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Generalized X-join of graphs and their automorphisms*

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Abstract

In this paper, we first introduce a new product of finite graphs as a generalization of the X-join of graphs. We then give necessary and sufficient conditions for a graph to be isomorphic to a generalized X-join. As a main result, we give necessary and sufficient conditions under which the full automorphism group of a generalized X-join is equal to the generalized wreath product of the automorphism groups of its factors.

Keywords: Automorphism, generalized wreath product, graph, lexicographic product, permutation group, X-join.

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1 Introduction

One of the main problems in the theory of graphs, known as the König problem, asks for a concrete characterization of all automorphism groups of graphs. In particular, the problem of computing a generating set of the automorphism group is equivalent to the graph isomorphism problem [9]. The automorphism groups of many graphs can be expressed in terms of the automorphism groups of their subgraphs. For instance, in most cases the automorphism groups of the graphs which are the lexicographic product of graphs are expressed in terms of the automorphism groups of their factors. The lexicographic product of graphs is one of the important products of graphs, defined by Harary in [7]. Sabidussi in [11] showed that under some conditions the automorphism group of the lexicographic product of two graphs

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Γ and Γ' can be expressed as the wreath product of the automorphism groups of Γ and Γ'. An important generalization of the lexicographic product is the X-join. It was introduced by Sabidussi as the graph formed from a given graph $\Gamma = (V, R)$ by replacing every vertex v of Γ by a graph B_v and joining the vertices of B_v with those of B_u whenever $uv \in R$ [11]. Note that the graphs B_v, $v \in V$, need not be mutually isomorphic. Hemminger in [8] gave necessary and sufficient conditions for the automorphism group of the X-join of graphs $\{B_v\}_{v\in V}$ to be the natural ones, i.e., those that are obtained by first permuting the graphs B_v, $v \in V$, according to a permutation of subscripts by an automorphism of Γ and then performing an arbitrary automorphism of each B_v. Note that Hemminger did not determine the structure of the automorphism group of the X-join of $\{B_v\}_{v\in V}$ in terms of automorphism groups of B_v, $v \in V$. It should be mentioned that the above results have been generalized to directed color graphs in [3]. If for a color digraph $C = (V, R)$ and a collection of color digraphs $\{D_c \mid c \in V\}$, each vertex c of C is replaced by a copy of D_c and all possible arcs of color k from D_c to $D_{c'}$ are included, if and only if there is an arc of color k from c to c' in C, we get the C-join of these color digraphs. The wreath product of two color digraphs C and D is the C-join of $\{D_c \mid c \in V\}$ where $D_c \cong D$ for every $c \in V$. In [3], all automorphism groups of digraphs that can be written as a wreath product have been determined.

In this paper we first give a generalization of the X-join of graphs (see Definition 2.1). This generalization, as a new operation on finite graphs, is a natural generalization of the X-join of graphs (a more algebraic way was considered by Weisfeiler [12, page 45] as the wreath product of a family of stable graphs with another stable graph). Also this new graph product generalizes the generalized wreath product of circulant digraph which defined in [2] (see Remark 2.9). It is also closely related with the wedge product of association schemes introduced and studied in [10] (see Remark 2.8). In Section 2 we give necessary and sufficient conditions under which a graph is isomorphic to a generalized X-join (see Theorem 2.4). But the main result of this paper deals with the connections between the automorphism group of a generalized X-join and the automorphism groups of its factors. For computing the automorphism group of the generalized X-join of graphs, we need a generalization of the wreath product of permutation groups. Recently, such a generalization, called the generalized wreath product, has been given in [1, 5]. We first show that under some conditions the automorphism group of the generalized X-join of graphs contains the generalized wreath product of the automorphism groups of their factors (Theorem 4.1). As a main result, we then give necessary and sufficient conditions under which the full automorphism group of the generalized X-join of graphs is equal to the generalized wreath product of the automorphism groups of their factors (Theorem 4.2). In particular, we determine the structure of the natural automorphism group of the X-join of graphs (Corollary 4.7).

Terminology and notation: Throughout this paper, by a graph $\Gamma = (V, R)$ we mean a finite undirected graph without multiple edges with the vertex set $V = V(\Gamma)$ and the edge set $R = E(\Gamma)$. We denote the complement of Γ by $\overline{\Gamma}$. If all pairs of vertices of a subgraph Γ' of Γ that are adjacent in Γ are also adjacent in Γ', then Γ' is an induced subgraph. For $X \subseteq V$ we write $\Gamma[X]$ for the subgraph of Γ induced by X and we also denote by $\Gamma(X)$ the graph with vertices X and edge set $E(\Gamma[X]) \cup \{(x, x) \mid x \in X\}$. For two graphs $\Gamma = (V, R)$ and $\Gamma' = (V', R')$, by a graph homomorphism $f\colon \Gamma \to \Gamma'$ we mean a mapping $f\colon V \to V'$ such that $(f(u), f(v)) \in R'$ whenever $(u, v) \in R$. In the case when $f\colon V \to V'$ is surjective, $f\colon \Gamma \to \Gamma'$ is called a graph epimorphism. Moreover, if $f\colon V \to V'$ is a bijection and $f^{-1}\colon \Gamma' \to \Gamma$ is also a graph homomorphism, then $f\colon \Gamma \to \Gamma'$ is called a graph isomorphism. Two graphs Γ and Γ' are called isomorphic if there exists a graph isomorphism between Γ and Γ'. In this case we write $\Gamma \simeq \Gamma'$. When $\Gamma = \Gamma'$ every graph isomorphism $f\colon \Gamma \to \Gamma$ is called a graph automorphism of Γ. The set of all graph automorphisms of Γ is denoted by $\mathrm{Aut}(\Gamma)$ and is called the automorphism group of Γ.

If Π is a partition of the vertices of a graph Γ, then the quotient graph Γ/Π is a graph with vertex set Π, for which distinct classes $X, X' \in \Pi$ are adjacent if some vertex in X is adjacent to a vertex of X'.

Let $\Gamma = (V, R)$ be a graph. The X-join of a set of graphs $\{B_x = (Y_x, E_x) \mid x \in V\}$ with Γ, denoted by $\Gamma[B_x]_{x\in V}$, is a graph $W = (Y, E)$ where $Y = \dot{\bigcup}_{x\in V} Y_x$ and

$$E = \{(y_x, y'_{x'}) \in Y_x \times Y_{x'} \mid (x, x') \in R, \text{or else } x = x' \text{ and } (y_x, y'_x) \in E_x\}.$$

If $B = (Y', E')$ and $B_x = B$ for every $x \in V$, we can identify $\dot{\bigcup}_{x\in V} Y_x$ with $Y' \times V$ and then the X-join of $\{B_x = (Y_x, E_x) \mid x \in V\}$ is the lexicographic product of Γ and B and is denoted by $\Gamma \circ B$.

We denote by K_n a complete graph with n vertices. For the graph theoretical terminology and notation that are not defined here, we refer the reader to [6].

For a finite set V, we denote by $\mathrm{Sym}(V)$ the group of all permutations of V. Every subgroup of $\mathrm{Sym}(V)$ is called a permutation group on V. For $F \leq \mathrm{Sym}(V)$ and $\Delta \subseteq V$, the setwise stabilizer of Δ in F is $F_{\{\Delta\}} = \{f \in F \mid \Delta^f = \Delta\}$ and the pointwise stabilizer of Δ in F is $F_{(\Delta)} = \{f \in F \mid\ x^f = x,\ \ \forall x \in \Delta\}$. We say that two permutation groups $F \leq \mathrm{Sym}(V)$ and $F' \leq \mathrm{Sym}(V')$ are permutation isomorphic if there exist a bijection $\lambda\colon V \to V'$ and a group isomorphism $\eta\colon F \to F'$ such that for every $f \in F$ and $v \in V$ we have $\lambda(v^f) = \lambda(v)^{\eta(f)}$.

By a system of blocks Π for a permutation group $F \leq \mathrm{Sym}(\Omega)$ we mean

(1) Π is a partition of Ω;

(2) for every $\Delta \in \Pi$ and every $f \in F$, $\Delta^f \cap \Delta = \varnothing$ or $\Delta^f = \Delta$.

If Π is a system of blocks for F and $\Delta \in \Pi$, by F^Δ we mean the group induced by the action of $F_{\{\Delta\}}$ on Δ. Then $F^\Delta/F_{(\Delta)} \leq \mathrm{Sym}(\Delta)$ is a permutation group.

2 A generalization of the X-join of graphs

In this section we first introduce a new product of graphs, called the generalized X-join of graphs. Then we give necessary and sufficient conditions under which a graph is isomorphic to a generalized X-join.

Definition 2.1. Let $\Gamma = (V, R)$ be a graph and Π be a partition of V. Suppose that for every $X \in \Pi$ we are given a graph $B_X = (Y_X, E_X)$ and a graph epimorphism $\pi_X\colon Y_X \to X$ from B_X onto $\Gamma(X)$. Put $Y = \dot{\bigcup}_{X\in\Pi} Y_X$ and $\pi = \dot{\bigcup}_{X\in\Pi} \pi_X$ where for every $y \in Y_X$, $\pi(y) := \pi_X(y)$. We define a graph W with vertex set Y and edge set E such that $(y, y') \in E$ if and only if

(1) either $(y, y') \in E_X$, for some $X \in \Pi$;

(2) or $(y, y') \in \pi_X^{-1}(x) \times \pi_{X'}^{-1}(x')$ where $X \neq X'$ and $(x, x') \in R$.

We call the graph $W = (Y, E)$ the *generalized X-join* of Γ and $\{B_X\}_{X\in\Pi}$ with respect to π, and we denote it by $\Gamma \circ_\pi \{B_X\}_{X\in\Pi}$. (See Figure 1.)

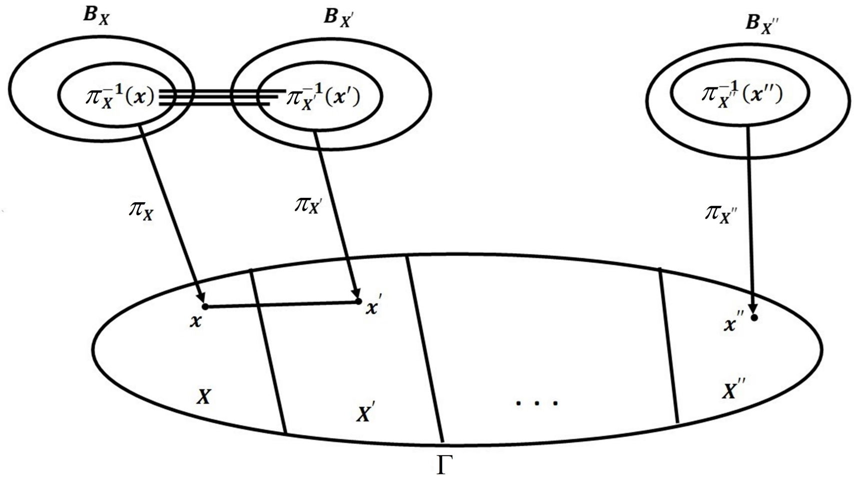

Figure 1: The generalized X-join of Γ and $\{B_X\}_{X\in\Pi}$.

