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# GENERALIZED THREE-MANIFOLDS WITH ZERO-DIMENSIONAL SINGULAR SET

by

DUŠAN REPOVŠ

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GENERALIZED THREE-MANIFOLDS WITH ZERO-DIMENSIONAL SINGULAR SET

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Dušan Repovš, Ph.D.
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Major Professor: Robert Christopher Lacher, Ph.D.

We study two "disjoint disks properties" in dimension 3 due to H.W.Lambert and R.B.Sher (Pacific.J.Math. 24 (1968) 511-518), the Dehn lemma property (DLP) and the map separation property (MSP). Theorem 1. Let G be a cell-like closed 0-dimensional upper semicontinuous decomposition of a 3-manifold M (possibly with boundary) with $N_G \subset \operatorname{int} M$. Then the following statements are equivalent: (i) M/G has the DLP; (ii) M/G has the MSP; (iii) M/G is a 3-manifold. Theorem 2. Let $\underline{C}$ be the class of all compact generalized 3-manifolds X with $\dim S(X) \leq 0$ and let $\underline{C}_0 \subset \underline{C}$ be the subclass of all $X \in \underline{C}$ with $S(X) \subset \{pt\}$ and $X \simeq S^3$. Then the following statements are equivalent: (i) The Poincaré conjecture in dimension three is true; (ii) If $X \in \underline{C}$ has the DLP or the MSP then $S(X) = \emptyset$; (iii) If $X \in \underline{C}_0$ has the DLP or the MSP then $S(X) = \emptyset$.

We also study neighborhoods of peripherally 1-acyclic compacta in nonorientable 3-manifolds. We prove a finiteness and a neighborhood theorem for such compacta and as an application extend a result of J.L.Bryant and R.C.Lacher concerning resolutions of almost $\mathbb{Z}_2$-acyclic images of orientable 3-manifolds (Math.Proc.Camb.Phil.Soc.

88 (1980) 311-320), to nonorientable 3-manifolds. Theorem 3. Let f be a closed, monotone mapping from a 3-manifold M onto a locally simply connected $\mathbb{Z}_2$-homology 3-manifold X. Suppose that there is a 0-dimensional set $Z \subset X$ such that $\check{H}^1(f^{-1}(x);\mathbb{Z}_2) = 0$ for all $x \in X - Z$. Then the set $C = \{x \in X \mid f^{-1}(x)$ is not cell-like$\}$ is locally finite in X. Moreover X has a resolution.

Included is an investigation of the basic properties of generalized 3-manifolds with boundary, a topics on which little study has been done so far, as well as some results on regular neighborhoods of compacta in 3-manifolds with applications to homotopic PL embeddings of compact polyhedra into 3-manifolds.

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## LIST OF SYMBOLS AND ABBREVIATIONS

[ ] ............. bibliographical reference

$\#\#$ .............. the proof (or discussion of it) is ended or omitted

* ............... a point

$\mathbb{R}^n$ ............... Euclidean n-space

$B^n$ ............... (closed) n-ball

$S^n$ ............... n-sphere

$I^n$ ............... n-cube ($I \times \ldots \times I$, n factors $I = [0,1]$ )

$\mathbb{Z}_+$ .............. nonnegative integers

$\mathbb{R}_+$ .............. nonnegative reals

$\mathrm{Fr}\, U$ ($\mathrm{C\ell}\, U$) ....... frontier (resp. closure) of U

$\overset{\circ}{X}$ ($\dot{X}$) ............ interior (resp. boundary) of X, a generalized manifold with boundary (See p.11)

$\mathrm{int}\, M$ ($\partial M$) ....... interior (resp. boundary) of M, a manifold with boundary

$X \twoheadrightarrow Y$ ........... surjective map

$X \rightarrowtail Y$ ........... injective map

$X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$ ........... bijective map

$[t]$ .............. the greatest integer not exceeding $t \in \mathbb{R}$

$S_f$ .............. singular set of a map f

$\Sigma_f$ .............. the image under f of $S_f$

PID ............... principal ideal domain

ANR ............... absolute neighborhood retract

ENR ............... euclidean neighborhood retract (See p.11)

DDP ............... disjoint disks property (See p.13)

DLP ............... Dehn's lemma property (See p.47)

MSP ............... map separation property (See p.48)

KF ............... Kneser Finiteness (See p.15)

dim X ............. covering dimension of X, a separable metrizable space

$g(\partial M)$ ............. total genus of $\partial M$, M a compact 3-manifold with boundary (See p.24)

k-UV, $UV^k$, $UV^\infty$ ..... UV properties (See p.7)

k-LC, $LC^k$, $LC^\infty$ ..... LC properties (See p.8)

k-uv(R), $uv^k(R)$, $uv^\infty(R)$ .. uv properties, R a PID (See p.8)

k-lc(R), $lc^k(R)$, $lc^\infty(R)$ .. lc properties, R a PID (See p.8)

1-LCC ............. locally simply co-connected (See p.8)

g(X,p) ............ genus of X at p, X a generalized 3-manifold (See p.15)

$H^*(\_;R), H_*(\_;R)$ .. singular (co)homology with coefficients in R

$\check{H}^*(\_;R), \check{H}_*(\_;R)$ .. Čech (co)homology with coefficients in R

$\sim$ ............... homology equivalence

$\simeq$ ($\underset{s}{\simeq}$) ........... (resp. simple) homotopy equivalence

$\approx$ .............. homeomorphism

$\underset{PL}{\cong}$ .............. PL isomorphism

## INTRODUCTION

Generalized manifolds have held an important position in topology ever since they were introduced in the 1930's. For low dimensions ($\leq 2$) their local algebraic properties are strong enough to imply that they are genuine manifolds. In higher dimensions they are interesting for at least two reasons: (i) they arise in many different classes (as quotient spaces of cell-like upper semicontinuous decompositions of manifolds, as manifold factors, as quotients of Lie group actions on manifolds, and as suspensions of homology spheres), and (ii) they have the same global algebraic properties possessed by manifolds (local orientability, duality). Recent success in higher dimensions -- a remarkably simple characterization of n-manifolds ($n \geq 5$) -- has stimulated an upsurge in interest in the geometric topology of generalized manifolds, so we first briefly review these results which, in turn, motivated our research in dimension three.

The definition of a topological n-manifold (without boundary) is simple -- this is a separable metrizable space that locally looks like $\mathbb{R}^n$ . However, when working with topological spaces it is quite often difficult to determine whether a certain construction is a topological manifold. Thus it would be desirable to have a short list of topological properties that are reasonably easy to check and that characterize topological manifolds. Such a list should not include e.g., homeomorphisms (since they are usually difficult to construct), or induction on dimension (since nice submanifolds are in general hard to find), or homogeneity (since the construction is usually already euclidean at some points so that homogeneity is precisely the problem), etc. His successful solution of J.W.Milnor's classical problem about the double suspension of homology 3-spheres [17 ;Ch.11] , led J.W.Cannon to conjecture that topological n-manifolds are precisely generalized n-manifolds satisfying a "minimal amount of general position" [16;Conjecture (1.3)] .

In higher dimensions ($n \geq 5$) this conjecture was proved soon thereafter, in two steps: (i) in 1977 R.D.Edwards showed that every resolvable generalized n-manifold ($n \geq 5$) with Cannon's disjoint disks property (DDP) [17 ;p.83] is a genuine n-manifold ([23] and [25 ;pp.118-122 ]), and (ii) in 1978 F.Quinn announced a proof that every generalized n-manifold ($n \geq 5$) is resolvable ([53] and [55 ; Theorem(1.1) ]). Exploring the latest remarkable results of S.Donaldson [21] and M.H.Freedman [27] in dimension four, Quinn proved the resolution conjecture also for dimension four [56 ;Theorem (2.6.1) ] . Therefore in order to get a characterization of 4-manifolds, an analogue of Edwards' shrinking theorem for this dimension should be developed.

This dissertation is a study of generalized 3-manifolds with 0-dimensional singular set and their possible role in characterization of 3-manifolds. Before summarizing our results we briefly review the work of others in this area.

M.G.Brin and D.R.McMillan,Jr. proved that, modulo the Poincaré conjecture, every compact generalized 3-manifold with 0-dimensional singular set has a resolution [12 ;Theorem 5 ], hence by J.L.Bryant and R.C.Lacher a conservative one [14;Theorem 1] , provided it satisfies a certain "torsion-free" hypothesis. This extra condition was inherited from Brin's extension of the Loop theorem and Dehn's lemma [10;Theorems 1-3] they used in their proof. T.L.Thickstun removed the "torsion-free" hypothesis from [10] and thus from [12] as well [61]. He later proved a positive result [62;Main Theorem] (obtained independently by R.J.Daverman) to the effect that such generalized 3-manifolds are images of "tame" generalized 3-manifolds (whose singular set has genus zero at each point), and consequently disentangled the Poincaré conjecture from the resolution theorem [61] . Another positive result is due to Bryant and Lacher who proved that every locally contractible $\mathbb{Z}_2$- acyclic image of a 3-manifold has a resolution (and is thus a generalized 3-manifold) [14;Theorem 2] . A refinement of their proof enabled them to omit the acyclicity hypothesis over a zero-dimensional set provided that the 3-manifold domain was orientable [14;Theorem 3] . In Chapter Two we prove that orientability is not necessary:

Theorem 2.7. Let f be a closed, monotone mapping from a 3-manifold M without boundary onto a locally simply connected $\mathbb{Z}_2$- homology 3-manifold X. Suppose that there is a 0-dimensional set $Z \subset X$ such that $\check{H}^1(f^{-1}(x);\mathbb{Z}_2) = 0$ for all $x \in X - Z$. Then the set $C = \{ x \in X \mid f^{-1}(x)$ is not cell-like $\}$ is locally finite in X. Moreover, X has a resolution.

M.Starbird introduced two notions of the disjoint disks property (DDP I and DDP II) for decompositions G of $\mathbb{R}^3$ (rather than for the quotient spaces $\mathbb{R}^3/G$) and he proved that for G a cell-like, upper semicontinuous 0-dimensional decomposition, satisfying either DDP I or DDP II, $\mathbb{R}^3/G \approx \mathbb{R}^3$ [60;Theorem (3.1)] . His result is useful for generalized 3-manifolds X which are already known to be a quotient $X = \mathbb{R}^3/G$ . A different approach was taken by Bryant and Lacher who showed that if in a compact generalized 3-manifold X the singular set S(X) lies in a compact, tamely embedded 0-dimensional set $Z \subset X$ (i.e., Z is 1-LCC in X) then X is a topological 3-manifold, provided X contains at most finitely many pairwise disjoint fake 3-cells [14;Theorem 4] . (This generalizes previous results of C.H.Edwards, Jr. [22;Theorem 1] and C.T.C.Wall [66;Corollaries 1 and 2] .) However, the condition "$S(X) \subset Z$ where Z is a closed 1-LCC subset of X" is not suitable since many potential singular sets may be wildly embedded in X. Professor Lacher suggested in July 1980 that instead, one should look for a disjoint disks property for generalized 3-manifolds X with 0-dimensional singular set such that it would imply first, the existence of a resolution $f:M \twoheadrightarrow X$ and second, the shrinkability of the associated cell-like decomposition $G = \{ f^{-1}(x) | x \in X \}$ of M. In Chapter Three we prove that a concept due to H.W. Lambert and R.B.Sher, called the map separation property (MSP) [44 ;p.514] characterizes the 3-manifold property in certain cases (modulo the Poincaré conjecture). We also study a similar concept from [44] called the Dehn's lemma property (DLP) and show it plays the same role as the MSP:

Theorem 3.8. Let G be a cell-like, closed 0-dimensional upper semi-continuous decomposition of a 3-manifold M (possibly with boundary) such that $\overline{N}_G \subset \operatorname{int} M$. Then the following statements are equivalent:

(i) M/G has the DLP;

(ii) M/G has the MSP;

(iii) M/G is a 3-manifold.

Theorem 3.10. Let $\underline{C}$ be the class of all compact generalized 3-manifolds X with $\dim S(X) \leq 0$ and let $\underline{C}_o \subset \underline{C}$ be the subclass of all $X \in \underline{C}$ with $S(X) \subset \{pt\}$ and such that $X \simeq S^3$. Then the following statements are equivalent:

(i) Poincaré conjecture in dimension three is true;

(ii) If $X \in \underline{C}$ has the DLP or the MSP then $S(X) = \emptyset$;

(iii) If $X \in \underline{C}_o$ has the DLP or the MSP then $S(X) = \emptyset$.

We conclude this introduction by a description of the organization of the dissertation. In Chapter One we collect most important facts about UV and LC properties, about upper semicontinuous decompositions of manifolds, and about generalized manifolds. In Chapter Two we investigate the nature of the neighborhoods of certain compacta in nonorientable 3-manifolds and prove a finiteness and a neighborhood theorem. We then use these results to prove Theorem (2.7). Also included is a comparative study of various kinds of acyclicity over $\mathbb{Z}_2$ for embeddings of compacta in 3-manifolds. In Chapter Three we introduce the DLP and the MSP, verify that every 3-manifold has both properties and then prove Theorems (3.8) and (3.10). We conclude by an application of the DLP/MSP to the study of isolated singularities. In Chapter Four we present a study of generalized 3-manifolds with boundary, an area where almost no research has been done yet. We prove several results analogous to those known for generalized 3-manifolds. In Chapter Five we collect some open problems and state some conjectures related to this topics. In the Appendix we study regular neighborhoods of compact polyhedra and prove some results concerning homotopic PL embeddings of compact polyhedra into 3-manifolds. We have included these results since they are related to (and were motivated by) those from Chapter Two.

# I. PRELIMINARIES

In this chapter we collect some important definitions and results from three subjects that underline our dissertation topics: UV, LC, and related properties, upper semicontinuous decompositions, and generalized manifolds. Standard references for other topics are: E.H.Spanier [59] for algebraic topology, C.P.Rourke and B.J.Sanderson [57] for PL topology, J.Hempel [32] for 3-manifolds, K.Borsuk [7] for ANR's, and R.C.Lacher [40] for cell-like mappings.

## 1. UV, LC, and Related Properties

A continuum is a compact and connected set. A compactum K in an ANR X has property k-UV (resp. $UV^k$; $UV^\infty$) ($k \in \mathbb{Z}_+$) if for each neighborhood $U \subset X$ of K there is a neighborhood $V \subset U$ of K such that any singular k-sphere in V is null-homotopic in U (resp. any singular j-sphere in V ($0 \leq j \leq k$) is null-homotopic in U; V is null-homotopic in U). An n-manifold is a topological n-manifold without boundary. A compact subset K of an n-manifold M is cellular in M if K is the intersection of a properly nested decreasing sequence of n-cells in M. A space X is cell-like if there exist a manifold N and an embedding $f:X \rightarrowtail N$ such that f(X) is cellular in N. For finite-dimensional compacta this is known to be equivalent to "X has property $UV^\infty$" [40;p.509]. A mapping (or a map) is a continuous map but not necessarily also PL. A map defined on a space (resp. an ANR; a manifold) X is monotone (resp. cell-like; cellular) if its point-inverses are continua (resp. cell-like sets; cellular sets) in X. A closed map is <u>proper</u> if its point-inverses are compact.

A compactum K in a manifold M is <u>point-like</u> if $M - K \approx M - \{pt\}$. A subset Z of a space X is <u>$\Pi_1$-negligible</u> if for each open set U in X the inclusion-induced homomorphism $\Pi_1(U - Z) \to \Pi_1(U)$ is 1 - 1. A space X is k-<u>LC</u> (resp. <u>$LC^k$</u>; <u>$LC^\infty$</u>) <u>at</u> $x \in X$ ($k \in \mathbb{Z}_+$) if for every neighborhood U in X of x there exists a neighborhood $V \subset U$ of x such that any singular k-sphere in V is null-homotopic in U (resp. any singular j-sphere in V ($0 \leq j \leq k$) is null-homotopic in U; V is null-homotopic in U). A subset $Z \subset X$ is 1-<u>LCC</u> (for "locally simply coconnected") if for every $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the inclusion-induced homomorphism $\Pi_1(V - Z) \to \Pi_1(U - Z)$ is 0. A compactum K in an ANR X has k-<u>uv(R)</u> (resp. <u>$uv^k(R)$</u>; <u>$uv^\infty(R)$</u>) <u>property</u> ($k \in \mathbb{Z}_+$, R a PID) if for each neighborhood $U \subset X$ of K there is a neighborhood $V \subset U$ of K such that any singular k-cycle in V is null-homologous in U (resp. any singular j-cycle in V ($0 \leq j \leq k$) is null-homologous in U; any singular j-cycle in V ($j \geq 0$) is null-homologous in U), with coefficients in R understood. The uv properties are related to Čech cohomology: if a compactum X has properties j-uv(R) (j = k-1,k; R a PID) then $\check{H}^k(X;R) = 0$ and conversely, if $\check{H}^j(X;R) = 0$ (j = k,k+1; R a PID) then X has property k-uv(R) [40;p.502] . A map defined on an ANR is <u>$UV^k$</u> (resp. <u>$uv^k(R)$</u>) ($k \in \mathbb{Z}_+$, R a PID) if its point-inverses have $UV^k$ (resp. $uv^k(R)$) property. The following two results will often be needed in our proofs: the first one is a consequence of the Vietoris-Smale-Begle theorems [40 ;pp.505-508] , while the second one is due to R.C.Lacher and D.R.McMillan, Jr. [43;Lemma (4.1)] :

Proposition 1.1. Suppose that $f:X \twoheadrightarrow Y$ is a proper $UV^{k-1}$ (resp. $uv^{k-1}(R)$) map ($k \in \mathbb{Z}_+$, $R$ a PID) and that $Y$ is $LC^k$ (resp. $lc^k(R)$). Then the inclusion-induced homomorphism $\Pi_q(X,*) \to \Pi_q(Y,*)$ (resp. $H_q(X;R) \to H_q(Y;R)$) is bijective for $0 \leq q \leq k-1$ and surjective for $q = k$. ****

Proposition 1.2. Let $M$ be a manifold, $V$ a connected open set in $M$, and suppose that the inclusion-induced homomorphism $H_1(V;\mathbb{Z}_2) \to H_1(M;\mathbb{Z}_2)$ is 0. Then $V$ is orientable. ****

## 2. Upper Semicontinuous Decompositions

Let $G = \{g \subset X\}$ be a decomposition of a space $X$ into compact (and connected) sets and let $\pi : X \twoheadrightarrow X/G$ be the corresponding quotient map, $H_G$ the collection of all nondegenerate (i.e., $g \neq *$) elements of $G$, and $N_G$ their union. A set $U \subset X$ is G-saturated if $U = \pi^{-1}\pi(U)$. A decomposition $G$ is upper semicontinuous if for each $g \in G$ and for each open neighborhood $U \subset X$ of $g$ there exists a G-saturated open neighborhood $V \subset U$ of $g$. Equivalently, $\pi$ is a closed map. A decomposition $G$ of a separable metrizable space $X$ is k-dimensional (resp. closed k-dimensional) ($k+1 \in \mathbb{Z}_+$) if $\dim \pi(N_G) = k$ (resp. $\dim \pi(cl(N_G)) = k$). A decomposition $G$ of a metric space $X$ is weakly shrinkable if for each $\varepsilon > 0$ and each neighborhood $U \subset X$ of $N_G$ there is a homeomorphism $h:X \to X$ such that $h \mid (X - U) = id$ and for each $g \in G$, $\operatorname{diam} h(g) < \varepsilon$. A decomposition $G$ of a metric space $X$ is shrinkable if for every $\varepsilon > 0$ and every G-saturated open cover $\Omega$ of $N_G$

there is a homeomorphism $h:X \to X$ such that

(i) $h | (X - \Omega^*) = id$, where $\Omega^* = \cup\{U \in \Omega\}$;

(ii) for each $g \in G$, $\operatorname{diam} h(g) < \varepsilon$;

(iii) for each $g \in G$ there is a $U \in \Omega$ such that $h(g) \cup g \subset U$.

<u>Theorem 1.3</u>. Let G be a cell-like upper semicontinuous decomposition of an n-manifold M. For n = 3 assume, in addition, that each $g \in G$ has a neighborhood in M embeddable in $\mathbb{R}^3$. Then G is shrinkable if and only if $M/G \approx M$.

<u>Proof</u>. Follows by Bing's Shrinking criterion [45] and Armentrout-Quinn-Siebenmann's Approximation theorem ([1], [56; Corollary (2.6.2)], [58]). ⁂

An upper semicontinuous decomposition G of an n-manifold M has a <u>defining sequence</u> if there is a sequence $\{M_i \mid i \geq 1\}$ of closed subsets $M_i \subset M$ with the following properties:

(i) for each i, each component of $M_i$ is a compact n-manifold with boundary;

(ii) for each i, $M_{i+1} \subset \operatorname{int} M_i$;

(iii) for each $g \in G$, $g \in H_G$ if and only if g is a nondegenerate component of $\bigcap_{i \geq 1} M_i$.

It is well-known (and easy to prove) that an upper semicontinuous decomposition of a PL manifold has a defining sequence if and only if it is closed 0-dimensional. In our studies of decompositions we shall mainly consider those decompositions which are definable by (homology) cubes with handles (cf. a paper of McMillan, Jr. [47]).

## 3. Generalized n-Manifolds

A space X is an *euclidean neighborhood retract* (ENR) if it is homeomorphic to a retract of an open subset of some $\mathbb{R}^n$. Equivalently, X is a separable, locally compact, finite-dimensional metrizable ANR. Let R be a PID. A Hausdorff space X is an *R-homology n-manifold* ($n \in \mathbb{N}$) if for each $x \in X$, $\check{H}^*(X,X-\{x\};R) \cong \check{H}^{n-*}(\{x\};R)$. A Hausdorff space X is an *R-homology n-manifold with boundary* ($n \in \mathbb{N}$) if for each $x \in X$, either $\check{H}^*(X,X-\{x\};R) \cong \check{H}^{n-*}(\{x\};R)$ or $\cong 0$. The subset $\dot{X} = \{x \in X \mid \check{H}^*(X,X-\{x\};R) \cong 0\}$ is called the *boundary of* X and $\mathring{X} = X - \dot{X}$ the *interior of* X.

*Lemma 1.4.* Let X be an ANR and R a PID.

(i) If X is an R-homology n-manifold then for each $x \in X$ and each $q \in \mathbb{Z}$

$$H_q(X,X-\{x\};R) \cong \begin{cases} R \; ; \; q = n \\ 0 \; ; \; q \neq n \end{cases}$$

(ii) If X is an R-homology n-manifold with boundary then for each $x \in X$ and each $q \in \mathbb{Z}$

$$H_q(X,X-\{x\};R) \cong \begin{cases} R \; ; \; q = n \text{ and } x \in \mathring{X} \\ 0 \; ; \text{ otherwise} \end{cases}$$

*Proof.* On the class of ANR's the Čech cohomology agrees with the singular cohomology. The conclusion now follows by the Universal Coefficients theorem. ⁑

*Lemma 1.5.* Let X be an R-homology n-manifold and an ANR, where R is a PID. Choose any $x \in X$ and let $i_*: H_q(X-\{x\};R) \to H_q(X;R)$ be the inclusion-induced homomorphism ($q \in \mathbb{Z}$). Then the following holds:

(i) if $q = n$ then $i_*$ is 1-1;

(ii) if $q = n-1$ then $i_*$ is onto;

(iii) if $q \neq n-1,n$ then $i_*$ is bijective.

Proof. Consider the homology sequence of the pair $(X,X-\{x\})$ over R and apply Lemma (1.4). ⁑⁑

A generalized n-manifold is an ENR that is also a $\mathbb{Z}$-homology n-manifold. A generalized n-manifold with boundary is an ENR X such that X is a $\mathbb{Z}$-homology n-manifold with boundary and $\dot{X}$ is a generalized (n-1)-manifold. Let X be a generalized n-manifold (possibly with boundary). The set $S(X) = \{x \in X \mid x$ has no neighborhood in X homeomorphic to an open subset of $B^n\}$ is the singular set of X, its complement $M(X) = X - S(X)$ is the manifold set of X. The points of $S(X)$ (resp. $M(X)$) are called the singularities (resp. manifold points). If $\dot{X} = \emptyset$ or $S(X) \subset \mathring{X}$ then $M(X)$ is a topological n-manifold.

Generalized manifolds arise as (i) the finite-dimensional quotient spaces of cell-like upper semicontinuous decompositions of manifolds; (ii) the manifold factors; (iii) the qoutients of the Lie group actions on manifolds; and (iv) the suspensions of $\mathbb{Z}$-homology spheres. A resolution of a generalized n-manifold X is a pair (M,f) where M is an n-manifold and $f:M \twoheadrightarrow X$ is a proper cell-like surjection. A resolution (M,f) of X is conservative if for each $x \in M(X)$, $f^{-1}(x) = *$ .

Theorem 1.6. Let X be a generalized n-manifold. If n = 3 assume the Poincaré conjecture and also that $\dim S(X) \leq 0$. Then X has a conservative resolution.

Proof. If $n \leq 2$ then $S(X) = \emptyset$ ([68;Theorems (IX.1.2) and (IX.2.3)]). If $n = 3$ then X has a resolution by T.L.Thickstun [61] hence by J. L.Bryant and R.C.Lacher [14;Theorem 1], a conservative resolution. For $n = 4$ the assertion was recently proved by F.Quinn ([56;Theorem (2.6.1) and Corollary (2.6.2)]). If $n \geq 5$ then X resolves by Quinn's Resolution theorem [55;Theorem (1.1)] and the assertion then follows by L.C.Siebenmann's Approximation theorem [58] . ⁑

A metric space X has the disjoint disks property (DDP) [17] if for every $\varepsilon > 0$ and every two maps $f_1, f_2 : B^2 \to X$ there are disjoint maps $g_1, g_2 : B^2 \to X$ such that $d(f_1, g_1) < \varepsilon > d(f_2, g_2)$.

Theorem 1.7. Topological n-manifolds ($n \geq 5$) are precisely the generalized n-manifolds satisfying the DDP.

Proof. Follows by R.D.Edwards' Shrinking theorem [23] and Theorems (1.3) and (1.6). ⁑

## 4. Generalized 3-Manifolds

Dimension three is in many respects peculiar mostly due to the unresolved status of the Poincaré conjecture. We list some of the most important facts. First, X cannot have "cone" singularities (which are common in higher dimensions) [42;p.84] and [14;p.311] :

Proposition 1.8. Let X be a generalized 3-manifold (possibly with boundary). Then no singularity of X can have an open cone neighborhood in X. ⁑

Corollary 1.9. Let X be a PL generalized 3-manifold (possibly with boundary). Then X is a 3-manifold. ⁑

Another property of generalized 3-manifolds peculiar for this dimension is a kind of algebraic finiteness, as it was observed by Bryant and Lacher [14;pp.312-313] :

Proposition 1.10. Let X be a compact generalized 3-manifold (resp. with a resolution). Then there exists an integer k such that among any k+1 pairwise disjoint $\mathbb{Z}_2$-homology 3-cells at least one is contractible (resp. a 3-cell). ⁑

So far we have made no assumption on the dimension of S(X). The following result of Brin and McMillan [12 ;Theorem 1 ] delineates the 0-dimensional singular set case as natural and closely related to the embedding problem for open 3-manifolds:

Proposition 1.11. Let X be a compact generalized 3-manifold with $\dim S(X) \leq 0$. Then the following statements are equivalent:

(i) X has a resolution;

(ii) M(X) embeds in a compact 3-manifold;

(iii) S(X) has a neighborhood $N \subset X$ such that $N \cap M(X)$ embeds in a compact orientable 3-manifold;

(iv) S(X) has a neighborhood $N \subset X$ such that $N \cap M(X)$ embeds in $\mathbb{R}^3$. ⁑

Let X be a generalized 3-manifold with 0-dimensional singular set. Then by [12 ;Lemma 1] every $p \in X$ has arbitrarily small compact generalized 3-manifold-with-boundary neighborhoods $N \subset X$ such that $\dot{N}$ is a compact orientable surface in M(X). We say that X has genus $\leq n$ at p if p has arbitrarily small such neighborhoods N with $\dot{N}$ a surface of genus $\leq n$. We say that X has genus n at p if X has genus

$\leq$n at p and doesn't have genus $\leq$n-1 at p. If X doesn't have genus $\leq$n at p for any n we say that X <u>has genus $\infty$ at</u> p. We shall denote the genus of X at p by g(X,p) [42] . We say that a generalized 3-manifold X satisfies <u>Kneser Finiteness</u> (KF) if for each compact subset $X_o \subset X$ there is an integer k such that $X_o$ contains at most k pairwise disjoint fake cubes. A sequence of pairwise disjoint compacta $\{C_i\}$ in a metric space X is a <u>null-sequence</u> if for every $\varepsilon > 0$ all but finitely many among $C_i$'s have diameter $< \varepsilon$.

It is not surprising that the Poincaré conjecture enters into the picture as soon as we try to resolve generalized 3-manifolds (just recall Proposition (1.11)). We consider an example which will be used later on in the dissertation. Suppose fake cubes exist and consider in $S^3$ a null-sequence of pairwise disjoint 3-cells $\{B_i\}$ converging to a point $p \in S^3$. Replace each $B_i$ by a fake cube $F_i$ and choose a metric in $W = (S^3 - \cup_{i=1}^{\infty} \operatorname{int} B_i) \cup (\cup_{i=1}^{\infty} F_i)$ so that $F_i$'s also converge to p.

<u>Proposition 1.12</u>. $W^3$ is a compact generalized 3-manifold with the following properties:

(i) $S(W) = \{p\}$; (We shall call such singularities "<u>soft singularities</u>".)

(ii) W doesn't have a resolution;

(iii) $W \simeq S^3$;

(iv) $g(W,p) = 0$.

<u>Proof</u>. (i) Follows by Kneser's Finiteness theorem [32;Lemma (3.14)].

(ii) Follows by Proposition (1.10).

(iii) Let $f:W \twoheadrightarrow S^3$ be the map which shrinks out all $F_i$'s. Then $f$ is cell-like hence by [40 ;Theorem (4.2)] a homotopy equivalence.

