# On identities of Watson type

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**Abstract**

We prove several identities of the type $\alpha(n) = \sum_{k=0}^{\infty} \beta \left( \frac{n - k(k+1)/2}{2} \right)$. Here, the functions $\alpha(n)$ and $\beta(n)$ count partitions with certain restrictions or the number of parts in certain partitions. Since Watson proved the identity for $\alpha(n) = Q(n)$, the number of partitions of $n$ into distinct parts, and $\beta(n) = p(n)$, Euler's partition function, we refer to these identities as Watson type identities. Our work is motivated by results of G. E. Andrews and the second author who recently discovered and proved new Euler type identities. We provide analytic proofs and explain how one could construct bijective proofs of our results.

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## 1 Introduction

Any positive integer $n$ can be written as a sum of one or more positive integers, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k. \tag{1.1}$$

When the order of integers $\lambda_i$ does not matter, this representation is known as an integer partition [1] and can be rewritten as

$$n = m_1 + 2m_2 + \cdots + nm_n,$$

where each positive integer $i$ appears $m_i$ times. If the order of integers $\lambda_i$ is important, then the representation (1.1) is known as a composition. For

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k,$$

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we have a descending composition. In the literature, partitions are often defined as descending compositions and this is also the convention used in this paper. We refer to $\lambda_1, \lambda_2, \ldots, \lambda_k$ as the parts of $\lambda$ and use the notation $\lambda \vdash n$ to denote a partition of $n$, i.e., a partition whose parts add up to $n$. We denote by $\ell(\lambda)$ the number of parts of $\lambda$, i.e.,

$$\ell(\lambda) = k \quad \text{or} \quad \ell(\lambda) = \sum_{j=1}^{n} m_j.$$

As usual, for a positive integer $n$, we denote by $p(n)$ the number of partitions of $n$ and we set $p(0) = 1$.

In 1936, Watson [24] computed tables of the number of partitions of $n$ into distinct parts $Q(n)$ and the number of partitions of $n$ into distinct odd parts $Q_{odd}(n)$ up to $n = 400$. He notes that his "computations were considerably simplified by the use of certain formulae of elliptic functions in conjunction with the existing table of values of $p(n)$, the number of unrestricted partitions of $n$, up to $n = 200$ which was constructed by MacMahon and published by Hardy and Ramanujan" [13] in 1918. Watson [24, p. 551] stated two identities whose developments lead to

$$Q(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{2}\right) \tag{1.2}$$

and

$$Q_{odd}(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{4}\right), \tag{1.3}$$

where $p(x) = 0$ when $x$ is not a nonnegative integer.

In 2016, the second author [17, Theorem 1] considered the identity (1.2) and obtained a method to compute the values of the partition function $p(n)$ that requires only the values of $p(k)$ with $k \leqslant n/2$, namely

$$p(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\infty} p(k) p\left(\frac{n - j(j+1)/2}{2} - k\right). \tag{1.4}$$

One year later, the identity (1.2) was used by the authors [6, Theorem 2.7] to prove the following parity result related to sums of partition numbers and squares in arithmetic progressions. For $n \geqslant 0$,

$$\sum_{16k+1 \text{ square}} p(n - k) \equiv 1 \pmod{2}$$

if and only if $48n + 1$ is a square.

Recently, Fu and Tang [10] generalized Vandervelde's bijection [23] and gave a combinatorial proof of the identity (1.2). A combinatorial proof of (1.3) can be found in [25] where the author uses abacus displays which were first introduced in [14]. We remark that [25] also refers to [16, Proposition 5.2] for a combinatorial proof of (1.2).

In this paper, motivated by these results, we investigate other identities of Watson type (1.2). To begin, we consider a recent paper [2] in which Andrews solved a problem of Beck and provided the following result: For all $n \geqslant 1$,

$$a_1(n) = b_1(n) = c_1(n),$$

where:

- $a_1(n)$ is the number of partitions of $n$ in which the set of even parts has only one element;
- $b_1(n)$ is the difference between the number of parts in all partitions of $n$ into odd parts and the number of parts in all partitions of $n$ into distinct parts;
- $c_1(n)$ is the number of partitions of $n$ in which exactly one part is repeated.

Shortly after that, inspired by Andrews's proof of this result, the second author [19] discovered and proved analytically an analogue of the identity (1.2) involving the number of parts in partitions.

