ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 15 (2018) 127–145
https://doi.org/10.26493/1855-3974.1059.4be
(Also available at http://amc-journal.eu)

Coordinatizing n_3 configurations

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Received 14 March 2016, accepted 7 January 2018, published online 13 June 2018

Abstract

Given an n_3 configuration, a one-point extension is a technique that constructs $(n+1)_3$ configurations from it. A configuration is *geometric* if it can be realized by a collection of points and straight lines in the plane. Given a geometric n_3 configuration with a planar coordinatization of its points and lines, a method is presented that uses a one-point extension to produce $(n+1)_3$ configurations from it, and then constructs geometric realizations of the $(n+1)_3$ configurations. It is shown that this can be done using only a homogeneous *cubic* polynomial in just *three* variables, *independent* of n. This transforms a computationally intractable problem into a computationally practical one.

Keywords: $(n,3)$*-configuration, geometric configuration, anti-Pappian, rational coordinatization, elliptic curve.*

Math. Subj. Class.: 51E20, 51E30

1 Projective configurations

A *projective configuration* consists of a set Σ of points and lines, and an incidence relation Π, such that two lines intersect in at most one point. We denote this by (Σ, Π). For example, a triangle with points A, B, C and lines a, b, c can be represented by the pair $(\{A, B, C, a, b, c\}, \{Ab, Ac, Ba, Bc, Ca, Cb\})$. A configuration (Σ, Π) can also be viewed as a bipartite incidence graph of points versus lines. We will always assume that the incidence graph of a configuration is connected. Excellent references on configurations are the recent books by Grünbaum [10], and by Pisanski and Servatius [18].

An n_3-configuration is a projective configuration with n points and n lines such that every line is incident with 3 points, and every point is incident with 3 lines. There is a unique

*This work is partially funded by a discovery grant from the Natural Sciences and Engineering Research Council of Canada.

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7_3-configuration, the Fano configuration, and a unique 8_3-configuration, the Möbius-Kantor configuration.

An n_3 configuration which can be represented by a collection of points and straight lines in the real or rational plane, such that all incidences are respected, and no two points or two lines coincide is termed a *geometric* n_3 configuration. In order to show that an n_3 configuration is geometric, the usual method is to assign suitable homogeneous coordinates to its points and lines. We call this a *coordinatization* of the configuration. A central problem [10] is to characterize which n_3 configurations are geometric configurations, and to find rational coordinatizations [4, 10, 21, 22, 23] of those that are geometric. Grünbaum [9], and [10] (p. 151) has conjectured that an (n_3) configuration that admits a real coordinatization also admits a rational coordinatization. He considers this the single most important outstanding problem in n_3 configurations [11]. Sturmfels and White [22, 23] have shown that all (11_3) and (12_3) configurations have rational coordinatizations. These configurations were originally discovered by Martinetti [17], and Daublebsky von Sterneck [6, 7]. Sturmfels and White and Bokowski [4, 22, 23] found rational coordinatizations by constructing systems of diophantine equations, and then using methods of computer algebra to solve them, in particular, Grassmannian algebras and Gröbner bases.

A coordinatization of an n_3 configuration is usually represented by homogeneous coordinates in the plane, e.g., let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be the homogeneous coordinates of two points, and let $L = (a, b, c)$ be the homogeneous coordinates of a line. Then P_1 and P_2 are incident with L if and only if $P_1 \cdot L = P_2 \cdot L = 0$. Equivalently, L is a multiple of $P_1 \times P_2$. Consequently there is an exterior algebra that the homogeneous coordinates generate. If there are n points and n lines, with $3n$ incidences, there are $6n$ variables, and numerous algebraic constraints that the coordinates must satisfy. Bokowski and Sturmfels [4] used computer-aided algebra to search for rational solutions to these algebraic constraints. Eventually the constraints can be manipulated to produce a homogeneous polynomial with at most $3n$ variables whose zeros characterize the coordinatizations. The polynomial has degree bounded by n. The difficulty of this work led Sturmfels and White [23] to suggest that the problem of finding rational coordinatizations of n_3 configurations may be *recursively undecidable*.

A simpler method of finding a coordinatizing polynomial, without the need of Gröbner bases and the exterior algebra, was presented in Kocay-Szypowski [15]. The degree of the polynomial is still bounded by n. This method was used in Kocay [13] to find a rational coordinatization of the Georges configuration, which is a (25_3) configuration. In Sturmfels and White [22, 23], ad-hoc methods were used to find rational roots of the coordinatizing polynomials for each of the (11_3) and (12_3) configurations. There are 31 (11_3) and 229 (12_3) configurations.

A note on homogeneous polynomials and their zeros: Homogeneous *quadratic* polynomials are well understood, see Conway [5]. It is the theory of quadratic forms. Cubic homogeneous polynomials are much more difficult. When there are three variables, they include the class of *elliptic curves* [20]. The rational points on an elliptic curve form a group. If there are one or more known rational points on the curve, then others can be found by combining them using the group operation. This generates a countable number of points. Mordell's theorem says that these groups are finitely generated, i.e., a finite number of starting points is needed to find the entire group. It does not say what the group is, or whether there are *any* rational points on the curve. And it does not provide a method to determine if there are any rational points on the curve. Because it is relatively easy to do computation in these groups, but simultaneously, there are theoretical difficulties in characterizing them, these groups are used in elliptic curve cryptographic systems [20]. Homogeneous polynomials of degree four or more are much more difficult, apparently not amenable to the same techniques. Thus the degree of the polynomial is important.

The purpose of this paper is to present an algorithm which can be used to construct real or rational coordinatizations of $(n+1)_3$-configurations from coordinatizations of n_3-configurations, by finding the roots (real or rational) of a *cubic homogeneous polynomial in three variables*. The use of a cubic homogeneous polynomial in three variables makes the formerly intractable problem of finding rational coordinatizations computationally practical and efficient. Some of the techniques are similar to methods used in the theory of elliptic curves [3, 20].

An elliptic curve is a cubic polynomial that can be expressed in the form

$$y^2 = ax^3 + bx^2 + cx + d$$

The rational points on an elliptic curve form a group. See [20] for further information on these groups.

Theorem 1.1 (Mordell's theorem)**.** *If a non-singular plane cubic curve has a rational point, then the group of rational points is finitely generated.*

Methods that originated with Diophantus [1] are used to find the rational roots of elliptic curves [20]. We use similar methods to construct coordinatizations of n_3-configurations. As there can be very many rational points on an elliptic curve, there can be also be very many different rational coordinatizations of an n_3 configuration. They are related in a way that is similar to the group operation of an elliptic curve. In general, it seems to be difficult to characterize when a rational coordinatization is possible. However the method presented here is very fast in practice, and can be automated.