(iv) Clear. ⁂

Proposition 1.13. Let $X$ be a generalized 3-manifold with $S(X) \subset Z$, where $Z \subset X$ is a closed, 0-dimensional set. Then the following statements are equivalent:

(i) $Z$ is 1-LCC in $X$;

(ii) $Z$ is $\Pi_1$-negligible;

(iii) For every $z \in Z$, $g(X,z) = 0$.

Furthermore, anyone of the statements (i)-(iii) implies that all singularities of $X$ are "soft"; i.e., $X$ is obtained from a 3-manifold by replacing null-sequences of pairwise disjoint 3-cells by null-sequences of pairwise disjoint fake cubes. The latter property -- that all singularities of $X$ are "soft" -- is strictly weaker than (i)-(iii) if Poincaré conjecture is false.

Proof. (i) $\Rightarrow$ (iii): See the proof of Theorem 4 in [14 ;pp.317-318] .

(iii) $\Rightarrow$ (ii) : Let $U \subset X$ be an open set and $J$ a loop in $U - Z$. Since $\dim Z = 0$ and $g(X,z) = 0$ for all $z \in Z$ there is a covering $V_1$, ..., $V_t$ of $Z \cap U$ with pairwise disjoint compact generalized 3-manifolds with boundary $\dot{V}_i = S^2 \subset M(X)$ for all $i$. (We may assume that $X$ is compact.) Suppose now that $J$ bounds a (singular) disk in $U$. With techniques described in details in the proofs of Theorems (3.1) and (3.9) we can make this disk locally PL near the surface $S = \bigcup_{i=1}^{t} \dot{V}_i$ , put it in general position with respect to $S$, and cut it off at $S$, thus pushing it into $U - Z$, or just get it off $V_i \cap Z$ for each $i$.

(ii) $\Rightarrow$(i): Let $x \in X$ be an arbitrary point and choose a neighborhood $U \subset X$ of x. Since X is an ANR it is 1-LC. Thus there is a neighborhood $V \subset U$ of x such that the inclusion-induced homomorphism $\pi_1(V) \to \pi_1(U)$ is 0. Since Z is $\pi_1$-negligible, the homomorphisms $\pi_1(V - Z) \to \pi_1(V)$ and $\pi_1(U - Z) \to \pi_1(U)$ are 1-1. Consider the commutative diagram:

$$\begin{array}{ccc} \pi_1(V - Z) & \xrightarrow{\ i_*\ } & \pi_1(U - Z) \\ \downarrow & \text{trivial} & \downarrow \\ \pi_1(V) & \xrightarrow{\text{map}} & \pi_1(U) \end{array}$$

Clearly, $i_* = 0$.

Assume now, say the statement (i). By [14 ;Theorem 4 ] no open subset $V \subset X$ has the KF unless $V \subset M(X)$. This implies X has only"soft" singularities.

The last assertion is demonstrated as follows: take any wild Cantor set in $S^3$, direct to it a nice null-sequence of pairwise disjoint 3-balls in $S^3$ and then replace each of them by a fake 3-cell. Denote the new space by Y. Clearly, Y is a generalized 3-manifold, S(Y) is precisely the chosen wild Cantor set, and by [14 ;Theorem 4 ] S(Y) cannot be 1-LCC in Y, so Y doesn't satisfy the statement (i) (hence neither (ii) and (iii)). ⁑

<u>Corollary 1.14</u>. Let X be a generalized 3-manifold satisfying KF and suppose that $S(X) \subset Z$, where Z is a closed, 0-dimensional set in X. Then X is a 3-manifold if and only if for every $x \in X$, $g(X,x) = 0$.

<u>Proof</u>. Follows by [14 ;Theorem 4 ] and Proposition (1.13). ⁑

We conclude by stating two important results due to T.L.Thickstun: his extension of the Loop theorem [61](see also [11 ;p.30] ) and his Resolution theorem [62] which considerably improves the $n = 3$ case of Theorem (1.6) -- most notably, it disentangles the Poincaré conjecture from (1.6).

<u>Theorem 1.15</u>. Let $X$ be a compact generalized 3-manifold-with-boundary neighborhood of the singular set of a generalized 3-manifold, where $\dot{X}$ is a 2-manifold. Let $C$ be a component of $\dot{X}$, let $N$ be a normal subgroup of $\pi_1(C)$, and let $J$ be a loop in $C$ that shrinks in $X$ but that has $[J] \notin N$. Then in any neighborhood of $J$ in $C$ there is a simple closed curve $K$ such that $[K] \notin N$ and $K$ shrinks in $X$. ⁑

<u>Theorem 1.16</u>. Let $X$ be a compact $\mathbb{Z}$-homology 3-manifold with boundary such that $\dim S(X) \leq 0$, $S(X) \subset \mathring{X}$, and $X$ satisfies KF. Then there exist a proper cell-like surjection $f:(Y,\dot{Y}) \twoheadrightarrow (X,\dot{X})$ from a compact generalized 3-manifold $Y$ with boundary, with only "soft" singularities. (So, in particular, if the Poincaré conjecture is true, $X$ has a resolution.) ⁑

# II. NEIGHBORHOODS OF COMPACTA IN NONORIENTABLE 3-MANIFOLDS

The main result of this chapter is Theorem (2.7) -- a generalization of a theorem of J.L.Bryant and R.C.Lacher [14;Theorem 3] on resolutions of $\mathbb{Z}_2$-homology 3-manifolds which are almost 1-acyclic (over $\mathbb{Z}_2$) images of orientable 3-manifolds, to $\mathbb{Z}_2$-homology 3-manifolds which are almost 1-acyclic images of nonorientable 3-manifolds. In the first two sections we develop two technical results -- a finiteness and a neighborhood theorem (Theorems (2.1) and (2.2)). We then use them in Section Three to prove Theorem (2.7). In the last section we present a comparative analysis of various kinds of 1-acyclicities for compacta in 3-manifolds. Some related results that were inspired by these findings are collected in the Appendix at the end of the dissertation.

## 1. A Finiteness Theorem

T.E.Knoblauch [35] proved that in a closed orientable 3-manifold there can be but a finite number of pairwise disjoint compact sets that do not have a neighborhood embeddable in $\mathbb{R}^3$. He also gave an example in [35]to show that this need not hold for nonorientable 3-manifolds. In the theorem below we give an additional condition under which the statement is true also in the nonorientable case.

<u>Theorem 2.1</u>. For every closed nonorientable 3-manifold M there exists an integer K such that if $X_1,\dots,X_{K+1} \subset M$ are pairwise disjoint compact sets and each $X_i$ has a neighborhood $U_i \subset M$ such that the inclusion-induced homomorphism $H_1(U_i - X_i; \mathbb{Z}_2) \longrightarrow H_1(M;\mathbb{Z}_2)$ is 0, then at least one $X_i$ has a neighborhood which embeds in $\mathbb{R}^3$.

<u>Proof</u>. We work with $\mathbb{Z}_2$ coefficients and we shall supress them from the notation. Let $X_1,\dots,X_n \subset M$ be pairwise disjoint compact sets and suppose that each $X_i$ has a neighborhood $U_i \subset M$ such that the inclusion-induced homomorphism $H_1(U_i - X_i) \longrightarrow H_1(M)$ is trivial and if $i \neq j$ then $U_i \cap U_j = \emptyset$. Let $X = \cup_{i=1}^{n} X_i$ and $U = \cup_{i=1}^{n} U_i$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \bigoplus_{i=1}^{n} H_1(U_i - X_i) & & \bigoplus_{i=1}^{n} H_1(U_i) & & & & \\
 & & \downarrow \cong & & \downarrow \cong & & & & \\
\cdots & \longrightarrow & H_1(U-X) & \xrightarrow{f} & H_1(U) & \xrightarrow{f'} & H_1(U,U-X) & \longrightarrow & \cdots \\
 & & \downarrow & \searrow^{e} & \downarrow p & & \cong \downarrow p' & & \\
\cdots & \longrightarrow & H_1(M-X) & \xrightarrow{F} & H_1(M) & \xrightarrow{F'} & H_1(M,M-X) & \longrightarrow & \cdots
\end{array}$$

where the horizontal sequences are exact and $p'$ is the excision isomorphism. Suppose that for some $u_i \in \operatorname{im}[\, H_1(U_i) \to H_1(U)\,]$ $\Sigma_{i=1}^{n} p(u_i) = 0$. Then $p(\,\Sigma_{i=1}^{n} u_i) = 0$ hence $f'(\,\Sigma_{i=1}^{n} u_i) = (p')^{-1} \circ F' \circ p(\,\Sigma_{i=1}^{n} u_i) = 0$ so $\Sigma_{i=1}^{n} u_i \in \ker f' = \operatorname{im} f$. Therefore $\Sigma_{i=1}^{n} u_i = f(v)$ for some $v \in H_1(U-X)$. Now, $v = \Sigma_{i=1}^{n} v_i$ where $v_i \in \operatorname{im}[\, H_1(U_i - X_i) \to H_1(U-X)\,]$ therefore $f(v_i) = u_i$ for each i since the $U_i$'s are pairwise disjoint. We conclude that $p(u_i) = pf(v_i) = e(v_i) = 0$ since by hypothesis e is trivial. Therefore the image of the inclusion-induced homomorphism $H_1(U) \to H_1(M)$ is the direct sum of the images of the inclusion-induced homomorphisms $H_1(U_i) \to H_1(M)$, $1 \leq i \leq n$. So if we let $b_1 = \operatorname{rank} H_1(M)$ then $n-b_1$ of the homomorphisms $H_1(U_i) \to H_1(M)$ are trivial and so $n-b_1$ neighborhoods $U_i$ are orientable by Proposition (1.2).

Consider the orientable 3-manifold double cover $\tilde{p}:\tilde{M} \to M$ of M. Let $k(\tilde{M})$ be the Knoblauch number of $\tilde{M}$ [35;Theorem 1]. Since every orientable neighborhood lifts in $\tilde{M}$ to two homeomorphic copies it follows that if $2(n-b_1) > k(\tilde{M})$ then some $X_i$ has a neighborhood which embeds in $\mathbb{R}^3$. We can now determine the number K from the equation $2(K-b_1)-k(\tilde{M}) = 0$ : $K = [\frac{1}{2}(a+\operatorname{rank} H_1(M;\mathbb{Z}_2)+1]+b_1$ where a is the maximal number of pairwise disjoint nonparallel incompressible twosided surfaces in $\tilde{M}$ and [t] denotes the largest integer not greater than t. That such an a always exists follows by Haken's Finiteness theorem [29;p.48]. (The proof in [29] is valid only for irreducible 3-manifolds [30]. For a proof of the general case see e.g., A.H.Wright's dissertation [69].) ⁑

Remark. Theorem (2.1) holds also over the integers: by the Universal Coefficients theorem we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & H_1(U_i-X_i;\mathbb{Z})\otimes\mathbb{Z}_2 & \xrightarrow{f} & H_1(U_i-X_i;\mathbb{Z}_2) & \to & \operatorname{Tor}(H_0(U_i-X_i;\mathbb{Z}),\mathbb{Z}_2) & \to 0 \\ & \downarrow j_*\otimes id & & \downarrow j_*' & & \downarrow & \\ 0 \to & H_1(M;\mathbb{Z})\otimes\mathbb{Z}_2 & \xrightarrow{g} & H_1(M;\mathbb{Z}_2) & \to & \operatorname{Tor}(H_0(M;\mathbb{Z}),\mathbb{Z}_2) & \to 0 \end{array}$$

Since $\operatorname{Tor}(H_0(U_i-X_i;\mathbb{Z});\mathbb{Z}_2) = 0 = \operatorname{Tor}(H_0(M;\mathbb{Z}),\mathbb{Z}_2)$, f and g are isomorphisms. Thus if $j_*=0$ then $j_*'=0$.

On the other hand Theorem (2.1) is false over $\mathbb{Z}_p$, p any odd prime number, as the following example illustrates: let $M = P^2 \times S^1$, where $P^2$ denotes the projective plane. For each $t \in S^1$ let $X_t = P^2 \times \{t\}$. Since $M-X_t$ contracts onto $P^2$ and since $H_1(P^2;\mathbb{Z}_p) = 0$ it follows that the inclusion-induced homomorphism $H_1(M-X_t;\mathbb{Z}_p) \to H_1(M;\mathbb{Z}_p)$ is trivial. However, no $X_t$ has a neighborhood embeddable in $\mathbb{R}^3$ since $P^2$ doesn't embedd in $\mathbb{R}^3$ [28 ;Theorem (27.11) ].

## 2. A Neighborhood Theorem

Let K be a continuum in a 3-manifold M. How nice a neighborhood can K have? For example, if K is cellular in M then K is the intersection of properly nested 3-cells, while if it is cell-like then K is the intersection of properly nested homotopy 3-cells with 1-handles [47 ;Theorem 3 ]. We describe below neighborhoods of almost 1-acyclic (over $\mathbb{Z}_2$) continua K.

Theorem 2.2. Let K be a continuum in the interior of a 3-manifold M with (possibly empty) boundary. Suppose that K does not separate its connected neighborhoods and that for every neighborhood $U \subset M$ of K there exists a neighborhood $V \subset U$ of K such that the inclusion-induced homomorphism $H_1(V-K;\mathbb{Z}_2) \to H_1(U;\mathbb{Z}_2)$ is trivial. Then $K = \bigcap_{i=1}^{\infty} N_i$ where each $N_i \subset \operatorname{int} M$ is a compact 3-manifold with boundary satisfying the following properties:

(i) for each i, $N_{i+1} \subset \operatorname{int} N_i$ ;

(ii) $N_i$ is obtained from a compact 3-manifold $Q_i$ with a 2-sphere boundary by adding to $\partial Q_i$ a finite number of orientable (solid) 1-handles;

(iii) for each i, the inclusion-induced homomorphism $H_1(\partial N_{i+1};\mathbb{Z}_2) \to H_1(N_i;\mathbb{Z}_2)$ is trivial;

(iv) there is a homeomorphism $h_i : N_i \to N_i$ such that $h_i \mid \partial N_i =$ id and $h_i(Q_i^*) = Q_{i+1}$ where $Q_i^* \subset \text{int}\, Q_i$ is formed by pushing $Q_i$ into int $Q_i$ along a collar of $\partial Q_i$.

Remark. Theorem (2.2):(i)-(iii) was proved for orientable 3-manifolds by D.R.McMillan, Jr. [49;Theorem 2] . A.H.Wright observed [70;Theorem 2 ] that McMillan's theorem generalizes to nonorientable 3-manifolds but he did not obtain orientable 1-handles. Neither of the two papers [49]and [70] gave details of the proof because it was enough to indicate necessary changes in the proof of an earlier result of McMillan [47;Theorem 2 ]. Theorem (2.2):(iv) was proved for orientable 3-manifolds by J.L.Bryant and R.C.Lacher [14;Lemma C ]. We have decided to present the details in order to explain the specific situation for nonorientable 3-manifolds. Our proof of (i)-(iii) is modelled after the proof of [49;Theorem 2 ] as outlined in the lecture notes of McMillan [48;pp.45-49 ] , from which we also quote the following folklore lemmas we shall need at several points in the dissertation (cf. [48;pp.7-8,p.49] ).

Lemma 2.3. Let K be a compact set in the interior of a 3-manifold M, $K \neq M$ and let $N \subset M$ be a neighborhood of K. Then there exists a compact polyhedron $U \subset \text{int}\, N$ with the following properties:

(i) each component of U is a 3-manifold with boundary;

(ii) each closed surface in U-K separates U-K;

(iii) $K \subset \text{int}\, U$. ** **

Let M be a compact 3-manifold with boundary and let $F_1, \ldots, F_m \subset \partial M$ be its boundary components. Then we define the total genus of

$\partial M$ to be the sum of the genera of $F_i$ $(1 \leq i \leq m)$: $g(\partial M) = \Sigma_{i=1}^{m} g_i$, $g_i$ = genus of $F_i$.

Lemma 2.4. Let M be a compact orientable 3-manifold with boundary and let $R = \mathbb{Z}_p$ or the rationals (p a prime). Let $i_*: H_1(\partial M;R) \rightarrow H_1(M;R)$ be the inclusion-induced homomorphism. Then $\mathrm{rank}_R(\mathrm{im}\, i_*) = g(\partial M)$. ⁂

Proof of Theorem (2.2). First, we shall prove that $K = \cap_{i=1}^{\infty} N_i$ where $N_i$ satisfy (i) and (ii). It will follow by hypotheses that we can find a subsequence of $\{N_i\}$ satisfying (iii). By choosing a further subsequence we shall demonstrate (iv). We shall supress the $\mathbb{Z}_2$ coefficients from the notation.

To prove (i)-(iii) it therefore suffices to show that given a neighborhood $U \subset M$ of K there is a compact 3-manifold neighborhood $N \subset U$ of K such that N is obtained from a compact 3-manifold Q with $\partial Q$ a 2-sphere, by attaching a finite number of orientable (solid) 1-handles to $\partial Q$. So let $U \subset M$ be a neighborhood of K. We may assume the following about U:

(1) U is a nonorientable connected compact 3-manifold with boundary;

(2) $K \subset \mathrm{int}\, U$;

(3) U-K is orientable and connected;

(4) each closed surface in U-K separates U-K.

The condition (3) follows by Proposition (1.2) since, for sufficiently small U's, the inclusion induces trivial homomorphisms $H_1(U-K) \longrightarrow H_1(M)$. The condition (iv) is provided by Lemma (2.3).

Let $n_o \in \mathbb{N}$ be Haken's number of U [29;p.48]. Using the hypothesis we can construct an ordered $(n_0+2)$-tuple $Y = \{V_o, V_1, \ldots, V_{n_o+1}\}$ of compact 3-manifolds with boundary such that:

(5) $V_o = U$;

(6) $V_{i+1} \subset \operatorname{int} V_i$;

(7) $\partial V_i$ is an orientable (possibly disconnected) two-sided closed 2-manifold;

(8) $H_1(\partial V_{i+1}) \to H_1(V_i)$ is trivial;

(9) $K \subset \operatorname{int} V_{n_o+1}$.

(Note, that (7) follows by (3) and (4).)

Define the complexity of Y to be the integer $c(Y) = \Sigma_{i=0}^{n_o+1} \Sigma_{n=0}^{\infty} (n+1)^2 g_i(n)$, where $g_i(n)$ is the number of components of $\partial V_i$ with genus n [47;p.130]. We shall show that in a finite number of steps we can improve Y so that it will still satisfy (5)-(8) (but not necessarily also (9)) and that for some $i \geq 1$, $\partial V_i$ will be a collection of 2-spheres. We shall achieve this by compressing $\partial Y = \bigcup_{i=0}^{n_o+1} \partial V_i$ in a careful manner to reduce the complexity c(Y) and then we shall apply Haken's Finiteness theorem [29].

The sequence of compressions that accomplish our goal is a sequence of modifications on Y (McMillan [47] calls them "simple moves") of two types: if a compression of $\partial V_i$ takes place along a disk contained in $V_i$ we say that we removed a 1-handle while if the compressing disk lies outside $\mathring{V}_i$ we say that we added a 2-handle. So suppose first that there is a disk $D \subset \operatorname{int} V_o$ such that $D \cap \partial Y = \partial D \subset \partial V_i$ for some $i \in \{1, \ldots, n_o+1\}$ and such that $\partial D$ bounds no disk in $\partial V_i$. So D either lies outside $V_i$ (in $\operatorname{int} V_{i-1}$) or inside $V_i$ (in $V_i - V_{i+1}$).

In the first case we add a 2-handle to $V_i$ while in the second case we remove a 1-handle from $V_i$. Denote the new $V_i$ and Y by $V_i'$ and $Y'$, respectively. Note that in both cases we did not change any $V_j$, $i \neq j$. By [47 ;Lemma 4] , $1 \leq c(Y') < c(Y)$ so by a finite number of compressions we get $Y^* = \{V_0^*,\dots,V_{n_0+1}^*\}$ which cannot be compressed in such a manner anymore. A routine "trading disks" argument now implies that each component of $\partial Y^*$ which is not a 2-sphere is incompressible.

We want to verify that $Y^*$ satisfies the conditions (5)-(8). We first note that if F is a boundary of a 3-manifold Z it still bounds after the compression: if we added a 2-handle then the new F will bound the manifold Z plus the "half-open" 3-cell attached via the 2-handle, while if we removed a 1-handle from Z then the new F will bound the manifold Z minus the "half-open" 3-cell removed via the 1-handle. Therefore $Y^*$ is well-defined.

Next, $Y^*$ satisfies (5) and (6) by our construction. To prove (7) we show that a compression of an orientable boundary of a 3-manifold Z always yields an orientable boundary: suppose first that $Z' = Z+(\text{2-handle})$ had nonorientable boundary. Then we could find a simple closed curve $J \subset \partial Z'$ such that J would reverse the orientation in $\partial Z'$. We could isotope J off the cocore of the 2-handle and hence off the entire handle and into $\partial Z$, thus showing $\partial Z$ to be nonorientable. Since removing a 1-handle from Z has the same effect on $\partial Z$ as adding a 2-handle to the complementary 3-manifold component bounded by $\partial Z$, the preceeding argument also proves that for $Z' = Z -(\text{1-handle})$, $\partial Z'$ stays orientable. Finally, the condition (8) follows by [47;Lemma B ] because we made the simplifications $V_i \longrightarrow V_i'$ without disturbing $V_j$, $i \neq j$.

We now prove that for some $k \in \{1,\ldots,n_o+1\}$, $\partial V_k^*$ is a collection of 2-spheres. If not, then by Haken's Finiteness theorem [29] for some $1 \leq p < q \leq n_o+1$ there exist components $S_1 \subset \partial V_p^*$ and $S_2 \subset \partial V_q^*$ that are topologically parallel and different from $S^2$. So there is an embedding $f: S_1 \times [0,1] \longrightarrow U$ such that $f(S_s \times \{s\}) = S_s$ where s=0;1. Let $X = f(S_1 \times [0,1])$. We may assume that no surface in $(\text{int}\, X) \cap \partial Y^*$ is parallel to $S_1$ in X. By [65;Corollary (3.2)] each incompressible surface in int X is parallel to $S_1$ in X. Therefore $(\text{int}\, X) \cap \partial Y^*$ consists entirely of 2-spheres. Also, X must be irreducible for if there were a 2-sphere in X which would not bound a 3-cell in X then it would be incompressible hence parallel to $S_1 \neq S^2$. Therefore X minus the interiors of a finite disjoint collection of 3-cells lies in $V_p^*$. Hence every 1-cycle in $S_1$ is homologous to a 1-cycle in $S_2$ thus it bounds in $V_p^*$ by (8). Since by Lemma (2.4) the image of the inclusion-induced homomorphism $H_1(\partial V_p^*) \longrightarrow H_1(V_p^*)$ has rank (as a vector space over $\mathbb{Z}_2$) equal to $g(\partial V_p^*)$, it follows by (7) that $S_1$ is a 2-sphere, a contradiction.

Let V be a 3-manifold among $V_i^*$ all of whose boundary components are 2-spheres. Clearly, (9) may no longer be true so we now take care of that. During the compressions, when we attached a 2-handle it may have happened that it passed through the space in U that was previously occupied by a 1-handle which was removed at an earlier stage. In such cases we require that the boundary of the 2-handle be in general position with respect to the boundary of the 1-handle.

In addition, we shall assume that the annulus removed from $\partial V_i'$ (recall $\partial V_i'$ is orientable so it contains no Möbius bands) in the k-th compression be disjoint from all 1-handles or 2-handles involved in the preceeding k-1 compressions. So if we now add to $\partial V$ all 1-handles that were removed from V during the compressions, we get several 1-handles attached to $\partial V$. Note that adding of an old 1-handle H to $\partial V$ may result in many new smaller 1-handles as H may run through several 2-handles that now occupy portions of its original place. (See Figure (2.1).)

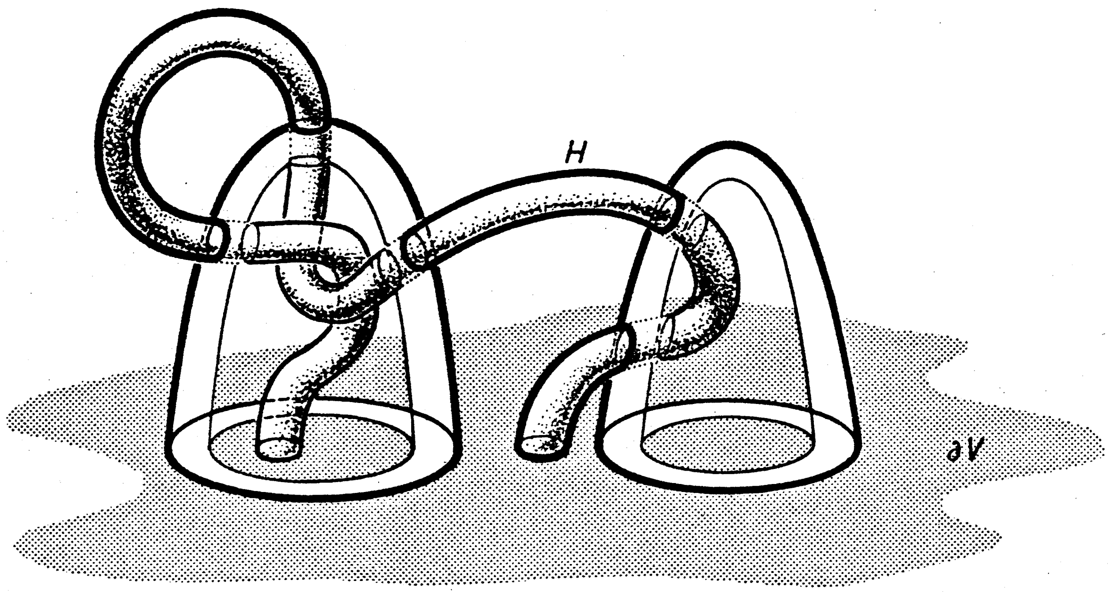



Figure 2.1.

Every resulting 1-handle is orientable. For suppose in reattaching the 1-handles sequentially we have added a nonorientable 1-handle. Then for every subsequent reattachment of the remaining 1-handles we have only one isotopy class of attaching maps [57 ;Theorem (3.34)] so we end up with a nonorientable surface. But this is impossible by (3) and (4). We may also assume that for every resulting 1-handle H both ends of H are attached to the same boundary component for otherwise we add H to V thus reducing the number of boundary components of V by one.

The 3-manifold N which we get from V by reattaching all 1-handles may be disconnected so we keep only the component which contains K. Thus N is obtained from a compact 3-manifold Q with $\partial Q$ a collection of 2-spheres by attaching a finite number of orientable 1-handles to $\partial Q$ so that every 1-handle has both ends on the same component of $\partial Q$. Let $p_i \in \Sigma_i$ (i=1;2) be arbitrary points on two distinct 2-sphere components $\Sigma_1$ and $\Sigma_2$ of $\partial Q$. Since K doesn't separate N there is a polygonal arc A in N-K joining $p_1$ and $p_2$. Suppose that A passes through a 1-handle H. We may assume that $A \cap H$ is just one arc meeting $\partial Q$ in only two points on $\Sigma_2$. Then $A \cap H$ can be replaced by another polygonal arc $B \subset N - \text{int}\, H$ attached to $\Sigma_2$. So we may assume that A doesn't pass through any of the 1-handles. Therefore by drilling tunnels we can effectively join the components of $\partial Q$ thus obtaining the desired neighborhood N. (See Figure (2.2).)

We can describe the structure of the neighborhoods N of K as follows: N = Q+(1-handles) where Q captures the "nonorientability" of K while the handles capture the "pathology" of K. (Figure (2.3))

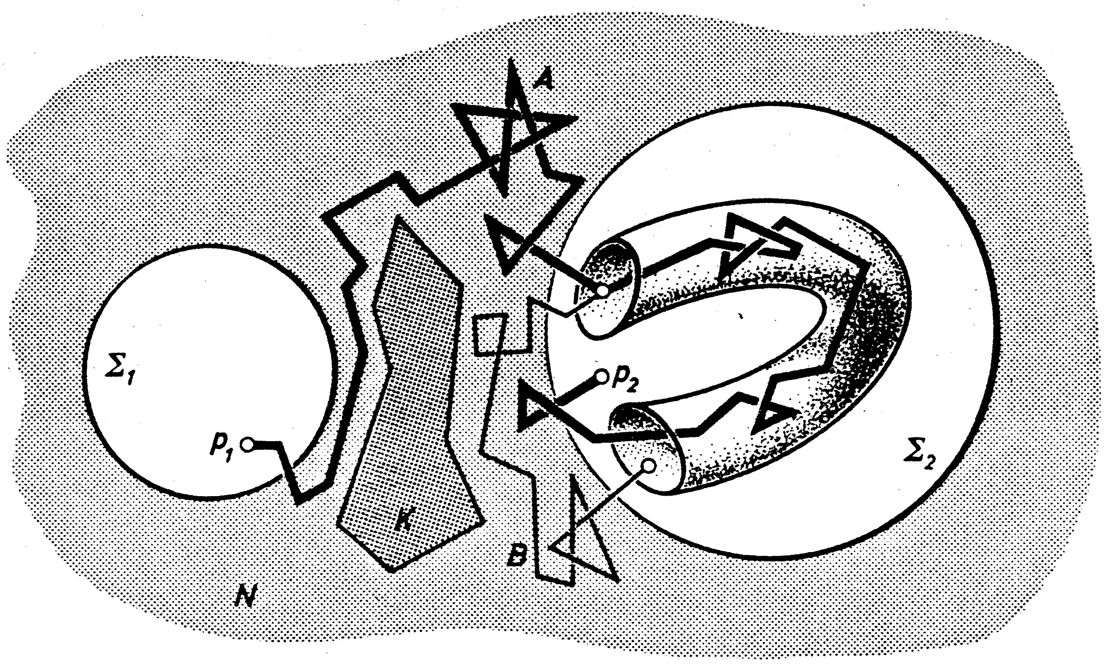



Figure 2.2.