**Theorem 1.1.** *For $n \geqslant 0$,*

$$b_1(n) = \sum_{k=0}^{\infty} S\left(\frac{n - k(k+1)/2}{2}\right),\tag{1.5}$$

*where $S(n)$ denotes the total number of parts in all partitions of n, with $S(x) = 0$ if x is not a positive integer.*

We remark that combinatorial proofs of $a_1(n) = b_1(n)$ and $c_1(n) = b_1(n)$ are given in [4] and, as a result of a generalization, in [26]. A combinatorial proof of a generalization of $a_1(n) = c_1(n)$ was initially given in [11]. Thus, a purely combinatorial proof of Theorem 1.1 follows from the combinatorial proof of either of the next two theorems which we present in Section 3.

**Theorem 1.2.** *Let $\alpha_j(n)$ denote the number of partitions of $n$ whose set of even parts consists of the single element $2j$ and let $S_j(n)$ be the number of parts equal to $j$ in all partitions of $n$. Then, for $n \geqslant 0$, we have*

$$\alpha_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n - k(k+1)/2}{2}\right).\tag{1.6}$$

**Theorem 1.3.** *Let $\gamma_j(n)$ denote the number of partitions of $n$ in which exactly one part is repeated and the repeated part is $j$. Then, for $n \geqslant 0$, we have*

$$\gamma_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n - k(k+1)/2}{2}\right).\tag{1.7}$$

Very recently, Andrews and the second author [3] proved that for all $n \geqslant 1$,

$$a_2(n) = (-1)^n b_2(n) = c_2(n),$$

where:

- $a_2(n)$ is the number of even parts in all partitions of $n$ into distinct parts;
- $b_2(n)$ is the difference between the number of partitions of $n$ into an odd number of parts in which the set of even parts has only one element and the number of partitions of $n$ into an even number of parts in which the set of even parts has only one element;
- $c_2(n)$ is the difference between the number of partitions of $n$ in which exactly one part is repeated and this part is odd and the number of partitions of $n$ in which exactly one part is repeated and this part is even.

Combinatorial proofs of $a_2(n) = c_2(n)$ and $(-1)^n b_2(n) = c_2(n)$ are given by the authors in [5]. We obtain a new analogue of the identity (1.2) which we prove both analytically and combinatorially in Section 4.

**Theorem 1.4.** *For $n \geqslant 0$,*

$$a_2(n) = \sum_{k=0}^{\infty} S_{o-e}\left(\frac{n - k(k+1)/2}{2}\right), \tag{1.8}$$

*where $S_{o-e}(n)$ denotes the difference between the number of odd parts and the number of even parts in all partitions of $n$, with $S_{o-e}(x) = 0$ if $x$ is not a positive integer.*

Let $S'(n)$ be the number of parts that appear at least once in a given partition of $n$, summed over all partitions of $n$, i.e., $S'(n)$ equals the number of different parts in all partitions of $n$. For example, $S'(5) = 12$ since the number of different parts in $(5), (4, 1)$, $(3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1)$ and $(1, 1, 1, 1, 1)$ is $1 + 2 + 2 + 2 + 2 + 2 + 1 = 12$. The following result in partition theory has been widely attributed to Richard Stanley, although it is a particular case of a more general result that had been established by Nathan Fine fifteen years earlier [12]: The number of parts equal to 1 in the partitions of $n$ is equal to $S'(n)$. Recently, the second author and Schmidt [20] provided a new identity for the number of parts equal to 1 in the partitions of $n$ involving a well-known object in multiplicative number theory: Euler's totient $\phi(n)$. We have the following analogue of the identity (1.2) which we prove both analytically and combinatorially in Section 5.

**Theorem 1.5.** *For $n \geqslant 0$,*

$$E(n) = \sum_{k=0}^{\infty} S'\left(\frac{n - k(k+1)/2}{2}\right),$$

*where $E(n)$ counts the partitions of $n$ with exactly one even part and $S'(x) = 0$ if $x$ is not a positive integer.*

Related to Theorem 1.5, we have the following result which we prove combinatorially in Section 6.

**Theorem 1.6.** *For $n \geqslant 0$,*

$$E(n) = \sum_{\lambda \in \mathcal{O}(n)} l_2(\lambda),$$

*where $\mathcal{O}(n)$ is the set of all integer partitions of $n$ into odd parts and*

$$l_2(\lambda) = \sum_{\substack{k=1 \\ m_k > 0}}^{n} \lfloor \log_2(m_k) \rfloor.$$

Let $S_2'(n)$ be the number of parts equal to 2 in all partitions of $n$ that do not contain 1 as a part. We have the following analogue of the identity (1.2) which we prove both analytically and combinatorially in Section 7.

