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There is a unique crossing-minimal rectilinear drawing of K_{18}*

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Abstract

We show that, up to order type isomorphism, there is a unique crossing-minimal rectilinear drawing of K_{18}. It is easily verified that this drawing does not contain any crossing-minimal drawing of K_{17}. Therefore this settles, in the negative, the following question from Aichholzer and Krasser: is it true that, for every integer $n \geq 4$, there exists a crossing-minimal drawing of K_n that contains a crossing-minimal drawing of K_{n-1}?

Keywords: Rectilinear crossing number, complete graphs, k-edges.

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1 Introduction

The *rectilinear crossing number* $\overline{\text{cr}}(G)$ of a graph G is the minimum number of edge crossings in a *rectilinear* drawing of G in the plane, i.e., a drawing of G in the plane where the vertices are points in general position and the edges are straight line segments. A drawing of G with exactly $\overline{\text{cr}}(G)$ crossings is *crossing-minimal*.

Determining the rectilinear crossing number $\overline{\text{cr}}(K_n)$ of the complete graph K_n is a well-known open problem in combinatorial geometry (see for instance [5, 11]). In [9] Aichholzer et al. determined the exact values of $\overline{\text{cr}}(K_n)$ for $13 \leq n \leq 17$. In that paper also the following question was raised.

Question 1.1. Is it true that, for every integer $n \geq 4$, there exists a crossing-minimal drawing of K_n that contains a crossing-minimal drawing of K_{n-1}?

The exact value of $\overline{\text{cr}}(K_n)$ is known for $n \leq 27$ and $n = 30$ (see [3, 7, 8, 9, 10]). The value of $\overline{\text{cr}}(K_{18}) = 1029$ was established in [8]. Crossing-minimal rectilinear drawings of K_n for this range of values of n can be found in [2] and [6]. In particular, from [6], we know that there are at least 37269 non-isomorphic crossing-minimal drawings of K_{17}.

Let θ denote the counterclockwise rotation of $2\pi/3$ around the origin, and let $W := \{(-51, 113), (6, 834), (16, 989), (18, 644), (18, 1068), (22, 211)\}$. From [2], we know that the 18-point set $W \cup \theta(W) \cup \theta^2(W)$ induces a crossing-minimal drawing of K_{18}. See Figure 1 for an illustration of such a point set.

Our main result is the following.

Theorem 1.2. *Up to order type isomorphism, there is a unique* 18*-point set whose induced rectilinear drawing of* K_{18} *has* $\overline{\text{cr}}(K_{18})$ *crossings.*

Let $\mathcal{D}$ be the (unique, in view of Theorem 1.2) crossing-minimal rectilinear drawing of K_{18}. It is easily verified that every subdrawing of $\mathcal{D}$ with 17 points has more than $\overline{\text{cr}}(K_{17}) = 798$ crossings. This settles Question 1.1 in the negative.

In the next section, we introduce the necessary notation and additional concepts required for the proof of Theorem 1.2. In Section 4 we prove Theorem 1.2.

2 k-edges, $(\leq k)$-edges, and 3-decomposability

Throughout this section, Q is a set of $n \geq 3$ points in general position in the plane. If p and q are distinct points of Q, then we denote by pq the directed line spanned by p and q, directed from p towards q. Furthermore, pq^+ and pq^- denote the set of points in Q on the right and left, respectively, of pq. Thus $Q = pq^- \cup \{p, q\} \cup pq^+$ for all $p, q \in Q$ with $p \neq q$.

Let $k \in \{0, 1, \ldots, \lfloor n/2 \rfloor - 1\}$. A *$k$-edge* of Q is a directed line spanned by two distinct points of Q, which leaves exactly k points of Q on one side. A *$(\leq k)$-edge* (respectively, a *$(> k)$-edge*) is an i-edge of Q with $0 \leq i \leq k$ (respectively, $k < i \leq \lfloor n/2 \rfloor - 1$). Let $E_k(Q)$, $E_{\leq k}(Q)$, and $E_{>k}(Q)$ denote, respectively, the set of k-edges, $(\leq k)$-edges and $(> k)$-edges of Q. We use $e_k(Q)$, $e_{\leq k}(Q)$, and $e_{>k}(Q)$ to denote, respectively, the number of elements in $E_k(Q)$, $E_{\leq k}(Q)$, and $E_{>k}(Q)$. Then $e_{\leq k}(Q) = \sum_{j=0}^{k} e_j(Q)$ and $e_{>k}(Q) = \binom{n}{2} - e_{\leq k}(Q)$.

The vector $\mathbf{E}_{\leq k}(Q) := (e_{\leq 0}(Q), e_{\leq 1}(Q), \ldots, e_{\leq \lfloor n/2 \rfloor - 1}(Q))$ is the *$(\leq k)$-edges vector* of Q. Finally, $e_{\leq k}(n)$ denotes the minimum of $e_{\leq k}(P)$ taken over all n-point sets P

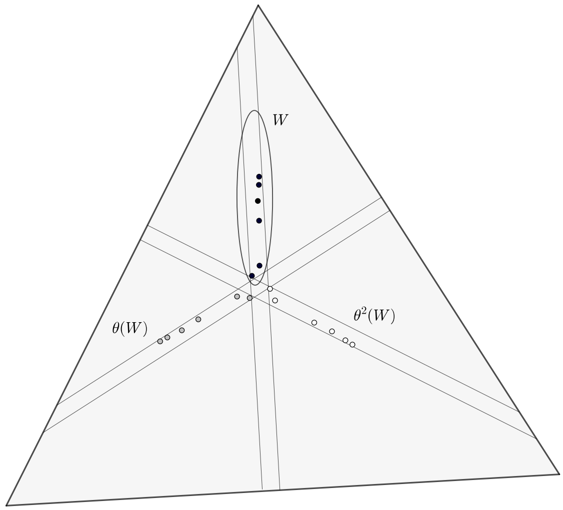

Figure 1: This is the 18-point set produced by the union of $W = \{(-51, 113), (6, 834), (16, 989), (18, 644), (18, 1068), (22, 211)\}$, $\theta(W)$ and $\theta^2(W)$. It is not difficult to see that P produces a crossing-minimal rectilinear drawing of K_{18}. The triangle and the six straight line segments show that P is 3-decomposable.

in the plane in general position. The exact determination of $e_{\leq k}(n)$ is another well known open problem in combinatorial geometry (see for instance [3, 4, 7, 8]).

The number of crossings in a rectilinear drawing of K_n and the number of k- and $(\leq k)$-edges in its underlying n-point set P are closely related by the following equality, independently proved in [4] and [12]:

$$\overline{\text{cr}}(P) = \sum_{k=0}^{\lfloor n/2 \rfloor -2} (n - 2k - 3)\, e_{\leq k}(P) - \frac{3}{4}\binom{n}{3} + \left(1 + (-1)^{n+1}\right)\frac{1}{8}\binom{n}{2}. \tag{2.1}$$

This equality allows us to fully determine the $(\leq k)$-edges vector of any 18-point set whose induced drawing attains the rectilinear crossing number of K_{18}.

Proposition 2.1. *If P is an 18-point set such that $\overline{\text{cr}}(P) = \overline{\text{cr}}(K_{18})$, then $\mathbf{E}_{\leq k}(P) = (3, 9, 18, 30, 45, 63, 87, 120, 153)$.*

Proof. Let Q be an 18-point set in the plane in general position. It is known (see [3] or [7]) that $\mathbf{E}_{\leq k}(Q) = (e_{\leq 0}(Q), e_{\leq 1}(Q), \ldots, e_{\leq 8}(Q))$ is bounded below entry-wise by $(3, 9, 18, 30, 45, 63, 87, 120, 153)$. On the other hand, from (2.1) we know that

$$\overline{\text{cr}}(Q) = -612 + 15 \cdot e_{\leq 0}(Q) + 13 \cdot e_{\leq 1}(Q) + \cdots + 1 \cdot e_{\leq 7}(Q).$$

From the coefficients of this equation and the fact that $e_{\leq 8}(Q) = 153$, it follows that if $e_{\leq k}(Q)$ is greater than the k-th component in the vector $(3, 9, 18, 30, 45, 63, 87, 120, 153)$, then $\overline{\text{cr}}(Q) > 1029$. □

Finally, we introduce a concept that captures a property shared by all known crossing-minimal rectilinear drawings of K_n, for n a multiple of 3. A point set Q is 3-*decomposable* if it can be partitioned into three equal-size sets A, B and C, such that (i) there exists a triangle T enclosing the point set Q; and (ii) the orthogonal projection of Q onto the three sides of T shows A between B and C on one side, B between C and A on the second side, and C between A and B on the third side. In such a case, we say that $\{A, B, C\}$ is a 3-*decomposition* of Q. For instance, $\{W, \theta(W), \theta^2(W)\}$ is a 3-decomposition of the 18-point set shown in Figure 1.

