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# 10 Do We Find High Energy Physics Inside (Almost) Every Solid or Fluid at Low Temperature?

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**Abstract.** It is an old idea of ours (H. B. Nielsen "Dual Models",section 6 "Catastrophe Theory Program", Scottish University Summer School, 1976) that a most general material with only translation symmetry, but otherwise no symmetries should generically (in general) have some small regions in quasi momentum space, where you "see" an approximate Weyl equation behavior. The Weyl equation is the relativistic equation for a (left handed) neutrino. This remark means that one could imagine, that there were behind the Standard Model of High energy physics, a very general crystal model with very little symmetry. Even for the Yang Mills or electrodynamics types fields a similar philosophy is possible. There are though some problems with this solid-state type of model beyond the Standard model, for which we thought have some remedy by means of homolumo gap effects.

By making use of relativistic quantum field theory on the lattice we predicted theoretically very high magneto-conduction due to Adler-Bell-Jackiw chiral anomaly effect – so called Nielsen-Ninomiya effect (or mechanism) in gapless parity violating material. Nowadays this kind of material such as chiral or Weyl semimetal and the effect are detected by experiments.

**Povzetek.** Avtorja obravnavata idejo HBN (H. B. Nielsen "Dual Models", razdelek 6 "Catastrophe Theory Program", Scottish University Summer School, 1976), da obstajajo v najbolj splošnem modelu za snov, ki ima le translacijsko simetrijo, majhna območja v prostoru kvazi gibalne količine, v katerih približno velja Weylova enačba. Ker velja Weylova enačba za relativistično gibanje (levoročnih) nevtrinov, predlagata, da razširjeni standardni model gledamo kot zelo splošen model za kristal z zelo malo simetrijami. Podoben pristop uporabita za primer elektromagnetnega polja in vsa Yang-Millsova polja. Težave, ki se pri tem pojavijo, omilita s "homo-lumo" vrzelmi.

Uporaba relativistične kvantne teorije polja na rešetki napove visoko magnetno prevodnost, ki jo sproži kiralna anomalija Adler-Bell-Jackiwa, ter s tem pojav Nielsen-Ninomiye: visoko magnetno prevodnost v snoveh, ki kršijo parnost, med obema pasovoma pa ni vrzeli. Te lastnosti materialov merijo v Weylovih (kiralnih) polkovinah.

Keywords: Weyl equation, homo-lumo gap

## Introduction

The authors, in particular H. B. N. have through many years the dream, that it is not important what the (most) fundamental laws of Nature might be, because almost certainly the same effective laws would come out anyway: This philosophy is called "Random Dynamics".

Inside a piece of matter - crystal, glass, ... - one should then at very low temperature according to this dream find the Standard Model.

Recently one is about to find Cases of Relativity-behaving Quasi-particles: A material, e.g. graphene, with such simulations of relativistic particles as we talk about.

Materials with relativistic particles simulated as quasiparticles may be very applicable to say high conductivity purposes,...

Some of our publications:

- H. B. Nielsen and M. Ninomiya, "No Go Theorem for Regularizing Chiral Fermions," Phys. Lett. **105B**, 219 (1981).
- H. B. Nielsen and M. Ninomiya, "Absence of Neutrinos on a Lattice, 1. Proof by homotopy theory" Nucl. Phys. B **185**, 20 (1981).
- H. B. Nielsen and M. Ninomiya, "Absence of Neutrinos on a Lattice. 2. Intuitive Topological Proof," Nucl. Phys. B **193**, 173 (1981).
- As for the initiation of Random Dynamics, See "Fundamentals of Quark Models". Proceedings: 17th Scottish Universities Summer School in Physics, St. Andrews, Aug 1976, I.M. Barbour, A.T. Davies (Glasgow U.);1977 - 588 pages; Edinburgh: SUSSP Publ. (1977);Conference: C76-08-01; Contributions: Dual Strings, Holger Bech Nielsen (Bohr Inst.). Aug 1974, 71 pp.;NBI-HE-74-15 In the last section the idea of "Random Dynamics " is introduced based on finding Weyl equation in "whatever".

**The present paper consists as part I and part II.**

**The part I: Relativity Theory found in solid state.**

**and**

**The part II "What comes beyond Topological Insulator – Nielsen-Ninomiya Effect (or Mechanism) due to ABJ Anomaly –"**

**Part I: Relativity-Theory found in Solid State Physics**

**I-1** Introduction
**I-2** Automatic: a pet-thought: Natural laws come by themselves! ("Random Dynamics")
**I-3** General: A very general world with (only) momentum conservation.
**I-4** Graphene: Example Graphene.
**I-5** Heusler: Half-metals, Heusler compounds.
**I-6** Wang: Thoughts about making materials having models of relativistic particles inside.
**I-7** Doubling: Nielsen - Ninomiya theorem about doubling of such relativistic particles unavoidably on the lattice. great future; hope of seeing high energy physics in low temperature materials not out, but not quite finished. material simulates relativistic quantum field theory.

## I-2 Automatic

**Our Old Work in 1976: Dreams Laws of Nature Automatic**

"Dual Strings. Fundamentals of Quark Models." by H. B. Nielsen, in Scottish University Summer School in Physics, St. Andrews, 1976 (There H.B.N. still mainly is talked on String theory, but at the end a general (fermion) Hamiltonian is studied.)

Assumed was **translational invariance**, at least with respect to a lattice say, and thus a (quasi) momentum conservation, but with respect to the **"internal degrees of freedom"** there is a **very general** theory, though assuming there being essentially a finite (discrete). system of states(representing possibly spin and band degrees of freedom.).

**(Trivial) Generic Considerations on Fermion Dispersion relations (1976)**. We ignore all conservation laws except for

- Energy conservation and Hamiltonian development.
- Momentum Conservation.
- Particle (number) conservation.
- Free approximation (first).
- Smoothness, (so that e.g $\mathbf{H}(\vec{p})$ is differentiable and continuous as function of $\vec{p}$.)
- Generic: i.e. no fine-tuned values of parameters,

and consider a single particle equation:

$$i\frac{\partial}{\partial t}\psi(\vec{p},t) = \mathbf{H}(\vec{p})\psi(\vec{p}), \tag{10.1}$$

where for each value of the momentum $\vec{p}$ the $\mathbf{H}(\vec{p})$ is a Hermitian matrix.

Relativity and Dimensionality of Space time being 3+1 come out Automatically!

A priori - with no fine-tuning (=generically) - the Fermi surface would put itself at separate eigenvalues; but if for some reason ( e.g. "homlumo-gap effect")

the Fermi-level were just where $n = 2$ levels meet, then in a small neighborhood the shape of the dispersion relations would be given by taking $\mathbf{H}(\tilde{\mathbf{p}})$ to be $n \times n = 2 \times 2$. We then Taylor expand

$$\mathbf{H}(\vec{p}) \approx \mathbf{H}(\vec{p}_0) + \sum_{a,\mu} \sigma^a V_a^\mu p_\mu + ... \tag{10.2}$$

where $\sigma^a$ are the Pauli-matrices and the unit matrix $\sigma^0 = \mathbf{1}$. The "vierbein" $V_a^\mu$ is a set of expansion coefficients for $\mathbf{H}(\vec{p})$ as function of the components $p_\mu$ (strictly speaking $\mu$ =1,2,3; here).

**Hermitian matrix, Provided Fermi-level at Degeneracy $n = 2$ leads to Weyl Equation in 3+1 Dimensions.**

In the old days we argued that in a general physics universe the **Hubble expansion** would finally lead to the Fermi-level approaching an $n = 2$ degenerate levels energy; but now H. B. N.'s Zagreb group - I.Andric, L. Jonke, D. Jurman, and HBN - have studied in general, what is called **"Homolumo-gap Effect"** meaning the by Jahn and Teller[1] first proposed effect, that the electrons filling the Fermi-sea would back react such as to increase the homolumo gap between the lowest unoccupied (LUMO) and the highest occupied (HOMO) state. This effect goes in the direction to make metals not occur, and make every materials become an insulator, but the gapless semiconductor may be too hard for the homolumo-gap effect to dispense with.

Note that this hope for getting automaticly a Weyl-equation like theory had, when using just Hermitean Hamiltonian marices and looking at the $n = 2$ degeneracy possibility, the consequence that there came only *three* spatial dimensions functioning the relativistic way, because there were only 3 Pauli matrices. Somehow arguing that the dimensions for which there are no Pauli matrices will lead to essentially zero velocity for the fermion/quasi-electron in these directions and that such dimensions will not be observed, we have come to 3+1 dimensions as an additional prediction from the very general starting theory!

With time-reversal symmetry imposed dimension prediction gets modified.

| Symmetry | Square | Pauli M. | Dimension | Field |
|---|---|---|---|---|
| TP | $(TP)^2 = 1$ | $\sigma_x, \sigma_z$ | 2+1 | Real |
| - | - | $\sigma_x, \sigma_y, \sigma_z$ | 3+1 | Complex |
| TP | $(TP)^2 = -1$ | 5 of them | 5+1 | Quaternions |

**Table 10.1.** The symmetry assumed in line 1 and 3 is the combination of time reversal T and parity P to TP, which leaves the momentum $\vec{p}$ invariant but is an antilinear operator effectively conjugating the complex numbers in the matrix. If then Fermi-level falls at $n = 2$ degenerate levels in addition to the Kramers-Kronig doubling in the 3rd case, one gets by Taylor expanding the $2 \times 2$ resolved into Pauli-matrices, and a generalized Weyl equation results corresponding to the in fourth column denote *space + time* dimensions. Actually the effective theory is naturally written in terms of the in column 5 mentioned division-algebra(= field).

**Fundamentally in many Dimensions, but in Most dimensions the Fermion Run with Zero Velocity, we Ignore them.**

In the for fundamental physics ideal situation of **no extra T or TP symmetry** the Hamiltonian matrix $\mathbf{H}(\vec{p})$ is just a generic(~ random) Hermitian matrix (with complex matrix elements), and it predicts at the two levels degenerate point - hoped to be favored at the Fermi-surface by either Hubble expansion or homolumo-gap-effect - that the Fermion only moves with appreciable velocity in as many spatial dimensions as there are Pauli-matrices. We hope that the dimensions in which the velocity gets zero, can/shall be ignored. If the zero-velocity dimensions are ignored, then we have remarkable agreement:

The number of dimensions in which the generic double degeneracy neighborhood has the fermions move just corresponds to experimental number of dimensions 3+1 and to having relativity and rotational invariance!

**If TP (or T) is good symmetry and $(TS)^2 = 1$ then $\mathbf{H}(\vec{p})$ must have real matrix elements.**

This is the case in which we in a crystal - with PT symmetry say - **completely ignore** the usual spin as being decoupled so as to be totally ignored.

In this case we get the effective dimensionality, if we ignore the zero-velocity directions:

$$\mathbf{2+1}$$

This means that the relativistic effective fermion should appear "generically" (automatically) even in only 2 spatial dimensions.

