & 0 & \gamma^T \\ \alpha & \beta & \gamma & A(G - u - v - w) \end{pmatrix}.$$

Since $N(u) = N(v) \cup N(w)$ and $N(v) \cap N(w) = \emptyset$, we have $\alpha = \beta + \gamma$. By letting the u-th row (resp. u-th column) minus the sum of the v-th and w-th rows (resp. the sum of the v-th and w-th columns) of $A(G)$, we get that $A(G)$ is congruent to

$$\begin{pmatrix} 0 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & 0 & \beta^T \\ 0 & 0 & 0 & \gamma^T \\ \mathbf{0} & \beta & \gamma & A(G - u - v - w) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & A(G - u) \end{pmatrix}.$$

Thus $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$ by Lemma 2.4. □

Remark 3.5. The vertex u is said to be a *congruent vertex of II-type* if there exist two non-adjacent vertices v and w such that $N(u)$ is a disjoint union of $N(v)$ and $N(w)$. Lemma 3.4 implies that if one congruent vertex of II-type is deleted from a graph then the positive and negative indices left unchanged, but the nullity reduces just one. Conversely, if there exist two non-adjacent vertices v and w such that $N(v)$ and $N(w)$ are disjoint, we can add a new vertex u that joins all the vertices in $N(v) \cup N(w)$ (briefly we refer to add a vertex of II-type in what follows), then the positive and negative inertia indices left unchanged, but the nullity adds just one. The graph transformation of deleting or adding vertices of II-type is called the (graph) *transformation of II-type*.

An induced quadrangle $C_4 = uvxy$ of G is called *congruent* if there exists a pair of independent edges, say uv and xy in C_4, such that $N(u) \setminus \{v, y\} = N(v) \setminus \{u, x\}$ and $N(x) \setminus \{y, v\} = N(y) \setminus \{x, u\}$, where uv and xy are called a pair of *congruent edges* of C_4. We call the vertices in a congruent quadrangle the *congruent vertices of III-type*.

Lemma 3.6. *Let u be a congruent vertex of III-type in a graph G. Then $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$.*

Proof. Let $C_4 = uvxy$ be the congruent quadrangle of G containing the congruent vertex u. Then $(0, 1, 0, 1, \alpha^T)$, $(1, 0, 1, 0, \alpha^T)$, $(0, 1, 0, 1, \beta^T)$, $(1, 0, 1, 0, \beta^T)$ are the row vectors

of $A(G)$ corresponding to the vertices u, v, x and y, respectively. Thus $A(G)$ can be presented by

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & & \alpha^T \\ 1 & 0 & 1 & 0 & & \alpha^T \\ 0 & 1 & 0 & 1 & & \beta^T \\ 1 & 0 & 1 & 0 & & \beta^T \\ \alpha & \alpha & \beta & \beta & A(G-u-v-x-y) \end{pmatrix}.$$

By letting the u-th row (resp. u-th column) minus the x-th row (resp. x-th column) of $A(G)$, and letting the v-th row (resp. v-th column) minus the y-th row (resp. y-th column) of $A(G)$, we obtain that $A(G)$ is congruent to

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & & \alpha^T - \beta^T \\ 0 & 0 & 0 & 0 & & \alpha^T - \beta^T \\ 0 & 0 & 0 & 1 & & \beta^T \\ 0 & 0 & 1 & 0 & & \beta^T \\ \alpha - \beta & \alpha - \beta & \beta & \beta & A(G-u-v-x-y) \end{pmatrix}.$$

Again, by letting the u-th row (resp. u-th column) minus the v-th row (resp. v-th column) of B, and adding the y-th row (resp. y-th column) to the v-th row (resp. v-th column) of B, we obtain that B is congruent to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & & \mathbf{0}^T \\ 0 & 0 & 1 & 0 & & \alpha^T \\ 0 & 1 & 0 & 1 & & \beta^T \\ 0 & 0 & 1 & 0 & & \beta^T \\ \mathbf{0} & \alpha & \beta & \beta & A(G-u-v-x-y) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & A(G-u) \end{pmatrix}.$$

Thus $p(G) = p(G-u)$, $n(G) = n(G-u)$ and $\eta(G) = \eta(G-u) + 1$ by Lemma 2.4. $\quad\square$

Remark 3.7. The Lemma 3.6 confirms that if a congruent vertex of III-type is deleted from a graph then the positive and negative inertia indices left unchanged, but the nullity reduces just one. Conversely, if we add a new vertex to a graph that consists of a congruent quadrangle with some other three vertices in this graph (briefly we refer to add a vertex of III-type in what follows) then the positive and negative inertia indices left unchanged, but the nullity adds just one. The graph transformation of deleting or adding vertices of III-type is called the (graph) *transformation of III-type*.

Remark 3.2, Remark 3.5 and Remark 3.7 provide us three transformations of graphs that keep the positive and negative inertia indices and change the nullity just one. By applying these transformations we will construct the graphs in $\mathcal{G}$. Let $\mathcal{G}_1$ be the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of I-type, $\mathcal{G}_2$ be the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of II-type and $\mathcal{G}_3$ be the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of III-type. At the end of this section, we would like to give an example to illustrate the constructions of the graphs in $\mathcal{G}_i$ $(i = 1, 2, 3)$.

Example 3.8. We know the path P_4, with spectrum $\mathrm{Spec}(P_4) = \{1.6180, 0.6180, -0.6180, -1.6180\}$, is a graph belonging to $\mathcal{H}$. By adding a vertex u of I-type to P_4 we obtain $H_1 \in \mathcal{G}_1$ (see Figure 2) where $\mathrm{Spec}(H_1) = \{1.8478, 0.7654, 0, -0.7654, -1.8478\}$,
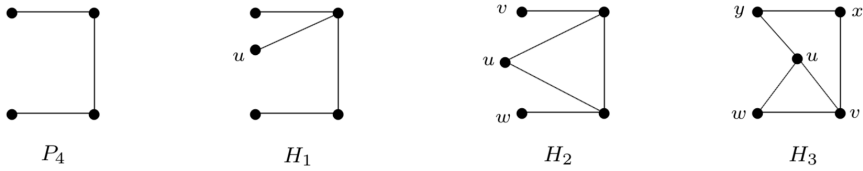
Figure 2: The graphs P_4, H_1, H_2 and H_3.

adding a vertex u of II-type to P_4 we obtain $H_2 \in \mathcal{G}_2$ where $\mathrm{Spec}(H_2) = \{2.3028, 0.6180, 0, -1.3028, -1.6180\}$. Finally, by adding a vertex u of III-type to P_4 we obtain $H_3 \in \mathcal{G}_3$, where $\mathrm{Spec}(H_3) = \{2.4812, 0.6889, 0, -1.1701, -2\}$. In fact, uv and xy is a pair of independent edges in H_3. Clearly, $N(u) \setminus \{v, y\} = N(v) \setminus \{u, x\} = \{w\}$ and $N(x) \setminus \{y, v\} = N(y) \setminus \{x, u\} = \emptyset$. Thus $C_4 = uvxy$ is a congruent quadrangle of H_3.

Clearly, $G = K_{1,2} \cup P_2$ is a non-connected graph in $\mathcal{G}$, and all such graphs we collect in $\mathcal{G}^- = \{G \in \mathcal{G} \mid G \text{ is disconnected}\}$. Additionally, H_1 and H_2 shown in Figure 2 are graphs with pendant vertex belonging to $\mathcal{G}$, and all such graphs we collect in $\mathcal{G}^+ = \{G \in \mathcal{G} \mid G \text{ is connected with a pendant vertex}\}$. In next section, we firstly determine the graphs in $\mathcal{G}^-$ and $\mathcal{G}^+$.

4   The characterization of graphs in $\mathcal{G}^-$ and $\mathcal{G}^+$

The following result completely characterizes the disconnected graphs of $\mathcal{G}$.

Theorem 4.1. *Let G be a graph of order $n \geq 5$. Then $G \in \mathcal{G}^-$ if and only if $G \cong K_s + K_t + K_1, H + K_1$ or $K_s + K_{n-s} \setminus e$ for $e \in E(K_{n-s})$, where $H \in \mathcal{H}$ is connected and $s + t = n - 1$, $s, t \geq 2$.*

Proof. All the graphs displayed in Theorem 4.1 have two positive and one zero eigenvalues by simple observation. Now we prove the necessity.

Let $G \in \mathcal{G}^-$, and $H_1, H_2, \ldots, H_k$ $(k \geq 2)$ the components of G. Since $\lambda_1(H_i) \geq 0$ for $i = 1, 2, \ldots, k$ and $\lambda_4(G) < 0$, G has two or three components and so $k \leq 3$.

First assume that $G = H_1 + H_2 + H_3$. It is easy to see that G has exactly one isolated vertex due to $\eta(G) = 1$ and $p(G) = 2$. Without loss of generality, let $H_3 \cong K_1$. Since $\lambda_3(G) = 0$ and $\lambda_1(H_i) > 0$ $(i = 1, 2)$, we have $\lambda_2(H_1) < 0$ and $\lambda_2(H_2) < 0$. By Theorem 2.7 (2), $G \cong K_s + K_t + K_1$ as desired, where $s + t = n - 1$ and $s, t \geq 2$.

Next assume that $G = H_1 + H_2$. If $H_1 \cong K_1$, then

$$\lambda_1(G) = \lambda_1(H_2) \geq \lambda_2(G) = \lambda_2(H_2) > \lambda_3(G) =$$
$$0 = \lambda_1(H_1) > \lambda_4(G) = \lambda_3(H_2) < 0.$$

Thus $H_2 \cong H \in \mathcal{H}$, and so $G \cong H + K_1$ as desired. If $|H_i| \geq 2$ for $i = 1, 2$, then one of $\lambda_2(H_1)$ and $\lambda_2(H_2)$ is equal to zero and another is less than zero because $\lambda_3(G) = 0$ and $\lambda_4(G) < 0$. Without loss of generality, let $\lambda_2(H_1) < 0$ and $\lambda_2(H_2) = 0$. We have $\lambda_3(H_1) \leq \lambda_2(H_1) < 0$, in addition, $\lambda_3(H_2) < 0$ since $\eta(G) = 1$. By Theorem 2.7 (2), $H_1 \cong K_s$ for some $s \geq 2$ and by Theorem 2.7 (1), $H_2 \cong K_{n-s} \setminus e$.

We complete this proof. □

In terms of Theorem 2.6, we will determine all connected graphs with a pendant vertex satisfying $p(G) = 2$ and $\eta(G) = d$ for any positive integer d.

Theorem 4.2. *Let G be a connected graph of order n with a pendant vertex. Then $p(G) = 2$ and $\eta(G) = d \geq 1$ if and only if $G \cong K_{1,r}(u) \odot^k K_{n_1,\ldots,n_l}$, where $r + n_1 + n_2 + \cdots + n_l - (l+1) = d$.*

Proof. Let $G = K_{1,r}(u) \odot^k K_{n_1,\ldots,n_l}$ and vu is a pendant edge of G. By deleting v and u from G we obtain $H = G - \{u,v\} = (r-1)K_1 \cup K_{n_1,\ldots,n_l}$. It is well known that $p(K_{n_1,\ldots,n_l}) = 1$ and $\eta(K_{n_1,\ldots,n_l}) = n_1 + \cdots + n_l - l$. From Lemma 2.3, we have

$$p(G) = p(H) + 1 = p(K_{n_1,\ldots,n_l}) + 1 = 2,$$
$$\eta(G) = \eta(H) = (r-1) + (n_1 + \cdots + n_l - l) = d.$$

Conversely, let G be a graph with a pendant vertex and $p(G) = 2$. By Theorem 2.6, we have $G \cong K_{1,r}(u) \odot^k K_{n_1,\ldots,n_l}$. According to the arguments above, we know that $\eta(G) = r + n_1 + n_2 + \cdots + n_l - (l+1) = d$. $\square$

From Theorem 4.2, it immediately follows the result that completely characterizes the graphs in $\mathcal{G}^+$.

Corollary 4.3. *A connected graph $G \in \mathcal{G}^+$ if and only if $G \cong K_{1,2}(u) \odot^k K_{n-3}$ or $G \cong K_{1,1}(u) \odot^k K_{n-2} \setminus e$ for $e \in E(K_{n-2})$.*

Proof. By Theorem 4.2, we have $G \in \mathcal{G}^+$ if and only if $G \cong K_{1,r}(u) \odot^k K_{n_1,\ldots,n_l}$, where $r + n_1 + n_2 + \cdots + n_l - (l+1) = 1$ and $r, l, n_1, \ldots, n_l \geq 1$. It gives two solutions: one is $r = 2$, $n_1 = n_2 = \cdots = n_l = 1$ and $l = n - 3$ which leads to $G \cong K_{1,2}(u) \odot^k K_{n-3}$; another is $r = 1, n_1 = 2, n_2 = \cdots = n_l = 1$ and $l = n - 2$ which leads to $G \cong K_{1,1}(u) \odot^k K_{n-2} \setminus e$ for $e \in E(K_{n-2})$. $\square$

Let $\mathcal{G}^*$ denote the set of all connected graphs in $\mathcal{G}$ without pendant vertices. Then $\mathcal{G} = \mathcal{G}^- \cup \mathcal{G}^+ \cup \mathcal{G}^*$. Therefore, in order to characterize $\mathcal{G}$, it remains to consider those graphs in $\mathcal{G}^*$.

5 The characterization of graphs in $\mathcal{G}^*$

First we introduce some symbols which will be persisted in this section. Let $G \in \mathcal{G}^*$. The eigenvalues of G can be arranged as:

$$\lambda_1(G) \geq \lambda_2(G) > \lambda_3(G) = 0 > \lambda_4(G) \geq \cdots \geq \lambda_n(G).$$

We choose $v^* \in V(G)$ such that $d_G(v^*) = \delta(G) = t$, and denote by $X = N_G(v^*)$ and $Y = V(G) - N_G[v^*]$. Then $t = |X| \geq 2$ since G has no pendant vertices. In addition, $|Y| > 0$ since otherwise G would be a complete graph. First we characterize the induced subgraph $G[Y]$ in the following result.

Lemma 5.1. $G[Y] \cong K_{n-t-1} \setminus e, K_1 + K_{n-t-2}$ or K_{n-t-1}.

Proof. First we suppose that Y is an independent set. If $|Y| \geq 3$, then $\lambda_4(G) \geq \lambda_4(G[Y \cup \{v^*\}]) = 0$ by Theorem 2.1, a contradiction. Hence $|Y| \leq 2$, and so $G[Y] \cong K_1$ or $G[Y] \cong K_2 \setminus e = 2K_1$.

Next we suppose that $G[Y]$ contains some edges. We distinguish the following three situations.

If $\lambda_2(G[Y]) > 0$, we have $p(G[Y]) \geq 2$. For any $x \in X$, the induced subgraph $G[\{v^*, x\} \cup Y]$ has a pendant vertex v^* by our assumption. By Lemma 2.2 and Lemma 2.3, we have $p(G) \geq p(G[\{v^*, x\} \cup Y]) = p(G[Y]) + 1 \geq 3$, a contradiction.

If $\lambda_2(G[Y]) < 0$, by Theorem 2.7 (2) we have $G[Y] \cong K_{n-t-1}$ as desired.

At last assume that $\lambda_2(G[Y]) = 0$. If $\lambda_3(G[Y]) < 0$, by Theorem 2.7 (1), we have $G[Y] \cong K_{n-t-1} \setminus e, K_1 + K_{n-t-2}$ as desired. If $\lambda_3(G[Y]) = 0$, by Lemma 2.3 we have $p(G[\{v^*, x\} \cup Y]) = p(G[Y]) + 1 = 2$ and $\eta(G[\{v^*, x\} \cup Y]) = \eta(G[Y]) \geq 2$, which implies that $\lambda_4(G) \geq \lambda_4(G[\{v^*, x\} \cup Y]) = 0$, a contradiction.