We begin with a *1-point extension* [14] in an n_3 configuration, which extends it to an $(n+1)_3$ configuration, and which leads to the coordinatization algorithm. This extension is different from Martinetti's extension [17], which is described in Grünbaum [10] (p. 89). As pointed out in [10], it is in general quite difficult to characterize exactly which configurations are generated by an inductive construction which produces an $(n+1)_3$ configuration from an n_3 configuration. This is true even if the construction can easily be described. In [14] the configurations that can be built using a 1-point extension are characterized.

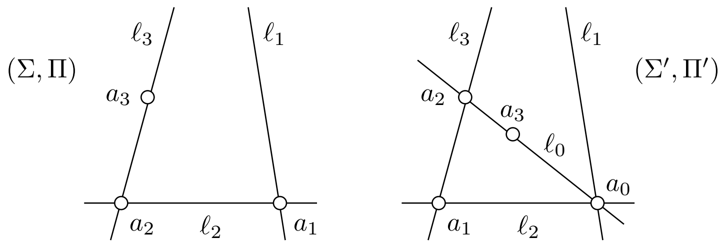
Theorem 1.2 (1-Point Extension)**.** *Let (Σ, Π) be an n_3-configuration. Let a_1, a_2, a_3 be 3 distinct points in Σ, and let ℓ_1, ℓ_2, ℓ_3 be 3 distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$ and $a_3 \in \ell_3$, where $a_3 \notin \ell_1$. We can represent this in tabular form as*

(Σ, Π)	ℓ_1	ℓ_2	ℓ_3	$\cdots$
	a_1	a_1	a_2	$\cdots$
	b_1	a_2	a_3	$\cdots$
	b_2	b_3	b_4	$\cdots$

where the dots indicate other points of the configuration. Here the points in each column are incident with the line at the top of the column. Let ℓ' be the third line containing a_1. Suppose further that if $\ell' \cap \ell_3 \neq \emptyset$, then $\ell' \cap \ell_3 = a_3$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. Define the new incidences as $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3\} \cup \{a_1\ell_3, a_2\ell_0, a_3\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

(Σ', Π')	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\cdots$
	a_2	a_0	a_1	a_1	$\cdots$
	a_3	b_1	a_0	a_2	$\cdots$
	a_0	b_2	b_3	b_4	$\cdots$

Here the dots represent exactly the same points as in the previous table. Then (Σ', Π') is an $(n+1)_3$-configuration. (Refer to Figure 1.)



(a) initial points and lines

(b) after 1-point extension

Figure 1: A 1-point extension with 3 points, before (a), after (b).

Example. The Fano configuration can be represented by the following table.

Fano	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	1	2	3	4	5	6	7
	2	3	4	5	6	7	1
	4	5	6	7	1	2	3

Choose ℓ_1, ℓ_2, ℓ_3 as indicated, and choose $a_1 = 2$, $a_2 = 3$, $a_3 = 6$, and let $a_0 = 8$. Notice that the third line containing a_1 is $\ell' = \ell_6$, which intersects ℓ_3 in $a_3 = 6$. Then by Theorem 1.2, the following table represents an 8_3-configuration, which is known to be unique.

8_3-config	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	3	1	2	2	4	5	6	7
	6	4	5	3	5	6	7	1
	8	8	8	4	7	1	2	3

The diagram of Figure 1 illustrates the 1-point extension schematically, showing the incidences altered by the extension. The method uses three points a_1, a_2, a_3 and three lines ℓ_1, ℓ_2, ℓ_3 sequentially incident, with a new point a_0 and line ℓ_0 added. It can be generalized to m points $a_1, a_2, \ldots, a_m$ and m lines $\ell_1, \ell_2, \ldots, \ell_m$ sequentially incident, see Kocay [14] for more details. This is indicated in Figure 2 for $m = 4$. When $m = 4$, the 1-point extension theorem has the following abridged form.

Theorem 1.3 (1-Point Extension with 4 points and 4 lines)**.** *Let (Σ, Π) be an n_3-configuration. Let a_1, a_2, a_3, a_4 be 4 distinct points in Σ, and let $\ell_1, \ell_2, \ell_3, \ell_4$ be 4 distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$, $a_3 = \ell_3 \cap \ell_4$, and $a_4 \in \ell_4$, where $a_3, a_4 \notin \ell_1, \ell_2$, and $a_1 \notin \ell_4$.*

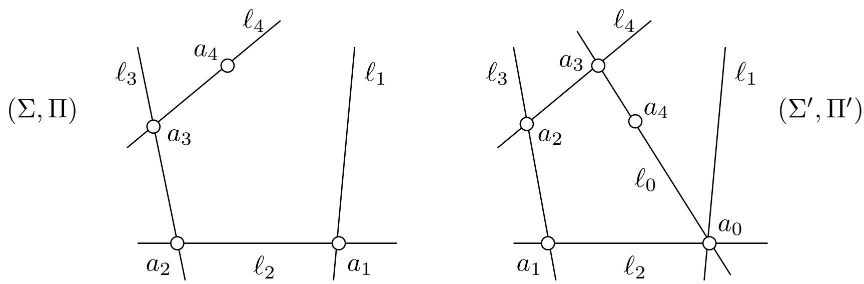
Let ℓ_1' be the third line containing a_1, and ℓ_2' be the third line containing a_2. Suppose further that if $\ell_1' \cap \ell_3 \neq \emptyset$, then $\ell_1' \cap \ell_3 = a_3$; and if $\ell_2' \cap \ell_4 \neq \emptyset$, then $\ell_2' \cap \ell_4 = a_4$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. Define the new incidences as $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3, a_4\ell_4\} \cup \{a_1\ell_3, a_2\ell_4, a_3\ell_0, a_4\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$.

Then (Σ', Π') is an $(n+1)_3$-configuration. (Refer to Figure 2.)

When one point extensions are generated by computer, it is necessary to name them, so that the extensions generated can be identified. We have used the following naming convention. Here a configuration (Σ, Π) is assumed, but is not explicitly indicated in the notation, as this will be clear from the context.

Definition 1.4. A 1-point extension using three lines ℓ_1, ℓ_2, ℓ_3 and three points a_1, a_2, a_3 is denoted $\mathrm{Ext}(\ell_1, \ell_2, \ell_3; a_1, a_2, a_3)$. A 1-point extension using four lines $\ell_1, \ell_2, \ell_3, \ell_4$ and four points a_1, a_2, a_3, a_4 is denoted $\mathrm{Ext}(\ell_1, \ell_2, \ell_3, \ell_4; a_1, a_2, a_3, a_4)$, and so forth.