As in [2], if $\{A, B, C\}$ is a 3-decomposition of Q, we define two types of edges. Let p and q be distinct points of Q. If $p, q \in A$, $p, q \in B$ or $p, q \in C$ then we call pq *monochromatic*; otherwise, pq is *bichromatic*. Let $E_k^{\text{mon}}(Q)$ and $E_k^{\text{bi}}(Q)$ denote the set of monochromatic and bichromatic k-edges of Q, respectively. As before, we use $e_k^{\text{mon}}(Q)$ and $e_k^{\text{bi}}(Q)$ to denote $|E_k^{\text{mon}}(Q)|$ and $|E_k^{\text{bi}}(Q)|$, respectively. Note that $e_k(Q) = e_k^{\text{mon}}(Q) + e_k^{\text{bi}}(Q)$. Now we partition the monochromatic edges of Q into three types. If $p, q \in A$, then we say that pq is an edge *of type aa*. Similarly, we define the edges of types *bb* and *cc*. For $x \in \{a, b, c\}$, we denote the number of monochromatic k-edges of type xx by $e_k^{xx}(Q)$. Then $e_k^{\text{mon}}(Q) = e_k^{aa}(Q) + e_k^{bb}(Q) + e_k^{cc}(Q)$.

3 Overview of the proof of Theorem 1.2

For the rest of this paper, P is an 18-point set in the plane in general position where the rectilinear crossing number of K_{18} is attained. That is, $\overline{\text{cr}}(P) = \overline{\text{cr}}(K_{18})$.

The first step in the proof, carried out in Section 4.1, consists of giving an algorithm that yields a canonical, unambiguous labelling of the points in P. The 18 points in P get labelled $x_0, x_1, \ldots, x_5, y_0, y_1, \ldots, y_5, z_0, z_1, \ldots, z_5$. Thus P gets naturally partitioned into three sets $X = \{x_0, x_1, x_2, x_3, x_4, x_5\}, Y = \{y_0, y_1, y_2, y_3, y_4, y_5\}$, and $Z := \{z_0, z_1, z_2, z_3, z_4, z_5\}$. As we shall prove shortly afterwards, $\{X, Y, Z\}$ happens to be a 3-decomposition of P.

Once we have laid out the foundation by giving a canonical labelling of the points of P, the rest of the proof consists of showing the following:

Lemma 3.1 (Implies Theorem 1.2)**.** *For each pair of distinct points $p, q \in P$, the set pq^+ is uniquely determined.*

Clearly Lemma 3.1 implies Theorem 1.2: if the lemma holds, then the unambiguity of the labelling of the points in P implies that P is unique up to order type isomorphism.

First we establish the lemma for the case in which pq is a (≤ 5)-edge. This is actually done in Section 4.1, where we give the algorithm to label the points in P. Indeed, the unambiguity in the labelling of the points in P is established in Proposition 4.3(1), and in order to prove this we need to prove simultaneously Proposition 4.3(2), which in particular implies Lemma 3.1 for the case in which pq is a (≤ 5)-edge.

We then move on to proving Lemma 3.1 for the case in which pq is a (>5)-edge, that is, when pq is either a 6-edge, or a 7-edge, or an 8-edge. As we shall see, even if this follows from elementary observations, the investigation of these cases is remarkably more involved than the case in which pq is a (≤ 5)-edge.

The first step towards the investigation of (>5)-edges is given in Section 4.2, where we prove that $\{X, Y, Z\}$ is a 3-decomposition of P. This allows us to classify each edge of P as either monochromatic or bichromatic, as we explained at the end of Section 2. Also in Section 4.2 we show that for each $k \in \{6, 7, 8\}$ it is easy to determine the number of bichromatic k-edges and the number of monochromatic k-edges.

After proving these elementary properties of P we move on to Section 4.3. This is the most technical and long part of the paper, and its purpose is to establish a collection of structural properties of P. On a first read it may be advisable to skip this section, and only come back to it whenever its main results are invoked in Sections 4.4 and 4.5.

Finally, in Section 4.4 (respectively, Section 4.5) we prove Lemma 3.1 for the case in which pq is a monochromatic (respectively, bichromatic) 6-edge, 7-edge, or 8-edge. As we shall see, using the structural results from Section 4.3 these tasks are reduced to a relatively straightforward case analysis.

For completeness, the conclusion of the proof is presented in Section 4.6.

4 Proof of Theorem 1.2

We recall that throughout this paper, P is an 18-point set in the plane in general position such that $\overline{\text{cr}}(P) = \overline{\text{cr}}(K_{18})$.

4.1 The algorithm to label the 18 points in P, and proof of Lemma 3.1 when pq is a (≤ 5)-edge

It follows from Proposition 2.1 that the convex hull of P has exactly 3 vertices. Without loss of generality (the whole set P may be rotated, if necessary) we may assume that all three vertices have distinct x-coordinates. Let x_0 denote the vertex with the smallest x-coordinate. As we travel counterclockwise along the convex hull starting from x_0, let y_0 be the first vertex we find, and let z_0 be the other vertex. See Figure 2.

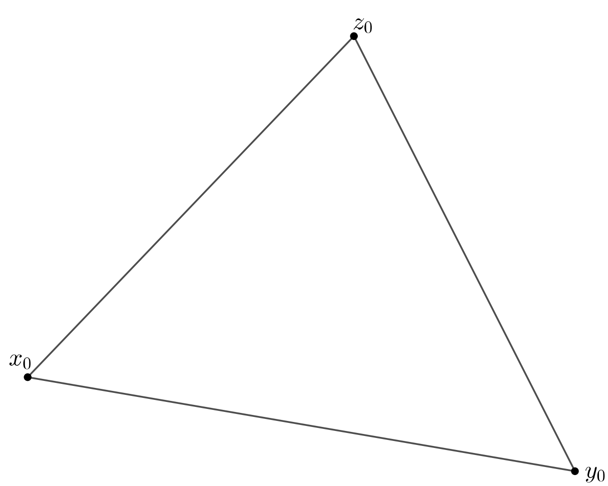

Figure 2: The convex hull of P.

Observation 4.1. $E_0(P) = \{x_0y_0, y_0z_0, x_0z_0\}$. □

We have already unambiguously determined a labelling for the three convex hull vertices of P. It remains to unambiguosly determine a labelling for the remaining 15 points of P.

For $j \in \{0, \dots, 5\}$, let $x_j^{\curvearrowright}$ denote the j-th point in P that we find as we rotate the line y_0x_0 clockwise around y_0 (we consider x_0 to be the 0-th point in P hit by the rotating line, so that $x_0^{\curvearrowright} = x_0$). We define $y_j^{\curvearrowright}$ and $z_j^{\curvearrowright}$ similarly, using z_0y_0 and x_0z_0 as the clockwise rotating lines, around z_0 and x_0, respectively. See Figure 3(a).

In an analogous manner, we let $x_j^{\curvearrowleft}$ denote the j-th point in P that we find as we rotate the line z_0x_0 counterclockwise around z_0 (again, we consider x_0 to be the 0-th point in P hit by the rotating line, so that $x_0^{\curvearrowleft} = x_0$). We define $y_j^{\curvearrowleft}$ and $z_j^{\curvearrowleft}$ similarly, using x_0y_0 and y_0z_0 as the counterclockwise rotating lines, around x_0 and y_0, respectively. See Figure 3(b).

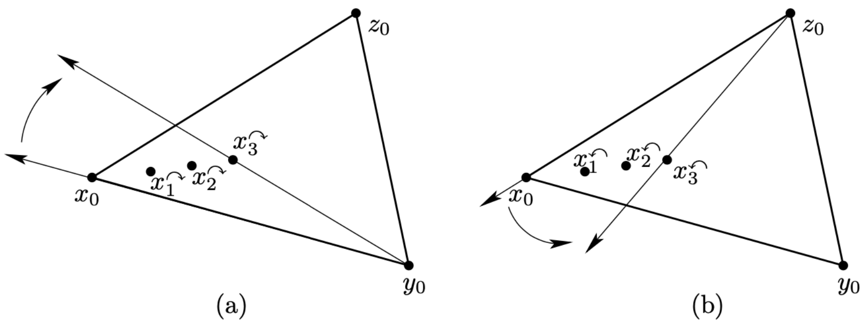

Figure 3: (a) As we rotate the line y_0x_0 clockwise around y_0, the third point in P we find is labelled $x_3^{\curvearrowright}$. The points $x_1^{\curvearrowright}$ and $x_2^{\curvearrowright}$ are also indicated. (b) As we rotate the line z_0x_0 counterclockwise around z_0, the third point in P we find is labelled $x_3^{\curvearrowleft}$. The points $x_1^{\curvearrowleft}$ and $x_2^{\curvearrowleft}$ are also indicated. By definition $x_0^{\curvearrowright} = x_0^{\curvearrowleft} = x_0$. Note that in this example $x_i^{\curvearrowright} = x_i^{\curvearrowleft}$ for $i = 0, 1, 2, 3$.