We complete this proof. $\qquad\square$

First assume that $Y = \{y_1\}$. If $G[X] = K_t$, then $G = K_n \setminus v^* y_1$. However $K_n \setminus v^* y_1 \notin \mathcal{G}^*$ since $p(K_n \setminus v^* y_1) = 1$. Thus there exist $x_1 \nsim x_2$ in X. Then $N_G(x_1) = N_G(x_2)$ and $N_G(v^*) = N_G(y_1)$. It follows that $\eta(G) \geq 2$ by Lemma 3.1. Next assume that $Y = \{y_1, y\}$ is an independent set. We have $N_G(v^*) = N_G(y_1) = N_G(y)$ since $d_G(y_1), d_G(y) \geq d_G(v^*) = \delta(G)$. Thus, by Lemma 3.1 we have $\eta(G) = \eta(G - y_1) + 1 = \eta(G - y_1 - y) + 2 \geq 2$. Thus we only need to consider the case that $G[Y]$ contains at least one edge. Concretely, we distinguish three situations in accordance with the proof of Lemma 5.1:

(a) $G[Y] \cong K_{n-t-2} + K_1$ in case of $\lambda_2(G[Y]) = 0$ and $\lambda_3(G[Y]) < 0$, where $n - t - 2 \geq 2$;

(b) $G[Y] \cong K_{n-t-1} \setminus e$ in case of $\lambda_2(G[Y]) = 0$ and $\lambda_3(G[Y]) < 0$, where $|Y| = n - t - 1 \geq 3$;

(c) $G[Y] \cong K_{n-t-1}$ in case of $\lambda_2(G[Y]) < 0$, where $|Y| = n - t - 1 \geq 2$.

In the following, we deal with situation (a) in Lemma 5.2, (b) in Lemma 5.3 and (c) in Lemma 5.4, 5.7 and Lemma 5.15. We will see that the graph $G \in \mathcal{G}^*$ illustrated in (a) and (b) can be constructed from some $H \in \mathcal{H}$ by the graph transformations of I-, II- and III-type, but (c) can not.

Lemma 5.2. *If $G[Y] \cong K_{n-t-2} + K_1$, where $n - t - 2 \geq 2$, then $G \in \mathcal{G}_1$.*

Proof. Since $G[Y]$ is isomorphic to $K_{n-t-2} + K_1$ ($n - t - 2 \geq 2$), Y exactly contains one isolated vertex of $G[Y]$, say y. We have $N_G(v^*) = N_G(y)$ and thus y is a congruent vertex of I-type. By Lemma 3.1, we have $p(G) = p(G - y)$ and $\eta(G) = \eta(G - y) + 1$. Notice that $G - y$ is connected, we have $G - y \in \mathcal{H}$, and so $G \in \mathcal{G}_1$. Such a graph G, displayed in Figure 3 (1), we call the v^*-*graph of I-type*. $\qquad\square$

In Figure 3 and Figure 5, two ellipses joining with one full line denote some edges between them. A vertex and an ellipse joining with one full line denote some edges between them, and with two full lines denote that this vertex joins all vertices in the ellipse. Two vertices join with same location of an ellipse denote that they have same neighbours in this ellipse.

It needs to mention that the v^*-graph of I-type characterized in Lemma 5.2, is a graph obtained from $H \in \mathcal{H}$ by adding a new vertex joining the neighbors of a minimum degree vertex of H.

For $S \subseteq V(G)$ and $u \in V(G)$, let $N_S(u) = N_G(u) \cap S$ and $N_S[u] = N_G[u] \cap S$.

Lemma 5.3. *Let $G[Y] \cong K_{n-t-1} \setminus e$, where $n - t - 1 \geq 3$ and $e = yy'$. Then $G \in \mathcal{G}_1$ if $N_X(y) = N_X(y')$ and $G \in \mathcal{G}_2$ otherwise.*
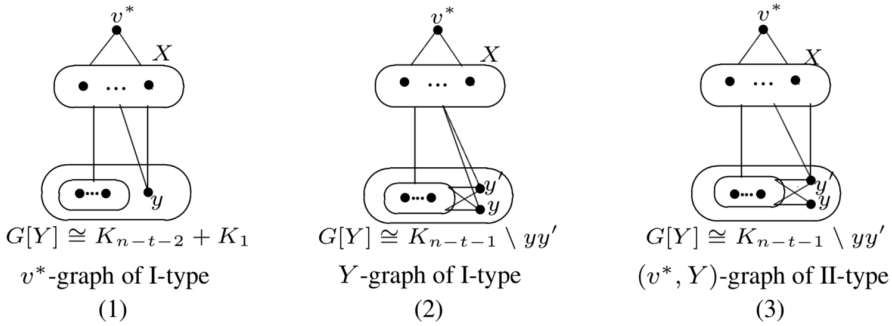
Figure 3: The structure of some graphs.

Proof. Since $n - t - 1 \geq 3$, there is $y^* \in Y$ other than y and y'. It is clear that $N_G(y) = N_X(y) \cup (Y \setminus \{y, y'\})$ and $N_G(y') = N_X(y') \cup (Y \setminus \{y, y'\})$, and thus $N_G(y) = N_G(y')$ if and only if $N_X(y) = N_X(y')$. We consider the following cases.

Case 1. $N_X(y) = N_X(y')$.

By assumption, $N_G(y) = N_G(y')$, thus y and y' are congruent vertices of I-type. By Lemma 3.1, we have $p(G) = p(G - y)$ and $\eta(G) = \eta(G - y) + 1$. Since $G - y$ is connected, we have $G - y \in \mathcal{H}$ and so $G \in \mathcal{G}_1$. Such a G, displayed in Figure 3 (2), we call the *Y-graph of I-type*.

Case 2. $N_X(y) \neq N_X(y')$.

First suppose that exactly one of $N_X(y)$ and $N_X(y')$ is empty, say $N_X(y) = \emptyset$ and $N_X(y') \neq \emptyset$. Then yy^* is a pendant edge of the induced subgraph $G[X \cup \{y, y', y^*, v^*\}]$. By Lemma 2.2 and Lemma 2.3, we have

$$2 = p(G) \geq p(G[X \cup \{y, y', y^*, v^*\}]) = p(G[X \cup \{y', v^*\}]) + 1 \geq 2.$$

Thus

$$p(G[X \cup \{y, y', y^*, v^*\}]) = 2 \quad \text{and}$$
$$p(G[X \cup \{y', v^*\}]) = 1.$$

We see that $\lambda_2(G[X \cup \{y', v^*\}]) = 0$ (since otherwise $\lambda_2(G[X \cup \{y', v^*\}]) < 0$ and then $G[X \cup \{y', v^*\}]$ is a complete graph, but $y' \nsim v^*$). If $\lambda_3(G[X \cup \{y', v^*\}]) = 0$, we have

$$\eta(G[X \cup \{y, y', y^*, v^*\}]) = \eta(G[X \cup \{y', v^*\}]) \geq 2,$$

which implies

$$\lambda_4(G) \geq \lambda_4(G[X \cup \{y, y', y^*, v^*\}]) = 0,$$

a contradiction. If $\lambda_3(G[X \cup \{y', v^*\}]) < 0$, then $G[X \cup \{y', v^*\}] \cong K_{t+2} \setminus e$ or $K_{t+1} + K_1$ by Theorem 2.7 (1). Notice that $G[X \cup \{y', v^*\}]$ is connected, we get $G[X \cup \{y', v^*\}] \cong K_{t+2} \setminus e$ where $e = v^*y'$. Thus $N_X(y') = X$ and so $N_G(y') = X \cup (Y \setminus \{y, y'\}) = N_G(v^*) \cup N_G(y)$ is a disjoint union. Additionally, $\{y', v^*, y\}$ is an independent set in G, we see that y' is a congruent vertex of II-type. Thus $p(G) = p(G - y')$ and $\eta(G) = $
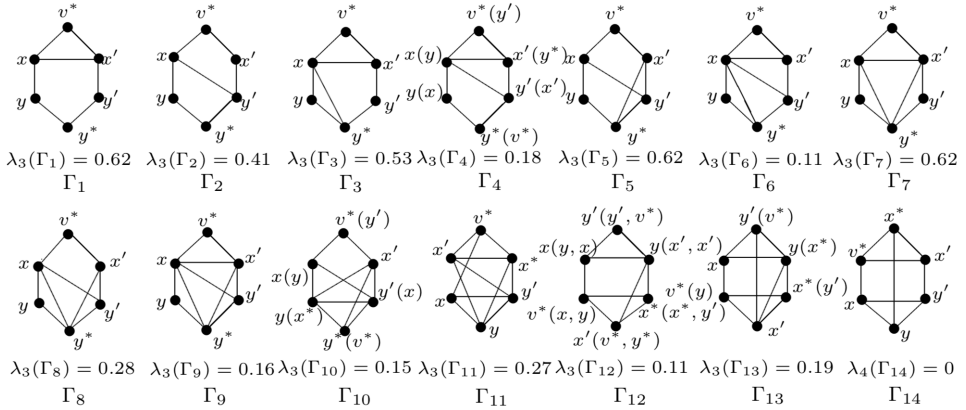
$\lambda_3(\Gamma_1) = 0.62$ $\lambda_3(\Gamma_2) = 0.41$ $\lambda_3(\Gamma_3) = 0.53$ $\lambda_3(\Gamma_4) = 0.18$ $\lambda_3(\Gamma_5) = 0.62$ $\lambda_3(\Gamma_6) = 0.11$ $\lambda_3(\Gamma_7) = 0.62$

   Γ_1 Γ_2 Γ_3 Γ_4 Γ_5 Γ_6 Γ_7

$\lambda_3(\Gamma_8) = 0.28$ $\lambda_3(\Gamma_9) = 0.16$ $\lambda_3(\Gamma_{10}) = 0.15$ $\lambda_3(\Gamma_{11}) = 0.27$ $\lambda_3(\Gamma_{12}) = 0.11$ $\lambda_3(\Gamma_{13}) = 0.19$ $\lambda_4(\Gamma_{14}) = 0$

   Γ_8 Γ_9 Γ_{10} Γ_{11} Γ_{12} Γ_{13} Γ_{14}

Figure 4: The graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_{14}$.

$\eta(G - y') + 1$ by Lemma 3.4. This implies that $G - y' \in \mathcal{H}$, and so $G \in \mathcal{G}_2$. Such a G, displayed in Figure 3 (3), we call the (v^*, Y)-*graph of II-type*.

Next suppose that $N_X(y), N_X(y') \neq \emptyset$, without loss of generality, assume that $N_X(y') \setminus N_X(y) \neq \emptyset$. Then there exists $x' \in N_X(y') \setminus N_X(y)$. Thus $x' \sim y'$ and $x' \nsim y$. Now by taking some $x \in N_X(y)$, we see that $C_6 = v^* x y y^* y' x'$ is a 6-cycle in G. Note that x may joins each vertex in $\{x', y', y^*\}$ and x' may joins y^*. By distinguishing different situations in according with the number of edges we have

$$G[v^*, x, y, y^*, y', x'] \cong \begin{cases} C_6 & \text{no edge;} \\ \Gamma_1 \text{ or } \Gamma_2 & \text{one edges;} \\ \Gamma_3, \Gamma_4 \text{ or } \Gamma_5 & \text{two edges;} \\ \Gamma_6, \Gamma_7 \text{ or } \Gamma_8 & \text{three edges;} \\ \Gamma_9 & \text{four edges.} \end{cases}$$

However C_6 and $\Gamma_1, \ldots, \Gamma_8$ and Γ_9 are all forbidden subgraphs of G (see Figure 4).

We complete this proof. $\square$

It remains to characterize the graph $G \in \mathcal{G}^*$ satisfying $G[Y] \cong K_{n-t-1}$. Such a graph G we call X-*complete* if $G[X]$ is also complete graph, and X-*imcomplete* otherwise. The following result characterizes the X-imcomplete graphs.

Lemma 5.4. *Let $G[Y] \cong K_{n-t-1}$, where $n - t - 1 \geq 2$, and G is X-imcomplete. Then $G \in \mathcal{G}_1$ if there exist two non-adjacent vertices $x_1 \nsim x_2$ in $G[X]$ such that $N_Y(x_1) = N_Y(x_2)$ and $G \in \mathcal{G}_3$ otherwise.*

Proof. Let $X = \{x_1, x_2, \ldots, x_t\}$ and $Y = \{y_1, y_2, \ldots, y_{n-t-1}\}$. Then $V(G) = \{v^*\} \cup X \cup Y$ and Y induces K_{n-t-1}. Let x and x' be two non-adjacent vertices in X. Since $d_G(x) \geq d_G(v^*)$ and $n - t - 1 \geq 2$, we have $|N_Y(x)| \geq 1$ and $|Y| \geq 2$, respectively. First we give some claims.

Claim 5.5. *If $x \nsim x'$ in $G[X]$ then one of $N_Y(x)$ and $N_Y(x')$ includes another. If $N_Y(x) \subset N_Y(x')$ then $|N_Y(x)| = 1$ and $N_Y(x') = Y$.*

Proof. On the contrary, let $y \in N_Y(x) \setminus N_Y(x')$ and $y' \in N_Y(x') \setminus N_Y(x)$, then $G[v^*, x, x', y, y'] \cong C_5$. Thus one of $N_Y(x)$ and $N_Y(x')$ includes another. Now assume that $N_Y(x) \subset N_Y(x')$. If $|N_Y(x)| \geq 2$, say $\{y, y'\} \subseteq N_Y(x)$, then $x' \sim y, y'$ and exists $y^* \in N_Y(x') \setminus N_Y(x)$. Thus $G[v^*, x, x', y, y', y^*] \cong \Gamma_{10}$ (see Figure 4). However $p(\Gamma_{10}) = 3$. Hence $|N_Y(x)| = 1$, and we may assume that $N_Y(x) = \{y\}$. If $N_Y(x') \neq Y$, then there exists $y' \in Y \setminus N_Y(x')$. Also, there exists $y^* \in N_Y(x') \setminus N_Y(x)$. We have $G[v^*, x, x', y, y', y^*] \cong \Gamma_4$ (see the labels in the parentheses of Figure 4), but $p(\Gamma_4) = 3$. Thus $N_Y(x') = Y$. $\square$

Claim 5.6. *If* $x \not\sim x'$ *in* $G[X]$ *then* $N_X(x) = N_X(x')$.

Proof. On the contrary, we may assume that $x^* \in N_X(x') \setminus N_X(x)$. Then $x^* \sim x'$ and $x^* \not\sim x$, thus $|N_Y(x)| \geq 2$ since $|N_G(x)| \geq t$. By Claim 5.5, we have $N_Y(x^*), N_Y(x') \subseteq N_Y(x)$. Then either $N_Y(x^*) = N_Y(x') = N_Y(x)$ or one of $N_Y(x^*)$ and $N_Y(x')$ is a proper subset of $N_Y(x)$ (without loss of generality, assume that $N_Y(x^*) \subset N_Y(x)$, and then $|N_Y(x^*)| = 1$ and $N_Y(x) = Y$ by Claim 5.5).

Suppose that $N_Y(x) = N_Y(x^*) = N_Y(x')$. Take $y, y' \in N_Y(x)$, we see that $G[v^*, x, x^*, x', y, y'] \cong \Gamma_{11}$ (see Figure 4). However $p(\Gamma_{11}) = 3$.