When the starting n_3 configuration has a real or rational coordinatization, we want to use its coordinatization to find a real or rational coordinatization of the resulting $(n+1)_3$ configuration. Both Theorems 1.2 and 1.3 are needed for the extension algorithm.



(a)
initial points and lines

(b)
after 1-point extension

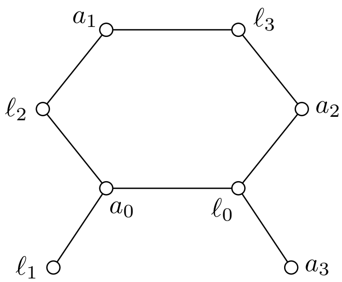
Figure 2: A 1-point extension with 4 points, before (a), after (b).

2 The coordinatization algorithm

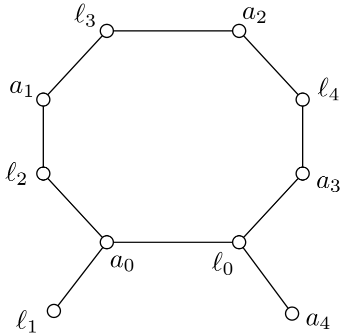
Let the points of a geometric n_3 configuration (Σ, Π) be $\{a_1, a_2, \ldots, a_n\}$ and let the lines be $\{\ell_1, \ell_2, \ldots, \ell_n\}$. Let the homogeneous coordinates of a_i be P_i, and the homogeneous coordinates of ℓ_i be L_i. These can be either real or rational. Then point a_i is incident on line ℓ_j if and only if $P_i \cdot L_j = 0$. Suppose that a 1-point extension is applied to (Σ, Π) to obtain an $(n+1)_3$ configuration (Σ', Π'), using three points and lines of (Σ, Π), as in Figure 1. We can assume that the points and lines are labelled so that the extension uses points a_1, a_2, a_3 and lines ℓ_1, ℓ_2, ℓ_3 as in Figure 1, and adds a_0 and ℓ_0.

Let G denote the incidence graph, also known as the *Levi* graph, of (Σ, Π). The subgraph induced by $\{a_1, a_2, a_3, \ell_1, \ell_2, \ell_3\}$ is a path of length five, since $a_3 \notin \ell_1$, and because the girth of the incidence graph must be at least six. After the extension, a_0 and ℓ_0 are added. Let G' be the new incidence graph. The subgraph now induced is illustrated in

Figure 3(a), since the girth of the incidence graph must be at least six. The significant feature of this subgraph is the hexagon induced by $\{a_0, a_1, a_2, \ell_0, \ell_2, \ell_3\}$. We now look for a *shortest* path Q in the incidence graph, not using any edges of the hexagon, from any one of $\{a_0, a_1, a_2\}$ to any one of $\{\ell_0, \ell_2, \ell_3\}$. This is easy to do using a breadth-first search of the incidence graph. Note that the shortest path may possibly contain a_3 and/or ℓ_1. Q must contain at least two internal vertices, i.e., one point and one line. Let the endpoints of Q be a_i and ℓ_j. If u is an internal vertex of Q, then u is not incident with the other vertices ℓ_k on the hexagon (where $k \neq j$), or there would either be a shorter path than Q, or else the girth requirement would not be satisfied. Similarly, u is not incident with the other vertices a_m on the hexagon (where $m \neq i$).



induced subgraph for Figure 1 (b)



induced subgraph for Figure 2 (b)

Figure 3: An induced subgraph of the incidence graph of (Σ', Π') of Figures 1 and 2.

We now have a *theta subgraph* in the incidence graph, that is, two vertices (a_i and ℓ_j), connected by three internally disjoint paths. When $m = 4$, the situation is similar. The vertices $a_1, a_2, a_3, a_4, \ell_1, \ell_2, \ell_3, \ell_4$ of Figure 2(b) determine a path of length 7 in the incidence graph G. After the extension, the subgraph of G' determined by Figure 2(b) is illustrated in Figure 3(b). It is necessary that this be an *induced* subgraph for the coordinatization algorithm. We now look for a *shortest* path Q in the incidence graph, not using any edges of the octagon, from any one of $\{a_0, a_1, a_2, a_3\}$ to any one of $\{\ell_0, \ell_2, \ell_3, \ell_4\}$. Let the endpoints of Q be a_i and ℓ_j. Once again we find that Q must contain at least two internal vertices, and again we have a theta-subgraph, Θ. The algorithm requires that this be an *induced* theta subgraph. The incidence graph is 3-regular, so that vertices a_i and ℓ_j are adjacent only to vertices of Θ. All other vertices of Θ are adjacent to *exactly one* vertex not in Θ. We now look for a coordinatization of (Σ', Π') such that all points and lines have the *same* coordinates as in (Σ, Π), *except* for the points and lines of Θ.

Let the homogeneous coordinates of a_i be (x, y, z), where x, y, z are real or rational indeterminates, according to whether the coordinatization of (Σ, Π) is real or rational. Then Θ contains three internally disjoint paths Q_1, Q_2, Q_3 from a_i to ℓ_j. We follow each path, and execute the following statements, assigning coordinates to its vertices in terms of x, y, z. For each vertex not in Θ, its homogeneous coordinates are those of (Σ, Π). These are known constants. The algorithm below constructs coordinates for the vertices of Θ in terms of x, y, z, by starting at a_i, and successively following each path Q_m of Θ to ℓ_j. Note that if L and L' are homogeneous coordinates of lines, then the cross product $L \times L'$ gives the homogeneous coordinates of the unique point which is the intersection of the two lines. Similarly $P \times P'$ gives the homogeneous coordinates of the unique line containing points with coordinates P and P'.

procedure FOLLOWPATH(a_i, ℓ_j, Q_m)
comment: follow a path Q_m of Θ from a_i to ℓ_j, assigning coordinates

$u \leftarrow a_i$
$v \leftarrow$ first vertex on path Q_m after a_i
while $v \neq \ell_j$
do
  if v is a point
    then
      let ℓ be the unique adjacent line not in Θ
      let L be the known coordinates of ℓ
      let L' be the assigned coordinates of u
      $P \leftarrow L \times L'$
      assign P as the coordinates of v
    else
      let a be the unique adjacent point not in Θ
      let P be the known coordinates of a
      let P' be the assigned coordinates of u
      $L \leftarrow P \times P'$
      assign L as the coordinates of v
  $u \leftarrow v$
  $v \leftarrow$ next vertex on path Q_m after u
comment: every vertex of Q_m except for ℓ_j now has coordinates assigned

Observation. *Once the algorithm* FOLLOWPATH() *has been executed for each path of Θ, all vertices of Θ except for ℓ_j have homogeneous coordinates assigned such that each coordinate is a* linear homogeneous function *of x, y, z.*

There are three vertices of Θ adjacent to ℓ_j. Let their coordinates be P, P' and P''. Define the polynomial $p(x, y, z) = P \cdot P' \times P''$.

