ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 14 (2018) 55–65
https://doi.org/10.26493/1855-3974.1283.456
(Also available at http://amc-journal.eu)

Tridiagonal pairs of q-Racah type, the Bockting operator ψ, and L-operators for $U_q(L(\mathfrak{sl}_2))$

Paul Terwilliger
Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706-1388 USA

Received 11 January 2017, accepted 18 March 2017, published online 5 May 2017

Abstract

We describe the Bockting operator ψ for a tridiagonal pair of q-Racah type, in terms of a certain L-operator for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$.

Keywords: Bockting operator, tridiagonal pair, Leonard pair.

Math. Subj. Class.: 17B37, 15A21

1 Introduction

In the theory of quantum groups there exists the concept of an L-operator; this was introduced in [20] to obtain solutions for the Yang-Baxter equation. In linear algebra there exists the concept of a tridiagonal pair; this was introduced in [13] to describe the irreducible modules for the subconstituent algebra of a Q-polynomial distance-regular graph. Recently some authors have connected the two concepts. In [1], [4] Pascal Baseilhac and Kozo Koizumi use L-operators for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$ to construct a family of finite-dimensional modules for the q-Onsager algebra $\mathcal{O}_q$; see [2, 3, 5, **?**] for related work. A finite-dimensional irreducible $\mathcal{O}_q$-module is essentially the same thing as a tridiagonal pair of q-Racah type [**?**, Section 12], [23, Section 3]. In [22, Section 9], Kei Miki uses similar L-operators to describe how $U_q(L(\mathfrak{sl}_2))$ is related to the q-tetrahedron algebra $\boxtimes_q$. A finite-dimensional irreducible $\boxtimes_q$-module is essentially the same thing as a tridiagonal pair of q-geometric type [16, Theorem 2.7], [14, Theorems 10.3, 10.4]. Following Baseilhac, Koizumi, and Miki, in the present paper we use L-operators for $U_q(L(\mathfrak{sl}_2))$ to describe the Bockting operator ψ associated with a tridiagonal pair of q-Racah type. Before going into detail, we recall some notation and basic concepts. Throughout this paper

E-mail address: terwilli@math.wisc.edu (Paul Terwilliger)

$\mathbb{F}$ denotes a field. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. For an $\mathbb{F}$-linear map $A : V \to V$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V | Av = \theta v\}$; in this case θ is called the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1. (See [13, Definition 1.1].) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions:

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{1.1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{1.2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We refer the reader to [12, 13, 17] for background on TD pairs, and here mention only a few essential points. Let A, A^* denote a TD pair on V, as in Definition 1.1. By [13, Lemma 4.5] the integers d and δ from (1.1) and (1.2) are equal; we call this common value the *diameter* of A, A^*. An ordering of the eigenspaces for A (resp. A^*) is called *standard* whenever it satisfies (1.1) (resp. (1.2)). Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [13, Lemma 2.4] the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. Until the end of this section fix a standard ordering $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) of the eigenspaces for A (resp. A^*). For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V_i^*). By construction $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{F}$. Moreover $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{F}$. By [13, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this constraint the solutions can be given in closed form [13, Theorem 11.2]. The "most general" solution is called q-Racah, and will be described shortly.

We now recall the split decomposition [13, Section 4]. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$. By [13, Theorem 4.6] the sum $V = \sum_{i=0}^d U_i$ is direct. By [13, Theorem 4.6] both

$$U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^*,$$
$$U_i + U_{i+1} + \cdots + U_d = V_0 + V_1 + \cdots + V_{d-i}$$

for $0 \leq i \leq d$. Let $I : V \to V$ denote the identity map. By [13, Theorem 4.6] both

$$(A - \theta_{d-i} I) U_i \subseteq U_{i+1}, \qquad (A^* - \theta^*_i I) U_i \subseteq U_{i-1} \tag{1.3}$$

for $0 \leq i \leq d$.

