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Variacijski način določitve enačbe mazanja ne-newtonske tanke plasti

A Variational Approach to the Establishment of a Lubrication Equation for a Non-Newtonian Thin Film

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V prispevku je predstavljen nov postopek izpeljave enačbe mazanja za ne-newtonsko tanko plast. Enačba Reynoldseve vrste je izpeljana neposredno iz varacijskega načela kot pogoj stacionarnosti. V postopku izpeljave smo uporabili tudi metodo istoležne motnje.

(Ključne besede: enačbe Reynolds, tok nenewtonski, mazanje, metode perturbacij)

This paper provides a new approach to deriving a lubrication equation for a non-Newtonian thin film. The Reynolds-type equation is directly deduced from a variational principle as a stationary condition. The homotopy perturbation method is also applied in the derivation procedure.

(Keywords: Reynolds-type equation, non-Newtonian flow, lubrication, perturbation methods)

0 INTRODUCTION

In the analysis of many lubrication problems, the most important thing is to establish a lubrication equation. Often, the number of governing equations can be reduced by a direct elimination of the pertinent variables. In this process of elimination some of the equations are in effect satisfied identically, and the error (in an approximate analysis) remains in the reduced set of governing equations. The standard approach to the establishment of the Reynolds equation is to integrate the governing equations over the film; the integration constants are determined from the boundary conditions. The procedure is easy for Newtonian lubrication, but it is very difficult for non-Newtonian conditions ([1] to [6]). Clearly, by a judicious identification of the integration constants we can arrive at different forms of Reynolds-type equations and therefore cast the error of the approximate analysis in the obtained lubrication equations.

Variational theory is a powerful tool in finite-element methods and when dealing with the cavitations in lubrications. Our aim in this paper is to deduce the required lubrication equation by minimizing the energy functional, and the obtained Reynolds-type equation, therefore, best serves to describe the lubrication problem. The technique developed provides a powerful mathematical tool for engineers in tribology.

1 OUTLINE OF THE DERIVATION

For a better illustration of the basic procedure, making the underlying idea clear and not clouded by unnecessarily complicated forms of mathematical expressions, we consider a one-dimensional lubrication problem as an illustrating example.

The geometrical configuration of a one-dimensional slider bearing, considering the squeeze-action effect, is shown in Figure 1. It is assumed that the flow is isothermal, incompressible and laminar, and that the lubricant inertia effect is small. According to the thin-film theory of hydrodynamic lubrication, the equations of continuity and motion in Cartesian coordinates reduce to:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

$$\frac{\partial p}{\partial x}=\frac{\partial \tau_{xz}}{\partial z} \quad (2)$$

$$\frac{\partial p}{\partial z}=0 \quad (3).$$

And the nonlinear constitutive relation is written in the following general form:

$$\tau_{xz}=\mu_0\frac{\partial u}{\partial z}+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1} \quad (4),$$

where μ_0 is the zero shear-rate viscosity, which is equivalent to the viscosity of Newtonian fluids, and a_n $(n\geq 1)$ denotes a nonlinear factor accounting for non-Newtonian effects.

In view of Eq.(4), Eq.(2) can be rewritten in the form:

$$\frac{\partial p}{\partial x}=\frac{\partial}{\partial z}\left[\mu_0\frac{\partial u}{\partial z}+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}\right] \quad (5).$$

The boundary conditions are illustrated in Fig.1.

The variational approach to lubrication has now become a popular tool for establishing various Reynolds-type equations for various non-Newtonian lubrication problems ([7] and [8]). This paper aims to establish a variational principle for the above problem; the Euler equations of the searched variational functional are Eqs. (1), (3) and (5). To this end, we apply the semi-inverse method [9], and establish a trial-functional in the following form

$$J(u,w,p)=\int_0^L\int_0^h L(u,w,p)dxdz \quad (6),$$

where u, w and p are independent variables, L is a trial-Lagrangean. There exists many approaches to the construction of a trial-Lagrangean, and illustrating examples can be found in Refs.[10] to [13]. In this paper we write the trial-Lagrangean in the form:

$$L(u,w,p,)=-p(\frac{\partial u}{\partial x}+\alpha\frac{\partial w}{\partial z})+(1-\alpha)w\frac{\partial p}{\partial z}+F(u) \quad (7),$$

where F is an unknown function of u and its derivatives, α is an arbitrary constant. The advantage of the above trial-Lagrangean lies in the fact that the stationary condition with respect to p is Eq.(1).