Observation 4.2. For each $j \in \{0, \dots, 5\}$, $y_0x_j^{\curvearrowright}$, $z_0x_j^{\curvearrowleft}$, $z_0y_j^{\curvearrowright}$, $x_0y_j^{\curvearrowleft}$, $x_0z_j^{\curvearrowright}$, and $y_0z_j^{\curvearrowleft}$ are all j-edges. □

The next statement is our first major result on the structure of P. In particular, it yields a labelling of all the points of P. As it happens, this proposition simultaneously establishes Lemma 3.1 for the case in which pq is a (≤ 5)-edge.

Proposition 4.3. *Let $j \in \{0, \dots, 5\}$. Then:*

(1) *For $u \in \{x, y, z\}$, $u_j^{\curvearrowright}$ and $u_j^{\curvearrowleft}$ are the same point, which will be denoted u_j;*

(2) *For all nonnegative integers m, n such that $m + n = j$, we have that*

   (a) *$E_j(P) = \{u_mv_n \mid m+n = j$ and $uv \in \{xy, yz, zx\}\}$. Moreover, for such values of m, n, and j the following holds:*

(b) $u_m v_n^+ = \{u_i \,|\, i < m\} \cup \{v_i \,|\, i < n\}$ *for any* $uv \in \{xy, yz, zx\}$.

Proof. We prove (1) and (2) by induction on j. Since $x_0^{\curvearrowright} = x_0^{\curvearrowleft} = x_0$, $y_0^{\curvearrowright} = y_0^{\curvearrowleft} = y_0$, and $z_0^{\curvearrowright} = z_0^{\curvearrowleft} = z_0$, it follows from Observations 4.1 and 4.2 that (1) and (2) are true for $j = 0$. Now we let $t \in \{0, 1, 2, 3, 4\}$ be an integer such that (1) and (2) hold for every j such that $0 \leq j \leq t$ (in particular, the points x_j, y_j, z_j are already defined for $0 \leq j \leq t$). We complete the proof by showing that then (1) and (2) hold for $j = t + 1$.

Let $X_t := \{x_0, \ldots, x_t\}$, $Y_t := \{y_0, \ldots, y_t\}$, $Z_t := \{z_0, \ldots, z_t\}$, and $P_t := X_t \cup Y_t \cup Z_t$. From the definitions involved, it follows that $|P_t| = 3(t + 1)$. First we establish an injection $\psi\colon P_t \to E_{t+1}(P)$.

Consider any point $x_i \in X_t$. It follows from the induction hypothesis that $x_i y_{t-i}$ is a t-edge, and that $x_i y_{t-i}^+ = \{x_r \mid r < i\} \cup \{y_r \mid r < t - i\}$.

Let $\overline{x}_i$ be the first point that we find as we rotate the line $x_i y_{t-i}$ counterclockwise around x_i. It is easy to see that the induction hypothesis implies that the rotating line hits $\overline{x}_i$ with its head, and so $x_i \overline{x}_i^+ = \{x_r \mid r < i\} \cup \{y_r \mid r < t - i + 1\}$. We define $\psi(x_i) = x_i \overline{x}_i$. In an analogous manner we define $\psi(y_i)$ and $\psi(z_i)$ for all $y_i \in Y_t$ and $z_i \in Z_t$. Since ψ defines a one-to-one relation and $|P_t| = 3(t + 1)$, it follows that $|\psi(P_t)| = 3(t + 1)$.

Let $E' := \{x_0 z_{t+1}^{\curvearrowright}, y_0 x_{t+1}^{\curvearrowright}, z_0 y_{t+1}^{\curvearrowright}\}$. Observation 4.2 implies that $E' \subset E_{t+1}(P)$. We note that $\psi(x_0) = x_0 y_{t+1}^{\curvearrowleft}$, $\psi(y_0) = y_0 z_{t+1}^{\curvearrowleft}$, and $\psi(z_0) = z_0 x_{t+1}^{\curvearrowleft}$. Using these observations and that $\{x_{t+1}^{\curvearrowright}, y_{t+1}^{\curvearrowright}, z_{t+1}^{\curvearrowright}\} \cap P_t = \emptyset$, it follows that $E' \cap \psi(P_t) = \emptyset$. On the other hand, from Proposition 2.1 it follows that $|E_{t+1}(P)| = 3(t + 2)$. Thus $E_{t+1}(P)$ is the disjoint union of $\psi(P_t)$ and E'.

By way of contradiction, suppose that $x_{t+1}^{\curvearrowright} \neq x_{t+1}^{\curvearrowleft}$. Then each point of $\{x_1, \ldots, x_t\}$ is contained in the interior of the quadrilateral bounded by $y_0 x_{t+1}^{\curvearrowright}, z_0 x_{t+1}^{\curvearrowleft}, z_0 x_0$, and $x_0 y_0$ (see Figure 4). From the induction hypothesis it follows that $x_{t+1}^{\curvearrowright} x_{t+1}^{\curvearrowleft} \notin E_{\leq t}(P)$. This and the fact that $x_{t+1}^{\curvearrowright} x_{t+1}^{\curvearrowleft} \notin \psi(P_t) \cup E'$ imply that $|x_{t+1}^{\curvearrowright} x_{t+1}^{\curvearrowleft}{}^-| \geq t + 2$. Then the interior of the triangle T bounded by $y_0 x_{t+1}^{\curvearrowright}, z_0 x_{t+1}^{\curvearrowleft}$, and $x_{t+1}^{\curvearrowright} x_{t+1}^{\curvearrowleft}$ is nonempty, and, moreover, it contains every element of $Q := x_{t+1}^{\curvearrowright} x_{t+1}^{\curvearrowleft}{}^- \setminus \{x_0, x_1, \ldots, x_t\}$. Let p be the first point of Q that $z_0 x_{t+1}^{\curvearrowleft}$ finds as it is rotated counterclockwise around $x_{t+1}^{\curvearrowleft}$. Then $x_{t+1}^{\curvearrowleft} p$ must be a $(t + 1)$-edge of P. On the other hand, it is immediately seen that $x_{t+1}^{\curvearrowleft} p \notin \psi(P_t) \cup E'$, contradicting that $E_{t+1} = \psi(P_t) \cup E'$.

This contradiction shows that $x_{t+1}^{\curvearrowright}$ and $x_{t+1}^{\curvearrowleft}$ are the same point. Analogous arguments show that $y_{t+1}^{\curvearrowright}$ and $y_{t+1}^{\curvearrowleft}$ are the same point, and that $z_{t+1}^{\curvearrowright}$ and $z_{t+1}^{\curvearrowleft}$ are the same point. This proves (1) for $j = t + 1$.

Now we show that (2) holds for $j = t + 1$. Note that at this point $x_{t+1}, y_{t+1}, z_{t+1}$ are all well-defined. For each $m \in \{0, 1, \ldots, t + 1\}$, we let $X_m := \{x_i \mid i \leq m\}$, $Y_m := \{y_i \mid i \leq m\}$, and $Z_m := \{z_i \mid i \leq m\}$.

Let $m \in \{0, 1, 2, \ldots, t + 1\}$. We shall show that

(i) $x_m y_{t+1-m}$ is in $E_{t+1}(P)$; and

(ii) $x_m y_{t+1-m}^+ = X_{m-1} \cup Y_{t-m}$.

By symmetry, analogous arguments show that: (i') $y_m z_{t+1-m}$ is in $E_{t+1}(P)$; (ii') $y_m z_{t+1-m}^+ = Y_{m-1} \cup Z_{t-m}$; (i'') $z_m x_{t+1-m}$ is in $E_{t+1}(P)$; and (ii'') $z_m x_{t+1-m}^+ = Z_{m-1} \cup X_{t-m}$. Note that these six assertions, together with the fact that $|E_{t+1}(P)| = 3(t + 2)$, imply (2).

Thus we complete the proof by showing (i) and (ii).