We now describe the q-Racah case. Pick a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We say that A, A^* has *q-Racah type* whenever there exist nonzero $a, b \in \mathbb{F}$ such that both

$$\theta_i = aq^{2i-d} + a^{-1} q^{d-2i}, \qquad \theta^*_i = bq^{2i-d} + b^{-1} q^{d-2i} \tag{1.4}$$

for $0 \leq i \leq d$. For the rest of this section assume that A, A^* has q-Racah type. For $1 \leq i \leq d$ we have $q^{2i} \neq 1$; otherwise $\theta_i = \theta_0$. Define an $\mathbb{F}$-linear map $K : V \to V$ such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i}. Thus

$$(K - q^{d-2i} I) U_i = 0 \qquad (0 \leq i \leq d). \tag{1.5}$$

Note that K is invertible. For $0 \leq i \leq d$ the following holds on U_i:

$$aK + a^{-1} K^{-1} = \theta_{d-i} I. \tag{1.6}$$

Define an $\mathbb{F}$-linear map $R : V \to V$ such that for $0 \leq i \leq d$, R acts on U_i as $A - \theta_{d-i} I$. By (1.6),

$$A = aK + a^{-1} K^{-1} + R. \tag{1.7}$$

By the equation on the left in (1.3),

$$RU_i \subseteq U_{i+1} \qquad (0 \leq i \leq d). \tag{1.8}$$

We now recall the Bockting operator ψ. By [8, Lemma 5.7] there exists a unique $\mathbb{F}$-linear map $\psi : V \to V$ such that both

$$\psi U_i \subseteq U_{i-1} \qquad (0 \leq i \leq d), \tag{1.9}$$

$$\psi R - R\psi = (q - q^{-1})(K - K^{-1}). \tag{1.10}$$

The known properties of ψ are described in [7, 8, **?**]. Suppose we are given A, A^*, R, K in matrix form, and wish to obtain ψ in matrix form. This can be done using (1.8), (1.9), (1.10) and induction on i. The calculation can be tedious, so one desires a more explicit description of ψ. In the present paper we give an explicit description of ψ, in terms of a certain L-operator for $U_q(L(\mathfrak{sl}_2))$. According to this description, ψ is equal to $-a$ times the ratio of two components for the L-operator. Theorem 5.4 is our main result.

The paper is organized as follows. In Section 2 we review the algebra $U_q(L(\mathfrak{sl}_2))$ in its Chevalley presentation. In Section 3 we recall the equitable presentation for $U_q(L(\mathfrak{sl}_2))$. In Section 4 we discuss some L-operators for $U_q(L(\mathfrak{sl}_2))$. In Section 5 we use these L-operators to describe ψ.

2 The quantum loop algebra $U_q(L(\mathfrak{sl}_2))$

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We will be discussing algebras. An algebra is meant to be associative and have a 1. Recall the field $\mathbb{F}$. Until the end of Section 4, fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

All tensor products are meant to be over $\mathbb{F}$.

Definition 2.1. (See [10, Section 3.3].) Let $U_q(L(\mathfrak{sl}_2))$ denote the $\mathbb{F}$-algebra with generators $E_i, F_i, K_i^{\pm 1}$ $(i \in \{0, 1\})$ and relations

$$\begin{aligned}
&K_i K_i^{-1} = 1, && K_i^{-1} K_i = 1, \\
&K_0 K_1 = 1, && K_1 K_0 = 1, \\
&K_i E_i = q^2 E_i K_i, && K_i F_i = q^{-2} F_i K_i, \\
&K_i E_j = q^{-2} E_j K_i, && K_i F_j = q^2 F_j K_i, \qquad i \neq j, \\
&E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
&E_i^3 E_j - [3]_q E_i^2 E_j E_i + [3]_q E_i E_j E_i^2 - E_j E_i^3 = 0, && \qquad i \neq j, \\
&F_i^3 F_j - [3]_q F_i^2 F_j F_i + [3]_q F_i F_j F_i^2 - F_j F_i^3 = 0, && \qquad i \neq j.
\end{aligned}$$

We call $E_i, F_i, K_i^{\pm 1}$ the *Chevalley generators* for $U_q(L(\mathfrak{sl}_2))$.

Lemma 2.2. (See [18, p. 35].) *We turn $U_q(L(\mathfrak{sl}_2))$ into a Hopf algebra as follows. The coproduct Δ satisfies*

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, && \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}, \\
&\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, && \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}.
\end{aligned}$$

The counit ε satisfies

$$\varepsilon(K_i) = 1, \qquad \varepsilon(K_i^{-1}) = 1, \qquad \varepsilon(E_i) = 0, \qquad \varepsilon(F_i) = 0.$$

The antipode S satisfies

$$S(K_i) = K_i^{-1}, \qquad S(K_i^{-1}) = K_i, \qquad S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i.$$

We now discuss the $U_q(L(\mathfrak{sl}_2))$-modules.

