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Compression ratio of Wiener index in 2-d rectangular and polygonal lattices

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Abstract

In this paper, we establish leading coefficient of Wiener index for open and closed 2-dimensional rectangular lattices, for various open and closed polygonal lattices, and for open and closed multidimensional cubes. These results enable us to establish compression ratio of Wiener index when number of rows and columns in the lattice tends to infinity.

Keywords: Graph theory, 2D rectangular and polygonal lattices, Wiener index, Compression ratio.

Math. Subj. Class.: 05C12, 92E10

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1 Introduction

Topological indices are very important in chemistry, since they can be used for modeling and prediction of many chemical properties. One of the most famous and the most researched indices is Wiener index. Topological indices are invariants defined on graphs representing various chemical compounds. For example, one such compound is graphene which is represented with hexagonal lattice graph. Also, nanotubes and nanotori received much attention recently, and they are represented with a graph which is rectangle shaped lattice with opposite sides identified.

Recent theoretical investigations point out that the minimization of distance-based graph invariants, namely the Wiener index W [7] and the topological efficiency index ρ recently introduced [3], provides the fast determination of the subsets of isomers with relative structural stability of a given chemical structure. This method has been applied to important classes of carbon hexagonal systems like fullerenes, graphene with nanocones and graphene. This elegant computational topological approach quickly sieves the most stable C_{66} cages among 4478 distinct isomers as reported in [8]. Moreover, the same method gives the correct numbers of NMR resonance peaks and relative intensities. The Wiener index has been computed for monodimensional infinite lattices to describe conductibilty features of conjugated polymers [1]. Present article reports about a relevant property - the compression factor [3] - of the topological invariants computed on infinite lattices. For different topological indices, there is quite much recent interest [4-6] in the ratio of the value of the index on open and closed lattices (i.e. nanotubes and nanotori). In this article, we investigate compression ratio of 2-dimensional and multidimensional rectangular lattices, and various 2-dimensional polygonal lattices.

The present paper is organized as follows. In the second section named 'Preliminaries', we introduce some basic notions and notation that will be used throughout the paper. In the third section, we establish the leading coefficient in Wiener index for open and closed 2-dimensional rectangular lattices, which leads us to compression ratio in asymptotic case. In the fourth section, we use the results from the second section to derive the same kind of result for hexagonal and similar lattices. In the fifth section, we establish the limit of compression ratio for d-dimensional rectangular cube. Finally, in the last section named 'Conclusion', we summarize the main results of this paper.

2 Preliminaries

In this paper, we consider only simple connected graphs. We will use the following notation: G for graph, $V(G)$ or just V for its set of vertices, $E(G)$ or just E for its set of edges. With N we will denote number of vertices in a graph. For two vertices $u, v \in V$, we define *distance* $d(u, v)$ of u and v as the length of shortest path connecting u and v. Given the notion of distance, Wiener index of a graph G is defined [9] as

$$W(G) = \sum_{u,v \in V} d(u, v).$$

Here, the pair of vertices u, v is unordered. In some literature, the summation goes over all ordered pairs (u, v) of vertices and then the sum needs to be multiplied by a half. Now, let us introduce some special kinds of graphs that will be of interest to us in this paper, namely open and closed lattice graphs. First, let $R_{n,k}$ be rectangular lattice containing $2n$ rows and $2kn$ columns of squares. Lattices $R_{3,1}$ and $R_{3,2}$ are shown in Figure 1.

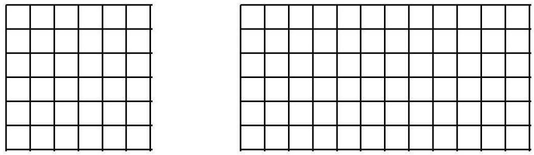

Figure 1: Lattices $R_{3,1}$ and $R_{3,2}$.