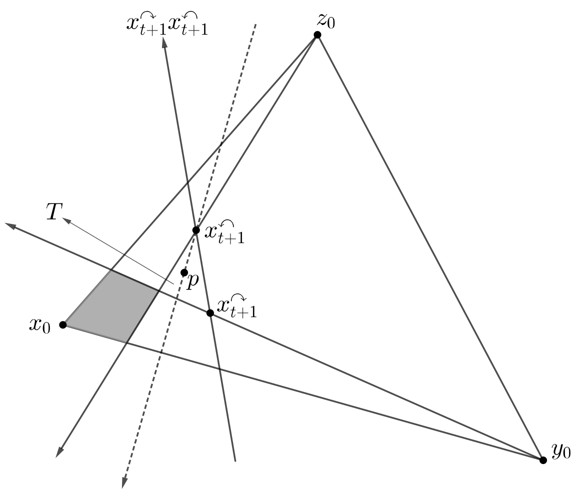

Figure 4: If $x^{\curvearrowright}_{t+1} \neq x^{\curvearrowleft}_{t+1}$, then the triangle T contains a point p such that $x^{\curvearrowleft}_{t+1}p$ is a $(t+1)$-edge of P.

Since $x_{t+1} = x^{\curvearrowleft}_{t+1} = x^{\curvearrowright}_{t+1}$, $y_{t+1} = y^{\curvearrowleft}_{t+1} = y^{\curvearrowright}_{t+1}$, and $z_{t+1} = z^{\curvearrowleft}_{t+1} = z^{\curvearrowright}_{t+1}$, it follows that (i) and (ii) hold whenever m is in $\{0, t+1\}$. Thus it suffices to prove (i) and (ii) for $1 \leq m \leq t$.

From the induction hypothesis we have that $x_{m-1}y^{+}_{t+1-m} = X_{m-2} \cup Y_{t-m}$ and $x_m y^{+}_{t-m} = X_{m-1} \cup Y_{t-m-1}$. Also note that $X_{m-1} \cup Y_{t-m} \subseteq x_m y^{+}_{t+1-m}$.

Let B denote the triangle bounded by the lines $x_{m-1}y_{t+1-m}$, $x_m y_{t-m}$, and $x_m y_{t+1-m}$ (see Figure 5). Let P_B be the set of points of P contained in the interior of B.

We claim that $P_B = \emptyset$. By way of contradiction, suppose this is not the case. Let $L = p_1 p_2 \cdots p_k$ be the lower chain of the convex hull of $P_B \cup \{x_m, y_{t+1-m}\}$. Then $p_1 = x_m$ and $p_k = y_{t+1-m}$, where (since $B \neq \emptyset$) $k \geq 3$. We note that $p_i p^{+}_{i+1} = X_{m-1} \cup Y_{t-m}$ for all $i = 1, 2, \ldots, k-1$. Thus each edge of L is a $(t+1)$-edge. We recall that $E_{t+1}(P) = \psi(P_t) \cup E'$. It is readily seen that no edge of L is in E', and so every edge of L is in $\psi(P_t)$. In particular, the line $p_2 p_3$ is in $\psi(P_t)$.

Recall that every edge in $\psi(P_t)$ is obtained by starting with a line $v_i w_{t-i}$ (for $v, w \in \{x, y, z\}, v \neq w$), counterclockwise rotating it around v_i, and recording the first point p in P hit by the rotating line: $\psi(v_i)$ is then the line $v_i p$. Thus, in particular $p_2 p_3$ is obtained in this way. Now if we reverse the process and clockwise rotate $p_2 p_3$ around p_2, the first point hit by the rotating line must be y_{t-m}. This implies that $p_2 = x_m$, contradicting that $p_1 = x_m$. We therefore conclude that $P_B = \emptyset$. Finally, note that $P_B = \emptyset$ immediately implies that $\psi(x_m) = x_m y_{t+1-m}$. Thus $x_m y_{t+1-m}$ is a $(t+1)$-edge. This proves (i). Moreover, as we observed above, $X_{m-1} \cup Y_{t-m} \subseteq x_m y^{+}_{t+1-m}$. Since $|X_{m-1} \cup Y_{t-m}| = t+1$, then $X_{m-1} \cup Y_{t-m} = x_m y^{+}_{t+1-m}$. Thus (ii) follows. $\square$

In view of Proposition 4.3(1), we have achieved our goal to unambiguously identify

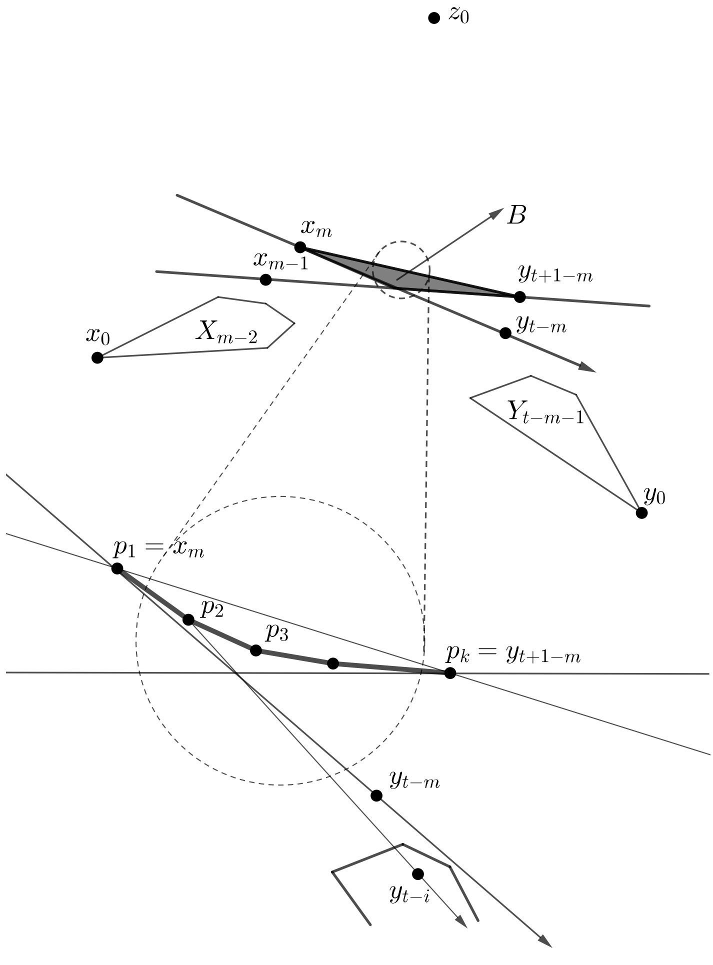

Figure 5: If the interior of the triangle B is nonempty, then every edge of the convex chain $p_1, p_2, \ldots, p_k$ is a $(t+1)$-edge.

(and label) all 18 points of P. For the rest of the paper, we let $X := \{x_0, x_1, \dots, x_5\}$, $Y := \{y_0, y_1, \dots, y_5\}$, and $Z := \{z_0, z_1, \dots, z_5\}$, where x_j, y_j, and z_j are as in Proposition 4.3, for $j = 0, 1, \dots, 5$.

4.2 $\{X, Y, Z\}$ is a 3-decomposition of P

If we rotate the line x_0z_0 clockwise along x_0, then for $j = 1, 2, \dots, 5$, the j-th point hit by the rotating line is z_j. If we rotate the line x_0y_0 counterclockwise along x_0, then for $j = 1, 2, \dots, 5$, the j-th point hit by the rotating line is y_j. It follows that the sixth point hit by the clockwise rotating line ℓ_x is in X, and the sixth point hit by the counterclockwise rotation line ℓ'_x is also in X (see Figure 6). These two points in X are obviously distinct (since $|X| > 2$), and so they define an infinite cone C_X with vertex x_0 (here by *cone with vertex* p we mean a pair of distinct directed rays, both with startpoint p). Note that C_X is the smallest infinite cone with vertex x_0 that contains X. See Figure 6.

We similarly find infinite cones C_Y (with vertex y_0) and C_Z (with vertex z_0).

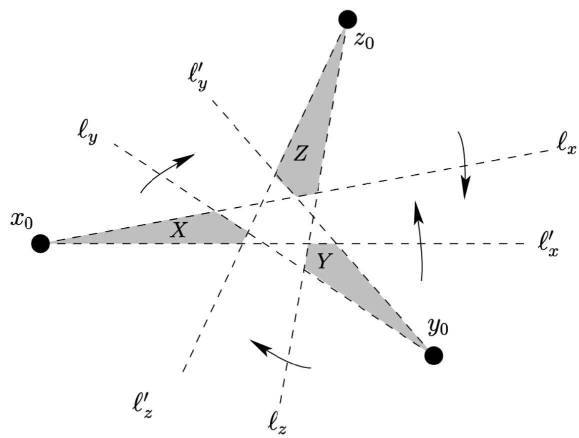

Figure 6: The sets X, Y, and Z are contained in the indicated (closed) shaded regions. The shaded region containing X is $\Delta_{X,Y} \cap \Delta_{X,Z}$.