Calculating the variation with respect to u, we have the following Euler equation:

$$\frac{\partial p}{\partial x}+\frac{\delta F}{\delta u}=0 \quad (8),$$

where $\delta F/\delta u$ is called the variational derivative with respect to u, defined as:

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}}-\frac{\partial}{\partial z}\frac{\partial F}{\partial u_{,z}} \quad (9).$$

The Euler equation (8) should satisfy one of the field equations, e.g., Eq.(5), so we set:

$$\frac{\delta F}{\delta u}=-\frac{\partial p}{\partial x}=-\frac{\partial}{\partial z}\left[\mu_0\frac{\partial u}{\partial z}+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}\right] \quad (10),$$

from which F can be identified as:

$$F=\frac{1}{2}\mu_0(\frac{\partial u}{\partial z})^2+\sum_{n=1}^{m}\frac{1}{2n+2}a_n(\frac{\partial u}{\partial z})^{2n+2} \quad (11).$$

By incorporating the boundary conditions, we finally obtain the following generalized variational principle:

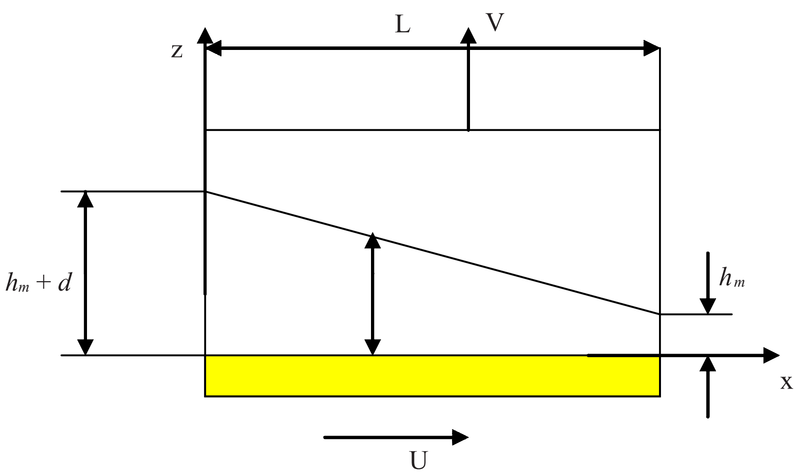

Fig.1 *Geometrical configuration of a wide slider bearing*

$$J(u,w,p)=\int_0^L\int_0^h\left\{\begin{array}{l}-p(\frac{\partial u}{\partial x}+\alpha\frac{\partial w}{\partial z})+(1-\alpha)w\frac{\partial p}{\partial z}+\\+\frac{1}{2}\mu_0(\frac{\partial u}{\partial z})^2+\sum_{n=1}^{m}\frac{1}{2n+2}a_n(\frac{\partial u}{\partial z})^{2n+2}\end{array}\right\}dxdz$$
$$-\int_0^L(1-\alpha)\left[w(h)-w(0)\right]pdx+\int_0^L U\,\tau_{xz}\big|_{z=0}\,dx \quad (12).$$

Note: at the boundary, Eq.(4) is considered as a constraint.