Suppose that $|N_Y(x^*)| = 1$ and $N_Y(x) = Y$. Let $N_Y(x^*) = \{y\}$ and there exists another $y' \in Y$. Then $G[v^*, x, x^*, x', y, y']$ is isomorphic Γ_{13} (see Figure 4) if $x' \sim y, y'$, or isomorphic to Γ_{12} (see Figure 4) if $x' \sim y$ and $x' \not\sim y'$, or isomorphic to Γ_{14} (see Figure 4) if $x' \not\sim y$ and $x' \sim y'$. However $p(\Gamma_{12}) = p(\Gamma_{13}) = 3$ and $\lambda_4(\Gamma_{14}) = 0$. We are done. $\square$

Now we distinguish the following cases to prove our result.

Case 1. There exist $x_1 \not\sim x_2$ such that $N_Y(x_1) = N_Y(x_2)$.

Since $x_1 \not\sim x_2$, we have $N_X(x_1) = N_X(x_2)$ by Claim 5.6, so $N_G(x_1) = N_G(x_2)$. Thus x_1 and x_2 are congruent vertices of I-type. By Lemma 3.1, $p(G) = p(G - x_1)$ and $\eta(G) = \eta(G - x_1) + 1$. Thus $G - x_1 \in \mathcal{H}$ and so $G \in \mathcal{G}_1$. Such a G, displayed in Figure 5 (1), we call the X-graph of I-type.

Case 2. For each pair of $x \not\sim x' \in X$, $N_Y(x) \neq N_Y(x')$.

By Claim 5.5, without loss of generality, assume that $N_Y(x) \subset N_Y(x')$ and then $N_Y(x) = \{y\}$ and $N_Y(x') = Y$. Thus $y \sim x, x'$ and furthermore we will show that $X \subseteq N_G(y)$. In fact, let $x^* \in X \setminus \{x, x'\}$ (if any), if $x \not\sim x^*$, we have $N_Y(x^*) \supseteq N_Y(x) = \{y\}$ by Claim 5.6. Thus $y \sim x^*$. Otherwise, $x \sim x^*$ and thus $x' \sim x^*$ since $N_X(x) = N_X(x')$ by Claim 5.6. Now take $y' \in Y \setminus \{y\}$. If $y \not\sim x^*$, then $G[v^*, x, x', x^*, y, y']$ is isomorphic to Γ_{12} (see the first labels in the parentheses of Figure 4) while $x^* \not\sim y'$, or isomorphic to Γ_{13} (see the labels in the parentheses of Figure 4) while $x^* \sim y'$, but $p(\Gamma_{12}) = p(\Gamma_{13}) = 3$. It follows that $N_G(y) = X \cup (Y \setminus \{y\})$ since Y induces a clique.

On the other hand, since $d_G(x) \geq |X| = t$, $x \not\sim x'$ and $N_Y(x) = \{y\}$, we have $N_X(x) = X \setminus \{x, x'\}$ and so $N_X(x') = X \setminus \{x, x'\}$ by Claim 5.6. Thus $N_G(x) = (X \setminus \{x, x'\}) \cup \{v^*, y\}$ and $N_G(x') = (X \setminus \{x, x'\}) \cup Y \cup \{v^*\}$. Hence the quadrangle $C_4 = xv^*x'y$ is congruent, where xv^* and $x'y$ is a pair of congruent edges of C_4. It gives that x, v^*, x', y are congruent vertices of III-type. By Lemma 3.6, we have $p(G) = p(G - x)$ and $\eta(G) = \eta(G - x) + 1$ thus $G - x \in \mathcal{H}$, and so $G \in \mathcal{G}_3$. Such a G, displayed in Figure 5 (2), we call the (v^*, X, Y)-graph of III-type.

We complete this proof. $\square$

X-graph of I-type (v^*, X, Y)-graph of III-type (X, Y)-graph of III-type
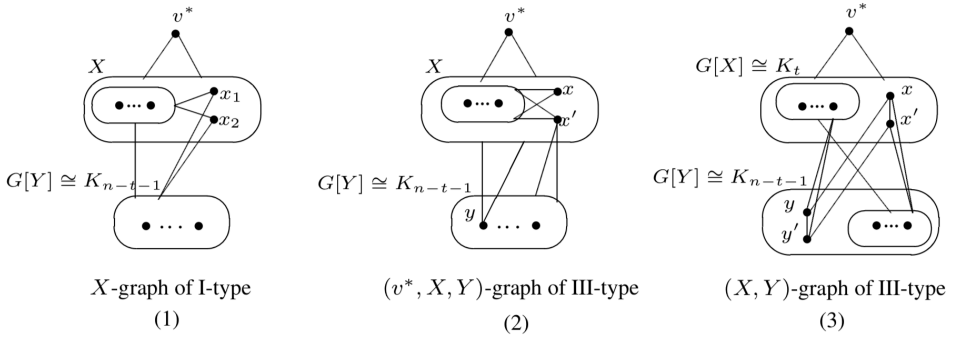(1)   (2)   (3)

Figure 5: The structure of some graphs.

At last we focus on characterizing X-complete graph $G \in \mathcal{G}^*$, i.e., $G[X] \cong K_t$ and $G[Y] \cong K_{n-t-1}$. A X-complete graph $G \in \mathcal{G}^*$ is called *reduced* if one of $N_Y(x_i)$ and $N_Y(x_j)$ is a subset of another for any $x_i \neq x_j \in X$ and *non-reduced* otherwise. Thus the X-complete graphs are partitioned into a disjoint union of the reduced and non-reduced X-complete graphs. Concretely, for a reduced X-complete graph $G \in \mathcal{G}^*$, we may assume that $\emptyset = N_Y(v^*) \subseteq N_Y(x_1) \subseteq N_Y(x_2) \subseteq \cdots \subseteq N_Y(x_t)$; for a non-reduced (X, Y)-complete graph $G \in \mathcal{G}^*$, there exist some $x \neq x' \in X$ such that $N_Y(x) \setminus N_Y(x') \neq \emptyset$ and $N_Y(x') \setminus N_Y(x) \neq \emptyset$. Such vertices x and x' are called *non-reduced vertices*. It remains to characterize the reduced and non-reduced X-complete graphs in what follows.

Lemma 5.7. *Let $G \in \mathcal{G}^*$ be a non-reduced X-complete graph and x, x' be non-reduced vertices. Then $G \in \mathcal{G}_3$.*

Proof. Since x, x' are non-reduced vertices, there exist $y \in N_Y(x) \setminus N_Y(x')$ and $y' \in N_Y(x') \setminus N_Y(x)$. Then x, x', y', y induces C_4 (see Figure 5 (3)). It suffices to verify that C_4 is congruent. Clearly, $N_G(x) \supset (X \setminus \{x\}) \cup \{v^*\}$ and $N_G(x') \supset (X \setminus \{x'\}) \cup \{v^*\}$. If there exists $y^* \in N_Y(x) \setminus N_Y(x')$ other than y, then $G[v^*, x, x', y', y, y^*] \cong \Gamma_{12}$ (see the second labels in the parentheses of Figure 4), however Γ_{12} is a forbidden subgraph of G. Hence $N_Y(x) \setminus N_Y(x') = \{y\}$. Similarly, $N_Y(x') \setminus N_Y(x) = \{y'\}$. On the other aspect, $x \in N_X(y) \setminus N_X(y')$ and $x' \in N_X(y') \setminus N_X(y)$. If there exists $x^* \in N_X(y) \setminus N_X(y')$ other than x, then $G[v^*, x, x', x^*, y, y'] \cong \Gamma_{10}$ (see the labels in the parentheses of Figure 4), however Γ_{10} is a forbidden subgraph of G. Hence $N_X(y) \setminus N_X(y') = \{x\}$. Similarly, $N_X(y') \setminus N_X(y) = \{x'\}$. Hence $N_X(y) \setminus \{x\} = N_X(y') \setminus \{x'\}$. Note that $N_G(y) \supset Y \setminus \{y\}$ and $N_G(y') \supset Y \setminus \{y'\}$, we have $N_G(y) \setminus \{y', x\} = (Y \setminus \{y, y'\}) \cup (N_X(y) \setminus \{x\}) = N_G(y') \setminus \{x', y\}$. Hence the quadrangle $C_4 = xx'y'y$ is congruent, where xx' and $y'y$ is a pair of congruent edges. It follows that x, x', y', y are congruent vertices of III-type. By Lemma 3.6, we have $p(G) = p(G - x)$ and $\eta(G) = \eta(G - x) + 1$. Thus $G - x \in \mathcal{H}$, and so $G \in \mathcal{G}_3$. Such a G, displayed in Figure 5 (3), we call the (X, Y)-graph of III-type.

We complete this proof. □

To characterize the reduced X-complete graph, we need the notion of canonical graph which is introduced in [7]. For a graph G, a relation ρ on $V(G)$ we mean that $u\rho v$ iff $u \sim v$ and $N_G(u) \setminus v = N_G(v) \setminus u$. Clearly, ρ is symmetric and transitive. In accordance with ρ,

the vertex set is decomposed into classes:

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_k, \tag{5.1}$$

where $v_i \in V_i$ and $V_i = \{x \in V(G) \mid x\rho v_i\}$. By definition of ρ, V_i induces a clique K_{n_i} where $n_1 + n_2 + \cdots + n_k = n = |V(G)|$, and vertices of V_i join that of V_j iff $v_i \sim v_j$ in G. We call the induced subgraph $G[\{v_1, v_2, \ldots, v_k\}]$ as the *canonical graph* of G, denoted by G_c. Thus $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$ is a *generalized lexicographic product* of G_c (by $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$).

Let G be a reduced X-complete graph. From (5.1) we have $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$, where $G_c = G[\{v_1, v_2, \ldots, v_k\}]$ and $V_i = \{x \in V(G) \mid x\rho v_i\}$ induces clique K_{n_i}. Without loss of generality, assume $v_1 = v^*$. Let $X_c = N_{G_c}(v_1)$ and $Y_c = \{v_2, v_3, \ldots, v_k\} \setminus X_c$. Clearly, $G_c[X_c]$ is a clique since X_c is a subset of X and X induces a clique in G. Furthermore, $G_c[Y_c]$ is a clique since Y_c is a subset of Y and Y induces a clique in G. Thus G_c is also a X_c-complete graph. Additionally, since G is reduced, G_c is also reduced. Let $t_c = d_{G_c}(v_1)$ and $X_c = \{x_1, x_2, \ldots, x_{t_c}\}$, $Y_c = \{y_1, y_2, \ldots, y_{k-t_c-1}\}$. We may assume $N_{Y_c}(v_1) \subset N_{Y_c}(x_1) \subset \cdots \subset N_{Y_c}(x_{t_c})$ and $N_{X_c}(y_1) \subset \cdots \subset N_{X_c}(y_{k-t_c-1})$. Therefore,

$$0 = |N_{Y_c}(v_1)| < |N_{Y_c}(x_1)| < \cdots < |N_{Y_c}(x_{t_c})| \leq |Y_c| = k - t_c - 1, \tag{5.2}$$

and

$$0 \leq |N_{X_c}(y_1)| < |N_{X_c}(y_2)| < \cdots < |N_{X_c}(y_{k-t_c-1})| \leq |X_c| = t_c. \tag{5.3}$$

From Equation (5.2), we have $t_c \leq k - t_c - 1$. Similarly, $k - t_c - 2 \leq t_c$ from Equation (5.3). Thus $k - 2 \leq 2t_c \leq k - 1$, and so $t_c = \lceil \frac{k}{2} \rceil - 1$.

If k is even, then $t_c = \frac{k}{2} - 1$. From Equation (5.2), we have $|N_{Y_c}(x_i)| = i$ for $i = 1, 2, \ldots, t_c$. Thus we may assume that

$$N_{Y_c}(v_1) = \emptyset,$$
$$N_{Y_c}(x_1) = \{y_{\frac{k}{2}}\},$$
$$\vdots$$
$$N_{Y_c}(x_{\frac{n}{2}-2}) = \{y_{\frac{k}{2}}, \ldots, y_3\},$$
$$N_{Y_c}(x_{\frac{n}{2}-1}) = \{y_{\frac{k}{2}}, \ldots, y_2\}.$$

This implies that $G \cong G_k$ where G_k is defined in Section 2. Similarly, $G \cong G_k$ if k is odd. Thus we obtain the following result.

Lemma 5.8. *Let G be a reduced X-complete graph. Then $G_c \cong G_k$ where $k \geq 2$ is determined in (5.1).*

Let $G \in \mathcal{G}^*$ be a reduced X-complete graph. The following lemma gives a characterization for G. First we cite a result due to Oboudi in [5].

Lemma 5.9 ([5]). *Let $G = G_3[K_{n_1}, K_{n_2}, K_{n_3}]$, where n_1, n_2, n_3 are some positive integers. Then the following hold:*

(1) If $n_1 = n_2 = n_3 = 1$, that is $G \cong P_3$, then $\lambda_3(G) = -\sqrt{2}$;

(2) If $n_1 = n_2 = 1$ and $n_3 \geq 2$, then $\lambda_3(G) = -1$;

(3) If $n_1 n_2 > 1$, then $\lambda_3(G) = -1$.

We know that any graph G is a generalized lexicographic product of its canonical graph, i.e., $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$. We also have $G_c = G_k$ if G is reduced X-complete by Lemma 5.8. Furthermore, the following result prove that $4 \leq k \leq 13$.

Lemma 5.10. *Let $G \in \mathcal{G}^*$ be a reduced X-complete graph. Then there exists $4 \leq k \leq 13$ such that $G = G_k[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$.*

Proof. By Lemma 5.8, $G = G_k[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$ for some k. If $k = 1$ or 2 then $G \cong K_n \notin \mathcal{G}^*$, and so $k \geq 3$. If $k = 3$, then $G = G_3[K_{n_1}, K_{n_2}, K_{n_3}]$. Thus $\lambda_3(G) < 0$ by Lemma 5.9, a contradiction. Hence $k \geq 4$. On the other hand, since $G_c = G_k$ is an induced subgraph of G, we have $\lambda_4(G_k) \leq \lambda_4(G) < 0$ by Theorem 2.1. Note that G_{14} is an induced subgraph of G_k (by Remark 2.8) for $k \geq 15$, we have $\lambda_4(G_k) \geq \lambda_4(G_{14}) = 0$. It implies that $k \leq 13$. $\square$

Next we consider the converse of Lemma 5.10. In other words, we will try to find the values of $n_1, \ldots, n_k$ such that $p(G_k[K_{n_1}, \ldots, K_{n_k}]) = 2$ and $\eta(G_k[K_{n_1}, \ldots, K_{n_k}]) = 1$, where $4 \leq k \leq 13$ and $n = n_1 + n_2 + \cdots + n_k$. For the simplicity, we use notation in [8] to denote

$$G_{2s}[K_{n_1}, \ldots, K_{n_{2s}}] = B_{2s}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}) \quad \text{and}$$
$$G_{2s+1}[K_{n_1}, \ldots, K_{n_{2s+1}}] = B_{2s+1}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}; n_{2s+1}).$$

By Remark 3.2 in [6], we know

$$H_0 = B_{2s}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s})$$
$$\cong B_{2s}(n_{s+1} \ldots, n_{2s}; n_1, \ldots, n_s) = H_0' \quad \text{and}$$

$$H_1 = B_{2s+1}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}; n_{2s+1})$$
$$\cong B_{2s+1}(n_{s+1}, \ldots, n_{2s}; n_1, \ldots, n_s; n_{2s+1}) = H_1'.$$

In what follows, we always take H_0 and H_1, in which $(n_1, \ldots, n_s)$ is prior to $(n_{s+1}, \ldots, n_{2s})$ in dictionary ordering, instead of H_0' and H_1'. For example we use $B_6(4, 3, 2; 4, 3, 1)$ instead of $B_6(4, 3, 1; 4, 3, 2)$ and $B_7(5, 3, 2; 5, 2, 4; 8)$ instead of $B_7(5, 2, 4; 5, 3, 2; 8)$.