With Genuine Spin=$\frac{1}{2}$ Electrons and Unbroken Time reversal, the "Quaternion Case" If T or TP good symmetries, and spin $\frac{1}{2}$ included, then $T^2 = (TP)^2 = -1$ we have generally doubling of all levels according to Kramers-Kronig rule.[2]

So double degeneracy is already there generally and nothing special. In this case we shall therefore instead consider that we can get 4 times degenerate levels sporadically. If we go to such a 4-times degenerate point in momentum space, we could elegantly go to a quaternion $2 \times 2$ matrices (quaternions are writable as $2 \times 2$ complex matrices, so that $2 \times 2$ quaternion matrices can be equivalent to $4 \times 4$ complex matrices with some restriction. Dimension of non-zero velocity directions:

$$\mathbf{5+1}$$

## I-3 Graphene

Graphene denotes the layer of carbon like the ones in graphite taken as seperate, i.e. it is 2(space)dimensional material. The quasi electrons running in the graphene layers actuall do show dispersion relations behaving how we above argued for the case with time reversal but ignoring the spin leading to the effective space time dimension 2+1.

On the following picture 10.1 one sees the lattice structure of graphene:

The next figure 10.2 is supposed to generally illustrate a metal, an insulator and a material with a Dirac-like quasi particle (on the figure 10.1).

Even just making a two-layer of graphene complicates the situation and the work by Gammelgaard on the next figure 10.3 illustrates a gap appearing:

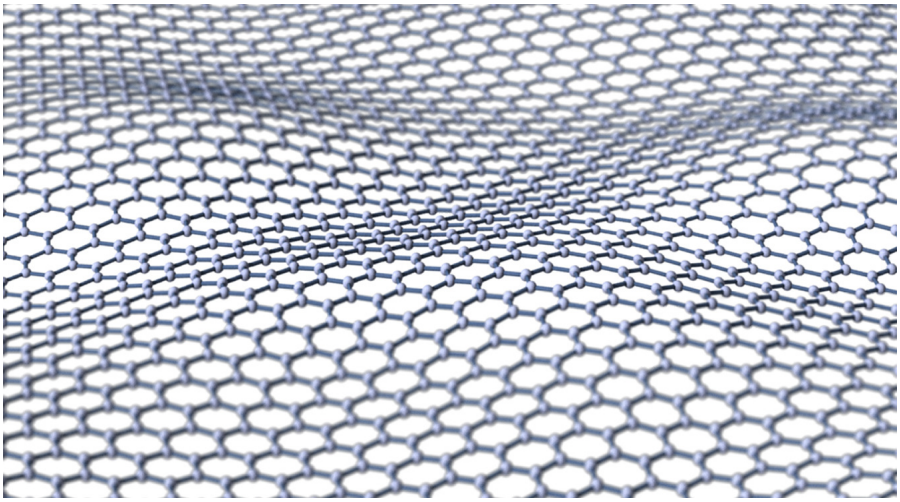

**Fig. 10.1.** (2+1)-dimensional Example is Graphene.

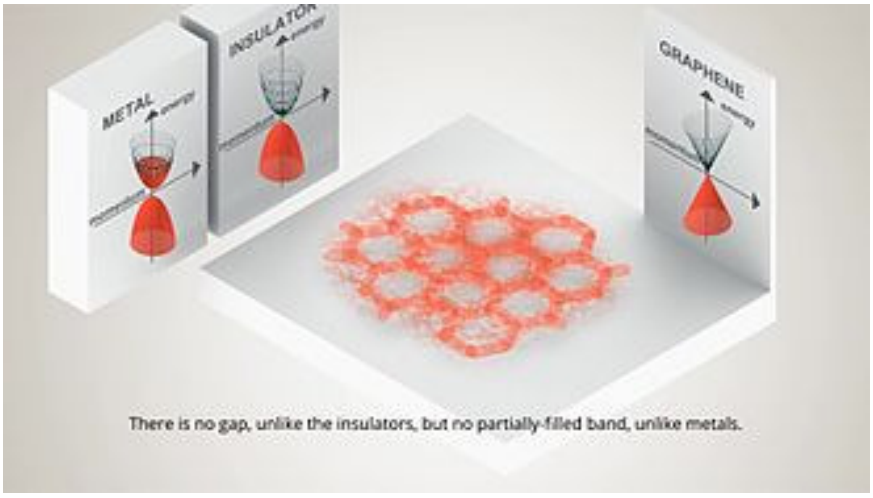


**Fig. 10.2.**

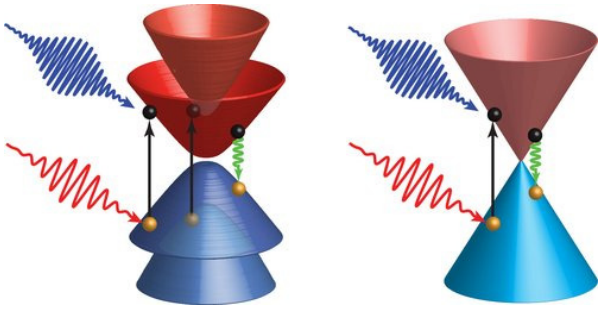

**Fig. 10.3.** Putting Double Layer Produces Gap.

The left dispersion law is for a double layer of graphene; the right for single layer. (Gammelgaard).

The next figures 10.4 illustrate calculation of the dispersion relations for quasi-electrons in graphene by the model described just below. Since we have a 2 space dimension material the energy can be the orbital direction up in the perspective while the two spatial momentum components form the basis plane of the three-dimensional perspective figure:

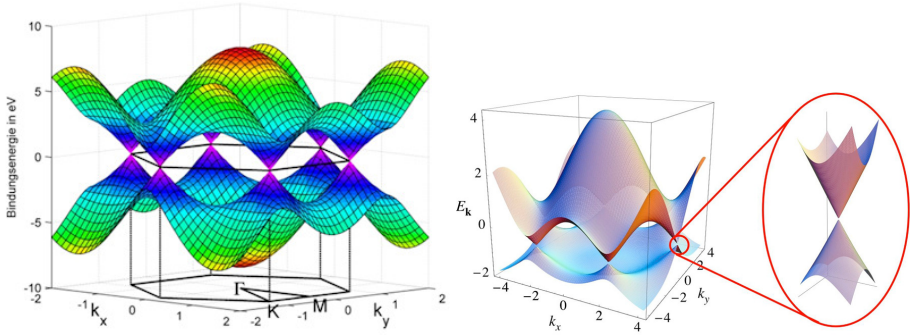


**Fig. 10.4.** .

The Dirac points are of course the points where two branches of the dispersion relation meet *with a cone shape.* (Fig. 10.5):

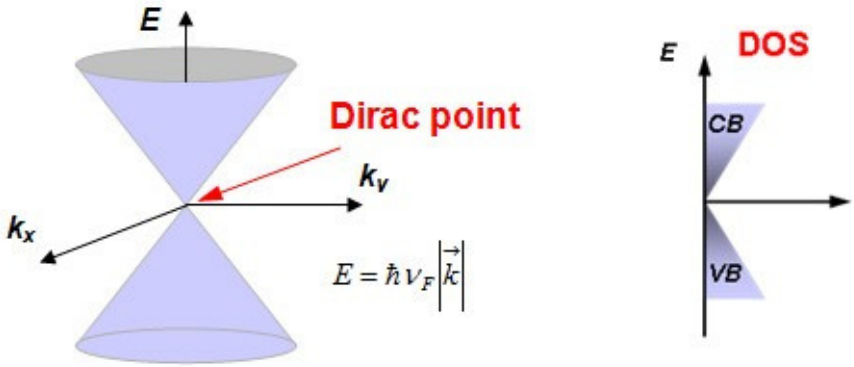


**Fig. 10.5.**

**Dispersion Relation of Graphene** The electronic properties of graphene can be described using a simple tight binding model. The electrons in the covalent bonds form deep fully filled valence bands, and thus their effects on the conductivity can be safely disregarded. The unhybridized p orbital is only slightly perturbed by the neighboring atoms. Therefore, the wave function of an electron in the system can be written as a Linear Combination of Atomic Orbitals (LCAO). Using

these orbitals as the basis set to represent the wave function, the Hamiltonian that governs the dynamics of the electron is given by:

$$H = \sum_i \epsilon_i |\psi_i><\psi_i| + \sum_l \sum_{\{<i|j>\}_l} t_l (|\psi_i><\psi_j| + |\psi_j><\psi_i|) \quad (10.3)$$

where$\epsilon_i$ represents the onsite energy at the atom, $|\psi_i>$ the i'th atomic orbital , $\{<i,j>\}_l$ the set of couples of lth-nearest neighbors, and $t_l$ the hopping parameter between them.

**In Graphene the Fermi- surface just Lies at the Double degenerate Point**

So in graphene by symmetry one really get a simulation of a 2+1 dimensional massless Weyl/Dirac fermion, also w.r.t. the placing of the fermi surface.

If we think of just the generic case of a very general theory there will typically be no reason why the fermi surface should be just at the Weyl point (with the double degeneracy).

We have, however, speculated on two mechanisms, which might make the fermi-surface be driven towards the degeneracy point:

- If the world in question has a strong Hubble expansion, then filled states above the degeneracy point would be gradually emptied and holes below the degeneracy point would be also gradually be expanded away/attenuated.
- "Homolumo-gap-effect" - meaning that the fermions act back onto the various degrees of freedom that can be adjusted in the lattice in which the fermions run. This back action will be so as to in the ground state arrange to lower the energies of filled fermi states. Thereby arise the so called Homolumo-gap, or rather it gets expanded by this back action "homolumo-gap-effect". In the case that we have degeneracy point that is somehow topologically stabilized, as one might say of the Weyl points discussed here, it may not be possible for the homolumo-gap-effect to really produce a gap. In stead we expect that it will only bring the fermi surface to coincide with the degeneracy point; that would namely lower the filled states as much as possible with the "topological ensurance" of the degeneracy point.

## I-4 Heusler

**Heusler Compound $Mn_2CoAl$ is a Spin Gapless Semicondutor**:

Siham Oardi, G.H. Fecher, C. Felser and J. Kübler (arXiv:1210.0148v1 [cond-mat.mtrl-sci], 29 Sep. 2012.) investigated the **Heusler compound $Mn_2CoAl$.** They gave the article the name **Realization of spin gapless semiconductors: the Heusler compound $Mn_2CoAl$.**

In halfmetallic ferromagnets you have so to speak metal as far as the electrons with one direction of the spin is concerned, but insulator w.r.t. to the elctrons with the opposite spin direction. Now it may further happen that we instead of the metallic we get a gapless semiconductor, namely if we have a degeneracy point as we discussed above. Once there is effectively only one spin of the electron one escapes the time reversal symmetry. Thus in such halfmettals there is a better chance to find Weyl points.