**Theorem 1.7.** *For $n \geqslant 5$,*

$$Q_2(n-4) = \sum_{k=0}^{\infty} S_2' \left( \frac{n - k(k+1)/2}{2} \right),$$

*where $Q_2(n)$ is the number of partitions of $n$ into distinct parts, none being $2$ and $S_2'(x) = 0$ if $x$ is not a positive integer.*

## 2   Review of a combinatorial proof of (1.2)

The combinatorial proof of (1.2) is key to the combinatorial proofs of all our statements. In [10], Fu and Tang give a beautiful bijective proof of (1.2). In this section we reformulate their bijection in a way that is much shorter and easier to convey.

Recall that Dyson [9] defined the rank of a partition $\lambda$ by $r(\lambda) = \lambda_1 - \ell(\lambda)$. The BG-rank of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$, denoted by $r_{bg}(\lambda)$, is defined in [7] as the excess in the number of odd-indexed odd parts over the number of even-indexed odd parts of $\lambda$, i.e.,

$$r_{bg}(\lambda) = \sum_{j=1}^{\ell(\lambda)} (-1)^{j+1} \operatorname{par}(\lambda_j),$$

where $\operatorname{par}(m) = 1$ if $m$ is odd and $0$, otherwise.

Start with a partition $\lambda$ with distinct parts and consider the shifted Young diagram of $\lambda$, i.e., the Young diagram in which row $i$ is shifted $i$ boxes to the right, $i = 1, 2, \ldots, \ell(\lambda)$. Remove the first $\ell(\lambda)$ columns of the shifted diagram and denote the conjugate of the resulting partition by $\nu$. We have $\ell(\nu) = r(\lambda)$. Suppose $r_{bg}(\lambda) = j \in \mathbb{Z}$. Recall [8] that the 2-core of a partition $\lambda$ is the partition whose Young diagram is obtained from the Young diagram of $\lambda$ by repeatedly removing removing pairs of adjacent squares. At each step, the resulting diagram must be a valid Young diagram. Then the 2-core of $\lambda$ is the staircase partition of size $j(2j-1)$.

Let $a$ equal the height of the 2-core. It is equal to $2j - 1$ if $j > 0$ and to $-2j$ if $j \leq 0$. Let $b = \ell(\lambda) - a$. Define a partition $\mu$ via its Young diagram as follows.
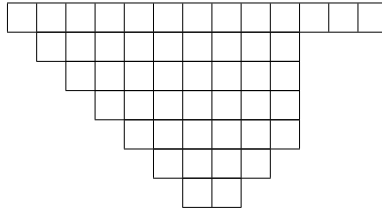
(i) If $b = 0$, all parts of $\nu$ have even multiplicity. Then $\mu$ is the partition obtained from $\nu$ by removing half the parts of each size.

(ii) If $b \neq 0$, set

$$d_0 = \begin{cases} \frac{b}{2} & \text{if } b \text{ is even} \\ a + \lceil \frac{b}{2} \rceil & \text{if } b \text{ is odd} \end{cases}$$

and define recursively $d_i = \nu_i - d_{i-1}$ for $i = 1, 2, \ldots, r(\lambda)$. To obtain the Young diagram of $\mu$, begin with a rectangle of size $\lceil \frac{b}{2} \rceil \times (a + \lceil \frac{b}{2} \rceil)$ (i.e., $\lceil \frac{b}{2} \rceil$ rows and $a + \lceil \frac{b}{2} \rceil$ columns). If $b$ is odd (respectively, even), for $i = 1, 2, \ldots$ append columns of length $d_{2i-1}$ (respectively, $d_{2(i-1)}$) to the right of the rectangle and rows of length $d_{2i}$ (respectively, $d_{2i-1}$) below the rectangle. In [10], it is shown that this is a bijection from the set of partitions with distinct parts and BG-rank $j$ to the set of partitions of $\frac{n - j(2j-1)}{2}$. Summing over all $j \in \mathbb{Z}$ gives (1.2).

**Example 2.1.** Let $\lambda = (13, 9, 8, 7, 6, 4, 2) \vdash 49$. We have $\lambda_1 = 13$ and $\ell(\lambda) = 7$. Then $r(\lambda) = 13 - 7 = 6$. Since the odd parts are the first, second and fourth parts, we have

$r_{bg}(\lambda) = -1$ and $a = 2$. Then, $b = \ell(\lambda) - 2 = 7 - 2 = 5$. The shifted Young diagram of $\lambda$ is given below.