Observation. *$p(x, y, z)$ is a* cubic homogeneous polynomial *in x, y, z.*

Note that by projective duality we could equally well follow the paths in the other direction, from ℓ_j to a_i, starting with (x, y, z) as the coordinates of ℓ_j.

Theorem 2.1. *If there is a coordinatization of (Σ', Π') such that all points and lines not in Θ have the same coordinates as in (Σ, Π), then the values of x, y, z must satisfy $p(x, y, z) = 0$.*

Proof. The three points incident on ℓ_j all belong to Θ, with coordinates P, P', P''. Therefore $P \cdot P' \times P'' = p(x, y, z) = 0$. Note that the coordinates of ℓ_j can be taken as any one of $P \times P'$, $P \times P''$ or $P' \times P''$. □

Thus, if there is a coordinatization of (Σ', Π') of the type we are looking for, we can find it by solving $p(x, y, z) = 0$ for x, y, z. In general, there will be many values (x, y, z) with $p(x, y, z) = 0$. They do not all give valid coordinatizations. According to the current coordinatization of (Σ, Π), we want the values to be either real or rational. We will use a method that originated with Diophantus (see [1]), as frequently used in the theory of elliptic curves [3, 20]. Now the groups defined by elliptic curves are used for cryptography, because it is relatively easy to calculate with them, but a characterization of the groups appears to be algorithmically intractable. A similar situation exists in the search for coordinatizations of n_3 configurations. But if we can find suitable values of x, y, z such that $p(x, y, z) = 0$, then a real or rational coordinatization of (Σ', Π') can be relatively easy to find. The method described below works very effectively.

Lemma 2.2. *Let ℓ be any one of the three lines adjacent to a_i in Θ, and let its coordinates be L. Let a be the unique point not in Θ adjacent to ℓ, and let its coordinates be P. If (x, y, z) is set equal to P, then $p(x, y, z) = 0$.*

Proof. If $(x, y, z) = P$, then $L = P \times (x, y, z) = (0, 0, 0)$. Each subsequent vertex on this path in Θ will have coordinates $(0, 0, 0)$, so that ℓ_j will also have coordinates $(0, 0, 0)$. Therefore $p(x, y, z) = 0$. □

As there are three lines in Θ adjacent to a_i, this gives three different points (x, y, z) with $p(x, y, z) = 0$. None of these give coordinatizations of (Σ', Π'), because $(0, 0, 0)$ is not a valid homogeneous coordinate. However, we can now proceed as follows.

Suppose that $p(x, y, z) = 0$, for some value $(x, y, z) = (u, v, w)$. The equation $p(x, y, z) = 0$ defines a *cubic* curve in the projective plane. The tangent line at point (u, v, w) has the equation $x\partial p/\partial x + y\partial p/\partial y + z\partial p/\partial z = 0$, where the partial derivatives are evaluated at (u, v, w). This is a *linear* equation in (x, y, z). As long as at least one partial derivative is non-zero, say $\partial p/\partial z$, we can solve for the associated variable, and obtain $z = -[x\partial p/\partial x + y\partial p/\partial y]/[\partial p/\partial z]$ along the tangent line. This is substituted into the cubic homogeneous polynomial $p(x, y, z) = 0$ to obtain $q(x, y) = 0$, where $q(x, y)$ is a cubic homogeneous polynomial in x, y. At this point, we can divide by y^3 to obtain the cubic polynomial $q(x/y, 1) = 0$ in one variable x/y. Now $q(x/y, 1) = 0$ has three roots, of which one, $x/y = u/v$, is already known (note: if $v = 0$, use $y/x = v/u$ instead). The tangent line has *double contact* (see [3]) with the curve $p(x, y, z) = 0$ at $(x, y, z) = (u, v, w)$. Therefore we can divide $q(x, y)$ by $vx - uy$ *twice* to obtain a *linear* homogenous equation $h(x, y) = 0$. The single root of $h(x, y)$ is then easy to find, even over the rational numbers. Combining this with the expression for z, we obtain another root $(x, y, z) = (u', v', w')$ of $p(x, y, z) = 0$.

This new value for (x, y, z) is now substituted into the coordinates for the vertices of Θ, and the coordinates (which are linear homogeneous functions of x, y, z) of all vertices of Θ are evaluated. It is then quickly determined whether this produces a valid coordinatization of (Σ', Π'). The conditions that must be satisfied are:

1. All points must have inequivalent homogeneous coordinates;
2. All lines must have inequivalent homogeneous coordinates;
3. $P \cdot L = 0$ only if point P is incident with line L.

If some points or lines coincide, or if unwanted incidences are produced, then the method can be repeated, starting from $(x, y, z) = (u', v', w')$. Either a new point (u'', v'', w'') will be found, or else a value previously found will recur, and so forth.

This can be done for each of the three lines in Θ adjacent to a_i, which frequently produces a number of valid coordinatizations of (Σ', Π').

There is still another possibility. The coordinates of any two of the three lines in Θ adjacent to a_i determine a line in the projective plane, intersecting the curve $p(x, y, z) = 0$

in two known points. The third point of intersection is then easy to find. This calculation allows a sequence of points satisfying $p(x, y, z) = 0$ to be found. We can then continue with the tangents from these points, or take any two known roots on the curve to find another. The number of points on the curve that can be generated from the starting values can be either finite or countably infinite, as this is the situation that holds for rational points on elliptic curves (see [20]).

We summarize this method as two theorems.

Theorem 2.3. *Let $(x, y, z) = (u, v, w)$ be a rational solution of the cubic homogeneous polynomial $p(x, y, z) = 0$. If at least one of $\partial p/\partial x, \partial p/\partial y, \partial p/\partial z$ evaluated at $(x, y, z) = (u, v, w)$ is non-zero, then the tangent line $x\partial p/\partial x + y\partial p/\partial y + z\partial p/\partial z = 0$ intersects the curve in another rational point.*

Theorem 2.4. *Let $(x, y, z) = (u_1, v_1, w_1)$ and $(x, y, z) = (u_2, v_2, w_2)$ be two rational solutions of the cubic homogeneous polynomial $p(x, y, z) = 0$. Then the line containing (u_1, v_1, w_1) and (u_2, v_2, w_2) intersects the curve in another rational point.*

In practice, we want at least two of the partial derivatives to be non-zero at $(x, y, z) = (u, v, w)$. For if two of them are zero, then solving for the third variable forces one of x, y, z to be zero. This invariably leads to a solution which does not give a valid coordinatization. (However, it can then be used to find another rational solution.)