The following figure 10.6 illustrates dispersion relation along a piecewise straight curve in momentum space for the two different spin directions along the magnetization axis for the compound $Mn_2CoAl$. The dispersion relation for the two different spin orientations are printed respectively red and blue:

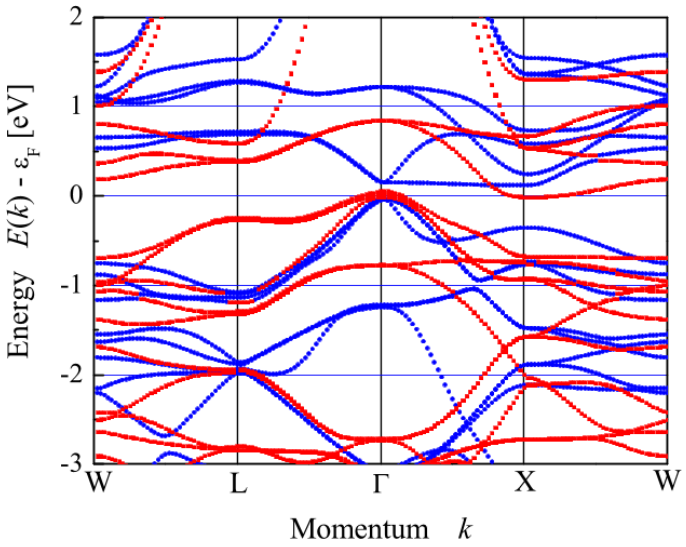


**Fig. 10.6.** Band structure of $Mn_2CoAl$, Majority spin red.

In the following figure 10.7 are then as function of temperature given some carrier properties of this material $Mn_2CoAl$:

On the following page from Lakhan Baisly et al. as figure 7 in their article we see the density of electron levels (DOS) for the two spin orientations seperately. In the in red shown DOS there can be seen crudely a gap, so for this spin orientation we have the insulator. For the other spin orientation - shown with the positive ordinate pointing upwards there is also a dip at the fermilevel, but now the DOS is going non-zero immediately by going away from the fermilevel. So for this spin we rather have the gapless semiconductor behavior.

The strong dependence of the conductivity as function of the magnetic field is just what one expects due to the Adler-Bell-Jackiw-anomaly-effect described more in part II of the present article below.

These figures are from:

Siham Ouardi et al. "Realization of Spin Gapless Semiconductors: The Heusler Compound $Mn_2CoAl$" DOI: 10.1103/PhysRevLett.110.100401.

**Zero Gap Material with Quadratic Energy Dispersion** (this is by fine tuning) $HgTe$ is one of the few materials wherin this quadratic dispersion law zero gap has been found, since 1950's.

$Pb_{1-x}Sn_xTe$, $Pb_{1-x}Sn_xSeandBi_xSb_{1-x}$ are zero-gap materials (with quadratic disp.).

But really one - Wang, Dou, and Zhang - expects that all narrow gap semiconductors by some doping or pressure could be tuned to have zero gap (with

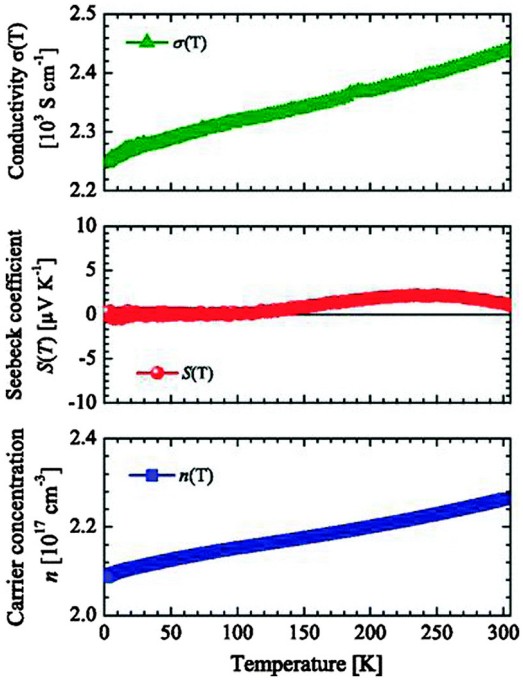


**Fig. 10.7.** Majority spin and Minority spin. Calculated with spin orbit coupling.

quadratic dispersion law). Then they call for finding a non-toxic material of this kind.

## I-5 Wang

Physical Chemistry; Chemical Physics

Controllable electronic and magnetic properties in a two-dimensional germanene heterostructure Run-wu Zhang, Wei-xiao Ji, Chang-wen Zhang,* Shengshi Li,b Ping Li, Pei-ji Wang, Feng Lia and Miao-juan Rena Author affiliations Abstract

The control of spin without a magnetic field is one of the challenges in developing spintronic devices. Here, based on first-principles calculations, we predict a new kind of ferromagnetic half-metal (HM) with a Curie temperature of 244 K in a two-dimensional (2D) germanene Van der Waals heterostructure (HTS). Its electronic band structures and magnetic properties can be tuned with respect to external strain and electric field. More interestingly, a transition from HM to bipolar-magnetic-semiconductor (BMS) to spin-gapless-semiconductor (SGS) in a HTS can be realized by adjusting the interlayer spacing. These findings provide a promising platform for 2D germanene materials, which hold great potential for application in nanoelectronic and spintronic devices.

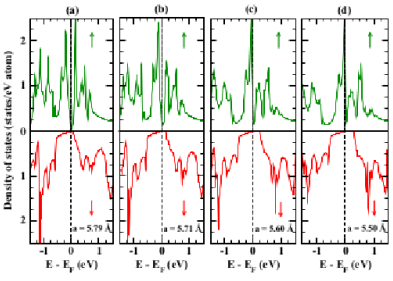


FIG. 7. (Color online) CFCG DOS versus $a$(Å): (a) 5.79 ($a_{\text{expt}}$), (b) 5.71, (c) 5.60, and (d) 5.50.

of the half-metallic $Co_2FeSi$ ($\approx 200$ S/cm at 300 K) [18] and $Co_2MnAl$ ($\approx 2000$ S/cm) [19].

## IV. ELECTRONIC-STRUCTURE CALCULATIONS

To further investigate the electronic properties of CFCG, we have performed first-principles electronic-structure calculations using spin-polarized density functional theory, as employed in Vienna *ab initio* simulation package (VASP) [20] based on a projected-augmented wave basis [21]. The exchange-correlation functional was based on the generalized gradient approximation (GGA). A $16 \times 16 \times 16$ Monkhorst-Pack $k$-point mesh was used for the Brillouin zone integration. We have used a plane-wave cutoff of 340 eV with the convergence criteria of 0.1 meV/cell (10 kbar) for energy (stress).

The prototype of a quaternary Heusler with the composition $XX'YZ$ and space group $F\text{-}43m$ (216) is LiMgPdSn. There are four nonequivalent configurations based on the occupation of various Wyckoff sites by different constituent elements. From the total energy calculations we found the configuration with Co at $X(0,0,0)$, Fe at $X'(1/2,1/2,1/2)$, Cr at $Y(1/4,1/4,1/4)$, and Ga at $Z(3/4,3/4,3/4)$ to be the most stable.

The spin-resolved dispersion and density of states (DOS) of CFCG in the most stable configuration with experimental lattice parameter ($a_{\text{expt}} = 5.79$ Å) are shown in Fig. 6. A closed band-gap character in the majority-spin state and a small open band gap (near the Fermi energy, $E_F$) in the minority-spin state suggest CFCG to behave as a spin gapless semiconductor. The valence band maximum for the minority-spin state is slightly above $E_F$, yielding a negligibly small DOS at $E_F$ ($\approx$0.011 states/eV atom), which arises from mixed contributions of $d$ bands from Co, Fe, and Cr. Though the DOS plot apparently shows a clear gap near $E_F$ in the minority channel, it is actually a disrupted energy gap as the values of DOS are not exactly zero ($\sim$0.003 states/eV/atom just above $E_F$) but negligibly small. With this consideration, we find a small band gap of $\sim$0.07 eV in the minority-spin states. A careful analysis of various bands crossing the $E_F$ for the minority-spin state (right panel) indicates that such small DOS at $E_F$ arises mainly from three bands, two of which are degenerate and are composed of $\sim$54% Co and $\sim$46% Fe $e_g$ subband characters (i.e., $d_{x2-y2}$ and $d_{3z2-1}$). The third band is contributed almost equally by $t_{2g}$ subbands (i.e., $d_{xy}, d_{yz}$, and $d_{xz}$) of Co (35%), Fe (29%), and Cr (32%).

Notably, some other Ga-based Heusler alloys [7,22,23] have also been shown to have similar bands crossing the Fermi level in the minority-spin states with very small values of DOS near $E_F$. These materials are predicted to be half-metallic, i.e., semiconducting in the minority-spin channel as in our case (with a negligibly small DOS at $E_F$) and metallic in the majority-spin channel.

To study the behavior of CFCG under pressure, we have calculated DOS in the most stable configuration versus lattice parameters below $a_{\text{expt}}$, which are plotted in Fig. 7. Notably, the behavior of CFCG changes from SGS to half-metallic with the decrease of the lattice parameter. In the majority-spin state, the DOS at $E_F$ increases significantly from almost zero value at $a_{\text{expt}}$ to a finite value (under pressure); see Fig. 8(a). In addition, the band gap ($\Delta E_g$) at $E_F$ in the minority-spin state increases with lattice parameters decreasing below $a_{\text{expt}}$, indicating the collective effect of transition. The majority DOS at $E_F$ and $\Delta E_g$ with varying lattice parameters are shown in Fig. 8(a).

Figure 8(b) shows the variation of total magnetic moment per formula unit and the individual atomic magnetic moments

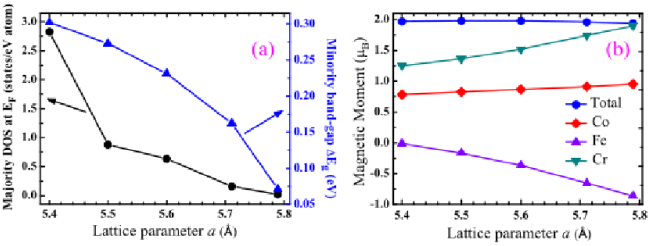


FIG. 8. (Color online) For CFCG, versus $a$ the (a) DOS at $E_F$ in the majority-spin state (left-hand scale) and band gap in the minority-spin state (right-hand scale) and (b) total and site magnetic moments.



**Fig. 10.8.**

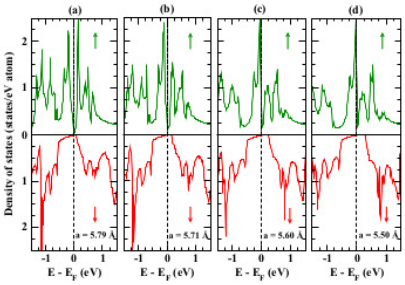


FIG. 7. (Color online) CFCG DOS versus $a$(Å): (a) 5.79 ($a_{\text{expt}}$), (b) 5.71, (c) 5.60, and (d) 5.50.

of the half-metallic $Co_2FeSi$ ($\approx 200$ S/cm at 300 K) [18] and $Co_2MnAl$ ($\approx 2000$ S/cm) [19].

### IV. ELECTRONIC-STRUCTURE CALCULATIONS

To further investigate the electronic properties of CFCG, we have performed first-principles electronic-structure calculations using spin-polarized density functional theory, as employed in Vienna *ab initio* simulation package (VASP) [20] based on a projected-augmented wave basis [21]. The exchange-correlation functional was based on the generalized gradient approximation (GGA). A $16 \times 16 \times 16$ Monkhorst-Pack $k$-point mesh was used for the Brillouin zone integration. We have used a plane-wave cutoff of 340 eV with the convergence criteria of 0.1 meV/cell (10 kbar) for energy (stress).

The prototype of a quaternary Heusler with the composition $XX'YZ$ and space group $F\text{-}43m$ (216) is LiMgPdSn. There are four nonequivalent configurations based on the occupation of various Wyckoff sites by different constituent elements. From the total energy calculations we found the configuration with Co at $X(0,0,0)$, Fe at $X'(1/2,1/2,1/2)$, Cr at $Y(1/4,1/4,1/4)$, and Ga at $Z(3/4,3/4,3/4)$ to be the most stable.