Lemma 2.3. (See [10, Section 4].) *There exists a family of $U_q(L(\mathfrak{sl}_2))$-modules*

$$\mathbf{V}(d, t) \qquad\qquad 0 \neq d \in \mathbb{N}, \qquad\qquad 0 \neq t \in \mathbb{F} \tag{2.1}$$

with this property: $\mathbf{V}(d, t)$ has a basis $\{v_i\}_{i=0}^d$ such that

$$\begin{aligned}
&K_1 v_i = q^{d-2i} v_i && (0 \leq i \leq d), \\
&E_1 v_i = [d - i + 1]_q v_{i-1} && (1 \leq i \leq d), \qquad E_1 v_0 = 0, \\
&F_1 v_i = [i + 1]_q v_{i+1} && (0 \leq i \leq d - 1), \qquad F_1 v_d = 0, \\
&K_0 v_i = q^{2i-d} v_i && (0 \leq i \leq d), \\
&E_0 v_i = t [i + 1]_q v_{i+1} && (0 \leq i \leq d - 1), \qquad E_0 v_d = 0, \\
&F_0 v_i = t^{-1} [d - i + 1]_q v_{i-1} && (1 \leq i \leq d), \qquad F_0 v_0 = 0.
\end{aligned}$$

The module $\mathbf{V}(d, t)$ is irreducible provided that $q^{2i} \neq 1$ for $1 \leq i \leq d$.

Definition 2.4. Referring to Lemma 2.3, we call $\mathbf{V}(d, t)$ an *evaluation module* for $U_q(L(\mathfrak{sl}_2))$. We call d the *diameter*. We call t the *evaluation parameter*.

Example 2.5. For $0 \neq t \in \mathbb{F}$ the $U_q(L(\mathfrak{sl}_2))$-module $\mathbf{V}(1, t)$ is described as follows. With respect to the basis v_0, v_1 from Lemma 2.3, the matrices representing the Chevalley generators are

$$E_1: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F_1: \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K_1: \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$
$$E_0: \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \qquad F_0: \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, \qquad K_0: \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

Lemma 2.6. (See [19, p. 58].) *Let U and V denote $U_q(L(\mathfrak{sl}_2))$-modules. Then $U \otimes V$ becomes a $U_q(L(\mathfrak{sl}_2))$-module as follows. For $u \in U$ and $v \in V$,*

$$\begin{aligned} K_i(u \otimes v) &= K_i(u) \otimes K_i(v), \\ K_i^{-1}(u \otimes v) &= K_i^{-1}(u) \otimes K_i^{-1}(v), \\ E_i(u \otimes v) &= E_i(u) \otimes v + K_i(u) \otimes E_i(v), \\ F_i(u \otimes v) &= u \otimes F_i(v) + F_i(u) \otimes K_i^{-1}(v). \end{aligned}$$

Definition 2.7. (See [11, p. 110].) Up to isomorphism, there exists a unique $U_q(L(\mathfrak{sl}_2))$-module of dimension 1 on which each $u \in U_q(L(\mathfrak{sl}_2))$ acts as $\varepsilon(u)I$, where ε is from Lemma 2.2. This $U_q(L(\mathfrak{sl}_2))$-module is said to be *trivial*.

Proposition 2.8. (See [22, Theorem 3.2].) *Assume that $\mathbb{F}$ is algebraically closed with characteristic zero, and q is not a root of unity. Let V denote a nontrivial finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module on which each eigenvalue of K_1 is an integral power of q. Then V is isomorphic to a tensor product of evaluation $U_q(L(\mathfrak{sl}_2))$-modules.*

3 The equitable presentation for $U_q(L(\mathfrak{sl}_2))$

In this section we recall the equitable presentation for $U_q(L(\mathfrak{sl}_2))$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. In a moment we will discuss some objects X_{ij}. The subscripts i, j are meant to be in $\mathbb{Z}_4$.