Once a valid coordinatization of (Σ', Π') has been found for a suitable value $(x, y, z) = (u, v, w)$, this process can be repeated, and more coordinatizations can be found. In general, numerous coordinatizations for a given configuration can be found in this way. They are inter-related through tangents to the cubic polynomial $p(x, y, z)$, and through lines containing pairs of rational solutions, similar to the relation between points of the group of rational points on an elliptic curve.

Example. We begin with a rational coordinatization of a (9_3) configuration, shown in Figure 4. This is the (9_3) configuration listed as $(9_3)_2$ in Figure 2.2.1 of [10], and as 9.2 in [2]. It is cyclic and self-dual, with an automorphism group of order 9. The two "parallel" lines ℓ_4 and ℓ_8 meet in point a_9 at infinity. Similarly ℓ_5 and ℓ_7 meet in a_8 at infinity, and lines ℓ_1 and ℓ_3 meet in a_2 at infinity. These three points at infinity are all contained in the line ℓ_6, which is the "line at infinity". The drawing is based on the rational coordinatization of the configuration given by the coordinates shown in Table 1.

Table 1: Rational coordinates of the 9_3 configuration of Figure 4.

$P_1 = (2, 4, -3)$	$L_1 = (1, 1, 2)$
$P_2 = (-1, 1, 0)$	$L_2 = (2, -1, 0)$
$P_3 = (1, 2, -3)$	$L_3 = (1, 1, 1)$
$P_4 = (1, 1, -1)$	$L_4 = (0, 3, 2)$
$P_5 = (0, 0, 1)$	$L_5 = (3, 0, 2)$
$P_6 = (1, 0, -1)$	$L_6 = (0, 0, 1)$
$P_7 = (2, 2, 3)$	$L_7 = (1, 0, 1)$
$P_8 = (0, 1, 0)$	$L_8 = (0, 1, 0)$
$P_9 = (1, 0, 0)$	$L_9 = (1, -1, 0)$

A 1-point extension using four points $\mathrm{Ext}(\ell_1, \ell_9, \ell_4, \ell_6; a_4, a_7, a_9, a_8)$, as in Figure 2, is then done. (This example using 4 points was chosen instead of one using 3 points,

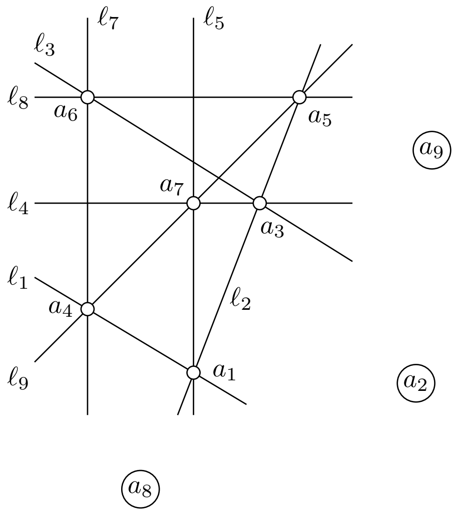


Figure 4: A 9_3 configuration.

because the resulting (10_3) configuration has a "nice" drawing.) Observe that $a_4 = \ell_1 \cap \ell_9$, $a_7 = \ell_9 \cap \ell_4$, $a_9 = \ell_4 \cap \ell_6$, and that $a_8 \in \ell_6$. The third line through a_4 is ℓ_7. It intersects ℓ_6 in a_8, as required for the 1-point extension. The result of the extension is the 10_3 configuration shown in Figure 5. It is $(10_3)_6$ in Grünbaum [10]. Lines ℓ_1, ℓ_3 and ℓ_6 in Figure 5 meet in point a_2 at infinity. Points and lines whose coordinates did not change from (9_3) are drawn in heavier lines. (But note that the scaling of the two diagrams may be slightly different.)

In order to find a rational coordinatization of it, we first find a theta subgraph by searching for a shortest path from one of a_4, a_7, a_9, a_{10} to one of $\ell_9, \ell_4, \ell_6, \ell_{10}$, where a_{10} and ℓ_{10} are the new point and line that were added. The theta subgraph is shown in Figure 6. It consists of the octagon of Figure 3(b) and the shortest path just found. This is partly indicated in Figure 5. The "corners" of the theta subgraph, a_4 and ℓ_{10}, are shaded light grey. With the aid of Figure 6, the paths can be traced out in Figure 5.

We now assign homogeneous coordinates (x, y, z) to ℓ_{10}, as it is one of the "corner" vertices of the theta subgraph, and using the coordinates of Table 1 for the points and lines not in the theta subgraph, we calculate coordinates for those of the theta subgraph in terms of (x, y, z). Each point or line of the theta subgraph (except for the "corner" vertices) is adjacent to exactly one line or point not in the theta subgraph. The adjacent vertices can be determined from Figure 5. The calculated homogeneous coordinates are linear homogeneous forms, shown in Table 2. Note that homogeneous coordinates can be multiplied by a constant without changing the configuration. Therefore sometimes the coordinates in Table 2. were multiplied by -1, or a common factor was removed from the individual coordinates in order to simplify them. We then find that $p(x, y, z) = L_9 \cdot L_7 \times L_4$, which is expanded to

$$p(x, y, z) = -4x^3 + 4x^2y + 4x^2z + 7xz^2 - 16xyz + 11yz^2 - 6z^3$$

where a common factor of six has been removed from each term. The partial derivatives

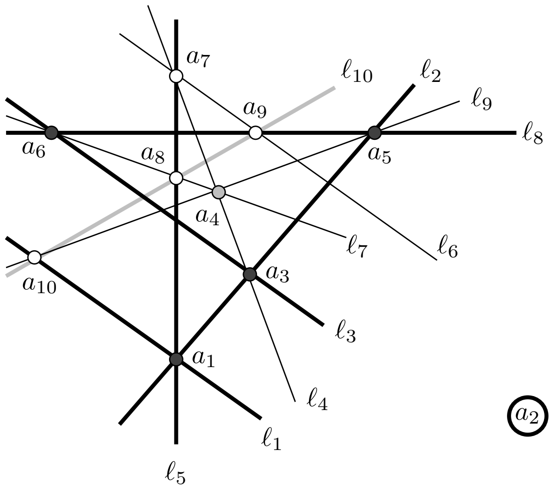


Figure 5: The extended 10_3 configuration.