The spin-resolved dispersion and density of states (DOS) of CFCG in the most stable configuration with experimental lattice parameter ($a_{\text{expt}} = 5.79$ Å) are shown in Fig. 6. A closed band-gap character in the majority-spin state and a small open band gap (near the Fermi energy, $E_F$) in the minority-spin state suggest CFCG to behave as a spin gapless semiconductor. The valence band maximum for the minority-spin state is slightly above $E_F$, yielding a negligibly small DOS at $E_F$($\approx$0.011 states/eV atom), which arises from mixed contributions of $d$ bands from Co, Fe, and Cr. Though the DOS plot apparently shows a clear gap near $E_F$ in the minority channel, it is actually a disrupted energy gap as the values of DOS are not exactly zero ($\sim$0.003 states/eV/atom just above $E_F$) but negligibly small. With this consideration, we find a small band gap of $\sim$0.07 eV in the minority-spin states. A careful analysis of various bands crossing the $E_F$ for the minority-spin state (right panel) indicates that such small DOS at $E_F$ arises mainly from three bands, two of which are degenerate and are composed of $\sim$54% Co and $\sim$46% Fe $e_g$ subband characters (i.e., $d_{x2\text{-}y2}$ and $d_{3z2\text{-}1}$). The third band is contributed almost equally by $t_{2g}$ subbands (i.e., $d_{xy}, d_{yz}$, and $d_{xz}$) of Co (35%), Fe (29%), and Cr (32%).

Notably, some other Ga-based Heusler alloys [7,22,23] have also been shown to have similar bands crossing the Fermi level in the minority-spin states with very small values of DOS near $E_F$. These materials are predicted to be half-metallic, i.e., semiconducting in the minority-spin channel as in our case (with a negligibly small DOS at $E_F$) and metallic in the majority-spin channel.

To study the behavior of CFCG under pressure, we have calculated DOS in the most stable configuration versus lattice parameters below $a_{\text{expt}}$, which are plotted in Fig. 7. Notably, the behavior of CFCG changes from SGS to half-metallic with the decrease of the lattice parameter. In the majority-spin state, the DOS at $E_F$ increases significantly from almost zero value at $a_{\text{expt}}$ to a finite value (under pressure); see Fig. 8(a). In addition, the band gap ($\Delta E_g$) at $E_F$ in the minority-spin state increases with lattice parameters decreasing below $a_{\text{expt}}$, indicating the collective effect of transition. The majority DOS at $E_F$ and $\Delta E_g$ with varying lattice parameters are shown in Fig. 8(a).

Figure 8(b) shows the variation of total magnetic moment per formula unit and the individual atomic magnetic moments

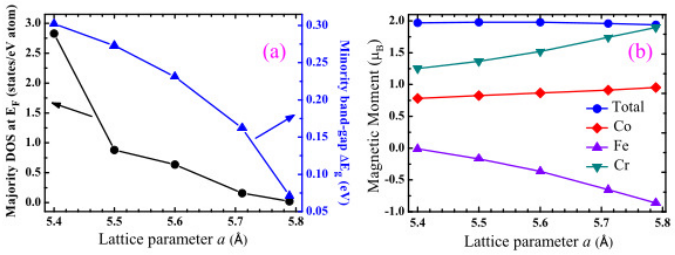


FIG. 8. (Color online) For CFCG, versus $a$ the (a) DOS at $E_F$ in the majority-spin state (left-hand scale) and band gap in the minority-spin state (right-hand scale) and (b) total and site magnetic moments.



**Fig. 10.9.** Hall conductivity as function of magnetic field.

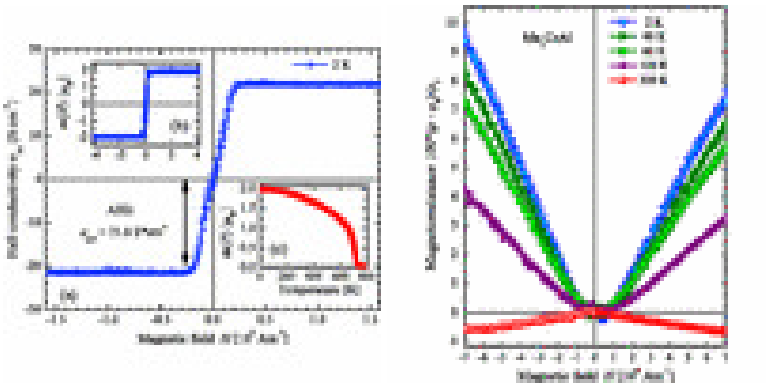

**Fig. 10.10.** Magnetoresistence as function of a magnetic field.

## I-6 Doubling

# Nielsen-Ninomiya's No-go theorem

The authors are very proud of, that we have shown a theorem saying:

**When one makes the mentioned "relativistic fermions of Weyl-type" (=chirale fermion) on a lattice (so e.g. in a crystal) then you always get equally many right-spinning and left-spinning Weyl-type particle(species).**

This theorem is a great challenge for those wanting to make a lattice model (with calculational purposes) for a theory with massless (or almost massless) quarks, let alone the Standard Model.

**By having 3 K +3 K′ Dirac-points of Compensating Handedness Our Doubling Theorem Realized in Graphene.**

## I-7 ABJ Anomaly

In the article

H. B. Nielsen and M. Ninomiya, "Adler-Bell-Jackiw Anomaly And Weyl Fermions In Crystal," Phys. Lett. 130B, 389 (1983). doi:10.1016/0370-2693(83)91529-0

we have put forward how to understand intuitively the Adler-Bell-Jackiw anomaly and how it should be possible to see it in crystals. Indeed now it has -presumably- been found in $Na_3Sb$ in its three dimensional form; at least the characteristic property that this anomaly can lead to a negative magnetoresistance seems justified for this material as should be seen from the following figure 10.12: It is clearly seen for the low temperatures that there is a dramatic peak in the resistance when the magnetic field is small, whereas the resistance becomes appreciably smaller when the magnetic field is switched on. The lower of the two figures shows the resistance in the direction of the magnetic field. It is indeed important that this increased conductivity goes in the direction of the magnetic field and thus there is a dependence of the magnetoresistance as a function also of the angle between the magnetic field and the direction of the electric field.

This subject will be explained in more detail in part II.

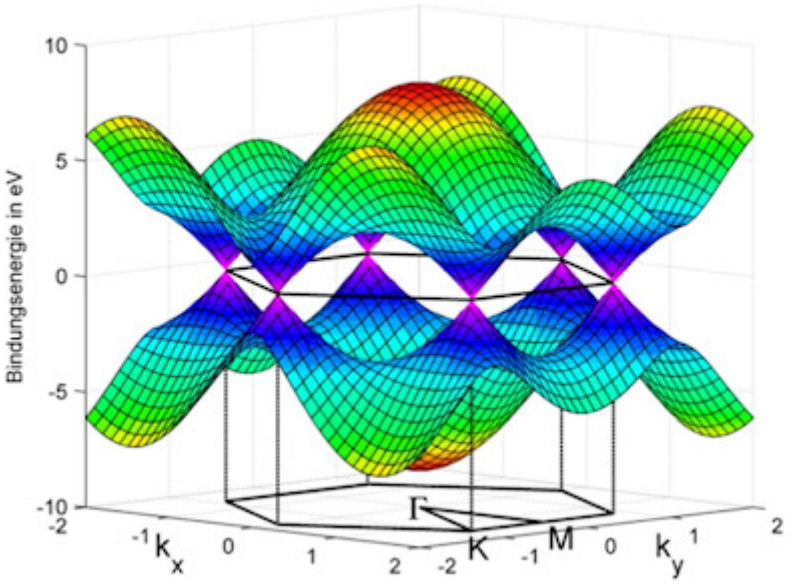


**Fig. 10.11.** Our Doubling Theorem Realized in Graphene.

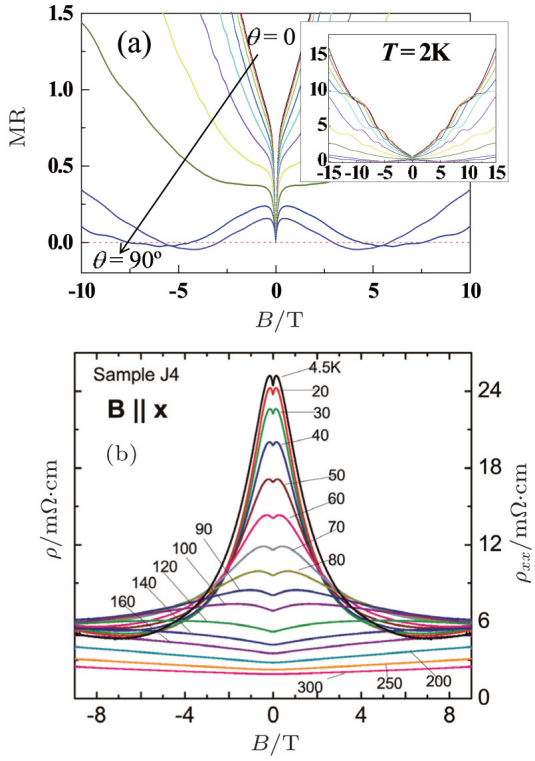


**Fig. 10.12.**

## I-8 Further

**Further Developments of Our "Random Dynamics"** Further speculations, calculations, supporting the idea of getting the Standard Model out as a - say low energy limit - of/from almost whatever the (most) "fundamental" physical laws (say complicated) might be:

- A low energy boson system - with only momentum conservation ... like the general fermion system considered - gives (in free approximation) free Maxwell equations.
- Remarkably: All species of particles in the Standard Model **except the Higgs boson** are eihter Yang-Mills particles or chiral fermions; so they would all be **massless except for effects due to the Higgs field!** This is just what one gets by asking for the low energy limit in the general theory!

## I-9 Conclusion

- Hope that the type of relativistic chiral fermions, one finds in high energy physics Standard Model in fact comes by itself - and even points to the right dimensionality 3 +1, which just is the right one-; but there are a couple of "small" problems (different species of particles have in first go different "maximal" velocities)
- Now adays the phenomenon is about being found in real materials, graphene etc. One can make relativity models chemically

It should be especially stressed that **the negative magneto-resistance due to the Adler Bell Jackiw anomaly has been seen in** $Na_3Sb$**.**

## II. What comes beyond Topological Insulator ?

–"Nielsen-Ninomiya Effect" due to Adler-Bell Jackiw chiral Anomaly–

## II-1 Introduction

In part I we mainly argued about "Gapless Semiconductor" "Topological Insulator" and this subject has been very rapidly developing presently.

We now, in this part II, argue chiefly a new application of relativistic quantum field theory. Specifically, We investigate in condensed matter (in nano-scale$\cong 10^{-9}$m) how the Relativistic Quantum field theory Effect can appear and can be detected in material science.

Theoretically this effect was predicted already 35 years ago in 1983 by the present authors

- (H. B. N and M. N.) in a High Energy Theoretical Physics journal, Physics Letters B Vol. 130, issue 6 p 389 (1983), entitled "The Adler-Bell-Jackiw anomaly and Weyl Fermions in a Crystal".