Lemma 3.1. (See [15, Theorem 2.1], [22, Proposition 4.2].) *The algebra $U_q(L(\mathfrak{sl}_2))$ has a presentation by generators*

$$X_{01}, \quad X_{12}, \quad X_{23}, \quad X_{30}, \quad X_{13}, \quad X_{31} \tag{3.1}$$

and the following relations:

$$X_{13}X_{31} = 1, \quad X_{31}X_{13} = 1, \quad \frac{qX_{01}X_{12} - q^{-1}X_{12}X_{01}}{q - q^{-1}} = 1, \quad \frac{qX_{12}X_{23} - q^{-1}X_{23}X_{12}}{q - q^{-1}} = 1,$$
$$\frac{qX_{23}X_{30} - q^{-1}X_{30}X_{23}}{q - q^{-1}} = 1, \quad \frac{qX_{30}X_{01} - q^{-1}X_{01}X_{30}}{q - q^{-1}} = 1, \quad \frac{qX_{01}X_{13} - q^{-1}X_{13}X_{01}}{q - q^{-1}} = 1,$$
$$\frac{qX_{31}X_{12} - q^{-1}X_{12}X_{31}}{q - q^{-1}} = 1, \quad \frac{qX_{23}X_{31} - q^{-1}X_{31}X_{23}}{q - q^{-1}} = 1, \quad \frac{qX_{13}X_{30} - q^{-1}X_{30}X_{13}}{q - q^{-1}} = 1,$$
$$X_{i,i+1}^3 X_{i+2,i+3} - [3]_q X_{i,i+1}^2 X_{i+2,i+3} X_{i,i+1} + [3]_q X_{i,i+1} X_{i+2,i+3} X_{i,i+1}^2 - X_{i+2,i+3} X_{i,i+1}^3 = 0.$$

An isomorphism with the presentation in Definition 2.1 sends

$$X_{01} \mapsto K_0 + q(q-q^{-1})K_0F_0, \qquad X_{12} \mapsto K_1 - (q-q^{-1})E_1,$$
$$X_{23} \mapsto K_1 + q(q-q^{-1})K_1F_1, \qquad X_{30} \mapsto K_0 - (q-q^{-1})E_0,$$
$$X_{13} \mapsto K_1, \qquad X_{31} \mapsto K_0.$$

The inverse isomorphism sends

$$E_1 \mapsto (X_{13} - X_{12})(q-q^{-1})^{-1}, \qquad E_0 \mapsto (X_{31} - X_{30})(q-q^{-1})^{-1},$$
$$F_1 \mapsto (X_{31}X_{23} - 1)q^{-1}(q-q^{-1})^{-1}, \qquad F_0 \mapsto (X_{13}X_{01} - 1)q^{-1}(q-q^{-1})^{-1},$$
$$K_1 \mapsto X_{13}, \qquad K_0 \mapsto X_{31}.$$

Note 3.2. For notational convenience, we identify the copy of $U_q(L(\mathfrak{sl}_2))$ given in Definition 2.1 with the copy given in Lemma 3.1, via the isomorphism given in Lemma 3.1.

Definition 3.3. Referring to Lemma 3.1, we call the generators (3.1) the *equitable generators* for $U_q(L(\mathfrak{sl}_2))$.

Lemma 3.4. (See [24, Theorem 3.4].) *From the equitable point of view the Hopf algebra $U_q(L(\mathfrak{sl}_2))$ looks as follows. The coproduct Δ satisfies*

$$\Delta(X_{13}) = X_{13} \otimes X_{13}, \qquad \Delta(X_{31}) = X_{31} \otimes X_{31},$$
$$\Delta(X_{01}) = (X_{01} - X_{31}) \otimes 1 + X_{31} \otimes X_{01}, \qquad \Delta(X_{12}) = (X_{12} - X_{13}) \otimes 1 + X_{13} \otimes X_{12},$$
$$\Delta(X_{23}) = (X_{23} - X_{13}) \otimes 1 + X_{13} \otimes X_{23}, \qquad \Delta(X_{30}) = (X_{30} - X_{31}) \otimes 1 + X_{31} \otimes X_{30}.$$

The counit ε satisfies

$$\varepsilon(X_{13}) = 1, \qquad \varepsilon(X_{31}) = 1, \qquad \varepsilon(X_{01}) = 1,$$
$$\varepsilon(X_{12}) = 1, \qquad \varepsilon(X_{23}) = 1, \qquad \varepsilon(X_{30}) = 1.$$

The antipode S satisfies

$$S(X_{31}) = X_{13}, \qquad S(X_{13}) = X_{31},$$
$$S(X_{01}) = 1 + X_{13} - X_{13}X_{01}, \qquad S(X_{12}) = 1 + X_{31} - X_{31}X_{12},$$
$$S(X_{23}) = 1 + X_{31} - X_{31}X_{23}, \qquad S(X_{30}) = 1 + X_{13} - X_{13}X_{30}.$$

4 Some L-operators for $U_q(L(\mathfrak{sl}_2))$

In this section we recall some L-operators for $U_q(L(\mathfrak{sl}_2))$, and describe their basic properties.