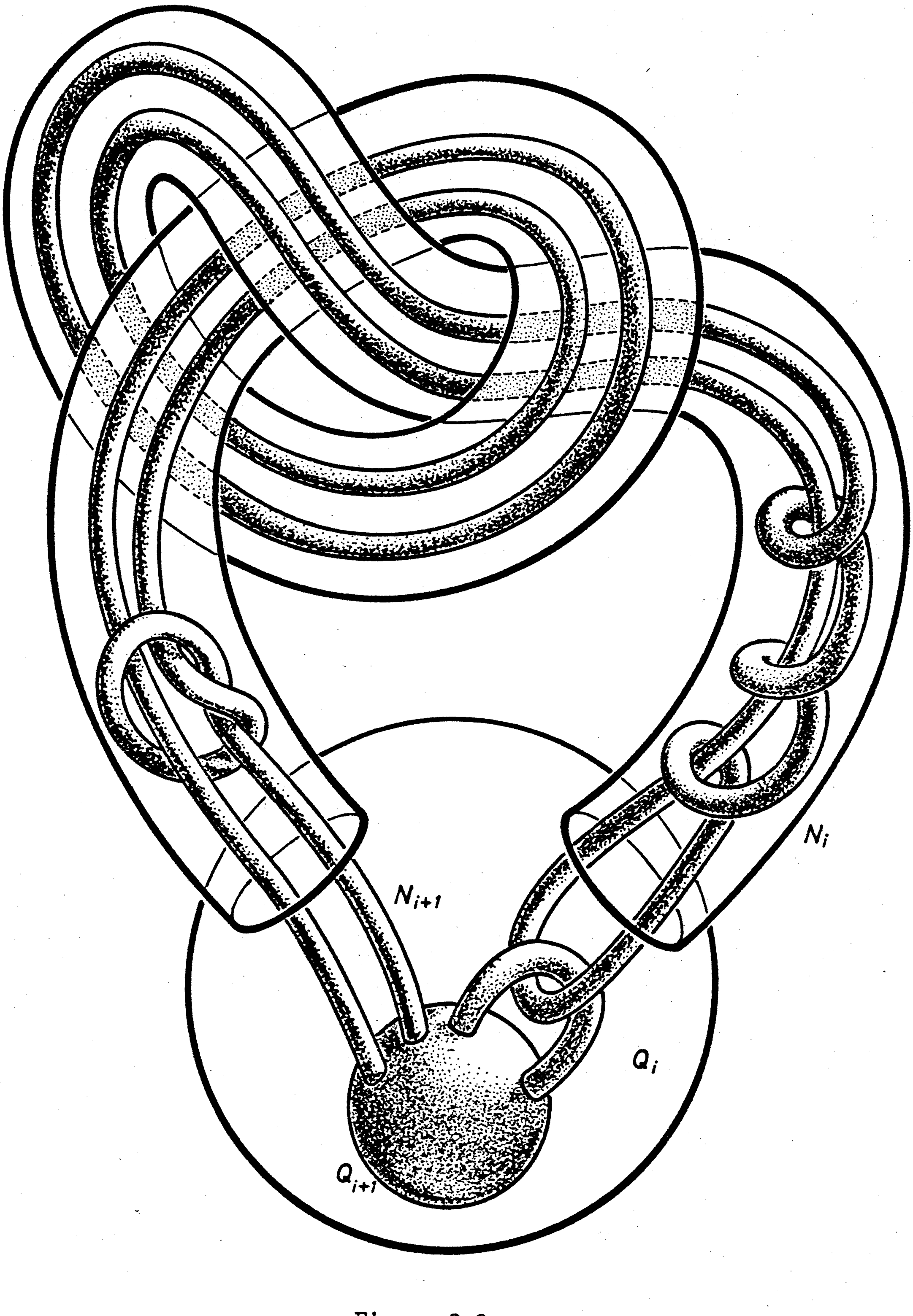



Figure 2.3.

It remains to prove (iv). So assume $K = \bigcap_{i=1}^{\infty} N_i$ where $N_i = Q_i +$ (1-handles), as in (i)-(iii). Let $K_i \subset \text{int}\, Q_i$ be a spine of $Q_i$. Let $\hat{Q}_i$ be the closed 3-manifold we obtain by attaching a 3-cell to $\partial Q_i$. For each $i \geq 1$, $N_1 = (N_1/K_i) \# \hat{Q}_i$ (the interior connected sum). Recall that $N_1$ is a nonorientable compact 3-manifold with boundary so by [32;(3.15),(3.17)] $N_1$ admits a unique normal, prime decomposition $N_1 = M_1 \# \dots \# M_n$, $M_i \neq S^2 \times S^1$. Consider normal, prime decompositions for $N_1/K_i$ and $\hat{Q}_i$ $(i \geq 1)$. Observe that every $N_1/K_i$ is orientable because $(N_1 - N_i) \cap K = \emptyset$ so $N_1 - N_i$ is orientable and $N_1/K_i \supset N_i/K_i =$ an orientable cube with handles. Therefore in a normal, prime decomposition $N_1/K_i = A_1 \# \dots \# A_p \# B_1 \# \dots \# B_q$ of $N_1/K_i$ ~~such that~~ $p \geq 0$ factors $A_i = S^2 \times S^1$. On the other hand $\hat{Q}_i$ must be nonorientable (since $N_1$ is) so in a normal, prime decomposition $\hat{Q}_i = C_1 \# \dots \# C_r$, every $C_i \neq S^2 \times S^1$. By [32;Lemma (3.17)] we may replace each $A_i$ by $P =$ the nonorientable $S^2$-bundle over $S^1$ to get a normal, prime decomposition for $N_1 = P \# \dots \# P \# B_1 \# \dots \# B_q \# C_1 \# \dots \# C_r$ (p factors P). It follows by the uniqueness of normal, prime decompositions that p+q+r=n and after a suitable permutation of the indices of summands each $C_i$ is homeomorphic to some $M_i$. Therefore among n+1 $\hat{Q}_i$'s at least two have the same prime summands (up to a homeomorphism). By choosing an appropriate subsequence of $\{Q_i\}$ we may henceforth assume that for each $i \leq j$ there is a homeomorphism $s_{ij} : Q_i \longrightarrow Q_j$.

We first construct $h_1$. Let $q_m : N_1 \to N_1/K_m$ be the quotient map where $K_m \subset \text{int}\, Q_m$ is a spine of $Q_m$ and let $i < j$. Then the identity on $\partial N_1$ induces a homeomorphism $t'_{ij} : \partial(N_1/K_i) \longrightarrow \partial(N_1/K_j)$ which makes the next diagram commutative:

$$\begin{array}{ccc} \partial N_1 & \xrightarrow{\quad id \quad} & \partial N_1 \\ \downarrow q_i|\partial N_1 & & \downarrow q_j|\partial N_1 \\ \partial(N_1/K_i) & \xrightarrow{\quad t'_{ij} \quad} & \partial(N_1/K_j) \end{array}$$

We shall show how to extend $t'_{ij}$ to a homeomorphism $t_{ij}:N_1/K_i \longrightarrow N_1/K_j$.

Take a simple closed curve $J\subset\partial(N_1/K_j)$ such that J is essential on $\partial(N_1/K_i)$ and null-homotopic in $N_1/K_i$. Since $N_1 = (N_1/K_i)\#\hat{Q}_i$ we can consider J also as an essential simple closed curve on $\partial N_1$ which is null-homotopic in $N_1$. Therefore $q_j(J)$ is a simple closed curve on $\partial(N_1/K_j)$ which is essential on $\partial(N_1/K_j)$ and null-homotopic in $N_1/K_j$). By Dehn's lemma, J (resp. $t'_{ij}(J)$) bounds an embedded disk $(D,\partial D)\subset(N_1/K_i,\partial(N_1/K_i))$ (resp. $(D',\partial D')\subset(N_1/K_j,\partial(N_1/K_j))$). By (ii) $N_1/K_m$ is a cube with (solid) 1-handles, so in finitely many steps we can cut $N_1/K_i$ along compressing disks D to get a 3-cell $R_i$. Extend $t'_{ij}$ over each D by mapping it to the corresponding compressing disk D' in $N_1/K_j$ described above. Finally, we can extend $t'_{ij}$ over the interior of $R_i$ to get $t_{ij}$.

Recall again that $N_1 = (N_1/K_m)\#\hat{Q}_m$ for all $m \geq 1$. Let $B_m\subset\hat{Q}_m$ and $C_m\subset\text{int}(N_1/K_m)$ be open 3-cells, m=i,j. Let $f_i: (\hat{Q}_i-B_i) \longrightarrow \partial((N_1/K_i)-C_i)$ be an attaching homeomorphism for the connected sum $(N_1/K_i)\#\hat{Q}_i$. If we define the attaching homeomorphism $f_j$ for $(N_1/K_j)\#\hat{Q}_j$ by $f_j = (s_{ij}|\partial)\circ f_i^{-1}\circ(t'_{ij})^{-1}$ then the diagram on the top of the next page will commute. (Note that because $Q_j$ is nonorientable any two attaching homeomorphisms $f_j$ are ambient isotopic [57;Ch.3] .)

$$\begin{array}{ccc} \partial((N_1/K_i)-B_i) & \xrightarrow{\;t'_{ij}\;} & \partial((N_1/K_j)-B_j) \\ \downarrow f_i & & \downarrow f_j \\ \partial Q_i & \xrightarrow{\;s_{ij}|\partial Q_i\;} & \partial Q_j \end{array}$$

Finally, define $h_{ij}: N_1 \longrightarrow N_1$ by $h_{ij}(x) = s_{ij}(x)$ if $x \in Q_i$ and $=$ $t_{ij}(x)$ if $x \in N_1 - Q_i$. Clearly, $h_{ij}|\,\partial N_1 = \mathrm{id}$ and $h_{ij}(Q_i^*) = Q_j^*$. The homeomorphism $h_1$ is the composition of $h_{1,2}$ and a homeomorphism of $N_1$ that is the identity outside a neighborhood of $\partial Q_2$ in $N_1$ and pushes $Q_2^*$ onto $Q_2$. We can get $h_i$, $i \geq 2$ in a similar way (see the proof of Lemma C in [14;pp.317-318] ). ⁂

Let K be a compact set in the interior of a 3-manifold M. We say that K can be <u>engulfed in</u> M if the interior of some punctured 3-ball in M contains K. A sequence $\{K_i\}$ of compact 3-manifolds with boundary is a W-<u>sequence</u> if for every i the following conditions hold:

(i) $K_i \subset \operatorname{int} K_{i+1}$ ;

(ii) the inclusion-induced homomorphism is trivial:

$$\Pi_1(K_i) \rightarrow \Pi_1(K_{i+1}).$$

An open 3-manifold M is called s <u>Whitehead manifold</u> if it can be expressed as $M = \bigcup_{i=0}^{\infty} K_i$ for some W-sequence of handlebodies [50;p.313].

An examination of the proofs in a recent paper of D.R.McMillan, Jr. and T.L.Thickstun [50] shows that the orientability hypothesis can be removed from all results in [50] if one uses Theorem (2.2) in the place of [49 ;Theorem 2 ]:

<u>Theorem 2.5</u>. Let M be a compact 3-manifold (possibly with boundary) and $K \subset \operatorname{int} M$ a compact subset. Then K can be engulfed in M if and

only if there is an open, connected neighborhood $U \subset M$ of $K$ such that $U$ embeds in $S^3$ and $H_1(U;\mathbb{Z}) = 0$. ⁂

Theorem 2.6. Let M be a compact 3-manifold (possibly with boundary). Then M contains no fake 3-cells if and only if each Whitehead manifold that embeds in int M also embeds in $S^3$. ⁂

## 3. A Resolution Theorem

J.L.Bryant and R.C.Lacher have proved that every locally contractible 1-acyclic over $\mathbb{Z}_2$ image X of a 3-manifold M without boundary admits a resolution. In particular, X is a generalized 3-manifold [14;Theorem 2] . A refinement of their proof enabled them to omit the acyclicity hypothesis over a 0-dimensional set provided M was orientable [14;Theorem 3] . We show below that orientability is not necessary. (We are referring to the case p=0 or 2 of [14 ;Theorem 3] only.)

Theorem 2.7. Let f be a closed, monotone mapping from a 3-manifold M without boundary onto a locally simply connected $\mathbb{Z}_2$-homology 3-manifold X. Suppose that there is a 0-dimensional set $Z \subset X$ such that $\check{H}^1(f^{-1}(x);\mathbb{Z}_2) = 0$ for all $x \in X-Z$. Then the set $C = \{ x \in X \mid f^{-1}(x)$ is not cell-like $\}$ is locally finite in X. Moreover, X has a resolution.

Proof. Again we supress the coefficients. Let $A = \{ x \in X \mid \check{H}^1(f^{-1}(x)) \neq 0 \}$. By [38 ;Theorem (4.1) ] A is locally finite in X. Let $B = \{ x \in X \mid f^{-1}(x)$ has no neighborhood in M embeddable in $\mathbb{R}^3 \}$. In order to show that B is locally finite in X it suffices by Theorem (2.1) to prove that for each $x \in X$, $f^{-1}(x)$ possesses a neighborhood $U \subset M$ such that $H_1(U-f^{-1}(x)) \to H_1(M)$ is trivial.

So let $x \in X$. Since A is locally finite in X there is a neighborhood $W \subset X$ of x such that $W \cap A \subset \{x\}$. By hypothesis, X is $LC^1$ so there is a connected neighborhood $W' \subset W$ of x such that any loop in W' is null-homotopic in W. Consider the following commutative diagram:

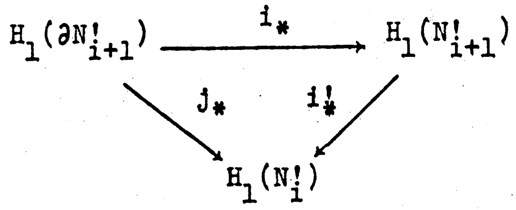
$$\begin{array}{ccc} H_1(f^{-1}(W')-f^{-1}(x)) & \xrightarrow{i'_*} & H_1(f^{-1}(W)-f^{-1}(x)) \\ \cong \downarrow f|_* & & \cong \downarrow f|_* \\ H_1(W'-\{x\}) & \longrightarrow & H_1(W-\{x\}) \\ \cong \downarrow j'_* & & \cong \downarrow j_* \\ H_1(W') & \xrightarrow{i_*} & H_1(W) \end{array}$$

where the horizontal homomorphisms are induced by inclusions, $f|_*$ are the isomorphisms of Proposition (1.1), and $j_*$ and $j'_*$ are the isomorphisms from the homology sequence of the pairs $(W,W-\{x\})$ and $(W',W'-\{x\})$, respectively. By hypothesis, $i_* = 0$ hence $i'_* = 0$. Thus we may apply Theorem (2.1) -- we conclude that B is locally finite in X.

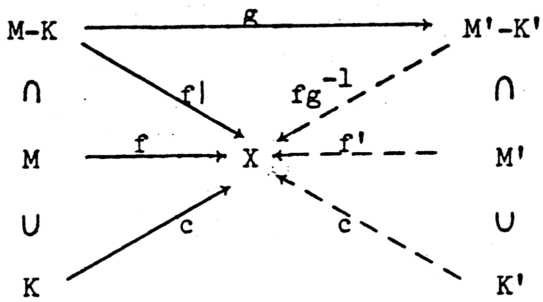
By Theorem (2.2), $f^{-1}(x)$ is definable by (orientable) cubes with handles for all $x \in X-B$, so by [49;Theorem 3], $f^{-1}(x)$ has the 1-UV property. Since cubes with handles have no higher homotopy, each $f^{-1}(x)$ has the $UV^{\infty}$ property and hence $C \subset B$ (cf. [40]). Therefore C is locally finite in X. Note that, in particular, by G.Kozlowski and J.J.Walsh [36], X-C (hence also X) is finite-dimensional.

It now remains to find a resolution for X. We construct it by improving f over the points of C. Observe first, that if $x \in X$ is an arbitrary point and $W \subset X$ is any of its neighborhoods then combining the isomorphism $H_o(f^{-1}(W)-f^{-1}(x)) \cong H_o(W-\{x\})$ given by Proposition (1.1), with the isomorphism $H_o(W-\{x\}) \cong H_o(W)$ given by Lemma (1.5), we can conclude that $f^{-1}(x)$ doesn't separate its connected neighborhoods in M. It follows by the arguments employed in the diagram on page 36 that $f^{-1}(x)$ satisfies the hypotheses of Theorem (2.2). So if we let $c \in C$ and put $K = f^{-1}(c)$ then $K = \bigcap_{i=1}^{\infty} N_i$ where $N_i$'s are the compact 3-manifolds with boundary described in the conclusions of Theorem (2.2). We shall use the notation from that theorem in the rest of the proof (i.e., $Q_i$, $Q_i^*$, and $h_i$).

(The following is modelled after [14;pp.316-317] .) Let $M' = M/Q_1^*$ and let $h_i':N_i \to N_i$ be a homeomorphism such that $h_i'|\partial N_i = id$ and $h_i'(Q_{i+1}) = Q_{i+1}^*$. Define a map $h_i^*:M \to M$ by letting $h_i^*(x) = h_i'h_i(x)$ for $x \in N_i$ and $= x$ otherwise. Then $h_i^*$ is a homeomorphism and $h_i^*(Q_i^*) = Q_{i+1}^*$. Let $g_o:M \to M'$ be the quotient map. Define inductively $g_i = g_{i-1}(h_i^*)^{-1}:M \to M'$, $i \geq 1$. Then $g_i|(M-N_i) = g_{i-1}|(M-N_i)$. Also, the only nondegenerate point-inverse of $g_i$ is $Q_{i+1}^*$. Indeed, $g_i = g_{i-1}(h_i^*)^{-1} = g_{i-2}(h_{i-1}^*)^{-1}(h_i^*)^{-1} = \ldots = g_o(h_1^*)^{-1}\ldots(h_i^*)^{-1}$, $h_j^*$ are homeomorphisms, the only nondegenerate point-inverse of $g_o$ is clearly $Q_1^*$, and $g_i^{-1}g_o(Q_1^*) = (h_i^*\ldots h_1^*g_o^{-1})g_o(Q_1^*) = (h_i^*\ldots h_1^*)(Q_1^*) = (h_i^*\ldots h_2^*)(Q_2^*) = \ldots = h_i^*(Q_i^*) = Q_{i+1}^*$. Let $N_i' = g_i(N_{i+1})$, $i \geq 0$, and $K' = \bigcap_{i=0}^{\infty} N_i'$. It follows by Theorem (2.2) that for every i, $\Pi_1(N_i')$ is free on finitely many generators. Also, considering the commutative diagram on the top of the next page we observe that $i_*$ is onto and $j_* = 0$ by the choice of $N_i$'s and $g_i$'s. Hence $i_*' = 0$, too. It follows by [49;Theorem 3] that K' is cell-like.




Define a map $g{:}M\text{-}K \to M'$ by letting $g = g_i$ on $M\text{-int}\,N_i$. Then $g(M\text{-}K) = M'\text{-}K'$ and $g$ is a homeomorphism. Finally, we let $f'{:}M' \to X$ be given by $f'(x) = fg^{-1}(x)$ for $x \in M'\text{-}K'$ and $= c$ for $x \in K'$. It is easy to see that $f'$ is a continuous, proper onto map. Since $K'$ is cell-like, $f$ is cell-like over $X\text{-}(C\text{-}\{c\})$:




The proof is now completed by repeating the above surgery over the rest of $C$ (i.e., over $C\text{-}\{c\}$.) ⁑

<u>An alternative proof of Theorem (2.7)</u>. Let $A = \{x \in X \mid \check{H}^1(f^{-1}(x) \neq 0\}$. By [14 ;Assertion 1 on p.315 ], $A$ is locally finite in $X$. Since by [48 ;Proposition 2 ] and Proposition (1.2), every $f^{-1}(x)$, $x \in X\text{-}A$, has an orientable neighborhood in $M$, it follows by [14 ;Assertion 3 on p.316 ] that $C\text{-}A$ is locally finite in $X\text{-}A$. It thus remains to show that no limit point of $C\text{-}A$ can belong to $A$. Let $a \in A$ and suppose that for a sequence $\{x_n\} \subset X\text{-}A$ , $\lim_{n\to\infty} x_n = a$. By [14 ;Assertion 2 on p. 316 ], every $f^{-1}(x)$ is strongly $\mathbb{Z}_2$-acyclic hence by [47 ;Theorem 2 ], the intersection of a nested sequence of $\mathbb{Z}_2$-homology 3-cells with

handles. Thus for each $n \geq 1$ there exists an orientable neighborhood $U_n \subset M$ of the continuum $f^{-1}(x_n)$ and a $\mathbb{Z}_2$-homology 3-cell with handles $H_n \subset U_n$ such that $f^{-1}(x_n) \subset \text{int } H_n$. We may also assume that if $i \neq j$ then $U_i \cap U_j = \emptyset$. It is a well-known corollary of Grushko-Neumann theorem [32 ;p.25] that in a compact 3-manifold there is but a finite number of pairwise disjoint $\mathbb{Z}_2$-homology 3-cells that fail to be genuine 3-cells [69]. Therefore, by [49;Theorem 3] all but a finite number among $f^{-1}(x_n)$ are 1-UV hence cell-like [40]. Thus $x_n \notin C$ for all but a finite number of indices n. Therefore the set C-A is locally finite in X. Consequently, C is locally finite in X. The construction of a resolution for X is now as in the preceeding proof. ⁑

The next corollary provides a partial converse in dimension 3 to a well-known fact that cell-like upper-semicontinuous decompositions of topological n-manifolds yield generalized n-manifolds (for $n \geq 4$ assume also the quotient space is finite-dimensional).

Corollary 2.8. Let G be a 0-dimensional upper semicontinuous decomposition of a closed 3-manifold M such that M/G is a generalized 3-manifold. Then the set $C = \{g \in G \mid g \text{ is not cell-like}\}$ is finite.

Proof. Since G is upper semicontinuous the quotient map $q: M \twoheadrightarrow M/G$ is closed and monotone. Let $Z = q(N_G)$. Then $\dim Z \leq 0$. The conclusion now follows immediately by Theorem (2.7). ⁑

Remarks. (1) The Hopf maps or the Bing map [13 ;p.48] show that if $q(N_G)$ is a 1-manifold then all nondegenerate elements $g \in G$ may fail to be cell-like. Thus the restriction dim G = 0 in Corollary (2.8) seems reasonable.

(2) Spine maps [13;p.48] show that the set C in Corollary (2.8) may have any finite number of elements even when $C = H_G$.

(3) The following modification of the classical construction of the Whitehead continuum [67] shows that all nondegenerate elements of G may fail to be cellular in M even when $q(N_G)$ is a Cantor set and G is cell-like. Let $\{T_i\}$ be the defining sequence for the Whitehead continuum. Keep $T_0$. Replace $T_1$ by two smaller solid tori $T_{00}$ and $T_{01}$ as shown in Figure (2.4).

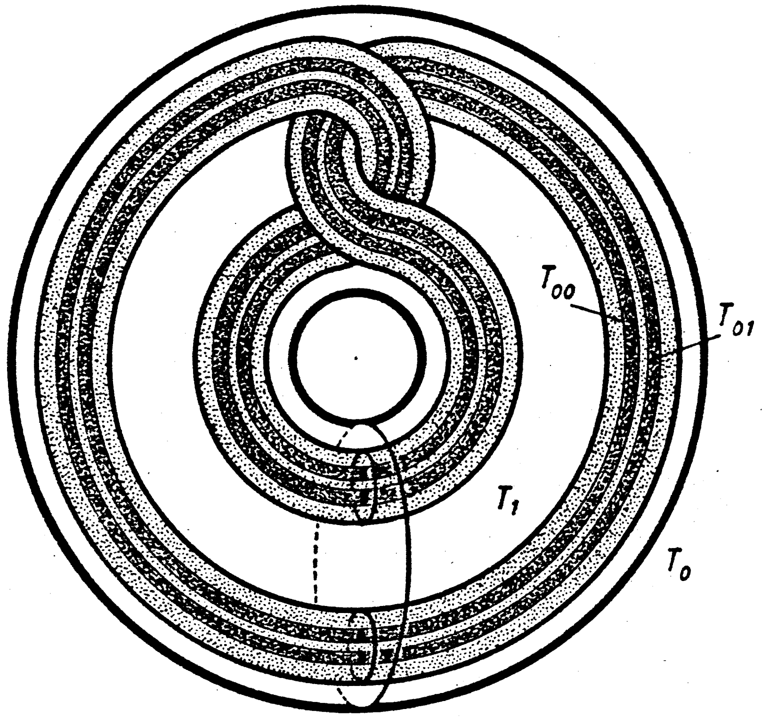



Figure 2,4.

In an analogous way replace $T_2$ by four smaller solid tori $T_{000}$, $T_{001} \subset \operatorname{int} T_{00}$ and $T_{010}, T_{011} \subset \operatorname{int} T_{01}$, etc. Let $Y = T_0 \cap (T_{00} \cup T_{01}) \cap (T_{000} \cup T_{001} \cup T_{010} \cup T_{011}) \cap \ldots$ and let G be the decomposition of $S^3$ into points and the components of Y.

## 4. Peripheral 1-Acyclicity

We wish to compare various concepts of 1-acyclicity we employed in the preceeding sections. Let K be a subset of an ANR X. We say that the inclusion $K \subset X$ is strongly (resp. weakly peripherally, strongly peripherally) 1-acyclic over R (R a PID) if for each neighborhood $U \subset X$ of K there is a neighborhood $V \subset U$ of K such that the inclusion-induced homomorphism $H_1(V;R) \to H_1(U;R)$ (resp. $H_1(V-K;R) \to H_1(U;R)$; $H_1(V-K;R) \to H_1(U-K;R)$) is trivial. It is well-known that strong 1-acyclicity does not depend upon the embedding of K into X and that furthermore, for R a field, it is equivalent to the condition $\check{H}^1(K;R) = 0$ [40;p.502] . The following example shows that the other two acyclicities may depend upon the embedding. Let $X = S^2 \times S^1$ and $K = S^2 \vee S^1$, and let $f: K \to \mathbb{R}^3$ and $g: \mathbb{R}^3 \to X$ be embeddings. Then $K \subset X$ is strongly peripherally 1-acyclic over any PID R (since X-K is an open 3-cell) while $(gf)(K) \subset X$ is not even weakly peripherally 1-acyclic for any PID R (just take $U = g(\mathbb{R}^3)$). It is not a coincidence that $\dim K = 2$ in this example for we prove in Theorem (2.11) that for $\dim K \leq 1$ and X an R-orientable 3-manifold, all three 1-acyclicities are equivalent so, in particular, independent of the embedding.

It is clear that strong peripheral 1-acyclicity implies weak peripheral 1-acyclicity. We now show that for compacta in 3-manifolds the two concepts are equivalent if $R = \mathbb{Z}_2$.

Theorem 2.9. Let K be a compact set in the interior of a 3-manifold M. Suppose that $K \subset M$ is weakly peripherally 1-acyclic over $\mathbb{Z}_2$. Then $K \subset M$ is strongly peripherally 1-acyclic over $\mathbb{Z}_2$.

Proof. We shall supress $\mathbb{Z}_2$ coefficients from the notation. Using the hypothesis we can express K as the intersection of a properly nested sequence of compact 3-manifolds $N_i \subset \operatorname{int} M$ with boundary such that all inclusion-induced homomorphisms $H_1(N_i - K) \to H_1(N_{i-1})$ are trivial.

Let $\alpha$ be a simple closed curve in $N_i - K$. Then there is an integer $j > i$ such that $\alpha \subset N_i - N_j$. Let $\Sigma \subset \partial N_{j+1}$ be a component of $\partial N_{j+1}$. Since $\Sigma$ is a closed 2-manifold it contains a bouquet T of finitely many simple closed curves so that $\Sigma - T$ is an open 2-cell. Let $\beta \subset T$ be one of these loops. Since $H_1(\partial N_{j+1}) \to H_1(N_j)$ is trivial, $\beta$ bounds a surface $\Gamma_\beta$ in $N_j$. Also, $\alpha$ bounds a surface $\Gamma_\alpha$ in $N_{i-1}$ since $H_1(N_i - K) \to H_1(N_{i-1})$ is trivial. Put the surfaces $\Gamma_\alpha$ and $\Gamma_\beta$ into general position. Let $p_1, \ldots, p_t \in \Gamma_\alpha \cap \beta$ be the points of the intersection, ordered in such a way that for each i, $p_i$ lies between $p_{i-1}$ and $p_{i+1}$ on $\beta$. Note also that each $p_i$ lies in $\operatorname{int} \Gamma_\alpha$ because $\partial \Gamma_\alpha \cap \beta = \alpha \cap \beta = \emptyset$. Let $A \subset N_{i-1}$ be a regular neighborhood of $\beta$ in $N_{i-1}$ and $C = \partial A$. Thus A can either be a solid torus or a solid Klein bottle. For each i, there is a disk $D_i \subset \operatorname{int} \Gamma_\alpha$ centered at $p_i$ such that $A \cap \Gamma_\alpha = \cup_{i=1}^{t} D_i$. Let $C_i \subset C$ be the annulus determined by the pair $(\partial D_{2i-1}, \partial D_{2i})$, $1 \le i \le [\frac{t}{2}]$, i.e., $D_{2i-1} \cup C_i \cup D_{2i}$ is the boundary of

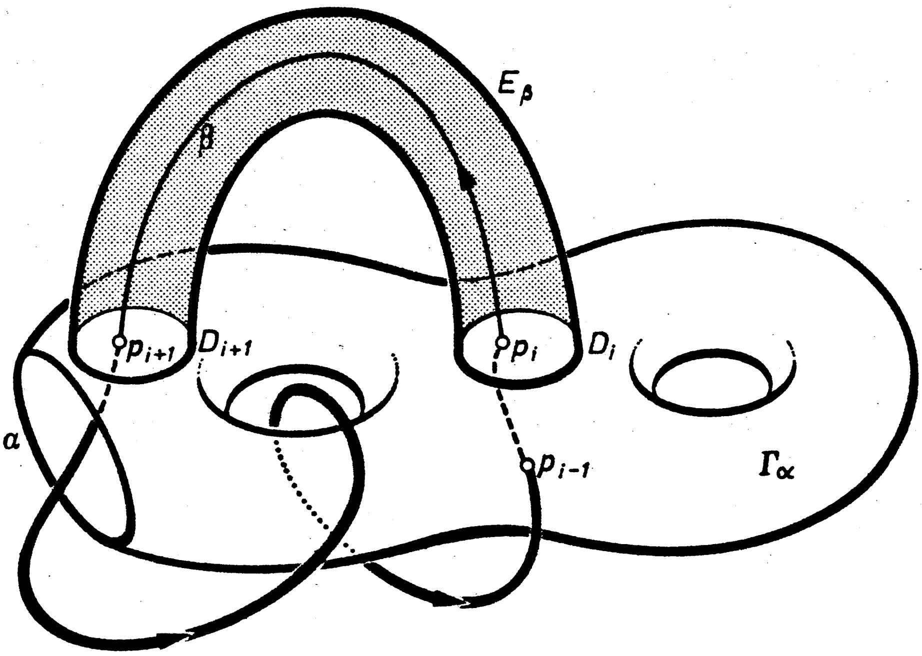



Figure 2.5.

the 3-cell $E_\beta$ which $D_{2i-1}$ and $D_{2i}$ cut off on A and which doesn't contain any $D_j$, $j \neq 2i-1, 2i$. (See Figure (2.5).)