- Prior to the above paper one of the authors (M. N) was invited to give talks in the International Workshop on "Lattice Field Theory" in Saclay, Paris and Subsequently held XXI International Conference on High Energy Physics, Paris July 26-31, 1982 (so called "Rochester Conference series"), where he talked about Weyl fermions on lattices and the ABJ-anomaly.

In solid material there offen appears crystal lattice structure. Thus we are forced to use lattice field theory which has been well developped in high energy physics. In this formulation the crucial facts for us are the following:

Suppose At each lattice site we put one Weyl fermion e.g. $\Psi_L$ (Left-handed one).

Our Nielsen-Ninomiya Theorem states that there should appear equally many right handed and left handed Weyl fermions - looking in momentum space at different momentum values -. In the simplest construction resulting from just "naively" replacing derivatives by differences on the lattice our theorem is implemented by there appearing $2^d$ species (d: space dimension). Therefore in 3 space dimensions it turs out that there should be 8 species of Weyl (or chiral) fermions. Furthermore 4 of them are left-handed $\Psi_L$ and rest 4 species are right-handed $\Psi_R$ chiral fermions.

That is to say on the lattice there should be pairwise (left-handed and right-handed) chiral fermions. Therefore we are not able to construct chiral theory with for instance only one handed fermion on the lattice. Thus it leads to the very important consequence in high energy physics. In reality the Standard Model or, unified model of, weak and electromagnetic interactions called "Glashow-Salam-Weinberg model", or "Standard Model" of Weak and Electromagnetic Interaction cannot be constructed on the lattice! The reason is that in the Standard Model all the fermions are left-handed chiral fermions, while no right-handed fermion at all. The experimental results performed so far are all well in agreement with the standard model predictions.

If one takes serious the proposal of a new law of nature by one of us and various collaborators, "Multiple Point Principle", one can even claim an indication for, that the Standard Model contrary to the expectation of many of our colleagues, should be valid up to an energy scale of the order of $10^{18} GeV$ (rather close to the Planck scale):

One of the authors (H. B. N.) made together with C. D. Froggatt a theoretical calculation of $m_H$ with recourse to the just mentioned "multiple point principle (MPP)". The value is in very good agreement with experimental value at LHC (Large Hadron Collider in CERN, Geneva) $m_H \sim 125 GeV$.

See e.g. H. B. Nielsen and M. Ninomiya "Degenerate vacua from unification of second law of thermodynamics with other laws; The derivation of Multiple point principle" Int. J. Mod. Phys. **A23** (2008) 919 DOI: 10.1142/S02177510839682, in which an argument for among other things is given MPP from a model with the action taken to be complex rather than real as it is normal.

If the Standard Model shall as from this suggestion from Multiple Point Principle etc. be valid only with tiny corrections if any almost up to the Planck scale, it would be even more mysterious that we could not put it on a lattice

because of its chiral particles. Really we could -it looks -hardly regularize it with any sensible cut off! Quite a mystery. [3]

2) ABJ anomaly on a lattice

Condensed matter researchers except for high energy physicists (including some nuclear theorists), may not have heard of the Adler-Bell-Jackiw or chiral anomaly. Therefore we briefly explained ABJ anomaly in continuum space in Appendix A.

Here we turn to our nano-scale material case. In the material there is a lattice structure Fig. 10.13.

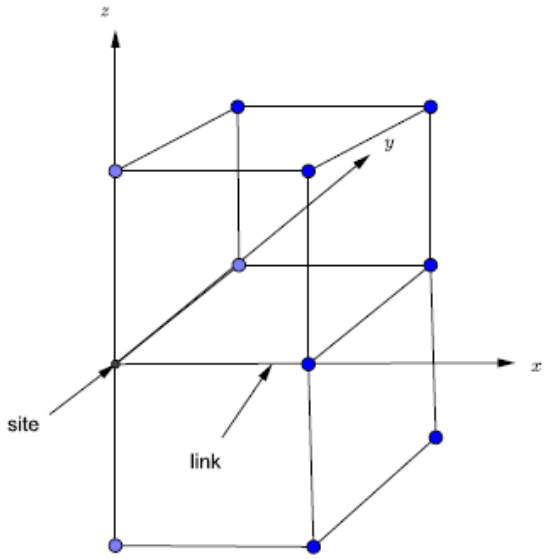


**Fig. 10.13.** Lattice structure.

In this 3 dimensional lattice on each sites we put one Weyl or Chiral electron e.g. $e_L$ (Left handed electron),then according to the Nielsen-Ninomiya Theorem, there should appear somehow so many of them, that there are equally many right haned and left handed ones. In fact we get in the simplest case 4 $e_L$ as well as 4 $e_R$.

To understand band structure, we go to the momentum space.

Note that due to the lattice translational invariance the momentum is conserved modulo multiple of the unit length of reciprocal lattice.

The Brillouin zone in the momentum space is topologically equivalent to the hypertorus $S^1 \times S^1 \times S^1$.

In such a topological structure of crystal lattice, the Adler-Bell-Jackiw anomaly explained for continuum spacetime in appendix B, is easily understood also , as was presented in PLB **130** n06, (1983) by the present authors.

## II-2 $1+1$ dimensional example

For simplicity, as an example the 1 space 1 time dimensional case is considered. Right chiral (Weyl) fermion obeys lattice Weyl eg.

$i\frac{\partial}{\partial t}\Psi_R(na) = \frac{i}{2a}[\Psi_R((n+1)a) - \Psi_L((n-1)a)]$ where $n = 0, \pm 1, \pm 2, \cdots$ denote sites and $a$ is a lattice space. This can be easily solved and the dispersion relation is given by $w = (\frac{1}{a})\sin pa$. Thus near $p = 0$ there is a RH (RH = right handed) species with the dispersion law $w \approx p$ and further there is a LH (LH =left handed) species near $\frac{\pi}{a}$ with the dispersion law

$w \approx -(p - \frac{\pi}{a})$.

These situations are illustrated in the following Fig. 10.14.

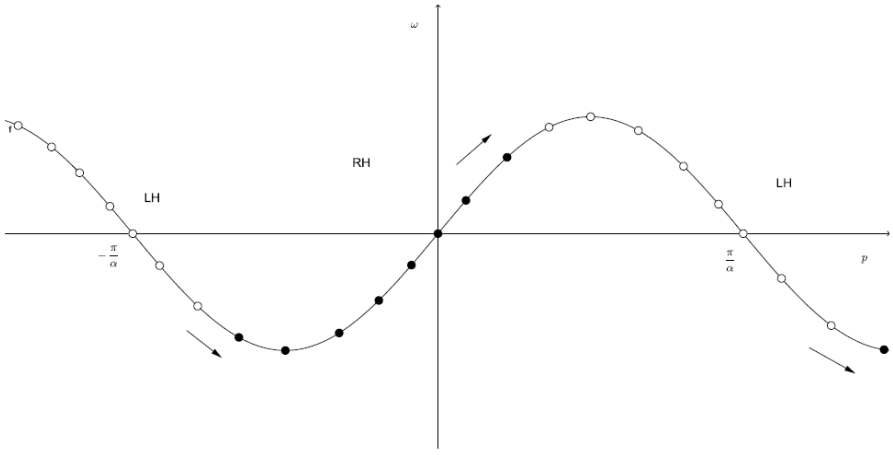


**Fig. 10.14.**

Note that due to topology of momentum space, there is a periodicity modulo $2\pi$. (e.g. points $p = -\frac{\pi}{a}$ and $\frac{\pi}{a}$ are identified)

## II-3 $3+1$ dimensional case

This $1+1$ dimension example clearly tells us, that in lattice theory there appear equal number of RH and LH chiral (or Weyl) fermion species (really in 1+1 dimension one should rather talk about right mover and left mover, because there is no genuine handedness in 1+1 dimensions) . It is not completely straightforward to generalize to $3+1$ dimensions, but with use of the appriate mathematics of homotopy (group) theory one make the analogous theorem in 3+1 or in even higher dimensions to the theorem in 1+1 that in a period real function has pass zero in positive and in negative direction equally many times per period.

### II-3 (a) Weyl (or chiral) Fermion

In generic chiral (Weyl) fermion theory which obeys

$i\dot{\Psi}(\vec{x}) = H\Psi(\vec{x}) = w\Psi(\vec{x})$

We assume that the generic Hamiltonian satisfy the following four conditions:

(1) Locality of interaction in the sense that $H(\vec{x}-\vec{y}) \to 0$ as $|\vec{x}-\vec{y})| \to large$ fast enough that the Fourie transform of $H(\vec{x})$ has continuous first derivative. (2) Translational invariance in the lattice (3) Hermiticiti of H (reality of S) (4) Furthermore an assumption is that the charge (=lepton number in our case) is bilinear in the fermion field.

Under these conditions in the generic H case we gave a rigorous proof in terms of the Homotopy theory in topology in 1981 (see, II-1).

### II-3 (b) Adler-Bell-Jackiw anomaly on a lattice

Let us go into the Adler-Bell-Jackiw (ABJ) anomaly on the lattice in the continuum spacetime. We reviewed this anomaly in continuum spacetime in Appendix B.

Here we argue for the lattice version of the ABJ anomaly. Firstly we as as an example let us explain the $1+1$ dimensional lattice Weyl (chiral) fermion. In the lattice RH chiral or Weyl electron system, we put on an external uniform electric field $E$ in $x$-direction denoted by $\dot{A}^1 = E$ in temporal gauge $(A^0 = 0)$. Then the Weyl eq. reads

$i\frac{\partial}{\partial t}\Psi_R(x) = (-i\frac{\partial}{\partial x} - \dot{A}^1)\Psi_R(x)$.

The dispersion law is given by $\omega(p) = p$.

In the classical eq. of the electron in the presence of the electric field is $\dot{p} = eE$ so that the RH electron in quantum theory is given by

$\dot{\omega} = \dot{p} = eE$.

Therefore the creation rate of the RH electrons per unit time and unit length is determined by a change of the Fermi surface that separates the filled and unifilled states as shown in Fig. 10.15.

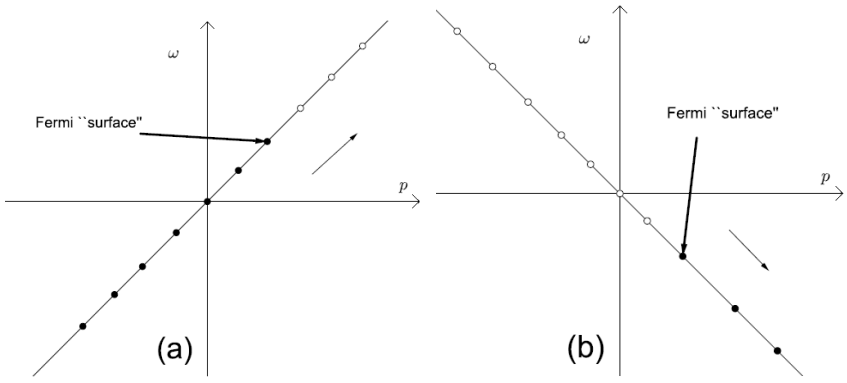


**Fig. 10.15.**

We denotes the quantization length L, then the density of states per unit momentum is given by $\frac{L}{2\pi}$. Therefore the rate of change of the RH electron number $N_R$ is given by $\dot{N}_R = \frac{L}{2\pi} \cdot \dot{\omega}_{fs}$

where $\dot{\omega}_{fs}$ denotes the rate of energy take up of the RH electron fermi surface per fermion, i.e. eE.