For $4 \leq k \leq 13$, let

$$\mathcal{B}_k(n) = \{G = B_k(n_1, \ldots, n_k) \mid n = n_1 + \cdots + n_k, n_i \geq 1\}.$$

Let $\mathcal{B}_k^+(n)$, $\mathcal{B}_k^{00}(n)$, $\mathcal{B}_k^0(n)$ and $\mathcal{B}_k^-(n)$ denote the set of graphs in $\mathcal{B}_k(n)$ satisfying $\lambda_3(G) > 0$ for $G \in \mathcal{B}_k^+(n)$, $\lambda_4(G) = \lambda_3(G) = 0$ for $G \in \mathcal{B}_k^{00}(n)$, $\lambda_4(G) < \lambda_3(G) = 0$ for $G \in \mathcal{B}_k^0(n)$ and $\lambda_3(G) < 0$ for $G \in \mathcal{B}_k^-(n)$, respectively. Clearly, $\mathcal{B}_k(n) = \mathcal{B}_k^+(n) \cup \mathcal{B}_k^{00}(n) \cup \mathcal{B}_k^0(n) \cup \mathcal{B}_k^-(n)$ is disjoint union and $G = G_k[K_{n_1}, K_{n_2}, \ldots, K_{n_k}] \in \mathcal{B}_k^0(n)$ if $G \in \mathcal{G}^*$ is a reduced X-complete graph by Lemma 5.10. In what follows, we further show that $n \leq 13$. First, one can verify the following result by using computer.

Lemma 5.11. $\mathcal{B}_k^0(14) = \emptyset$ *for $4 \leq k \leq 13$ (it means that there are no reduced X-complete graphs of order* 14).

Proof. For $4 \leq k \leq 13$, the k-partition of 14 gives a solution $(n_1, n_2, \ldots, n_k)$ of the equation $n_1 + n_2 + \cdots + n_k = 14$ that corresponds a graph $G = B_k(n_1, n_2, \ldots, n_k) \in \mathcal{B}_k(14)$. By using computer, we exhaust all the graphs of $\mathcal{B}_k(14)$ to find that there is no any graph $G \in \mathcal{B}_k(14)$ with $\lambda_4(G) < \lambda_3(G) = 0$. It implies that $\mathcal{B}_k^0(14) = \emptyset$. $\square$

In [6], Oboudi gave all the integers $n_1, \ldots, n_k$ satisfying $\lambda_2(B_k(n_1, \ldots, n_k)) > 0$ and $\lambda_3(B_k(n_1, \ldots, n_k)) < 0$ for $4 \leq k \leq 9$. For simplicity, we only cite this result for $k = 5$ and the others are listed in Appendix B.

Theorem 5.12 ([6]). *Let $G = B_5(n_1, n_2; n_3, n_4; n_5)$, where n_1, n_2, n_3, n_4, n_5 are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_5(a, w; 1, 1; 1)$;   *(6) $B_5(a, 1; x, w; 1)$;*   *(11) $B_5(x, w; 1, d; 1)$;*

(2) $B_5(a, x; 1, d; 1)$;   *(7) $B_5(a, 1; x, y; e)$;*   *(12) $B_5(x, w; 1, 1; e)$;*

(3) $B_5(a, x; 1, y; z)$;   *(8) $B_5(a, 1; 1, d; e)$;*   *(13) $B_5(1, b; 1, d; 1)$;*

(4) $B_5(a, x; 1, 1; e)$;   *(9) $B_5(w, x; y, 1; e)$;*   *(14) $B_5(1, b; 1, x; y)$;*

(5) $B_5(a, 1; c, 1; e)$;   *(10) $B_5(x, b; 1, 1; 1)$;*   *(15) $B_5(1, x; 1, y; e)$;*

(16) 63 specific graphs: 13 graphs of order 10, 25 graphs of order 11, and 25 graphs of order 12,

where $a, b, c, d, e, x, y, z, w$ are some positive integers such that $x \leq 2$, $y \leq 2$, $z \leq 2$ and $w \leq 3$.

Lemma 5.13. *Let $G \in \mathcal{B}_k(n)$, where $4 \leq k \leq 9$ and $n \geq 14$. If $G \notin \mathcal{B}_k^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$.*

Proof. We prove this lemma by induction on n. If $n = 14$, since $G \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$, our result is obviously true by taking $\Gamma = G$. Let $n \geq 15$ and $G' \in \mathcal{B}_k(n-1)$ be an induced subgraph of G. If $G' \notin \mathcal{B}_k^-(n-1)$, then G' has an induced subgraph $\Gamma \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$ by induction hypothesis, and so does G. Hence it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_k(n-1) \setminus \mathcal{B}_k^-(n-1)$ for $n \geq 15$ in the following. We will prove that there exists $G' \in \mathcal{B}_5(n-1) \setminus \mathcal{B}_5^-(n-1)$ for $n \geq 15$, and it can be similarly proved for the other k which we keep in the Appendix B.

Let $G = B_5(n_1, n_2; n_3, n_4; n_5) \in \mathcal{B}_5(n)$. Then one of

$$H_1 = B_5(n_1 - 1, n_2; n_3, n_4; n_5), \qquad H_2 = B_5(n_1, n_2 - 1; n_3, n_4; n_5),$$
$$H_3 = B_5(n_1, n_2; n_3 - 1, n_4; n_5), \qquad H_4 = B_5(n_1, n_2; n_3, n_4 - 1; n_5) \text{ and}$$
$$H_5 = B_5(n_1, n_2; n_3, n_4; n_5 - 1)$$

must belong to $\mathcal{B}_5(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_5^-(n-1)$ for $i = 1, 2, \ldots, 5$. Then H_i is a graph belonging to (1)–(15) in Theorem 5.12 since $|H_i| = n - 1 \geq 14$.

First we consider H_1. If H_1 is a graph belonging to (1) of Theorem 5.12, then $H_1 = B_5(a, w; 1, 1; 1)$ where $n_1 - 1 = a, n_2 = w, n_3 = n_4 = n_5 = 1$, and hence $G = B_5(a+1, w; 1, 1; 1) \in \mathcal{B}_5^-(n)$, a contradiction. Similarly, H_1 cannot belong to (2)–(8) of Theorem 5.12. If H_1 is a graph belonging to (9) of Theorem 5.12, then $H_1 = B_5(w, x; y, 1; e)$ where $n_1 - 1 = w, n_2 = x, n_3 = y, n_4 = 1, n_5 = e$. Since $w \leq 3$, we have $n_1 \leq 4$. If $n_1 < 4$ then $w + 1 \leq 3$ and $G = B_5(w + 1, x; y, 1; e) \in \mathcal{B}_5^-(n)$, a contradiction. Now assume that $n_1 = 4$. Then $H_1 = B_5(3, x; y, 1; e)$. Since $x, y \in \{1, 2\}$, we have $G \in \{B_5(4, 1; 1, 1; e), B_5(4, 2; 1, 1; e), B_5(4, 1; 2, 1; e), B_5(4, 2; 2, 1; e)\}$. However $B_5(4, 1; 1, 1; e), B_5(4, 2; 1, 1; e), B_5(4, 1; 2, 1; e)$ belong to (4), (5) of Theorem 5.12 which contradicts our assumption. Thus $G = B_5(4, 2; 2, 1; e)$. By Theorem 5.12, $G =$

$B_5(4, 2; 2, 1; e) \notin \mathcal{B}_5^-(n)$, and also its induced subgraph $B_5(4, 2; 2, 1; e-1) \notin \mathcal{B}_5^-(n-1)$, a contradiction. Hence H_1 belongs to (10) – (15) of Theorem 5.12, from which we see that $n_1 - 1$ is either x or 1. Thus $n_1 \leq 3$ due to $x \leq 2$.

By the same method, we can verify that $n_2 \leq 3$ if $H_2 \in \mathcal{B}_5^-(n-1)$; $n_3 \leq 3$ if $H_3 \in \mathcal{B}_5^-(n-1)$; $n_4 \leq 3$ if $H_4 \in \mathcal{B}_5^-(n-1)$ and $n_5 \leq 2$ if $H_5 \in \mathcal{B}_5^-(n-1)$. Hence $n = n_1 + \cdots + n_5 \leq 14$, a contradiction. We are done. □

Lemma 5.14 ([6])**.** *If* $n \geq 14$, *then* $\mathcal{B}_k^-(n) = \emptyset$ *for* $10 \leq k \leq 13$.

Lemma 5.15. *Given* $4 \leq k \leq 13$, $\mathcal{B}_k^0(n) = \emptyset$ *for* $n \geq 14$ (*it means that there are no reduced* X-*complete graphs of order* $n \geq 14$).

Proof. Let $G \in \mathcal{B}_k^0(n)$ and $n \geq 14$. Then $\lambda_4(G) < \lambda_3(G) = 0$. First we assume that $4 \leq k \leq 9$. Since $G \notin \mathcal{B}_k^-(n)$, G has an induced subgraphs $\Gamma \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$ by Lemma 5.13. Thus $\lambda_3(\Gamma) \geq 0$. Furthermore, we have $\lambda_3(\Gamma) = 0$ since otherwise $0 < \lambda_3(\Gamma) \leq \lambda_3(G)$. Additionally, $\lambda_4(\Gamma) \leq \lambda_4(G) < 0$, we have $\Gamma \in \mathcal{B}_k^0(14)$, contrary to Lemma 5.11. Next we assume that $10 \leq k \leq 13$. By deleting $n - 14$ vertices from G, we may obtain an induced subgraph $\Gamma \in \mathcal{B}_k(14)$. By Lemma 5.14, we have $\lambda_3(\Gamma) \geq 0$, and then $\lambda_3(\Gamma) = 0$ by the arguments above. Additionally, $\lambda_4(\Gamma) \leq \lambda_4(G) < 0$, we have $\Gamma \in \mathcal{B}_k^0(14)$ which also contradicts Lemma 5.11. □

By Lemma 5.15, we know that, for any reduced X-complete graph $G \in \mathcal{G}^*$, there exists $4 \leq k \leq 13$ and $n \leq 13$ such that $G \in \mathcal{B}_k^0(n)$. Let

$$\mathcal{B}^* = \{G = B_k(n_1, n_2, \ldots, n_k) \in \mathcal{B}_k^0(n) \mid 4 \leq k \leq 13 \text{ and } n \leq 13\}.$$

Thus $G \in \mathcal{G}^*$ is a reduced X-complete graph if and only if $G \in \mathcal{B}^*$.

Remark 5.16. Clearly, $\mathcal{B} = \cup_{4 \leq k \leq 13, n \leq 13} \mathcal{B}_k(n)$ contains finite graphs. By using computer we can exhaust all the graphs of $\mathcal{B}$ to find out the graphs in $\mathcal{B}^*$. We list them in Table 1.

Recall that $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of I, II, III-type, respectively. Summarizing Lemmas 5.2, 5.3, 5.4, 5.15 and Theorem 4.2, finally we give the characterization of the connected graphs in $\mathcal{G}$.

Theorem 5.17. *Let* G *be a connected graph of order* $n \geq 5$. *Then* $G \in \mathcal{G}$ *if and only if* G *is isomorphic to one of the following graphs listed in (1), (2) and (3):*

(1) $K_{1,2}(u) \odot^k K_{n-3}$ *or* $K_{1,1}(u) \odot^k K_{n-2} \setminus e$ *for* $e \in E(K_{n-2})$;

(2) *the graphs belonging to* $\mathcal{G}_1, \mathcal{G}_2$ *or* $\mathcal{G}_3$;

(3) *the* 802 *specific graphs belonging to* $\mathcal{B}^*$ *some of which we list in Table 1.*

If G^* is obtained from $G \in \mathcal{G}$ by adding one vertex of I, II or III-type, then the positive and negative indices of G^* left unchanged, but the nullity adds just one. Repeating this process, we can get a class of graphs which has two positive eigenvalues and s zero eigenvalues, where $s \geq 2$ is any integer. However, by using the I, II and III-type (graph) transformations, we can not get all such graphs. For example, $H = B_{10}(1, 1, 2, 3, 2; 1, 1, 1, 1, 1)$ is a graph satisfying $p(H) = 2$ and $\eta(H) = 2$ that can not be constructed by above (graph) transformation. Hence the characterization of graphs with $p(H) = 2$ and $\eta(H) = s$ (especially $\eta(H) = 2$) is also an attractive problem.

Table 1: All graphs of $\mathcal{B}^*$.

k	$\mathcal{B}^*$	Number
4	$B_4(3,2;3,2)$; $B_4(4,3;2,2)$, $B_4(4,3;3,1)$; $B_4(5,4;2,1)$, $B_4(5,2;2,3)$, $B_4(3,4;2,3)$, $B_4(4,1;3,4)$, $B_4(5,2;4,1)$; $B_4(7,3;2,1)$, $B_4(4,6;2,1)$, $B_4(7,2;2,2)$, $B_4(3,6;2,2)$, $B_4(4,2;2,5)$, $B_4(3,3;2,5)$, $B_4(7,2;3,1)$, $B_4(3,6;3,1)$, $B_4(6,1;3,3)$, $B_4(6,1;4,2)$.	18
5	$B_5(2,2;2,2;1)$; $B_5(2,3;1,2;2)$, $B_5(3,3;2,1;1)$; $B_5(3,4;1,1;2)$, $B_5(3,4;1,2;1)$, $B_5(1,3;1,3;3)$, $B_5(2,2;1,3;3)$, $B_5(2,4;2,1;2)$, $B_5(4,2;3,1;1)$; $B_5(4,5;1,1;1)$, $B_5(2,5;1,1;3)$, $B_5(4,3;1,1;3)$, $B_5(1,4;1,2;4)$, $B_5(3,2;1,2;4)$, $B_5(2,5;1,3;1)$, $B_5(4,3;1,3;1)$, $B_5(1,4;1,4;2)$, $B_5(3,2;1,4;2)$, $B_5(5,2;2,1;2)$, $B_5(3,1;2,3;3)$, $B_5(3,1;2,5;1)$, $B_5(4,1;3,2;2)$; $B_5(3,7;1,1;1)$, $B_5(6,4;1,1;1)$, $B_5(2,7;1,1;2)$, $B_5(6,3;1,1;2)$, $B_5(2,4;1,1;5)$, $B_5(3,3;1,1;5)$, $B_5(2,7;1,2;1)$, $B_5(6,3;1,2;1)$, $B_5(1,6;1,2;3)$, $B_5(5,2;1,2;3)$, $B_5(1,3;1,2;6)$, $B_5(2,2;1,2;6)$, $B_5(1,6;1,3;2)$, $B_5(5,2;1,3;2)$, $B_5(2,4;1,5;1)$, $B_5(3,3;1,5;1)$, $B_5(2,2;1,6;2)$, $B_5(2,7;2,1;1)$, $B_5(7,2;2,1;1)$, $B_5(4,2;2,1;4)$, $B_5(2,3;2,1;5)$, $B_5(5,1;2,3;2)$, $B_5(5,1;2,4;1)$, $B_5(3,2;3,1;4)$, $B_5(6,1;3,2;1)$.	47
6	See Table 2 of Appendix A	138
7	See Table 3 of Appendix A	161
8	See Table 4 of Appendix A	205
9	See Table 5 of Appendix A	124
10	See Table 6 of Appendix A	78
11	$B_{11}(1,1,1,2,1;1,1,1,1,1;1)$, $B_{11}(2,1,1,1,1;1,1,1,1,1;1)$; $B_{11}(1,1,1,1,3;1,1,1,1,1;1)$, $B_{11}(1,1,1,2,2;1,1,1,1,1;1)$, $B_{11}(1,1,2,1,2;1,1,1,1,1;1)$, $B_{11}(1,1,2,2,1;1,1,1,1,1;1)$, $B_{11}(1,1,3,1,1;1,1,1,1,1;1)$, $B_{11}(1,2,1,1,2;1,1,1,1,1;1)$, $B_{11}(1,2,2,1,1;1,1,1,1,1;1)$, $B_{11}(1,3,1,1,1;1,1,1,1,1;1)$, $B_{11}(2,1,1,1,2;1,1,1,1,1;1)$, $B_{11}(2,2,1,1,1;1,1,1,1,1;1)$, $B_{11}(1,1,1,1,2;1,1,1,1,1;2)$, $B_{11}(1,1,2,1,1;1,1,1,1,1;2)$, $B_{11}(1,2,1,1,1;1,1,1,1,1;2)$, $B_{11}(1,1,1,1,1;1,1,1,1,1;3)$, $B_{11}(1,1,1,1,2;1,1,1,1,2;1)$, $B_{11}(1,1,1,2,1;1,1,1,1,2;1)$, $B_{11}(1,1,2,1,1;1,1,1,1,2;1)$, $B_{11}(1,2,1,1,1;1,1,1,1,2;1)$, $B_{11}(1,1,2,1,1;1,1,1,2,1,1;1)$, $B_{11}(1,2,1,1,1;1,1,2,1,1;1)$, $B_{11}(2,1,1,1,1;1,1,1,2,1,1;1)$, $B_{11}(1,2,1,1,1;1,2,1,1,1;1)$.	24
12	$B_{12}(1,1,1,1,1,2;1,1,1,1,1,1)$, $B_{12}(1,1,1,1,2,1;1,1,1,1,1,1)$, $B_{12}(1,1,1,2,1,1;1,1,1,1,1,1)$, $B_{12}(1,1,2,1,1,1;1,1,1,1,1,1)$, $B_{12}(1,2,1,1,1,1;1,1,1,1,1,1)$, $B_{12}(2,1,1,1,1,1;1,1,1,1,1,1)$.	6
13	$B_{13}(1,1,1,1,1,1;1,1,1,1,1,1;1)$.	1

References

[1] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Application*, volume 87 of *Pure and Applied Mathematics*, Academic Press, New York, 1980.