We recall some notation. Let Δ denote the coproduct for a Hopf algebra H. Then the opposite coproduct Δ^{op} is the composition

$$\Delta^{\mathrm{op}}: \quad H \xrightarrow[\Delta]{} H \otimes H \xrightarrow[r\otimes s \mapsto s \otimes r]{} H \otimes H.$$

Definition 4.1. (See [22, Section 9.1].) Let V denote a $U_q(L(\mathfrak{sl}_2))$-module and $0 \neq t \in \mathbb{F}$. Consider an $\mathbb{F}$-linear map

$$L: \quad V \otimes \mathbf{V}(1,t) \to V \otimes \mathbf{V}(1,t).$$

We call this map an *L-operator for V with parameter t* whenever the following diagram commutes for all $u \in U_q(L(\mathfrak{sl}_2))$:

$$\begin{array}{ccc} V \otimes \mathbf{V}(1,t) & \xrightarrow{\Delta(u)} & V \otimes \mathbf{V}(1,t) \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L} \\ V \otimes \mathbf{V}(1,t) & \xrightarrow[\Delta^{\mathrm{op}}(u)]{} & V \otimes \mathbf{V}(1,t) \end{array}$$

Definition 4.2. (See [22, Section 9.1].) Let V denote a $U_q(L(\mathfrak{sl}_2))$-module and $0 \neq t \in \mathbb{F}$. Consider any $\mathbb{F}$-linear map

$$L: \quad V \otimes \mathbf{V}(1,t) \to V \otimes \mathbf{V}(1,t). \tag{4.1}$$

For $r, s \in \{0, 1\}$ define an $\mathbb{F}$-linear map $L_{rs} : V \to V$ such that for $v \in V$,

$$L(v \otimes v_0) = L_{00}(v) \otimes v_0 + L_{10}(v) \otimes v_1, \tag{4.2}$$
$$L(v \otimes v_1) = L_{01}(v) \otimes v_0 + L_{11}(v) \otimes v_1. \tag{4.3}$$

Here v_0, v_1 is the basis for $\mathbf{V}(1,t)$ from Lemma 2.3.

Lemma 4.3. *Referring to Definition 4.2, the map* (4.1) *is an L-operator for V with parameter t if and only if the following equations hold on V:*

$$K_1 L_{00} = L_{00} K_1, \qquad K_1 L_{01} = q^{-2} L_{01} K_1,$$
$$K_1 L_{10} = q^2 L_{10} K_1, \qquad K_1 L_{11} = L_{11} K_1;$$

$$L_{00} E_1 - q E_1 L_{00} = L_{10}, \qquad L_{01} E_1 - q E_1 L_{01} = L_{11} - L_{00} K_1,$$
$$L_{10} E_1 - q^{-1} E_1 L_{10} = 0, \qquad L_{11} E_1 - q^{-1} E_1 L_{11} = -L_{10} K_1;$$

$$F_1 L_{00} - q^{-1} L_{00} F_1 = L_{01}, \qquad F_1 L_{01} - q L_{01} F_1 = 0,$$
$$F_1 L_{10} - q^{-1} L_{10} F_1 = L_{11} - K_0 L_{00}, \qquad F_1 L_{11} - q L_{11} F_1 = -K_0 L_{01};$$

$$K_0 L_{00} = L_{00} K_0, \qquad K_0 L_{01} = q^2 L_{01} K_0,$$
$$K_0 L_{10} = q^{-2} L_{10} K_0, \qquad K_0 L_{11} = L_{11} K_0;$$

$$L_{00} E_0 - q^{-1} E_0 L_{00} = -t L_{01} K_0, \qquad L_{01} E_0 - q^{-1} E_0 L_{01} = 0,$$
$$L_{10} E_0 - q E_0 L_{10} = t L_{00} - t L_{11} K_0, \qquad L_{11} E_0 - q E_0 L_{11} = t L_{01};$$

$$F_0 L_{00} - q L_{00} F_0 = -t^{-1} K_1 L_{10}, \qquad F_0 L_{01} - q^{-1} L_{01} F_0 = t^{-1} L_{00} - t^{-1} K_1 L_{11},$$
$$F_0 L_{10} - q L_{10} F_0 = 0, \qquad F_0 L_{11} - q^{-1} L_{11} F_0 = t^{-1} L_{10}.$$