We now do the following surgery on $\Gamma_\alpha$: replace each pair $(D_{2i-1}, D_{2i})$ by the annulus $C_i$, $1 \leq i \leq [\frac{t}{2}]$. Denote the new surface by $\Gamma_\alpha^*$ If t is even then $\Gamma_\alpha^* \cap \beta = \emptyset$ while if it is odd then $\Gamma_\alpha^* \cap \beta = \{p_t\}$ . Suppose t were odd. Consider $X = \Gamma_\alpha^* \cap \Gamma_\beta$ and let $A \subset X$ be the component containing the point $p_t$. Then A is a compact 1-manifold hence an arc. Plainly, $p_t \in \partial A$. Let $q \in \partial A$ be the other endpoint. Now, $q \notin \Gamma_\alpha^*$ because $\partial\Gamma_\alpha^* \cap \Gamma_\beta = \alpha \cap \Gamma_\beta = \emptyset$ since $\Gamma_\beta \subset N_j$ and $\alpha \subset N_i - N_j$. Also, $q \notin \partial\Gamma_\beta$ because $(\partial\Gamma_\beta - \{p_t\}) \cap \Gamma_\alpha^* = (\beta \cap \Gamma_\alpha^*) - \{p_t\} = \emptyset$. Since $q \in \partial\Gamma_\alpha^* \cup \partial\Gamma_\beta$ this yields a contradiction to our hypothesis that t was odd.

We can therefore assume that $\alpha$ bounds a surface $\Gamma$ in $N_{i-1}$ such that for every loop $\beta \subset T$, $\Gamma \cap \beta = \emptyset$. Thus if $\Gamma$ hits $N_{j+1}$ at all, it enters through open disks in $\partial N_{j+1}$ and so it can be cut off at $\partial N_{j+1}$. Hence $\alpha$ bounds a surface in $N_i - N_{j+1} \subset N_i - K$. ⁑

<u>Theorem 2.10</u>. Let R be a PID and let K be a compact set in the interior of an R-orientable 3-manifold M. Suppose that K is strongly 1-acyclic over R. Then $K \subset M$ is strongly peripherally 1-acyclic over R.

<u>Proof</u>. We shall supress the coefficients from the notation. Let $V \subset U \subset M$ be neighborhoods of K such that the inclusion-induced homomorphism $H_1(V) \to H_1(U)$ is trivial. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & \cdots \longrightarrow & H_2(V,V-K) & \longrightarrow & H_1(V-K) & \longrightarrow & H_1(V) & \longrightarrow & \cdots \\
 & \nearrow^{f} & \downarrow \cong & & \downarrow j_*' & & \downarrow j_* & & \\
\check{H}^1(K) & & & & & & & & \\
 & \searrow_{f'} & & & & \xrightarrow{i_*} & & & \\
 & \cdots \longrightarrow & H_2(U,U-K) & \longrightarrow & H_1(U-K) & \longrightarrow & H_1(U) & \longrightarrow & \cdots
\end{array}$$

where the horizontal sequences are from the homology sequence of the pairs (V,V-K) and (U,U-K), and f,f' are the Alexander duality isomorphisms. By [40 ;p.502 ] , $\check{H}^1(K) = 0$ hence $i_*$ is a monomorphism. Since $j_* = 0$ we can therefore conclude that $j_*' = 0$. ⁑

The converse of Theorem (2.10) is false: let $M = (S^2 \times S^1) - B$ where $B \subset S^2 \times S^1$ is the interior of a 3-cell. Then $K = S^2 \vee S^1$ is a spine of M so that $M - K = S^2 \times [0,1)$. Therefore, K is strongly peripherally 1-acyclic over any PID R. On the other hand K certainly is not strongly 1-acyclic over any PID R. Note that in this example

$\dim K = 2$. The next theorem asserts that there can be no counterexample with $\dim K \le 1$.

<u>Theorem 2.11</u>. Let R be a PID and let K be a compact set in the interior of an R-orientable 3-manifold M. Suppose that $\dim K \le 1$. Then the following statements are equivalent:

(i) K is strongly 1-acyclic over R;

(ii) $K \subset M$ is strongly peripherally 1-acyclic over R;

(iii) $K \subset M$ is weakly peripherally 1-acyclic over R.

<u>Proof</u>. We shall supress the coefficients from the notation. We already know that (i) $\Rightarrow$(ii) by Theorem (2.10), while (ii) $\Rightarrow$ (iii) is clear. We show (iii) $\Rightarrow$ (i): let $V \subset U \subset M$ be neighborhoods of K such that the inclusion-induced homomorphism $H_1(V-K) \to H_1(U)$ is trivial. Let z be a 1-cycle in V. By [39;Lemma (2.1)] z is homologous to a 1-cycle $z^* \in Z_1(V-K)$. By hypothesis, $z^* \sim 0$ in U hence $z \sim 0$ in U, as well. ⁑⁑

<u>Theorem 2.12</u>. Let K be a compact set in the interior of a 3-manifold M. Then the following statements are equivalent:

(i) $K \subset M$ is weakly peripherally 1-acyclic over $\mathbb{Z}_2$;

(ii) $K \subset M$ is strongly peripherally 1-acyclic over $\mathbb{Z}_2$;

(iii) There exists a neighborhood $W \subset M$ of K such that each simple closed curve in W-K is $\mathbb{Z}_2$-homologous to zero in M-K.

<u>Remark</u>. Let W be an open neighborhood of K such as in (iii) above. Then by [49;Lemma 1] K is strongly 1-acyclic over $\mathbb{Z}_2$ if and only if, in addition, each simple closed curve in W is $\mathbb{Z}_2$-homologous to

zero in M. This gives us a good measure of the (possible) difference between the two acyclicities (over $\mathbb{Z}_2$).

Proof of Theorem (2.12). We only need to prove (iii) $\Rightarrow$(ii) since (ii) $\Rightarrow$ (iii) is clear and (i)$\Leftrightarrow$(ii) follows by Theorem (2.9). So let $U \subset M$ be a neighborhood of K. We may assume that $U \subset W$, that U is a compact 3-manifold with boundary, and that $K \subset \text{int}\, U$. Let $S \subset \partial U$ be a component of $\partial U$. Then there is a bouquet $T \subset S$ of simple closed curves such that $S - \cup\{J \in T\}$ is an open 2-cell. By hypothesis, each curve $J \in T$ bounds a surface $S_J$ in M-K. Let $V = \text{int}\, U - \cup\{S_J \mid J \in T\}$ and let J* be a simple closed curve in V-K. Then J* bounds a surface S* in M-K. Using the same argument as in the proof of Theorem (2.9) we can show that J* bounds a surface S' in $M-(K \cup (\cup\{S_J \mid J \in T\}))$ and hence enters S through open disks and can thus be cut off on S. We may therefore assume that $S' \subset U-K$. This shows that every 1-cycle in V-K bounds in U-K. ※

Corollary 2.13. Let K be a compact set in the interior of a 3-manifold M and suppose that $H_1(M-K;\mathbb{Z}_2) = 0$. Then $K \subset M$ is strongly peripherally 1-acyclic over $\mathbb{Z}_2$.

Proof. Apply Theorem (2.12) with W = M. ※

## III. A DISJOINT DISKS PROPERTY FOR 3-MANIFOLDS

The main results of this chapter are Theorems (3.8) and (3.10) -- we show that the map separation property (MSP), a concept due to H.W.Lambert and R.B.Sher [44] is an appropriate analogue of J.W. Cannon's disjoint disks property (DDP) for the class of compact generalized 3-manifolds with 0-dimensional singular set, modulo the Poincaré conjecture. In the first section we introduce the MSP and a similar concept from [44], called the Dehn's lemma property (DLP) and we prove that 3-manifolds have both properties. In the second section we prove the main results. We conclude the chapter by an application of Thickstun's extension of the Loop theorem (Theorem (1.15)) to the study of isolated singularities

### 1. Dehn Disks in 3-Manifolds

We recall that a mapping means only a continuous hence not necessary PL map. A mapping f of a disk (resp. disk with holes) D into a space X is called a Dehn disk (resp. Dehn disk with holes) if $S_f \cap \partial D = \emptyset$, where $S_f = cl\ \{ x \in D \mid f^{-1}(f(x)) \neq x \}$ is the singular set of f. Also, define $\Sigma_f = f(S_f)$. A space X is said to have the Dehn's lemma property (DLP) [44] if for every Dehn disk $f: D \longrightarrow X$ and every neighborhood $U \subset X$ of $\Sigma_f$ there exists an embedding $F: D \longrightarrow X$ such that $F(D) \subset f(D) \cup U$ and $F(\partial D) = f(\partial D)$. A space X is said to have the map separation property (MSP) [44] if given any collection of Dehn disks $f_1,\dots,f_k:D \longrightarrow X$ such that if $i \neq j$ then $f_i(\partial D) \cap f_j(D) = \emptyset$, and given a neighborhood $U \subset X$ of $\bigcup_{i=1}^{k} f_i(D)$ there exist mappings $F_1,\dots,F_k:D \longrightarrow U$ such that for each $i$, $F_i|\partial D = f_i|\partial D$ and if $i \neq j$ then $F_i(D) \cap F_j(D) = \emptyset$.

Lambert and Sher say in [44] that "it is a well-known (and useful) fact that $S^3$ has the DLP and the MSP" but they give no proof or reference [44;p.514] . We prove below that every 3-manifold (possibly with boundary) has both properties (by (3.2) and (3.7)). This result follows by the following stronger result:

Theorem 3.1. Let $f:D \longrightarrow M$ be a Dehn disk in a 3-manifold M (possibly with boundary) and $U \subset M$ a neighborhood of $\Sigma_f$. Then there exists an embedding $F:D \longrightarrow M$ such that

(i) $F(D)-U = f(D)-U$;

(ii) $F|\partial D = f|\partial D$.

Corollary 3.2. Every 3-manifold (possibly with boundary) has the DLP. ⁂

The proof of Theorem (3.1) relies heavily on two deep results from 3-manifolds topology -- R.H.Bing's Surface Approximation theorem [5] and D.W.Henderson's extension of Dehn lemma [33 ;Theorem (IV.3)] .

Theorem 3.3.(R.H.Bing [5] ) Let P be a compact surface in a 3-manifold M, $N \subset P$ a closed subset, and let $f:P \longrightarrow \mathbb{R}_+$ be an arbitrary map, Assume that at each point $x \in N$, P is locally PL in M at x. Then there exists a surface $P^* \subset M$ and a homeomorphism $h:P \rightarrowtail\!\!\!\!\twoheadrightarrow P^*$ such that:

(i) $h|N:N \longrightarrow P^*$ is the inclusion;

(ii) for every $x \in P$, $d(x,h(x)) \leq f(x)$;

(iii) for every $x \in P$, with $f(x) > 0$, $P^*$ is locally PL in M at $h(x)$. ⁑

Theorem 3.4. (D.W.Henderson [33]) Let M be a 3-manifold and $f:D \to M$ a PL disk with $S_{f|\partial D} = \emptyset$. Then for every $\varepsilon > 0$ there exists a PL embedding $F:D \to M$ such that

(i) $F(D)-N_\varepsilon(\Sigma_f) = f(D)-N_\varepsilon(\Sigma_f)$, where $N_\varepsilon$ is the $\varepsilon$-neighborhood of $\Sigma_f$;

(ii) $F|\partial D = f|\partial D$. ⁑

Corollary 3.5. (Bing's extension of Dehn's lemma [15; Theorem (4.5.4)]) Let $f:D \to M$ be a Dehn disk in a 3-manifold M (possibly with boundary) and $U \subset M$ an open neighborhood of $f(\mathrm{int}\, D)$. Then there is a homeomorphism F of D into $f(\partial D) \cup U$ such that F is locally PL except (possibly) on $\partial D$.

Proof. Follows by Theorems (3.1) and (3.3). ⁑

Proof of Theorem (3.1). We first consider the case when $f(D) \subset \mathrm{int}\, M$. Here is an outline of the proof: Put $S_f$ inside pairwise disjoint PL disks with holes $C_1, \ldots, C_m \subset f^{-1}(U)$. Let $C = \cup_{i=1}^{m} C_i$. Assume that on some neighborhood of $\partial C$, f is a locally PL embedding.

Step 1. Consider the surface $H = f(D-\mathrm{int}\, C)$. Use Theorem (3.3) to make H PL.

Step 2. Consider the singular surface $L = f(C)$. Use Zeeman's Relative Simplicial Approximation theorem [72] to make L polyhedral.

Step 3. Now $H \cup L$ is a desired PL Dehn disk. Apply Theorem

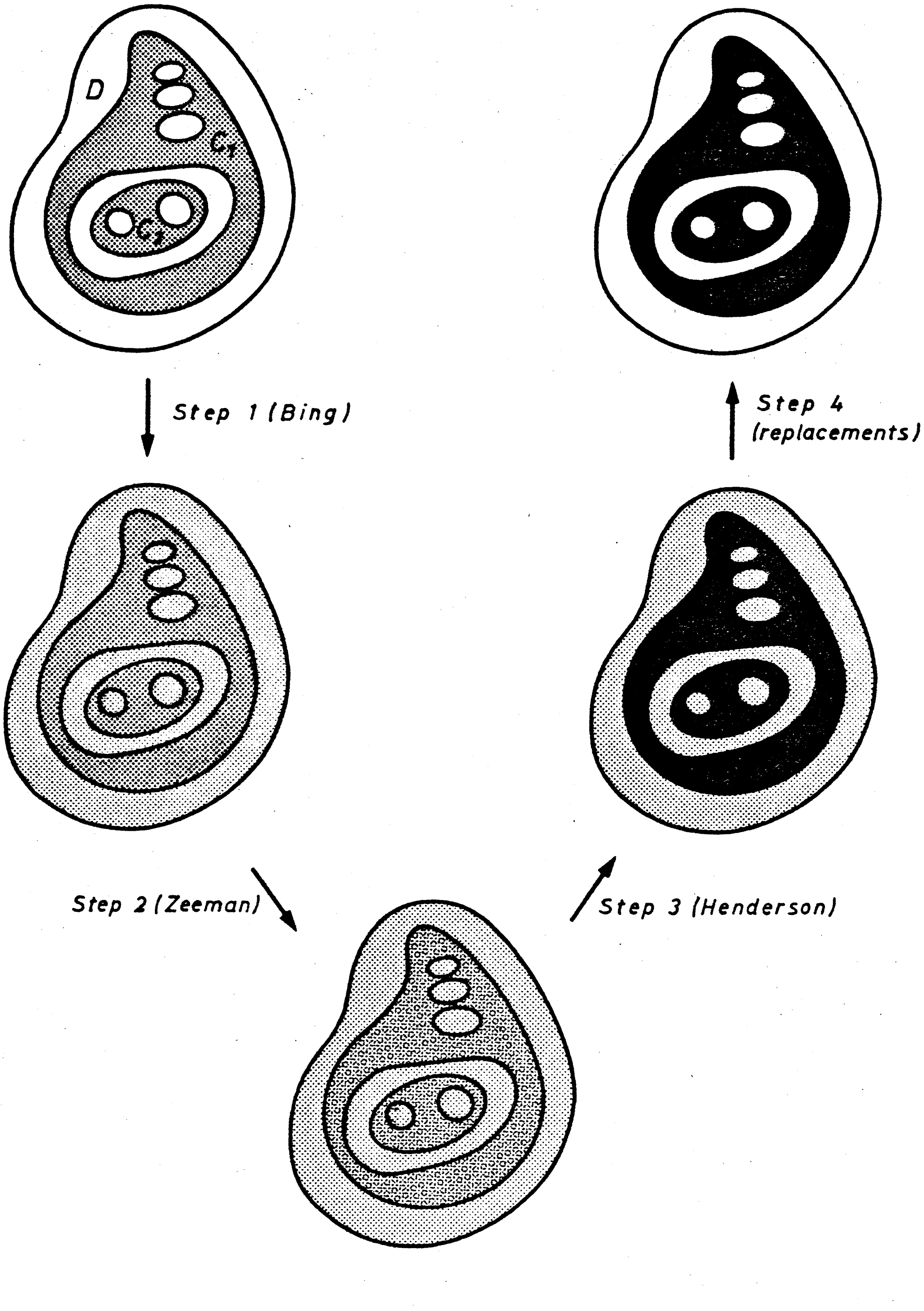

D
$C_1$
$C_1$
Step 1 (Bing)
Step 4
(replacements)
Step 2 (Zeeman)
Step 3 (Henderson)

Figure 3.1.

(3.4) to get an embedded disk $T \subset M$.

Step 4. Replace the portions of T which lie outside U by corresponding pieces of H. (See Figure (3.1).)

In general, the curves from $f(\partial C)$ are going to be "wildly" embedded in M so additional care must be taken to improve f near $\partial C$. This is achieved by using four concentric families of pairwise disjoint PL disks with holes rather than just one such family (our C).

Now, the details. Let $U' = f^{-1}(U)$. By [15; Theorem (4.8.3)], there exist families $\{A_i^{(j)} \mid 1 \leq i \leq t\}$, $1 \leq j \leq 4$, of pairwise disjoint PL disks with holes in $U'$ such that:

(1) for each i,j, $A_i^{(j)} \subset \operatorname{int} A_i^{(j+1)}$;

(2) $S_f \subset \operatorname{int} B_1$;

where $B_j = \cup_{i=1}^{t} A_i^{(j)}$. Let k=1;2. By (1) and (2), $f|(D-\operatorname{int} B_{2k-1})$ is an embedding hence $f(D-\operatorname{int} B_{2k-1})$ is closed in M thus $V_k = U- f(D-\operatorname{int} B_{2k-1})$ is open in M and $V_1 \subset V_2 \subset U$. Let $V_k' = f^{-1}(V_k)$. Then:

(3) $S_f \subset V_1' \subset \operatorname{int} B_1$;

(4) $B_2 \subset V_2' \subset \operatorname{int} B_3$.

Let $K \subset L \subset D-U'$ be PL annuli such that $\partial D = \partial L \cap \partial K$. (See Figure (3.2).)

Apply Theorem (3.3) to replace f by a Dehn disk $f_1: D \to M$ with the following properties:

(5) $f_1|(D-D_1) = f|(D-D_1)$;

(6) $f_1|D_1$ is locally PL;

(7) $S_{f_1} = S_f$;

where $D_1 = \operatorname{int}(B_4-B_1)$. Apply Theorem (3.3) again to get a Dehn disk $f_2: D \to M$ such that:

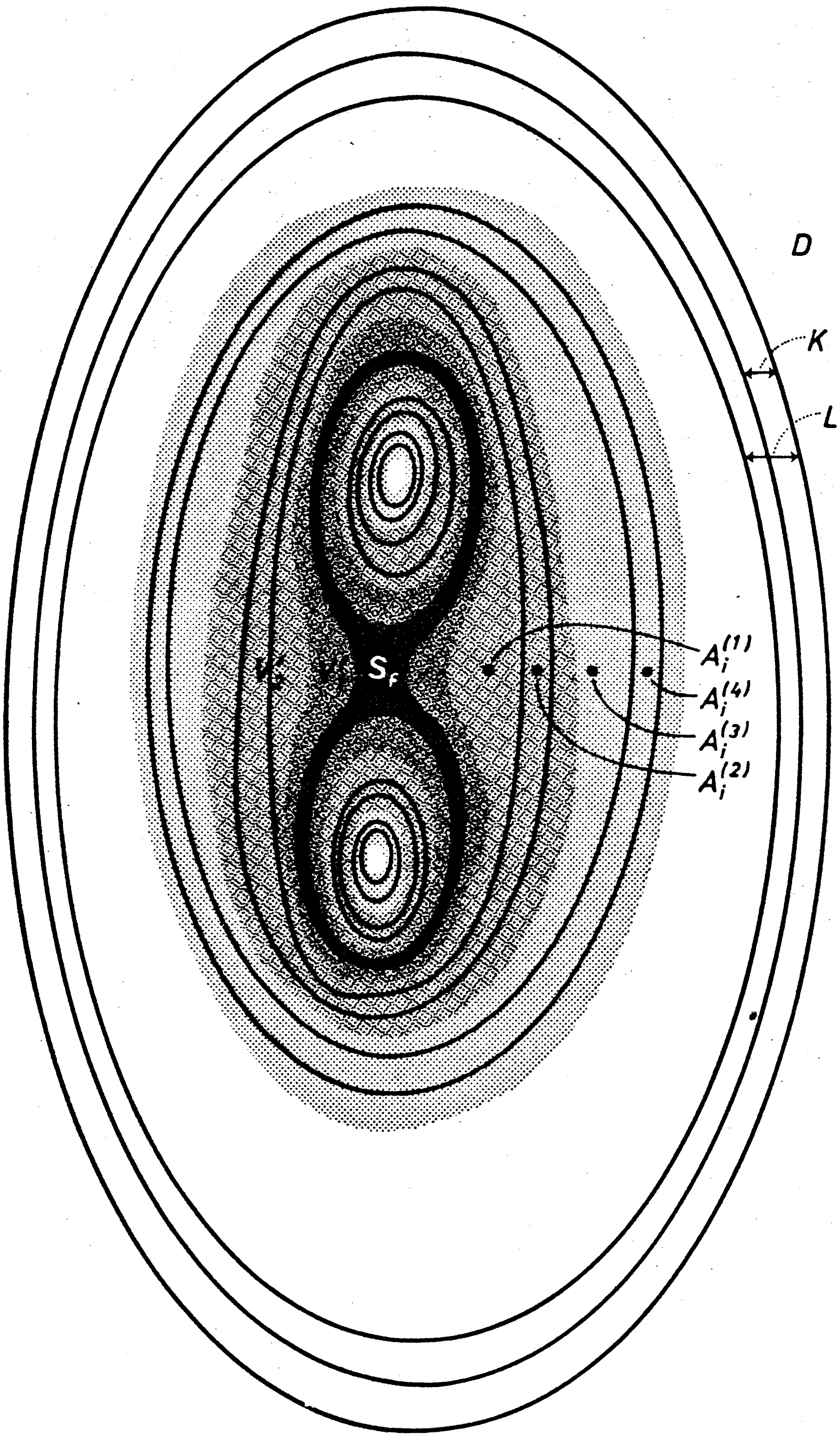



Figure 3.2.

(8) $f_2|(D-\text{int } L) = f_1|(D-\text{int } L)$;

(9) $f_2|\text{int } L$ is locally PL;

(10) $S_{f_2} = S_{f_1}$.

Remark. We could have gotten the map $f_2$ from f in just one step rather than going via $f_1$. However, we shall need $f_1$ in assembling the final map F (See Figure (3.4).)

Another application of Theorem (3.3) yields a Dehn disk $f_3$:D → M such that:

(11) $f_3|D_2 = f_2|D_2$;

(12) $f_3|(D-D_2)$ is locally PL;

(13) $S_{f_3} = S_{f_2}$;

where $D_2 = K \cup B_3$.

Remark. If for some $j \in \{1,2,3,4\}$ the simple closed curves $f(\partial B_j) \subset M$ and $f(\partial K) \subset M$ are nicely embedded in M we can skip $f_1$ and $f_2$ and just apply Theorem (3.3) to $f|(D-\text{int } B_j)$ to get $f_3$. However, if this isn't the case then we must get $f_1$ and $f_2$ first to make certain that $f(\partial(\overline{D-D_2}))$ is nicely embedded in M.

By Zeeman's Relative Simplicial Approximation theorem [72] there is a Dehn disk $f_4$:D → M such that:

(14) $f_4|(D-\text{int } B_2) = f_3|(D-\text{int } B_2)$;

(15) $f_4|(D-(\text{int } K \cup \partial D)$ is PL;

(16) $S_{f_4} \subset V_2'$.

By Theorem (3.4) there is an embedding $f_5$:D → M such that:

(17) $f_5|\text{int } D$ is locally PL;

(18) $f_5|K = f_4|K$;

(19) $f_5(D)-V_2 = f_4(D)-V_2$.

In particular, by (4), (5), (8), (11), (14), (18), and (19):

(20) $f_4(D\text{-int } B_3) \subset f_5(D) \subset f_4(D) \cup V_2$.

Note, however, that in general, $f_4$ and $f_5$ do not agree pointwisely, not even on D-int $B_3$.

We wish to know what regions of D are mapped by $f_5$ onto $f_4(D\text{-int}B_3)$. Let $C = f_5^{-1}f_4(D\text{-int } B_3)$. By (20), C is well-defined and non-empty. There exist pairwise disjoint PL disks with holes $\{E_i \mid 1 \leq i \leq r\}$ such that

(21) $D\text{-int } B_3 = \bigcup_{i=1}^{r} E_i$.

By (16), $f_4(D\text{-int } B_3)$ is a collection of disks with holes, namely $f_4(E_i)$'s hence by (20) so is $C = \bigcup_{i=1}^{r} f_5^{-1}f_4(E_i)$. Define $F:D \longrightarrow M$ by

$$(22)\ F(x) = \begin{cases} f_1\bullet(f_4|(D\text{-int } B_3))^{-1}\bullet f_5\ (x) & ;\ x \in C \\ f_5(x) & ;\ x \in D\text{-}(\text{int } C \cup \partial D) \end{cases}$$

The map F is well-defined: each $x \in C$ lies in precisely one disk with holes $f_5^{-1}f_4(E_i)$, so $f_5(x)$ lies in $f_4(E_i)$. Now, by (16), $f_4|(D\text{- int } B_3)$ is an embedding, therefore $f_4^{-1}$ is well-defined over $f_4(D\text{-int } B_3)$. Also, by (8), (11), (14), and (21):

(23) $f_1|\ \partial B_3 = f_4|\ \partial B_3$

hence for every $x \in \partial C\text{-}\ \partial D$: $f_1\bullet(f_4|_{(D\text{-int } B_3)})^{-1}\bullet f_5(x) = f_1\bullet(f_4|_{\partial B_3})^{-1}\bullet f_5(x) = id\bullet f_5(x) = f_5(x)$ so F is well-defined. By (3), (7), and (20)-(23), F is an embedding and by (5), (8), (11), (14), (19), and (20)-(23), $F(D)\text{-}U = f(D)\text{-}U$ as desired. (See Figure (3.3).)

<u>Remark</u>. The disk F(D) is thus obtained from $f_5(D)$ by glueing together the pieces $f_5(D\text{-int } C \cup \partial D)$ and $f_1(D\text{-int } B_3)$ using the homeomorphism $f_4^{-1}\bullet f_5$ on $\partial C - \partial D$. (See Figure (3.4).)

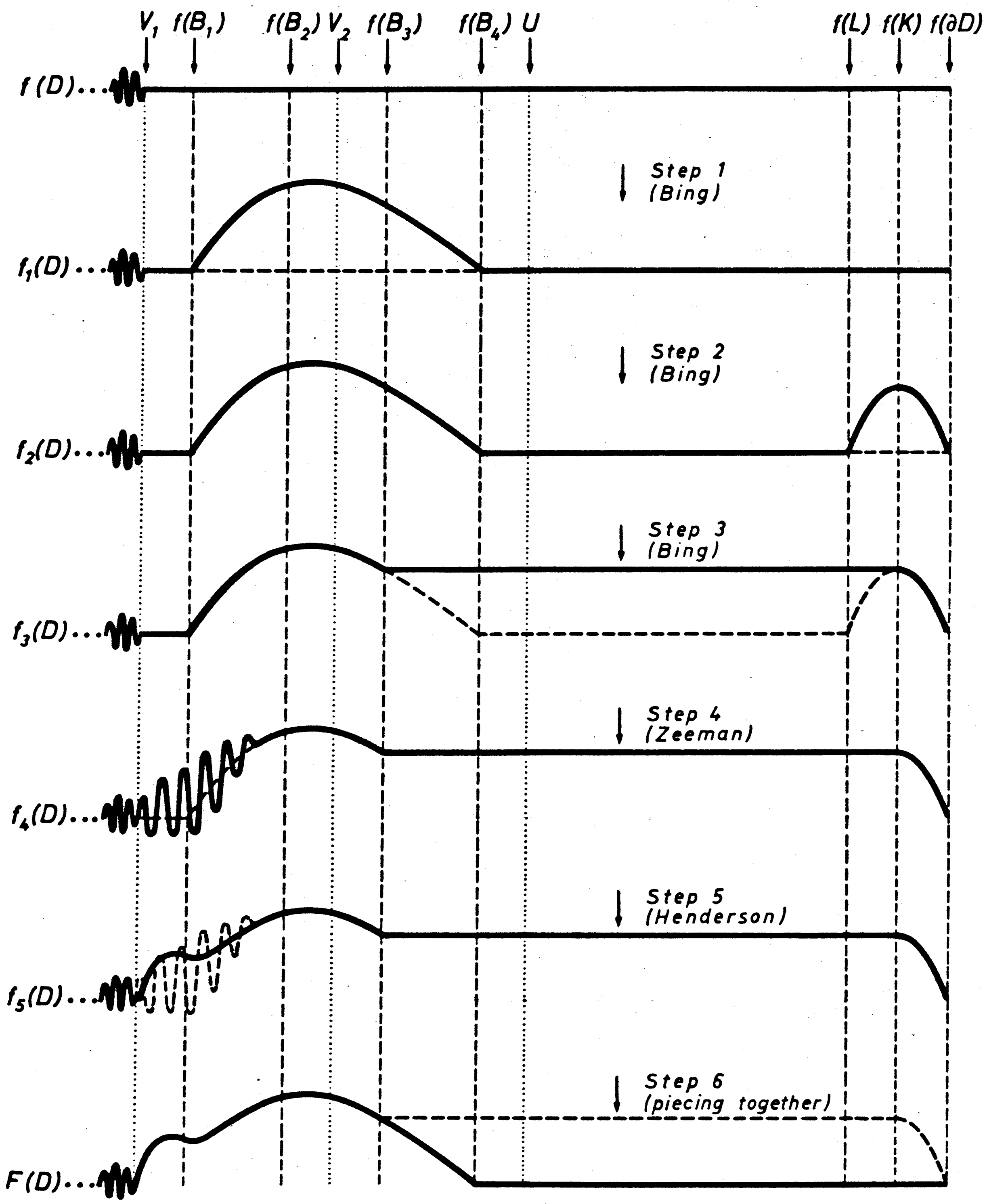

V1 f(B1) f(B2) V2 f(B3) f(B4) U f(L) f(K) f(∂D)
f(D)...
Step 1 (Bing)
f1(D)...
Step 2 (Bing)
f2(D)...
Step 3 (Bing)
f3(D)...
Step 4 (Zeeman)
f4(D)...
Step 5 (Henderson)
f5(D)...
Step 6 (piecing together)
F(D)...