After removing the first $\ell(\lambda) = 7$ columns and conjugating, we obtain the partition $\nu = (7, 6, 5, 1, 1, 1)$. Since $b = 5$ is odd, $d_0 = a + \lceil \frac{b}{2} \rceil = 5$. We calculate recursively

$$d_1 = \nu_1 - d_0 = 7 - 5 = 2,$$
$$d_2 = \nu_2 - d_1 = 6 - 2 = 4,$$
$$d_3 = \nu_3 - d_2 = 5 - 4 = 1,$$
$$d_4 = \nu_4 - d_3 = 1 - 1 = 0,$$
$$d_5 = \nu_5 - d_4 = 1 - 0 = 1,$$
$$d_6 = \nu_6 - d_5 = 1 - 1 = 0.$$

We start with a rectangle of size $\lceil \frac{b}{2} \rceil \times (a + \lceil \frac{b}{2} \rceil) = 3 \times 5$ and append columns of size $d_1, d_3$, and $d_5$ (i.e., columns of size $2, 1$, and $1$) to the right of the rectangle and rows of size $d_2, d_4$, and $d_6$ (i.e., rows of size $4, 0$, and $0$) below the rectangle to obtain the Young diagram of the partition $\mu = (8, 6, 5, 4) \vdash \frac{49 - (-1)(-2-1)}{2} = 23$.

## 3   Combinatorial proofs of Theorems 1.2 and 1.3

In this section we use the combinatorial proof of (1.2) reviewed in the previous section to derive combinatorial proofs of Theorems 1.2 and 1.3. Then, summing over $j \geqslant 1$ and using the combinatorial proofs of $b_1(n) = a_1(n)$ and $b_1(n) = c_1(n)$, we obtain two slightly different combinatorial proofs of Theorem 1.1. For the combinatorial proofs of $b_1(n) = a_1(n)$ and $b_1(n) = c_1(n)$, which are fairly straight forward, we refer the reader to [4] or [26]. We do not repeat the argument here.

First, we introduce some notation. For any partition $\lambda$ and any positive integer $j$ we denote by $m_j$ the multiplicity of $j$ in $\lambda$. We denote by $p(n, j, t)$ the number of partitions of $n$ such that $m_j \geqslant t$. Removing $t$ parts equal to $j$ from a partition of $n$ with $m_j \geqslant t$ gives a partition of $n - jt$. Conversely, adding $t$ parts equal to $j$ to a partition of $n - jt$ gives a partition of $n$ with $m_j \geqslant t$, Thus,

$$p(n, j, t) = p(n - jt).$$

As noted in the introduction, we denote by $S_j(n)$ the number of parts equal to $j$ in all partitions of $n$. Then

$$S(n) = \sum_{j \geqslant 1} S_j(n).$$

Let $\mathcal{A}(n)$ be the set of partitions of $n$ such that the set of even parts has exactly one element and let $\mathcal{C}(n)$ be the set of partitions of $n$ in which exactly one part is repeated.

*Proof of Theorem 1.2.* Recall that $\alpha_j(n)$ denotes the number of partitions in $\mathcal{A}(n)$ in which the even part is $2j$. Let $\alpha_j^{(t)}(n)$ be the number of partitions in $\mathcal{A}(n)$ with $m_{2j} = t$. The above argument using removing/adding $t$ parts equal to $2j$ shows that $\alpha_j^{(t)}(n) = Q(n - 2jt)$. Therefore,

$$\alpha_j(n) = \sum_{t \geqslant 1} \alpha_j^{(t)}(n) = \sum_{t \geqslant 1} Q(n - 2jt).$$

From (1.2), we have

$$Q(n - 2jt) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{2} - jt\right).$$

For any $n \geqslant 0$, to determine $S_j(n)$ we count, in order, the first appearance of $j$ in all partitions of $n$, then the second appearance of $j$ in all partitions of $n$, and so on. The number of the $t^{\text{th}}$ appearance of $j$ in all partitions of $n$ equals $p(n, j, t)$. Thus,

$$S_j(n) = \sum_{t \geqslant 1} p(n, j, t) = \sum_{t \geqslant 1} p(n - jt). \tag{3.1}$$

Then,

$$\alpha_j(n) = \sum_{t \geqslant 1} Q(n - 2jt) = \sum_{t \geqslant 1} \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{2} - jt\right)$$

and thus

$$\alpha_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n - k(k+1)/2}{2}\right). \qquad \square$$

Summing (1.6) for $j \geqslant 1$, we obtain

$$a_1(n) = \sum_{k=0}^{\infty} S\left(\frac{n - k(k+1)/2}{2}\right).$$

Since there are purely combinatorial proofs of (1.2) and $a_1(n) = b_1(n)$, this gives a combinatorial proof of Theorem 1.1.