Let us denote vertices of $R_{n,k}$ with integer coordinates (i,j) for $i = 0, \ldots, 2nk$ and $j = 0, \ldots, 2n$ as if the lattice was placed in first quadrant of Cartesian coordinate system. Therefore

$$V(R_{n,k}) = \{v_{i,j} : i = 0, \ldots, 2nk \text{ and } j = 0, \ldots, 2n\}.$$

Open lattice is a graph ${}^O R_{n,k}$ obtained from $R_{n,k}$ by deleting vertices $v_{0,j}$ and $v_{i,0}$. Closed lattice is a graph ${}^C R_{n,k}$ obtained from $R_{n,k}$ by identifying vertices $v_{0,j}$ and $v_{2nk,j}$, and also vertices $v_{i,0}$ and $v_{i,2n}$. Therefore, open and closed lattice graphs have the same number of vertices which is

$$\left|V({}^O R_{n,k})\right| = \left|V({}^C R_{n,k})\right| = 4n^2 k.$$

Further, for a fixed integer k let us consider polygons P_k with $4k+2$ vertices. Let $L_{n,k}$ be rectangle shaped lattice consisting of $2n$ rows of polygons P_k, with rows containing n and $n-1$ polygons alternatively, such that two neighboring polygons from one row share exactly two vertices, while two neighboring polygons from different rows share exactly $k+1$ vertices. For $k=1$, lattice $L_{n,k}$ is actually hexagonal lattice with $2n$ rows and n columns of hexagons. Note that lattice $L_{n,k}$ can be considered as subgraph of rectangular lattice $R_{n,k}$ assigning to polygons the size of $2k$ square cells. Therefore, we can use the same vertex notation in $L_{n,k}$ as in $R_{n,k}$. Lattices $L_{3,1}$ and $L_{3,2}$ for $k=1$ and $k=2$ are shown in Figure 2. Again, open and closed lattices ${}^O L_{n,k}$ and ${}^C L_{n,k}$ can be obtained as for rectangular lattice. Finally, a d-dimensional rectangular cube R_n^d is defined in a following manner: set of vertices $V(R_n^d)$ is defined with

$$V(R_n^d) = \{(x_1, \ldots, x_d) : x_i \in \{0, 1, \ldots, n\}\},$$

and two vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are connected with an edge if and only if there is a coordinate j such that $x_i = y_i$ for every $i \in \{1, \ldots, d\} \setminus \{j\}$, and for j-th coordinate holds $|y_j - x_j| = 1$. Open lattice ${}^O R_n^d$ is a graph obtained from R_n^d by deleting all vertices that have at least one zero coordinate. Closed lattice ${}^C R_n^d$ is a graph obtained from R_n^d by identifying every pair of vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ such that $x_i = y_i$ for every $i \in \{1, \ldots, d\} \setminus \{j\}$ and for j-th coordinate holds $x_j = 0$ and $y_j = n$. Obviously, the number of vertices in ${}^O R_n^d$ and ${}^C R_n^d$ is the same and it equals n^d.

Now, if ${}^O G$ is an open lattice graph and ${}^C G$ its closed version, compression ratio of G is defined [3] as

$$cr(G) = \frac{W({}^C G)}{W({}^O G)}.$$

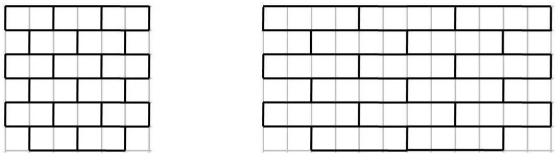

Figure 2: Lattices $L_{3,1}$ and $L_{3,2}$.

Obviously, Wiener index depends on the size of the lattice i.e. on n and k, and therefore compression ratio depends on them too. Our goal is to establish the limit of compression ratio for $R_{n,k}$, $L_{n,k}$ and R_n^d when n tends to infinity for a fixed k.

3 Compression ratio of $R_{n,k}$

For $k = 1$ (i.e. for square shaped rectangular lattices), the result $cr(R_{n,k}) = \frac{3}{4}$ was already obtained in reference [4]. The result $\frac{3}{4}$ for bidimensional square is the same as the result for monodimensional lattices (polymer chains) obtained in [3]. We will here derive the same result for $k > 1$, i.e. for some rectangle shaped lattices. Let k be fixed integer. We have following theorems.