Figure 3.3.

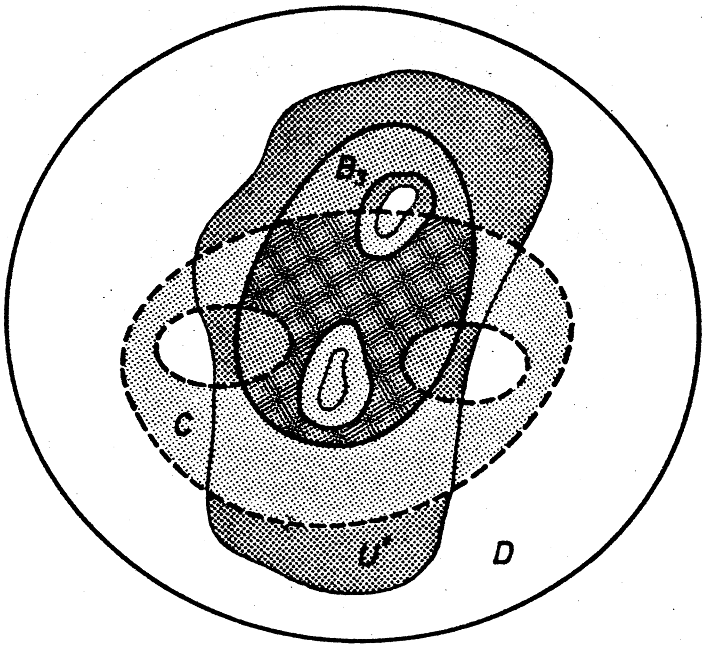



Figure 3.4.

It remains to consider the case when $f(D) \cap \partial M \neq \emptyset$. Attach a collar $C_0 = \partial M \times [0,1]$ to $\partial M$ and extend the neighborhood U over $C_0$ in the obvious way -- let $U' = U \cup ((U \cap \partial M) \times [0,1])$. Let $M' = M \cup_{\partial M} C_0$. Apply the preceeding case to the 3-manifold $M'$ to get an embedding $F':D \to M$: such that $F'(D)-U' = f(D)-U'$ and $F'|\partial D = f|\partial D$. The disk $F'(D)$ may now hit $M'-M$ so we wish to push it in M by a nice ambient PL isotopy with support in $U'$. Note that by taking a PL collar h: $\partial M \times [0,1] \to M$ of $\partial M$ in M we get a "product structure" in $M'$ close to $\partial M$, i.e., $C_0 \cup h(\partial M \times [0,1])$ is PL homeomorphic to $\partial M \times [-1,1]$ where we identify $\partial M$ with $\partial M \times \{0\}$. We can now construct the desired ambient PL isotopy $H_t:M' \times [0,1] \to M'$ by pushing $F'(D)$ from

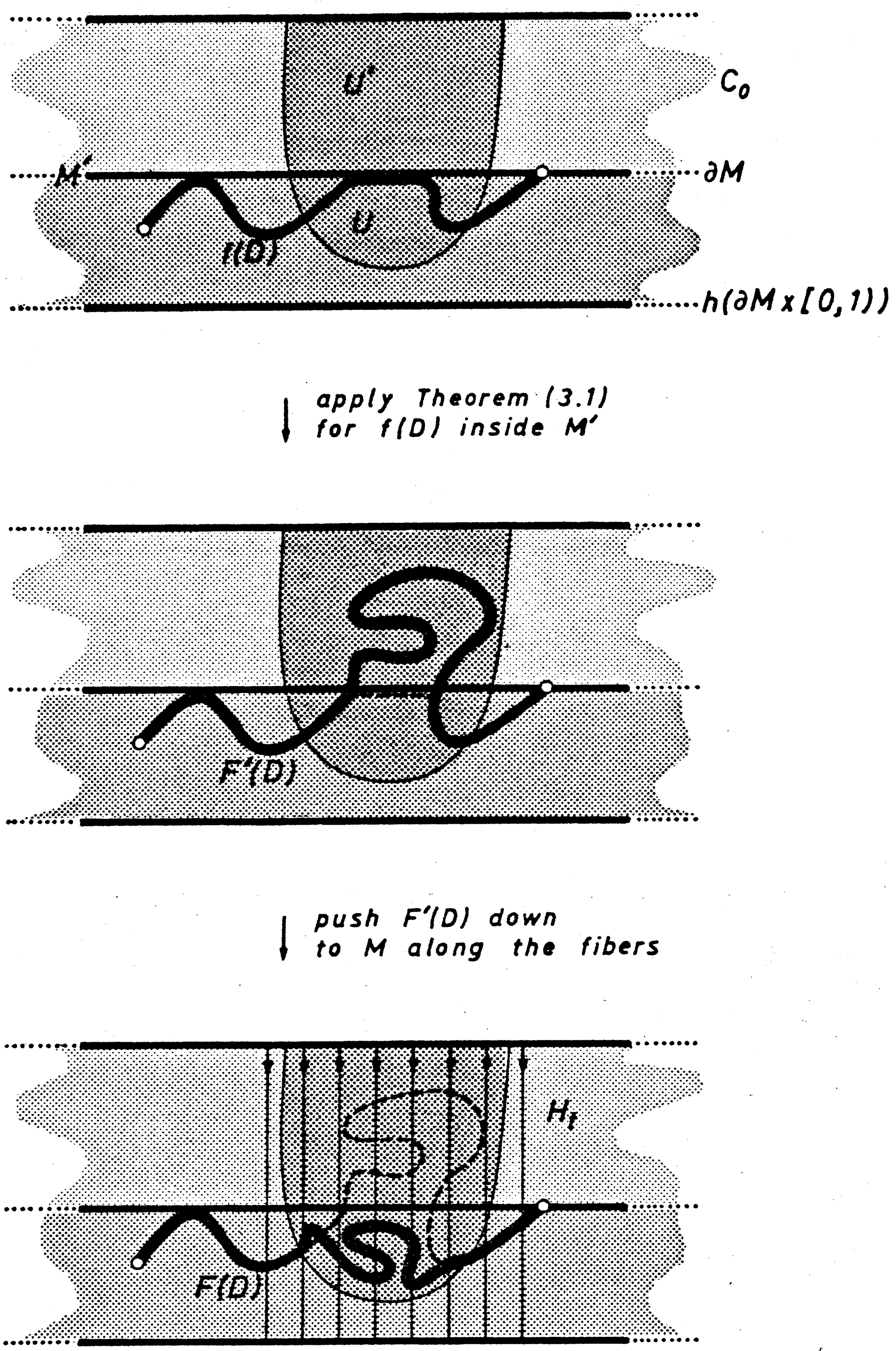

U'
$C_0$
M'
∂M
U
f(D)
h(∂M x [0,1))
apply Theorem (3.1)
for f(D) inside M'
F'(D)
push F'(D) down
to M along the fibers
$H_t$
F(D)

Figure 3.5.

$M'$-$M$ down to $M$ by means of stretching down the fibers of the product $\partial M \times [-1,1]$ . Finally, let $F: D \to f(D) \cup U$ be given by $F = H_1 F'$. (See Figure (3.5).) **

<u>Theorem 3.6</u>. Let $f_1, \ldots, f_k : D \to M$ be Dehn disks in a 3-manifold $M$ (possibly with boudary) such that if $i \neq j$ then $f_i(\partial D) \cap f_j(D) = \emptyset$. Then for every neighborhood $U \subset M$ of $\bigcup_{i=1}^{k} f_i(D)$ there exist embeddings $F_1, \ldots, F_k : D \to U$ such that:

(i) for each $i$, $F_i | \operatorname{int} D : \operatorname{int} D \to U$ is locally PL;

(ii) for each $i$, $F_i | \partial D = f_i | \partial D$;

(iii) if $i \neq j$ then $F_i(D) \cap F_j(D) = \emptyset$.

<u>Corollary 3.7</u>. Every 3-manifold (possibly with boundary) has the MSP. **

<u>Proof of Theorem (3.6)</u>. By an argument similar to the one in the preceeding proof we may assume that for each $i$, $f_i(D) \subset \operatorname{int} M$. We use induction on $k$. For $k=1$ the assertion follows by Theorems (3.1) and (3.3). Assume now that the assertion is true for all $k \leq n$ and consider the case $k=n+1$. By the inductive hypothesis there are embeddings $F_1, \ldots, F_n$ to $U - f_{n+1}(\partial D)$ satisfying (i)-(iii) and $f_{n+1}$ can be replaced by an embedding $f^*_{n+1} : D \to U - (\bigcup_{i=1}^{n} F_i(\partial D))$ such that $f'_{n+1} | \operatorname{int} D$ is locally PL, $f'_{n+1}$ is in general position with respect to the surface $S = \bigcup_{i=1}^{n} F_i(D)$, and $f'_{n+1} | \partial D = f_{n+1} | \partial D$. Hence $f'_{n+1}(D) \cap S$ is a finite collection of pairwise disjoint PL simple closed curves. Starting off with an innermost one (on the surface S) of these curves, we can eventually cut $f'_{n+1}(D)$ off S inside the neighborhood U thus obtaining the desired embedding $F_{n+1}$. **

## 2. Recognizing 3-Manifolds

The next result is an improvement of a theorem of H.W.Lambert and R.B.Sher [44] who proved our result for the case when G is a point-like, closed 0-dimensional upper semicontinuous decomposition of $S^3$. (Consequently, their conclusion (iii) was $S^3/G \approx S^3$.)

Theorem 3.8. Let G be a cell-like, closed 0-dimensional upper semicontinuous decomposition of a 3-manifold M. If $\partial M \neq \emptyset$ assume that $\overline{N_G} \subset \text{int}\, M$. Then the following statements are equivalent:

(i) M/G has the DLP;

(ii) M/G has the MSP;

(iii) M/G is a 3-manifold.

Remark. The ideas Lambert and Sher used to prove their result in [44] can easily be adapted to prove Theorem (3.8) for the case when every $g \in G$ has a neighborhood in M embeddable in $\mathbb{R}^3$ (and (iii) then reads $M/G \approx M$). Here's how this would go: By [64;Lemma (2.5)] it suffices that given $\varepsilon > 0$ and a neighborhood $U \subset M$ of $N_G$ we find a homeomorphism $h: M \to M$ that shrinks all elements of G to a size less than $\varepsilon$ and stays the identity off U. By [47;Theorem 3 ] there are pairwise disjoint cubes with handles $F_1, \ldots, F_k \subset U$ such that $N_G \subset \bigcup_{i=1}^{k} \text{int}\, F_i$. Let $W_1, \ldots, W_k \subset U$ be pairwise disjoint open neighborhoods of $F_1, \ldots, F_k$, respectively. Restrict our attention to $F_1 \subset W_1$ and let $C_1 = N_G \cap F_1$. As far as $F_1$ is concerned it suffices to find a homeomorphism $h_1: M \to M$ that shrinks $C_1$ and stays the identity off $W_1$. We get $h_1$ as the composition of two homeomorphisms $f_1, g_1: M \to M$. The first one, $f_1: M \to M$ shrinks $F_1$ towards its 1-dimensional spine so that

$f_1(F_1)$ can be split up into adjacent 3-cell "chambers" of size $< \varepsilon/2$ ( Figure (3.6)).Pull this chamber partition up in $F_1$. It is now clear that if $g \in G$ lies in at most two adjacent chambers it will get shrunk under $f_1$ to a size less than $\varepsilon/2 + \varepsilon/2 = \varepsilon$. So it now remains to make each $g \in G$ meet at most one wall of these "chambers". By going to M/G and using the DLP (or the MSP) we can recover new walls in $F_1 \subset M$ as illustrated in Figure (3.7) on p.63.Pick any homeomorphism $g_1$ of M which maps new walls on the old ones and rests off $F_1$. Finally, let $h_1 = f_1 g_1$.

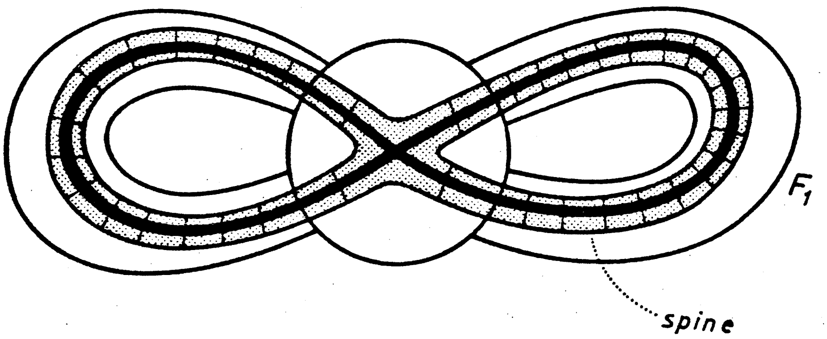



Figure 3.6.

This proof will not work for Theorem (3.8). For, the best we can say about G is that by [47 ;Theorem 3] it is definable by homotopy cubes with handles. So if there are fake cubes, some of the chambers in the partition of $F_1$ above may fail to be 3-cells hence no homeomorphism $g_1$ can be produced. A different approach is called for.

Proof of Theorem (3.8). The implications (iii) $\Rightarrow$(i) and (iii) $\Rightarrow$ (ii) follow by Corollaries (3.2) and (3.7), respectively. We prove (i) $\Rightarrow$(iii) and (ii) $\Rightarrow$(iii) simultaneously. So assume M/G has either the DLP or the MSP.

Assertion 1. If every $g \in G$ has a neighborhood in M embeddable in $\mathbb{R}^3$ then $M/G \approx M$.

As we have already observed above the proof from [44] will work except that instead of [52 ;Theorem (2.1)] one should use an improvement due to R.C.Lacher [40 ;Lemma A on p.506] .

Assertion 2. If $G_o = \{ g \in G \mid g$ has no neighborhood embeddable in $\mathbb{R}^3 \}$ then $\pi(G_o)$ is locally finite in M/G, where $\pi : M \twoheadrightarrow M/G$ is the quotient map.

If M is orientable apply [35 ;Theorem 1 ] and if it is not apply Theorem (2.1).

Assertion 3. For every $g \in G$ and every neighborhood $U \subset M$ of g there is a homotopy 3-cell $H \subset U$ such that $g \subset \text{int}\, H$.

We may assume that U is G-saturated. By [47 ;Theorem 3 ], G is definable by homotopy cubes with handles hence there is a homotopy cube with handles $H \subset U$ such that $g \subset \text{int}\, H$. By going further in the defining sequence for G we may assume that on some neighborhood $N \subset U$ of $\partial H$ the restriction $\pi|N : N \longrightarrow M/G$ is an embedding. The idea of the proof is to use the DLP or the MSP to cut the handles of H along pairwise disjoint compressing disks which miss g. We detect such disks as follows.

Assume first that M/G has the DLP. Let $C_1$ and $C_2$ be disjoint simple closed curves on $\partial H$ such that they are null-homotopic in H but not on $\partial H$. By Dehn's lemma there exist embeddings $f_1, f_2 : (D, \partial D) \longrightarrow (H, \partial H)$ such that $f_i(\partial D) = C_i$, i=1;2. By running a ribbon in U-int H between slightly expanded disks $f_1(D)$ and $f_2(D)$ we get an embedding $f : D \longrightarrow U$ such that for disjoint subdisks $D_1, D_2 \subset \text{int}\, D$, $f|D_i = f_i$, i=1;2 and $f(D-(D_1 \cup D_2)) \subset U-H$. Since by our choice $\pi|N : N \longrightarrow M/G$ is an embedding it follows that $\pi f : D \longrightarrow \pi(U)$ is a Dehn disk and that $\Sigma_{\pi f} = \Sigma_{\pi f_1} \cup \Sigma_{\pi f_2}$. Therefore $\Sigma_{\pi f} \subset \pi(\text{int}\, H)$ so using the DLP we can get an embedding $F : D \longrightarrow \pi f(D) \cup \pi(\text{int}\, H)$ such that $F(\partial D) = \pi f(\partial D)$. Let $q_i : D \longrightarrow \pi(H)$ be the subdisks of F(D) bounded by $\pi f_i(\ D)$, i=1;2. Note that $q_1(D) \cap q_2(D) = \emptyset$ so there exist disjoint neighborhoods $W_i \subset \pi(U)$ of $q_i(D)$. Let $V_i = \pi^{-1}(W_i)$. By [40 ; Lemma A on p.506], $q_i$ lifts to a Dehn disk $Q_i : D \longrightarrow V_i \cap H$, i=1;2. By Theorems (3.1) and (3.3) we can assume that $Q_i$ is a locally PL embedding. Since $V_1 \cap V_2 = \emptyset$ one of the disks $Q_i(D)$ will miss g hence by cutting along it we get a homotopy cube with one handle less, H*, which contains g in its interior. (See Figure (3.8).) In continuing this process one must be careful to choose the new pair of simple closed curves $C_1^*$, $C_2^*$ away from the intersections of $N_G$ with H*. That is because in doing the compressions we may have

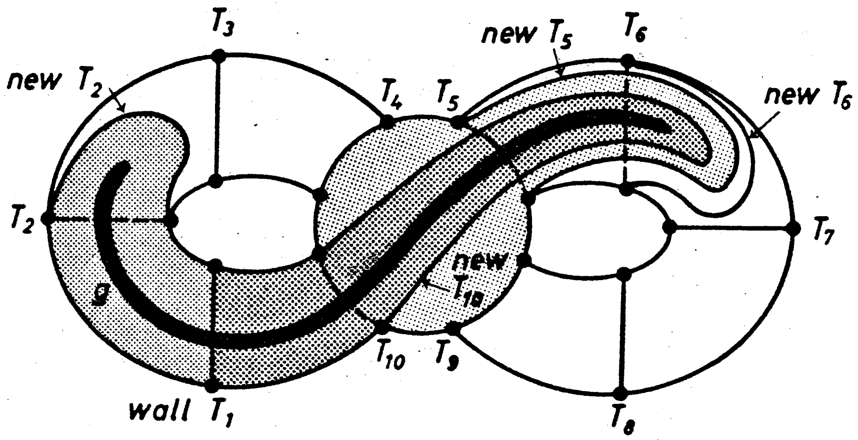



Figure 3.7.

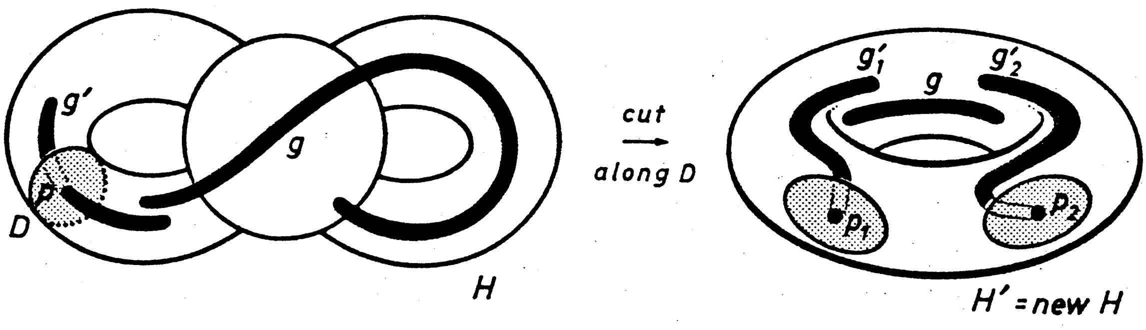



Figure 3.8.

hit some elements of $H_G - \{g\}$ so now $\partial H^* \cap N_G$ may no longer be empty. Since any possible intersections lie inside the two copies of the compressing disk on $\partial H^*$, we can always push $C_1^*$ and $C_2^*$ off $H_G \cap \partial H^*$ if necessary. This way $\pi : M \twoheadrightarrow M/G$ is an embedding on a neighborhood of $C_i^*$, i=1;2. (See Figure (3.9).)

If instead of the DLP we have the MSP for M/G the procedure is similar. We do not need to introduce f for it suffices to consider $f_i$, i=1;2. Use the MSP to separate the Dehn disks $\pi f_1, \pi f_2 : D \longrightarrow \pi(H)$. The rest of the argument stays the same.

We now finish off the proof of the theorem, first for the case when $\partial M = \emptyset$. By Assertion 2, $G = G_o \cup G_1$ where $G_1 = G - G_o$ and the set $\pi(G_o)$ is locally finite in M/G. Consider $M_o = M/G_o$ and let $\pi_o : M \to M_o$ be the corresponding quotient map. Since the elements of G are cell-like, $M_o$ is a generalized 3-manifold. Clearly, $S(M_o) \subset \pi_o(G_o)$, where $S(M_o)$ is the singular set of $M_o$. Also, $M_o$ satisfies KF by Proposition (1.10) since it is resolvable.

Assertion 4. For every $p \in M_o$, $g(M_o,p) = 0$.

If $p \notin \pi_o(G_o)$ then $p \notin S(M_o)$ so the assertion is clear. Let $p\ \pi_o(G_o)$. By Assertion 2, there is a neighborhood $U \subset M_o$ of p such that $U \cap \pi_o(G_o) = \{p\}$. Let $V = \pi_o^{-1}(U)$. By Assertion 3, there is a homotopy cube $H \subset V$ such that $\pi_o^{-1}(p) \subset \text{int}\, H$ and $\partial H \cap (\cup\{g \in G_o\}) = \emptyset$. Therefore, $\pi_o(\partial H)$ is a 2-sphere so $\pi_o(H)$ is the desired neigh-

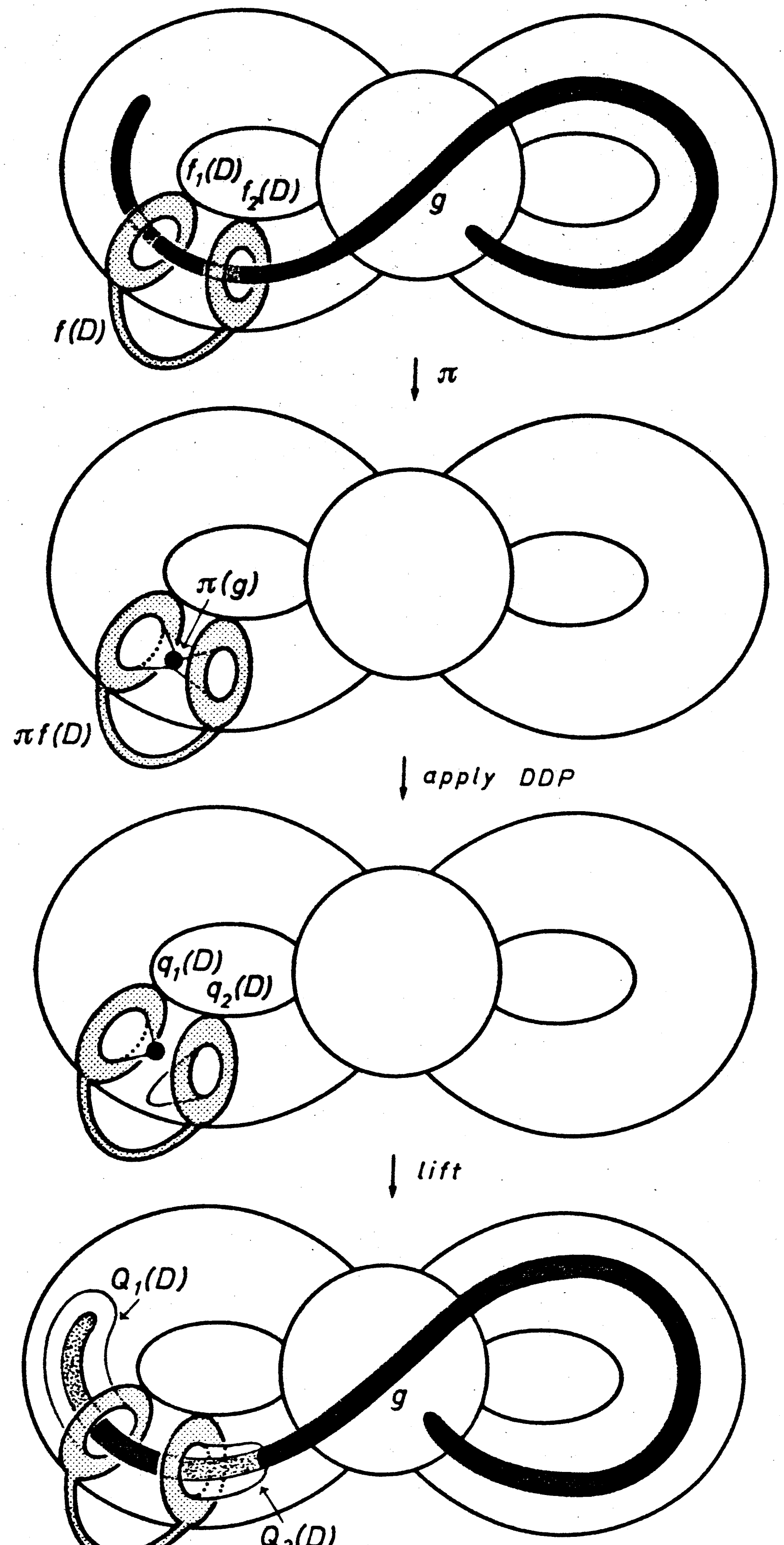



Figure 3.9.

borhood of p.

It now follows by Assertion 4 and Corollary (1.14) that $S(M_o) = \emptyset$ since $\dim S(M_o) \leq \dim \pi_o(G_o) \leq 0$. Thus $M_o$ is a 3-manifold. Consider $G_1^* = G_1 \cup \pi_o(G_o)$ as a decomposition of $M_o$. By Assertions 2 and 3, the decomposition $G_1^*$ is cellular, closed 0-dimensional, and upper semicontinuous. Also, $M_o/G_1^* \approx (M/G_o)/G_1^* \approx M/G$ so $M_o/G_1^*$ has the DLP (the MSP, respectively). By Assertion 1, $M_o/G_1^*$ is homeomorphic to $M_o$ so M/G is homeomorphic to $M_o$ and thus is a 3-manifold. This completes the proof if $\partial M = \emptyset$.

In the case when $\partial M \neq \emptyset$ we consider the double DM of M (i.e., we identify two copies of M along $\partial M$ using the identity map) and apply the preceeding arguments to the to the decomposition DG, the double of G. The proofs of all assertions go through the same although we are not claiming that in general, the hypothesis "M/G has the DLP/MSP" implies "DM/DG has the DLP/MSP". The point is that we do not need that much to prove Assertions 1-4. Of course, in our case it eventually turns out that DM/DG has the DLP and the MSP since we prove that DM/DG is a 3-manifold. **

<u>Theorem 3.9</u>. Let X be a generalized 3-manifold with 0-dimensional singular set such that for every $x \in X$, $g(X,x) = 0$. Then X has the DLP and the MSP.

<u>Proof</u>. We first prove the DLP. Let $f:D \to X$ be a Dehn disk. We first show that we may assume $f(\partial D) \cap S(X) = \emptyset$. Indeed, by hypothesis there is a neighborhood $N \subset D$ of $\partial D$ such that $S_f \cap N = \emptyset$. Thus $N \cap f^{-1}(S(X))$ is 0-dimensional so there is a PL simple closed curve $J \subset N - f^{-1}(S(X))$

such that J is ambient PL isotopic in N to $\partial D$. Let $E \subset D$ be the sub-disk of D bounded by J and consider the Dehn disk $f' = f|E: E \to X$. If we can show how to find an embedding $F': E \to f(E) \cup U$, where $U \subset X$ is a neighborhood of $\Sigma_f = \Sigma_{f'}$ such that $F'(J) = f'(J)$, then by defining $F: D \to X$ to be f on D-E and $F'$ on E we get the desired embedding.

So assume that $f(\partial D) \cap S(X) = \emptyset$ and that $f(\partial D) \subset X-U$. By [12 ;Lemma 1] and by Theorem (3.3) we can find a collection $N_1, \ldots, N_k \subset X$ of pairwise disjoint compact generalized 3-manifold-with-boundary neighborhoods of $S(X) \cap f(D)$ such that:

(1) for each i, $\dot{N}_i$ is a locally PL 2-sphere;

(2) $\dot{H} \cap S(X) = \emptyset$;

(3) $S(X) \cap (f(D)-U) \subset H_1 \subset X-(f(\partial D) \cup U)$;

(4) $S(X) \cap f(D) \cap U \subset H_2 \subset U$;

where $H_1 = \bigcup_{i=1}^{p} N_i$, $H_2 = \bigcup_{i=p+1}^{k} N_i$, and $H = H_1 \cup H_2$. Then by (2), $f(D) \cap H \subset M(X)$.