Proof. This is routinely checked. □

Example 4.4. (See [21, Appendix], [22, Proposition 9.2].) Referring to Definition 4.2, assume that V is an evaluation module $\mathbf{V}(d, \mu)$ such that $q^{2i} \neq 1$ for $1 \leq i \leq d$. Consider the matrices that represent the L_{rs} with respect to the basis $\{v_i\}_{i=0}^d$ for $\mathbf{V}(d, \mu)$ from Lemma 2.3. Then the following are equivalent:

(i) the map (4.1) is an L-operator for V with parameter t;

(ii) the matrix entries are given in the table below (all matrix entries not shown are zero):

operator	$(i,i-1)$-entry	(i,i)-entry	$(i-1,i)$-entry
L_{00}	0	$\frac{q^{1-i}-\mu^{-1}tq^{i-d}}{q-q^{-1}}\xi$	0
L_{01}	$[i]_q q^{1-i}\xi$	0	0
L_{10}	0	0	$[d-i+1]_q q^{i-d}\mu^{-1}t\xi$
L_{11}	0	$\frac{q^{i-d+1}-\mu^{-1}tq^{-i}}{q-q^{-1}}\xi$	0

Here $\xi \in \mathbb{F}$.

Lemma 4.5. (See [22, Proposition 9.3].) *Let U and V denote $U_q(L(\mathfrak{sl}_2))$-modules, and consider the $U_q(L(\mathfrak{sl}_2))$-module $U \otimes V$ from Lemma 2.6. Let $0 \neq t \in \mathbb{F}$. Suppose we are given L-operators for U and V with parameter t. Then there exists an L-operator for $U \otimes V$ with parameter t such that for $r, s \in \{0,1\}$,*

$$L_{rs}(u \otimes v) = L_{r0}(u) \otimes L_{0s}(v) + L_{r1}(u) \otimes L_{1s}(v) \qquad u \in U, \quad v \in V. \tag{4.4}$$

Proof. For $r, s \in \{0,1\}$ define an $\mathbb{F}$-linear map $L_{rs} : U \otimes V \to U \otimes V$ that satisfies (4.4). Using (4.4) and Lemma 2.6 one checks that the L_{rs} satisfy the equations in Lemma 4.3. The result follows by Lemma 4.3. □

Corollary 4.6. *Adopt the notation and assumptions of Proposition 2.8. Then for $0 \neq t \in \mathbb{F}$ there exists a nonzero L-operator for V with parameter t.*

Proof. By Proposition 2.8 along with Example 4.4 and Lemma 4.5. □

5 TD pairs and L-operators

In Section 1 we discussed a TD pair A, A^* on V. We now return to this discussion, adopting the notation and assumptions that were in force at the end of Section 1. Recall the scalars q, a, b from (1.4). Recall the map K from above (1.5).

Proposition 5.1. (See [17, p. 103].) *Assume that $\mathbb{F}$ is algebraically closed with characteristic zero, and q is not a root of unity. Then the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \qquad A^* = bX_{23} + b^{-1}X_{30}.$$

Proof. This is how [17, p. 103] looks from the equitable point of view. □

Note 5.2. The $U_q(L(\mathfrak{sl}_2))$-module structure from Proposition 5.1 is not unique in general.

We now investigate the $U_q(L(\mathfrak{sl}_2))$-module structure from Proposition 5.1. Recall the map R from above (1.7).

Lemma 5.3. *Assume that the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \qquad A^* = bX_{23} + b^{-1}X_{30}.$$

On this module,

(i) *R looks as follows in the equitable presentation:*

$$R = a(X_{01} - X_{31}) + a^{-1}(X_{12} - X_{13}). \tag{5.1}$$

(ii) *R looks as follows in the Chevalley presentation:*

$$R = (q - q^{-1})(aqK_0F_0 - a^{-1}E_1). \tag{5.2}$$

Proof. (i) In line (1.7) eliminate A, K, K^{-1} using the assumptions of the present lemma. (ii) Evaluate the right-hand side of (5.1) using the identifications from Lemma 3.1 and Note 3.2. □

We now present our main result. Recall the Bockting operator ψ from (1.9), (1.10).