Theorem 3.1. *For the lattice graph ${}^O R_{n,k}$, Wiener index $W({}^O R_{n,k})$ is a polynomial in n of degree 5 whose leading coefficient equals*

$$\frac{16}{3}k^2\left(1+k\right).$$

Proof. For vertices $v_{i,j}$ and $v_{p,q}$ of ${}^O R_{n,k}$ holds

$$d\left(v_{i,j}, v_{p,q}\right) = |p-i| + |q-j|.$$

Obviously,

$$W({}^O R_{n,k}) = \sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn}\sum_{q=1}^{2n} d(v_{i,j}, v_{p,q}) - \sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{q=j}^{2n} d(v_{i,j}, v_{i,q}).$$

The second (subtracted) sum does not influence leading term in n, therefore we can neglect it. To avoid absolute value, we can rewrite the first sum as

$$W({}^O R_{n,k}) \approx 2\sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn}\sum_{q=j}^{2n} d(v_{i,j}, v_{p,q}) - \sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn} d(v_{i,j}, v_{p,j}).$$

Again, the second (subtracted) sum does not influence leading term in n, therefore we can neglect it. Now we have

$$W({}^O R_{n,k}) \approx 2\sum_{i=1}^{2kn}\sum_{j=1}^{2n}\sum_{p=i}^{2kn}\sum_{q=j}^{2n} (p-i+q-j)$$

and the result then follows by easy calculation. $\square$

Theorem 3.2. *For the lattice graph ${}^{C}R_{n,k}$, Wiener index $W({}^{C}R_{n,k})$ is a polynomial in n of degree 5 whose leading coefficient equals*

$$4k^2\,(1+k)\,.$$

Proof. Obviously, all vertices in ${}^{C}R_{n,k}$ have the same sums of distances to all other vertices. Therefore, to obtain $W({}^{C}R_{n,k})$ it is enough to calculate distances from one vertex ($v_{1,1}$ is easiest for calculation) to all other vertices. Since ${}^{C}R_{n,k}$ is a torus, to do that we will calculate the sum of distances from $v_{1,1}$ to $v_{p,q}$ where $1 \leq p \leq kn$ and $1 \leq q \leq n$ and multiply it by 4. Now, the obtained number should be multiplied by number $2n \cdot 2kn$ of vertices in the lattice, and then divided by 2 since each distance was counted twice. Therefore we have

$$W({}^{C}R_{n,k}) \approx \frac{1}{2} \cdot 2n \cdot 2kn \cdot 4 \cdot \sum_{p=1}^{kn} \sum_{q=1}^{n} (p-1+q-1)\,,$$

and the result now follows by direct calculation.

□

Corollary 3.3. *Holds*

$$\lim_{n\to\infty} cr(R_{n,k}) = \frac{3}{4}.$$

4 Compression ratio of $L_{n,k}$

This section is devoted to the exact determination of the compression factors for the 2-dimensional polygonal lattice $L_{n,k}$. We start from a numerical example devoted to the case of the graphene lattice. Figure 3 shows the rectangular $L_{3,1}$ portion of this hexagonal system.

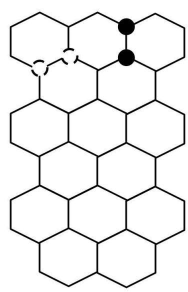

Figure 3: View of $L_{3,1}$ hexagonal lattice with bold vertices common to neighboring polygons along one row, whereas the dotted ones are shared by two neighboring polygons from two different rows.

The numerical determination of invariants $W({}^{O}G)$ and $W({}^{C}G)$ for this infinite graph is based on the results summarized in Table 1 where, for an increasing number of vertices N, values of both descriptors are listed. The exact polynomial forms are given in Table 1 for ${}^{O}W$ and ${}^{C}W$ with leading terms ${}^{O}W \approx \frac{2N^{\frac{5}{2}}}{5}$ and ${}^{C}W \approx \frac{7N^{\frac{5}{2}}}{24}$ respectively, producing

N	${}^{O}\mathbf{W} = \frac{1}{15}\left(6N^{\frac{5}{2}} - 5N^{\frac{3}{2}} - N^{\frac{1}{2}}\right)$	${}^{C}\mathbf{W} = \frac{1}{24}\left(7N^{\frac{5}{2}} - 4N^{\frac{3}{2}}\right)$
36	3 038	2 232
100	39 666	29 000
196	214 214	156 408
324	753 882	550 152
484	2 057 902	1 501 368
676	4 746 690	3 462 472
900	9 710 998	7 083 000

Table 1: Exact polynomial forms for the Wiener index of the open (${}^{O}W$) and closed (${}^{C}W$) rectangular graphene lattices $L_{n,1}$ with N vertices.

the value $cr(L_{n,1}) = \frac{35}{48}$ for the compression factor of rectangular graphene. More details about the numerical determination of various topological descriptors of the graphene rectangular lattices are given in [2].

