

The pairing strategies of the 9-in-a-row game

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Abstract

One of the most useful strategies for proving Breaker’s win in Maker-Breaker Positional Games is to find a pairing strategy. In some cases there are no pairing strategies at all, in some cases there are unique or almost unique strategies. For the k -in-a-row game, the case $k = 9$ is the smallest (sharp) for which there exists a Breaker winning pairing (paving) strategy. One pairing strategy for this game was given by Hales and Jewett.

In this paper we show that there are other winning pairings for the 9-in-a-row game, all have a very symmetric torus structure. While describing these symmetries we prove that there are only a finite number of non-isomorphic pairings for the game (around 200 thousand), which can be also listed up by a computer program. In addition, we prove that there are no “irregular”, non-symmetric pairings. At the end of the paper we also show a pairing strategy for a variant of the 3-dimensional k -in-a-row game.

Keywords: Positional games, k -in-a-row game, pairing strategies, symmetries.

Math. Subj. Class.: 05C65, 05C15

1 Introduction

5-in-a-row is one of the most well known positional games, and its study inspired several deep results in this field. For a very thorough introduction of these, see Beck [2, 3]. In the classical version two players take the squares of a gridpaper (integer lattice), alternately, and the first who achieves five in a row, i.e. five consecutive squares in a vertical, horizontal or diagonal direction, wins the game. John Nash [4] invented the “strategy stealing” argument showing that in these type of games the first player either wins the game or it is a draw. It explains the notion of the so-called *Maker-Breaker* (M-B) version of a game; in these, Maker wins by achieving the original goal, while Breaker wins by preventing Maker

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to doing so. There is a connection between the normal (Maker-Maker) and the M-B versions of a game: If the first player wins the normal game, she wins the M-B one, as well. If Breaker wins the M-B game, then the second player can draw the normal game. However, the reverse statements are not true, see the Tic-Tac-Toe game. In summary, the M-B version is easier for Maker because she does not need to act Breaker's moves.

Allis et al. [1] solved the 5-in-a-row game for the 19×19 and 15×15 boards: the first player wins. However, the case of infinite board is still open (in the normal version). It is natural to ask then what happens in the k -in-a-row game, where the winning condition is to get k consecutive squares in a row. The first result in that direction is due to C. Shannon and H. Pollak [4] who showed that Breaker wins the 9-in-a-row. Later A. Hales and R. Jewett even gave a winning pairing strategy for Breaker. A. Brouwer, under the pseudonym T. G. L. Zetters published in [11] (as a solution to a problem by Guy and Selfridge [6]) that Breaker wins the 8-in-a-row on the infinite board. The cases $k = 6, 7$ are still open, although it is widely believed that those are both draws. (Of course for $k \leq 4$ Maker wins easily.) On the other hand, it was shown that there are no pairing strategies for $k \leq 8$, see [5, 10].

The concept of pairing strategies were useful for other games, the best known are the Harray games. Here Maker's goal is to get a given polyomino on the infinite board; most cases solved by A. Blass, see [4]. The notorious case of "Snaky" is still unsolved, although there are partial results for it. Csernenszky et al. [5] proved a relative existence theorem of pairing strategies for the Snaky: if a pairing is good for the Snaky, it is good for the polyomino P_5 that consists of five consecutive squares vertically or horizontally (but not diagonally). They also managed to give *all* possible pairings for P_5 , there are two of those and those are not appropriate for Snaky.

As we mentioned, there is a Hales-Jewett winning pairing strategy for Breaker in the 9-in-a-row, see [3, 4] and also in Figure 1. The easiest way to see that the Hales-Jewett pairing blocks all lines of the board is that this pairing is an extension of a *domino pairing* from the 8×8 torus to the whole board. The torus lines consist of not only the rows and columns, but all diagonals, continuing on the opposite side when reaching the border of the 8×8 board.

Since no one has published different pairings for the 9-in-a-row,¹ the highly symmetric structure of the Hales-Jewett pairing, and the other examples of uniqueness or quasi uniqueness of pairings in similar problems; it is natural to think this is the only possible solution.