Here is an outline of the proof. First, we want to make f(D) meet $\dot{H}$ "transversely". But f may not be (even locally) PL so we must improve it to be (locally) PL near $\dot{H}$. We do this as follows: close to $\dot{H}_2$ we use the Simplicial Approximation theorem while close to $\dot{H}_1$ we use Theorem (3.3) in order to keep f an embedding in that region. By applying general position in M(X) we can make f(D) meet $\dot{H}$ transversely and then we can either cut it off at $\dot{H}_1$ (by standard "cut and paste" techniques) or "push" $f(D) \cap \dot{H}_1$ into M(X) (replacing annuli of $f(D) \cap \dot{H}_1$ by "nicer" annuli in $H_1 \cap M(X)$) while intersections of f(D) with $\dot{H}_2$ are dealt with in a different manner, again making f(D) lie in M(X). Apply Theorem (3.1) to get an embedding $F':D \to M(X)$ such that $F'|\partial D = f|\partial D$ and $F'(D)-U = f(D)-U$. Finally, replace the portions which $\dot{H}_1$ cuts off $F'(D)$ by $f(D) \cap H_1$ and thus obtain the desired embedding $F:D \to X$ with $F(D) \subset f(D) \cup U$ and $F(\partial D) = f(\partial D)$. (See Figures (3.10), (3.11), and (3.12).)

Now the details. By [15;Theorem (4.8.3)], there exists a collection $B_1,\ldots,B_k \subset \text{int}\, D$ of pairwise disjoint PL disks with holes such that:

(5) for each i, $f^{-1}(\dot{N}_i) \subset \text{int}\, B_i$ ;

(6) $A_1 \subset D-f^{-1}(U)$;

(7) $A_2 \subset f^{-1}(U-S(X))$;

where $A_1 = \bigcup_{i=1}^{p} B_i$ and $A_2 = \bigcup_{i=p+1}^{k} B_i$. Let $A = A_1 \cup A_2$. Applying Theorem (3.3) to $f|A_1:A_1 \to X$ and the Simplicial Approximation theorem to $f|A_2:A_2 \to X$ we replace f by a map $f_1:D \to X$ with the following properties:

(8) $f_1|\text{int}\, A$ is locally PL;

(9) $f_1|(D-\text{int}\, A) = f|(D-\text{int}\, A)$ ;

(10) $S_{f_1} \subset U$.

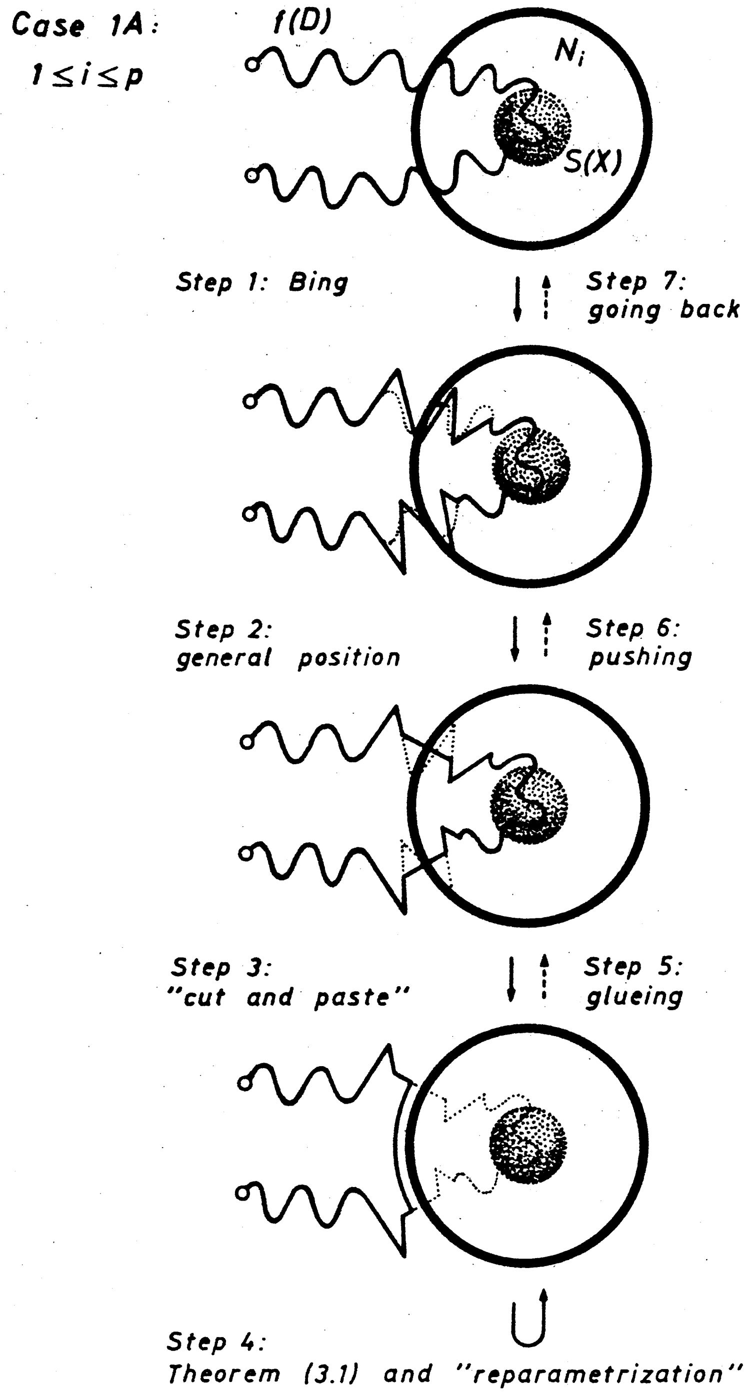
Note that by (2), $\dot{H} \subset M(X)$ hence we can apply general position in M(X) to get a map $f_2:D \to X$ such that:

(11) $f_2$ is ambient isotopic (in X) to $f_1$;

(12) $f_2$ is in general position with respect to $\dot{H}$;

(13) $f_2|(D-\text{int}\, A) = f_1|(D-\text{int}\, A)$;

(14) $S_{f_2} \subset U$.



Case 1A:
1 ≤ i ≤ p
f(D)
N_i
S(X)
Step 1: Bing
Step 7: going back
Step 2: general position
Step 6: pushing
Step 3: "cut and paste"
Step 5: glueing
Step 4: Theorem (3.1) and "reparametrization"


Figure 3.10.

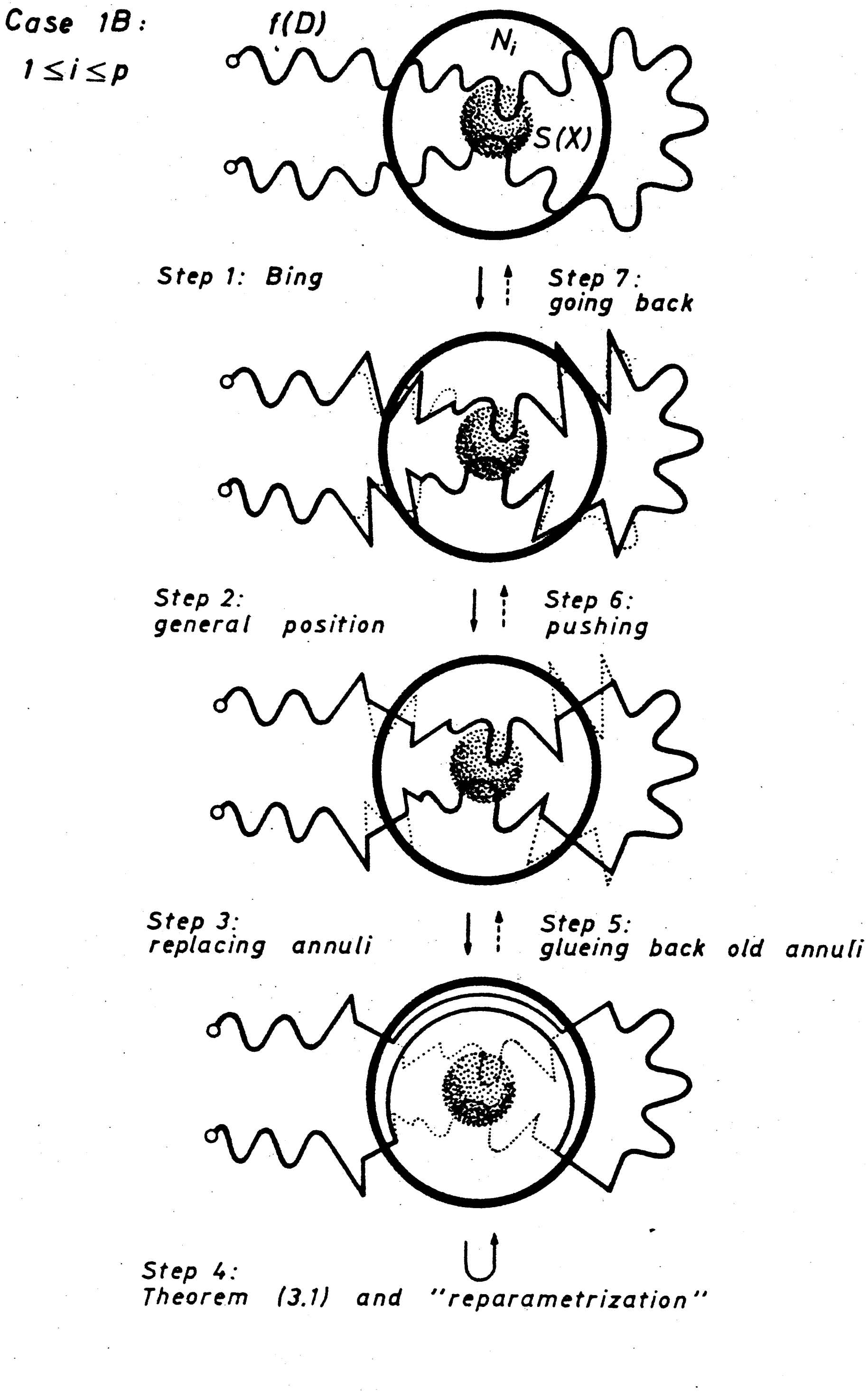

Case 1B:
1 ≤ i ≤ p
f(D)
N_i
S(X)
Step 1: Bing
Step 7: going back
Step 2: general position
Step 6: pushing
Step 3: replacing annuli
Step 5: glueing back old annuli
Step 4: Theorem (3.1) and "reparametrization"

Figure 3.11.

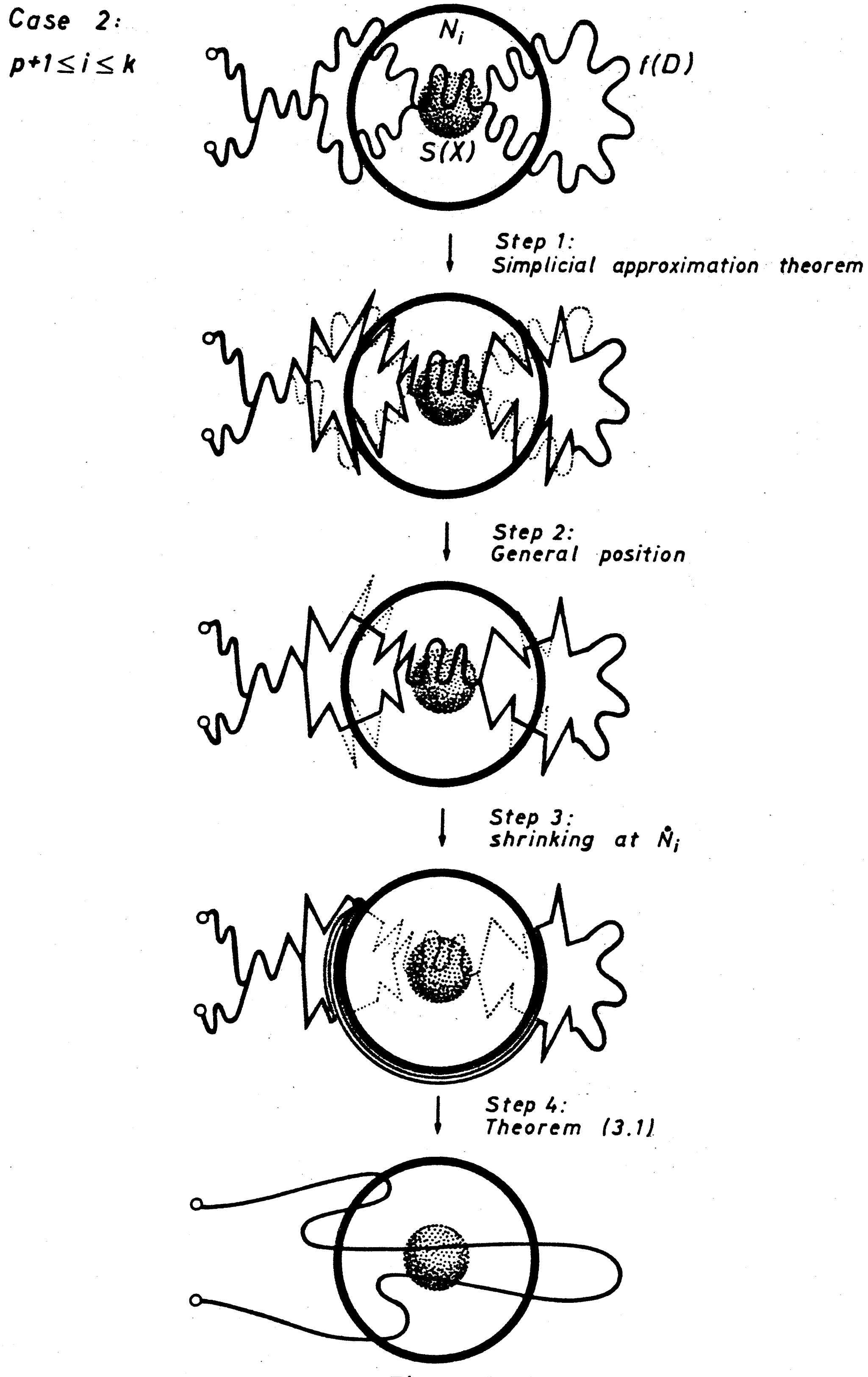

Case 2:
p+1≤i≤k
N_i
f(D)
S(X)
Step 1:
Simplicial approximation theorem
Step 2:
General position
Step 3:
shrinking at N̊_i
Step 4:
Theorem (3.1)

Figure 3.12.

Let $i \in \{1,\ldots,p\}$. By (12) and (14), $f_2(D) \cap \dot{N}_i$ is a collection of pairwise disjoint PL simple closed curves $J_j^{(i)}$, $L_j^{(i)}$, or $K_j^{(i)}$ which bound (in $N_i$) either disks $D_j^{(i)}$ or annuli $A_j^{(i)}$, respectively:

(15) $f_2(D) \cap \dot{N}_i = \{J_1^{(i)},\ldots,J_{t_i}^{(i)}, L_1^{(i)}, K_1^{(i)},\ldots,L_{s_i}^{(i)}, K_{s_i}^{(i)}\}$ ;

(16) $f_2(D) \cap \dot{N}_i = \{D_1^{(i)},\ldots,D_{t_i}^{(i)}, A_1^{(i)},\ldots,A_{s_i}^{(i)}\}$ ;

(17) for each i,j: $J_j^{(i)} = \partial D_j^{(i)}$;

(18) for each i,j: $L_j^{(i)} \cup K_j^{(i)} = \partial A_j^{(i)}$;

(19) for each i,p,q: $D_p^{(i)} \cap D_q^{(i)} = D_p^{(i)} \cap A_q^{(i)} =$
$= A_p^{(i)} \cap A_q^{(i)} = \emptyset$.

In order to get $f(D) \cap N_i$ inside M(X) we perform the following surgery: we replace every disk $D_j^{(i)}$ (resp. annulus $A_j^{(i)}$) by another disk $\bar{D}_j^{(i)}$ (resp. annulus $\bar{A}_j^{(i)}$) with the following properties:

(20) for each i,j: $\bar{D}_j^{(i)} \subset N_i \cap M(X)$ and $J_j^{(i)} = \partial \bar{D}_j^{(i)}$;

(21) for each i,j: $\bar{A}_j^{(i)} \subset N_i \cap M(X)$ and $L_j^{(i)} \cup K_j^{(i)} = \partial \bar{A}_j^{(i)}$;

(22) for each i,p,q: $\bar{D}_p^{(i)} \cap \bar{D}_q^{(i)} = \bar{D}_p^{(i)} \cap \bar{A}_q^{(i)} = \bar{A}_p^{(i)} \cap \bar{A}_q^{(i)} = \emptyset$.

( See Figure (3.13 ).)

Now let $i \in \{p+1,\ldots,k\}$. By (1), (2), (4), (6), (8), and (12) we may assume that $\dot{N}_i - f(D) \neq \emptyset$ (for otherwise we may take slightly smaller $N_i$ obtained by pushing $N_i$ into int $N_i$ along a collar $C \subset M(X) \cap N_i$ on $\dot{N}_i$). Thus by shrinking out the disk $\dot{N}_i - \{p_i\}$, where $p_i \in \dot{N}_i - f(D)$ is an arbitrary point, we may assume that $f(D) \cap \dot{N}_i$ is just a point. (See Figure (3.14 ).)

By performing the replacements described in the preceeding two paragraphs we obtain a map $f_3 : D \to X$ with the following properties:

(23) $f_2(D) \cap N_i$ can be recovered from $f_3(D)$ by replacing each

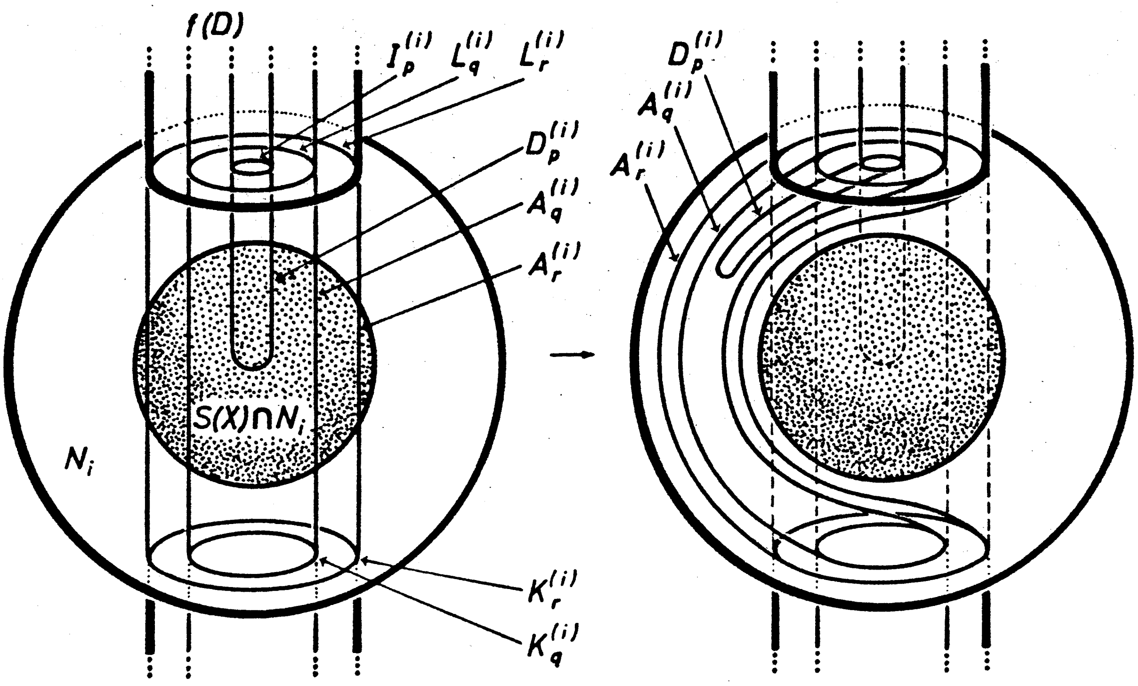



Figure 3.13.

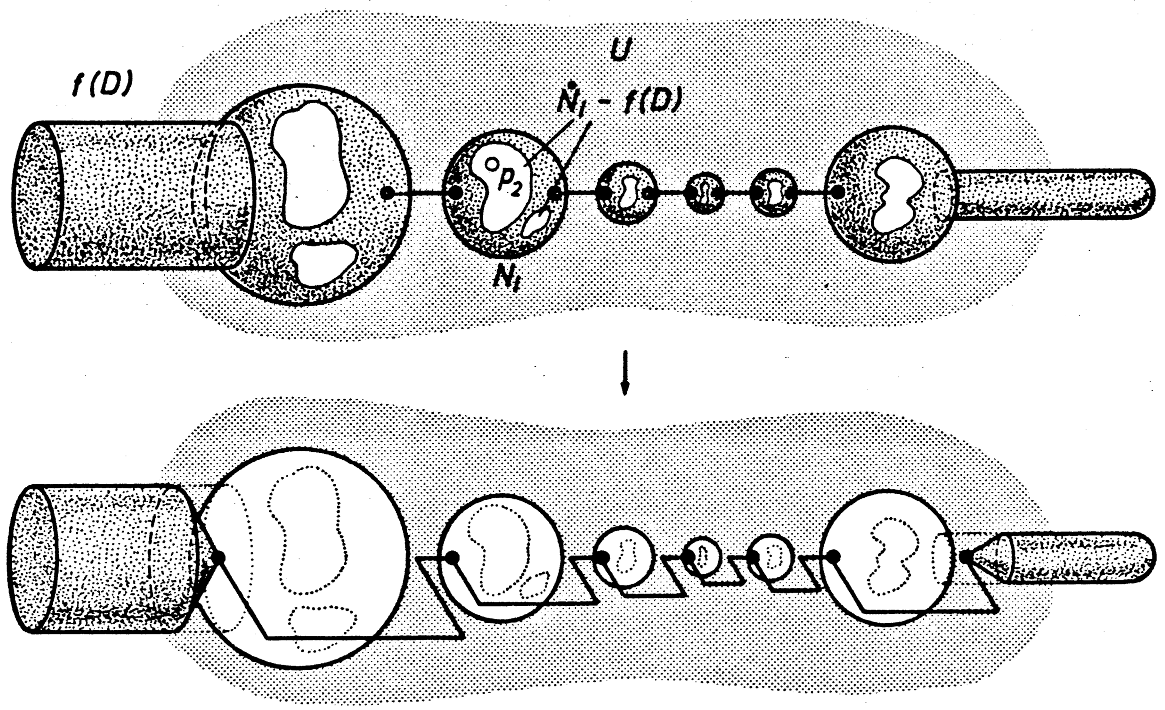



Figure 3.14.

$\bar{D}_j^{(i)}$ (or $\bar{A}_j^{(i)}$) by $D_j^{(i)}$ (resp. $A_j^{(i)}$);

(24) $f_3(D) \cap H_2 = \{x_{p+1},\dots,x_k\}$, where $x_i \in \dot{N}_i$, $p+1 \le i \le k$;

(25) $f_3(D) \subset M(X)$;

(26) $S_{f_3} \subset U$;

(27) $f_3$ is a Dehn disk (not necessarily PL);

(28) $f_3(D\text{-int}\,A) \subset f_2(D) \cup U$;

(29) $f_3|\partial D = f_2|\partial D$;

(30) $f_3(D-(H \cup U)) = f_2(D-(H \cup U))$;

because (23) follows by (20) and (21); (24)-(26) by (14)-(22); (28) by (13)-(21), and (28)-(30) follow by the construction of $f_3$.

By (25) we can apply Theorem (3.1) to get an embedding $F_1 : D \to X$ such that:

(31) $F_1(D)-U = f_3(D)-U$;

(32) $F_1|\partial D = f_3|\partial D$.

Next, replace the disks $\bar{D}_j^{(i)}$ (resp. annuli $\bar{A}_j^{(i)}$) by the disks $\bar{D}_j^{(i)}$ (resp. annuli $\bar{A}_j^{(i)}$). Since whenever F and $f_3$ agree they agree (in general) only pointwisely, we must do some "reparametrization" (in a similar way as it was done in the proof of Theorem (3.1)). For each i,j, let $\hat{D}_j^{(i)} = F_1^{-1}(\bar{D}_j^{(i)})$ and $\hat{A}_j^{(i)} = F_1^{-1}(\bar{A}_j^{(i)})$. By (26), (28), and (31), there exist PL homeomorphisms $u_{ij}, v_{ij} : D \to D$ such that the diagrams

$$\begin{array}{ccc} \hat{D}_j^{(i)} & \xrightarrow{F_1|} & \bar{D}_j^{(i)} \\ & u_{ij} \searrow \quad \nearrow f_3| & \\ & f_3^{-1}(\bar{D}_j^{(i)}) & \end{array} \qquad \begin{array}{ccc} \hat{A}_j^{(i)} & \xrightarrow{F_1|} & \bar{A}_j^{(i)} \\ & v_{ij} \searrow \quad \nearrow f_3| & \\ & f_3^{-1}(\bar{A}_j^{(i)}) & \end{array}$$

commute.

Define $F_2: D \longrightarrow X$ by

$$F_2(x) = \begin{cases} f_2 u_{ij}(x) & ; \ x \in f_3^{-1}(\bar{D}_j^{(i)}) \text{ for some } i,j \\ f_2 v_{ij}(x) & ; \ x \in f_3^{-1}(\bar{A}_j^{(i)}) \text{ for some } i,j \\ F_1(x) & ; \text{ otherwise.} \end{cases}$$

Then by (26)-(32), we have the following properties for $F_2$:

(33) $S_{F_2} = \emptyset$;

(34) $F_2(D) \subset f_2(D) \cup U$;

(35) $F_2(D) - U = f_2(D) - U$;

(36) $F_2 | \partial D = f_2 | \partial D$.

By (11) there is an embient isotopy $K_t: X \times I \longrightarrow X$ from $f_2$ to $f_1$. Let $F_3 = K_1 F_2$. Then by (33)-(36) we have that:

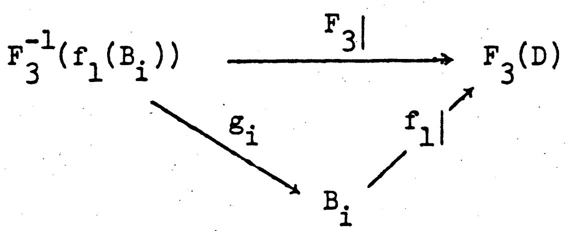
(37) $S_{F_3} = \emptyset$;

(38) $F_3(D) \subset f_1(D) \cup U$;

(39) $F_3(D) - U = f_1(D) - U$;

(40) $F_3 | \partial D = f_3 | \partial D$.

Let $g_i: D \longrightarrow D$ be a homeomorphism that makes the diagram




commute for each $i \in \{1, \ldots, p\}$. By (9), the map $F: D \longrightarrow X$ given by

$$F(x) = \begin{cases} f g_i(x) & ; \ x \in B_i \text{ for some } 1 \leq i \leq p \\ F_3(x) & ; \text{ otherwise} \end{cases}$$

is well-defined. By (37)-(40) we have that $S_F = \emptyset$, $F(D) \subset f(D) \cup U$ (in fact, we have more -- $F(D) - U = f(D) - U$), and $F | \partial D = f | \partial D$. This completes the proof that X has the DLP.

We now prove that X has the MSP, too. Let $f_1,\dots,f_k:D \to X$ be Dehn disks, $U \subset X$ a neighborhood of $\bigcup_{i=1}^{k} f_i(D)$, and suppose that if $i \neq j$ then $f_i(\partial D) \cap f_j(D) = \emptyset$. As before, we may assume that for each i, $f_i(\partial D) \cap S(X) = \emptyset$. Since X was already shown to have the DLP we may also assume that all $f_i$'s are embeddings. Cover $S(X) \cap (\bigcup_{i=1}^{k} f_i(D))$ by a collection of pairwise disjoint generalized 3-manifolds with boundary $N_1,\dots,N_t \subset U$ such that for each i, $\dot{N}_i$ is a locally PL 2-sphere and $\dot{N}_i \cap S(X) = \emptyset$ ([12;Lemma 1] and Theorem (3.3)). We may also make $N_i$'s small enough as to be sure that for no j is $f_j(\partial D) \cap N_i \neq \emptyset$. Let $P = \bigcup_{i=1}^{t} \dot{N}_i$. As before, we can apply Theorem (3.3) close to P in order to make P meet each $f_j(D)$ transversely. Then we can cut each $f_j(D)$ off at P (working within $M(X) \cap U$ all the time) and thus get a new Dehn disk $f_j':D \to X$ with $f_j'|\partial D = f_j|\partial D$. Since $f_j'(D) \subset M(X)$ we can apply Corollary (3.7) to get $f_j'$'s disjoint and yet still inside U and keeping their boundaries fixed. Since $f_j$ and $f_j'$ agree on the boundary, this completes the proof. ⁑⁑

<u>Theorem 3.10</u>. Let $\underline{C}$ be the class of all compact generalized 3-manifolds X with $\dim S(X) \leq 0$ and let $\underline{C}_o \subset \underline{C}$ be the subclass of all $X \in \underline{C}$ which have $\leq 1$ singularity and are also homotopy equivalent to $S^3$. Then the following statements are equivalent:

(i) Poincaré conjecture in dimension three is true;

(ii) If $X \in \underline{C}$ has the DLP or the MSP then $S(X) = \emptyset$;

(iii) If $X \in \underline{C}_o$ has the DLP or the MSP then $S(X) = \emptyset$.

<u>Proof</u>. (i)$\Rightarrow$(ii): If Poincaré conjecture is true then X has a conservative resolution $f:M \to X$ by Theorem (1.6). Let $G = \{ f^{-1}(x) \mid x \in X\}$.

be the associated cell-like, closed 0-dimensional upper semicontinuous decomposition of M. It follows by Theorem (3.8) that S(X) = ∅.

(ii) ⇒ (iii): Clear.