It is easy to prove that all Euler equations satisfy the governing equations (1), (3) and (5), and the boundary conditions. Calculating the first variation with respect to *u,w,* and *p*, we have:

$$\delta J(u,w,p)=\int_0^L\int_0^h\left\{\begin{array}{l}\frac{\partial p}{\partial x}\delta u-\frac{\partial u}{\partial x}\delta p+\frac{\partial p}{\partial z}\delta w-\frac{\partial w}{\partial z}\delta p-\\-\frac{\partial}{\partial z}\left[\mu_0(\frac{\partial u}{\partial z})^2+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}\right]\delta u\end{array}\right\}dxdz$$
$$+\int_0^L(1-\alpha)\,w\big|_0^h\,\delta p dx-\int_0^L u\delta\left[\mu_0(\frac{\partial u}{\partial z})^2+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}\right]\Bigg|_{z=0}dx$$
$$-\int_0^L(1-\alpha)\left[w(h)-w(0)\right]\delta p dx+\int_0^L U\,\delta\tau_{xz}\big|_{z=0}\,dx=0. \quad (13).$$

Considering the independence of δ*u*, δ*v*, and δ*p*, and noting the constraint , Eq.(4), we obtain, as steady conditions, Eqs.(1), (3), (5), and boundary conditions *w*=*w*(*h*) at *z*=*h,* *w*=*w*(0) at *z*=0, and *u*=*U* at *z*=0.

It is easy and straightforward to obtain the various constrained functionals from the above generalized functional by enforcing some of the field equations into the functional (13). We will not write all of them, only the most important one:

$$J(p)=\int_0^L\int_0^h\left\{\frac{1}{2}\mu_0(\frac{\partial u}{\partial z})^2+\sum_{n=1}^{m}\frac{1}{2n+2}a_n(\frac{\partial u}{\partial z})^{2n+2}\right\}dxdz+$$
$$+\int_0^L\left[w(h)-w(0)\right]pdx+\int_0^L U\,\tau_{xz}\big|_{z=0}\,dx \quad (14).$$

It is easy to prove $\delta^2 J>0$, so the functional, Eq.(14), is a minimum energy principle.

From Eq.(5) we have:

$$\mu_0\frac{\partial u}{\partial z}+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}=z\frac{\partial p}{\partial x}+C \quad (15),$$

where *C* is an integration constant, it can be approximately identified by the assumption $t_{xz}=0$ at $z=h/2$. Under such a condition we have:

$$\mu_0\frac{\partial u}{\partial z}+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}=(z-\frac{h}{2})\frac{\partial p}{\partial x} \quad (16).$$

We should obtain an explicit expression for $\partial u/\partial z$. To this end, we illustrate a new homotopy perturbation method ([14] to [19]) to solve the expression approximately. The basic idea of the homotopy perturbation method is to construct a homotopy in the form:

$$(\mu_0+\beta)\frac{\partial u}{\partial z}-(z-\frac{h}{2})\frac{\partial p}{\partial x}=-\varepsilon\left[\beta\frac{\partial u}{\partial z}+\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}\right] \quad (17),$$

where ε is a homotopy parameter, and β is a linearized factor defined as:

$$\sum_{n=1}^{m}a_n(\frac{\partial u}{\partial z})^{2n+1}=\beta\frac{\partial u}{\partial z} \quad (18),$$

and can be identified by the method of weighted residuals, which requires:

$$\int_0^A x(\sum_{n=1}^{m}a_n x^{2n+1}-\beta x)dx=0 \quad (19),$$

where *A* is a fixed value for a definite problem.

It is obvious that when $\varepsilon=0$, Eq.(17) becomes a linearized equation, and when $\varepsilon=1$, we recover the nonlinear equation. The homotopy perturbation method applies the homotopy parameter to expand the solution in the form:

$$\frac{\partial u}{\partial z}=(\frac{\partial u}{\partial z})_0+\varepsilon(\frac{\partial u}{\partial z})_1+\varepsilon^2(\frac{\partial u}{\partial z})_2+\cdots \quad (20).$$

By setting $\varepsilon=1$, we obtain an approximate expression for $\partial u/\partial z$.

To illustrate the procedure, we are concerned here with finding the real root of the polynomial equation:

$$x^5+x-1=0 \quad (21).$$

Construct a homotopy:

$$(1+\beta)x-1=\varepsilon(\beta x-x^5) \quad (22),$$

where the linearized factor β can be determined from the relation:

$$\int_0^1 x(\beta x-x^5)dx=0 \quad (23),$$

leading to the result $\beta=3/7$. By perturbation technique, we obtain an approximate solution to order 2: $x=0.75999$.