In the following we show that the X-join of graphs is a special case of the generalized X-join of graphs.

Example 2.2. Let $\Gamma = (V, R)$ be a graph and Π be a partition of V such that for every $X \in \Pi$, $X = \{x\}$ for some $x \in V$. Suppose that $\{B_x = (Y_x, E_x) \mid x \in V\}$ is a set of graphs. Define a graph epimorphism $\pi_x \colon Y_x \to X$ from B_x onto $\Gamma(X)$ such that $\pi_x(y_x) = x$ for every $y_x \in Y_x$. Then the generalized X-join of Γ and $\{B_x\}_{x\in\Pi}$ with respect to $\pi = \dot{\bigcup}_{x\in V}\pi_x$ is a graph with vertices $Y = \dot{\bigcup}_{x\in V} Y_x$ and the edge set E such that $(y_x, y'_{x'}) \in E$ if and only if

(1) either $x = x'$ and $(y_x, y'_x) \in E_x$;

(2) or $x \neq x'$ and $(x, x') \in R$.

One can see that in this case $\Gamma \circ_\pi \{B_x\}_{x\in V} = \Gamma[B_x]_{x\in V}$, the X-join of graphs $\{B_x\}_{x\in V}$.

Example 2.3. Let $\Gamma = (V, R)$ be the graph in Figure 2. Consider the partition $\Pi = \{X, X', X''\}$ of V where $X = \{1, 2\}, X' = \{3, 4\}$, and $X'' = \{5, 6\}$. Suppose that $B_X = (Y_X, E_X)$, $B_{X'} = (Y_{X'}, E_{X'})$, and $B_{X''} = (Y_{X''}, E_{X''})$ are the graphs in Figure 2 with vertices $Y_X = \{a, b, c\}$, $Y_{X'} = \{d, e, f\}$, and $Y_{X''} = \{g, h, i, k\}$, respectively.

Now define the graph epimorphisms $\pi_X \colon B_X \to \Gamma(X)$, $\pi_{X'} \colon B_{X'} \to \Gamma(X')$, and $\pi_{X''} \colon B_{X''} \to \Gamma(X'')$ as follows:

$$\begin{cases} \pi_X(a) = \pi_X(b) = 1 \\ \pi_X(c) = 2 \end{cases}$$

$$\begin{cases} \pi_{X'}(e) = \pi_{X'}(d) = 3 \\ \pi_{X'}(f) = 4 \end{cases}$$

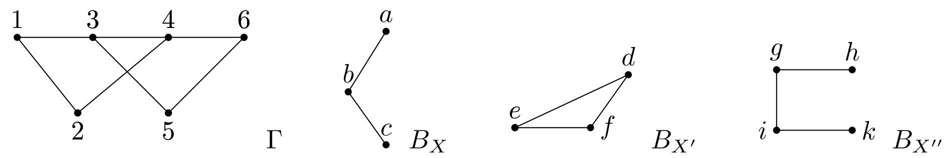

Figure 2: Graph Γ and set of graphs $\{B_X\}_{X\in\Pi}$.

$$\begin{cases}\pi_{X''}(g) = \pi_{X''}(h) = 5\\ \pi_{X''}(i) = \pi_{X''}(k) = 6.\end{cases}$$

Then the generalized X-join of Γ and $\{B_X, B_{X'}, B_{X''}\}$ with respect to π is the graph in Figure 3.

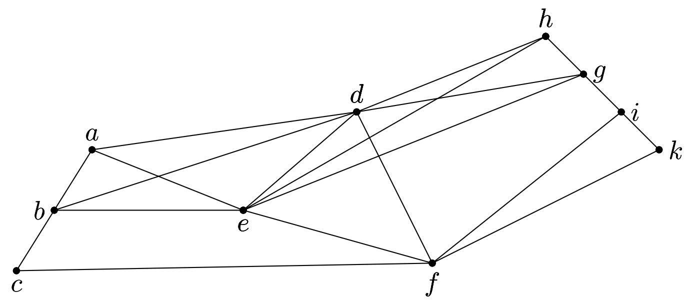

Figure 3: Graph $W = \Gamma \circ_\pi \{B_X, B_{X'}, B_{X''}\}$.

Let $\Gamma = (V, R)$ be a graph and let $A, B \subseteq V$. We say that A is *externally related* with respect to B, if every vertex $v \in B$ that is adjacent to at least one element in A is adjacent to all vertices of A. Moreover, if B is also externally related with respect to A, we say that A and B are *externally related to each other*.

Suppose that $W = (Y, E)$ is the generalized X-join of $\Gamma = (V, R)$ and $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ with respect to π. Then we can define two equivalence relations E_0 and E_1 on Y as follows:

$$(u, v) \in E_0 \quad \Leftrightarrow \quad u, v \in \pi_X^{-1}(x), \quad \text{for some } X \in \Pi \text{ and } x \in X; \tag{2.1}$$
$$(u, v) \in E_1 \quad \Leftrightarrow \quad u, v \in Y_X, \quad \text{for some } X \in \Pi. \tag{2.2}$$

Clearly, $E_0 \subseteq E_1$. In the following we give a characterization of the generalized X-join of graphs in terms of the equivalence relations E_0 and E_1.

Theorem 2.4. *A graph $W = (Y, E)$ is a generalized X-join of graphs if and only if there exist two equivalence relations E_0 and E_1 on Y such that*

(i) $E_0 \subseteq E_1$;

(ii) *for every equivalence class P of E_0 which is contained in a equivalence class Q of E_1, P is externally related with respect to every equivalence class of E_0 which is not in Q.*

Proof. Suppose that $W = (Y, E)$ is the generalized X-join of Γ and $\{B_X\}_{X \in \Pi}$ with respect to π. Then as we saw above, there are two equivalence relations E_0 and E_1 on Y such that $E_0 \subseteq E_1$. Since for every $x \in X$ and $x' \in X'$ where $X \neq X'$, $\pi_X^{-1}(x)$ and $\pi_{X'}^{-1}(x')$ are externally related to each other, it follows that condition (ii) holds.

Now suppose that there exist two equivalence relations E_0 and E_1 on Y such that conditions (i) and (ii) hold. Let Y/E_0 and Y/E_1 be the sets of the equivalence classes of E_0 and E_1 on Y, respectively. Let Γ be the quotient graph W/E_0. Moreover, for every $U \in Y/E_1$, let U_0 be the equivalence classes of E_0 which are contained in U and B_{U_0} be the subgraph of W induced by U. Since $E_0 \subseteq E_1$, $\{U_0 \mid U \in Y/E_1\}$ gives a partition Π on Y/E_0. Then for every $U \in Y/E_1$ we can define a graph epimorphism π_{U_0} from the graph B_{U_0} onto $\Gamma(U_0)$. Suppose that W' is the generalized X-join of Γ and $\{B_{U_0}\}_{U_0 \in \Pi}$ with respect to π. Then $V(W') = Y$ and it follows from condition (ii) that the set of edges of W and W' are the same. Thus $W = W'$ and so W is a generalized X-join of graphs. □

Remark 2.5. The following example shows that unlike the X-join of graphs, a graph can be represented as a generalized X-join of graphs, but not a unique way. This means that if W and W' are two isomorphic generalized X-join of graphs then it is not necessarily true that the factors of W and W' are isomorphic.

Example 2.6. Consider the graph $W = (Y, E)$ in Figure 4. If we consider two equivalence relations $E_0 \subseteq E_1$ such that $Y/E_0 = \{\{a\}, \{b, c\}, \{d, e, f, g\}, \{h\}\}$ and $Y/E_1 = \{\{a, b, c\}, \{d, e, f, g, h\}\}$, then one can see that conditions (i) and (ii) of Theorem 2.4 hold. So it follows from Theorem 2.4 that $W = \Gamma \circ_\pi \{B_X, B_{X'}\}$ where the graphs Γ, B_X and $B_{X'}$ are shown in Figure 5. On the other hand, consider the graphs Γ', B'_X and $B'_{X'}$ that are shown in Figure 6. Set $\Pi' = \{X = \{1, 2, 3\}, X' = \{4, 5\}\}$ and define the graph epimorphisms $\pi'_X \colon B'_X \to \Gamma'(X)$ by

$$\begin{cases} a \longrightarrow 1 \\ b, c \longrightarrow 2 \\ h \longrightarrow 3 \end{cases}$$

and $\pi'_{X'} \colon B'_{X'} \to \Gamma'(X')$ by

$$\begin{cases} d, e \longrightarrow 4 \\ f, g \longrightarrow 5 \end{cases}.$$

Then one can see that W is the generalized X-join $\Gamma' \circ_{\pi'} \{B'_X, B'_{X'}\}$ with respect to π'.

The following lemma that gives a sufficient condition under which two generalized X-join are isomorphic, is straightforward and therefore left to the reader.

Lemma 2.7. *Let $W = \Gamma \circ_\pi \{B_X\}_{X \in \Pi}$ and $W' = \Gamma' \circ_{\pi'} \{B_{X'}\}_{X' \in \Pi'}$. Suppose that the following conditions hold.*

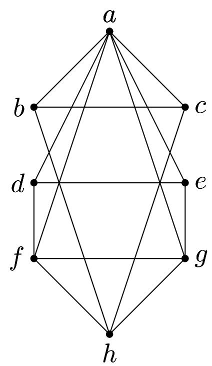

Figure 4: Graph W.

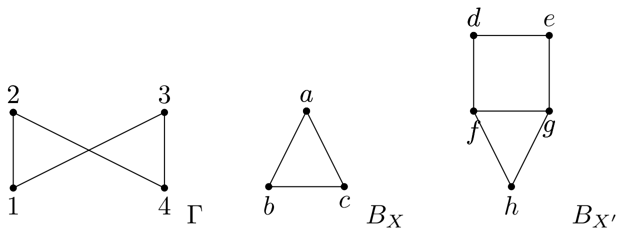

Figure 5: Graph Γ and set of graphs $\{B_X, B_{X'}\}$.