(iii) ⇒ (i): Suppose the Poincaré conjecture is false. Consider the construction $W^3$ from Proposition (1.12). Then $W \in \underset{\sim}{C}_0$. On the other hand, W has the DLP and the MSP since g(W,x) = 0 for all $x \in W$, by Theorem (3.9). Contradiction, since S(W) ≠ ∅. ⁂

## 3. Isolated Singularities

In this section we give an application of the DLP (MSP) to studying isolated singularities in generalized 3-manifolds. The proof is an application of Thickstun's extension of the Loop theorem (Theorem (1.15)) for compact generalized 3-manifolds with 0-dimensional singular set.

Theorem 3.11. Let X be a generalized 3-manifold satisfying KF and suppose that X has the DLP or the MSP (in fact, it suffices to assume the MSP only for pairs of Dehn disks). Then X has no isolated singularities.

Remarks. If Poincaré conjecture is true this follows by Theorem (3.10), provided dim S(X) ≤ 0 and that a "complete" MSP is assumed. Suppose now that fake cubes exist. Then one cannot drop any of the hypotheses from Theorem (3.11): the example of M.G.Brin [9] (or the example of Brin and D.R.McMillan,Jr.[12]) has S(X) ={*} and satisfies KF; on the other hand the example W from Proposition (1.12) has S(W) ={*} and also the DLP and the MSP by Theorem (3.9).

Corollary 3.12. Let M be an open 3-manifold with finitely many ends and $\hat{M}$ its Freudenthal compactification. Then the following statements are equivalent:

(i) $\hat{M}$ is a 3-manifold;

(ii) $\hat{M}$ is an $LC^2$ $\mathbb{Z}$-homology 3-manifold, satisfies KF and has either the DLP or the MSP.

Proof. (i)$\Rightarrow$(ii): Follows by Corollaries (3.2) and (3.7) and by Kneser's Finiteness theorem.

(ii)$\Rightarrow$(i): $\hat{M}$ is clearly finite-dimensional hence it is an ENR as soon as it is $LC^\infty$ at the points $p_1,\ldots,p_t$ of compactifications (assume that M has t ends). Since for each i, $\hat{M}$ is always 0-LC at $p_i$ and since $\hat{M}$ deforms onto a Freudenthal compactification of a locally finite 2-dimensional polyhedron with t ends, it suffices to show that $\hat{M}$ is $LC^2$ at each $p_i$. The assertion now follows by Theorem (3.11). ⁑⁑

Proof of Theorem (3.11). Here is the idea of the proof: by Corollary (1.14) it suffices to show that every point $p \in X$ which has a neighborhood $U \subset X$ such that $U \cap S(X) \subset \{p\}$, satisfies the condition that $g(X,p) = 0$. This is done using standard disk-trading techniques from 3-manifolds topology except that instead of the classical Loop theorem we must invoke Theorem (1.15) and the classical Dehn lemma is replaced here by the DLP (resp. MSP) combined with Theorem (3.3). The latter is done as follows: whenever we want to perform a cut along a compressing disk D which hits p we may use the DLP (or the MSP) on two "close" copies of D to make one of them miss p so that the cut can be performed in $M(X)$.

Now the details. Let $p \in X$ and let $U \subset X$ be an open neighborhood of $p$ such that $U \cap S(X) \subset \{p\}$. By [12;Lemma 1] there is a compact orientable connected generalized 3-manifold $N \subset U$ with boundary a compact orientable 2-manifold such that $p \in \text{int}\, N$. Since $X$ is an ENR it is locally contractible [7 ;Theorem ( V.10.3)] so we may assume that $N$ is null-homotopic in $U$. Let $c = \Sigma_{n=0}^{\infty} (n+1)^2 g(n)$ where $g(n)$ is the number of components of $\dot{N}$ with genus $n$ [47;p.130]. Choose $N$ so that $c$ is minimal. We shall show that $c = 0$. So suppose that $c > 0$. Then there is a boundary component $C \subset \dot{N}$ with positive genus ;$C$ is a 2-sphere with $k > 0$ handles since $\dot{N}$ is orientable. Let $L: \partial B^2 \to C$ be an essential simple closed curve. By our choice of $N$ the inclusion-induced homomorphism $\Pi_1(\dot{N}) \to \Pi_1(U)$ is trivial hence there is an extension $f: B^2 \to U$ of $L$ over $B^2$. Using methods similar to those employed in the proof of Theorem (3.9) we can assume that $f$ is locally PL near $C$ and that it is in general position with respect to $C$, because $C \subset M(X)$. Thus we may assume $f^{-1}(C)$ is a finite collection of pairwise disjoint PL simple closed curves in $B^2$, one of them being $\partial B^2$. Let $J \subset \text{int}\, B^2$ be an innermost such curve and let $E \subset f(B^2)$ be the (singular) subdisk bounded by $f(J)$. There are three possibilities.

<u>Case 1.</u> $f(J)$ is inessential on $C$. Then $f(J)$ bounds a (singular) disk $E' \subset C$. Exchanging $E$ with $E'$ we can go to the next innermost curve.

<u>Case 2.</u> $f(J)$ is essential on $C$ and $E \subset U - \text{int}\, N$. Since $U - \text{int}\, N \subset M(X)$ we can use Dehn's lemma to attach a 2-handle to $N$ after we

have made E locally PL by Theorem (3.3). This reduces c which, in turn, contradicts the minimality of c. Hence this case cannot occur.

Case 3. $F(J)$ is essential on C and $E \subset N$. By Theorem (1.15), $f(J)$ can be replaced by a simple closed curve $J' \subset C$ such that $J'$ is nontrivial on C but bounds a Dehn disk in N. Let $R \subset C$ be a regular neighborhood of $J'$ in C and let $J_1$ and $J_2$ be two simple closed curves boundary components of R. Then $J_i$ bounds a Dehn disk $D_i$ in N for each i = 1;2. Assume first, that X has the MSP. Then we can get $D_1$ and $D_2$ disjoint in N -- denote them by $D_1^*$ and $D_2^*$, respectively. Thus one of them will miss p, say $p \notin D_1^*$. By Theorem (3.3) we can make $D_1^*$ locally PL -- denote it by $D_1^{**}$. Then by cutting $\dot{N}$ along $D_1^{**}$ we reduce the complexity c which, in turn, again contradicts its minimality. Hence this case cannot occur either. (See Figure (3.15).) If instead of the MSP we have the DLP the argument is similar -- join the Dehn disks $D_1$ and $D_2$ by a ribbon in U-int N to get a Dehn disk D. Apply the DLP to get an embedded disk $D^*$ such that $D^*$-int N = D-int N and that $\partial D^* = \partial D$. This replaces $D_1$ and $D_2$ by embedded disjoint subdisks $D_1^*, D_2^* \subset D^* \cap N$ so one of them, say $D_1^*$, misses p. The rest of the argument is now as before: apply Theorem (3.3) and cut $\dot{N}$ along $D_1^{**}$ to reduce the complexity c.

We conclude that indeed c = 0 hence $g(N,p) = 0$. Since N satisfies KF and since $S(N) \subset \{p\}$ it follows by Corollary (1.14) that N is a 3-manifold. In particular, $p \in M(X)$. ****

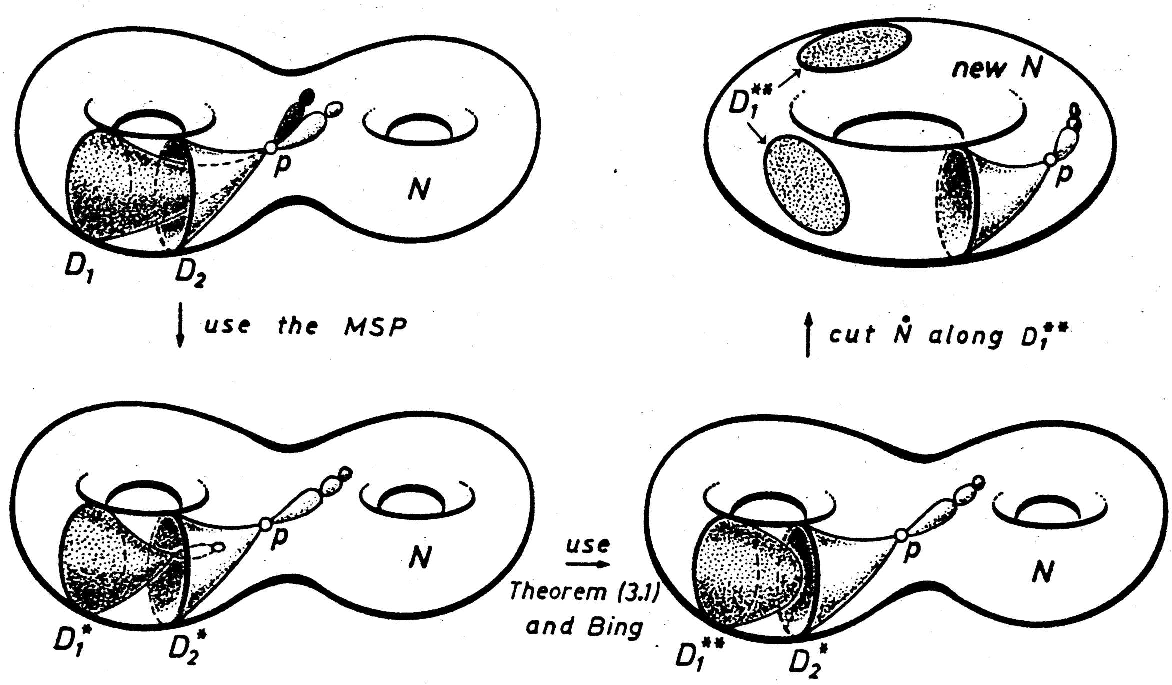



Figure 3.15.

Remark. Suppose that X is a compact generalized 3-manifold with dim S(X) $\leq$ 0, satisfying KF and having the DLP or the MSP. If S(X) $\neq \emptyset$ then X has the following properties:

(i) X admits no resolution ([14 ;Theorem 1 ] and Theorem (3.8));

(ii) S(X) is wildly embedded in X (Proposition (1.13) and Corollary (1.14));

(iii) S(X) has no isolated points (Theorem (3.11)).

# IV. GENERALIZED 3-MANIFOLDS WITH BOUNDARY

Little investigation has been done concerning generalized manifolds with boundary. In this chapter we present some results which are most of the time analogues of those known for generalized manifolds.

Let R be a PID and consider an R-homology n-manifold X with boundary. We first observe that $\dot{X}$ need not be an R-homology (n-1)-manifold (as it would be the case with topological manifolds with boundary). A simple example is the interior of any n-manifold with boundary together with just one point of its boundary ($n > 1$). It may also happen that $\dot{X}$ is an R-homology (n-1)-manifold with boundary. The next proposition gives a criterion for determing the boundary points of $\dot{X}$:

<u>Proposition 4.1</u>. Let X be an ANR and an R-homology n-manifold with boundary, R a PID. Suppose that $p \in \dot{X}$ and that $H_*(\dot{X}-\{p\};R) \cong H_*(\dot{X};R)$. Then $p \in (\dot{X})^{\boldsymbol{\cdot}}$.

<u>Proof</u>. We supress the coefficients. Consider the homology sequence of the triple $(X,\dot{X},\dot{X}-\{p\})$ over R:

$$\cdots \xrightarrow{i_*} H_{q+1}(X,\dot{X}-\{p\}) \xrightarrow{j_*} H_{q+1}(X,\dot{X}) \xrightarrow{\Delta} H_q(\dot{X},\dot{X}-\{p\}) \xrightarrow{i_*} H_q(X,\dot{X}-\{p\}) \longrightarrow$$
$$\xrightarrow{j_*} H_q(X,\dot{X}) \longrightarrow \cdots$$

Since $H_*(\dot{X}) \cong H_*(\dot{X}-\{p\})$ it follows by [59 ;Lemma 6 on p.202] that

$H_*(X,\dot{X}-\{p\}) \cong H_*(X,\dot{X})$. Hence $\text{im}\,\Delta_* = 0 = \ker i_*$ so $H_*(\dot{X},\dot{X}-\{p\}) \cong \ker j_* = 0$ thus by Lemma (1.4), $p \in (\dot{X})^{\bullet}$. ##

By Corollary (1.9) true generalized 3-manifolds with boundary cannot have a PL structure. The next result is from [59 ;p.277] :

Proposition 4.2. Let X be a PL R-homology n-manifold with boundary, R a PID. Let K be a triangulation of X. Then $\dot{X}$ is a subpolyhedron of X, $|\dot{K}| = \dot{X}$, K is a pseudo n-manifold with boundary and Lefshetz duality holds, i.e., $H^*(X,\dot{X};R) \cong H_{n-*}(\mathring{X})$ and $H_*(X,\dot{X};R) \cong H^{n-*}(\mathring{X})$. ##

It is not difficult to manufacture examples of generalized manifolds with boundary as the next proposition shows:

Proposition 4.3. Let X be a compact generalized n-manifold with $S(X) \subset Z$, where $Z \subset X$ is a closed, 0-dimensional set. Then there exists an n-cell $B \subset X$ for which $Z \subset \partial B$, $S(Y) = Z$, and Y is a compact generalized n-manifold with boundary, where $Y = X - \text{int}\,B$.

Proof. Let $B_0 \subset X-Z$ be any tamely embedded n-cell. We get B from $B_0$ by pushing out from $B_0$ wildly embedded (in X) "feelers" towards the points of Z. ##

Example 4.4. Let $X = S^3$ and B = thickened one half of the Fox-Artin wild arc [26 ;Example (3.1)] . Then $S(Y) = \{p\}$ = the wild point of the arc. (See Figure (4.1).)

Lemma 4.5. Let X and Y be generalized n-manifolds with boundary and suppose that there exists a homeomorphism $h:\dot{X} \to \dot{Y}$. Then $X \cup_h Y$ is a generalized n-manifold.

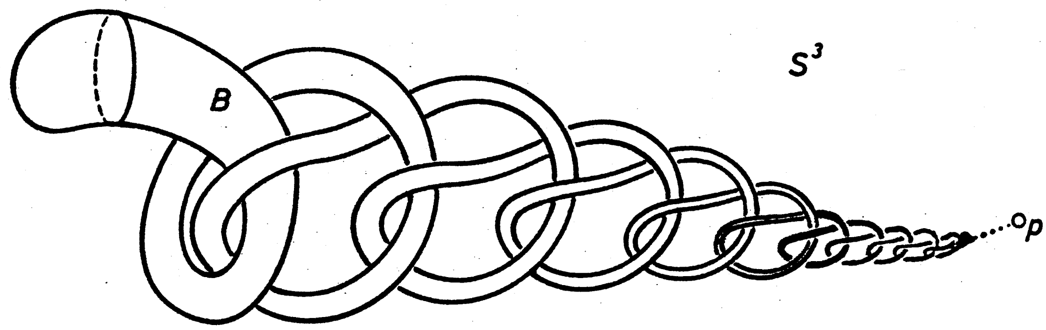



Figure 4.1.

Proof of Lemma (4.5). Since $\dot{X}$ is an ENR so is $X \cup_h Y$ by [7 ;Theorem (IV.6.1)]. It therefore suffices to show that $X \cup_h Y$ is a $\mathbb{Z}$-homology n-manifold. The argument we give below is valid over any PID R. We shall supress the coefficients. Consider the Mayer-Vietoris sequence for the pairs $(X,X-\{p\})$ and $(Y,Y-\{h(p)\})$. (By the Excision theorem it suffices to consider only the case when $p \in \dot{X}$, cf. also Lemma (1.4).):

$$\ldots \longrightarrow H_q(X,X-\{p\}) \oplus H_q(Y,Y-\{h(p)\}) \longrightarrow H_q(X \cup_h Y,(X \cup_h Y)-\{p\}) \longrightarrow$$
$$H_{q-1}(\dot{X},\dot{X}-\{p\}) \longrightarrow H_{q-1}(X,X-\{p\}) \oplus H_{q-1}(Y,Y-\{h(p)\}) \longrightarrow \ldots$$

Since $p \in \dot{X}$ and $h(p) \in \dot{Y}$ it follows by Lemma (1.4) that $H_*(X,X-\{p\}) \cong 0 \cong H_*(Y,Y-\{h(p)\})$. Also, $\dot{X}$ is a generalized (n-1)-manifold hence:

$H_q(X \cup_h Y, (X \cup_h Y)-\{p\}) \cong H_{q-1}(\dot{X}, \dot{X}-\{p\}) \cong R$ if $q = n$ and $\cong 0$ if $q \neq n$. The assertion now follows by Lemma (1.4). ⁂

We now turn to dimension three. First, we prove an analogue of the Finiteness theorem of J.L.Bryant and R.C.Lacher (Proposition (1.10)): Propositions (4.6) and (4.8).

Proposition 4.6. For every compact generalized 3-manifold with boundary X there is an integer $k_o$ such that among any $k_o+1$ pairwise disjoint $\mathbb{Z}_2$-homology 3-cells in X at least one is contractible.

Proof. By Lemma (4.5) the double DX of X is a generalized 3-manifold so there exists the Bryant-Lacher number $n_o$ for DX (Proposition (1.10)). Let $k_o = [\frac{1}{2}(n_o+1)]$. ⁂

Let X be a generalized n-manifold with boundary. We say that X has a resolution if there is a pair (M,f) where M is an n-manifold with boundary and $f: M \to X$ is a proper cell-like onto map such that $f(\partial M) \subset \dot{X}$.

Lemma 4.7. Suppose that X is a generalized n-manifold with boundary. If X has a resolution then X has a conservative resolution.

Proof. Apply Theorem (1.6) to $\mathring{X}$. (Note that the hypothesis concerning the Poincaré conjecture is necessary only for the existence of a resolution in dimension three.). ⁂

Proposition 4.8. Let X be a compact generalized 3-manifold with boundary. Suppose that X has a resolution. Then there exists an integer $k_o$ such that among any $k_o+1$ pairwise disjoint $\mathbb{Z}_2$-homology 3-cells there is at least one genuine 3-cell.

Proof. Let $k_1$ be the Kneser number of M [32;Lemma (3.14)] and $k_2$ the number determined for X by Proposition (4.6). Let $k_o = k_1+k_2$ and consider an arbitrary $(k_o+1)$-tuple $F_1,\ldots,F_{k_o+1} \subset X$ of pairwise disjoint $\mathbb{Z}_2$-homology 3-cells. By pushing each $F_i$ into int $F_i$ along a collar on $\partial F_i$ we may assume that each $F_i$ lies in $M(X) \cap \mathring{X}$. The assertion now follows by Proposition (4.6), Lemma (4.7), and Kneser Finiteness in M. ⁑

Proposition 4.9. Suppose that X is a compact generalized n-manifold with boundary. If n = 3 assume that dim $S(X) \leq 0$ and that the Poincaré conjecture in this dimension is true. Then X has a resolution.

Proof. Let Y = X + C, where $C = \dot{X} \times I$ is a collar on $\dot{X}$. By the arguments employed in the proof of Lemma (4.5) and by [7 ;Theorem (IV.6.1)], Y is a generalized n-manifold with boundary. Also, $S(Y) \subset S(X)$. That Y resolves now follows by [62;Main Theorem] if n=3, by [56;Theorem (2.6.1)] if n=4, and by [55 ;Theorem (1.1)] if $n \geq 5$. (If $n \leq 2$, $S(X) = \emptyset$ by [68;Theorems (IX.1.2) and (IX.2.3)].) So there is a proper cell-like surjection $f:(M,\partial M) \twoheadrightarrow (Y,\dot{Y})$ from an n-manifold with boundary. Let $g:Y \twoheadrightarrow X$ be the collapse of Y onto X along the fibers $\{x\} \times I$ $(x \in \dot{X})$ of the collar C. Then g is clearly cell-like so by [40;p.511], $gf:(M,\partial M) \twoheadrightarrow (X,\dot{X})$ is a resolution of X. ⁑

Remark. Suppose that the Poincaré conjecture is false. Let $X^*$ be the example described in Proposition (1.11) and let $B \subset X^*$ be a nicely embedded PL 3-cell in $X^*$ such that the limit point p lies on $\partial B$. Let $X \doteq X^*$-int B. (See Figure (4.2).) Then X is a compact generalized 3-manifold with boundary, $p \in \dot{X}$, $S(X) = \{p\}$, and X doesn't admit a

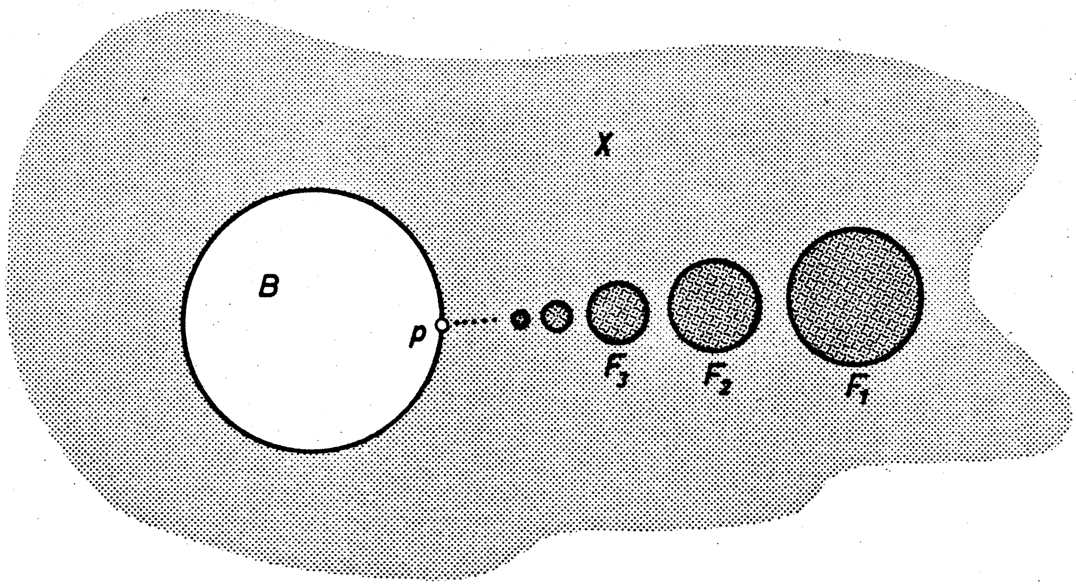



Figure 4.2.

resolution, by (4.8). This example together with (4.9) yields:

Corollary 4.10. Let $\underline{C}$ be the class of all compact generalized 3-manifolds with boundary such that $\dim S(X) \leq 0$ and let $\underline{C}_o \subset \underline{C}$ be the subclass of all $X \in \underline{C}$ such that $S(X) \subset *$ and X is a homotopy 3-cell. Then the following statements are equivalent:

(i) Poincaré conjecture in dimension three is true;

(ii) If $X \in \underline{C}$ then X has a resolution;

(iii) If $X \in \underline{C}_o$ then X has a resolution. ** **

Let X be a compact generalized 3-manifold with boundary and suppose that the double DX of X is a 3-manifold. Then X need not be a 3-manifold: e.g., R.H.Bing proved that the double of the Alexander solid horned sphere yields $S^3$ [4] . But we can prove something:

Proposition 4.11. Let X be a compact generalized 3-manifold with boundary such that DX is a 3-manifold. Then X has no isolated singularities.

Proof. Since $\dot{X}$ is a closed surface in a closed 3-manifold DX it follows by a results due to O.G.Harrold and E.E.Moise [31] that $\dot{X}$ can be wild at each point at most from one side in DX. But in DX the two sides are "symmetric". Hence $\dot{X} \subset DX$ is 1-LCC so the assertion now follows by Proposition (1.13) and Theorem (3.9). ⁂

The following is a generalization to generalized 3-manifolds with boundary of a results due to Bryant and Lacher [14;Theorem 4] :

Proposition 4.12. Let X be a compact generalized 3-manifold with boundary and suppose that $S(X) \subset Z$, where $Z \subset X$ is a closed, 0-dimensional set in X, Z is 1-LCC in X. Then X is a 3-manifold if and only if it has the KF.

Proof. The "if" direction is Kneser finiteness theorem. So assume now that X has the KF. Cover Z with pairwise disjoint 2-cells in $\dot{X}$ (note, that by [14 ;Theorem 4] we may assume that $S(X) \subset \dot{X}$) so that the boundaries of these 2-cells lie in M(X). Because of the 1-LCC condition these boundaries bound some Dehn disks in X hence real PL disks (apply Dehn's lemma in M(X)). The methods of the proof of Theorem 4 in [14] now yield the desired conclusion. ⁂

Proposition 4.12. Let X be a generalized n-manifold with boundary and Y = X+C, where C is a collar on $\dot{X}$. Suppose that Y is an n-manifold with boundary. Then the double DX of X is also an n-manifold.

Proof. Let G be the decomposition of DY, the double of Y, into points and fibers of the two adjacent copies of the collar C. Then G is a shrinkable, cell-like upper semicontinuous decomposition. By hypothesis, DY is an n-manifold so by Theorem (1.3), $DY \approx DY/G$. Since $DX \approx DY/G$, the assertion follows. ⁂

# V. EPILOGUE

In the last chapter we review some open problems which are related to results presented in the dissertation. First, we consider the resolution problem for $\mathbb{Z}_2$-homology 3-manifolds. Theorem (2.7) implies (so does already [14 ;Theorem 3] ) that a locally simply connected $\mathbb{Z}_2$-acyclic image X of a 3-manifold M is a generalized 3-manifold. R.J.Daverman and J.J.Walsh [20] constructed an example of an upper semicontinuous decomposition G of $S^3$ with the following properties:

(i) each $g \in G$ is strongly $\mathbb{Z}$-acyclic but not cell-like;

(ii) $S^3/G$ is a $\mathbb{Z}$-homology 3-manifold;

(iii) $S^3/G$ is $lc^{\infty}(\mathbb{Z})$;

(iv) $S^3/G$ is not an ANR.

Note, that $H_G = G$. By Theorem (2.7), (i) implies that $S^3/G$ is not even 1-LC, since if it were 1-LC on just an open set $U \subset S^3/G$ then almost all $g \in G$ which are mapped into U would have to be cell-like. Thus one cannot drop the 1-LC condition from Theorem (2.7). However, one can try to weaken the hypothesis on M: let's assume that M is only a generalized 3-manifold. By Proposition (1.1), X is still a $\mathbb{Z}_2$-homology 3-manifold. What is not clear is whether X is also a $\mathbb{Z}$-homology 3-manifold, i.e. is every ENR which is a $\mathbb{Z}_2$-homology 3-manifold necessarily also a $\mathbb{Z}$-homology 3-manifold [42]?

If instead of $\mathbb{Z}_2$ we have $\mathbb{Z}_p$, p any odd prime, then the answer to this question is negative. For, let X be the suspension of the projective plane, $X = \Sigma P^2$. Then X is locally contractible and since it's a finite dimensional simplicial complex it follows by [7; Theorem (V.10.3)] that X is an ENR. Next, we show that X is a $\mathbb{Z}_p$-homology 3-manifold. We shall supress the coefficients $\mathbb{Z}_p$. By Lemma (1.4) and by the Excision theorem it suffices to show that $H_q(\Sigma P^2,\Sigma P^2-\{a_i\}) \cong \mathbb{Z}_p$ if $q = 3$ and $\cong 0$ if $q \neq 3$, where $a_1, a_2 \in \Sigma P^2$ are the suspension points. From the Mayer-Vietoris sequence for the triple $(\Sigma P^2,\Sigma P^2-\{a_1\},\Sigma P^2-\{a_2\})$ we have that $H_{q+1}(\Sigma P^2) \cong H_q(P^2)$ for every q . From the homology sequence of the pair $(\Sigma P^2,\Sigma P^2-\{a_i\})$ we have that $\tilde{H}_q(\Sigma P^2) \cong H_q(\Sigma P^2,\Sigma P^2-\{a_i\})$ for all $q \geq 0$ hence we can conclude that $H_q(\Sigma P^2,\Sigma P^2-\{a_i\}) \cong \tilde{H}_{q-1}(P^2)$ for all q. Since p was an odd prime, the assertion follows. Now, the only two singularities are $a_1$ and $a_2$. Since each has an open cone neighborhood in X it follows by Proposition (1.8) that X can't be a generalized 3-manifold hence not a $\mathbb{Z}$-homology 3-manifold. Note that this particular example doesn't work for p = 2 since $H_2(\Sigma P^2,\Sigma P^2-\{a_i\};\mathbb{Z}_2) \cong \mathbb{Z}_2 \neq 0$. In fact, we can prove the following more general observation:

<u>Proposition 5.1</u>. Let X be the suspension of a closed surface and suppose that X is a $\mathbb{Z}_2$-homology 3-manifold. Then X is a 3-manifold.

<u>Proof</u>. Clearly, X is an ENR. By the Excision theorem it has the local $\mathbb{Z}$-homology of $\mathbb{R}^3$ everywhere except maybe at the suspension points $a_1$ and $a_2$. By hypothesis X is a $\mathbb{Z}_2$-homology 3-manifold so by Lemma (1.4), $H_*(X,X-\{a_i\};\mathbb{Z}_2) \cong \tilde{H}_*(S^3;\mathbb{Z}_2)$ hence by the Universal

Coefficients theorem $H_*(X,X-\{a_i\};\mathbb{Z}) \cong \tilde{H}_*(S^3;\mathbb{Z})$. Thus X is a generalized 3-manifold so by Proposition (1.8), $a_i \in M(X)$, i=1;2. This proves that $S(X) = \emptyset$. ⁑

We remark here that a negative answer to Lacher's question qouted above would yield an example of a $\mathbb{Z}_2$-homology 3-manifold that is not a $\mathbb{Z}_2$-acyclic image of any 3-manifold. For if it were then by Theorem (2.7), X would be a generalized 3-manifold. Therefore such a negative example would yield a counterexample to the resolution conjecture for $\mathbb{Z}_2$-homology 3-manifolds.

Next, we wish to state two problems concerning the DLP and the MSP. Let X be a compact generalized 3-manifold with $\dim S(X) \leq 0$ and satisfying the KF. By Theorem (1.16) there is a compact generalized 3-manifold Y and a cell-like map f from Y onto X. Also, $\dim S(Y) \leq 0$, $g(Y,y) = 0$ for all $y \in Y$, and all singularities of Y are "soft" (in the sense of Proposition (1.13)).