Theorem 5.4. *Assume that the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \qquad A^* = bX_{23} + b^{-1}X_{30}.$$

Consider an L-operator for V with parameter a^2. Then on V,

$$\psi = -a(L_{00})^{-1}L_{01} \tag{5.3}$$

provided that L_{00} is invertible.

Proof. Let $\widehat{\psi}$ denote the expression on the right in (5.3). We show $\psi = \widehat{\psi}$. To do this, we show that $\widehat{\psi}$ satisfies (1.9), (1.10). Concerning (1.9), by Lemma 4.3 the equation $K_0\widehat{\psi} = q^2\widehat{\psi}K_0$ holds on V. By Lemma 3.1, Note 3.2, and the construction, we obtain $K_0 = X_{31} = K$ on V. By these comments $K\widehat{\psi} = q^2\widehat{\psi}K$ on V. By this and (1.5) we obtain $\widehat{\psi}U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. So $\widehat{\psi}$ satisfies (1.9). Next we show that $\widehat{\psi}$ satisfies (1.10). Since L_{00} is invertible and $K_0K_1 = 1$ it suffices to show that on V,

$$L_{00}(\widehat{\psi}R - R\widehat{\psi}) = (q - q^{-1})L_{00}(K_0 - K_1). \tag{5.4}$$

By this and (5.2) it suffices to show that on V,

$$aqL_{00}(\widehat{\psi}K_0F_0 - K_0F_0\widehat{\psi}) - a^{-1}L_{00}(\widehat{\psi}E_1 - E_1\widehat{\psi}) + L_{00}(K_1 - K_0) = 0. \tag{5.5}$$

We examine the terms in (5.5). By Lemma 4.3 and the construction, the following hold on V:

$$\begin{aligned} L_{00}\widehat{\psi}K_0F_0 &= -aL_{01}K_0F_0 \\ &= -aq^{-2}K_0L_{01}F_0 \\ &= -aq^{-1}K_0(F_0L_{01} - a^{-2}L_{00} + a^{-2}K_1L_{11}) \end{aligned}$$

and

$$\begin{aligned} L_{00}K_0F_0\widehat{\psi} &= K_0L_{00}F_0\widehat{\psi} \\ &= q^{-1}K_0(a^{-2}K_1L_{10} + F_0L_{00})\widehat{\psi} \\ &= q^{-1}K_0(a^{-2}K_1L_{10}\widehat{\psi} - aF_0L_{01}) \end{aligned}$$

and

$$\begin{aligned} L_{00}\widehat{\psi}E_1 &= -aL_{01}E_1 \\ &= -a(qE_1L_{01} + L_{11} - L_{00}K_1) \\ &= -a(qE_1L_{01} + L_{11} - K_1L_{00}) \end{aligned}$$

and

$$\begin{aligned} L_{00}E_1\widehat{\psi} &= (L_{10} + qE_1L_{00})\widehat{\psi} \\ &= L_{10}\widehat{\psi} - qaE_1L_{01} \end{aligned}$$

and

$$L_{00}K_1 = K_1L_{00}, \qquad\qquad L_{00}K_0 = K_0L_{00}.$$

To verify (5.5), evaluate its left-hand side using the above comments and simplify the result using $K_0K_1 = 1$. The computation is routine, and omitted. We have shown that $\widehat{\psi}$ satisfies (1.10). The result follows. □

Acknowledgment

The author thanks Sarah Bockting-Conrad and Edward Hanson for giving this paper a close reading and offering valuable suggestions. The author also thanks Pascal Baseilhac for many conversations concerning quantum groups, L-operators, and tridiagonal pairs.

References

[1] P. Baseilhac, An integrable structure related with tridiagonal algebras, *Nuclear Phys. B* **705** (2005), 605–619, doi:10.1016/j.nuclphysb.2004.11.014.

[2] P. Baseilhac and S. Belliard, The half-infinite XXZ chain in Onsager's approach, *Nuclear Phys. B* **873** (2013), 550–584, doi:10.1016/j.nuclphysb.2013.05.003.

[3] P. Baseilhac and K. Koizumi, A deformed analogue of Onsager's symmetry in the XXZ open spin chain, *J. Stat. Mech. Theory Exp.* **10** (2005), P10005, doi:10.1088/1742-5468/2005/10/p10005.