(1) *There exists a graph isomorphism* $\alpha\colon \Gamma \to \Gamma'$ *which maps every partition class* $X \in \Pi$ *onto a partition class* $X' \in \Pi'$*;*

(2) *For every* $X \in \Pi$*, there exist graph isomorphisms* $\beta_{XX'}\colon B_X \to B_{X'}$ *with* $X' = X^\alpha$ *such that the following diagram is commutative.*

$$\begin{array}{ccc} Y_X & \xrightarrow{\beta_{XX'}} & Y_{X'} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_{X'}} \\ X & \xrightarrow{\alpha} & X' \end{array}$$

Then $\psi\colon \dot{\bigcup}_{X\in\Pi} Y_X \to \dot{\bigcup}_{X'\in\Pi'} Y_{X'}$ *defined by* $y_X \to \beta_{XX'}(y_X)$ *is a graph isomorphism between* W *and* W'.

Remark 2.8. The generalized X-join is closely related with the wedge product of association schemes. The wedge product of association schemes which provides a way to construct new association schemes from old ones has been given in [10]. In the following we give the relationship between the relations of a wedge product of symmetric association schemes and the generalized X-join.

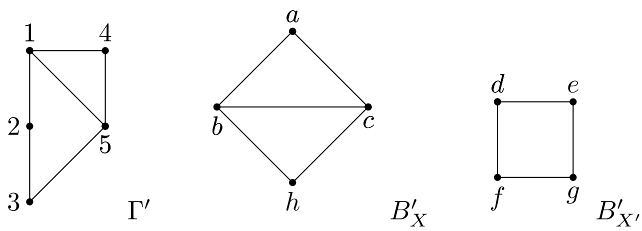

Figure 6: Graph Γ' and set of graphs $\{B'_X, B'_{X'}\}$.

Suppose (V, G) is an association scheme and E is an equivalence relation on V such that it is a union of some relations $R_0, R_1, \dots, R_t$ of G. Put $D = \{R_0, R_1, \dots, R_t\}$ and suppose Σ is the set of equivalence classes of E. For every $X \in \Sigma$, let $D_X = \{g_X \mid g \in D\}$ where $g_X = g \cap X \times X$. Moreover, assume that

(1) there is a set of association schemes $\{(Y_X, B_X) \mid X \in \Sigma\}$ such that all Y_X are pairwise disjoint and for every $X \in \Sigma$ there exists a scheme normal epimorphism $\pi_X \colon Y_X \cup B_X \to X \cup D_X$.

(2) for every $X, X' \in \Sigma$, there exists an algebraic isomorphism $\varphi_{XX'} \colon B_X \to B_{X'}$ such that the diagram

$$\begin{array}{ccc} B_X & \xrightarrow{\varphi_{XX'}} & B_{X'} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_{X'}} \\ D_X & \xrightarrow{\varepsilon_{XX'}} & D_{X'} \end{array}$$

is commutative, where $\varepsilon_{XX'}(g_X) = g_{X'}$.

Put $Y := \dot{\bigcup}_{X\in\Sigma} Y_X$, $\pi := \dot{\bigcup}_{X\in\Sigma} \pi_X$ and for every $b \in B_X$, $\widetilde{b} = \bigcup_{X'\in\Sigma} \varphi_{XX'}(b_X)$. Moreover, for every $g \in G$ put

$$\overline{g} = \bigcup_{\substack{(x,x')\in g\cap X\times X',\\ X,X'\in\Sigma, X\neq X'}} \psi_X^{-1}(x) \times \psi_{X'}^{-1}(x').$$

Fix $Z \in \Sigma$. Put $\widetilde{B_Z} = \{\widetilde{b} \mid b \in B_Z\}$. Then it follows from [10, Theorem 2.2] that the pair $(Y, \widetilde{B_Z} \cup (\overline{G} \setminus \overline{D}))$ is an association scheme, is called the *wedge product* of $(Y_X, B_X), X \in \Sigma$, and (V, G). Now let $g \in G \setminus D$ and $b_X \in B_X$ such that $\pi_X(b_X) = g_X$. Then one can see that the graph with vertices Y and the edge set $\overline{g} \cup \widetilde{b}$ is the generalized X-join of g and $\{\varphi_{XX'}(b_X)\}_{X'\in\Sigma}$ with respect to π.

Remark 2.9. The generalized wreath product of Cayley digraphs on abelain groups was first introduced in [2] and an entire section of the recent book [4, Section 5] is devoted to their study. A Cayley digraph $Cay(G, S)$ of G with connection set S is a generalized wreath product if there are subgroups $1 < K \leq L < G$ such that $S \setminus L$ is a union of cosets of K. In the following we show that the generalized X-join generalizes the generalized wreath product. To see this, let $Cay(G, S)$ be a generalized wreath product on abelian group G such that $1 \notin S$ and $S = S^{-1}$. Let $V = \{g_1, \ldots, g_t\}$ be a set of left coset representatives of K in G and Γ be the subgraph of G induced on V. Suppose that $\{a_0 = 1, a_1, \ldots, a_m\}$ is a set of left coset representatives of L in G. For every $0 \leq i \leq m$, let $X_i = \{g_j \in V \mid g_j \in a_i L\}$. Then $\Pi = \{X_0, X_1, \ldots, X_m\}$ is a partition of V. Put $B_0 = Cay(L, L \cap S)$ and for every $1 \leq i \leq m$, let $B_i = \phi_i(B_0)$ where $\phi_i \colon B_0 \to B_i$ is a graph isomorphism defined by $\phi_i(l) = a_i l$ for every $l \in L$. Then there is the graph epimorphism $\pi_i \colon B_i \to \Gamma(X_i)$ such that $\pi_i(g_j K) = g_j$. Now let $W = (G, E)$ be the generalized X-join of Γ and $\{B_0, B_1, \ldots, B_m\}$ with respect to π where $\pi = \dot{\bigcup}_{i=0}^{m} \pi_i$. We show that for every $x, y \in G$, $xy \in E$ if and only if $xy^{-1} \in S$. Clearly, if $x, y \in a_i L$, then $xy \in E$ if and only if $xy^{-1} \in S \cap L$. If $x \in g_r K \subseteq a_i L$ and $y \in g_s K \subseteq a_j L$ where $i \neq j$, then $xy^{-1} \in g_r g_s^{-1} K$. Then $xy \in E$ if and only if $g_r g_s^{-1} \in S \setminus L$ if and only if $g_r g_s^{-1} K \subseteq S \setminus L$, since $S \setminus L$ is a union of cosets of K. So in this case $xy \in E$ if and only if $xy^{-1} \in S \setminus L$. Thus we conclude that $W = (G, E) = Cay(G, S)$. This means that $Cay(G, S)$ is a generalized X-join.

3 Generalized wreath product, definition and construction

The generalized wreath product of permutation groups has been defined in [1, 5]. Since in the next section we need to construct the generalized wreath product of the automorphism group of graphs, here we have a look at the definition of this product which has been given in [1].

Let $\Gamma = (V, R)$ be a graph and $F = \mathrm{Aut}(\Gamma)$. Suppose that Π is a system of blocks for F. Moreover, suppose that we are given a set of graphs $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ such that the following conditions hold.

(G1) If for some $f \in F$, $X^f = X'$, then $B_X \simeq B_{X'}$,

(G2) If Δ is an orbit of F on Π, then for some $X \in \Delta$, there exists a graph epimorphism $\pi_X \colon Y_X \to X$ from B_X onto $\Gamma(X)$ and there exists an epimorphism $\eta_X \colon \mathrm{Aut}(B_X) \to F^X / F_{(X)}$ such that

$$\pi_X(y^l) = (\pi_X(y))^{\eta_X(l)}, \ \ \forall y \in Y_X, l \in \mathrm{Aut}(B_X).$$

By condition $(G1)$, if there exists $f_{XX'} \in F$ such that $X^{f_{XX'}} = X'$, we have a graph isomorphism $\phi_{XX'} \colon Y_X \to Y_{X'}$ from graph B_X onto $B_{X'}$. Then $\psi_{XX'} \colon \mathrm{Aut}(B_X) \to \mathrm{Aut}(B_{X'})$ defined by

$$\psi_{XX'}(\alpha) = \phi_{XX'} \alpha \phi_{XX'}^{-1}, \ \ \forall\, \alpha \in \mathrm{Aut}(B_X),$$

is an isomorphism from $\mathrm{Aut}(B_X)$ onto $\mathrm{Aut}(B_{X'})$.
Moreover, by condition $(G2)$, $\Lambda_X = \{\pi_X^{-1}(x) \mid x \in X\}$ is a system of blocks for $\mathrm{Aut}(B_X)$,

$$\overline{\eta_X} \colon \mathrm{Aut}(B_X)/K_X \to F^X / F_{(X)}$$

is an isomorphism, and $\mathrm{Aut}(B_X)/K_X \leq \mathrm{Sym}(\Lambda_X)$ and $F^X/F_{(X)} \leq \mathrm{Sym}(X)$ are permutation isomorphic where $K_X = \ker(\eta_X)$.

Lemma 3.1. *Let $X' \in \Delta$ with $X' \neq X$. Then*

(1) *there exists a graph epimorphism* $\pi_{X'}$ *from* $B_{X'}$ *onto* $\Gamma(X')$ *such that the following diagram is commutative, where* $\xi_{XX'}\colon X \to X'$*, given by* $\xi_{XX'}(x) = x^{f_{XX'}}$ *for every* $x \in X$.

$$\begin{array}{ccc} Y_X & \xrightarrow{\phi_{XX'}} & Y_{X'} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_{X'}} \\ X & \xrightarrow{\xi_{XX'}} & X' \end{array}$$

(2) *there exists an epimorphism* $\eta_{X'}\colon \operatorname{Aut}(B_{X'}) \to F^{X'}/F_{(X')}$ *such that the following diagram is commutative, where* $\rho_{XX'}\colon F^X/F_{(X)} \to F^{X'}/F_{(X')}$*, defined by* $\rho_{XX'}(hF_{(X)}) = f_{XX'}hf_{XX'}^{-1}F_{(X')}$ *for every* $hF_{(X)} \in F^X/F_{(X)}$.

$$\begin{array}{ccc} \operatorname{Aut}(B_X) & \xrightarrow{\psi_{XX'}} & \operatorname{Aut}(B_{X'}) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_{X'}} \\ F^X/F_{(X)} & \xrightarrow{\rho_{XX'}} & F^{X'}/F_{(X')} \end{array}$$

Proof. (1) If we define $\pi_{X'} = \xi_{XX'}\pi_X\phi_{XX'}^{-1}$, then $\pi_{X'}\colon Y_{X'} \to X'$ is a graph epimorphism from $B_{X'}$ onto $\Gamma(X')$ such that the diagram mentioned above is commutative.