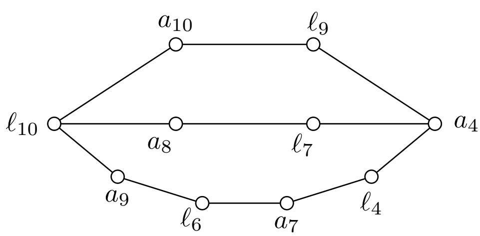


Figure 6: A theta subgraph in the 10_3 configuration.

are

$$\begin{aligned}\partial p/\partial x &= -12x^2 + 8xy + 8xz - 16yz + 7z^2\\ \partial p/\partial y &= 4x^2 - 16xz + 11z^2\\ \partial p/\partial z &= 4x^2 - 16xy + 14xz + 22yz - 18z^2\end{aligned}$$

We have three known solutions to $p(x, y, z) = 0$, namely

$$\begin{aligned}(x, y, z) &= L_1 = (1, 1, 2) \text{ (which makes } P_{10} = (0, 0, 0)),\\ (x, y, z) &= L_5 = (3, 0, 2) \text{ (which makes } P_8 = (0, 0, 0)),\\ (x, y, z) &= L_6 = (0, 0, 1) \text{ (which makes } P_9 = (0, 0, 0)).\end{aligned}$$

The tangent line at $(x, y, z) = L_1 = (1, 1, 2)$ has equation $2x + 4y - 3z = 0$. Solving for $2x = -4y + 3z$, substituting this into $p(x, y, z) = 0$, and removing common factors gives

$$q(y, z) = 4y^3 - 4y^2z + yz^2$$

The point L_1 on the tangent line has $(y, z) = (1, 2)$ so that $q(y, z)$ is divisible twice by $2y - z$. We find that

$$q(y, z) = 6y(2y - z)^2$$

Therefore the third point of intersection of the tangent with $p(x, y, z) = 0$ occurs when $y = 0$. Then since $2x + 4y - 3z = 0$, we can take $z = 2$, and obtain $2x = -4y + 3z = 6$, giving $(x, y, z) = (3, 0, 2)$. This does not give a valid solution, as it makes $P_8 = (0, 0, 0)$.

Table 2: Homogeneous coordinates for the theta subgraph.

L_{10}	$= (x, y, z)$	
P_{10}	$= L_{10} \times L_1$	$= (2y - z, z - 2x, x - y)$
P_8	$= L_{10} \times L_5$	$= (2y, 3z - 2x, -3y)$
P_9	$= L_{10} \times L_8$	$= (-z, 0, x)$
L_9	$= P_{10} \times P_5$	$= (z - 2x, z - 2y, 0)$
L_7	$= P_8 \times P_6$	$= (2x - 3z, -y, 2x - 3z)$
L_6	$= P_9 \times P_2$	$= (x, x, z)$
P_7	$= L_6 \times L_5$	$= (-2x, 2x - 3z, 3x)$
L_4	$= P_7 \times P_3$	$= (4x - 3z, x, 2x - z)$

We then try the tangent line at $(x, y, z) = L_5 = (3, 0, 2)$, which has equation $2x + y - 3z = 0$. Solve for $y = 3z - 2x$ and substitute this into $p(x, y, z)$ to obtain

$$q(x, z) = 4x^3 - 16x^2z + 21xz^2 - 9z^3$$

The known solution is $(x, z) = (3, 2)$, so that this is divisible twice by $2x - 3z$, giving

$$q(x, z) = (x - z)(2x - 3z)^2$$

We find that the third intersection point with the curve $p(x, y, z) = 0$ occurs when $x = z$. Without loss of generality, we take $(x, y, z) = (1, 1, 1)$. If we then calculate the coordinates, we find that L_{10} and L_6 both have coordinates $(1, 1, 1)$, which is not acceptable.

However, this gives another rational point on the curve, so we find the tangent line at $(x, y, z) = (1, 1, 1)$. It is $-5x - y + 6z = 0$. We substitute $y = 6z - 5x$ into $p(x, y, z)$ to obtain

$$q(x, z) = 2x^3 - 9x^2z + 12xz^2 - 5z^3$$

The known solution is $(x, z) = (1, 1)$, so that this is divisible twice by $x - z$, giving $q(x, z) = (2x - 5z)(x - z)^2$. The third point of intersection is therefore $(x, y, z) = (5, -13, 2)$. This value of (x, y, z) is then found to give a valid coordinatization of the 10_3 configuration found. The coordinates that result are shown in Table 3.

At this point, the algorithm could continue, and find the tangent line at $(x, y, z) = (5, -13, 2)$ to look for more rational coordinatizations. Or the known rational points on the curve could be taken two at a time, as the line containing two points intersects the curve in a third rational point, and so forth. In practice, very many rational coordinatizations can be found in this way from a single theta subgraph of a single one-point extension of a geometric configuration.

Table 3: Rational coordinates of the 10_3 configuration of Figure 5.

$P_1 = (2, 4, -3)$	$L_1 = (1, 1, 2)$
$P_2 = (-1, 1, 0)$	$L_2 = (2, -1, 0)$
$P_3 = (1, 2, -3)$	$L_3 = (1, 1, 1)$
$P_4 = (-14, -4, 27)$	$L_4 = (14, 5, 8)$
$P_5 = (0, 0, 1)$	$L_5 = (3, 0, 2)$
$P_6 = (1, 0, -1)$	$L_6 = (5, 5, 2)$
$P_7 = (-10, 4, 15)$	$L_7 = (4, 13, 4)$
$P_8 = (-26, -4, 39)$	$L_8 = (0, 1, 0)$
$P_9 = (-2, 0, 5)$	$L_9 = (-2, 7, 0)$
$P_{10} = (-14, -4, 9)$	$L_{10} = (-5, 13, -2)$

We now start from the 10_3 configuration of Figure 5, with the rational coordinatization given in Table 3. There is a one-pont extension $\mathrm{Ext}(\ell_{10}, \ell_6, \ell_3; a_9, a_2, a_6)$ that can be done, resulting in an 11_3 configuration. Its incidence table is given in Table 4. This configuration is isomorphic to configuration $(11_3)X$ in Martinetti [17]. The new point and line added are a_{11} and ℓ_{11}. We use a theta subgraph to find a rational coordinatization of it. The theta subgraph consists of the three paths $[a_9, \ell_3, a_2, \ell_{11}]$, $[a_9, \ell_6, a_{11}, \ell_{11}]$, $[a_9, \ell_8, a_6, \ell_{11}]$.