[2] C. Godsil and G. Royle, *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathemat-*

ics, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9.

[3] H. Ma, W. Yang and S. Li, Positive and negative inertia index of a graph, *Linear Algebra Appl.* **438** (2013), 331–341, doi:10.1016/j.laa.2012.07.014.

[4] M. R. Oboudi, Bipartite graphs with at most six non-zero eigenvalues, *Ars Math. Contemp.* **11** (2016), 315–325, doi:10.26493/1855-3974.749.264.

[5] M. R. Oboudi, On the third largest eigenvalue of graphs, *Linear Algebra Appl.* **503** (2016), 164–179, doi:10.1016/j.laa.2016.03.037.

[6] M. R. Oboudi, Characterization of graphs with exactly two non-negative eigenvalues, *Ars Math. Contemp.* **12** (2017), 271–286, doi:10.26493/1855-3974.1077.5b6.

[7] M. Petrović, On graphs with exactly one eigenvalue less than -1, *J. Comb. Theory Ser. B* **52** (1991), 102–112, doi:10.1016/0095-8956(91)90096-3.

[8] M. Petrović, Graphs with a small number of nonnegative eigenvalues, *Graphs Combin.* **15** (1999), 221–232, doi:10.1007/s003730050042.

[9] J. H. Smith, Symmetry and multiple eigenvalues of graphs, *Glasnik Mat. Ser. III* **12** (1977), 3–8, https://books.google.com/books?id=I1nZvGaOcscC&pg=PA3.

[10] A. Torgašev, On graphs with a fixed number of negative eigenvalues, *Discrete Math.* **57** (1985), 311–317, doi:10.1016/0012-365x(85)90184-0.

[11] G. Yu, L. Feng and H. Qu, Signed graphs with small positive index of inertia, *Electron. J. Linear Algebra* **31** (2016), 232–243, doi:10.13001/1081-3810.1976.

Appendix A Five tables

Appendix A contains 5 tables, in which there are 706 specific graphs: 4 graphs of order 10, 32 graphs of order 11, 150 graphs of order 12, and 520 graphs of order 13.

Table 2: $k = 6$.

n	$\mathcal{B}^*$
10	$B_6(1,2,2;1,2,2)$, $B_6(2,2,1;1,2,2)$;
11	$B_6(1,3,3;1,1,2)$, $B_6(2,3,2;1,1,2)$, $B_6(3,3,1;1,1,2)$, $B_6(1,3,3;1,2,1)$, $B_6(2,3,2;1,2,1)$, $B_6(3,3,1;1,2,1)$, $B_6(2,1,1;1,3,3)$, $B_6(3,2,1;2,1,2)$, $B_6(2,2,2;2,2,1)$, $B_6(3,1,2;3,1,1)$;
12	$B_6(1,4,4;1,1,1)$, $B_6(2,4,3;1,1,1)$, $B_6(3,4,2;1,1,1)$, $B_6(4,4,1;1,1,1)$, $B_6(1,2,4;1,1,3)$, $B_6(1,4,2;1,1,3)$, $B_6(2,2,3;1,1,3)$, $B_6(2,4,1;1,1,3)$, $B_6(3,2,2;1,1,3)$, $B_6(4,2,1;1,1,3)$, $B_6(1,3,1;1,2,4)$, $B_6(2,1,2;1,2,4)$, $B_6(3,1,1;1,2,4)$, $B_6(1,4,2;1,3,1)$, $B_6(2,2,3;1,3,1)$, $B_6(2,4,1;1,3,1)$, $B_6(3,2,2;1,3,1)$, $B_6(4,2,1;1,3,1)$, $B_6(2,1,2;1,4,2)$, $B_6(3,1,1;1,4,2)$, $B_6(2,3,3;2,1,1)$, $B_6(4,1,3;2,1,1)$, $B_6(4,3,1;2,1,1)$, $B_6(2,3,2;2,1,2)$, $B_6(3,2,2;2,1,2)$, $B_6(4,1,2;2,1,2)$, $B_6(2,3,1;2,1,3)$, $B_6(4,1,1;2,1,3)$, $B_6(3,1,3;2,2,1)$, $B_6(3,2,2;2,2,1)$, $B_6(3,3,1;2,2,1)$, $B_6(3,1,1;2,2,3)$, $B_6(2,3,1;2,3,1)$, $B_6(3,2,2;3,1,1)$, $B_6(4,2,1;3,1,1)$, $B_6(4,1,1;4,1,1)$;
13	$B_6(1,3,6;1,1,1)$, $B_6(1,6,3;1,1,1)$, $B_6(2,3,5;1,1,1)$, $B_6(2,6,2;1,1,1)$, $B_6(3,3,4;1,1,1)$, $B_6(3,6,1;1,1,1)$, $B_6(4,3,3;1,1,1)$, $B_6(5,3,2;1,1,1)$, $B_6(6,3,1;1,1,1)$, $B_6(1,2,6;1,1,2)$, $B_6(1,6,2;1,1,2)$, $B_6(2,2,5;1,1,2)$, $B_6(2,6,1;1,1,2)$, $B_6(3,2,4;1,1,2)$, $B_6(4,2,3;1,1,2)$, $B_6(5,2,2;1,1,2)$, $B_6(6,2,1;1,1,2)$, $B_6(1,2,3;1,1,5)$, $B_6(1,3,2;1,1,5)$, $B_6(2,2,2;1,1,5)$, $B_6(2,2,5;1,2,1)$, $B_6(2,6,1;1,2,1)$, $B_6(3,2,4;1,2,1)$, $B_6(4,2,3;1,2,1)$, $B_6(5,2,2;1,2,1)$, $B_6(2,3,1;1,1,5)$, $B_6(3,2,1;1,1,5)$, $B_6(1,2,6;1,2,1)$, $B_6(1,6,2;1,2,1)$, $B_6(6,2,1;1,2,1)$, $B_6(1,5,1;1,2,3)$, $B_6(2,1,4;1,2,3)$, $B_6(3,1,3;1,2,3)$, $B_6(4,1,2;1,2,3)$, $B_6(5,1,1;1,2,3)$, $B_6(2,1,1;1,2,6)$, $B_6(1,5,1;1,3,2)$, $B_6(2,1,4;1,3,2)$, $B_6(3,1,3;1,3,2)$, $B_6(4,1,2;1,3,2)$, $B_6(5,1,1;1,3,2)$, $B_6(2,2,2;1,5,1)$, $B_6(2,3,1;1,5,1)$, $B_6(3,2,1;1,5,1)$, $B_6(2,1,1;1,6,2)$, $B_6(2,2,5;2,1,1)$, $B_6(2,5,2;2,1,1)$, $B_6(3,1,5;2,1,1)$, $B_6(3,2,4;2,1,1)$, $B_6(3,3,3;2,1,1)$, $B_6(3,4,2;2,1,1)$, $B_6(3,5,1;2,1,1)$, $B_6(4,2,3;2,1,1)$, $B_6(4,3,2;2,1,1)$, $B_6(5,2,2;2,1,1)$, $B_6(6,1,2;2,1,1)$, $B_6(6,2,1;2,1,1)$, $B_6(2,2,4;2,1,2)$, $B_6(2,5,1;2,1,2)$, $B_6(3,1,4;2,1,2)$, $B_6(3,2,3;2,1,2)$, $B_6(6,1,1;2,1,2)$, $B_6(2,2,3;2,1,3)$, $B_6(3,1,3;2,1,3)$, $B_6(2,2,2;2,1,4)$, $B_6(3,1,2;2,1,4)$, $B_6(2,2,1;2,1,5)$, $B_6(3,1,1;2,1,5)$, $B_6(2,5,1;2,2,1)$, $B_6(4,2,2;2,2,1)$, $B_6(5,1,2;2,2,1)$, $B_6(5,2,1;2,2,1)$, $B_6(3,1,3;2,2,2)$, $B_6(4,1,2;2,2,2)$, $B_6(5,1,1;2,2,2)$, $B_6(3,1,2;2,2,3)$, $B_6(4,1,2;2,3,1)$, $B_6(4,2,1;2,3,1)$, $B_6(3,1,2;2,3,2)$, $B_6(4,1,1;2,3,2)$, $B_6(3,1,2;2,4,1)$, $B_6(3,2,1;2,4,1)$, $B_6(3,1,1;2,4,2)$, $B_6(3,3,2;3,1,1)$, $B_6(3,4,1;3,1,1)$, $B_6(6,1,1;3,1,1)$, $B_6(3,3,1;3,2,1)$, $B_6(4,2,1;3,2,1)$, $B_6(5,1,1;3,2,1)$, $B_6(4,1,1;3,3,1)$.

Table 3: $k = 7$.

n	$\mathcal{B}^*$
10	$B_7(2,2,1;1,1,2;1)$, $B_7(2,1,2;2,1,1;1)$;
11	$B_7(3,3,1;1,1,1;1)$, $B_7(2,1,3;1,1,1;2)$, $B_7(2,2,2;1,1,2;1)$, $B_7(2,1,2;1,1,2;2)$, $B_7(1,2,1;1,1,3;2)$, $B_7(2,1,1;1,1,3;2)$, $B_7(1,2,3;1,2,1;1)$, $B_7(1,2,2;1,2,2;1)$, $B_7(2,1,1;1,2,3;1)$, $B_7(2,2,2;2,1,1;1)$, $B_7(3,2,1;2,1,1;1)$, $B_7(3,1,1;3,1,1;1)$;
12	$B_7(1,3,4;1,1,1;1)$, $B_7(3,1,4;1,1,1;1)$, $B_7(3,3,2;1,1,1;1)$, $B_7(1,2,4;1,1,1;2)$, $B_7(2,2,3;1,1,1;2)$, $B_7(2,4,1;1,1,1;2)$, $B_7(3,2,2;1,1,1;2)$, $B_7(4,2,1;1,1,1;2)$, $B_7(1,1,4;1,1,1;3)$, $B_7(3,1,2;1,1,1;3)$, $B_7(1,3,3;1,1,2;1)$, $B_7(2,2,3;1,1,2;1)$, $B_7(3,1,3;1,1,2;1)$, $B_7(1,1,3;1,1,2;3)$, $B_7(1,3,1;1,1,2;3)$, $B_7(3,1,1;1,1,2;3)$, $B_7(1,3,2;1,1,3;1)$, $B_7(3,1,2;1,1,3;1)$, $B_7(1,2,2;1,1,3;2)$, $B_7(1,3,1;1,1,4;1)$, $B_7(3,1,1;1,1,4;1)$, $B_7(2,1,4;1,2,1;1)$, $B_7(2,2,3;1,2,1;1)$, $B_7(2,3,2;1,2,1;1)$, $B_7(4,2,1;1,2,1;1)$, $B_7(2,1,2;1,2,1;3)$, $B_7(1,2,2;1,2,2;2)$, $B_7(1,3,1;1,2,2;2)$, $B_7(2,4,1;1,2,1;1)$, $B_7(2,1,2;1,2,2;2)$, $B_7(3,1,1;1,2,2;2)$, $B_7(2,1,2;1,2,3;1)$, $B_7(1,3,2;1,3,1;1)$, $B_7(3,1,1;1,3,2;1)$, $B_7(2,3,2;2,1,1;1)$, $B_7(2,3,1;2,1,1;2)$, $B_7(4,1,1;2,1,1;2)$, $B_7(3,2,1;2,2,1;1)$, $B_7(3,1,1;2,2,1;2)$;
13	$B_7(1,2,6;1,1,1;1)$, $B_7(1,5,3;1,1,1;1)$, $B_7(2,1,6;1,1,1;1)$, $B_7(2,2,5;1,1,1;1)$, $B_7(2,3,4;1,1,1;1)$, $B_7(2,4,3;1,1,1;1)$, $B_7(2,5,2;1,1,1;1)$, $B_7(2,6,1;1,1,1;1)$, $B_7(3,2,4;1,1,1;1)$, $B_7(3,3,3;1,1,1;1)$, $B_7(4,2,3;1,1,1;1)$, $B_7(5,1,3;1,1,1;1)$, $B_7(5,2,2;1,1,1;1)$, $B_7(6,2,1;1,1,1;1)$, $B_7(1,1,6;1,1,1;2)$, $B_7(1,4,3;1,1,1;2)$, $B_7(2,3,3;1,1,1;2)$, $B_7(2,4,2;1,1,1;2)$, $B_7(5,1,2;1,1,1;2)$, $B_7(1,3,3;1,1,1;3)$, $B_7(2,3,2;1,1,1;3)$, $B_7(1,2,3;1,1,1;4)$, $B_7(2,2,2;1,1,1;4)$, $B_7(2,3,1;1,1,1;4)$, $B_7(3,2,1;1,1,1;4)$, $B_7(1,1,3;1,1,1;5)$, $B_7(2,1,2;1,1,1;5)$, $B_7(1,2,5;1,1,2;1)$, $B_7(1,5,2;1,1,2;1)$, $B_7(2,1,5;1,1,2;1)$, $B_7(2,2,4;1,1,2;1)$, $B_7(5,1,2;1,1,2;1)$, $B_7(1,1,5;1,1,2;2)$, $B_7(1,2,4;1,1,2;2)$, $B_7(1,3,3;1,1,2;2)$, $B_7(1,4,2;1,1,2;2)$, $B_7(1,5,1;1,1,2;2)$, $B_7(5,1,1;1,1,2;2)$, $B_7(1,2,3;1,1,2;3)$, $B_7(1,3,2;1,1,2;3)$, $B_7(1,2,2;1,1,2;4)$, $B_7(1,1,2;1,1,2;5)$, $B_7(1,2,1;1,1,2;5)$, $B_7(2,1,1;1,1,2;5)$, $B_7(1,2,4;1,1,3;1)$, $B_7(1,5,1;1,1,3;1)$, $B_7(2,1,4;1,1,3;1)$, $B_7(5,1,1;1,1,3;1)$, $B_7(1,1,4;1,1,3;2)$, $B_7(1,2,3;1,1,3;2)$, $B_7(1,2,3;1,1,4;1)$, $B_7(2,1,3;1,1,4;1)$, $B_7(1,2,2;1,1,5;1)$, $B_7(2,1,2;1,1,5;1)$, $B_7(1,2,1;1,1,6;1)$, $B_7(2,1,1;1,1,6;1)$, $B_7(1,5,2;1,2,1;1)$, $B_7(3,2,3;1,2,1;1)$, $B_7(4,1,3;1,2,1;1)$, $B_7(4,2,2;1,2,1;1)$, $B_7(1,4,2;1,2,1;2)$, $B_7(2,3,2;1,2,1;2)$, $B_7(3,2,2;1,2,1;2)$, $B_7(4,1,2;1,2,1;2)$, $B_7(1,3,2;1,2,1;3)$, $B_7(2,2,2;1,2,1;3)$, $B_7(2,3,1;1,2,1;3)$, $B_7(3,2,1;1,2,1;3)$, $B_7(1,2,2;1,2,1;4)$, $B_7(1,5,1;1,2,2;1)$, $B_7(2,1,4;1,2,2;1)$, $B_7(3,1,3;1,2,2;1)$, $B_7(4,1,2;1,2,2;1)$, $B_7(5,1,1;1,2,2;1)$, $B_7(1,2,2;1,2,2;3)$, $B_7(2,1,1;1,2,2;4)$, $B_7(2,1,3;1,2,3;1)$, $B_7(3,1,3;1,3,1;1)$, $B_7(3,2,2;1,3,1;1)$, $B_7(2,2,2;1,3,1;2)$, $B_7(2,3,1;1,3,1;2)$, $B_7(3,1,2;1,3,1;2)$, $B_7(3,2,1;1,3,1;2)$, $B_7(2,1,3;1,3,2;1)$, $B_7(3,1,2;1,3,2;1)$, $B_7(2,1,2;1,3,2;2)$, $B_7(2,1,1;1,3,2;3)$, $B_7(2,1,3;1,4,1;1)$, $B_7(2,2,2;1,4,1;1)$, $B_7(2,3,1;1,4,1;1)$, $B_7(3,2,1;1,4,1;1)$, $B_7(2,1,2;1,4,1;2)$, $B_7(2,1,2;1,4,2;1)$, $B_7(2,1,1;1,4,2;2)$, $B_7(2,1,1;1,5,2;1)$, $B_7(2,4,2;2,1,1;1)$, $B_7(2,5,1;2,1,1;1)$, $B_7(6,1,1;2,1,1;1)$, $B_7(2,2,1;2,1,1;4)$, $B_7(3,1,1;2,1,1;4)$, $B_7(2,4,1;2,2,1;1)$, $B_7(5,1,1;2,2,1;1)$, $B_7(2,3,1;2,2,1;2)$, $B_7(2,2,1;2,2,1;3)$, $B_7(2,3,1;2,3,1;1)$, $B_7(3,2,1;2,3,1;1)$, $B_7(4,1,1;2,3,1;1)$, $B_7(3,1,1;2,4,1;1)$.