*Proof of Theorem 1.3.* Recall that $\gamma_j(n)$ denotes the number of partitions in $\mathcal{C}(n)$ in which the repeated part is $j$ and, for $t \geqslant 1$ we denote by $\gamma_j^{(t)}(n)$ the number of partitions in $\mathcal{C}(n)$ such that $m_j = t$. Then, $\gamma_j^{(t)}(n)$ equals the number of partitions of $n - tj$ into distinct parts such that $j$ does not appear as a part. To any partition of $n - (2t + 1)j$ into distinct parts such that $j$ does not appear as a part, add a part equal to $j$ to obtain a partition of $n - 2tj$ into distinct parts such that $j$ appears as a part. Therefore,

$$\gamma_j^{(2t)}(n) + \gamma_j^{(2t+1)}(n) = Q(n - 2tj)$$

and

$$\gamma_j(n) = \sum_{t \geqslant 1} Q(n - 2tj).$$

Then, the proof of Theorem 1.2 gives a combinatorial argument for

$$\gamma_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n - k(k+1)/2}{2}\right).$$

□

Summing (1.7) over $j \geqslant 1$, we have

$$c_1(n) = \sum_{k=0}^{\infty} S\left(\frac{n - k(k+1)/2}{2}\right).$$

Using the combinatorial proof for $c_1(n) = b_1(n)$ in [4], this gives a second combinatorial proof of Theorem 1.1.

## 4   Proofs of Theorem 1.4

### 4.1   An analytic proof

We consider the following factorization for a special case of Lambert series [18]:

$$\sum_{n=1}^{\infty} \frac{q^n}{1 + q^n} = (q; q)_\infty \sum_{n=1}^{\infty} S_{o-e}(n)q^n.$$

According to [3], we have

$$\sum_{n=0}^{\infty} a_2(n)q^n = (-q; q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1 + q^{2n}}$$

$$= (-q; q)_\infty (q^2; q^2)_\infty \sum_{n=1}^{\infty} S_{o-e}(n)q^{2n}$$

$$= \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{n=1}^{\infty} S_{o-e}(n)q^{2n}.$$

Considering the theta identity [1, p. 23, Eq. (2.2.13)]

$$\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

the proof follows by equating the coefficients of $q^n$ in

$$\sum_{n=0}^{\infty} a_2(n)q^n = \left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)\left(\sum_{n=1}^{\infty} S_{o-e}(n)q^{2n}\right).$$

### 4.2   A combinatorial proof

Recall that [5] provides a combinatorial proof for $a_2(n) = c_2(n)$. Using the notation of Theorem 1.3, we have

$$c_2(n) = \sum_{j \geqslant 1}(\gamma_{2j-1}(n) - \gamma_{2j}(n))$$

and the proof of Theorem 1.3 provides a combinatorial argument for

$$c_2(n) = \sum_{j \geqslant 1} \sum_{k=0}^{\infty} \left( S_{2j-1}\left(\frac{n-k(k+1)/2}{2}\right) - S_{2j}\left(\frac{n-k(k+1)/2}{2}\right) \right)$$

$$= \sum_{k=0}^{\infty} S_{o-e}\left(\frac{n-k(k+1)/2}{2}\right).$$

Using the combinatorial proof for $c_2(n) = a_2(n)$ in [5], this gives a combinatorial proof of Theorem 1.4.

## 5 Proofs of Theorem 1.5

### 5.1 An analytic proof

We remark that the sequence $E(n)$ is known as sequence A038348 [21] and can be found in the On-Line Encyclopedia of Integer Sequence [22]. The generating function function for $E(n)$ is given by

$$\sum_{n=0}^{\infty} E(n)q^n = \frac{q^2}{1-q^2} \cdot \frac{1}{(q;q^2)_\infty}.$$

On the other hand, according to [20], the generating function for $S'(n)$ is given by

$$\sum_{n=0}^{\infty} S'(n)q^n = \frac{q}{1-q} \cdot \frac{1}{(q;q)_\infty}.$$

Thus we can write

$$\sum_{n=0}^{\infty} E(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \cdot \frac{q^2}{1-q^2} \cdot \frac{1}{(q^2;q^2)_\infty}$$

$$= \left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} S'(n)q^{2n} \right)$$

and the proof of the theorem follows by equating the coefficients of $q^n$.