(2) Define $\eta_{X'} = \rho_{XX'}\eta_X\psi_{XX'}^{-1}$. Then $\eta_{X'}\colon \operatorname{Aut}(B_{X'}) \to F^{X'}/F_{(X')}$ is an epimorphism such that the above diagram is commutative. □

Now suppose that a graph $\Gamma = (V, R)$ and a set of graphs $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ satisfy conditions $(G1)$ and $(G2)$. Set $Y = \dot{\bigcup}_{X\in\Pi} Y_X$. Since for every $X \in \Pi$, $K_X \leq \operatorname{Aut}(B_X)$ it follows that the action of $\prod_{X\in\Pi} K_X$ on Y defined by

$$y^k := y^{k_X}, \quad y \in Y_X,\ k = \prod_{X\in\Pi} k_X \in K$$

is faithful. Set $K = \prod_{X\in\Pi} K_X$. Then $K \leq \operatorname{Sym}(Y)$.

Moreover, every element of F can be also considered as an element of $\operatorname{Sym}(Y)$. In fact, for every $g \in F$ we can associate $\overline{g} \in \operatorname{Sym}(Y)$. To do this, let $y_X \in Y_X$ and T_X be a set of left coset representatives for K_X in $\operatorname{Aut}(B_X)$ such that $\mathrm{id}_{Y_X} \in T_X$. Let $g \in F$. We associate to g an element $\overline{g} \in \operatorname{Sym}(Y)$ as follows:

(i) if $X^g = X$, then $(y_X)^{\overline{g}} = (y_X)^t$ where $\overline{\eta_X}(tK_X) = gF_{(X)}$ for some $t \in T_X$;

(ii) if $X^g = X'$, then $(y_X)^{\overline{g}} = \phi_{XX'}((y_X)^t)$ where $\overline{\eta_X}(tK_X) = f_{XX'}^{-1}gF_{(X)}$ for some $t \in T_X$.

Set $\overline{F} = \{\overline{g} \mid g \in F\}$. Clearly, $\overline{F} \subseteq \operatorname{Sym}(Y)$ and $\langle K, \overline{F}\rangle \leq \operatorname{Sym}(Y)$. According to [1, Definition 2.1], the permutation group $\langle K, \overline{F}\rangle$, is the *generalized wreath product* of $\{\operatorname{Aut}(B_X)\}_{X\in\Pi}$ and F. We denote it by $F \circ \{\operatorname{Aut}(B_X)\}_{X\in\Pi}$.

Remark 3.2. It should be mentioned that the generalized wreath product of $\{\operatorname{Aut}(B_X)\}_{X\in\Pi}$ and F is independent of the choice of representatives T_X for every $X \in \Pi$. To see this, let $y_X \in Y_X$ and T'_X be a set of left coset representatives for K_X in $\operatorname{Aut}(B_X)$ such that

$\mathrm{id}_{Y_X} \in T'_X$ and $T'_X \neq T_X$. Let $g \in F$ and $\widehat{g}$ be an element of $\mathrm{Sym}(Y)$ associated with g by the above argument. If $X^g = X$, then $(y_X)^{\widehat{g}} = (y_X)^{t'}$ where $\overline{\eta_X}(t'K_X) = gF_{(X)}$ for some $t' \in T'_X$. Since $t' = tk_X$, for some $t \in T_X$ and $k_X \in K_X$, we have

$$(y_X)^{\widehat{g}} = (y_X)^{t'} = (y_X)^{tk_X} = (y_X)^{\overline{g}k_X}.$$

Similarly, if $X^g = X'$, then $(y_X)^{\widehat{g}} = \phi_{XX'}((y_X)^{t'})$ where $\overline{\eta_X}(t'K_X) = f^{-1}_{XX'}gF_{(X)}$ for some $t' \in T'_X$. If $t' = tk_X$ for some $t \in T_X$ and $k_X \in K_X$, then

$$(y_X)^{\widehat{g}} = \phi_{XX'}((y_X)^{t'}) = \phi_{XX'}((y_X)^{tk_X}) = \phi_{XX'}((y_X)^{\overline{g}k_X}).$$

Then we conclude that

$$< \widehat{F}, K >=< \overline{F}, K >,$$

where $\widehat{F} = \{\widehat{f} \mid f \in F\}$. This shows that $F \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} =< \widehat{F}, K >$.

Example 3.3. Let Γ and $\{B_X, B_{X'}, B_{X''}\}$ be the graphs in Figure 7. Then

$$\mathrm{Aut}(\Gamma) = \{\mathrm{id}_V, (14)(25)(36), (23), (56), (23)(56), (2635)(14), (2536)(14), (26)(35)(14)\}$$

and

$$\Pi = \{X = \{2,3\}, X' = \{5,6\}, X'' = \{1,4\}\}$$

is a system of blocks for $F = \mathrm{Aut}(\Gamma)$. Put $f_{XX'} = (14)(25)(36)$. Since $X^{f_{XX'}} = X'$ we have the following graph isomorphism from B_X onto $B_{X'}$.

$$\begin{aligned} \phi_{XX'}\colon Y_X &\to Y_{X'} \\ a &\longrightarrow a' \\ b &\longrightarrow b' \\ c &\longrightarrow c' \\ d &\longrightarrow d' \end{aligned}$$

So condition (G1) holds, because, $\{X, X'\}$ and $\{X''\}$ are the orbits of F on Π. Moreover, there exist the graph epimorphisms $\pi_X\colon B_X \to \Gamma(X)$, $\pi_{X'}\colon B_{X'} \to \Gamma(X')$, and $\pi_{X''}\colon B_{X''} \to \Gamma(X'')$ such that $\{\pi_X^{-1}(x) \mid x \in X\} = \{\{a,c\},\{b,d\}\}$, $\{\pi_{X'}^{-1}(x) \mid x \in X'\} = \{\{a',c'\},\{b',d'\}\}$, and $\{\pi_{X''}^{-1}(x) \mid x \in X''\} = \{\{b'',c''\},\{a'',d''\}\}$. If we define the epimorphisms $\eta_X\colon \mathrm{Aut}(B_X) \to F^X/F_{(X)}$, $\eta_{X'}\colon \mathrm{Aut}(B_{X'}) \to F^{X'}/F_{(X')}$, and $\eta_{X''}\colon \mathrm{Aut}(B_{X''}) \to F^{X''}/F_{(X'')}$ by

$$\begin{cases} \eta_X(\mathrm{id}_{Y_X}) = F_{(X)} \\ \eta_X((ab)(cd)) = (23)F_{(X)} \end{cases}$$

$$\begin{cases} \eta_{X'}(\mathrm{id}_{Y_{X'}}) = F_{(X')} \\ \eta_{X'}((a'b')(c'd')) = (56)F_{(X')} \end{cases}$$

and

$$\begin{cases} \eta_{X''}(\mathrm{id}_{Y_{X''}}) = F_{(X'')} \\ \eta_{X''}((a''b'')(c''d'')) = (14)(25)(36)F_{(X'')} \end{cases}$$

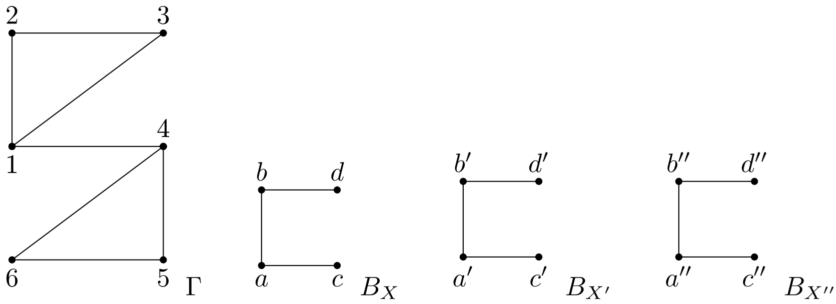

Figure 7: Graph Γ and set of graphs $\{B_X\}_{X\in\Pi}$.

then it is easy to verify that condition $(G2)$ holds.
Put $Y = \{a, b, c, d, a', b', c', d', a'', b'', c'', d''\}$. Consider element $g = (2536)(14) \in F$. Since $X = \{2, 3\}$, $X' = \{5, 6\}$, and $X'' = \{1, 4\}$ we have $X^g = X', X'^g = X$ and $X''^g = X''$. Now we associate to g, an element $\overline{g}$ such that

(i) $(y_X)^{\overline{g}} = \phi_{XX'}(y_X)$, since $\overline{\eta_X}(K_X) = f^{-1}_{XX'}gF_{(X)} = (56)F_{(X)} = F_{(X)}$;

(ii) $(y_{X'})^{\overline{g}} = \phi_{X'X}((y_{X'})^{(a'b')(c'd')})$, since $\overline{\eta_{X'}}((a'b')(c'd')K_{X'}) = f^{-1}_{X'X}gF_{(X')} = (56)F_{(X')}$;

(iii) $(y_{X''})^{\overline{g}} = (y_{X''})^{(a''b'')(c''d'')}$, since $\overline{\eta_{X''}}((a''b'')(c''d'')K_{X''}) = gF_{(X'')} = (14)(25)(36)F_{(X'')}$.

Then $\overline{g} = (aa'bb')(cc'dd')(a''b'')(c''d'')$. Similarly,

(1) if $g = (23)$ then $\overline{g} = (ab)(cd)$;

(2) if $g = (56)$ then $\overline{g} = (a'b')(c'd')$;

(3) if $g = (23)(56)$ then $\overline{g} = (ab)(cd)(a'b')(c'd')$;

(4) if $g = (2635)(14)$ then $\overline{g} = (ab'ba')(cd'dc')(a''b'')(c''d'')$;

(5) if $g = (14)(25)(36)$ then $\overline{g} = (aa')(bb')(cc')(dd')(a''b'')(c''d'')$;

(6) if $g = (14)(26)(35)$ then $\overline{g} = (ab')(ba')(cd')(dc')(a''b'')(c''d'')$.

Since K_X, $K_{X'}$ and $K_{X''}$ are trivial groups, it follows that

$$\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} = \langle \mathrm{id}_Y, (ab)(cd), (a'b')(c'd'), (aa'bb')(cc'dd')(a''b'')(c''d''), (ab'ba')(cd'dc')(a''b'')(c''d''), (aa')(bb')(cc')(dd')(a''b'')(c''d''), (ab')(ba')(cd')(dc')(a''b'')(c''d'')\rangle.$$

4 Automorphism group of the generalized X-join of graphs

In this section we show that the automorphism group of some graphs which are isomorphic to a generalized X-join can be expressed in terms of the generalized wreath product of automorphism groups of its factors.

Theorem 4.1. *With the notation above, suppose that a graph $\Gamma = (V, R)$ and a set of graphs $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ satisfy the conditions $(G1)$ and $(G2)$. Then*

$$\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} \leq \mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi}).$$

Proof. Let $W = (Y, E)$ be the generalized X-join of Γ and $\{B_X\}_{X\in\Pi}$ with respect to π, and $H = \langle K, \overline{F}\rangle$ be the generalized wreath product of $\{\mathrm{Aut}(B_X)\}_{X\in\Pi}$ and $\mathrm{Aut}(\Gamma)$.