Now turn back to our main procedure, after we get the explicit expression of $\partial u/\partial z$, we submit the obtained result into the minimum variational principle, Eq.(14). To illustrate the procedure, we consider the special case of Eq.(4):

$$\tau_{xz}=\mu_0\frac{\partial u}{\partial z}+\mu_0\gamma(\frac{\partial u}{\partial z})^3 \quad (24).$$

The cubic constitutive model can be applied to dilatant fluids for γ>0, and pseudoplastic fluids for γ<0. In the case when γ is a small parameter, we have:

$$\frac{\partial u}{\partial z}=\frac{1}{\mu_0}(z-\frac{h}{2})\frac{\partial p}{\partial x}-\frac{\gamma}{\mu_0^3}(z-\frac{h}{2})^3(\frac{\partial p}{\partial x})^3+O(\gamma^2) \quad (25).$$

Now substituting Eq.(25) into the functional (14):

$$J(p)=\int_0^L\int_0^h\left\{\frac{1}{2}\mu_0\left[\frac{1}{\mu_0}(z-\frac{h}{2})\frac{\partial p}{\partial x}-\frac{\gamma}{\mu_0^3}(z-\frac{h}{2})^3(\frac{\partial p}{\partial x})^3\right]^2\right\}dxdz+$$
$$+\int_0^L\int_0^h\left\{\frac{1}{4}\mu_0\gamma\left[\frac{1}{\mu_0}(z-\frac{h}{2})\frac{\partial p}{\partial x}-\frac{\gamma}{\mu_0^3}(z-\frac{h}{2})^3(\frac{\partial p}{\partial x})^3\right]^4\right\}dxdz$$
$$+\int_0^L\left[w(h)-w(0)\right]pdx-\int_0^L\frac{h}{2}U\frac{\partial p}{\partial x}dx, \quad (26)$$

integrating from z=0 to z=h, and neglecting higher-order terms of γ, we have the following functional:

$$J(p)=\int_0^L\frac{1}{2}\frac{1}{\mu_0}\left[\frac{1}{12}h^3(\frac{\partial p}{\partial x})^2-\frac{3\gamma}{320\mu_0^3}h^5(\frac{\partial p}{\partial x})^4\right]dx+$$
$$+\int_0^L\left[w(h)-w(0)\right]pdx-\int_0^L\frac{h}{2}U\frac{\partial p}{\partial x}dx \quad (27).$$

Calculating the variation with respect to p, we have the following Reynolds-type equation:

$$-\frac{\partial}{\partial x}\left[\frac{h^3}{12\mu_0}(\frac{\partial p}{\partial x})-\frac{3\gamma h^5}{80\mu_0^3}(\frac{\partial p}{\partial x})^3\right]+\left[w(h)-w(0)\right]+\frac{1}{2}\frac{\partial}{\partial x}(Uh)=0 \quad (28).$$

Note that :

$$w(h)-w(0)=\frac{\partial h}{\partial t} \quad (29).$$

We, therefore, have the following final form:

$$\frac{\partial}{\partial x}\left[\frac{h^3}{\mu_0}(\frac{\partial p}{\partial x})-\frac{9\gamma h^5}{20\mu_0^3}(\frac{\partial p}{\partial x})^3\right]=12\frac{\partial h}{\partial t}+6\frac{\partial}{\partial x}(Uh) \quad (30).$$

The obtained Reynolds-type equation, Eq.(30), has a very similar structure to the Reynolds equation for Newtonian lubrication.

2 CONCLUSION

Non-Newtonian lubricants are commonly found in engineering. Here we illustrate the utility of a derivation of a Reynolds-type equation from a variational functional. The technique developed can be readily applied to any other lubrication problems, and the present paper can be used as a paradigm for the establishment of a Reynolds-type equation for various lubrication problems.

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