Now, we want to establish the compression ratio of $L_{n,k}$. We will use the fact that $L_{n,k}$ can be considered as the subgraph of $R_{n,k}$. Therefore, distances between vertices in $L_{n,k}$ for some pairs of vertices are equal as in $R_{n,k}$, while for some other pairs of vertices are greater than in $R_{n,k}$. We will establish for which pairs of vertices the distance is greater in $L_{n,k}$ than in $R_{n,k}$, and how much greater. We will not establish the exact value of Wiener index for ${}^{O}L_{n,k}$ or ${}^{C}L_{n,k}$ as that would be tedious for all the possible cases, but we will neglect some quantities which do not influence the leading coefficient.

Theorem 4.1. *For the lattice graph* ${}^{O}L_{n,k}$*, Wiener index* $W({}^{O}L_{n,k})$ *is a polynomial in* n *of degree* 5 *whose leading coefficient equals*

$$\frac{16}{15}k^2\left(7k+5\right).$$

Proof. Since $L_{n,k}$ is a subgraph of $R_{n,k}$ that means distances in $L_{n,k}$ are equal or greater than in $R_{n,k}$. Therefore

$$W({}^{O}L_{n,k}) = W({}^{O}R_{n,k}) + \Delta.$$

If we establish leading coefficient in Δ, then by combining that result with Theorem 3.1 we obtain desired result. Therefore, we are interesting in establishing Δ, i.e. for which pairs of vertices the distance in $L_{n,k}$ is greater than in $R_{n,k}$ and how much greater. For a vertex $v_{i,j}$ lattice $L_{n,k}$ can be divided into four areas as illustrated with Figure 4. Vertices in areas left and right to $v_{i,j}$ have the same distance to $v_{i,j}$ in $L_{n,k}$ as in $R_{n,k}$, while the vertices in areas up and down to $v_{i,j}$ have the greater distance to $v_{i,j}$ in $L_{n,k}$ than in $R_{n,k}$. For easier calculation, we will approximate zig-zag lines that divide $L_{n,k}$ into areas with lines

$$q = q_1(p) = j - \frac{1}{k}(p-i),$$
$$q = q_2(p) = j + \frac{1}{k}(p-i).$$

as also illustrated in Figure 4, and we denote upper and lower areas with $L^{A}_{n,k}$ and left and right areas with $L^{B}_{n,k}$. With such an approximation, we make an error in some vertices near

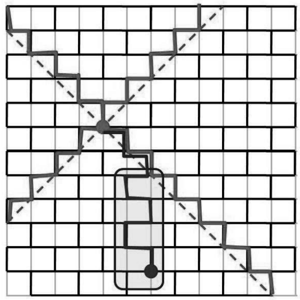

Figure 4: Division of lattice $L_{n,k}$ into areas for a vertex $v_{i,j}$.

the lines, but since number of such vertices is linear in n, that error does not influence the leading term of Δ and we can neglect it. Now that we established the pairs of vertices for which the distance is greater, we want to establish how much greater. Obviously, the trouble is if we have to go vertically, since some vertical edges are missing now. Therefore, we go vertically as little as we can (the shortest path is illustrated in Figure 4), and for each vertical step the path is k edges longer in $L_{n,k}$ than in $R_{n,k}$. Now, to calculate all these exactly, we have to divide into cases, regarding the position of $v_{i,j}$ (since then lines $q_1(p)$ and $q_2(p)$ intersect boundaries in different sides, which influences calculation). The division into cases is illustrated with Figure 5. There are four areas, but upper and lower