Therefore we obtain RH electron creation is given by $\dot{N}_R = \frac{e}{2\pi}E$ per unit length (namely for L=1). This is the ABJ anomaly.

Thus the chiral charge $Q_R$ defined as the total number of RH particles (over the fermisea minus the number of holes) is not conserved: $\dot{Q}_R = \dot{N}_R = \frac{e}{2\pi}E$ In the same manner the annihilation rate of LH electrons with $\omega = -p$ is derived as $\dot{N}_L = -\frac{e}{2\pi}E$

This means that creation rate of the LH anti-electron is given as

$\dot{N}_L = \frac{e}{2\pi}E$

By adding both, the anomaly of the Dirac electrons is

$\dot{N}_R + \dot{N}_L = \frac{e}{\pi}E$, and thus

$\dot{Q}_5 = \frac{e}{\pi}E$

To proceed to the $3+1$ dimension case, we should calculate the energy levels in the presence of an external uniform magnetic field, e.g. in the $z$-direction so that $A^2 = Hx$, and $A^\mu = 0$ otherwise. Thus we consider the equation for the two component RH electron field $\Psi_R$

$\left[i\frac{\partial}{\partial t} - (\overrightarrow{p} - e\overrightarrow{A})\overrightarrow{\sigma}\right]\Psi_R(x) = 0$

This eq. can be solved by introducing an auxiliary field $\Phi$ as

$\Psi_R = \left[i\frac{\partial}{\partial t} + (\overrightarrow{p} - e\overrightarrow{A})\overrightarrow{\sigma}\right]\Phi.$

Thus the eq. for $\Phi$ is given by

$\left[i\frac{\partial}{\partial} - (\overrightarrow{p} - e\overrightarrow{A})\overrightarrow{\sigma}\right]\cdot\left[i\frac{\partial}{\partial} + (\overrightarrow{p} - e\overrightarrow{A})\overrightarrow{\sigma}\right]\Phi = 0$

This eq. reduces to the harmonic oscillation tpe eq,

$\left[-(\frac{\partial}{\partial x'})^2 + (eH)^2(x' + \frac{p_2}{eH}) + (p_3)^2 + eH\sigma_3\right]\Phi = \omega^2\Phi$ with $\sigma_3 = \pm 1$

The energy eigenvalues $\omega$ are given by the Landau levels as follows

$\omega(n, \sigma_3, p_3) = \pm\left[2eH(n + \frac{1}{2}) + (p_3)^2 + (eH\sigma_3)\right]^{\frac{1}{2}}$ with $n = 0, 1, 2, \cdots$, except for the $n = 0$ and $\sigma_3 = -1$ mode. Here

$\omega(n = 0, \sigma = -1, p_3) = \pm p_3$.

The eigenfunction is of the form

$\Phi_{n\sigma_3}(x) = N_{n\sigma_3}(x) \times \exp(-ip_2x^2 - ip_3x^3) \times \exp(-\frac{1}{2}eH(x' + \frac{p_2}{eH})^2) \times H_m(x' + \frac{p_2}{eH})\chi(\sigma_3)$

where $N_{n\sigma_3}$ is normalization constant and $\chi(\sigma_3)$ denotes the eigenfunctions of Pauli spin $\sigma_3 : \chi(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Thus the solution of the eq. for Two-component RH electron $\Psi_R$ becomes the relations $\Psi_R^{(n+1,\sigma_3=-1)} = \frac{N_{n+1,\sigma_3=-1}}{N_{n,\sigma_3=1}}\Psi_R^{n,\sigma_3=1}$

for $n = 0, 1, 2, \cdots$

The zero mode $n = 0$ is

$\Psi_R^{(n=0,\sigma_3=-1)} = 0$ with $\omega = -p_3$.

Therefore the ground state energy of $\Psi_R$ is given by $\omega(n = 0, \sigma_3 = -1, p_3) = p_3$ The energy eigenvalue for the other modes are

$\omega(n=0,\sigma_3,p_3)=\pm\left[2eH(n+\frac{1}{2})+(p_3)^2+eH\sigma_3\right]^{\frac{1}{2}}$
These dispersion laws are depicted in the Fig. 10.16.

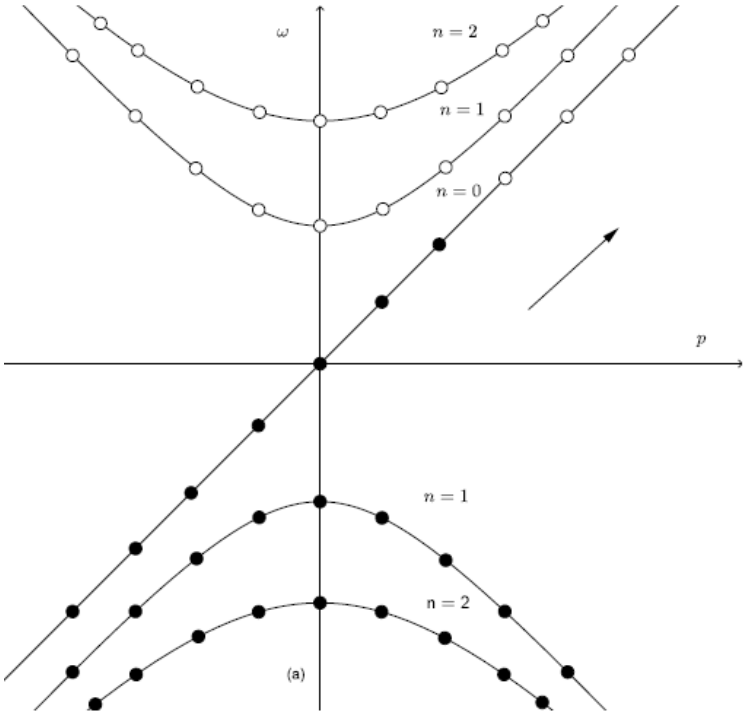


**Fig. 10.16.**

In the next step an external uniform electric field E is turned on along the same direction parallel to H. For the zero mode $(n=0,\sigma_3=-1)$ the dispersion law is the same as that for $1+1$ dimensions. Thus the creation rate of the particles is calculated in a similar manner.

We should note that the electric field E is switched on adiabatically,and there is no particle creation in the $n\neq 0$ modes. The density of the state in momentum space in the magnetic field direction is for quantization length L $L\frac{eH}{4\pi^2}$, and thus the creation rate (=the ABJ anomaly) is expressed as

$$\begin{aligned}\dot{N}_R &= \tfrac{1}{L}\tfrac{LeH}{4\pi^2}\omega_{fs}\ \ (n=0,\sigma_3=-1,P_3)\\ &= \tfrac{e^2}{4\pi^2}EH\\ &= \dot{Q}_R\end{aligned}$$

For the LH electrons annihilation rate of LH anti electron is

$$\dot{N}_L=-\frac{e^2}{4\pi}EH$$

and the creation rate of the LH anti particle is given by

$$\begin{aligned}\dot{\bar{N}}_L &= \tfrac{e^2}{4\pi^2} EH \\ &= \dot{Q}_L\end{aligned}$$

In the case of the Dirac electron

$$\begin{aligned}\dot{N}_R + \dot{\bar{N}}_L &= \tfrac{e^2}{2\pi^2} EH \\ &= \dot{Q}_5\end{aligned}$$

**II-3 (c) Generic Case**

We again look at a generic case of which Hamiltonian is given by N×N local Hermitian matrix. The N discrete energy eigenvalues are determined by the following eigenvalue eq.

$$\sum_{l=1}^{N} H_{kl}(\vec{p})\Psi_l^{(i)}(\vec{p}) = \omega_i \Psi_k(\vec{p}) \;\; (i = 1, \cdots, N)$$

Here we assume that the ith level $\Psi_i(\vec{p})$ and $(i+1)$th level are degenerate. The eigenvalue $\omega_i(\vec{p})$ are assumed to be degenerate with the $(i+1)$ level at several different points in momentum space, which are denoted as $(\omega_d(\vec{p}_d), \vec{p}_d)$ in the dispersion space $(\omega(\vec{p}), \vec{p})$. The ith and $(i+1)$th levels are described by d submatrix $H^{(2)}(\vec{p})$: it has the ith and $(i+1)$th entries of N×N matrix H.

We then expand $H^{(2)}(\vec{p})$ in powers of $(\vec{p} - \vec{p}_d)$ around are of the degenerate point $(\omega_d(\vec{p}_d), p_d)$. In the expansion of $H^{(2)}(\vec{p})$ is given

$$H^{(2)}(\vec{p}) = H^{(2)}(\vec{p}_d) + (\vec{p} - \vec{p}_d)\frac{\partial H^{(2)}(\vec{p})}{\partial \vec{p}}|_{\vec{p}=\vec{p}_d} + O((\vec{p} - \vec{p}_d)^2).$$

The derivative term is expressed by the Pauli matrices $(1_- + \sigma_\alpha)$, $(\alpha = 1, 2, 3)$ and $1_- = 2 \times 2$ unit matrix, as

$$\frac{\partial H^{(2)}}{\partial \vec{p}_k}|_{\vec{p}=\vec{p}_d} = a_k(\vec{p}_d)1_- + V_k^\alpha(\vec{p}_d)\sigma_\alpha$$

Here V are the constants depending on $\vec{p}_d$. Thus near $\vec{p} = \vec{p}_d$, $H^{(2)}(\vec{p})$ takes the form

$$H^{(2)}(\vec{p}) = \omega_d 1_- + (\vec{p} - \vec{p}_d)\vec{a}1_- + (\vec{p} - \vec{p}_d)_k V_\alpha^k \sigma^\alpha$$

The eigenvalue eq. of the ith and $(1+i)$th energy eigenvalues near $\vec{p} = \vec{p}_d$ $H^{(2)}(\vec{p})u = \omega u$.

This is rewritten by using a new set of variables

$$\hat{p} = \vec{p} - \vec{p}_d, \;\; p^0 = \omega - \omega_d - \hat{p}\vec{a}$$

as

$$\hat{p}V\vec{\sigma} \;\; u = p^0 u$$

If we introduce

$$K^0 = p^0 \text{ and } k = \pm\hat{p}\overrightarrow{V}$$

Were $\pm$ correspond to the sign of det V. For simplicity we may take as an example $V_{k\alpha} = v\delta_{k\alpha}$ $(k, \alpha = 1, 2, 3)$.

The above eigenvalue eq. becomes

$$\overrightarrow{k}\overrightarrow{\sigma}u = \pm k^0 u$$

Where the dispersion law $(k^0)^2 = v^2k^2$. Thus, it is $\omega^2 = v^2p^2$

In this way RH and LH Weyl eq. describes the 2 energy levels near degeneracy point in $(\omega(\overrightarrow{p}), \overrightarrow{p})$ space correspond to a species of Weyl fermions contained in the theory. Our theorem tells that RH and LH degeneracy points appear necessarily as a pair because of the Brillouin zero structure (topology). The theorem was proved by only topological arguments together with locality, as was shown our papers in 1981. The doubling of the Weyl fermions are illustrated in Fig. II-4 (page 18).