Now $C_X \cup C_Y$ divide the plane into several regions, three of which are bounded. Two of these bounded regions are triangles: one triangle $\Delta_{X,Y}$ with x_0 as a vertex and another triangle $\Delta_{Y,X}$ with y_0 as a vertex; the other one is a quadrilateral. The entire set X is contained in $\Delta_{X,Y}$, and the entire set Y is contained in $\Delta_{Y,X}$. By considering the pair C_X, C_Z (respectively, C_Y, C_Z), we obtain triangles $\Delta_{X,Z}$ and $\Delta_{Z,X}$ (respectively, $\Delta_{Y,Z}$ and $\Delta_{Z,Y}$). Thus $X \subseteq \Delta_{X,Y} \cap \Delta_{X,Z}$, $Y \subseteq \Delta_{Y,X} \cap \Delta_{Y,Z}$, and $Z \subseteq \Delta_{Z,X} \cap \Delta_{Z,Y}$. Hence the situation is as illustrated in Figure 6. In this figure, each of $\Delta_{X,Y} \cap \Delta_{X,Z}$, $\Delta_{Y,X} \cap \Delta_{Y,Z}$, and $\Delta_{Z,X} \cap \Delta_{Z,Y}$ is a quadrilateral, although it is easy to see that any (or all) of them may be a triangle.

In view of this, it follows immediately that there is a triangle that witnesses the following:

Proposition 4.4. *P is 3-decomposable, with 3-decomposition $\{X, Y, Z\}$.* □

As we mentioned in Section 2, knowing that $\{X, Y, Z\}$ is a 3-decomposition of P

allows us to classify each edge of P as either monochromatic or bichromatic: for $p, q \in P$, the edge pq is *monochromatic* if p and q belong to the same set of the 3-decomposition $\{X, Y, Z\}$. Otherwise, pq is *bichromatic*.

We close this section by noting that using the 3-decomposability of P it is easy to determine the number of bichromatic and monochromatic k-edges in P, for each $k \in \{0, \ldots, 8\}$.

Indeed, since P is 3-decomposable it follows from [2, Claim 1] that $e^{\text{bi}}_{\leq k}(P) = 3\binom{k+2}{2}$ for $k \in \{0, \ldots, 5\}$, $e^{\text{bi}}_{\leq 6}(P) = 81$, and $e^{\text{bi}}_{\leq 7}(P) = 99$. Also note that $e^{\text{bi}}_{\leq 8}(P)$ is the total number of bichromatic edges of P, namely $3 \cdot 6 \cdot 6 = 108$. Using that $e^{\text{bi}}_{j}(P) = e^{\text{bi}}_{\leq j}(P) - e^{\text{bi}}_{\leq j-1}(P)$ for $j \in \{1, \ldots, 8\}$, we obtain the following.

Proposition 4.5. $e^{\text{bi}}_{k}(P) = 3(k+1)$ *for* $k \in \{0, \ldots, 5\}$, $e^{\text{bi}}_{6}(P) = 18$, $e^{\text{bi}}_{7}(P) = 18$, *and* $e^{\text{bi}}_{8}(P) = 9$.

To obtain $e^{\text{mon}}_{6}(P), e^{\text{mon}}_{7}(P)$ and $e^{\text{mon}}_{8}(P)$, we note that Proposition 2.1 implies that $e_6(P) = 24, e_7(P) = 33$, and $e_8(P) = 33$. Since $e_j(P) = e^{\text{bi}}_{j}(P) + e^{\text{mon}}_{j}(P)$ for $j = 0, \ldots, 8$, Proposition 4.5 implies the following.

Corollary 4.6. $e^{\text{mon}}_{k}(P) = 0$ *for* $k \in \{0, \ldots, 5\}$, $e^{\text{mon}}_{6}(P) = 6$, $e^{\text{mon}}_{7}(P) = 15$, *and* $e^{\text{mon}}_{8}(P) = 24$.

4.3 Structural properties of P

4.3.1 Determination of $e^{uu}_{k}(P)$ for any $u \in \{x, y, z\}$ and any $k \in \{0, \ldots, 8\}$

Let $u \in \{x, y, z\}$, $i \in \{1, 2, \ldots, 5\}$, and ℓ_u, ℓ'_u be the directed rays forming the cone C_U with vertex u_0 mentioned in the arguments leading to Proposition 4.4. See Figure 6. From now on, we shall use u^i_0 to denote the i-th point of P that ℓ_u finds when it is rotated clockwise around of u_0 until it reaches ℓ'_u. Clearly, $\{u^1_0, u^2_0, \ldots, u^5_0\} = \{u_1, u_2, \ldots, u_5\}$.

Our next observation is evident, but useful.

Observation 4.7. Let v_1, v_2, and v_3 be three distinct points in P, and let $\ell := v_1v_2$ and $\ell' := v_1v_3$. Let $P_1 := \ell^- \cap \ell'^+$, $P_2 := \ell^+ \cap \ell'^+$, and $P_3 := \ell^- \cap \ell'^-$. Then P_1, P_2, and P_3 are pairwise disjoint subsets of P. See Figure 7. For $i = 1, 2, 3$, let r_i be the number of points in P_i. If $P \setminus \{v_1, v_2, v_3\}$ is the disjoint union of P_1, P_2, and P_3, and p_i is the i-th point of P_1 that ℓ finds when it is rotated counterclockwise around v_1 until it reaches ℓ', then v_1p_i is a j-edge of P for $j = \min\{r_2 + i, 16 - (r_2 + i)\}$.

The next observation is immediate from the definition of u^i_0 and Observation 4.7.

Observation 4.8. Let $u \in \{x, y, z\}$. Then

(1) $u_0u^1_0$ and $u_0u^5_0$ are both 6-edges,

(2) $u_0u^2_0$ and $u_0u^4_0$ are 7-edges and they are the only 7-edges of the type u_0u, and

(3) $u_0u^3_0$ is an 8-edge.

Claim 4 in [2] implies that $e^{uu}_{8}(P) \leq 8$ for each $u \in \{x, y, z\}$. Using this, together with Observation 4.8 and Corollary 4.6, we obtain the following.

Proposition 4.9. *Let* $u \in \{x, y, z\}$. *Then* $e^{uu}_{k}(P) = 0$ *for* $k \in \{0, \ldots, 5\}$, $e^{uu}_{6}(P) = 2$, $e^{uu}_{7}(P) = 5$, *and* $e^{uu}_{8}(P) = 8$.

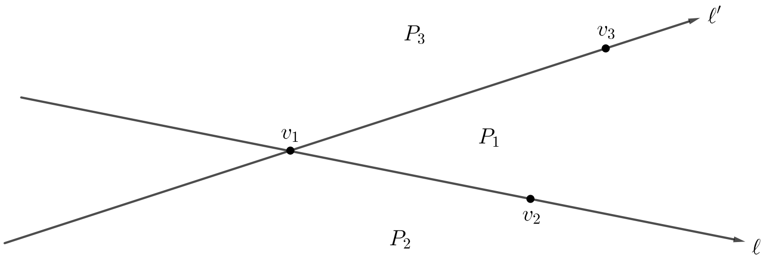

Figure 7: The i-th point of $P_1 := \ell^- \cap \ell'^+$ that ℓ finds when it is rotated counterclockwise around v_1 is a j-edge for $j = \min\{r_2 + i, 16 - (r_2 + i)\}$, where r_2 denotes the number of points of $P_2 := \ell^+ \cap \ell'^+$.

The next corollary follows immediately from Observation 4.8(1) and Proposition 4.9.

Corollary 4.10. $E_6^{\text{mon}}(P) = \{x_0x_0^1, x_0x_0^5, y_0y_0^1, y_0y_0^5, z_0z_0^1, z_0z_0^5\}$. *Moreover, any other monochromatic edge must belong to* $E_7^{\text{mon}}(P) \cup E_8^{\text{mon}}(P)$.

4.3.2 Determination of the convex hull of U for $U \in \{X, Y, Z\}$ and related facts

Let u be any element of $\{x, y, z\}$. One of the main goals in this subsection is to show that the triangle formed by u_0, u_4 and u_5 contains in its interior the remaining u's, namely, u_1, u_2 and u_3. We also prove other statements about the relative position of the elements of U. Almost all these assertions will be used in the subsequent steps later on.