It turned out that this belief is very far from reality, as we found lots of new ones and will exhibit a few in the following sections. Another belief was that all pairings must be *torical* extensions of their 8×8 section. Somehow surprisingly, this belief is not true either; there are lot of solutions which are connected to the 16×16 torus, but are *not* extensions of the pairings of a 8×8 torus.

In the next sections, we define pairings precisely and give some conditions for their existence and structure. We will show that *all* solutions are either the extension of the pairings of a 8×8 torus (there are 194 543 non-isomorphic ones) or some combinations of those resulting in 16×16 toric solutions.

At the end of the paper, we prove a special case of the conjecture of Kruczek and Sundberg [8] about the existence of pairings in higher dimensions.

¹According to [4], Selfridge also produced a Hales-Jewett pairing, but that pairing is either not different from the known Hales-Jewett pairing or left unpublished.

Since in our paper k -in-a-row type games play an important role, we define \mathcal{H}_k , the hypergraph of the k -in-a-row games.

Definition 1.1. The vertices of the **k -in-a-row hypergraph** \mathcal{H}_k are the squares of the infinite (chess)board, i.e., the infinite square grid. The edges of the hypergraph \mathcal{H}_k are the k -element sets of consecutive squares in a row horizontally, vertically or diagonally. We refer to the whole infinite rows as *lines*.

2 Pairing strategies

Given a hypergraph $\mathcal{H} = (V, E)$, where $V = V(\mathcal{H})$ and $E = E(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}) = \{S : S \subseteq V\}$ are the set of vertices and edges, respectively. A bijection $\rho: X \rightarrow Y$, where $X, Y \subset V(\mathcal{H})$ and $X \cap Y = \emptyset$ is a **pairing** on the hypergraph \mathcal{H} . An $(x, \rho(x))$ pair **blocks** an $A \in E(\mathcal{H})$ edge, if A contains both elements of the pair. If the pairs of ρ block all edges, we say that ρ is a **good pairing** of \mathcal{H} .

Pairings are one way to show that Breaker has a winning strategy in hypergraph games. A good pairing ρ for a hypergraph \mathcal{H} can be turned to a winning strategy for Breaker in the M-B game on \mathcal{H} : following ρ on \mathcal{H} in a M-B game, for every $x \in X \cup Y$ element chosen by Maker, Breaker chooses $\rho(x)$ or in case of $x \in Y$ vice versa (if $x \notin X \cup Y$, then Breaker can choose an arbitrary vertex). Hence Breaker can block all edges and wins the game. Since our main topic is the 9-in-a-row game, this will be the first illustration to pairings and pairing strategies.

The following result is due to Hales and Jewett [4, 7]:

Theorem 2.1. *Breaker wins the 9-in-a-row M-B game by a pairing strategy, i.e., there exists a good pairing for the 9-in-a-row.*

Proof. Figure 1 is an extension of a pairing of an 8×8 torus (framed), where the pairs have a periodicity 8 in every line. Hence, the pairing blocks all 9-in-a-row edges. \square

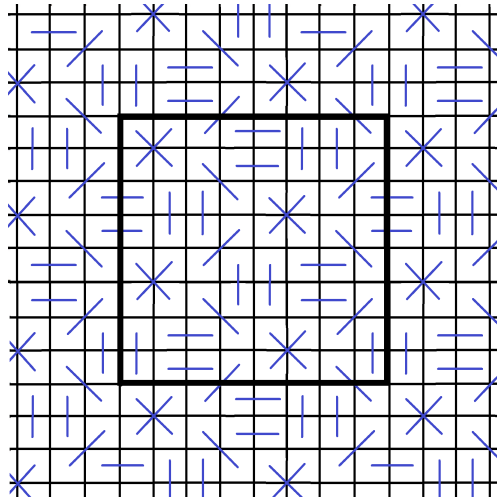


Figure 1: Hales-Jewett pairing blocks the 9-in-a-row.

A pairing is a **domino pairing** or rather a **match(-stick) pairing** on the square grid, if all pairs consist of only neighboring cells (horizontally, vertically or diagonally), called **dominoes**. Note that the pairing in Figure 1 is a domino pairing.

A counting type proposition [5] showed that there is no good pairing strategy for the k -in-a-row hypergraph, if $k < 9$. We will use this proposition, so we formulate the exact statement.