There are many rational coordinatizations that result. One of them is shown in Table 5. We see that the integer coordinates are getting bigger. This is the single greatest obstacle that the algorithm has to deal with. One of the questions that needs to be addressed is how to limit the number of digits in the integers that arise. It is very easy for integer overflow to occur after several successive extensions have been done.

The Desargues configuration cannot be obtained by a 1-point extension (see [14]). The "anti-Pappian" (see [8, 16]) is the only non-geometric 10_3 configuration. Rational coordinatizations of *all* the other (10_3) configurations, can be easily found using one-point extensions of the (9_3) configurations in this way. Then rational coordinatizations of all the (11_3) configurations can be found from the (10_3) configurations, which then extend to coordinatizations of all the (12_3) configurations. The author has written a computer program to generate coordinatizations from a theta subgraph in a one point extension. It produces *thousands* of them very quickly. Currently the program has to be run individually for each starting configuration, and the resulting output files must be individually collated and then tested for isomorphisms.

Table 4: The 11_3 configuration extended from Figure 5.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}
1	1	2	3	1	7	4	5	4	8	2
2	3	3	4	7	9	6	6	5	10	6
10	5	9	7	8	11	8	9	10	11	11

Table 5: Rational coordinates of the 11_3 configuration extended from Table 3.

$P_1 = (2, 4, -3)$	$L_1 = (1, 1, 2)$
$P_2 = (16, -34, 9)$	$L_2 = (2, -1, 0)$
$P_3 = (1, 2, -3)$	$L_3 = (28, 19, 22)$
$P_4 = (-14, -4, 27)$	$L_4 = (14, 5, 8)$
$P_5 = (0, 0, 1)$	$L_5 = (3, 0, 2)$
$P_6 = (-136, 4, 123)$	$L_6 = (27, -15, 22)$
$P_7 = (-10, 4, 15)$	$L_7 = (4, 13, 4)$
$P_8 = (-26, -4, 39)$	$L_8 = (1, 34, 0)$
$P_9 = (-748, 22, 933)$	$L_9 = (-2, 7, 0)$
$P_{10} = (-14, -4, 9)$	$L_{10} = (-5, 13, -2)$
$P_{11} = (-64, -14, 69)$	$L_{11} = (37, 28, 40)$

3 In practice

Given an n_3 configuration (Σ, Π), it is relatively easy to write a computer algorithm that searches for all possible one-point extensions $\mathrm{Ext}(\ell_1, \ell_2, \ell_3; a_1, a_2, a_3)$ or $\mathrm{Ext}(\ell_1, \ell_2, \ell_3, \ell_4; a_1, a_2, a_3, a_4)$, and extends (Σ, Π) to an $(n+1)_3$ configuration (Σ', Π'), in all possible ways. We also want a coordinatization of (Σ', Π') when (Σ, Π) is geometric. For *each* extension (Σ', Π') found, the coordinatization algorithm of the previous section can be used to look for a coordinatization of (Σ', Π'). There are various situations that one has to be aware of when programming this.

1. The polynomial $p(x, y, z)$ is a cubic homogeneous polynomial in three variables. Sometimes a cubic polynomial will factor into the product of three linear homogeneous polynomials, or a linear and quadratic polynomial. In these cases the algorithm will not succeed. This happens occasionally in practice. It will usually be detected when the tangent is found. Not every extension (Σ', Π') has a coordinatization extended using a given theta subgraph. However, another theta subgraph can be chosen in this case.
2. The tangent at $(x, y, z) = (u, v, w)$ is a linear homogeneous polynomial. It may be identically 0. In this case the extension does not succeed.
3. The tangent at $(x, y, z) = (u, v, w)$ may be be a monomial, e.g., $x = 0$. This does not tend to produce valid coordinatizations.

4. Suppose that the tangent at $(x, y, z) = (u, v, w)$ is $ax+by+cz = 0$. Solving for one variable, e.g., $cz = -ax - by$ and substituting this into $p(x, y, z)$ gives the reduced polynomial $q(x, y) = 0$, which is divisible twice by $vx - uy$. It can happen that $q(x, y)/(vx - uy)^2$ is a monomial, e.g., $q(x, y) = x(vx - uy)^2$. This gives $x = 0$, from which we find the solution $(x, y, z) = (0, -c, b)$. This frequently occurs as a special case.
5. The general case is when $q(x, y)$ factors into $(vx - uy)^2(rx + sy)$. In this case the solution is $cx = cs, cy = -cr$ and $cz = -as + br$, or equivalently, $(cs, -cr, -as + br)$ is taken as the solution. The majority of solutions fall into this case.
6. The algorithm stores an array of solutions $(x, y, z) = (u, v, w)$ to $p(x, y, z) = 0$. Initially there are three such points (u, v, w) known, and they are known not to give valid coordinatizations of (Σ', Π'). They are placed on the array of solutions. For *each* (u, v, w) on the array, the tangent is used to find another possible solution, which is *appended* to the array. The solutions on the array are then taken in pairs (u_1, v_1, w_1) and (u_2, v_2, w_2), to find more solutions, which are also appended to the array. The algorithm proceeds to build an array of all solutions (u, v, w) that can be obtained by these methods. This is similar to generating the elements of a group. Typically a potentially infinite number of solutions will be found, so that a limit must be placed on the maximum number allowed. The algorithm can stop with the first valid coordinatization found, or it can look for some maximum number of valid coordinatizations. It can easily find *thousands* of valid integer coordinatizations. However, the values of the integers u, v, w rapidly become enormous if a large number of coordinatizations is required, causing integer overflow even when 64-bit integers are used. The author has programmed it to find a maximum of three valid coordinatizations for each extension (Σ', Π') found, using 64-bit integers, and using only one theta subgraph. More theta subgraphs could be chosen.

If (Σ, Π) is an n_3 configuration, there will be various $(n+1)_3$ configurations that can be produced from it by one-point extensions. If (Σ', Π') is such an $(n+1)_3$ configuration, then there are usually very many different extensions of (Σ, Π) that give rise to an isomorphic (Σ', Π'). Each extension will have up to three coordinatizations found. And this same (Σ', Π') may also arise by a one-point extension from another n_3 configuration, which will also produce numerous coordinatizations of (Σ', Π'). The result is thousands of integer coordinatizations for (Σ', Π') when $n = 10, 11$ or 12. Graph isomorphism software is used to distinguish and recognize the various configurations that are produced. The author has used the software of [12], although others could also be used. The configuration is represented by its Levi graph, with an initial partition of vertices into points and lines.

This method of finding coordinatizations is *much* simpler than that of [22, 23] because it only requires finding the roots of cubic homogeneous polynomials with three variables, whereas [23] states that solving their general diophantine equations for (n_3) configurations is likely to be recursively undecidable.