Table 4: $k = 8$.

n	$\mathcal{B}^*$
11	$B_8(1,2,1,2;1,1,1,2)$, $B_8(2,2,1,1;1,1,1,2)$, $B_8(1,2,2,1;1,1,2,1)$, $B_8(1,2,1,1;1,1,2,2)$, $B_8(2,1,2,1;1,2,1,1)$, $B_8(2,1,1,1;1,2,1,2)$;
12	$B_8(1,1,3,3;1,1,1,1)$, $B_8(1,3,1,3;1,1,1,1)$, $B_8(1,3,3,1;1,1,1,1)$, $B_8(2,1,3,2;1,1,1,1)$, $B_8(2,3,1,2;1,1,1,1)$, $B_8(3,1,3,1;1,1,1,1)$, $B_8(3,3,1,1;1,1,1,1)$, $B_8(1,1,2,3;1,1,1,2)$, $B_8(1,2,2,2;1,1,1,2)$, $B_8(1,3,2,1;1,1,1,2)$, $B_8(2,1,2,2;1,1,1,2)$, $B_8(2,2,2,1;1,1,1,2)$, $B_8(3,1,2,1;1,1,1,2)$, $B_8(1,1,1,3;1,1,1,3)$, $B_8(1,3,1,1;1,1,1,3)$, $B_8(2,1,1,2;1,1,1,3)$, $B_8(3,1,1,1;1,1,1,3)$, $B_8(1,1,3,2;1,1,2,1)$, $B_8(1,2,2,2;1,1,2,1)$, $B_8(1,3,1,2;1,1,2,1)$, $B_8(2,1,3,1;1,1,2,1)$, $B_8(2,2,2,1;1,1,2,1)$, $B_8(2,3,1,1;1,1,2,1)$, $B_8(2,1,1,1;1,1,2,3)$, $B_8(1,1,3,1;1,1,3,1)$, $B_8(1,3,1,1;1,1,3,1)$, $B_8(1,2,1,3;1,2,1,1)$, $B_8(1,2,2,2;1,2,1,1)$, $B_8(1,2,3,1;1,2,1,1)$, $B_8(2,2,1,2;1,2,1,1)$, $B_8(2,2,2,1;1,2,1,1)$, $B_8(3,2,1,1;1,2,1,1)$, $B_8(2,1,1,1;1,2,2,2)$, $B_8(2,1,1,2;1,3,1,1)$, $B_8(3,1,1,1;1,3,1,1)$, $B_8(2,1,1,1;1,3,2,1)$, $B_8(2,2,1,2;2,1,1,1)$, $B_8(3,1,2,1;2,1,1,1)$, $B_8(2,2,1,1;2,1,1,2)$, $B_8(3,1,1,1;2,1,1,2)$, $B_8(2,1,2,1;2,1,2,1)$, $B_8(2,2,1,1;2,1,2,1)$, $B_8(3,1,1,1;3,1,1,1)$;
13	$B_8(1,1,2,5;1,1,1,1)$, $B_8(1,1,5,2;1,1,1,1)$, $B_8(1,2,1,5;1,1,1,1)$, $B_8(1,2,2,4;1,1,1,1)$, $B_8(1,2,3,3;1,1,1,1)$, $B_8(1,2,4,2;1,1,1,1)$, $B_8(1,2,5,1;1,1,1,1)$, $B_8(1,3,2,3;1,1,1,1)$, $B_8(1,3,3,2;1,1,1,1)$, $B_8(1,4,2,2;1,1,1,1)$, $B_8(1,5,1,2;1,1,1,1)$, $B_8(1,5,2,1;1,1,1,1)$, $B_8(2,1,2,4;1,1,1,1)$, $B_8(2,1,5,1;1,1,1,1)$, $B_8(2,2,1,4;1,1,1,1)$, $B_8(2,2,2,3;1,1,1,1)$, $B_8(2,2,3,2;1,1,1,1)$, $B_8(2,2,4,1;1,1,1,1)$, $B_8(2,3,2,2;1,1,1,1)$, $B_8(2,3,3,1;1,1,1,1)$, $B_8(2,4,2,1;1,1,1,1)$, $B_8(2,5,1,1;1,1,1,1)$, $B_8(3,1,2,3;1,1,1,1)$, $B_8(3,2,1,3;1,1,1,1)$, $B_8(3,2,2,2;1,1,1,1)$, $B_8(3,2,3,1;1,1,1,1)$, $B_8(3,3,2,1;1,1,1,1)$, $B_8(4,1,2,2;1,1,1,1)$, $B_8(4,2,1,2;1,1,1,1)$, $B_8(4,2,2,1;1,1,1,1)$, $B_8(5,1,2,1;1,1,1,1)$, $B_8(5,2,1,1;1,1,1,1)$, $B_8(1,1,1,5;1,1,1,2)$, $B_8(1,1,4,2;1,1,1,2)$, $B_8(1,2,3,2;1,1,1,2)$, $B_8(1,2,4,1;1,1,1,2)$, $B_8(1,5,1,1;1,1,1,2)$, $B_8(2,1,1,4;1,1,1,2)$, $B_8(2,1,4,1;1,1,1,2)$, $B_8(2,2,3,1;1,1,1,2)$, $B_8(3,1,1,3;1,1,1,2)$, $B_8(4,1,1,2;1,1,1,2)$, $B_8(5,1,1,1;1,1,1,2)$, $B_8(1,1,3,2;1,1,1,3)$, $B_8(1,2,3,1;1,1,1,3)$, $B_8(2,1,3,1;1,1,1,3)$, $B_8(1,1,2,2;1,1,1,4)$, $B_8(1,2,2,1;1,1,1,4)$, $B_8(2,1,2,1;1,1,1,4)$, $B_8(1,2,1,1;1,1,1,5)$, $B_8(2,1,1,1;1,1,1,5)$, $B_8(1,1,2,4;1,1,2,1)$, $B_8(1,1,5,1;1,1,2,1)$, $B_8(1,2,1,4;1,1,2,1)$, $B_8(1,2,2,3;1,1,2,1)$, $B_8(1,5,1,1;1,1,2,1)$, $B_8(2,1,2,3;1,1,2,1)$, $B_8(2,2,1,3;1,1,2,1)$, $B_8(2,2,2,2;1,1,2,1)$, $B_8(3,1,2,2;1,1,2,1)$, $B_8(3,2,1,2;1,1,2,1)$, $B_8(3,2,2,1;1,1,2,1)$, $B_8(4,1,2,1;1,1,2,1)$, $B_8(4,2,1,1;1,1,2,1)$, $B_8(1,1,2,3;1,1,2,2)$, $B_8(1,1,3,2;1,1,2,2)$,

n	$\mathcal{B}^*$
13	$B_8(1,1,4,1;1,1,2,2)$, $B_8(2,1,1,3;1,1,2,2)$, $B_8(2,1,2,2;1,1,2,2)$, $B_8(2,1,3,1;1,1,2,2)$, $B_8(3,1,1,2;1,1,2,2)$, $B_8(3,1,2,1;1,1,2,2)$, $B_8(4,1,1,1;1,1,2,2)$, $B_8(1,1,3,1;1,1,2,3)$, $B_8(2,1,2,1;1,1,2,3)$, $B_8(1,2,1,3;1,1,3,1)$, $B_8(2,1,2,2;1,1,3,1)$, $B_8(2,2,1,2;1,1,3,1)$, $B_8(3,1,2,1;1,1,3,1)$, $B_8(3,2,1,1;1,1,3,1)$, $B_8(2,1,1,2;1,1,3,2)$, $B_8(2,1,2,1;1,1,3,2)$, $B_8(3,1,1,1;1,1,3,2)$, $B_8(1,2,1,2;1,1,4,1)$, $B_8(2,1,2,1;1,1,4,1)$, $B_8(2,2,1,1;1,1,4,1)$, $B_8(2,1,1,1;1,1,4,2)$, $B_8(1,2,1,1;1,1,5,1)$, $B_8(1,3,2,2;1,2,1,1)$, $B_8(1,4,1,2;1,2,1,1)$, $B_8(1,4,2,1;1,2,1,1)$, $B_8(2,1,1,4;1,2,1,1)$, $B_8(2,3,2,1;1,2,1,1)$, $B_8(2,4,1,1;1,2,1,1)$, $B_8(3,1,1,3;1,2,1,1)$, $B_8(4,1,1,2;1,2,1,1)$, $B_8(5,1,1,1;1,2,1,1)$, $B_8(1,2,3,1;1,2,1,2)$, $B_8(1,3,2,1;1,2,1,2)$, $B_8(1,4,1,1;1,2,1,2)$, $B_8(1,2,2,1;1,2,1,3)$, $B_8(1,3,1,2;1,2,2,1)$, $B_8(1,4,1,1;1,2,2,1)$, $B_8(2,1,1,3;1,2,2,1)$, $B_8(2,2,1,2;1,2,2,1)$, $B_8(2,3,1,1;1,2,2,1)$, $B_8(3,1,1,2;1,2,2,1)$, $B_8(3,2,1,1;1,2,2,1)$, $B_8(4,1,1,1;1,2,2,1)$, $B_8(2,1,1,2;1,2,3,1)$, $B_8(2,2,1,1;1,2,3,1)$, $B_8(3,1,1,1;1,2,3,1)$, $B_8(2,1,1,1;1,2,3,2)$, $B_8(2,1,1,1;1,2,4,1)$, $B_8(1,3,1,2;1,3,1,1)$, $B_8(1,3,2,1;1,3,1,1)$, $B_8(2,3,1,1;1,3,1,1)$, $B_8(2,2,1,1;1,3,2,1)$, $B_8(2,2,1,1;1,4,1,1)$, $B_8(2,1,1,1;1,5,1,1)$, $B_8(2,1,1,4;2,1,1,1)$, $B_8(2,1,2,3;2,1,1,1)$, $B_8(2,1,3,2;2,1,1,1)$, $B_8(2,1,4,1;2,1,1,1)$, $B_8(2,2,2,2;2,1,1,1)$, $B_8(2,2,3,1;2,1,1,1)$, $B_8(2,3,2,1;2,1,1,1)$, $B_8(2,4,1,1;2,1,1,1)$, $B_8(3,1,1,3;2,1,1,1)$, $B_8(3,1,2,2;2,1,1,1)$, $B_8(3,2,1,2;2,1,1,1)$, $B_8(3,2,2,1;2,1,1,1)$, $B_8(3,3,1,1;2,1,1,1)$, $B_8(4,1,1,2;2,1,1,1)$, $B_8(4,2,1,1;2,1,1,1)$, $B_8(5,1,1,1;2,1,1,1)$, $B_8(2,1,1,3;2,1,1,2)$, $B_8(2,1,2,2;2,1,1,2)$, $B_8(2,1,3,1;2,1,1,2)$, $B_8(2,2,2,1;2,1,1,2)$, $B_8(3,1,1,2;2,1,1,2)$, $B_8(2,1,2,1;2,1,1,3)$, $B_8(2,1,2,2;2,1,2,1)$, $B_8(3,1,1,2;2,1,2,1)$, $B_8(3,2,1,1;2,1,2,1)$, $B_8(4,1,1,2;2,1,2,1)$, $B_8(3,1,1,2;2,1,2,2)$, $B_8(3,1,1,1;2,1,3,1)$, $B_8(2,2,2,1;2,2,1,1)$, $B_8(2,3,1,1;2,2,1,1)$, $B_8(3,1,1,2;2,2,1,1)$, $B_8(3,2,1,1;2,2,1,1)$, $B_8(4,1,1,1;2,2,1,1)$, $B_8(3,1,1,1;2,2,2,1)$, $B_8(3,1,1,1;2,3,1,1)$, $B_8(3,2,1,1;3,1,1,1)$.