For a hypergraph \mathcal{H} let $d_2(\mathcal{H})$ (briefly d_2) be the greatest number of edges that can be blocked by two vertices of \mathcal{H} , i.e., d_2 is the maximal co-degree.

Proposition 2.2 ([5]). *If there is a good pairing ρ for the hypergraph $\mathcal{H} = (V, E)$, then $d_2|X|/2 \geq |\mathcal{G}|$ must hold for all $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

Proof. We will refer to X as a sub-board. The edges of \mathcal{G} can be blocked only by pairs coming from X . There are at most $|X|/2$ such pairs of ρ on the sub-board of size $|X|$. Since a pair blocks maximum d_2 edges, $|X|/2$ pairs block maximum $d_2|X|/2$. So, if there are more edges on the sub-board, there cannot be a good pairing. \square

With the help of Proposition 2.2, we can conclude that there is no pairing strategy for \mathcal{H}_k if $k < 9$. In the hypergraph \mathcal{H}_k , $d_2 = k - 1$ because a pair blocks at most $k - 1$ edges and this happens if and only if the pair is a domino. If X is an $n \times n$ sub-board for sufficiently large n , then $|\mathcal{G}| = 4n^2 + O(n)$ because four edges start from every square (a vertical, a horizontal and two diagonal, except at the border). By Proposition 2.2 we get $(k - 1)n^2/2 \geq 4n^2 + O(n)$; that is, $k \geq 9 + O(1/n)$. One can even compute $O(n)$ exactly: $O(n) = -48n + 128$.

Hales and Jewett gave a pairing for $k = 9$, see [4] or Figure 1. However, there are neither different solutions nor claims of the uniqueness of the Hales-Jewett pairing in the literature. Our main goal is to decide about these questions.

3 Conditions for good pairings of the 9-in-a-row

Consider an $n \times n$ square sub-board of the infinite board. Proposition 2.2 gives $(k - 1)n^2/2 \geq 4n^2 + O(n)$ which implies $k \geq 9 + O(1/n)$. It suggests that one must use the pairs “optimally” to block \mathcal{H}_9 that is a pair should block the maximum possible edges of \mathcal{H}_9 . We make the notion of optimality more precise as follows.

Definition 3.1. A pairing is optimal if:

1. Every pair blocks exactly $k - 1$ edges.
2. There are no overblockings, i.e., every edge is blocked by exactly one pair.
3. There is no empty square, i.e., every square is contained in a pair.

Corollary 3.2. *Let us consider an optimal good pairing for \mathcal{H}_9 . This pairing is then a domino pairing in which the dominoes are following each other by 8-periodicity in each line and all squares are covered by a pair.*

Proof. The first point of Definition 3.1 implies that the pairing is a domino pairing, while the second gives the 8-periodicity since otherwise it would cause either overblocking or resulting in an unblocked edge. The lack of empty squares is just the repetition of the third condition. \square

Definition 3.3. We call a square of a pairing **anomaly** where the 8-periodicity is violated, a non-domino type pair or an empty square appears in the pairing.

Of course the Hales-Jewett pairing is anomaly-free.

Remark 3.4. Because of the $O(n)$, there might be anomalies even in a good pairing of \mathcal{H}_9 .² However, in Section 6 we will show that a good pairing of \mathcal{H}_9 is always anomaly-free.

The first step towards this is the following lemma:

Lemma 3.5. *For every good pairing of \mathcal{H}_9 there is an arbitrarily big, anomaly-free square sub-board.*

Proof. Let us take any $n \times n$ sub-board X and cut it up to smaller $\sqrt{n}/100 \times \sqrt{n}/100$ sub-boards. According to Proposition 2.2, there are at most $48n - 128$ anomalies in X . Hence, most of the $10000n$ sub-squares of X must be anomaly-free. \square