We show that for every $h \in H$ and $u, v \in Y$, if $(u, v) \in E$, then $(u^h, v^h) \in E$. To do this, we assume that $(u, v) \in E$ and we consider the following cases.

Case 1. Suppose that $h = \prod_{X\in\Pi} k_X \in K$.

(i) If $u, v \in Y_X$ for some $X \in \Pi$, then since for every $X \in \Pi$, $k_X \in \mathrm{Aut}(B_X)$ we have $(u, v)^h = (u^h, v^h) = (u^{k_X}, v^{k_X}) \in E_X$.

(ii) If $u \in Y_X$ and $v \in Y_{X'}$ for some X and X' in Π where $X \neq X'$, then $(x, x') = (\pi_X(u), \pi_{X'}(v)) \in R$ and since $(u^{k_X}, v^{k_{X'}}) \in \pi_X^{-1}(x)\times\pi_{X'}^{-1}(x')$ we have $(u, v)^h = (u^h, v^h) = (u^{k_X}, v^{k_{X'}}) \in E$.

Case 2. Suppose that $h = \overline{g}$ for some $g \in F$ and $u, v \in Y_X$ for some $X \in \Pi$.

(i) If $X^g = X$, then since $(u, v)^{\overline{g}} = (u^{\overline{g}}, v^{\overline{g}}) = (u^t, v^t)$ where $\overline{\eta_X}(tK_X) = gF_{(X)}$ for some $t \in \mathrm{Aut}(B_X)$, we have $(u, v)^h = (u^h, v^h) = (u^t, v^t) \in E$.

(ii) If $X^g = X'$ for some $X' \in \Pi$, then since $(u, v)^{\overline{g}} = (u^{\overline{g}}, v^{\overline{g}}) = (\phi_{XX'}(u^t), \phi_{XX'}(v^t))$ where $\overline{\eta_X}(tK_X) = f_{XX'}^{-1} gF_{(X)}$ for some $t \in \mathrm{Aut}(B_X)$ we have $(u, v)^h = (u^h, v^h) = (\phi_{XX'}(u^t), \phi_{XX'}(v^t)) \in E$.

Case 3. Let $h = \overline{g}$ for some $g \in F$, $u \in Y_X$ and $v \in Y_{X'}$ for some $X, X' \in \Pi$ where $X \neq X'$. In this case since $(x, x') = (\pi_X(u), \pi_{X'}(v)) \in R$ and $g \in \mathrm{Aut}(\Gamma)$ we have $(x^g, x'^g) \in R$. Then the following cases arise.

(i) If $X^g = X$ and $X'^g = X'$, then $(u, v)^{\overline{g}} = (u^{\overline{g}}, v^{\overline{g}}) = (u^t, v^{t'})$ where $\overline{\eta_X}(tK_X) = gF_{(X)}$ and $\overline{\eta_{X'}}(t'K_{X'}) = gF_{(X')}$. Since $\pi_X(u^t) = \pi_X(u)^{\eta_X(t)} = x^g$ and $\pi_{X'}(v^{t'}) = \pi_{X'}(v)^{\eta_{X'}(t')} = x'^g$ we have

$$(u^t, v^{t'}) \in \pi_X^{-1}(x^g) \times \pi_{X'}^{-1}(x'^g).$$

Then $(u^h, v^h) = (u^t, v^{t'}) \in E$.

(ii) If $X^g = X$ and $X'^g = X''$, then $(u, v)^{\overline{g}} = (u^{\overline{g}}, v^{\overline{g}}) = (u^t, \phi_{X'X''}(v^{t'}))$ where $\overline{\eta_X}(tK_X) = gF_{(X)}$ and $\overline{\eta_{X'}}(t'K_{X'}) = f_{X'X''}^{-1} gF_{(X')}$. Then $\pi_X(u^t) = \pi_X(u)^{\eta_X(t)} = x^g$ and by condition $(G2)$ we have

$$\begin{aligned}\pi_{X''}(\phi_{X'X''}(v^{t'})) &= \xi_{X'X''}(\pi_{X'}(v^{t'}))\\ &= \xi_{X'X''}(\pi_{X'}(v)^{\eta_{X'}(t')})\\ &= f_{X'X''}\overline{\eta_{X'}}(t'K_{X'})(\pi_{X'}(v))\\ &= gF_{(X')}(\pi_{X'}(v))\\ &= (\pi_{X'}(v))^g\\ &= x'^g.\end{aligned}$$

So $(u^t, \phi_{X'X''}(v^{t'})) \in \pi_X^{-1}(x^g) \times \pi_{X''}^{-1}(x'^g)$ and then $(u^h, v^h) = (u^t, \phi_{X'X''}(v^{t'})) \in E$.

(iii) If $X^g = X'$ and $X'^g = X''$, then $(u,v)^{\overline{g}} = (u^{\overline{g}}, v^{\overline{g}}) = (\phi_{XX'}(u^t), \phi_{X'X''}(v^{t'}))$ where $\overline{\eta_X}(tK_X) = f_{XX'}^{-1} g F_{(X)}$ and $\overline{\eta_{X'}}(t'K_{X'}) = f_{X'X''}^{-1} g F_{(X')}$. From condition $(G2)$ we have

$$\begin{aligned}\pi_{X'}(\phi_{XX'}(u^t)) &= \xi_{XX'}(\pi_X(u^t)) \\ &= \xi_{XX'}(\pi_X(u)^{\eta_X(t)}) \\ &= f_{XX'}\overline{\eta_X}(tK_X)(\pi_X(u)) \\ &= gF_{(X)}(\pi_X(u)) \\ &= (\pi_X(u))^g \\ &= x^g.\end{aligned}$$

Similarly, $\pi_{X''}(\phi_{X'X''}(v^{t'})) = x'^g$. Then

$$(\phi_{XX'}(u^t), \phi_{X'X''}(v^{t'})) \in \pi_{X'}^{-1}(x^g) \times \pi_{X''}^{-1}(x'^g)$$

and so $(u^h, v^h) = (\phi_{XX'}(u^t), \phi_{X'X''}(v^{t'})) \in E$.

(iv) If $X^g = X''$ and $X'^g = X'''$, then $(u,v)^{\overline{g}} = (u^{\overline{g}}, v^{\overline{g}}) = (\phi_{XX''}(u^t), \phi_{X'X'''}(v^{t'}))$ where $\overline{\eta_X}(tK_X) = f_{XX''}^{-1} g F_{(X)}$ and $\overline{\eta_{X'}}(t'K_{X'}) = f_{X'X'''}^{-1} g F_{(X')}$. Then an argument similar to that given in (iii) shows that $\pi_{X''}(\phi_{XX''}(u^t)) = x^g$ and $\pi_{X'''}(\phi_{X'X'''}(v^{t'})) = x'^g$. So $(\phi_{XX''}(u^t), \phi_{X'X'''}(v^{t'})) \in \pi_{X''}^{-1}(x^g) \times \pi_{X'''}^{-1}(x'^g)$ and hence $(u^h, v^h) = (\phi_{XX''}(u^t), \phi_{X'X'''}(v^{t'})) \in E$.

Then we conclude that $\langle K, \overline{F} \rangle \subseteq \mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi})$. Thus

$$\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} \leq \mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi}). \qquad \square$$

The inclusion $\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} \leq \mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi})$ in the above theorem may be proper. For example

$$D_8 = \mathrm{Aut}(K_2) \circ \mathrm{Aut}(K_2) < \mathrm{Aut}(K_2 \circ K_2) = S_4,$$

where S_4 is the symmetric group on 4 elements $V = \{1,2,3,4\}$ and $D_8 = \{\mathrm{id}_V, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$ is the dihedral group of order 8; see [6, Chapter 10]. In the following we give necessary and sufficient conditions under which the above inclusion is proper.

Theorem 4.2. *With the notation above, suppose that the graph $\Gamma = (V, R)$ and the set of graphs $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ satisfy the conditions $(G1)$ and $(G2)$. Let $W = (Y, E)$ be the generalized X-join of Γ and $\{B_X\}_{X\in\Pi}$ with respect to π and let $E_0 \subseteq E_1$ be the equivalence relations defined in (1) and (2). Then the inclusion*

$$\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} \leq \mathrm{Aut}(W)$$

is proper if and only if there exist equivalence relations $E_0' \subseteq E_1'$ on Y such that the following conditions hold.

(i) $E'_0 \subsetneq E_0$ *and* $E'_1 \subsetneq E_1$, *and* $W = \Gamma' \circ_{\pi'} \{A_Z\}_{Z\in\Pi'}$, *where the graph* Γ' *is the quotient graph* W/E'_0, Π' *is a partition of* $V(\Gamma')$, $\{A_Z\}_{Z\in\Pi'}$ *are the subgraphs of* W *induced by the equivalence classes of* E'_1, $\pi' = \dot{\bigcup}_{Z\in\Pi'}\pi'_Z$, *and* $\{{\pi'}_Z^{-1}(x) \mid x \in V(\Gamma'), Z \in \Pi'\}$ *is the set of equivalence classes of* E'_0.

(ii) *There exist* $Z \neq Z' \in \Pi' - \Pi$ *and a graph isomorphism* $\phi\colon Y_Z \to Y_{Z'}$ *from* A_Z *onto* $A_{Z'}$, *such that* ϕ *preserves equivalence classes of* E'_0 *contained in* Y_Z.

(iii) *For every equivalence class* S' *of* E'_0 *contained in* Y_Z, *if* $W[S']$ *is a union of connected components of* $W[S]$ *for some* $S \in W/E_0$, *where* $S' \subsetneq S$ *then* $\pi'_Z(S')$ *and* $\pi'_{Z'}(\phi(S'))$ *are nonadjacent, otherwise* $\pi'_Z(S')$ *and* $\pi'_{Z'}(\phi(S'))$ *are adjacent. In both cases* $\pi'_Z(S')$ *and* $\pi'_{Z'}(\phi(S'))$ *have the same neighbors in* $V(\Gamma') \setminus Z \cup Z'$.

(iv) *For each two distinct equivalence classes* S'_1 , S'_2 *of* E'_0 *that are contained in* Y_Z, S'_1 *and* $\phi(S'_2)$ *are adjacent if and only if* $\phi(S'_1)$ *and* S'_2 *are adjacent.*

Proof. Suppose that there exists $\phi \in \mathrm{Aut}(W) \setminus \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi}$. Let U be the set of all elements of Y that are moved by ϕ. Since $Y/E_1 = \{Y_X \mid X \in \Pi\}$ and $Y/E_0 = \bigcup_{X\in\Pi} \Lambda_X$, where $\Lambda_X = \{\pi_X^{-1}(x) \mid x \in X\}$, are two system of blocks for $\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi}$, then there exist

(1) $X \in \Pi$ such that $U_X = U \cap Y_X \neq \emptyset$;

(2) $X' \in \Pi$ such that $X \neq X'$ and $\phi(U_X) \subseteq Y_{X'}$. Note that if $X = X'$, then the restriction of ϕ to Y_X is an automorphism of B_X and $\phi(U_X) \subseteq Y_X$. But since $\phi \notin \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi}$ we must have at least two equivalence classes $S_1, S_2 \in E_0$ such that a part of S_1 is moved by the automorphism ϕ to a part of S_2. This contradicts the fact that $\Lambda_X = \{\pi_X^{-1}(x) \mid x \in X\}$ is a system of blocks for $\mathrm{Aut}(B_X)$.