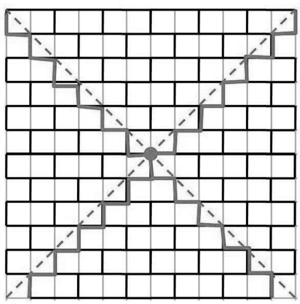

Figure 5: Division into cases with respect to the position of $v_{i,j}$ in the lattice.

are equal up to symmetry, and also left and right. Again, we will approximate division with lines

$$j = j_1(i) = n - \frac{1}{k}(i - kn),$$
$$j = j_2(i) = n + \frac{1}{k}(i - kn)$$

and denote upper and lower areas with $L_{n,k}^{(1)}$ and left and right areas with $L_{n,k}^{(2)}$. This approximation again produces an error in calculating $W(L_{n,k})$ but not great enough to influence the leading term, therefore we can again neglect it. Before we proceed note that lines we introduced can also be expressed as $i = i_1(j)$, $i = i_2(j)$, $p = p_1(q)$ and $p = p_2(q)$. Now we distinguish two cases.

CASE I: Let $v_{i,j} \in L^{(1)}_{n,k}$. For an arbitrary vertex $v_{p,q}$, we are interested in establishing the difference in $d(v_{i,j}, v_{p,q})$ between $L_{n,k}$ and $R_{n,k}$. The difference is greater than zero only if $v_{p,q} \in L^A_{n,k}$. To calculate the difference Δ_1 that occurs for pairs of vertices in this case, we have to divide $L^A_{n,k}$ into 4 subareas A_1, A_2, A_3, A_4 as illustrated in Figure 6. Now

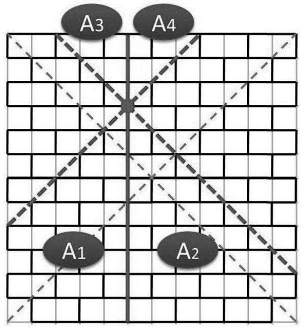

Figure 6: The division of the area $L^A_{n,k}$ into subareas in the case $v_{i,j} \in L^{(1)}_{n,k}$.

we have

$$\Delta(A_1) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=1}^{i}\sum_{q=1}^{q_2(p)} (q_2(p) - q) \cdot k,$$

$$\Delta(A_2) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=i}^{2kn}\sum_{q=1}^{q_1(p)} (q_1(p) - q) \cdot k,$$

$$\Delta(A_3) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=p_1(2n)}^{i}\sum_{q=q_1(p)}^{2n} (q - q_1(p)) \cdot k,$$

$$\Delta(A_4) = 2\sum_{j=n}^{2n}\sum_{i=i_1(j)}^{i_2(j)}\sum_{p=i}^{p_2(2n)}\sum_{q=q_2(p)}^{2n} (q - q_2(p)) \cdot k.$$

Therefore $\Delta_1 = \Delta(A_1)+\Delta(A_2)+\Delta(A_3)+\Delta(A_4)$ and by direct calculation we establish that Δ_1 is a polynomial in n of degree 5 with leading coefficient being $\frac{14}{5}k^3$.

CASE II: Let $v_{i,j} \in L^{(2)}_{n,k}$. Again, the difference $d(v_{i,j}, v_{p,q})$ is greater than zero only if $v_{p,q} \in L^A_{n,k}$. Again, we divide $L^A_{n,k}$ into four areas $A_1, \ldots, A_4$ as shown in Figure 7. Now, we calculate

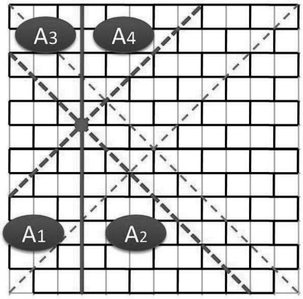

Figure 7: The division of the area $L_{n,k}^{A}$ into subareas in the case $v_{i,j} \in L_{n,k}^{(2)}$.

$$\Delta(A_1) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=1}^{i}\sum_{q=1}^{q_2(p)}(q_2(p)-q)\cdot k,$$

$$\Delta(A_2) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=i}^{p_1(1)}\sum_{q=1}^{q_1(p)}(q_1(p)-q)\cdot k,$$

$$\Delta(A_3) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=1}^{i}\sum_{q=q_1(p)}^{2n}(q-q_1(p))\cdot k,$$

$$\Delta(A_4) = 2\sum_{i=1}^{kn}\sum_{j=j_2(i)}^{j_1(i)}\sum_{p=i}^{p_2(2n)}\sum_{q=q_2(p)}^{2n}(q-q_2(p))\cdot k.$$