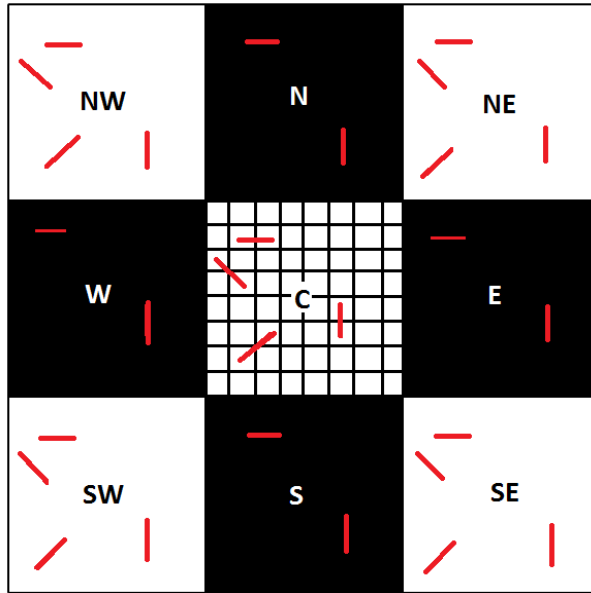
From now on we describe the structure of anomaly-free pairings of \mathcal{H}_9 . Let us divide a good pairing of \mathcal{H}_9 into 8×8 sub-boards and designate one that we call *Central square*, shortly C . We keep only the (domino) pairs touching C and examine where pairs should be on the neighboring 8×8 sub-boards of C . In order to talk about these sub-boards we call the 8×8 sub-boards Eastern (E), North-Eastern (NE) etc., while for the individual squares of the sub-boards the usual algebraic chess notations ($A1$ to $H8$) are used, see Figure 2.

Lemma 3.6. *Suppose we have an anomaly-free good pairing of \mathcal{H}_9 and we have nine 8×8 squares, (C, E, NE, \dots) as above. The horizontal and vertical dominoes touching the Central square C appear on the same places in all eight neighboring sub-boards of C . The diagonal dominoes also must appear on the sub-boards NE, NW, SW, SE in the same places. However, while the diagonal pattern of C may extend to the other four sub-boards, namely the E, S, W, N , it cannot be guaranteed. That is: the whole plane is the periodic copies either of the 8×8 sub-board C or the 16×16 square consisting of the sub-boards C, S, SE, E .*

Proof. It suffices to check the following five steps. We designate a general square in a 8×8 sub-board by $Xi \in \{A1, \dots, H8\}$ according to the chess notation. If a domino d covers the same pair of squares e.g., in the C and E square, we say that d extends to E from C .

1. Because of the 8-periodicity of the domino pairs on horizontal (vertical) lines, the pairs of C extend uniquely to the same places of W and E (N and S respectively). The slope $+1$ diagonal dominoes extend similarly to SW and NE , while the slope -1 diagonal dominoes to SE and NW .
2. To see the horizontal (vertical) extension of dominoes to N and S (W and E respectively) we need a little case study. We have already seen that the vertical dominoes of C extend to north and south. Suppose for example that there is a vertical domino v at the Xi square of C . If the Xi square of W is covered by a slope $+1$ (or -1)

²A pairing with anomalies might be called “quasi-crystal” referring to the highly symmetric, crystal like appearance of known anomaly-free examples such as the Hales-Jewett or the ones shown in [5].

Figure 2: The extension of a pairing of C .

diagonal domino, then the 8-periodicity implies that the Xi square of N (or S) is also covered by a diagonal domino. This is a contradiction because we know from the previous point, that the Xi square of N is covered by a copy of the vertical domino v . The same is true for the sub-board E . If the Xi square of W (or E) is covered by a horizontal domino, then C should contain the copy of that horizontal domino at Xi by 8-periodicity, which is also a contradiction. We get that the vertical domino v in C extends to W and E , moreover, by 8-periodicity v extends to SW , NW , NE , SE , too. So, we have seen that the vertical dominoes of C extend to all its eight neighboring sub-boards. The same is true for the horizontal dominoes of C .

3. Let us check the diagonal dominoes. At the first and second step all slope $+1$ diagonal dominoes of C extend to SW and NE . Since there are no empty squares or overblockings, the remaining squares in SW and NE can be covered only by -1 slope diagonal dominoes. The same is true for $+1$ slope diagonal dominoes in SE and NW . That is so far, all dominoes of C extend to the SW , SE , NE , NW , furthermore, the vertical and horizontal dominoes of C extend to S , E , N and W .
4. We can see that the diagonal dominoes of C do not necessarily extend to the sub-boards S , E , N , W (colored by black in Figure 2). However, by 8-periodicity the diagonal pairs of E extend to S , N and W , that is the black sub-boards S , E , N , W have the exactly same structure of pairs. \square

Remark 3.7. The diagonal dominoes of C may extend to the sub-boards S , E , N , W , and then all 8×8 sub-boards of the infinite board are the exact copy of C . However, it is possible that there are two different diagonal structures on the whole board, one in the

C, NW, NE, SE and SW types 8×8 sub-boards (colored by white in Figure 2) and a different diagonal structure in the sub-boards S, E, N and W (black ones). We will see a few examples in the next section.