<u>Question 5.2</u>. Let $f: Y \to X$ be as above and suppose that X has either the DLP or the MSP. Does Y satisfy the KF?

If the answer is "yes" then first, by Corollary (1.13), Y is a 3-manifold so it X has a resolution and is thus, by Theorem (3.8), itself a 3-manifold. This would answer in the affirmative the next question, which tries to disentangle the Poincaré conjecture from Theorem (3.10) (as Theorem (1.16) does with the n = 3 case of Theorem (1.6)):

<u>Question 5.3</u>. Suppose that X is a compact generalized 3-manifold with $\dim S(X) \leq 0$, satisfying the KF, and with the DLP or the MSP. Is $S(X) = \emptyset$?

Note, that by Theorem (3.8) it would suffice to only show that X has a resolution.

We also wish to discuss some open problems concerning generalized 3-manifolds with boundary. The following obstruction arises at once:

Question 5.4. Let M be a compact 3-manifold with boundary and $f:M \twoheadrightarrow X$ a cell-like mapping onto an ANR X. Let $DX = X \cup X/R$, where R is the equivalence relation on $X \times X$ given by: $xRy \Longleftrightarrow f^{-1}(x) \cap \partial M = f^{-1}(y) \cap \partial M \neq \emptyset$ $(x,y \in X)$. Is then the associated mapping $Df:DM \twoheadrightarrow DX$ also cell-like?

If the answer is affirmative then we get an analogue of Proposition (1.11):(i) $\Rightarrow$ (ii).

Proposition 5.5. Let X be a resolvable compact generalized 3-manifold with boundary and suppose that $\dim S(X) \leq 0$. If the answer to Question (5.4) is affirmative then M(X) embedds in some closed 3-manifold.

Proof. Let $f:(M,\partial M) \twoheadrightarrow (X,\dot{X})$ be a resolution of X. Then $Df:DM \twoheadrightarrow DX$ is a resolution of DX, by (5.4). It follows by Proposition (1.11) that M(DX) embedds in the interior of some compact 3-manifold hence so does $M(X) \subset M(DX)$. ⁑

Another useful consequence of a possible affirmative answer to Question (5.4) would be the following analogue of Theorem (2.7):

Proposition 5.6. Let X be a locally simply connected $\mathbb{Z}_2$-homology 3-manifold with boundary. Suppose that there exist a 3-manifold M with boundary and a closed, monotone map $f:M \twoheadrightarrow X$ such that for every $x \in X-Z$, $\check{H}^1(f^{-1}(x);\mathbb{Z}_2) = 0$, where $Z \subset X$ is some 0-dimensional set. If the answer to Question (5.4) is affirmative then X has a resolution.

Proof. Let $Y = X+C$, where C is a collar on $\dot{X}$. Then Y is a generalized 3-manifold with boundary, Y is 1-LC, and $S(Y) \subset S(X)$. Assuming that (5.4) is true we can extend f over $N = M+D$, where D is a collar on $\partial M$, fiberwise to get a map $g:N \twoheadrightarrow Y$ satisfying the hypotheses we required initially for M,X, and f. By similar methods as those applied in the proof of Theorem (2.7) one can now show that Y has a resolution $h_1:P \twoheadrightarrow Y$. Let $h_2:Y \twoheadrightarrow X$ be the collapse of C onto $\dot{X}$ (along the fibers of C). Then $h = h_2h_1:P \twoheadrightarrow X$ is a resolution of X. (See Figure (5.1).) ⁑

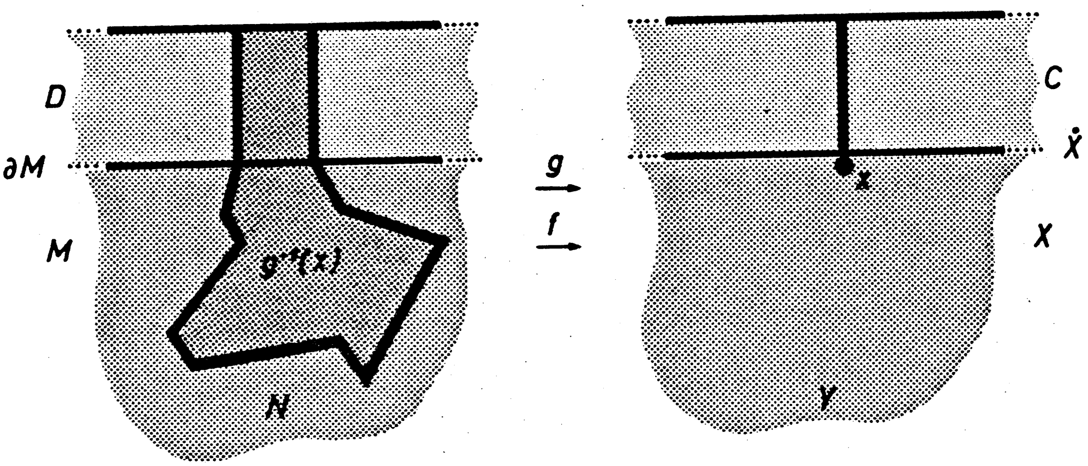



Figure 5.1.

Question 5.7. Let X be a compact generalized 3-manifold with boundary and suppose that $\dim S(X) \leq 0$. Are the following statements equivalent:

(i) X embeds in the interior of a compact 3-manifod with boundary;

(ii) X plus a collar on $\dot{X}$ is a 3-manifold with boundary.

Question 5.8. Let X be a compact generalized 3-manifold with boundary and suppose that $\dim S(X) \leq 0$. Suppose also that M(X) embeds in the interior of a compact 3-manifold with boundary. Does then X have a resolution?

Example 5.9. Let X be a compact generalized n-manifold with boundary and let Y be X plus a collar C on $\dot{X}$. Suppose that Y is an n-manifold with boundary. Then

(i) $S(X) \subset \dot{X}$;

(ii) X has a resolution (just collapse Y onto X along the fibers of C).

The converse need not be true. E.g., let A be a noncellular arc with one wild point $p^*$, $A \subset B^n$, $A \cap \partial B^n = \{p\}$ [26] . Then $X = B^n/A$ is a compact generalized n-manifold with boundary and satisfies both properties (i) and (ii). If Y = X+C were an n-manifold with boundary then A would necessary have to be cellular.One can thus only conclude that Y is a compact generalized n-manifold with boundary, $S(Y) \subset S(X)$, and that Y resolves if and only if X has a resolution. (See Figure (5.2) on p.97.)

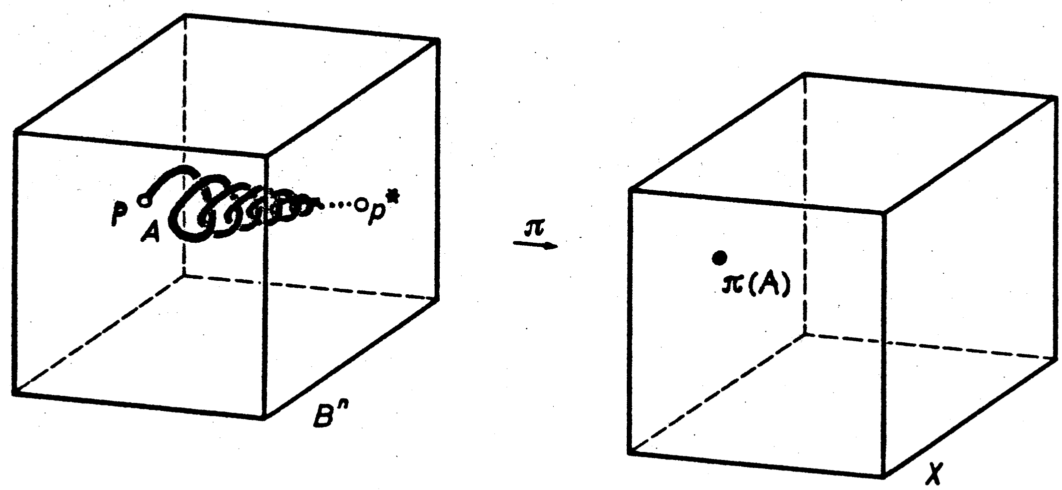



Figure 5.2.

<u>Question 5.10</u>. Suppose that X is a resolvable generalized n-manifold with boundary. Does X have a conservative resolution?

<u>Question 5.11</u>. Suppose that X is a compact generalized 3-manifold with boundary and that there is a proper cell-like onto map $f:(M,\partial M) \twoheadrightarrow (X,\dot{X})$ from a compact 3-manifold M with boundary onto X. Suppose furthermore that $f|\partial M:\partial M \twoheadrightarrow \dot{X}$ is proper and cell-like, $f^{-1}(\dot{X}) = \partial M$, and that $S(X) \subset \dot{X}$. Is then X a 3-manifold with boundary?

<u>Question 5.12</u>. Let X be a generalized 3-manifold with boundary with $\dim S(X) \leq 0$ and satisfying the KF. Assume that X has either the DLP or the MSP. Is then X resolvable? If "yes", is then X a 3-manifold with boundary?

For resolvable compact generalized 3-manifolds X with $\dim S(X) \leq 0$ the properties DLP and MSP were equivalent, by Theorem (3.8). Let DDP I and DDP II be Starbird's disjoint disks properties [60] .

Question 5.13. Let G be a cell-like 0-dimensional upper semicontinuous decomposition of a 3-manifold M. What implications exist between the following statements:

(i) M/G has the DLP;

(ii) M/G has the MSP;

(iii) M/G has the DDP I;

(iv) M/G has the DDP II;

(v) M/G is a 3-manifold.

Throughout the dissertation we have been dealing mostly with generalized manifolds with small singular sets. It has been known for many years that, e.g. in dimension three examples exist where S(X) is the whole space X (such an example is due to K.W.Kwun [37]). No results are available concerning resolutions of generalized 3-manifolds X with $\dim S(X) \geq 1$ except [14 ;Theorem 1].

Question 5.14. Let X be a compact generalized 3-manifold with dim S(X) = 1. Suppose that M(X) embedds in the interior of a compact 3-manifold with boundary. Does X have a resolution?

Question 5.15. Let X be a compact generalized 3-manifold satisfying the KF and with $S(X) \subset A$, where $A \subset X$ is a locally homotopically unknotted arc in X. Is X a 3-manifold?

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## APPENDIX: REGULAR NEIGHBORHOODS OF COMPACT POLYHEDRA IN 3-MANIFOLDS

Strong peripheral 1-acyclicity is a homology version of McMillan's <u>cellularity criterion</u> (CC) [46] while weak peripheral 1-acyclicity is a homology version of his <u>weak cellularity criterion</u> (WCC) [46]. It is therefore interesting to observe that by Theorem (2.9) the two acyclicities are equivalent over $\mathbb{Z}_2$ while CC is clearly a stronger property than WCC (just consider any noncellular arc in $S^3$). The example $M = S^2 \times S^1$, $K = S^2 \vee S^1$ from [39] shows that WCC is not a topological property of K (i.e., it may depend on the embedding, as might CC). Same example confirms this for peripheral acyclicities (p.41). On the other hand we proved in Theorem (2.11) that the peripheral 1-acyclicities do not depend on the embedding (if K is compact and M is a 3-manifold) provided $\dim K \leq 1$. This is an analogue of the result of Lacher [39] to the effect that if K is a codimension $\geq 2$ compact subset of a PL n-manifold M, $n \neq 4$, then WCC is equivalent to the 1-UV property and thus independent of embedding [39 ;p.499] . In [39] Lacher asked the following (to our knowledge still open) question: Suppose that f is an embedding of a compact set K into an n-manifold N, homotopic to the inclusion $K \subset N$. If $K \subset N$ has WCC does $f(K) \subset N$ have the same property? In this appendix we answer in the affirmative Lacher's question for PL embeddings of polyhedra in 3-manifolds.

Theorem A.1. Suppose that $f:K \to M$ is a PL embedding of a compact polyhedron in the interior of a 3-manifold. Suppose that f is homotopic to the inclusion $K \subset M$ and that $K \subset M$ has WCC (resp. is peripherally 1-acyclic over $\mathbb{Z}_2$). Then $f(K) \subset M$ has WCC (resp. is peripherally 1-acyclic over $\mathbb{Z}_2$).

Lemma A.2. Let $f_1, f_2 : K \to \operatorname{int} M$ be homotopic PL embeddings of a compact polyhedron K in a PL m-manifold M. Let $N_i \subset \operatorname{int} M$ be a regular neighborhood of $f_i(K)$ in M. Then $\chi(\partial N_1) = \chi(\partial N_2)$.

Proof. Let $r \neq 0$ be any even natural number satisfying $r \geq 2k-m+3$, where $k = \dim K$ and $m = \dim M$. Choose a triangulation of $M \times \mathbb{R}^r$ consistent with the one on M and define PL embeddings $F_i : K \to M \times \mathbb{R}^r$ by $F_i = f_i \times 0$, i=1;2. Since $f_i$'s are homotopic there is a homotopy H: $K \times I \to M \times \mathbb{R}^r$ from $F_1$ to $F_2$. Define a map $H^* : K \times I \to M \times \mathbb{R}^r \times I$ by $H^*(x,t) = (H_t(x),t)$ for each $(x,t) \in K \times I$. By [72] we may assume that $H^*$ is PL. Since $2k-m+2 < r$, it follows that $2(k+1)-(m+r+1) < 2k+2-m-(2k-m+2)-1 = -1 < 0$, so by general position [57;Theorem (5.4)] we may assume that $H^*$ is a PL embedding hence a concordance [34;p.128]. Now, $(m+r)-3 \geq m+(2k-m+3)-3 = 2k \geq k$, so by [34 ;Corollary (1.4)], $F_1$ and $F_2$ are ambient PL isotopic. Let $N_i^* = N_i \times B^r$, i=1;2. Then by [57 ;Corollary (3.29)], $N_i^*$ is a regular neighborhood of $F_i(K)$ in $M \times \mathbb{R}^r$, so $N_1^* \underset{PL}{\cong} N_2^*$. In particular, $\partial N_1^* \underset{PL}{\cong} \partial N_2^*$. Now, $\partial N_i^* = (\partial N_i \times B^r) \cup_{(\partial N_i \times S^{r-1})} (N_i \times S^{r-1})$ thus the Euler characteristic of $\partial N_i^*$ is:

$$\chi(\partial N_i^*) = \chi(\partial N_i \times B^r) + \chi(N_i \times S^{r-1}) - \chi(\partial N_i \times S^{r-1}) =$$
$$= \chi(\partial N_i) + \chi(N_i)\chi(S^{r-1}) - \chi(\partial N_i)\chi(S^{r-1}) = \chi(\partial N_i)$$

where the second equality follows by the product formula for the Euler characteristic and the third equality by the fact that r is always even hence $\chi(S^{r-1}) = 0$. Consequently, $\chi(\partial N_1) = \chi(\partial N_1^*) = \chi(\partial N_2^*) = \chi(\partial N_2)$. $\ast\ast$

Lemma A.3. Let K be a compact polyhedron in the interior of a 3-manifold M. Then the following statements are equivalent:

(i) K is weakly peripherally 1-acyclic over $\mathbb{Z}_2$ in M;

(ii) $K \subset M$ has WCC;

(iii) For every regular neighborhood $N \subset M$ of K, $\partial N$ is a collection of 2-spheres.

Proof. (i) $\Rightarrow$ (iii) Let $N \subset M$ be a regular neighborhood of a weakly peripherally 1-acyclic (over $\mathbb{Z}_2$) compact polyhedron K. Then there is a regular neighborhood $N^* \subset \operatorname{int} N$ of K such that the inclusion-induced homomorphism $H_1(N^*-K;\mathbb{Z}_2) \to H_1(N;\mathbb{Z}_2)$ is trivial. Since $c\ell(N-N^*) = \partial N^* \times [0,1]$ and $N^*-K = \partial N^* \times [0,1) \simeq N^*$ it follows that every 1-cycle in $\partial N^*$ bounds (over $\mathbb{Z}_2$) in $N^*$. By Lemma (2.4), $g(\partial N^*) = 0$. By Lemma (2.3), $\partial N^*$ is orientable hence $\partial N^*$ is a collection of 2-spheres.

(iii) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) Let $V \subset U \subset M$ be neighborhoods of K such that the inclusion $i: V-K \to U$ induces a zero homomorphism $i_\#$ on fundamental groups. Consider the following commutative diagram

$$\begin{array}{ccc}
\pi_1(V-K) & \xrightarrow{\quad i_\# \quad} & \pi_1(U) \\
\downarrow h & & \downarrow h' \\
H_1(V-K;\mathbb{Z}_2) & \xrightarrow{\quad i_* \quad} & H_1(U;\mathbb{Z}_2)
\end{array}$$

where h,h' are the Hurewicz epimorphisms. Since $i_{\#} = 0$ it follows that $i_* = 0$, too. ##

Proof of Theorem (A.1). Let $N \subset \operatorname{int} M$ be a regular neighborhood of K and $N^* \subset \operatorname{int} M$ a regular neighborhood of f(K). By hypothesis and by Lemma (1.2), $\partial N$ is orientable, and by Lemma (A.3), $\partial N$ is a collection of 2-spheres. By Lemma (A.2) so is then $\partial N^*$. Another application of Lemma (A.3) now yields the conclusion. ##

We continue with a result concerning pairs of polyhedra in 3-manifolds. So let (K,L) be a compact polyhedral pair in the interior of a 3-manifold M. By Theorem (2.9) and Lemma (A.3), the properties "$K \subset M$ has WCC", "$K \subset M$ is strongly peripherally 1-acyclic over $\mathbb{Z}_2$", and "$K \subset M$ is weakly peripherally 1-acyclic over $\mathbb{Z}_2$" are equivalent. In the next result "X has property P" will mean "X has any of these three properties (hence all three)":

Theorem A.4. Suppose that K and L are of the same simple homotopy type. Then K has property P if and only if L has property P.

Lemma A.5. Let N be a neighborhood of a compact polyhedron $K \subset \operatorname{int} M$, where M is a PL n-manifold. Then N is a regular neighborhood of K if and only if N is a compact n-manifold with boundary and N is of the same simple homotopy type as K.

Proof. The only if part is well-known [57;Corollary (3.30)] . So we prove the other implication. There is a sequence of expansions and collapses (in M) that transform N into K. We may do the expansions first [18 ;Exercise (4.D)] . It suffices to give the proof for the case when there is just one expansion. Let $N' \supset N$ be N after the expansion and let $N^* \subset \operatorname{int} N$ be N pushed in int N along a collar on $\partial N$ (keeping K fixed). Since $N'$ collapses onto K, $N'$ is a regular neighborhood of K in M [57 ;Corollary (3.30)] . Also, $N'$ collapses onto $N^*$ and N collapses onto $N^*$ hence both $N'$ and N are regular neighborhoods of $N^*$ in M. It follows by the uniqueness theorem [57 ;Theorem (3.24)] that there is an ambient PL isotopy of M carrying N onto $N'$ with support in $M-N^*$. Hence N is a regular neighborhood of K in M. (See Figure (A.1).) ⁑

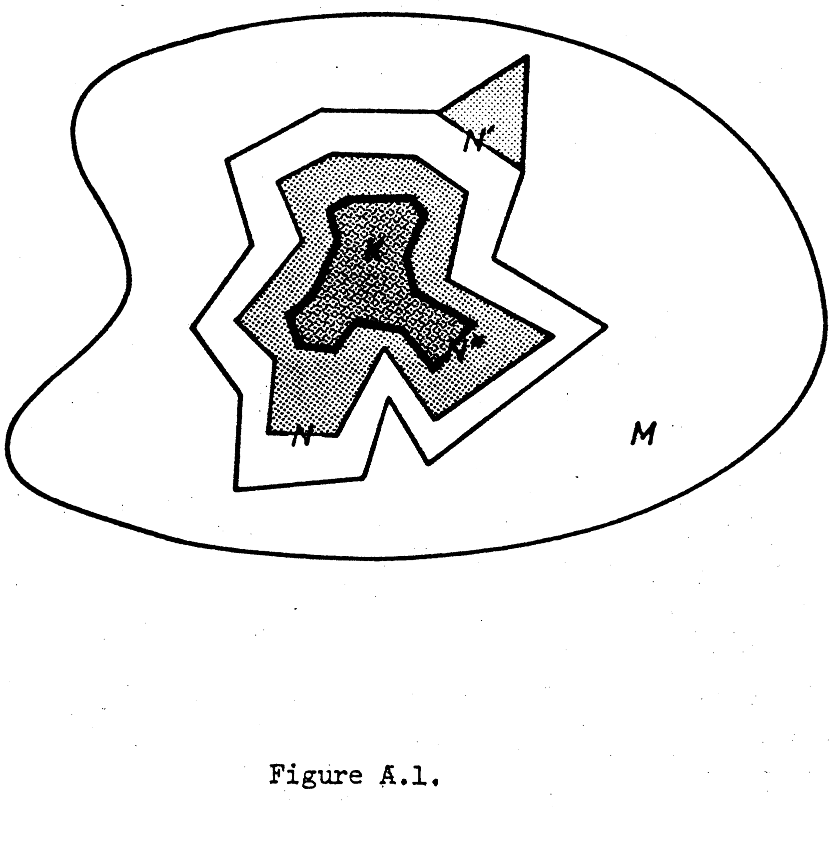


Figure A.1.

Proof of Theorem (A.4). We only prove the necessity. The other implication is proved similarily. Let (A,B) be a regular neighborhood of (K,L) in M. By Lemma (A.5), A is a regular neighborhood of L and thus homeomorphic to B [57 ;Theorem (3.24) ] . By Lemma (A.3), $\partial A$ is a collection of 2-spheres hence so is $\partial B$. The conclusion now follows by another application of Lemma (A.3). **

In the second part of the appendix we wish to discuss a related question concerning regular neighborhoods of homotopically PL embedded compact polyhedra in 3-manifolds.

Question A.6. Let K be a compact polyhedron with $H_2(K; \mathbb{Z}_2) = 0$ and let $f_1, f_2 : K \to M$ be homotopic PL embeddings of K into a 3-manifold M. Let $N_i \subset M$ be a regular neighborhood of $f_i(K)$ in M (i=1,2). Is then $N_1 \underset{PL}{\cong} N_2$?

Consider the following example: let $M = S^3$, $K = S^1 \vee S^2 \vee S^1$, and let $f_i$ be the two standard PL embeddings --$f_1(K)$ has both $S^1$'s attached at the same side of $S^2$, while $f_2(K)$ has one $S^1$ on each of the two sides of $S^2$, Clearly, $f_1 \simeq f_2$. However, $N_1 \ncong N_2$ because $\partial N_1 = S^2 \cup$ (double torus) while $\partial N_2 = (S^1 \times S^1) \cup (S^1 \times S^1)$. This example shows why condition on $H_2(K)$ is necessary. It is known that Dunce hat [71] can be PL embedded in $S^4$ in such a manner that the boundary of the corresponding regular neighborhood is not even simply connected. Hence (A.6) has a negative answer in dim 4.

Proposition A.7. Suppose the answer to Question (A.6) is affirmative. Then the following statements are equivalent:

(i) The Poincaré conjecture in dimension three is true;

(ii) The spine of any homotopy 3-cell PL embeds in $\mathbb{R}^3$.

Proof. (i) $\Rightarrow$ (ii): This implication is independent of (A.6) and is obvious.

(ii) $\Rightarrow$ (i): Suppose $K \subset \text{int}\, F$ is a spine of a homotopy 3-cell $F$. Let $B \subset \text{int}\, F$ be a nicely embedded 3-cell. By hypothesis there is a PL embedding $f: K \to \text{int}\, B$. Let $C \subset \text{int}\, B$ be a regular neighborhood of $f(K)$ in $\text{int}\, B$. Since $F$ is contractible, $f$ is homotopic to the inclusion $K \; F$. Also, $F$ is a regular neighborhood of $K$ [57 ;Corollary (3.30)] . If the answer to Question (A.6) is affirmative then $F = C$ so $C$ is a homotopy 3-cell. Since $C \subset \text{int}\, B = \mathbb{R}^3$, $C$ is a real 3-cell hence so is $F$. ⁑⁑

Theorem A.8. The answer to Question (A.6) is affirmative if K satisfies any of the following conditions:

(i) $\dim K \leq 1$;

(ii) $K$ is a compact surface with boundary;

(iii) $K$ is a closed surface and $\partial N_1$ is a 2-sphere.

Proof. We may assume that $K$ is connected. First, assume (i). By [32 ;Theorem (2.4)], $N_i$ is a 3-cell with $n_i$ (possibly nonorientable) 1-handles, $n_i \in \mathbb{N}$ . Since $f_1$ is homotopic to $f_2$, we have that $\Pi_1(N_1) \cong \Pi_1(N_2)$ hence $n_1 = n_2$. Suppose now that, say $N_1$ is orientable and $N_2$ nonorientable. Then there is an orientation-reversing loop $J$ in $N_2$. Since $f_1 \simeq f_2$ and $N_i$ collapses onto $f_i(K)$, there is a loop $J^*$ in $N_1$ with $[J^*] = [J] \in \Pi_1(M)$ hence $J^*$ also reverses the orientation. This yields a contradiction to the assumption that $N_1$ is orientable. It now follows by [32;Theorem (2.2)], that $N_1 \underset{PL}{\cong} N_2$.

Assume next that K satisfies (ii). Then there exists a bouquet T of simple closed curves on K such that K collapses onto T. The conclusion now follows by case (i) above and [57;Corollary (3.29)].

Finally, assume (iii). Then $N_1$ is an I-bundle over $f_1(K)$ so $\partial N_1$ is a double covering of $f_1(K)$ hence $f_1(K)$ is either $S^2$ or $P^2$, the projective plane. Thus K is either $S^2$ or $P^2$.

If $K = S^2$ then $f_i(K)$ is necessarily two-sided since there are no one-sided 2-spheres in 3-manifolds. (Indeed, suppose $S \subset M^3$ is a 2-sphere in a 3-manifold M and consider a regular neighborhood $N \subset M$ of S. Then N is a product I-bundle since S is simply connected and hence no loop on S can reverse the orientation in M. Thus S must be two-sided.) Consequently, $N_i = f_i(K) \times I$ hence $N_1 \underset{PL}{\cong} N_2$.

If $K = P^2$ we first consider the case when M is orientable. Then both embeddings $f_i(K)$ are one-sided since otherwise $f_i(K) \times I$ would represent a nonorientable 3-submanifold of M, an impossibility. Hence $N_i$ is a twisted I-bundle over $P^2$. It is known that over $P^2$ there is but one twisted I-bundle (up to a PL homeomorphism). Consequently, $N_1 \underset{PL}{\cong} N_2$.

Finally, assume that M is nonorientable and that $K = P^2$. If both embeddings $f_i(K)$ are one-sided apply the preceeding argument. If both embeddings $f_i(K)$ are two-sided then $N_i = f_i(K) \times I$ hence $N_1 \underset{PL}{\cong} N_2$. So it remains to consider the case when, say $f_1(K)$ is one -sided and $f_2(K)$ is two-sided. Consider the orientable double cover $\tilde{p}: \tilde{M} \twoheadrightarrow M$ of M. Then $f_1(K)$ lifts in $\tilde{M}$ to two disjoint homeomorphic copies while $\tilde{p}^{-1} f_2(K)$ is connected (and double covers $f_2(K)$). Since $f_1 \simeq f_2$ the number of components of $\tilde{p}^{-1} f_i$ in $\tilde{M}$ agree. Contradiction. ⁂

We conclude the appendix with some further comments regarding Question (A.6). Note, first, that by Lemma (A.2), the boundaries of the regular neighborhoods are always homeomorphic, since they are both connected (because $H_2(K:\mathbb{Z}_2) = 0$ implies that $N_i - f_i(K)$ is connected) and they are either both orientable or both nonorientable (because $f_1 \simeq f_2$).

Professor W.Heil suggested the following question which -- if the answer is affirmative -- would yield a negative answer to Question (A.6):

<u>Question A.9</u>. Let $K_1$ = the square knot, $K_2$ = the granny knot, and let $M_i$ be the corresponding knot space, i.e. $M_i = S^3$ - (open tubular neighborhood of $K_i$ in $S^3$). It is well-known that $M_1 \not\approx M_2$. However, do $M_1$ and $M_2$ collapse to homeomorphic spines?

There are certain grounds for a belief that the answer to (A.9) is negative: although the fundamental groups of $M_i$ are isomorphic, the peripheral systems are different, i.e. the diagram below can never be completed to a commutative one -- and <u>that</u> obstruction could determine the spine of $M_i$.

$$\begin{array}{ccc}
\Pi_1(\partial M_1) & \rightarrowtail & \Pi_1(M_1) \\
\vdots\downarrow & & \cong\downarrow \\
\Pi_1(\partial M_2) & \rightarrowtail & \Pi_1(M_2)
\end{array}$$

## VITA

Dušan Repovš was born on November 30, 1954 in Ljubljana, Yugoslavia. He graduated from the University of Ljubljana in the Fall of 1977 with Mathematics as a major and Mechanics as a minor. He wrote his B.Sc.Thesis on Borsuk's shape theory for compacta, under Professor J.Vrabec. In September 1978 he entered the FSU Graduate School. He passed the Preliminary Doctoral Examination in September 1979. In June and July of 1980 he attended the Topology Summer Conference at the University of Texas at Austin. In September 1980 he accepted an assistantship at the College of Mechanical Engineering (University of Ljubljana). In January 1981 he attended the Shape Theory and Geometric Topology Winter Postgraduate School in Dubrovnik, Yugoslavia. The Summer of 1981 he spent back at FSU. In September 1981 he was invited to present his joint research with Professor R.C.Lacher at a topology conference in Oberwolfach, West Germany. Next month he was drafted for a 12-months military service in his home country, after which he resumed his position at the College.

He is a member of the American Mathematical Society, Pi Mu Epsilon, and the Society of Mathematicians, Physicists and Astronomers of Yugoslavia.

He is married to the former Barbara Hvala from Ljubljana, a 1982 graduate of the University of Ljubljana (Philosophy/Sociology). They have a 5 years old son Uroš Peter and they live in Ljubljana.