(3) at least one equivalence class S of E_0 such that $S \cap U_X \subsetneq S$. Indeed, if $U_X = \bigcup_{i=1}^t S_i \subsetneq Y_X$ where every S_i is an equivalence class of E_0, then by (2), $\phi(U_X) \subseteq Y_{X'}$ for some $X' \neq X$ and $\phi(U_X)$ is a union of some equivalence classes of E_0 which are contained in $Y_{X'}$. This means that the vertices $\pi_X(S_1), \ldots, \pi_X(S_t)$ of X can be moved to the vertices $\pi_{X'}(\phi(S_1)), \ldots, \pi_{X'}(\phi(S_t))$ of X'. This contradicts the fact that Π is a system of blocks for $\mathrm{Aut}(\Gamma)$.

Put $V_{X'} = \phi(U_X)$. Let $S_1, S_2, ..., S_t$ be the equivalence classes of E_0 contained in Y_X such that for every $1 \leq i \leq t$,

$$S_i^X = S_i \cap U_X \neq \emptyset.$$

Then for at least one i, $S_i^X \subsetneq S_i$. Moreover, we have the following.

(a) The restriction of ϕ to U_X gives an isomorphism between $W[U_X]$ and $W[V_{X'}]$, the subgraphs of W induced by U_X and $V_{X'}$.

(b) For each i the vertices in $S_i^X \cup \phi(S_i^X)$ have the same neighbors in $Y \setminus (U_X \cup V_{X'})$. Indeed, suppose that $u \in S_i^X$ and w is a neighbor of u. Suppose that $T_1, \ldots, T_t$ are equivalence classes of E_0 such that $\phi(S_i^X) \cap T_i \neq \emptyset$ and $v_i \in T_i \setminus \phi(S_i^X)$. If $w \in Y \setminus (Y_X \cup Y_{X'})$, then $\phi(w)$ is adjacent to all vertices of T_i, specially v_i. So w and $\phi^{-1}(v_i) = v_i$ are adjacent. Thus w is adjacent to all vertices of $\phi(S_i^X)$.

Moreover, if $w \in Y_X \setminus U_X$, then since w is adjacent to u, $\phi(w) = w$ is adjacent to $\phi(u)$. Then w is adjacent to all vertices of $\phi(S_i^X)$. Similarly, if $w \in Y_{X'} \setminus V_{X'}$, then w is adjacent to all vertices of $\phi(S_i^X)$. Hence we conclude that $S_i^X \cup \phi(S_i^X)$ have the same neighbors in $Y \setminus (U_X \cup V_{X'})$.

(c) If $W[S_i^X]$ is a union of connected components of $W[S_i]$, then S_i^X and $\phi(S_i^X)$ are nonadjacent; otherwise by the definition of W, $S_i \setminus S_i^X$ and $\phi(S_i^X)$ are adjacent and since $\phi \in \mathrm{Aut}(W)$, S_i^X and $S_i \setminus S_i^X$ are adjacent and this contradicts the hypothesis that $W[S_i^X]$ is a union of connected components of $W[S_i]$. Also if $W[S_i^X]$ and $W[S_i \setminus S_i^X]$ are adjacent, then since $\phi \in \mathrm{Aut}(W)$, $S_i \setminus S_i^X$ and $\phi(S_i^X)$ must be adjacent and the definition of W implies that $\phi(S_i^X)$ and S_i are externally related to each other. Moreover, $S_i \setminus S_i^X$ and S_i^X are also externally related to each other.

(d) For two different equivalence classes S_1 and S_2 of E_0 with $S_1^X, S_2^X \neq \emptyset$, if S_1 and $\phi(S_2^X)$ are adjacent then from the definition of W it follows that S_1 and $\phi(S_2^X)$ are externally related to each other. Moreover, since $\phi \in \mathrm{Aut}(W)$ we must have $\phi(S_1^X)$ and S_2 are also externally related to each other. Similarly, if S_2 and $\phi(S_1^X)$ are adjacent then $\phi(S_2^X)$ and S_1 are externally related to each other.

Now we consider two equivalence relations $E_0' \subseteq E_1'$ on Y such that

$$Y/E_1' = \{U_X, V_{X'}, Y_X \setminus U_X, Y_{X'} \setminus V_{X'}, Y/E_1 \setminus \{Y_X, Y_{X'}\}\},$$

and the equivalence classes of E_0' are equal

$$S_i^X, S_i \setminus S_i^X, \phi(S_i^X), T_i \setminus \phi(S_i^X), \quad 1 \leq i \leq t,$$

and $Y/E_0 \setminus \{S_i, T_i \mid 1 \leq i \leq t\}$, where for every i, T_i is an equivalence class of E_0 such that $\phi(S_i^X) \subseteq T_i$.
From statements (b) and (c) we conclude that the condition (ii) of Theorem 2.4 holds and so $W = \Gamma' \circ_{\pi'} \{A_X\}_{X \in \Pi'}$, where Γ' is the quotient graph W/E_0', Π' is a partition of $V(\Gamma')$ induced by Y/E_1', and $\{A_X\}_{X \in \Pi'}$ are the subgraphs of W induced by the equivalence classes of E_1'. So (i) holds. If we denote by A_X and $A_{X'}$ the subgraphs of W induced by U_X and $V_{X'}$, respectively, then the restriction of ϕ to U_X gives a graph isomorphism between A_X and $A_{X'}$. Clearly, ϕ preserves the equivalence classes of E_0' contained in U_X. Thus condition (ii) of theorem holds. Moreover, (b), (c) and the definition of π' imply that condition (iii) holds. Finally, condition (iv) follows from statement (d).

Conversely, suppose that there exist equivalence relations $E_0' \subseteq E_1'$ on Y such that conditions (i) – (iv) hold. Let U_Z and $U_{Z'}$ be the vertex sets of A_Z and $A_{Z'}$, respectively. Assume that $\phi\colon U_Z \longrightarrow U_{Z'}$ is the graph isomorphism from A_Z onto $A_{Z'}$. We define a bijection $\psi\colon Y \longrightarrow Y$ as follows:

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \in U_Z, \\ \phi^{-1}(v) & \text{if } v \in U_{Z'}, \\ v & \text{if } v \notin \{U_Z, U_{Z'}\}. \end{cases}$$

We claim that $\psi \in \mathrm{Aut}(W)$. To do this, we suppose that $(u, v) \in E$ and we consider the following cases.

(1) If $u, v \in Y \setminus U_Z \cup U_{Z'}$, then clearly $(\psi(u), \psi(v)) = (u, v) \in E$.

(2) If $u, v \in U_Z$, then since ϕ is a graph isomorphism it follows that $(\psi(u), \psi(v)) = (\phi(u), \phi(v)) \in E$. Similarly, if $u, v \in U_{Z'}$ we have $(\psi(u), \psi(v)) = (\phi^{-1}(u), \phi^{-1}(v)) \in E$.

(3) If $u \in U_Z$ and $v \in Y \setminus U_Z \cup U_{Z'}$, then since v is a neighbor of u it follows from (iii) that v is also a neighbor of $\phi(u)$. Hence $(\psi(u), \psi(v)) = (\phi(u), v) \in E$.

(4) If $u \in U_Z$ and $v \in U_{Z'}$ such that for some equivalence class S' of E_0', $u \in S'$ and $v \in \phi(S')$, then the definition of W implies that all vertices in S' are adjacent to all vertices in $\phi(S')$ and so $(\psi(u), \psi(v)) = (\phi(u), \phi^{-1}(v)) \in E$.

(5) If $u \in U_Z$ and $v \in U_{Z'}$ such that for two equivalence classes S_1' and S_2' of E_0', $u \in S_1'$ and $v \in \phi(S_2')$, then by the definition of W, all vertices in S_1' are adjacent to all vertices in $\phi(S_2')$. On the other hand, it follows from (iv) that $\phi(S_1')$ and S_2' are adjacent. So all vertices of $\phi(S_1')$ are adjacent to all vertices of S_2' and thus $(\psi(u), \psi(v)) = (\phi(u), \phi^{-1}(v)) \in E$.

Hence $\psi \in \mathrm{Aut}(W)$. Since $E_0' \subsetneq E_0$ and $Z, Z' \in \Pi' - \Pi$, and ψ preserves the equivalence classes of E_0' we conclude that $\psi \in \mathrm{Aut}(W) \setminus \mathrm{Aut}(\Gamma) \circ \{Aut(B_X)\}_{X \in \Pi}$. □

Example 4.3. Suppose that Γ is the graph in Figure 8 with vertices $V = \{1, 2, 3, 4, 5, 6\}$. Then one can see that

$$F = \mathrm{Aut}(\Gamma) = \{\mathrm{id}_V, (12)(56)(34), (13)(24), (23)(56)(14)\}$$

and

$$\Pi = \{X = \{1, 2\}, X' = \{3, 4\}, X'' = \{5, 6\}\},$$

is a system of blocks for F. Moreover, $F^X = F^{X'} = \{\mathrm{id}_V, (12)(56)(34)\}$, $F^{X''} = F$, $F_{(X)} = F_{(X')} = \{\mathrm{id}_V\}$, and $F_{(X'')} = \{\mathrm{id}_V, (13)(24)\}$. Suppose that B_X, $B_{X'}$, and $B_{X''}$ are the graphs in Figure 8 with vertices $Y_X = \{a, b, c, d\}$, $Y_{X'} = \{a', b', c', d'\}$, $Y_{X''} = \{a'', b'', c'', d''\}$, respectively.