Definition 3.8. A pairing of the infinite board (or of an anomaly-free sub-board) is **k-toric** if it is an extension of a $k \times k$ torus, but not for a smaller value.

Now we can summarize the previous lemmas and remarks in one central theorem:

Theorem 3.9. *Suppose we have an anomaly-free good pairing of \mathcal{H}_9 . Then that pairing is either 8-toric or 16-toric.*

Proof. Lemma 3.6 and Remark 3.7 gives the proof of the Theorem. \square

Observation 3.10. There are 8-toric good pairings of \mathcal{H}_9 that are not isomorphic to the Hales-Jewett pairing.

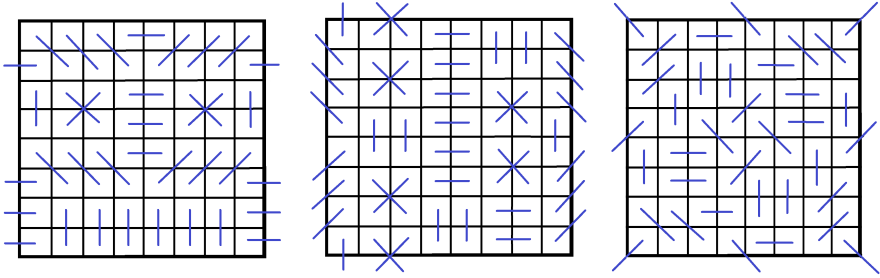


Figure 3: Some other pairings for 9-in-a-row.

Proof. The extensions of the 8×8 pairings in Figure 3 to the infinite board result in three different 8-toric pairings. Note that the pairings on the left have reflectional symmetry, while the pairing on the right has rotational symmetry. \square

It is somehow surprising that there exist also some 16-toric pairings of \mathcal{H}_9 . To understand their structure we refine the argument of the proof of Lemma 3.9 in the next section.

4 Diagonal alternating cycles

The 8-toric and 16-toric good pairings of \mathcal{H}_9 can be considered as special perfect matchings of graphs based on \mathcal{H}_9 . The vertex sets are the basic tori, and each vertex is connected to the eight neighbors of the square it represents. A domino of a pairing is an edge, and the whole pairing is not only a perfect matching but has the additional property that it contains exactly one edge (domino) from each torus line.

It is well known that the union of two perfect matchings on the same vertex set consist of parallel edges and alternating cycles. So if we take the (graph theoretic) union of two good pairings (e.g. of C and W) which have the same horizontal and vertical edges, then the non-trivial alternating cycles contain only diagonal edges. Identifying the vertices in the case of non-isomorphic G_C and G_W the system of diagonal alternating cycles gives the possible ways to get the 16-toric good pairings.

We arrive to the following simple corollary.

Corollary 4.1. *If there exists a 16-toric good pairing for \mathcal{H}_9 , then we can derive two 8-toric good pairings from it (in case of non-isomorphic G_C and G_W) differing only in some diagonal cycles.*

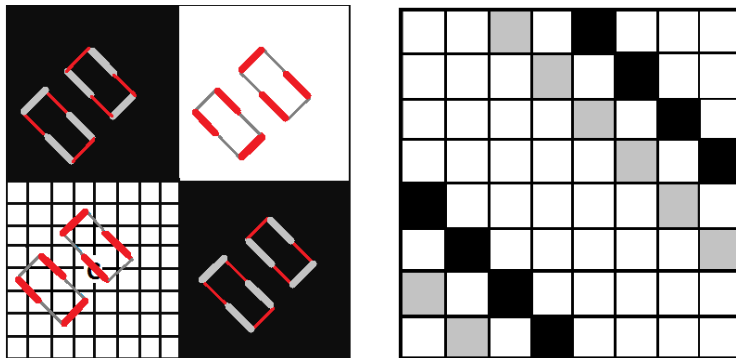


Figure 4: Diagonal alternating cycles give 16-toric pairing (left) and some -1 slope diagonal torus lines (right).