### 5.2 A combinatorial proof

We first follow [11] to prove the following Euler type identity.

**Proposition 5.1.** *Let $n \geqslant 1$. Then, the number of partitions with exactly one even part equals the number of partitions in which exactly one part is repeated with multiplicity 2 or 3.*

Before we prove the proposition, we introduce some notation. Recall that we denote by $\mathcal{O}(n)$ the set of partitions of $n$ into odd parts. We denote by $\mathcal{D}(n)$ the set of partitions of $n$ into distinct parts. In Section 3 we defined $\mathcal{C}(n)$ to be the set of partitions of $n$ in which exactly one part is repeated. Let $\mathcal{T}(n)$ be the subset of $\mathcal{C}(n)$ consisting of partitions of $n$ in which the repeated part has multiplicity 2 and let $\mathcal{T}'(n)$ be the subset of $\mathcal{C}(n)$ consisting of partitions of $n$ in which the repeated part has multiplicity 3. Let $c_3(n) = |\mathcal{T}(n)|$ and $c_4(n) = |\mathcal{T}'(n)|$. Moreover, let $\mathcal{E}(n)$ be the set of partitions of $n$ with exactly one even part.

*Proof of Proposition 5.1.* Consider the following transformation

$$\psi \colon \mathcal{E}(n) \to \mathcal{T}(n) \cup \mathcal{T}'(n).$$

Let $\mu \in \mathcal{E}(n)$ and suppose the even part is $2^k m$ with $k \geqslant 1$ and $m$ odd. Denote by $\overline{\mu}$ the partition consisting of the single part $2^k m$ and by $\tilde{\mu}$ the partition consisting of the remaining parts of $\mu$. Thus $\tilde{\mu}$ is a partition into odd parts. Let $\overline{\lambda} = (2^{k-1}m, 2^{k-1}m)$ and $\tilde{\lambda}$ be the partition with distinct parts obtained from $\tilde{\mu}$ after applying Glaisher's bijection (i.e., after merging equal parts repeatedly). Define $\psi(\mu) = \overline{\lambda} \cup \tilde{\lambda}$, the partition obtained by listing the parts of $\overline{\lambda}$ and $\tilde{\lambda}$ in non-increasing order. Then, in $\psi(\mu)$, the part $2^{k-1}m$ is the only repeated part and its multiplicity is 2 or 3. Thus, $\psi(\mu) \in \mathcal{T}(n) \cup \mathcal{T}'(n)$.

Conversely, if $\lambda \in \mathcal{T}(n) \cup \mathcal{T}'(n)$ suppose the repeated part is $t$. Then the multiplicity of $t$ in $\lambda$ is 2 or 3. Let $\overline{\lambda} = (t, t)$ and $\tilde{\lambda}$ be the partition consisting of the remaining parts of $\lambda$ (one of which could be $t$). Let $\overline{\mu} = (2t)$, a partition consisting of a single even part, and $\tilde{\mu}$ be the partition obtained from $\tilde{\lambda}$ after applying the inverse of Glaisher's bijection (i.e., split even parts repeatedly until all parts are odd). Then, $\psi^{-1}(\lambda) = \overline{\mu} \cup \tilde{\mu}$ is a partition in $\mathcal{E}(n)$.

Thus, $\psi$ is a bijection and $E(n) = c_3(n) + c_4(n)$.                    □

Next we complete the proof of Theorem 1.5.

*Combinatorial Proof of Theorem 1.5.* Let $d_j(n)$ denote the number of partitions in $\mathcal{T}(n) \cup \mathcal{T}'(n)$ with $m_j > 1$. Then $m_j = 2$ or 3. We have $c_3(n) + c_4(n) = \sum_{j \geqslant 1} d_j(n)$.

From the proof of Theorem 1.3, we have

$$d_j(n) = Q(n - 2j) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{2} - j\right).$$

Recall that $p\left(\frac{n-k(k+1)/2)}{2} - j\right)$ counts the number of first appearances of $j$ in all partition of $\frac{n-k(k+1)/2}{2}$. Since $E(n) = c_3(n) + c_4(n)$, summing over $j \geqslant 1$, gives a combinatorial proof of the theorem when $S'(n)$ equals the number of different parts in all partitions of $n$.