Now consider the graph epimorphisms $\pi_X \colon B_X \to \Gamma(X)$, $\pi_{X'} \colon B_{X'} \to \Gamma(X')$, and $\pi_{X''} \colon B_{X''} \to \Gamma(X'')$ as the following:

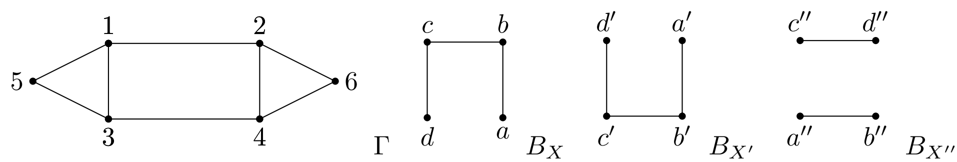

Figure 8: Graph Γ and set of graphs $\{B_X\}_{X \in \Pi}$.

$$\begin{cases} \pi_X(a) = \pi_X(b) = 1 \\ \pi_X(c) = \pi_X(d) = 2 \end{cases}$$

$$\begin{cases}\pi_{X'}(a') = \pi_{X'}(b') = 3\\ \pi_{X'}(c') = \pi_{X'}(d') = 4\end{cases}$$

and

$$\begin{cases}\pi_{X''}(a'') = \pi_{X''}(b'') = 6\\ \pi_{X''}(c'') = \pi_{X''}(d'') = 5\end{cases}.$$

Since $\mathrm{Aut}(B_X) = \{\mathrm{id}_{Y_X}, (bc)(ad)\}$, $\mathrm{Aut}(B_{X'}) = \{\mathrm{id}_{Y_{X'}}, (b'c')(a'd')\}$, and

$$\mathrm{Aut}(B_{X''}) = \{\mathrm{id}_{Y_{X''}}, (a''b''), (c''d''), (a''b'')(c''d''), (a''c''b''d''), (a''d''b''c''), (a''c'')(b''d''), (a''d'')(b''c'')\},$$

we can define epimorphisms $\eta_X\colon \mathrm{Aut}(B_X) \to F^X/F_{(X)}$, $\eta_{X'}\colon \mathrm{Aut}(B_{X'}) \to F^{X'}/F_{(X')}$, and $\eta_{X''}\colon \mathrm{Aut}(B_{X''}) \to F^{X''}/F_{(X'')}$ by

$$\begin{cases}\eta_X(\mathrm{id}_{Y_X}) = \mathrm{id}_V\\ \eta_X((bc)(ad)) = (12)(56)(34)F_{(X)}\end{cases}$$

$$\begin{cases}\eta_{X'}(\mathrm{id}_{Y_{X'}}) = \mathrm{id}_V\\ \eta_{X'}((b'c')(a'd')) = (12)(56)(34)F_{(X')}\end{cases}$$

and

$$\begin{cases}\eta_{X''}(\mathrm{id}_{Y_{X''}}) = \eta_{X''}((a''b'')) = \eta_{X''}((c''d'')) = \eta_{X''}((a''b'')(c''d'')) = \mathrm{id}_V\\ \eta_{X''}((a''c''b''d'')) = \eta_{X''}((a''d''b''c'')) = \eta_{X''}((a''c'')(b''d'')) = \eta_{X''}((a''d'')(b''c'')) =\\ (12)(56)(34)F_{(X'')}\end{cases}$$

Then K_X and $K_{X'}$ are trivial groups and

$$K_{X''} = \{\mathrm{id}_{Y_{X''}}, (a''b''), (c''d''), (a''b'')(c''d'')\}.$$

Let $T_X = \{\mathrm{id}_{Y_X}, (bc)(ad)\}$, $T_{X'} = \{\mathrm{id}_{Y_{X'}}, (b'c')(a'd')\}$, and $T_{X''} = \{\mathrm{id}_{Y_{X''}}, (a''c'')(b''d'')\}$. Put $f_{XX'} = (13)(24)$. Then the elements $(12)(56)(34), (13)(24)$ and $(23)(56)(14)$ in $\mathrm{Aut}(\Gamma)$ are associated to $(bc)(ad)(b'c')(a'd')(a''c'')(b''d'')$, $(aa')(bb')(cc')(dd')$ and $(bc')(ad')(cb')(da')(a''c'')(b''d'')$, respectively. Then we have

$$\mathrm{Aut}(\Gamma)\circ\{\mathrm{Aut}(B_X)\}_{X\in\Pi} = \langle \mathrm{id}_Y, (aa')(bb')(cc')(dd'), (bc')(ad')(cb')(da')(a''c'')(b''d''), (a''b''), (c''d''), (bc)(ad)(b'c')(a'd')(a''c'')(b''d'')\rangle.$$

Now let $W = (Y, E)$ be the generalized X-join of Γ and $\{B_X\}_{X\in\Pi}$ with respect to π. (See Figure 9.) Consider the equivalence relations E'_0 and E'_1 on Y with the following classes,

$$Y/E'_0 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a'\}, \{b'\}, \{c'\}, \{d'\}, \{a'', b''\}, \{c'', d''\}\}$$

$$Y/E'_1 = \{\{a, d\}, \{b, c\}, \{a', d'\}, \{b', c'\}, \{a'', b'', c'', d''\}\}.$$

Then one can see that

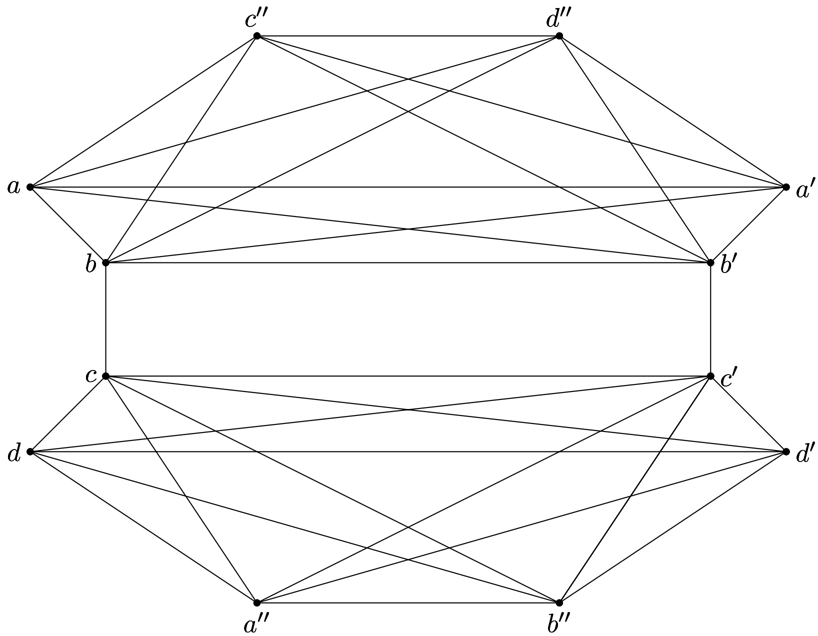

Figure 9: Graph $W = \Gamma \circ_\pi \{B_X\}_{X \in \Pi}$.

(1) $\Gamma \circ_\pi \{B_X\}_{X\in\Pi} = \Gamma' \circ_{\pi'} \{A_Z\}_{Z\in\Pi'}$, where Γ' is the quotient graph W/E'_0 with vertices $V(\Gamma') = \{1, 2, \ldots, 10\}$, and $\{A_Z\}_{Z\in\Pi'}$ are the subgraphs of W induced by the equivalence classes of E'_1, and $\pi' = \dot{\bigcup}_{Z\in\Pi'} \pi'_Z$ maps $\{a\}, \{b\}, \{c\}, \{d\}, \{a'\}$, $\{b'\}, \{c'\}, \{d'\}, \{c'', d''\}, \{a'', b''\}$ onto $1, 2, \ldots, 10$, respectively. (See Figure 10.)

(2) Put $Y_Z = \{b, c\}$ and $Y_{Z'} = \{b', c'\}$ and let $A_Z = W[Y_Z]$ and $A_{Z'} = W[Y_{Z'}]$. Then $\phi\colon Y_Z \to Y_{Z'}$ such that $\phi(b) = b'$ and $\phi(c) = c'$ is a graph isomorphism from A_Z onto $A_{Z'}$. Clearly, ϕ preserves the equivalence classes of E'_0 contained in Y_Z.

(3) Y_Z contains two equivalence classes $S'_1 = \{b\}$ and $S'_2 = \{c\}$ of E'_0 such that $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$ where $S_1 = \{a, b\} \in Y/E_0$ and $S_2 = \{c, d\} \in Y/E_0$. Moreover, $W[S_1]$ is connected and $\pi'_Z(b)$ and $\pi'_{Z'}(\phi(b))$ are adjacent and have the same neighbors in $V(\Gamma') \setminus \{Z \cup Z'\}$, where $Z = \{\pi'_Z(b), \pi'_Z(c)\}$ and $Z' = \{\pi'_{Z'}(b'), \pi'_{Z'}(c')\}$. Similarly, $W[S_2]$ is connected and $\pi'_Z(c)$ and $\pi'_{Z'}(\phi(c))$ are adjacent and have the same neighbors in $V(\Gamma') \setminus \{Z \cup Z'\}$.

(4) The vertex b is nonadjacent to $\phi(c)$ and vertex c is nonadjacent to $\phi(b)$.

Then the conditions of Theorem 4.2 hold. So

$$\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} \lneqq \mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi}).$$

In the following as a main result, we give necessary and sufficient conditions under which the full automorphism group of the generalized X-join of graphs is equal to the generalized wreath product of the automorphism groups of their factors.

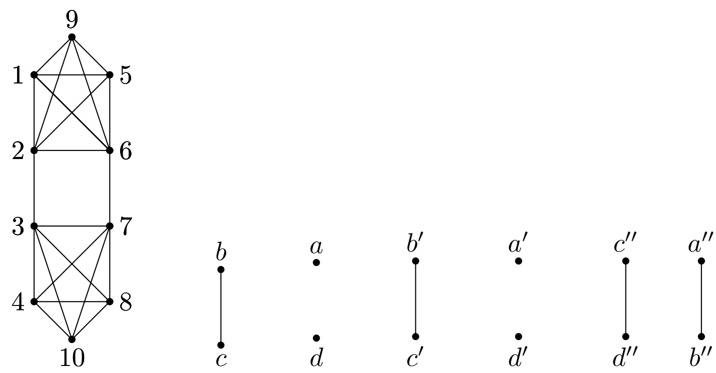

Figure 10: Graph Γ' and set of graphs $\{A_Z\}_{Z\in\Pi'}$.

Corollary 4.4. *Suppose that $W = \Gamma \circ_\pi \{B_X\}_{X\in\Pi}$ is such that the graph $\Gamma = (V, R)$ and the set of graphs $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ satisfy the conditions $(G1)$ and $(G2)$. Then $\mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi}) = \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi}$ if and only if there are no equivalence relations $E'_0 \subseteq E'_1$ on Y satisfying the conditions (i), (ii), (iii), and (iv) of* Theorem 4.2.