So far, the author has used this method to produce integer coordinatizations of all the geometric (10_3), (11_3) and (12_3) configurations. As n increases, the integer coordinates rapidly tend to have more and more digits, so that it is necessary to filter them somewhat to limit the number of digits in the coordinates. If fixed size integers are used (e.g. 64 bits), overflow can soon occur, which limits the number of coordinatizations found. It is advantageous to choose a coordinatization of (Σ, Π) to extend from, whose coordinates are "small" integers. Very many coordinatizations of (Σ, Π) are then obtained. This is the case with $n = 10, 11, 12, 13$, where thousands of coordinatizations are easily found. If multi-precision integer arithmetic is used, it is likely that coordinatizations can be found for nearly any fixed n.

The number of distinct (13_3) configurations is 2036 (see [10], p. 69). One of these is a Fano-type configuration, as described in [14], and therefore does not arise as a one point extension. Using ad-hoc methods, the author has shown that it is geometric, and in fact has a rational coordinatization. The other 2035 (13_3) configurations can all be constructed as one point extensions of (12_3) configurations. All of them are geometric, and all have rational coordinatizations. The coordinatization algorithm finds many integer coordinatizations of them. One of them was much more difficult than all the others, requiring integer coordinates with up to 22 digits in the intermediate calculations, and 13 in the final coordinates. For this one configuration, the algorithm was carried out by hand using *Maple* [24] as a calculator with unlimited precision. Maple was also used for constructing a coordinatization of the Fano-type configuration. The description of the coordinatizations is too long to include here. An article containing the details is currently in preparation.

4 Additional coordinatizations

Suppose that (Σ, Π) is an n_3 configuration for which an integer coordinatization is known. We would like to find more integer coordinatizations. One method is this.

1. Find an induced theta subgraph Θ in the incidence graph of (Σ, Π). This is most easily done by finding an induced cycle of reasonable length, and then finding a suitable path across the cycle. The path must have odd length.
2. The vertices not in Θ are to keep their current coordinates. One of the vertices of degree three in Θ is chosen to have coordinates (x, y, z), with values to be determined.
3. The algorithm FOLLOWPATH() is used to assign coordinates that are homogeneous linear forms to the vertices of Θ of degree two. A polynomial $p(x, y, z)$ is constructed using the second vertex of degree three of Θ. Solutions of $p(x, y, z) = 0$ are found as in the previous section.

This allows us to find "related" coordinatizations of (Σ, Π). The author has used this method to produce many rational coordinatizations of the (9_3) configurations, which can then be used as starting points for the generation and coordinatization of the (10_3) configurations and beyond. A given Θ may not produce any additional coordinatizations. In general, different choices of Θ will produce different results. This method is less reliable that the extension method of the previous section. The reason seems to be that the polynomial $p(x, y, z)$ frequently has large integer coefficients, resulting in solutions which lead to integer overflow. For some configurations (Σ, Π), no additional coordinatizations are found like this. For others, it gives dozens of new coordinatizations.

5 Real coordinatizations – the anti-Pappian

The previous sections are concerned with using one-point extensions to find rational coordinatizations of n_3 configurations. Theorems 2.3 and 2.4 also apply to real coordinatizations. The *anti-Pappian* [8, 16, 19] is the only (10_3) configuration that is not geometric. It cannot be coordinatized over *any* field, as shown in [8, 16]. However, it can be coordinatized over the quaternions [16].

The anti-Pappian can be obtained by a one-point extension from a geometric (9_3) configuration (it is $(9_3)_3$ in [10] and 9.1 in [2], a self-dual configuration with an automorphism group of order 12). When the extension algorithm is applied to find a coordinatization, it is necessary to divide polynomials. It is easy to divide polynomials with integer coefficients, as the division is always exact. However, when a computer works with real numbers, they are represented as floating point numbers, and round-off error is always present. Consequently division will always leave a non-zero remainder, which is usually very small, even when the division is theoretically exact. A suitably small number is then replaced by zero, e.g. 10^{-9}. When $P_i \cdot L_j$ is evaluated to test for incidence of a point and line, the result will usually not be exactly zero, due to round-off error, even if they are incident. So if $P_i \cdot L_j$ is sufficiently close to zero, it must be considered to be zero. Thus, it is possible to have a point and line not *exactly* incident, but *very, very* close to incident, for example, $|P_i \cdot L_j| \leq 10^{-9}$. Thus, a *near-coordinatization* can be found. *Every* real coordinatization found using floating point numbers is in fact a near-coordinatization.

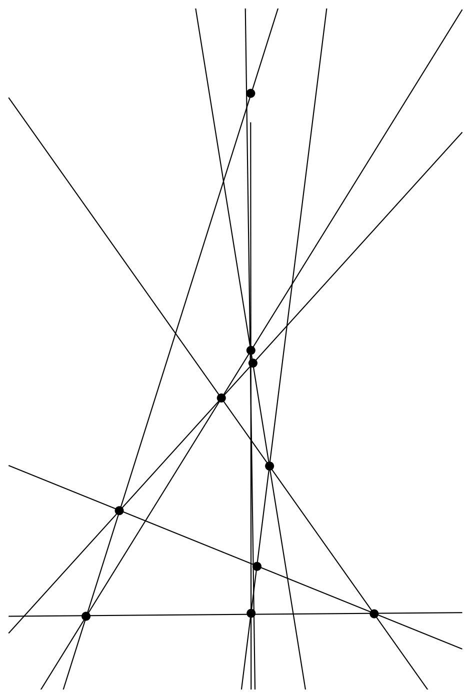


Figure 7: A near-coordinatization of the anti-Pappian configuration.

When the coordinatization algorithm is applied to the extension that produces the anti-Pappian, several near-coordinatizations are found, even though the anti-Pappian cannot be coordinatized over the reals. One of them is shown in Figure 7.

Question. Let ε be a small positive real value, and let Δ be a fixed positive real value, e.g., $\Delta = 1$. How small can ε be chosen so that there is a near-coordinatization of the anti-Pappian configuration such that $|P_i \cdot L_j| \leq \varepsilon$ for all points P_i and lines L_j which are incident, and $|P_i \cdot L_j| \geq \Delta$ if P_i and L_j are non-incident?

Grünbaum [10] (p. 151) also asks whether there are any n_3 configurations with $n > 10$ which are non-geometric? One place to look for them is the *Fano-type* configurations of [14], as they cannot be constructed using a one point extension, and so are not accessible to the cubic-polynomial-based coordinatization algorithm. The smallest Fano-type configuration is the unique (7_3). The next one is a (13_3) configuration (which *is* geometric). Then (14_3).

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