Table 5: $k = 9$.

n	$\mathcal{B}^*$
11	$B_9(2,1,2,1;1,1,1,1;1)$, $B_9(2,1,1,1;1,1,1,2;1)$, $B_9(1,1,2,1;1,1,2,1;1)$, $B_9(2,1,1,1;2,1,1,1;1)$;
12	$B_9(1,2,1,3;1,1,1,1;1)$, $B_9(1,2,3,1;1,1,1,1;1)$, $B_9(2,1,2,2;1,1,1,1;1)$, $B_9(2,2,2,1;1,1,1,1;1)$, $B_9(3,2,1,1;1,1,1,1;1)$, $B_9(1,1,1,3;1,1,1,1;2)$,

continued from previous page

n	$\mathcal{B}^*$
12	$B_9(1,1,3,1;1,1,1,1;2)$, $B_9(2,1,1,2;1,1,1,1;2)$, $B_9(3,1,1,1;1,1,1,1;2)$, $B_9(1,2,1,2;1,1,1,2;1)$, $B_9(2,1,1,2;1,1,1,2;1)$, $B_9(1,1,2,1;1,1,1,2;2)$, $B_9(1,2,1,1;1,1,1,3;1)$, $B_9(1,1,2,2;1,1,2,1;1)$, $B_9(1,2,1,2;1,1,2,1;1)$, $B_9(2,1,1,1;1,1,2,1;2)$, $B_9(1,2,1,1;1,1,2,2;1)$, $B_9(2,1,1,1;1,1,2,2;1)$, $B_9(1,2,1,1;1,1,3,1;1)$, $B_9(3,1,1,1;1,2,1,1;1)$, $B_9(2,1,1,1;1,2,2,1;1)$, $B_9(2,2,1,1;2,1,1,1;1)$;
13	$B_9(1,1,1,5;1,1,1,1;1)$, $B_9(1,1,2,4;1,1,1,1;1)$, $B_9(1,1,3,3;1,1,1,1;1)$, $B_9(1,1,4,2;1,1,1,1;1)$, $B_9(1,1,5,1;1,1,1,1;1)$, $B_9(1,2,2,3;1,1,1,1;1)$, $B_9(1,2,3,2;1,1,1,1;1)$, $B_9(1,3,2,2;1,1,1,1;1)$, $B_9(1,4,1,2;1,1,1,1;1)$, $B_9(1,4,2,1;1,1,1,1;1)$, $B_9(2,1,1,4;1,1,1,1;1)$, $B_9(2,1,2,3;1,1,1,1;1)$, $B_9(2,1,3,2;1,1,1,1;1)$, $B_9(2,2,2,2;1,1,1,1;1)$, $B_9(2,3,1,2;1,1,1,1;1)$, $B_9(2,3,2,1;1,1,1,1;1)$, $B_9(2,4,1,1;1,1,1,1;1)$, $B_9(3,1,1,3;1,1,1,1;1)$, $B_9(3,2,1,2;1,1,1,1;1)$, $B_9(4,1,1,2;1,1,1,1;1)$, $B_9(5,1,1,1;1,1,1,1;1)$, $B_9(1,1,2,3;1,1,1,1;2)$, $B_9(1,1,3,2;1,1,1,1;2)$, $B_9(1,2,2,2;1,1,1,1;2)$, $B_9(1,3,1,2;1,1,1,1;2)$, $B_9(1,3,2,1;1,1,1,1;2)$, $B_9(2,2,1,2;1,1,1,1;2)$, $B_9(2,3,1,1;1,1,1,1;2)$, $B_9(1,1,2,2;1,1,1,1;3)$, $B_9(1,2,1,2;1,1,1,1;3)$, $B_9(1,2,2,1;1,1,1,1;3)$, $B_9(2,2,1,1;1,1,1,1;3)$, $B_9(1,1,1,2;1,1,1,1;4)$, $B_9(1,1,2,1;1,1,1,1;4)$, $B_9(2,1,1,1;1,1,1,1;4)$, $B_9(1,1,1,4;1,1,1,2;1)$, $B_9(1,1,2,3;1,1,1,2;1)$, $B_9(1,1,3,2;1,1,1,2;1)$, $B_9(1,1,4,1;1,1,1,2;1)$, $B_9(1,2,2,2;1,1,1,2;1)$, $B_9(1,2,3,1;1,1,1,2;1)$, $B_9(1,3,2,1;1,1,1,2;1)$, $B_9(1,4,1,1;1,1,1,2;1)$, $B_9(2,1,1,3;1,1,1,2;1)$, $B_9(1,1,1,3;1,1,1,2;2)$, $B_9(1,1,2,2;1,1,1,2;2)$, $B_9(1,2,1,2;1,1,1,2;2)$, $B_9(1,2,2,1;1,1,1,2;2)$, $B_9(1,3,1,1;1,1,1,2;2)$, $B_9(1,1,1,2;1,1,1,2;3)$, $B_9(1,2,1,1;1,1,1,2;3)$, $B_9(1,1,1,3;1,1,1,3;1)$, $B_9(1,1,2,2;1,1,1,3;1)$, $B_9(1,1,3,1;1,1,1,3;1)$, $B_9(1,2,2,1;1,1,1,3;1)$, $B_9(1,1,2,1;1,1,1,4;1)$, $B_9(1,1,2,3;1,1,2,1;1)$, $B_9(1,4,1,1;1,1,2,1;1)$, $B_9(2,1,1,3;1,1,2,1;1)$, $B_9(2,2,1,2;1,1,2,1;1)$, $B_9(2,3,1,1;1,1,2,1;1)$, $B_9(3,1,1,2;1,1,2,1;1)$, $B_9(3,2,1,1;1,1,2,1;1)$, $B_9(4,1,1,1;1,1,2,1;1)$, $B_9(1,3,1,1;1,1,2,1;2)$, $B_9(2,2,1,1;1,1,2,1;2)$, $B_9(1,2,1,1;1,1,2,1;3)$, $B_9(1,1,2,2;1,1,2,2;1)$, $B_9(2,1,1,2;1,1,2,2;1)$, $B_9(1,2,1,1;1,1,2,2;2)$, $B_9(2,1,1,2;1,1,3,1;1)$, $B_9(2,2,1,1;1,1,3,1;1)$, $B_9(3,1,1,1;1,1,3,1;1)$, $B_9(2,1,1,1;1,1,3,2;1)$, $B_9(2,1,1,1;1,1,4,1;1)$, $B_9(1,2,2,2;1,2,1,1;1)$, $B_9(1,3,1,2;1,2,1,1;1)$, $B_9(1,3,2,1;1,2,1,1;1)$, $B_9(2,1,1,3;1,2,1,1;1)$, $B_9(2,2,1,2;1,2,1,1;1)$, $B_9(2,3,1,1;1,2,1,1;1)$, $B_9(3,1,1,2;1,2,1,1;1)$, $B_9(1,2,1,2;1,2,1,1;2)$, $B_9(1,2,2,1;1,2,1,1;2)$, $B_9(2,1,1,2;1,2,1,1;2)$, $B_9(2,2,1,1;1,2,1,1;2)$, $B_9(2,1,1,1;1,2,1,1;3)$, $B_9(1,2,2,1;1,2,1,2;1)$, $B_9(1,3,1,1;1,2,1,2;1)$, $B_9(1,3,1,1;1,2,2,1;1)$, $B_9(2,1,1,2;1,2,2,1;1)$, $B_9(2,2,1,1;1,2,2,1;1)$, $B_9(2,1,1,2;1,3,1,1;1)$, $B_9(2,2,1,1;1,3,1,1;1)$, $B_9(2,1,1,1;1,3,1,1;2)$, $B_9(2,1,1,1;1,4,1,1;1)$, $B_9(2,3,1,1;2,1,1,1;1)$, $B_9(2,2,1,1;2,2,1,1;1)$.

Table 6: $k = 10$.

n	$\mathcal{B}^*$
12	$B_{10}(1,1,2,1,2;1,1,1,1,1)$, $B_{10}(1,2,1,2,1;1,1,1,1,1)$, $B_{10}(2,1,2,1,1;1,1,1,1,1)$, $B_{10}(1,1,1,1,2;1,1,1,1,2)$, $B_{10}(1,2,1,1,1;1,1,1,1,2)$, $B_{10}(2,1,1,1,1;1,1,1,1,2)$, $B_{10}(1,1,2,1,1;1,1,1,2,1)$, $B_{10}(1,2,1,1,1;1,1,1,2,1)$, $B_{10}(1,1,2,1,1;1,1,2,1,1)$, $B_{10}(2,1,1,1,1;1,2,1,1,1)$;
13	$B_{10}(1,1,1,1,4;1,1,1,1,1)$, $B_{10}(1,1,1,2,3;1,1,1,1,1)$, $B_{10}(1,1,1,3,2;1,1,1,1,1)$, $B_{10}(1,1,1,4,1;1,1,1,1,1)$, $B_{10}(1,1,2,2,2;1,1,1,1,1)$, $B_{10}(1,1,2,3,1;1,1,1,1,1)$, $B_{10}(1,1,3,2,1;1,1,1,1,1)$, $B_{10}(1,1,4,1,1;1,1,1,1,1)$, $B_{10}(1,2,1,1,3;1,1,1,1,1)$, $B_{10}(1,2,1,2,2;1,1,1,1,1)$, $B_{10}(1,2,2,1,2;1,1,1,1,1)$, $B_{10}(1,2,2,2,1;1,1,1,1,1)$, $B_{10}(1,2,3,1,1;1,1,1,1,1)$, $B_{10}(1,3,1,1,2;1,1,1,1,1)$, $B_{10}(1,3,2,1,1;1,1,1,1,1)$, $B_{10}(1,4,1,1,1;1,1,1,1,1)$, $B_{10}(2,1,1,1,3;1,1,1,1,1)$, $B_{10}(2,1,1,2,2;1,1,1,1,1)$, $B_{10}(2,1,1,3,1;1,1,1,1,1)$, $B_{10}(2,1,2,2,1;1,1,1,1,1)$, $B_{10}(2,2,1,1,2;1,1,1,1,1)$, $B_{10}(2,2,1,2,1;1,1,1,1,1)$, $B_{10}(2,2,2,1,1;1,1,1,1,1)$, $B_{10}(2,3,1,1,1;1,1,1,1,1)$, $B_{10}(3,1,1,1,2;1,1,1,1,1)$, $B_{10}(3,1,1,2,1;1,1,1,1,1)$, $B_{10}(3,2,1,1,1;1,1,1,1,1)$, $B_{10}(4,1,1,1,1;1,1,1,1,1)$, $B_{10}(1,1,1,2,2;1,1,1,1,2)$, $B_{10}(1,1,1,3,1;1,1,1,1,2)$, $B_{10}(1,1,2,2,1;1,1,1,1,2)$, $B_{10}(1,1,3,1,1;1,1,1,1,2)$, $B_{10}(1,2,2,1,1;1,1,1,1,2)$, $B_{10}(2,1,1,2,1;1,1,1,1,2)$, $B_{10}(1,1,1,2,1;1,1,1,1,3)$, $B_{10}(1,1,2,1,1;1,1,1,1,3)$, $B_{10}(1,1,1,2,2;1,1,1,2,1)$, $B_{10}(1,1,1,3,1;1,1,1,2,1)$, $B_{10}(1,1,2,2,1;1,1,1,2,1)$, $B_{10}(1,2,1,1,2;1,1,1,2,1)$, $B_{10}(2,1,1,1,2;1,1,1,2,1)$, $B_{10}(2,1,1,2,1;1,1,1,2,1)$, $B_{10}(2,2,1,1,1;1,1,1,2,1)$, $B_{10}(3,1,1,1,1;1,1,1,2,1)$, $B_{10}(1,1,2,1,1;1,1,1,2,2)$, $B_{10}(2,1,1,1,1;1,1,1,2,2)$, $B_{10}(2,1,1,1,1;1,1,1,3,1)$, $B_{10}(1,2,1,1,2;1,1,2,1,1)$, $B_{10}(1,2,2,1,1;1,1,2,1,1)$, $B_{10}(1,3,1,1,1;1,1,2,1,1)$, $B_{10}(2,1,1,1,2;1,1,2,1,1)$, $B_{10}(2,1,1,2,1;1,1,2,1,1)$, $B_{10}(2,2,1,1,1;1,1,2,1,1)$, $B_{10}(3,1,1,1,1;1,1,2,1,1)$, $B_{10}(1,2,1,1,1;1,1,2,2,1)$, $B_{10}(2,1,1,1,1;1,1,2,2,1)$, $B_{10}(1,2,1,1,1;1,1,3,1,1)$, $B_{10}(2,1,1,1,1;1,1,3,1,1)$, $B_{10}(1,2,1,1,2;1,2,1,1,1)$, $B_{10}(1,2,2,1,1;1,2,1,1,1)$, $B_{10}(1,3,1,1,1;1,2,1,1,1)$, $B_{10}(2,2,1,1,1;1,2,1,1,1)$, $B_{10}(2,1,1,1,1;1,2,2,1,1)$, $B_{10}(2,1,1,1,2;2,1,1,1,1)$, $B_{10}(2,1,1,2,1;2,1,1,1,1)$, $B_{10}(2,1,2,1,1;2,1,1,1,1)$, $B_{10}(2,2,1,1,1;2,1,1,1,1)$, $B_{10}(3,1,1,1,1;2,1,1,1,1)$.

Appendix B   Some theorems and lemmas

Theorem B.1 ([6])**.** *Let* $G = B_4(a_1, a_2; a_3, a_4)$, *where* a_1, a_2, a_3, a_4 *are some positive integers. Then* $\lambda_2(G) > 0$ *and* $\lambda_3(G) < 0$ *if and only if* G *is isomorphic to one of the following graphs:*

(1) $B_4(a, b; 1, d)$;   *(3)* $B_4(a, 1; c, 1)$;   *(5)* $B_4(a, 1; x, d)$;   *(7)* $B_4(w, x; y, d)$;

(2) $B_4(a, x; y, 1)$;   *(4)* $B_4(a, 1; w, x)$;   *(6)* $B_4(w, b; x, 1)$;   *(8)* $B_4(x, b; y, d)$;

(9) 25 *specific graphs:* 5 *graphs of order* 10, 10 *graphs of order* 11, *and* 10 *graphs of order* 12,

where a, b, c, d, x, y, w *are some positive integers such that* $x \leq 2, y \leq 2$ *and* $w \leq 3$.

Lemma B.2. *Let* $G \in \mathcal{B}_4(n)$, *where* $n \geq 14$. *If* $G \notin \mathcal{B}_4^-(n)$, *then* G *has an induced subgraph* $\Gamma \in \mathcal{B}_4(14) \setminus \mathcal{B}_4^-(14)$.

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_4(n-1) \setminus \mathcal{B}_4^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_4(n_1, n_2; n_3, n_4) \in \mathcal{B}_4(n)$. Then one of

$$H_1 = B_4(n_1 - 1, n_2; n_3, n_4), \qquad H_2 = B_4(n_1, n_2 - 1; n_3, n_4),$$
$$H_3 = B_4(n_1, n_2; n_3 - 1, n_4) \quad \text{and} \qquad H_4 = B_4(n_1, n_2; n_3, n_4 - 1)$$

must belong to $\mathcal{B}_4(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_4^-(n-1)$ $(i = 1, 2, 3, 4)$. Then H_i is a graph belonging to $(1)-(8)$ in Theorem B.1 since $n \geq 15$.

First we consider H_1. If H_1 is a graph belonging to (1) of Theorem B.1, then $H_1 = B_4(a, b; 1, d)$ where $n_1 - 1 = a$, $n_2 = b$, $n_3 = 1$ and $n_4 = d$, hence $G = B_4(a + 1, b; 1, d) \in \mathcal{B}_4^-(n)$, a contradiction. Similarly, H_1 cannot belong to $(2)-(5)$ of Theorem B.1. Hence H_1 is belong to $(6)-(8)$ of Theorem B.1 from which we see that $n_1 - 1$ is either w or x. Thus $n_1 \leq 4$ due to $w \leq 3$ and $x \leq 2$.