Now $\Delta_2 = \Delta(A_1)+\Delta(A_2)+\Delta(A_3)+\Delta(A_4)$ and by direct calculation we establish that Δ_2 is a polynomial in n of degree 5 with leading coefficient being $\frac{22}{15}k^3$.
Therefore, we conclude that Δ is a polynomial in n of degree 5 with leading coefficient being equal to

$$\frac{1}{2}\left(\frac{14}{5}k^3+\frac{22}{15}k^3\right)=\frac{32}{15}k^3.$$

Now the result follows from this and Theorem 3.1.

□

Theorem 4.2. *For the lattice graph* ${}^{C}L_{n,k}$, *Wiener index* $W({}^{C}L_{n,k})$ *is a polynomial in* n *of degree* 5 *whose leading coefficient equals*

$$\frac{4}{3}k^2\left(4k+3\right).$$

Proof. Let us introduce the same vertex notation in ${}^{C}L_{n,k}$ as in ${}^{C}R_{n,k}$ (we can do that as ${}^{C}L_{n,k}$ is a subgraph of ${}^{C}R_{n,k}$). Since, ${}^{C}L_{n,k}$ is a subgraph of ${}^{C}R_{n,k}$ we have

$$W\left({}^{C}L_{n,k}\right)=W\left({}^{C}R_{n,k}\right)+\Delta.$$

Therefore, if we establish Δ, the result will follow from Theorem 3.2. Since all vertices in ${}^{C}L_{n,k}$ are equivalent in the sense that they have the same distances to all other vertices, it is enough to calculate difference in distances for one vertex, and then multiply it by number of vertices, and divide by two since each difference is thus calculated twice. It is easiest if we calculate for $v_{1,1}$. Since ${}^{C}L_{n,k}$ is a torus, we will calculate the difference in distance from $v_{1,1}$ to $v_{p,q}$ where $1 \leq p \leq kn$ and $1 \leq q \leq n$ and multiply it by 4. In that area difference in distances is greater than 0 only if $1 \leq p \leq kn$ and $q_1(p) \leq q \leq n$ where

$$q_1(p) = \frac{1}{k} \cdot p.$$

Therefore,

$$\Delta \approx \frac{1}{2} \cdot 2kn \cdot 2n \cdot 4 \cdot \sum_{p=1}^{kn} \sum_{q=q_1(p)}^{n} (q - q_1(p)).$$

By direct calculation, we obtain that Δ is a polynomial in n of degree 5 with leading coefficient being

$$\frac{4}{3}k^3.$$

Now the result follows from Theorem 3.2.

$\square$

Corollary 4.3. *Holds*

$$\lim_{n\to\infty} cr(L_{n,k}) = \frac{5\,(4k+3)}{4\,(7k+5)}.$$

Now, we can derive from this result compression ratio for some specific lattices. For example, lattice $L_{n,1}$ is hexagonal lattice with $2n$ rows and n columns of hexagons. Therefore, from Corollary 4.3 follows that compression factor for such lattice equals

$$\lim_{n\to\infty} cr(L_{n,1}) = \frac{5\,(4+3)}{4\,(7+5)} = \frac{35}{48} = 0.729\,17$$

confirming the result numerically derived in [2]. For $k = 2$, lattice $L_{n,k}$ is a lattice consisting of 10-gons, with $2n$ rows and n columns of rectangular unit cells with size $2k = 4$ squares as in Figure 2. From Corollary 4.3 follows that compression factor for such lattice equals

$$\lim_{n\to\infty} cr(L_{n,2}) = \frac{5\,(4\cdot 2+3)}{4\,(7\cdot 2+5)} = \frac{55}{76} = 0.723\,68.$$

Corollary 4.4. *Holds*

$$\lim_{n\to\infty} cr(L_{n,k}) = \lim_{n\to\infty} cr(R_{n,k}) - \frac{k}{4(7k+5)}.$$

Invariant $\lim_{n\to\infty} cr(L_{n,k})$ *reaches its maximum value* $\lim_{n\to\infty} cr(L_{n,1}) = \frac{35}{48}$, *whereas for large* k *the limit* $\lim_{n\to\infty} cr(L_{n,k})$ *constantly decreases toward its lower limit* $\frac{5}{7}$.