On the other hand, from (3.1), we have that the number of parts equal to 1 in all partitions of $n$ is $S_1(n) = \sum_{t \geqslant 1} p(n - t)$. This gives the combinatorial proof of the theorem when $S'(n)$ is viewed as the number of parts equal to 1 in all partitions of $n$.                    □

## 6    Combinatorial proof of Theorem 1.6

Let $b_3(n)$ be the difference between the total number of parts in the partitions of $n$ into distinct parts and the total number of *different* parts in the partitions of $n$ into odd parts. Thus, $b_3(n)$ is the difference between the number of parts in all partitions in $\mathcal{D}(n)$ and the number of different parts in all partitions in $\mathcal{O}(n)$ (i.e., parts counted without multiplicity).

**Definition 6.1.** Given a partition $\lambda \in \mathcal{O}(n)$, suppose the multiplicity of $i$ in $\lambda$ is $m_i$. If $i$ appears in $\lambda$, we define the *binary order of magnitude* of the multiplicity of $i$ in $\lambda$, denoted $\mathrm{bomm}_\lambda(i)$, to be the number of digits in the binary representation of $m_i$.

Note that, if $m_i > 0$, then $\mathrm{bomm}_\lambda(i) = \lfloor \log_2(m_i) \rfloor + 1$.

**Example 6.2.** If $\lambda = (5, 3, 3, 3, 3, 3, 1) \vdash 21$, we have $m_3(\lambda) = 5$. Since the binary representation of 5 is 101, we have $\mathrm{bomm}_\lambda(3) = 3$.

Let $b_4(n)$ denote the difference between the number of parts in all partitions in $\mathcal{O}(n)$, each counted as many times as its bomm, and the number of parts in all partitions in $\mathcal{D}(n)$. Since the number of parts in all partitions in $\mathcal{D}(n)$ equals the number of 1 in all binary representations of all multiplicities in all partitions of $\mathcal{O}(n)$, it follows that $b_4(n)$ equals the number of 0 in all binary representations of all multiplicities in all partitions of $\mathcal{O}(n)$.

**Example 6.3.** Let $n = 7$. We have $\mathcal{D}(7) = \{(7), (6, 1), (5, 2), (4, 3), (4, 2, 1)\}$ and the number of parts in $\mathcal{D}(7)$ equals 10. Denote by $z_i(\lambda)$ the number of 0 in the binary representation of $m_i(\lambda)$. In Table 1 we list the partitions in $\mathcal{O}(n)$ with the relevant data (omitting the subscript $\lambda$).

Table 1: Partitions in $\mathcal{O}(7)$ and their multiplicity statistics.

| $\lambda$ | $m_i(\lambda)$ in binary | $\text{bomm}_\lambda(i)$ | $z_i(\lambda)$ |
|---|---|---|---|
| $(7)$ | $m_7 = 1$ | $\text{bomm}(7) = 1$ | $z_7 = 0$ |
| $(5, 1, 1)$ | $m_5 = 1, m_1 = 10$ | $\text{bomm}(5) = 1, \text{bomm}(1) = 2$ | $z_5 = 0, z_1 = 1$ |
| $(3, 3, 1)$ | $m_3 = 10, m_1 = 1$ | $\text{bomm}(3) = 2, \text{bomm}(1) = 1$ | $z_3 = 1, z_1 = 0$ |
| $(3, 1, 1, 1, 1)$ | $m_3 = 1, m_1 = 100$ | $\text{bomm}(3) = 1, \text{bomm}(1) = 3$ | $z_3 = 0, z_1 = 2$ |
| $(1, 1, 1, 1, 1, 1, 1)$ | $m_1 = 111$ | $\text{bomm}(1) = 3$ | $z_1 = 0$ |

Thus $b_4(7) = 1 + 1 + 2 + 2 + 1 + 1 + 3 + 3 - 10 = 4$, which equals the sum of $z$ in the right column of the table above.

As shown in [4] combinatorially, we have $c_3(n) = b_3(n)$ and $c_4(n) = b_4(n)$. Together with the combinatorial proof of Theorem 1.5, this gives a combinatorial argument for the identity

$$b_3(n) + b_4(n) = \sum_{k=0}^{\infty} S'\left(\frac{n - k(k+1)/2}{2}\right), \quad \forall n \geqslant 0. \tag{6.1}$$

It follows directly from the definition of $b_3(n)$ and $b_4(n)$ that $b_3(n) + b_4(n)$ equals the number of parts in all partitions in $\mathcal{O}(n)$, where each part $i$ is counted with multiplicity $\text{bomm}_\lambda(i) - 1 = \lfloor \log_2(m_i) \rfloor$ in each partition $\lambda$ in which it appears.