By the same method, we can verify that $n_2 \leq 3$ if $H_2 \in \mathcal{B}_4^-(n-1)$; $n_3 \leq 4$ if $H_3 \in \mathcal{B}_4^-(n-1)$ and $n_4 \leq 3$ if $H_4 \in \mathcal{B}_4^-(n-1)$. Hence $n = n_1 + \cdots + n_4 \leq 14$, a contradiction. We are done. $\square$

Theorem B.3 ([6])**.** *Let* $G = B_6(a_1, a_2, a_3; a_4, a_5, a_6)$, *where* $a_1, \ldots, a_6$ *are some positive integers. Then* $\lambda_2(G) > 0$ *and* $\lambda_3(G) < 0$ *if and only if* G *is isomorphic to one of the following graphs:*

(1) $B_6(a, x, c; 1, 1, 1)$;   *(6)* $B_6(x, b, 1; y, 1, 1)$;   *(11)* $B_6(1, b, 1; 1, e, 1)$;

(2) $B_6(a, 1, c; 1, e, 1)$;   *(7)* $B_6(x, y, 1; 1, e, 1)$;   *(12)* $B_6(1, b, 1; 1, x, y)$;

(3) $B_6(a, 1, c; 1, x, y)$;   *(8)* $B_6(x, y, 1; 1, 1, f)$;   *(13)* $B_6(1, x, y; 1, 1, f)$;

(4) $B_6(a, 1, c; 1, 1, f)$;   *(9)* $B_6(x, 1, c; y, 1, f)$;

(5) $B_6(a, 1, 1; x, e, 1)$;   *(10)* $B_6(1, b, x; 1, 1, 1)$;

(14) 145 *specific graphs:* 22 *graphs of order* 10, 54 *graphs of order* 11, *and* 69 *graphs of order* 12,

where a, b, c, d, e, f, x, y *are some positive integers such that* $x \leq 2$ *and* $y \leq 2$.

Lemma B.4. *Let $G \in \mathcal{B}_6(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_6^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_6(14) \setminus \mathcal{B}_6^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_6(n-1) \setminus \mathcal{B}_6^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_6(n_1, n_2, n_3; n_4, n_5, n_6) \in \mathcal{B}_6(n)$. Then one of

$$H_1 = B_6(n_1 - 1, n_2, n_3; n_4, n_5, n_6), \qquad H_2 = B_6(n_1, n_2 - 1, n_3; n_4, n_5, n_6),$$
$$H_3 = B_6(n_1, n_2, n_3 - 1; n_4, n_5, n_6), \qquad H_4 = B_6(n_1, n_2, n_3; n_4 - 1, n_5, n_6),$$
$$H_5 = B_6(n_1, n_2, n_3; n_4, n_5 - 1, n_6) \quad \text{and} \quad H_6 = B_6(n_1, n_2, n_3; n_4, n_5, n_6 - 1)$$

must belong to $\mathcal{B}_6(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_6^-(n-1)$ ($i = 1, 2, \ldots, 6$). Then H_i is a graph belonging to (1)–(13) in Theorem B.3 since $n \geq 15$.

Let us consider H_3. If H_3 is a graph belonging to (1) of Theorem B.3, then $H_3 = B_6(a, x, c; 1, 1, 1)$ where $n_1 = a$, $n_2 = x$, $n_3 - 1 = c$, $n_4 = n_5 = n_6 = 1$, hence $G = B_6(a, x, c + 1; 1, 1, 1) \in \mathcal{B}_6^-(n)$, a contradiction. Similarly, H_3 cannot belong to (2)–(4) and (9) of Theorem B.3. If H_3 is a graph belonging to (10) of Theorem B.3, then $H_3 = B_6(1, b, x; 1, 1, 1)$, where $n_1 = 1$, $n_2 = b$, $n_3 - 1 = x$, $n_4 = n_5 = n_6 = 1$. Since $x \leq 2$, we have $n_3 \leq 3$. If $n_3 < 3$ then $x + 1 \leq 2$ and $G = B_6(1, b, x + 1; 1, 1, 1) \in \mathcal{B}_6^-(n)$, a contradiction. Now assume that $n_3 = 3$. Then $H_3 = B_6(1, b, 2; 1, 1, 1)$, and so $G = B_6(1, b, 3; 1, 1, 1)$. By Theorem B.3, $G \notin \mathcal{B}_6^-(n)$, and also its induced subgraph $B_6(1, b - 1, 3; 1, 1, 1) \notin \mathcal{B}_6^-(n-1)$, a contradiction. Similarly, H_3 cannot belong to (13) of Theorem B.3. Hence H_3 is belong to (5)–(8) and (11)–(12) of Theorem B.3 from which we see that $n_3 - 1 \leq 1$. Thus $n_3 \leq 2$.

By the same method, we can verify that $n_1 \leq 3$ if $H_1 \in \mathcal{B}_6^-(n-1)$; $n_2 \leq 3$ if $H_2 \in \mathcal{B}_6^-(n-1)$; $n_4 \leq 2$ if $H_4 \in \mathcal{B}_6^-(n-1)$; $n_5 \leq 2$ if $H_5 \in \mathcal{B}_6^-(n-1)$ and $n_6 \leq 2$ if $H_6 \in \mathcal{B}_6^-(n-1)$. Hence $n = n_1 + \cdots + n_6 \leq 14$, a contradiction. We are done. $\square$

Theorem B.5 ([6])**.** *Let $G = B_7(a_1, a_2, a_3; a_4, a_5, a_6; a_7)$, where $a_1, \ldots, a_7$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_7(a, 1, x; 1, e, 1; 1)$;   (4) $B_7(x, y, 1; 1, e, 1; g)$;   (7) $B_7(1, b, 1; 1, e, 1; g)$;

(2) $B_7(a, 1, 1; 1, e, 1; g)$;   (5) $B_7(x, 1, 1; y, 1, 1; g)$;   (8) $B_7(1, 1, c; 1, 1, f; 1)$;

(3) $B_7(a, 1, 1; 1, 1, x; 1)$;   (6) $B_7(1, b, x; 1, 1, 1; g)$;

(9) 143 specific graphs: 18 graphs of order 10, 52 graphs of order 11, and 73 graphs of order 12,

where $a, b, c, d, e, f, g, x, y$ are some positive integers such that $x \leq 2$ and $y \leq 2$.

Lemma B.6. *Let $G \in \mathcal{B}_7(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_7^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_7(14) \setminus \mathcal{B}_7^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_7(n-1) \setminus \mathcal{B}_7^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_7(n_1, n_2, n_3; n_4, n_5, n_6; n_7) \in \mathcal{B}_7(n)$. Then one of

$$H_1 = B_7(n_1 - 1, n_2, n_3; n_4, n_5, n_6; n_7),$$
$$H_2 = B_7(n_1, n_2 - 1, n_3; n_4, n_5, n_6; n_7),$$
$$H_3 = B_7(n_1, n_2, n_3 - 1; n_4, n_5, n_6; n_7),$$

$$H_4 = B_7(n_1, n_2, n_3; n_4 - 1, n_5, n_6; n_7),$$
$$H_5 = B_7(n_1, n_2, n_3; n_4, n_5 - 1, n_6; n_7);$$
$$H_6 = B_7(n_1, n_2, n_3; n_4, n_5, n_6 - 1; n_7) \text{ and}$$
$$H_7 = B_7(n_1, n_2, n_3; n_4, n_5, n_6; n_7 - 1)$$

must belong to $\mathcal{B}_7(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_7^-(n-1)$ $(i = 1, 2, \ldots, 7)$. Then H_i is a graph belonging to $(1)-(8)$ in Theorem B.5 since $n \geq 15$.

Let us consider H_1. If H_1 is a graph belonging to (1) of Theorem B.5, then $H_1 = B_7(a, 1, x; 1, e, 1; 1)$ where $n_1 - 1 = a$, $n_2 = 1$, $n_3 = x$, $n_4 = 1$, $n_5 = e$, $n_6 = n_7 = 1$, hence $G = B_7(a + 1, 1, x; 1, e, 1; 1) \in \mathcal{B}_7^-(n)$, a contradiction. Similarly, H_1 cannot belong to $(2)-(3)$ of Theorem B.5. If H_1 is a graph belonging to (4) of Theorem B.5, then $H_1 = B_7(x, y, 1; 1, e, 1; g)$, where $n_1 - 1 = x$, $n_2 = y$, $n_3 = n_4 = 1$, $n_5 = e$, $n_6 = 1$ and $n_7 = g$. Since $x \leq 2$, we have $n_1 \leq 3$. If $n_1 < 3$ then $x + 1 \leq 2$ and $G = B_7(x + 1, y, 1; 1, e, 1; g) \in \mathcal{B}_7^-(n)$, a contradiction. Now assume that $n_1 = 3$. Then $H_1 = B_7(2, y, 1; 1, e, 1; g)$, and so $G = B_7(3, y, 1; 1, e, 1; g)$. Since $y \in \{1, 2\}$, we have $G \in \{B_7(3, 1, 1; 1, e, 1; g), B_7(3, 2, 1; 1, e, 1; g)\}$. However $B_7(3, 1, 1; 1, e, 1; g)$ belongs to (2) of Theorem B.5 which contradicts our assumption. Thus $G = B_7(3, 2, 1; 1, e, 1; g)$. By Theorem B.5, $G \notin \mathcal{B}_7^-(n)$, and also its induced subgraph $B_7(3, 2, 1; 1, e - 1, 1; g)$ or $B_7(3, 2, 1; 1, e, 1; g - 1)$ is not in $\mathcal{B}_7^-(n - 1)$, a contradiction. Similarly, H_1 cannot belong to (5) of Theorem B.5. Hence H_1 belongs to $(6)-(8)$ of Theorem B.5 from which we see that $n_1 - 1 \leq 1$. Thus $n_1 \leq 2$.

By the same method, we can verify that $n_2 \leq 2$ if $H_2 \in \mathcal{B}_7^-(n - 1)$; $n_3 \leq 2$ if $H_3 \in \mathcal{B}_7^-(n - 1)$; $n_4 \leq 2$ if $H_4 \in \mathcal{B}_7^-(n - 1)$; $n_5 \leq 2$ if $H_5 \in \mathcal{B}_7^-(n - 1)$, $n_6 \leq 2$ if $H_6 \in \mathcal{B}_7^-(n - 1)$ and $n_7 \leq 2$ if $H_7 \in \mathcal{B}_7^-(n - 1)$. Hence $n = n_1 + \cdots + n_7 \leq 14$, a contradiction. We are done. $\qquad\square$

Theorem B.7 ([6]). *Let* $G = B_8(a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8)$, *where* $a_1, \ldots, a_8$ *are some positive integers. Then* $\lambda_2(G) > 0$ *and* $\lambda_3(G) < 0$ *if and only if* G *is isomorphic to one of the following graphs:*

(1) $B_8(a, 1, 1, d; 1, 1, g, 1)$;   *(2)* $B_8(1, b, 1, 1; 1, f, 1, 1)$;

(3) 134 *specific graphs:* 12 *graphs of order* 10, 42 *graphs of order* 11, *and* 80 *graphs of order* 12,

where a, b, d, f, g *are some positive integers.*

Lemma B.8. *Let* $G \in \mathcal{B}_8(n)$, *where* $n \geq 14$. *If* $G \notin \mathcal{B}_8^-(n)$, *then* G *has an induced subgraph* $\Gamma \in \mathcal{B}_8(14) \setminus \mathcal{B}_8^-(14)$.

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_8(n - 1) \setminus \mathcal{B}_8^-(n - 1)$ for $n \geq 15$ in the following.

Let $G = B_8(n_1, n_2, n_3, n_4; n_5, n_6, n_7, n_8) \in \mathcal{B}_8(n)$ and

$$H_1 = B_8(n_1 - 1, n_2, n_3, n_4; n_5, n_6, n_7, n_8),$$
$$H_2 = B_8(n_1, n_2 - 1, n_3, n_4; n_5, n_6, n_7, n_8),$$
$$H_3 = B_8(n_1, n_2, n_3 - 1, n_4; n_5, n_6, n_7, n_8),$$
$$H_4 = B_8(n_1, n_2, n_3, n_4 - 1; n_5, n_6, n_7, n_8),$$
$$H_5 = B_8(n_1, n_2, n_3, n_4; n_5 - 1, n_6, n_7, n_8),$$

$$H_6 = B_8(n_1, n_2, n_3, n_4; n_5, n_6 - 1, n_7, n_8),$$
$$H_7 = B_8(n_1, n_2, n_3, n_4; n_5, n_6, n_7 - 1, n_8) \quad \text{and}$$
$$H_8 = B_8(n_1, n_2, n_3, n_4; n_5, n_6, n_7, n_8 - 1).$$

If $n_3 \geq 3$, then $H_3 \in \mathcal{B}_8(n-1) \setminus \mathcal{B}_8^-(n-1)$ by Theorem B.7 as desired. If $n_3 = 2$, then at least one of $n_1, n_2, n_4, n_5, n_6, n_7, n_8$ is greater than 1 since $n \geq 15$, say n_2. Thus $H_2 \in \mathcal{B}_8(n-1) \setminus \mathcal{B}_8^-(n-1)$ by Theorem B.7 as desired. Hence let $n_3 = 1$. Similarly, let $n_5 = n_8 = 1$. Thus one of H_1, H_2, H_4, H_6, H_7 must belong to $\mathcal{B}_8(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_8^-(n-1)$ $(i = 1, 2, 4, 6, 7)$. Then H_i is a graph belonging to $(1) - (2)$ in Theorem B.7 since $n \geq 15$.

Let us consider H_1. If H_1 is a graph belonging to (1) of Theorem B.7, then $H_1 = B_8(a, 1, 1, d; 1, 1, g, 1)$; where $n_1 - 1 = a$, $n_2 = n_3 = 1$, $n_4 = d$, $n_5 = n_6 = 1$, $n_7 = g$ and $n_8 = 1$, hence $G = B_8(a+1, 1, 1, d; 1, 1, g, 1) \in \mathcal{B}_8^-(n)$, a contradiction. Hence H_1 belongs to (2) of Theorem B.7 from which we see that $n_1 = 2$ due to $n_1 - 1 = 1$.

By the same method, we can verify that $n_i = 2$ if $H_i \in \mathcal{B}_8^-(n-1)$ for $i = 2, 4, 6, 7$. Hence $n = n_1 + \cdots + n_8 \leq 13$, a contradiction. We are done. $\qquad\square$

Theorem B.9 ([6]). *Let* $G = B_9(a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8; a_9)$, *where* $a_1, \ldots, a_9$ *are some positive integers. Then* $\lambda_2(G) > 0$ *and* $\lambda_3(G) < 0$ *if and only if* G *is isomorphic to one of the following graphs:*

(1) $B_9(1, b, 1, 1; 1, f, 1, 1; k)$;

(2) 59 *specific graphs:* 3 *graphs of order* 10, 17 *graphs of order* 11, *and* 39 *graphs of order* 12,

where b, f, k *are some positive integers.*

Lemma B.10. *Let* $G \in \mathcal{B}_9(n)$, *where* $n \geq 14$. *If* $G \notin \mathcal{B}_9^-(n)$, *then* G *has an induced subgraph* $\Gamma \in \mathcal{B}_9(14) \setminus \mathcal{B}_9^-(14)$.

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_9(n-1) \setminus \mathcal{B}_9^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_9(n_1, n_2, n_3, n_4; n_5, n_6, n_7, n_8; n_9) \in \mathcal{B}_9(n)$. On the contrary, suppose that every induced subgraphs $G' \in \mathcal{B}_9(n-1)$ of G belongs to $\mathcal{B}_9^-(n-1)$. If $n_1 \geq 3$, then $H_1 = B_9(n_1 - 1, n_2, n_3, n_4; n_5, n_6, n_7, n_8; n_9) \notin \mathcal{B}_9^-(n-1)$ by Theorem B.9, a contradiction. If $n_1 = 2$, then at least one of $n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9$ is greater than 1 since $n \geq 15$, say n_2. Thus $H_2 = B_9(n_1, n_2 - 1, n_3, n_4; n_5, n_6, n_7, n_8; n_9) \notin \mathcal{B}_9^-(n-1)$ by Theorem B.9, a contradiction. Hence $n_1 = 1$. Similarly, $n_3 = n_4 = n_5 = n_7 = n_8 = 1$. But now $G = B_9(1, n_2, 1, 1; 1, n_6, 1, 1; n_9) \in \mathcal{B}_9^-(n)$ by Theorem B.9, a contradiction. We are done. $\qquad\square$