Theorem 4.2. *An 8-toric solution C gives a 16-toric solution if and only if another 8-toric solution W exists, differing in some diagonal dominoes, such that their union gives a system of diagonal alternating cycles. There are only two possible systems of diagonal alternating cycles which are shown in Figure 5; the left and middle ones.*

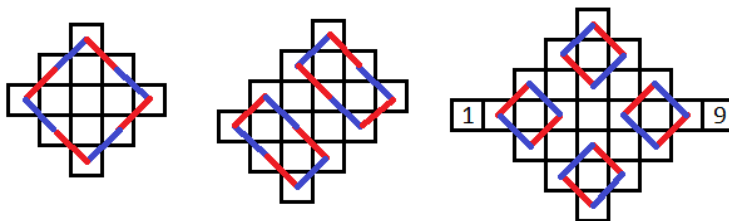


Figure 5: The diagonal alternating cycles.

Proof. Since there is exactly one domino in each torus line of an arbitrarily chosen 8×8 square sub-board of a 8-toric solution, then the alternating cycles coming from the diagonal dominoes of the union of C and W must meet the torus lines either in zero or two dominoes. (If they meet only in one, then there will be an unblocked torus line in C or W . Meeting more than two times would mean overblocking.)

An easy case study gives that only the systems of diagonal alternating cycles of Figure 5 may come into consideration. However, the third one would make a horizontal line (namely the 1-9) impossible to be blocked by a domino. \square

Remark 4.3. There are only two different systems of alternating cycles, but it is possible that there is more than one such system in one 16-toric pairing. In that case we can deduce

more than two (four or eight) 8-toric pairings from that 16-toric one. An example is shown in the right of Figure 6.

Observation 4.4. There exist good pairings for \mathcal{H}_9 containing the first or the second type of (the systems of) diagonal alternating cycles.

Proof. In Figure 6, one can see examples of the statement. Taking bold (thin) pairs of the alternating cycle for C (W) we get a 16-toric pairing. Naturally this 16-toric pairing is *not* 8-toric. \square

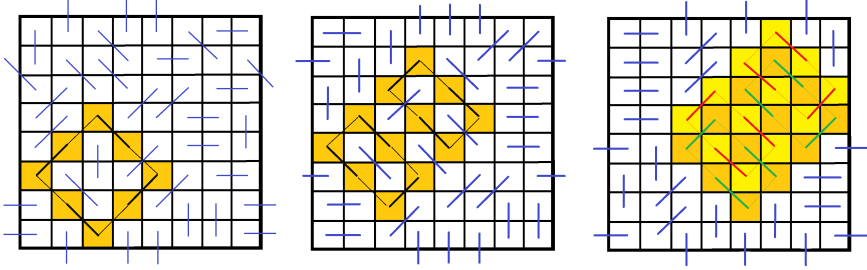


Figure 6: Examples of the alternating circles.

5 Pairings of the 8 torus

We have seen that pairings on the anomaly-free sub-boards are either 8-toric or 16-toric. Since the 16-toric solutions can be reduced to 8-toric ones or conversely, they can be constructed from 8-toric solutions we examine only the later ones in detail.

Definition 5.1. The 8×8 Maker-Breaker torus game is played on the 64 squares of the discrete torus, where there are 32 winning sets; the eight rows and columns and the diagonal torus lines of slope ± 1 (see the right side of Figure 4). We will call \mathcal{T}_8 the hypergraph of that game.

Observation 5.2. An arbitrary good 8-toric pairing for \mathcal{H}_9 provides a good pairing for \mathcal{T}_8 .

Remark 5.3. The reverse is not true, since \mathcal{T}_8 has good pairings which are not domino types. However, considering only domino pairings we can always extend a good pairing of \mathcal{T}_8 into a good 8-toric pairing of \mathcal{H}_9 .

To find all good domino pairings for the 8×8 torus is a finite task, which is not hard using a computer. However, one has to check the torus symmetries to list the non-isomorphic pairings, which gives the difficulty of the problem. The number of non-isomorphic domino type good pairings is 194 543, which turns out to be a prime. The pairings themselves can be downloaded at the page [9].