## II-4 Parity non-invariant zero-gap material

We assume that we have found a parity non invariant material (i.e. a crystal should be of non-centrosymmetric symmetry; e.g. BiTeI form a non-centrosymmetric crystal. Best might be a triclinic pedial class with no point symmetry at all.) with zero-gap, which can be simulated by a Weyl, fermion theory with a dispersion law $\omega^2 = v^2p^2$. The effect analogous to the ABJ anomaly gives rise to a peculiar behavior of the conductivity of the electric current in the presence of the magnetic field. It is enough to consider one conduction band $\omega_i$.

The valence band $\omega_{i+1}$ (negative energy state) is assumed to be completely filled. In the absence of external field, the single electron distribution function in the thermodynamical equilibrium is of the form $f_0(\overrightarrow{p}) = [1 + \exp[(\omega(p) - u)/kT]]^{-1}$

In the presence of E and $H = 0$ there occurs a small deviation from thermodynamical equilibrium so that $f = f_0 + \delta f$, and the E field accelerates the electrons in the same direction and then

$$\left(\frac{\partial f}{\partial t}\right)_{\text{drift}} = eE\frac{\partial f}{\partial p_z}.$$

At the same time the accelerated electrons get scattered back into some states in the same cone. We assume that f fills back into $f_0$ exponentially with a relaxation time $\tau_0$ so that $\delta f \propto e^{-\frac{\tau}{\tau_0}}$

Then

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{1}{\tau_0}(f - f_0)$$

Therefore the steady state condition is $\left(\frac{\partial f}{\partial t}\right)_{\text{drift}} = -\left(\frac{\partial f}{\partial t}_{\text{coll}}\right)$ (Boltzmann eq.).

The sol. of this is in the lowest order in E

$$f(\overrightarrow{p}) = f_0(\omega) + eE\tau_0\frac{\partial f(\omega)}{\partial p_z}$$

Then the longitudinal current density is given by

$$J_0 = \frac{1}{L^3}\sum_{\vec{p}}(-e)v_z f(\vec{p})(\#\text{deg. pts})$$

Where $v_z = \frac{\partial\omega}{\partial p_z}$ and (#deg. pt) denotes the number of deg. pts (= degeneracy points).

In the low temperature approximation $f_0(\omega) = \theta(\mu - \omega)$ so that

$$J_0 = \frac{1}{6\pi^2}e^2 E(\frac{\mu^2}{\nu})\tau_0(\#\text{deg. pt})$$

the relaxation time is given in terms of transition probability of electron from the state with$\vec{p}$ into one with$\vec{p}'$, $W(\vec{p} \to \vec{p}')$ by

$$\frac{1}{\tau_0} = \frac{1}{L^3}\sum_{\vec{p}'}\frac{p_z - p'_Z}{p_z}W(\vec{p} \to \vec{p}')$$

We assume that the interaction between the electron and the ionized impurities is given by the screened Coulomb potential (pot.) of the from

$$V(\vec{x}) = \left(\frac{4\pi e^2}{k}\right)\frac{e^{-\frac{|\vec{x}|}{\gamma_0}}}{|\vec{x}|}$$

With the screening length $\gamma_0$ and $k$ the dielectric constant. Computing $\tau_0$ in the first order perturbation we obtain the current as

$$J_0 = \frac{4e^2 E}{3\pi\eta_I}\left(\frac{k}{4\pi e^2}\right)^2\left(\frac{\mu^4}{\nu^2}\right)\left[\ln(1+\beta) - \frac{\beta}{1+\beta}\right]^{-1}(\#\text{deg. pt})$$

With $\beta = \frac{2\pi k\nu}{e^2}(\#\text{deg. pt})$ and $\eta_I$ the density of impurity.

Next compute the magneto-conductivity when H parallel to E is so strong that only the lowest states $n = 0, \sigma_3 = -1$ with dispersion law $\omega = \nu p_z$ or $\omega = -\nu p_z$ near the RH and LH degeneracy point are filled the ABJ anomaly effect will cause the movement in the momentum space of electrons from the lowest Landau level $(n = 0, \sigma_3 = -1)$ at the one deg. pt. (=degeneracy point) in the LH cone to the corresponding one $(n = 0, \sigma_3 = -1)$ in the RH cone (at the RH deg.pt.). Thus these moved electrons will give raise to a deviation from the thermodynamical equilibrium, that can be expressed by the different chemical potentials for the electrons at the RH degeneracy pt., $\mu_R$ and at the LH one $\mu_L$. If one had calculated the relaxation time in the approximation where only one degeneracy point at a time was relevant -such as we did above in the $H = 0$ case–we would have found $\frac{1}{\tau} = 0$. This comes out of such a calculation due to the energy conservation factor $\delta(\omega - \omega') = \frac{1}{\nu}\delta(p_z - p'_z)$ contained in $W(P_Z - P'_Z)$ which makes (23) give $\frac{1}{\tau} = 0$. However we cannot neglect scattering processes involving two degeneracy point.

## II-5 Transfer from LH to RH cones by Adler-Bell-Jackiw Anomaly

The mechanism for the electric current with both E can H switched on peculiarly different from the one with a negligibly weak H. In the presence of strong H the

lattice anomaly of the ABJ anomaly takes place: transfer of the particles from the LH degeneracy pt. to the RH one acts as a drift term, i.e. $\dot{N}|_{drift}$ in the Boltzmann equation. On the other hand for negligible $H$ each degeneracy points act independently. By the ABJ anomaly the Fermi energy level $\mu_R$ in the RH cone goes up compared to that of the $H = 0$ case $\mu$ and $\mu_L$ in the LH cone is lowered. See Fig. 10.16 (a) ($1+1$ dim. case) and Fig. 10.17 ($3+1$ dim. case)

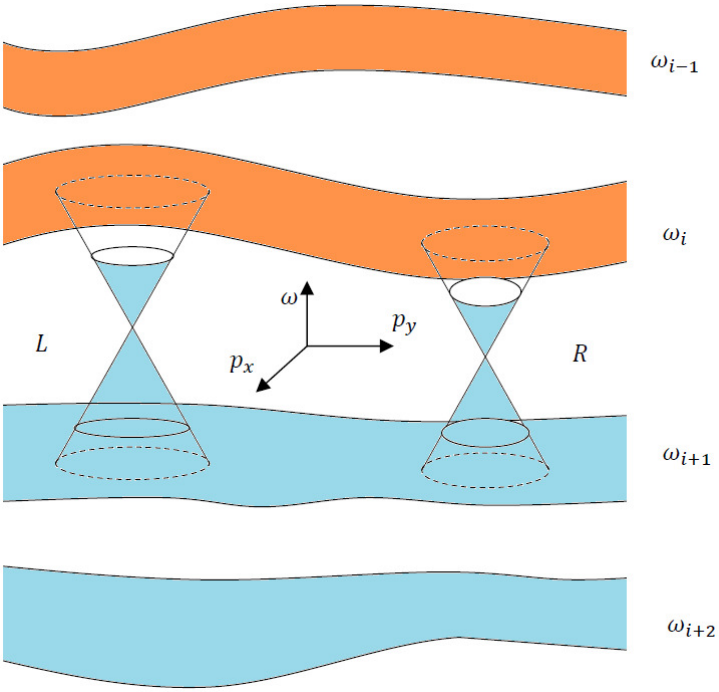


**Fig. 10.17.**

In order that the system is in the steady state

**the excess electrons by the ABJ anomaly in the RH cone must be scattered back to the another state**.

But they can not be scattered back into the state in the same come! because, as was explained above $\tau = \infty$.

Therefore they must transfer into the states in another cone; that is from the RH cone into the LH cone.

We may call this the intercone scattering and we denote the corresponding relaxation time by $\tau_I$. If the intercone transition probability $W(p_z \to p'_z)$ from RH cone into the LH cone is calculated, then the collision term is given by

$$\dot{N}_R|_{coll} = \frac{2}{L}\sum_{p_Z}[f(p_Z) - f_0(p_z)]\frac{1}{L}\sum_{p'_z} W(p_z \to p'_z)$$
$$\equiv -\frac{p'_z}{\tau_I}(N_R - N_R^0)$$

Here $N_R$ and $N_R^0$ denote the total electron numbers in the RH cone above the degeneracy energy in the $H \neq 0$ and $H = 0$ cases respectively. Thus $\frac{1}{\tau_I} = \frac{2eH}{(2\pi)^2}\frac{1}{L}\sum_{p_3'} W(p_z - p_z')$

The generation of a current associated with the ABJ anomaly can be shown by the following energy conservation argument. ABJ anomaly indicates that electrons are transferred from the LH cone into the RH cone by the rate of $\frac{e^2EH}{(2\pi)^2}$ per unit Time,per unit volume.: Notice that the dispersion law is continuous and the RH and LH cones are connected smoothly as shown Fig. 10.17.

Since the Fermi level energies are $\mu_R > \mu_L$ the transfer costs the energy $\frac{e^2}{(2\pi)^2}EH(\mu_R - \mu_L)$. This energy must be taken from the E field by the presence of a current $J_A$ determined by the energy balance as

$EJ_A = \frac{e^2}{(2\pi)^2}eH(\mu_R - \mu_L)$

At the zero temperature, in the RH cone

$f_0(\omega) = \theta(\mu_R - \omega)$ and thus

$$N_R = \frac{1}{L^3}\sum_{p_y p_z} f_0(\omega)\frac{eH}{(2\pi)^2}\frac{\mu_R}{v} \cong N_R^0 + (\mu_R - \mu)\frac{\partial N_R}{\partial \mu}$$

Inserting this into Boltzmann eq.

$\dot{N}_R|_{drift} = -\dot{N}_R|_{col}$

We obtain $\mu_R - \mu_L = ev E\tau_I$

Therefore $J_A = ev\frac{e^2}{(2\pi)^2}EH\tau_I$(#deg.pt.) Here the subscript A stands for the anomalous current the one associated with the analogue of the ABJ anomaly. In the definition of $\tau_I$ we may approximate $W(\overrightarrow{p_z} \to p_z') \cong W(\overrightarrow{p} - \overrightarrow{p'})$

So that $W(p_Z - p_Z') \cong (\frac{4\pi^2}{k})^2\eta_I\left[(\overrightarrow{p} - \overrightarrow{p}')^2 + \frac{1}{\gamma_H^2}\right]^{-2} 2\pi\delta(\omega - \omega')$ with $\frac{1}{\gamma_H^2} = \frac{EH}{kv}$(#deg.pt.). According to $\hat{\overrightarrow{p}} \equiv \overrightarrow{p} - \overrightarrow{p}_d$, $p^0 = \omega - \omega_d - \hat{p}\,\overrightarrow{a}$, we have $\overrightarrow{p} - \overrightarrow{p}' = \overrightarrow{p}_d - \overrightarrow{p}_d' + \hat{\overrightarrow{p}} - \hat{p}'$

where $\hat{\overrightarrow{p}}$ and $\hat{\overrightarrow{p}}'$ are oscillating around $\overrightarrow{p}_d$ and $\overrightarrow{p}_d'$: since they are order of $(eH)^{\frac{1}{2}}$. We may ignore the oscillatory part $(\hat{\overrightarrow{p}} - \hat{\overrightarrow{p}}')$ and $\frac{1}{\gamma_H^2}$ term in the denominator of $W(p_Z - p_Z')$ when compared to the distance of the RH and LH deg. pts $\overrightarrow{p}_d - \overrightarrow{p}_d'$. In this approximation we obtain

$J_A = \frac{e^2v^2E}{2\pi\eta_I}\left(\frac{k}{4\pi^2}\right)^2(\overrightarrow{p}_d - \overrightarrow{p}_d')^4$(#deg.pt.)