Proposition 4.11. *Let* $u \in \{x, y, z\}$. *If* $u_1 \in \{u_0^2, u_0^4\}$, *then there are at least three* 7*-edges of type* uu *involving* u_1 *but not* u_0.

Proof. We prove the proposition for the case $u = x$. The cases $u = y$ and $u = z$ are handled in a totally analogous manner.

Suppose that $x_1 = x_0^4$. Then $\ell := x_0x_1$ leaves x_0^1, x_0^2 and x_0^3 on its left halfplane (and x_0^5 on its right halfplane). Let z' be the first z that ℓ finds when it is rotated counterclockwise around x_1, and let $\ell' = x_1z'$. See Figure 8. By Corollary 4.10, we know that $\{x_1x_0^1, x_1x_0^2, x_1x_0^3\} \subset E_7^{\text{mon}}(P) \cup E_8^{\text{mon}}(P)$. Then ℓ finds each of x_0^1, x_0^2 and x_0^3 before it reaches z'. This and Observation 4.7 imply that at most one of $x_1x_0^1, x_1x_0^2, x_1x_0^3$ is an 8-edge and, by Corollary 4.10, the other two must be 7-edges.

From the way in that the x's were labelled it follows that x_0^5 is the first x that ℓ finds when it is rotated clockwise around x_1, and so $x_1x_0^5$ is a (≤ 7)-edge. This and Corollary 4.10 imply that $x_1x_0^5$ is the third required 7-edge. The case $x_1 = x_0^2$ can be handled in an analogous manner (y's play the role of z's). □

Proposition 4.12. *Let* $u \in \{x, y, z\}$ *and* $\{p, q\} = \{x, y, z\} \setminus \{u\}$. *Suppose that* $\{q_0, \dots, q_5\} \subset u_0u_0^{3-}$ *and that* $\{p_0, \dots, p_5\} \subset u_0u_0^{3+}$. *Then:*

(A1) $u_1 \notin \{u_0^1, u_0^5\}$*;*

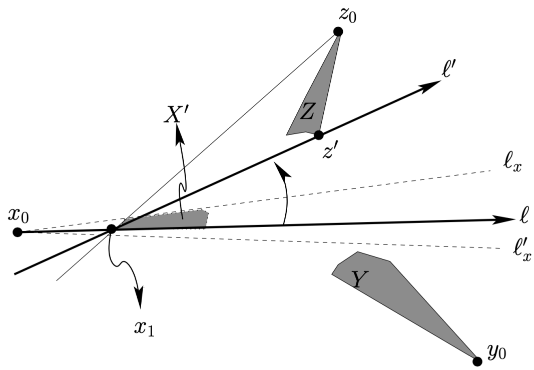

Figure 8: Here x_0x_1 is a 7-edge.

(A2) *there are at least two* 7*-edges of type* uu *involving* u_1 *but not* u_0*;*

(A3) $u_2 \notin \{u_0^1, u_0^5\}$*;*

(A4) *each of* u_3u_4, u_3u_5 *and* u_4u_5 *is an* 8*-edge;*

(A5) $\{u_0^1, u_0^5\} = \{u_4, u_5\}$*;*

(A6) *the triangle formed by* u_0, u_4 *and* u_5 *is the convex hull of* U*; and*

(A7) *if* $u_5 \in u_0u_4^+$*, then* $u_0u_4^- = \{q_0, \ldots, q_5\}$ *and* $u_0u_5^+ = \{p_0, \ldots, p_5\}$*. Otherwise,* $u_0u_4^+ = \{p_0, \ldots, p_5\}$ *and* $u_0u_5^- = \{q_0, \ldots, q_5\}$ *.*

Proof. By rotating P if necessary, and exchanging appropriately the labels x, y, and z, we can assume, without any loss of generality, that $u = x, p = y, q = z$ and that X, Y and Z are placed as in Figure 9.

(A1): Seeking a contradiction, suppose that $x_0^1 = x_1$. Let v be the first point that x_0x_1 finds when it is rotated clockwise around x_1 as shown in Figure 9(a). Note that $v \in Y$, as otherwise $v \in \{x_2, x_3, x_4, x_5\}$ and $x_1v^- = Z$. Then x_1v is a 6-edge, contradicting Corollary 4.10. Let x' be the last element of $\{x_2, x_3, x_4, x_5\}$ that x_1v finds when it is rotated clockwise around x_1. Since $v \in Y$, then x_1x' must be a (≤ 6)-edge, contradicting Proposition 4.9. The case $x_0^5 = x_1$ can be handled in an analogous manner (with the roles of Z and Y interchanged).

(A2): From (A1) we know that x_0x_1 leaves at least one x in each side. By definition of x_1, the points x_2, x_3, x_4 and x_5 must be contained in $X' := X \cap x_1z_0^+ \cap x_1y_0^-$, see Figure 9(b). Let x' be the last element of $\{x_2, x_3, x_4, x_5\}$ that x_0x_1 finds when it is rotated clockwise around x_1 as shown in Figure 9(b). Note that x_1x' must be a (≤ 7)-edge. Since $x' \neq x_0$, then Proposition 4.9 implies that x_1x' must be a 7-edge. Similarly, if we rotate x_0x_1 in the other direction, then we can find the other 7-edge involving x_1 but not x_0.

(A3): Seeking a contradiction, suppose that $x_2 = x_0^1$. Then x_0x_2 is a 6-edge, and x_3, x_4 and x_5 are contained in $X'' := X \cap x_0^1y_0^-$. See Figure 10(b). From Observation 4.7, we know that at most one of x_2x_3, x_2x_4, x_2x_5 is an 8-edge. This and Corollary 4.10 imply

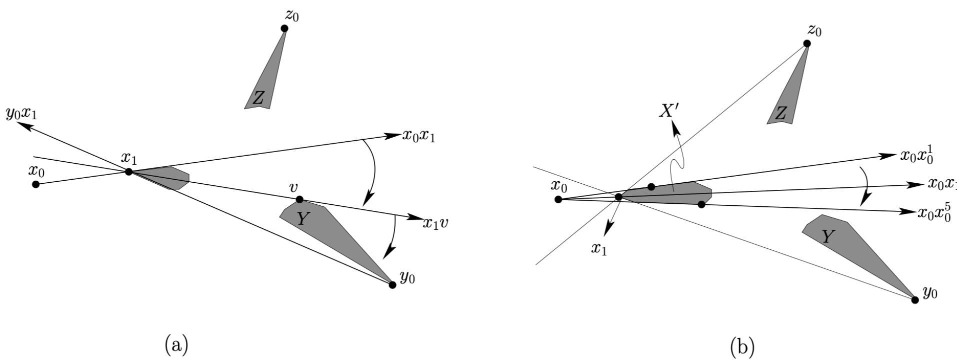

Figure 9: (a) x_0x_1 cannot be a 6-edge. (b) There are at least two 7-edges of type xx involving x_1 but not x_0.

that at least two of x_2x_3, x_2x_4, x_2x_5 are 7-edges. This together with Observation 4.8(2) and (A2) imply that $e_7^{xx}(P) \geq 6$, which contradicts Proposition 4.9. The case $x_2 = x_0^5$ can be handled in an analogous manner.

(A4): From Corollary 4.10, Observation 4.8(1), and (A1), we know that x_0x_1 is a 7-edge or an 8-edge. First suppose that x_0x_1 is a 7-edge. Then $x_1 \in \{x_0^2, x_0^4\}$. This together with Propositions 4.9 and 4.11 and Observation 4.8(2) imply that each element of $E_7^{xx}(P)$ contains at least one of x_0 or x_1. This fact and Proposition 4.9 imply that x_3x_4, x_3x_5 and x_4x_5 are 8-edges, as required.

Now, we suppose that x_0x_1 is an 8-edge. Then $x_2 \in x_0x_1^-$ or $x_2 \in x_0x_1^+$. We only analyze the case $x_2 \in x_0x_1^-$ (the other case is symmetric). Then we must have that $X' := X \cap x_0x_1^+ \cap x_2y_0^-$ contains exactly two elements x', x'' of $\{x_3, x_4, x_5\}$, see Figure 10(a). Now we rotate x_2y_0 clockwise around x_2 until it be parallel to x_0x_1. See Figure 10(a). From Observation 4.7 and Corollary 4.10, we know that at least one of x_2x', x_2x'' is a 7-edge. Such a 7-edge plus the four 7-edges provided by Observation 4.8(2) and (A2) give us, 5, the total number of 7-edges of P. This and Proposition 4.9 imply that x_3x_4, x_3x_5, x_4x_5 are 8-edges, as required.