5 Compression ratio of R_n^d

In this section, we will establish the limit of compression ratio of d-dimensional rectangular cube R_n^d when n tends to infinity, and we will show that it does not depend on dimension d.

Theorem 5.1. *Holds*

$$\lim_{n\to\infty} cr\left(R_n^d\right) = \frac{3}{4}.$$

Proof. Let us denote $[n] = \{1, 2, \ldots, n\}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{W\left({}^O R_n^d\right)}{n^{2d+1}} &= \lim_{n\to\infty} \frac{1}{n^{2d+1}} \left(\sum_{(x_1,\ldots x_d)\in[n]^d} \sum_{(y_1,\ldots y_d)\in[n]^d} \sum_{1\le i\le d} |y_i - x_i| \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{2d+1}} \left(d \cdot n^{2d-2} \sum_{1\le i\le j\le n} (j-i) \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{3}} \left(d \cdot \sum_{1\le i\le j\le n} (j-i) \right) = \\
&= \frac{d}{6.}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lim_{n\to\infty} \frac{W\left({}^C R_n^d\right)}{n^{2d+1}} &= \lim_{n\to\infty} \frac{1}{n^{2d+1}} \left(\frac{1}{2} \cdot n^d \cdot 2^d \cdot \sum_{(x_1,\ldots,x_d)\in[n/2]^d} \sum_{1\le i\le d} (x_i - 1) \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{d+1}} \left(2^{d-1} \cdot d \cdot \left(\frac{n}{2}\right)^{d-1} \sum_{1\le i\le \frac{n}{2}} (i-1) \right) = \\
&= \lim_{n\to\infty} \frac{1}{n^{2}} \left(d \cdot \sum_{1\le i\le \frac{n}{2}} (i-1) \right) = \\
&= \frac{d}{8}.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} cr(R_n^d) = \lim_{n\to\infty} \frac{W\left({}^C R_n^d\right)}{W\left(R_n^d\right)} = \lim_{n\to\infty} \frac{\frac{W\left({}^C R_n^d\right)}{n^{2d+1}}}{\frac{W\left(R_n^d\right)}{n^{2d+1}}} = \frac{\frac{d}{8}}{\frac{d}{6}} = \frac{3}{4}.$$

□

6 Conclusion

In this paper, we studied compression ratio for open and closed 2-dimensional rectangular lattices $R_{n,k}$ and 2-dimensional polygonal lattices $L_{n,k}$. For that purpose, we established that leading coefficient in $W({}^O R_{n,k})$ equals $\frac{16}{3} k^2 (1+k)$ (Theorem 3.1), while in $W(^C R_{n,k})$ equals $4k^2\,(1+k)$ (Theorem 3.2), which yields $\lim_{n\to\infty} cr(R_{n,k}) = \frac{3}{4}$ (Corollary 3.3). Also, we established that leading coefficient in $W(^O L_{n,k})$ equals $\frac{16}{15}k^2$ $(7k+5)$ (Theorem 4.1), while leading coefficient in $W(^C L_{n,k})$ equals $\frac{4}{3}k^2\,(4k+3)$ (Theorem 4.2), which yields $\lim_{n\to\infty} cr(L_{n,k}) = \frac{5(4k+3)}{4(7k+5)}$ (Corollary 4.3). Lattice $L_{n,1}$ is hexagonal lattice with $2n$ rows and n columns of hexagons, for which therefore holds $\lim_{n\to\infty} cr(L_{n,1}) = \frac{35}{48}$. Lattice $L_{n,2}$ is 10-gonal lattice with $2n$ rows and n columns, for which therefore $\lim_{n\to\infty} cr(L_{n,2}) = \frac{5(8+3)}{4(14+5)} = \frac{55}{76}$. Finally, we established that for d-dimensional rectangular cube holds $\lim_{n\to\infty} cr\left(R_n^d\right) = \frac{3}{4}$ (Theorem 5.1).

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