Proof. This follows immediately from Theorem 4.2. □

Example 4.5. Let $W = (Y, E)$ be the graph in Figure 11. It is easy to see that W is the graph $\Gamma \circ_\pi \{B_X, B_{X'}, B_{X''}\}$ where Γ and $\{B_X, B_{X'}, B_{X''}\}$, and $\pi = \pi_X \cup \pi_{X'} \cup \pi_{X''}$ are given in Example 3.3. Moreover, $Y/E_0 = \{\{a, c\}, \{b, d\}, \{a', c'\}, \{b', d'\}, \{b'', c''\}, \{a'', d''\}\}$ and $Y/E_1 = \{\{a, c, b, d\}, \{a', c', b', d'\}, \{b'', c'', a'', d''\}\}$. Since there are no equivalence relations $E'_0 \subseteq E'_1$ on Y that satisfy the conditions (i), (ii), (iii), and (iv) of Theorem 4.2, it follows that

$$\begin{aligned}\mathrm{Aut}(W) = \mathrm{Aut}(\Gamma)\circ\{\mathrm{Aut}(B_X)\}_{X\in\Pi} = \ &\langle \mathrm{id}_Y, (ab)(cd), (aa'bb')(cc'dd')(a''b'')(c''d''),\\ &(a'b')(c'd'), (ab'ba')(cd'dc')(a''b'')(c''d''),\\ &(aa')(bb')(cc')(dd')(a''b'')(c''d''),\\ &(ab')(ba')(cd')(dc')(a''b'')(c''d'')\rangle.\end{aligned}$$

The next corollary follows directly from Theorem 4.2.

Corollary 4.6. *Suppose that the graph $\Gamma = (V, R)$ and the set of graphs $\{B_X = (Y_X, E_X) \mid X \in \Pi\}$ satisfy the conditions $(G1)$ and $(G2)$. Let $W = (Y, E)$ be the generalized X-join of Γ and $\{B_X\}_{X\in\Pi}$ with respect to π and let $E_0 \subseteq E_1$ be the equivalence relations defined in (1) and (2). Then*

$$\mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_X)\}_{X\in\Pi} = \mathrm{Aut}(\Gamma \circ_\pi \{B_X\}_{X\in\Pi})$$

if W is uniquely determined by E_0 and E_1.

Corollary 4.7 (See [8, Theorem 2.10])**.** *Let $\Gamma = (V, R)$ be a graph and $\{B_x \mid x \in V\}$ be a set of graphs such that $B_x \simeq B_{x'}$ whenever $x^f = x'$ for some $f \in \mathrm{Aut}(\Gamma)$. Then*

$$\mathrm{Aut}(\Gamma[B_x]_{x\in V}) = \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_x)\}_{x\in V}$$

if and only if

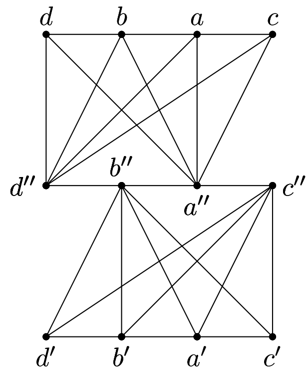

Figure 11: Graph W.

(1) *B_x is connected if there exists at least one vertex $w \in V$ such that x and w are nonadjacent and have the same neighbors in V,*

(2) *$\overline{B_x}$ is connected if there exists at least one vertex $w \in V$ such that x and w are adjacent and have the same neighbors in $V \setminus \{x, w\}$.*

Proof. By Example 2.2, $W = \Gamma[B_x]_{x \in V} = \Gamma \circ_\pi \{B_x\}_{x \in V}$ where $\Pi = \{\{x\} \mid x \in V\}$ and $\pi_x \colon Y_x \to X$ is a graph epimorphism from B_x onto $\Gamma(X)$ such that $\pi_x(y_x) = x$ for every $y_x \in Y_x$. In this case $E_0 = E_1$ and $F^X = F_{(X)}$. If we define

$$\eta_X := \mathrm{Aut}(B_X) \to F^X / F_{(X)}$$

by $\eta_X(\alpha) = 1_{F^X/F_{(X)}}$ for every $\alpha \in \mathrm{Aut}(B_X)$, then η_X is an epimorphism and condition (G2) holds. Then it follows from Corollary 4.4 that

$$\mathrm{Aut}(\Gamma[B_x]_{x \in V}) = \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_x)\}_{x \in V}$$

if and only if there is no equivalence relation E_0' on Y satisfying the conditions (i), (ii), (iii), and (iv) of Theorem 4.2.

Now suppose that $\mathrm{Aut}(\Gamma[B_x]_{x \in V}) = \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_x)\}_{x \in V}$ and there exist $x, w \in V$ such that x and w are nonadjacent and have the same neighbors in V. If B_x is disconnected then B_w is also disconnected and we can define an equivalence relation E_0' on Y such that the equivalence classes of E_0' are $Y_z, z \notin \{x, w\}$, together with the connected components of B_x and B_w. Since x and w are nonadjacent and have the same neighbors in V, one can see that the conditions (i), (ii), (iii), and (iv) of Theorem 4.2 hold, a contradiction. So B_x is connected. Moreover, suppose that there exist $x, w \in V$ such that x and w are adjacent and have the same neighbors in $V \setminus \{x, w\}$. If $\overline{B_x}$ is disconnected, then there exist at least two subsets $S_1, S_2 \subset Y_x$ such that all vertices of S_1 are adjacent to all vertices of S_2. Similarly, there exist subsets $S_1', S_2' \subset Y_w$ with the property that all vertices of S_1' are adjacent to all vertices of S_2'. Then we can define an equivalence relation E_0' on Y such that S_1, S_2, S_1', and S_2' together with $Y_z, z \notin \{x, w\}$ are its equivalence classes. One can see that in this case the conditions (i), (ii), (iii), and (iv) of Theorem 4.2 hold and thus again we have a contradiction.

Conversely, suppose that conditions (1) and (2) hold and suppose on the contrary that

$$\varphi \in \mathrm{Aut}(\Gamma[B_x]_{x\in V}) \setminus \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_x)\}_{x\in V}.$$

Then there is an equivalence relation E_0' on Y satisfying the conditions (i), (ii), (iii), and (iv) of Theorem 4.2. It follows from condition (i) that $W = \Gamma' \circ_{\pi'} \{A_z\}_{z\in V(\Gamma')}$, where the graph Γ' is the quotient graph W/E_0' and $\{A_z\}_{z\in V(\Gamma')}$ are the subgraphs of W induced by the equivalence classes of E_0'. It follows from (ii) that there exist $x, w \in V$ such that the equivalence classes of E_0' contain $Y_z \subsetneq Y_x$ and $Y_{z'} = \varphi(Y_z) \subsetneq Y_w$. By (iii) if $B_x[Y_z]$ is a union of connected components of B_x, then z and z' are nonadjacent and all of their neighbors are exactly the same in $V(\Gamma')$, otherwise z and z' are adjacent and have the same neighbors in $V(\Gamma') \setminus \{z, z'\}$. This implies that if B_x is disconnected then z and z' are nonadjacent and all vertices in Y_z and all vertices in $Y_{z'}$ have the same neighbors in $Y \setminus (Y_z \cup Y_{z'})$. Since $Y_z \subsetneq Y_x$ and $Y_{z'} \subsetneq Y_w$ it follows that x and w must be nonadjacent and have the same neighbors in V, which contradicts (1). Moreover, if B_x is connected, since z and z' are adjacent and have the same neighbors in $V(\Gamma') \setminus \{z, z'\}$ it follows that all vertices in Y_z are adjacent to all vertices of $Y_{z'}$ and all vertices in Y_z and all vertices in $Y_{z'}$ have the same neighbors in $Y \setminus (Y_z \cup Y_{z'})$. Then all vertices in Y_x are adjacent to all vertices of Y_w. So x and w must be adjacent and have the same neighbors in $V \setminus \{x, w\}$. Furthermore, since all vertices in Y_z are adjacent to all vertices of $Y_x \setminus Y_z$ it follows that $\overline{B_x}$ is disconnected, which contradicts (2). Thus we have

$$\mathrm{Aut}(\Gamma[B_x]_{x\in V}) = \mathrm{Aut}(\Gamma) \circ \{\mathrm{Aut}(B_x)\}_{x\in V}. \qquad \square$$

5 Conclusion

A generalization of the X-join of graphs has been introduced and necessary and sufficient conditions under which a graph is isomorphic to a generalized X-join has been given. A generating set for the automorphism groups of a class of graphs which are isomorphic to a generalized X-join has been computed.

Since the generalized X-join of graphs is a natural generalization of the X-join of graphs, the results on the X-join or lexicographic product of graphs can be also studied for the generalized X-join of graphs.

References

[1] J. Bagherian, Schurity of the wedge product of association schemes and generalized wreath product of permutation groups, *Discrete Math.* **343** (2020), 10, doi:10.1016/j.disc.2020.112084, id/No 112084, https://doi.org/10.1016/j.disc.2020.112084.

[2] S. Bhoumik, T. Dobson and J. Morris, On the automorphism groups of almost all circulant graphs and digraphs, *Ars Math. Contemp.* **7** (2014), 499–518, doi:10.26493/1855-3974.315.868, https://doi.org/10.26493/1855-3974.315.868.

[3] E. Dobson and J. Morris, Automorphism groups of wreath product digraphs, *Electron. J. Comb.* **16** (2009), research paper r17, 30, doi:10.37236/106, https://doi.org/10.37236/106.

[4] T. Dobson, A. Malnič and D. Marušič, *Symmetry in Graphs*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2022, doi:10.1017/9781108553995, https://doi.org/10.1017/9781108553995.

[5] S. A. Evdokimov and I. N. Ponomarenko, Schurity of S-rings over a cyclic group and generalized wreath product of permutation groups, *St. Petersbg. Math. J.* **24** (2013), 431–460, doi:10.1090/S1061-0022-2013-01246-5, https://doi.org/10.1090/S1061-0022-2013-01246-5.

[6] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product of Graphs*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, 2nd edition, 2011.

[7] F. Harary, On the group of the composition of two graphs, *Duke Math. J.* **26** (1959), 29–36, doi:10.1215/S0012-7094-59-02603-1, https://doi.org/10.1215/S0012-7094-59-02603-1.

[8] R. L. Hemminger, The group of an X-join of graphs, *J. Comb. Theory* **5** (1968), 408–418, doi:10.1016/S0021-9800(68)80017-1, https://doi.org/10.1016/S0021-9800(68)80017-1.

[9] R. Mathon, A note on the graph isomorphism counting problem, *Inf. Process. Lett.* **8** (1979), 131–132, doi:10.1016/0020-0190(79)90004-8, https://doi.org/10.1016/0020-0190(79)90004-8.

[10] M. Muzychuk, A wedge product of association schemes, *Eur. J. Comb.* **30** (2009), 705–715, doi:10.1016/j.ejc.2008.07.008, https://doi.org/10.1016/j.ejc.2008.07.008.

[11] G. Sabidussi, The lexicographic product of graphs, *Duke Math J.* **26** (1961), 573–578, doi:10.1215/S0012-7094-61-02857-5, https://doi.org/10.1215/S0012-7094-61-02857-5.

[12] B. Weisfeiler, *On Construction and Identifcation of Graphs*, volume 558 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.