**Example 6.4.** The total number of distinct parts in all partitions in $\mathcal{O}(7)$ equals 8. Then $b_3(7) = 10 - 8 = 2$ and $b_3(7) + b_4(7) = 2 + 4 = 6$ which equals $0 + 0 + 1 + 1 + 0 + 0 + 2 + 2$, the number of parts in all partitions in $\mathcal{O}(7)$, where each part $i$ is counted with multiplicity $\text{bomm}_\lambda(i) - 1$ in each partition $\lambda$ in which it appears.

Therefore, we have a combinatorial proof of Theorem 1.6.

## 7 Proofs of Theorem 1.7

### 7.1 An analytic proof

The sequence $Q_2(n)$ is known as sequence A015744 [15] and can be found in the On-Line Encyclopedia of Integer Sequences [22]. Since $(-q; q)_\infty = \frac{1}{(q;q^2)_\infty}$, the generating function for $Q_2(n)$ can be written as

$$\sum_{n=0}^{\infty} Q_2(n)q^n = \frac{1}{1 + q^2} \cdot \frac{1}{(q; q^2)_\infty}.$$

On the other hand, according to [20], the generating function for $S_2'(n)$ is given by

$$\sum_{n=0}^{\infty} S_2'(n)q^n = \frac{q^2}{1-q^2} \cdot \frac{1}{(q^2;q)_\infty}.$$

We can write

$$\sum_{n=0}^{\infty} Q_2(n-4)q^n = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \cdot \frac{q^4}{1+q^2} \cdot \frac{1}{(q^2;q^2)_\infty}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \cdot \frac{q^4}{1-q^4} \cdot \frac{1}{(q^4;q^2)_\infty}$$

$$= \left( \sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left( \sum_{n=0}^{\infty} S_2'(n)q^{2n} \right)$$

and the proof follows by equating the coefficients of $q^n$.

### 7.2   A combinatorial proof

Let $Q_2'(n)$ denote the number of partitions of $n$ into distinct parts containing 2 as a part. If $\lambda \in \mathcal{D}(n)$ has 2 as a part, removing 2 we obtain a partition counted by $Q_2(n-2)$. Conversely, if $\mu \in \mathcal{D}(n-2)$ does not have 2 as a part, adding a part equal to 2 we obtain a partition counted by $Q_2'(n)$. Thus, $Q_2'(n) = Q_2(n-2)$. Since $Q(n) = Q_2(n) + Q_2'(n)$, it follows that $Q_2(n) = Q(n) - Q_2(n-2)$. Recursively, we have

$$Q_2(n) = \sum_{j \geqslant 0} (-1)^j Q(n-2j). \tag{7.1}$$

Here, $Q(x) = 0$ if $x$ is negative. We rewrite (7.1) as

$$Q_2(n-4) = \sum_{t \geqslant 1} Q(n-4t) - \sum_{t \geqslant 1} Q(n-2-4t).$$

From the proof of Theorem 1.2, we have

$$Q_2(n-4) = \alpha_2(n) - \alpha_2(n-2)$$

$$= \sum_{k=0}^{\infty} S_2 \left( \frac{n-k(k+1)/2}{2} \right) - \sum_{k=0}^{\infty} S_2 \left( \frac{n-k(k+1)/2}{2} - 1 \right).$$

If $\lambda \vdash m-1$, adding a part equal to 1, we obtain a partition $\mu$ of $m$ containing 1. The number of parts equal to 2 is the same in $\lambda$ and in $\mu$. Therefore, $S_2(m) - S_2(m-1) = S_2'(m)$. This completes the proof of the theorem.

## 8   Concluding remarks

We presented several Watson type identities of the same shape as identity (1.2)

$$Q(n) = \sum_{k=0}^{\infty} p \left( \frac{n-k(k+1)/2}{2} \right)$$

and provided both analytic and combinatorial proofs for our results. Since the identity above has the companion identity (1.3) given by

$$Q_{odd}(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{4}\right)$$

it would be interesting to find Watson type identities of this shape. Because there is a combinatorial proof for identity (1.3), there is hope that such new identities can be proved combinatorially.

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