(A5): Seeking a contradiction, suppose that $\{x_0^1, x_0^5\} \neq \{x_4, x_5\}$. Then (A1) and (A3) imply that $x_3 = x_0^1$ or $x_3 = x_0^5$. Again, by symmetry it is enough to analyze the case $x_3 = x_0^1$. Clearly, both x_1 and x_2 are contained in the triangle formed by $x_0x_3, x_0x_0^5$ and x_3y_0; and x_4, x_5 are contained in $X'' := X \cap x_0^1y_0^-$, see Figure 10(b).

Now we rotate x_0x_3 clockwise around x_3 until it reaches x_3y_0, and note that such a rotation hits x_4 and x_5. From Observation 4.7 we know that at most one of x_3x_4 or x_3x_5 is 8-edge, contradicting (A4).

(A6): This follow directly from (A5) and the way in that the x's were labelled.

(A7): Suppose that $x_5 \in x_0x_4^+$. Then (A5) implies that $x_0^1 = x_4$ and $x_0^5 = x_5$. The required equalities follow from the definition of x_0^1 and x_0^5 and the hypotheses $Z \subset x_0x_0^{3-}$ and $Y \subset x_0x_0^{3+}$. Similarly, we can deduce that $x_0x_5^- = Z$ and $x_0x_4^+ = Y$ whenever $x_5 \in x_0x_4^-$. □

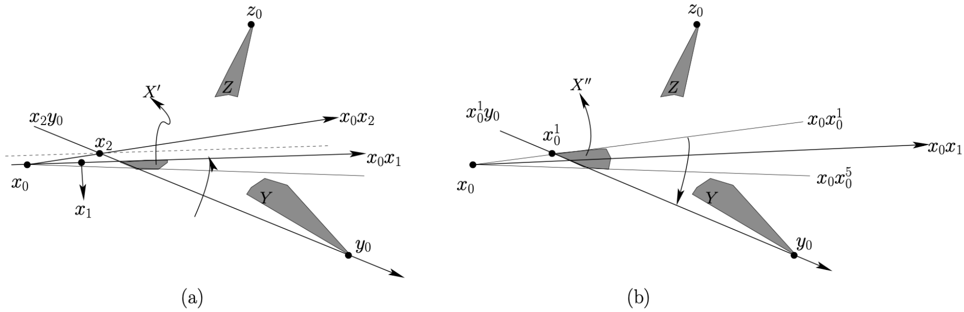

Figure 10: (a) x_3x_4, x_3x_5, x_4x_5 are 8-edges. The dotted straight line containing x_2 is parellel to x_0x_1. (b) $\{x_0^1, x_0^5\} = \{x_4, x_5\}$, and hence x_0x_4 and x_0x_5 are 6-edges.

4.3.3 Determination of the position of u_5 with respect to u_0u_4 for each $u \in \{x, y, z\}$

Our main goal in this subsection is to show that $u_5 \in u_0u_4^+$ for each $u \in \{x, y, z\}$. In order to prove this, we need to establish some auxiliary statements that will also be used later on.

Let $u, v \in \{x, y, z\}$ with $u \neq v$. We will say that u_4u_5 *splits* the v's to mean that u_4u_5 separates $\{u_0, \dots, u_3\} \cup \{v_0, \dots, v_3\}$ from the rest of the points of $P \setminus \{u_4, u_5\}$.

Proposition 4.13. *Let* $\{u, v, w\} = \{x, y, z\}$*, and suppose that* u_0u_5 *separates* u_4 *from the* v*'s. Then* u_4u_5 *splits the* v*'s.*

Proof. By rotating and/or reflecting P along u_0u_4, if necessary, we can assume that $u = x, v = y, w = z$ and that X, Y and Z are placed as in Figure 6.

Since x_0x_5 separates x_4 from the y's, then $x_5 \in x_0x_4^+$. From Proposition 4.3 we know that $x_4y_0^+ = \{x_0, x_1, x_2, x_3\}$. Then (A6) implies that $x_4y_0^+ \subseteq x_4x_5^+$. If we rotate x_4y_0 counterclockwise around x_4 until it reaches x_0x_4, then $x_5 \in x_0x_4^+$, (A6), and Observation 4.7 together imply that at most one element of $\{x_4x_5\} \cup \{x_4y \big| y \in Y\}$ is an 8-edge. This and (A4) imply that such an 8-edge must be x_4x_5. Then (A6) implies that x_4x_5 leaves exactly four y's on its right. Moreover, from Proposition 4.3 it is easy to see that y_0 and y_1 are in $x_4x_5^+$. Let y_i and y_j be two elements of Y in $x_4x_5^-$. Without loss of generality, we can assume that $i < j$. Then $2 \leq i < j \leq 5$.

From (A6) we know that the triangle formed by y_0, y_4, and y_5 is the convex hull of Y. This implies that $j \in \{4, 5\}$. Seeking a contradiction, suppose that $i \in \{2, 3\}$.

Let T be the triangle formed by x_0x_5, x_0y_i and x_4x_5. See Figure 11(b). By the way y's were labelled, we know that if $y_r \in T$ then $r \in \{i+1, \dots, 5\} \setminus \{j\}$. Let y' be the first point in $Y \cap T$ that y_ix_0 finds when it is rotated clockwise around y_i. See Figure 11(b). Then y_iy' is a $(\leq i+4)$-edge because the points of P lying in the left side of y_iy' is a subset of $\{x_0, \dots, x_3, y_0, \dots, y_{i-1}\}$. If $i = 2$, then y_iy' is a (≤ 6)-edge which does not involve y_0, contradicting Corollary 4.10. Finally, if $i = 3$, then y_iy' is a (≤ 7)-edge, contradicting (A4). □

Observation 4.14. From Proposition 4.13 we know that if $\{u, v, w\} = \{x, y, z\}$, then u_4u_5 splits the v's or the w's.

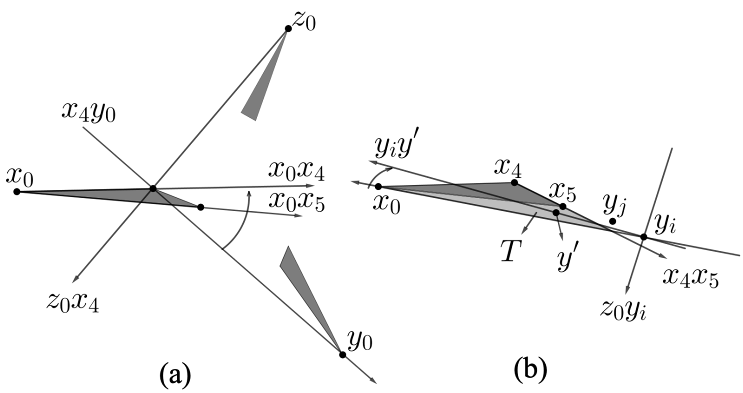

Figure 11: (a) If $x_4 \in x_0x_5^-$, then x_4x_5 splits Y. (b) There are exactly two y's, namely y_i and y_j, in $x_4x_5^-$.

Proposition 4.15. *Let u and v be two distinct elements of $\{x, y, z\}$. If u_4u_5 splits the v's, and v_3v_5 leaves u_0 and v_2 on the same side, then v_4v_5 splits the u's.*

Proof. By rotating P if necessary and exchanging appropriately the labels x, y and z, we can assume that $u = x$ and that X, Y and Z are placed as in Figure 6.

CASE 1: Suppose that x_4x_5 splits the y's. Then we need to show that if y_3y_5 leaves x_0 and y_2 on the same side, then y_4y_5 splits the x's.

From (A4), we know that y_4y_5 is an 8-edge, and from (A6), that y_4y_5 is in the convex hull of Y.

First, we show that $y_5 \in y_0y_4^-$. By way of contradiction, suppose this is not the case. Then $y_5 \in y_0y_4^+$ and the triangle formed by y_0, y_4 and y_5 looks like in Figure 12(a). Since x_4x_5 splits the y's, then x_4x_5 separates y_3 from y_4 and y_5, and so all the x's are on the left side of both y_3y_4 and y_3y_5. In particular, $x_0 \in y_3y_5^-$, and hence $y_2 \in y_3y_5^-$. Then $y_2 \in y_3y_5^- \cap z_0y_3^-$, and so y_2 is contained in the triangle Q formed by y_0y_4, z_0y_3, y_3y_5. See Figure 12(b). Since y_3y_4 is an 8-edge by (A4), and y_3y_4 leaves $\{y_0, y_2, x_0, \ldots, x_5\}$ on its left, then it leaves y_1, y_5 on its right. This and the fact that $y_1 \in z_0y_3^-$ imply that y_1 is contained in the triangle R formed by z_0y_3, y_3y_4, y_0y_5. See Figure 12(b). Then y_2y_4 and y_0y_1 are 7-edges. This, together with Observation 4.8(2) and Proposition 4.11, implies $e_7^{yy}(P) \geq 6$, the required contradiction. Thus, we can conclude that y_0y_4 leaves the x's and y_5 on the same side, and the desired result follows from Proposition 4.13.