6 There is no quasi crystal pairing for the infinite board

We have an open problem left: are there any pairings for \mathcal{H}_9 with anomalies? Note that on a $n \times n$ sub-board there can be $O(n)$ anomalies which might result in infinitely many (and possibly untraceable) solutions. Fortunately, this is not the case as we will see.

Lemma 6.1. *A given anomaly-free pairing of a large enough square sub-board can be extended uniquely to the whole plane.*

Proof. We have seen that all anomaly-free pairings of a square sub-board is an extension of a domino pairing of either a 8×8 or a 16×16 torus. Continuing the extension to the whole plane gives a good pairing. \square

Lemma 6.2. *Let us assume that a pairing of the whole plane is an extension of an anomaly-free half-plane R . Then the whole pairing is anomaly-free.*

Proof. To prove by contradiction, assume that we have an extension AL containing anomalies. Let AF be the anomaly-free extension of the half-plane pairing that exists by Lemma 6.1. Obviously AL is not equal to AF .

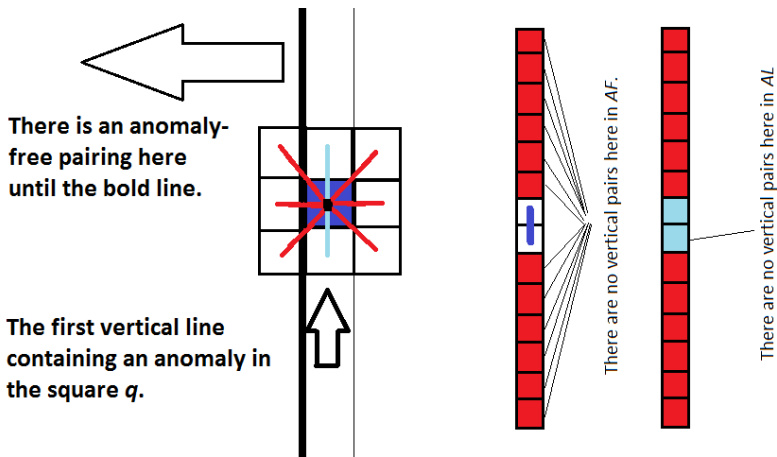


Figure 7: There are no quasicrystals.

Let us take a square with an anomaly which is one of the closest to R , and denote it by q . As it is pictured in Figure 7 we may assume that the border line of the half-plane R is vertical and the pairing AL is anomaly-free left to the square q . Let $AF(q)$ be the domino covering the square q in AF . If $AF(q)$ is placed horizontally and q is the right half of it, then AL does not contain the domino $AF(q)$ at the square q which leaves a 9-in-a-row edge unblocked by AL . A similar argument to diagonally placed dominoes shows that $AF(q)$ can be nothing but a vertical domino. Let us take the six squares above and below $AF(q)$. Because of 8-periodicity, there are no other squares containing a vertical pair in AF covering these 12 squares, but there must be a half of a vertical pair on those places (e.g. in s) in AL , because of the blocking condition. The domino $AF(s)$ is either horizontal or diagonal, and since $AF(s)$ is not in AL , it results in an unblocked horizontal or diagonal edge in AL . \square

To answer the main question at the beginning of this section, we will need the ideas of the previous lemma.

Theorem 6.3. *An anomaly-free pairing of a big enough square sub-board extends uniquely and anomaly-free to the whole board.*

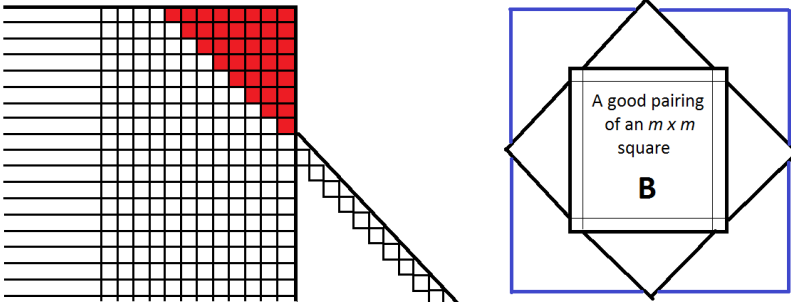


Figure 8: The extension of an anomaly-free pairing.