We then obtain the ratio of the conductivity that is defined by $f = \sigma E$ as $\frac{\sigma_A}{\sigma_0} = \frac{3}{16}\left(\frac{v}{\mu}\right)^4\left[\ln(1+\beta) - \frac{\beta}{1+\beta}\right](\overrightarrow{p}_d - \overrightarrow{p}_d')^4$

By these results, for the intercom relation time $\tau_I$ the electrons must travel a "long distance" in momentum space. Thus $\tau_I$ is expected to be a large value compared to $\tau_0$ for $H = 0$. Therefore $\frac{\sigma_A}{\sigma_0}$ given above is large.

## II-6 Further arguments

So far we have presented our own theoretical predictions in 1983 although we believed sooner or later our predicted "Nielsen-Ninomiya" mechanism (or effect)

will be proved by experiment. Indeed after almost 35 year later Princeton University group led by Prof. N. Phuan Ong and R. Cava, found chiral anomaly in crystalline material. This surprising news in science community appeared in an article by Catherine Zandonella,

- office of the Dean of Research, in Science, September 3, 2015 entitled Research at Princeton: Long-sought chiral anomaly detected in crystalline material (science).
  At the almost same time, scientist's article entitled.
- "Evidence for the chiral anomaly in the Dirac semimetal Na3Bi" By J. Xiong Satya K. Kushwaha, Tian Liang, J. W. Kritzan, M. Hirsehberger, Wulin Wang, R. J. Cava, X. P. Oug, Science Express, 03 , September 2015. and
- "Signature of the chiral anomaly in a Dirac semimetal – a current plume steered by J. Xiong, S. K. Kushwaraha, T. Liang, J. W. Krizan, Wudi Wang, R. J. Cava and N. P. Ong

Since then the works on this subject is really under rapidly developing mainly in Experiments, also theories: e. g. Dirac cones,and Weyl semimetals. We believe in the rather near future we shall see some machines using "Nielsen-Ninomiya Mechanism (or Effect). See e.g also [4].

## II-7 Conclusions

In the present article we present the viewpoint at two exceptional high energy theoretical physicists new eras of condensed matter.

In the first part I we mainly considered "Topological Insulator" from random dynamics point of view. The essential point is that in generic Fermion dispersion relations i.e. in (almost) all solids or fluids at low temperature we can derive the recently found properties of Topological insulators such as graphene etc.

In the 2nd point II, we present what comes beyond topological insulator.

We believe that the Adler-Bell-Jackiw anomaly effect in the chiral non invariant gapless material, causes that

- magnetic conductance is enhanced very much (ideally permanent current)
- Chiral electron (chiral fermion in general) in lattice of the gapless material runs with a fixed speed. (This fixed speed is what in the relativity theory analogue is the speed of light.) This is so, because we by analogy can apply the relativistic quantum field theory.

To make any apparatus using the above theory will be widely opened to not only condensed matter, but chemistry, beyond artificial division such as, physics chemistry engineering etc.

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## Appendix A

We consider electron in quantum field theory (Relativistic quantum mechanics.) We present only necessary properties in Appendix A

The electron in the relativistic quantum field theory it is usually described as Dirac field $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$ where $\Psi_L$ and $\Psi_R$ are 2 component fields. Now the electron has intrinsic spin $\overrightarrow{S}$. Thus electron has the angular momentum then $\overrightarrow{J}$, whose value are half integers, and the spin components is $\overrightarrow{S}$ take values $\pm\frac{1}{2}$.

For massless fermions the right $\Psi_R$ and the left $\Psi_L$ componets in the (free) Dirac equation gets seperated, and we actually even find that the spin direction is the same as that of electron movement for the right components $\Psi_R$ and the opposite for the left components $\Psi_L$. Let us start with the Dirac field such as an electron in the quantum field theory. The electron has intrinsic spin $\frac{1}{2}$ of fermion obeying the free Dirac eq.

$(i\gamma^\mu \partial_\mu - m_e)\Psi_D = 0 \quad (II-1)$

thereafter we ignore electron mass unless described. Our notation is that of the textbook of Bjorken-Drell " Relativistic Quantum Fields". For our purpose we list up relevant notations below

- The $3+1$ dimensional flat space metric (tensor):
  $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
- The $\gamma$ matrices are
  $\gamma^0 = \begin{pmatrix} 1\_ & 0 \\ 0 & -1\_ \end{pmatrix}$
  were $1\_ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  and $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ $i = 1,2,3$
  Here $\sigma^i$ denotes $2\times 2$ Pauli matrices and $1\_ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
  Furthermore
  $\gamma^5 = \gamma_5 = \begin{pmatrix} 0 & 1\_ \\ 1\_ & 0 \end{pmatrix}$ (note $(\gamma^5)^2 = 1$)
- The 4 component Dirac field is denoted as
  $\Psi_D(p,s)$
  and when there is no interactions obeys the free Dirac eq. as

$(i\gamma^\mu \partial_\mu - m)\Psi = 0$
where $\partial_\mu = \frac{\partial}{\partial x^\mu}$, in momentum representation
$p^\mu = i\frac{\partial}{\partial x_\mu}$
$\Psi_D(\overrightarrow{p}, s)$.
Here s denote intrinsic spin $|s| = \frac{1}{2}$
$\Psi_D(\overrightarrow{p}, \overrightarrow{s})$ obeys
$\not{p}\Psi_D(p, s) = 0$
with 4 component Dirac field we may describe

$$\Psi_D(p, s) = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}(p, s)$$

Where $\Psi_L$ and $\Psi_R$ are 2 component spinor respectively the eigenvalue solution of free Dirac eq. $(II - 1)$ is the form of
$\Psi(p, s) = \pm\sqrt{\overrightarrow{p}^2 + m^2}$

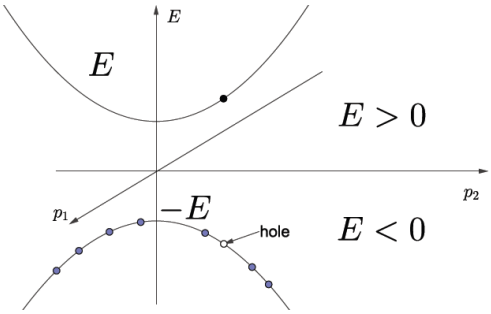



**Fig. 10.18.** Dirac's "hole theory".

We adopt the Dirac's "hole theory". In this theory often used in condensed matter as dispersion relation, the negative states are all filled, while the hole in the Dirac sea is antiparticle, i.e. positron $e^+$.

In solid state physics where one has say a crystal lattice, which from the quantum field theory is discretized, so therefore we are interested in discretizing the quantum field theory here. The Dirac fermion wave function $\Psi_D(\overrightarrow{p}, s)$ has 4-components: 2 degree of freedom as that energy can have plus or minus. Furthermore the electron has intrinsic spin of which value is $|s| = \frac{1}{2}$. In the massless case spin/(vector) direction can be either the direction of the electron motion or the opposite. We then define for describing "chirality". It is usually distinguished by this quantity. That is to say $\gamma_5\Psi = +1 \text{or} -1$. Customary $+1$ is named Left moving- and $-1$ case is Right moving-Weyl or chiral fermion denoted $\Psi_L$ and $\Psi_R$ respectively. (The Lorentz or Poincare group of spacetime in $3+1$ dim Hermann Weyl investigated in detail and the basis is 2 component spinor called Weyl spinors $\Psi_L$ and $\Psi_R$. In terms of these 4 component Dirac field $\Psi$ such handed components can be constructed ($\Psi_D = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$)

## Appendix B

We are now ready to discuss about Adler-Bell-Jacklin anomaly. In quantum field theory there are various symmetries. One of the most interesting symmetries is chiral (or axial) symmetry. That is the interaction of Dirac field $\Psi_D$ with electromagnetic field $A_\mu$ is given by

$$S = \int d^4x \bar{\Psi}(x) \left[ i\gamma^\mu \left( \partial_\mu + ieA_\mu(x) \right) \right] \Psi_D(x) \quad (*)$$

in the case of massless electron., where $\bar{\Psi}_D = \Psi^\dagger \gamma^0$. It has chiral symmetry which may be obvious, if we rewrite $(*)$ in terms of $\Psi_L$ and $\Psi_R$ as the Dirac eq. can be written as

$$\begin{pmatrix} 0 & i(\partial_0 + \sigma^i(\partial_i + ieA_i)) \\ i(\partial_0 - \sigma^i(\partial_i + ieA_i)) & 0 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0.$$

In this way the equations of $\Psi_L$ and $\Psi_R$ are separately given by the following Weyl equations

$$i(\partial_0 - \sigma^i(\partial_i + A_i))\Psi_L = 0$$
and
$$i(\partial_0 + \sigma^i(\partial_i + A_i))\Psi_R = 0.$$

In these forms it is evident that the theories are invariant under the following infinitesimal Weyl transformations

$$\Psi_L \to (1 - i\alpha^i \tfrac{\sigma_i}{2} - \beta^i \tfrac{\sigma_i}{2})\Psi_L$$
$$\Psi_R \to (1 - i\alpha^i \tfrac{\sigma_i}{2} + \beta^i \tfrac{\sigma_i}{2})\Psi_R$$

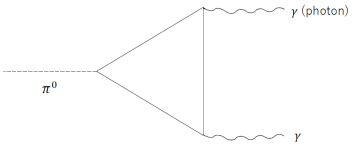
Where $\alpha^i$ and $\beta^i$ $(i = 1, 2, 3)$ are infinitesimal transformation parameters, restricted to leave the normalization of the Weyl fields invariant. This Weyl or Chiral transformation is broken due to quantum effect in quantum field theory. There were several suggestive articles, but explicit manifestation is presented by

- S. Adler, Phys. Rev. 177 (1969) 2426
  and
- J. S. Bell and R. Jackiw Nuovo Cimento 60A (1969) 4.

Furthermore the method of path integral formulation this ABJ anomaly is due to nor-invariance of the path integral measure

- K. Fujikawa Phys. Rev. Lett. 42 1195 (1979)

Phenomenologically this ABJ anomaly is really important. It has been observed by experiments. $\Pi^0$ meson decays into 2 photons. When we approximate $\Pi^0$ as being massless, this decay process is expressed as the following diagram, triangle diagram of Feynman diagram




If chiral symmetry is not broken, this diagram turns out to give zero. Thus this decay is not allowed. However, experimentally this decay process certainly exists. This is the evidence that Adler-Bell-Jackiw anomaly does exist. In high energy, physics the ABJ anomaly is expressed as the non-conservation chiral current $J^5_\mu$ such that

$\partial^\mu J^5_\mu = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$

Here the chiral current $J^5_\mu$ is defined as

$J^5_\mu = \lim_{\epsilon\to 0}\left\{\bar{\Psi}(x+\frac{\epsilon}{2})\gamma_\mu\gamma^5 \exp\left[-ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dz A(z)\right]\Psi(x+\frac{\epsilon}{2})\right\}$

We might perform the calculation to show that the above triangle diagram is non-zero due to the ABJ anomaly. But we have instead in subsection 10 alluded to a derivation of the ABJ-anomaly by using how particles are pumped up or down from or to the fermi-sea (in high energy physics the Dirac sea).

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