[4] P. Baseilhac and K. Koizumi, A new (in)finite-dimensional algebra for quantum integrable models, *Nuclear Phys. B* **720** (2005), 325–347, doi:10.1016/j.nuclphysb.2005.05.021.

[5] P. Baseilhac and K. Koizumi, Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory, *J. Stat. Mech. Theory Exp.* **9** (2007), P09006, doi:10.1088/1742-5468/2007/09/p09006.

[6] P. Baseilhac and K. Shigechi, A new current algebra and the reflection equation, *Lett. Math. Phys.* **92** (2010), 47–65, doi:10.1007/s11005-010-0380-x.

[7] S. Bockting-Conrad, Two commuting operators associated with a tridiagonal pair, *Linear Algebra Appl.* **437** (2012), 242–270, doi:10.1016/j.laa.2012.02.007.

[8] S. Bockting-Conrad, Tridiagonal pairs of q-Racah type, the double lowering operator ψ, and the quantum algebra $U_q(\mathfrak{sl}_2)$, *Linear Algebra Appl.* **445** (2014), 256–279, doi:10.1016/j.laa.2013.12.007.

[9] S. Bockting-Conrad and P. Terwilliger, The algebra $U_q(\mathfrak{sl}_2)$ in disguise, *Linear Algebra Appl.* **459** (2014), 548–585, doi:10.1016/j.laa.2014.07.022.

[10] V. Chari and A. Pressley, Quantum affine algebras, *Commun. Math. Phys.* **142** (1991), 261–283, `https://projecteuclid.org/euclid.cmp/1104248585`.

[11] V. Chari and A. N. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1995.

[12] T. Ito, K. Nomura and P. Terwilliger, A classification of sharp tridiagonal pairs, *Linear Algebra Appl.* **435** (2011), 1857–1884, doi:10.1016/j.laa.2011.03.032.

[13] T. Ito, K. Tanabe and P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: A. Barg and S. Litsyn (eds.), *Codes and Association Schemes*, American Mathematical Society, Providence, Rhode Island, volume 56 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pp. 167–192, 2001, papers from the DIMACS Workshop held at Rutgers University, Piscataway, NJ, November 9 – 12, 1999.

[14] T. Ito and P. Terwilliger, The q-tetrahedron algebra and its finite dimensional irreducible modules, *Comm. Algebra* **35** (2007), 3415–3439, doi:10.1080/00927870701509180.

[15] T. Ito and P. Terwilliger, Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$, *Ramanujan J.* **13** (2007), 39–62, doi:10.1007/s11139-006-0242-4.

[16] T. Ito and P. Terwilliger, Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations, *J. Algebra Appl.* **6** (2007), 477–503, doi:10.1142/s021949880700234x.

[17] T. Ito and P. Terwilliger, The augmented tridiagonal algebra, *Kyushu J. Math.* **64** (2010), 81–144, doi:10.2206/kyushujm.64.81.

[18] J. C. Jantzen, *Lectures on Quantum Groups*, volume 6 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, Rhode Island, 1996.

[19] C. Kassel, *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1995, doi:10.1007/978-1-4612-0783-2.

[20] P. P. Kulish and E. K. Sklyanin, Solutions of the Yang-Baxter equation, *J. Sov. Math.* **19** (1982), 1596–1620, doi:10.1007/bf01091463.

[21] P. P. Kulish and N. Yu. Reshetikhin, Quantum linear problem for the sine-Gordon equation and higher representations, *J. Sov. Math.* **23** (1983), 2435–2441, doi:10.1007/bf01084171.

[22] K. Miki, Finite dimensional modules for the q-tetrahedron algebra, *Osaka J. Math.* **47** (2010), 559–589, `https://projecteuclid.org/euclid.ojm/1277298918`.

[23] P. Terwilliger, Two relations that generalize the q-Serre relations and the Dolan-Grady relations, in: A. N. Kirillov, A. Tsuchiya and H. Umemura (eds.), *Physics and Combinatorics 1999*, World Scientific Publishing, River Edge, New Jersey, pp. 377–398, 2001, doi: 10.1142/9789812810199_0013, proceedings of the International Workshop held at Nagoya University, Nagoya, August 23 – 27, 1999.

[24] P. Terwilliger, The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$, *J. Algebra* **298** (2006), 302–319, doi:10.1016/j.jalgebra.2005.11.013.