Proof. Fix a good pairing for \mathcal{H}_9 and take an $m \times m$ sub-board B that is anomaly-free; this exists by Lemma 3.5. The pairing on B extends anomaly-free to a large part of the right side of B , like in Lemma 6.2. The extension surely contains the right-angled triangle whose hypotenuse length is $m - 16$, and touches the right side of B , see Figure 8. The argument of Lemma 6.2 does not work next to the top and the bottom of B , since there are no diagonal dominoes there in B which were used before.

Doing the same trick to extend the pairing on the other sides of B , that results in a bigger (the size is about $(\sqrt{2}m - 16) \times (\sqrt{2}m - 16)$) rotated square. Repeating this procedure, we can see that the anomaly-free pairing of B is forced to extend to the whole plane. \square

7 A pairing strategy in 3D

Kruczek and Sundberg [8] conjectured upper bounds matching with the lower bound of Proposition 2.2 for k -in-a-row type games in d dimension.

Conjecture 7.1 ([8]). *In the Maker-Breaker game on \mathbb{Z}^d where there is a finite set $S \subset \mathbb{Z}^d$ of winning line direction-vectors, Breaker has a pairing strategy that allows him to win if the length of each winning line is at least $2|S| + 1$, i.e., Breaker has a winning pairing-strategy for the game k -in-a-row if $k \geq 2|S| + 1$.*

The special case of the *plane* gives back that Breaker has winning pairing strategy in the k -in-a-row if and only if $k \geq 9$. The higher dimensional versions are mainly open. One possible form of the M-B game in 3-dimension is when the winning directions are given by 13 vectors: $\{(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (0, 1, -1), (1, 0, -1), (1, -1, 0), (1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$. Here Proposition 2.2 implies that k should be at least 27 to have a good pairing. According to Conjecture 7.1, we may expect good pairings for $k = 27$.

We have examined a related problem in 3-dimension. If the directions of winning lines are given by three vectors: $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$, then one expects pairing strategies if $k \geq 7$. (In other words, this is a Harary-type game [4] in 3-dimension, where the winning polyomino is the P_7 , i.e. the seven connected consecutive cubes in a row.)

In fact, a computer search confirms this expectation, see Figure 9. This is a domino pairing of 3-dimensional torus type, we give the pairing on the $6 \times 6 \times 6$ torus in layers. The horizontal and vertical pairs of the same layer are obvious, while the pairs between the layers are denoted by points and circles.

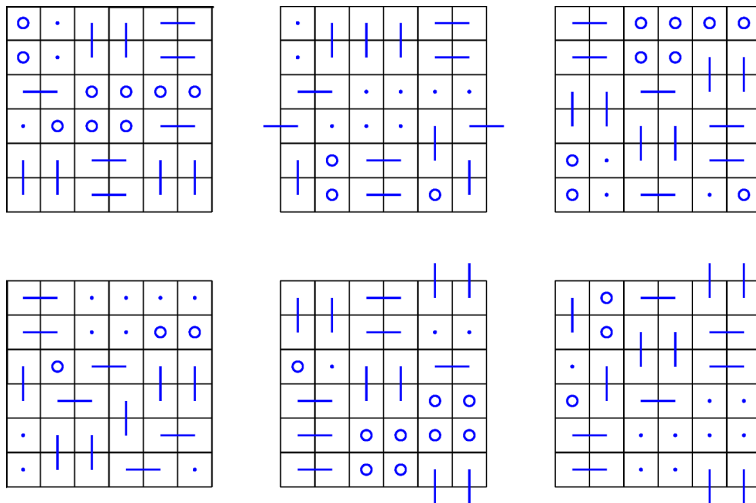


Figure 9: A good pairing of the 3D 7-in-a-row.

8 Conclusion

We have shown some new pairings for \mathcal{H}_9 . We have proved that a good pairing for \mathcal{H}_9 is either 8-toric or 16-toric. There are 194 543 pairings which are 8-toric, and it is possible to construct 16-toric good pairings of \mathcal{H}_9 from some of those. There are no good pairings on the infinite board containing anomalies.

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