CASE 2: x_4x_5 splits the z's. Follow the same argument as in CASE 1 with left, right, $y, z, -$ and $+$ in place of right, left, $z, y, +$ and $-$, respectively. $\square$

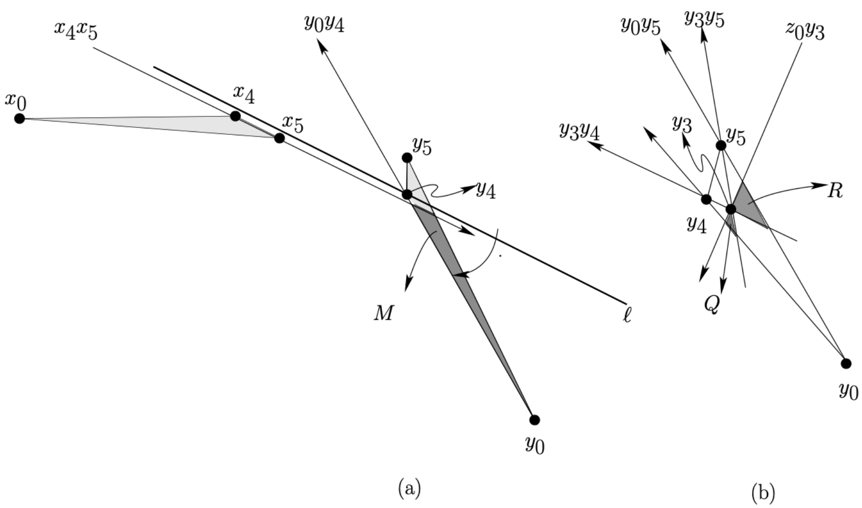

Figure 12: (a) $y_5 \in y_0y_4^+$. (b) y_2 is in Q and y_1 is in R.

Proposition 4.16. *If* $u \in \{x, y, z\}$*, then* u_3u_5 *leaves* u_0 *and* u_1 *on the same side.*

Proof. As in Proposition 4.15, we can assume that $u = x$ and that X, Y and Z are placed as in Figure 6. From (A4), we know that x_3x_5 is an 8-edge. Seeking a contradiction, suppose that x_3x_5 separates x_0 from x_1.

First, suppose that x_4x_5 splits the y's. Since P is placed as in Figure 6, then $x_0 \in x_3x_5^+$, and hence, $x_1 \in x_3x_5^-$. Then $x_3x_5^+ = \{x_0, x_2\} \cup Y$, or equivalently, $x_3x_5^- = \{x_1, x_4\} \cup Z$. This fact has two immediate consequences. The first one is that $x_1 = x_0^2$. This fact and Proposition 4.11 imply that there are at least three 7-edges of type xx involving x_1 but not x_0. The second consequence is that x_2 is in the triangle X' (see Figure 13) formed by x_1y_0, x_3x_5 and x_0x_5, and hence x_2x_5 must be a 7-edge too. The existence of these four 7-edges together with those in Observation 4.8(2) imply $e_7^{xx}(P) \geq 6$, contradicting Proposition 4.9.

Now suppose that x_4x_5 splits the z's. Again, since P is placed as in Figure 6, then $x_0 \in x_3x_5^-$ and $x_1 \in x_3x_5^+$. By similar arguments as above, we can deduce that $x_1 = x_0^4$ and that x_2x_5 is a 7-edge. As before, $x_1 = x_0^4$ and Proposition 4.11 imply that there are at least three 7-edges of type xx involving x_1 but not x_0. The existence of these four 7-edges together with those in Observation 4.8(2) imply $e_7^{xx}(P) \geq 6$, contradicting Proposition 4.9. □

Proposition 4.17. *There is a* $u \in \{x, y, z\}$ *such that* u_3u_5 *separates* u_4 *from the other* u*'s.*

Proof. From Proposition 4.16 we know that u_3u_5 leaves u_0 and u_1 on the same side for each $u \in \{x, y, z\}$. Seeking a contradiction, we suppose that u_3u_5 separates $\{u_0, u_1\}$ from $\{u_2, u_4\}$ for each $u \in \{x, y, z\}$.

By rotating and/or reflecting P if necessary, and exchanging appropriately the labels x, y and z, we can assume that X, Y and Z are placed as in Figure 11(a), and that x_4x_5 splits the y's. An immediate consequence of these assumptions and our hypothesis is that (i) x_3x_5 leaves to z_0 and x_2 on its left side.

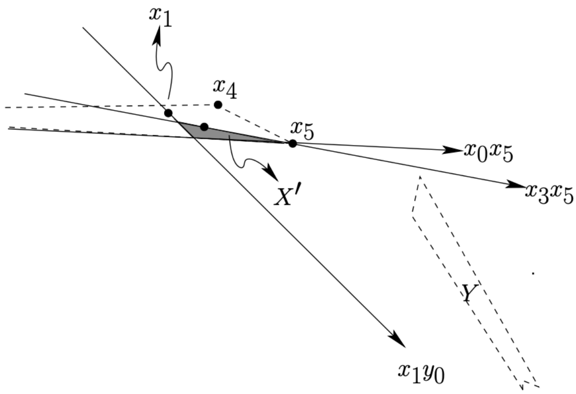

Figure 13: Here $x_1 \in x_3x_5^-$.

Now suppose that $y_5 \in y_0y_4^+$. Then y_0y_5 separates y_4 from the z's, and from Proposition 4.13 we know that y_4y_5 splits the z's. In particular, the triangle formed by y_0, y_4 and y_5 must be as in Figure 12(a) and $y_0 \in y_3y_5^+$. By supposition, y_3y_5 separates y_0 from y_2, and so y_3y_5 leaves to x_0 and y_2 on its left side. This last and Proposition 4.15 imply that y_4y_5 splits the x's too, which is impossible. Thus we conclude that $y_5 \in y_0y_4^-$. This and Proposition 4.13 imply that y_4y_5 splits the x's. This fact and our supposition imply that (ii) y_3y_5 leaves to z_0 and y_2 on its right side.

If z_4z_5 splits the x's, then (i) and Proposition 4.15 imply that x_4x_5 splits the z's, which contradicts that x_4x_5 splits the y's. Similarly, if z_4z_5 splits the y's, then (ii) and Proposition 4.15 imply that y_4y_5 splits the z's, again contradicting that y_4y_5 splits the x's. □

Proposition 4.18. *Let u be an element in $\{x, y, z\}$ that satisfies the property in* Proposition 4.17*, and suppose that u_4u_5 splits the v's, where $v \in \{x, y, z\} \setminus \{u\}$. Then the following hold:*

(B1) *u_3u_5 separates v_5 from the other v's;*

(B2) *v_4v_5 splits the w's, where $\{w\} = \{x, y, z\} \setminus \{u, v\}$;*

(B3) *v_1v_4 and v_2v_4 are both 7-edges; and*

(B4) *v_3v_5 separates v_4 from the other v's.*

Proof. Without loss of generality, we can assume that $u = x, v = y$, and X, Y and Z are placed according to Figure 11. Indeed, we can get such requirements by rotating and/or reflecting P, and by exchanging appropriately the labels x, y and z.

(B1): From our assumptions and the hypothesis we know that x_4 is the only x in $x_3x_5^-$. Since x_3x_5 is an 8-edge, then exactly one element y^* of Y is in $x_3x_5^-$. From (A6), we know that such a y^* is one of y_4 or y_5. If $y^* = y_4$, then, by the way $y_0, \dots, y_5$ were labelled, we have that y_5 must be contained in the triangle S of Figure 14. Then y_4y_5 leaves x_3, x_4, x_5 and all the z's on its right side. This implies that y_4y_5 cannot be an 8-edge, contradicting (A4). This contradiction implies that (B1) holds.

(B2): From (B1), we know that $y^* = y_5$ is the only element of Y in $x_3x_5^-$. If $y_5 \in y_0y_4^-$, then y_4 must be contained in the region R of Figure 14. This implies that each element in $(X \cup Y) \setminus \{x_4, y_4, y_5\}$ lies in $y_4y_5^-$, contradicting (A4) that $y_4y^* = y_4y_5$ is an 8-edge. Thus we have that $y_5 \in y_0y_4^+$. This fact and Proposition 4.13 imply that y_4y_5 splits the z's, as required.

In view of (B2) and our previous assumptions, for the rest of the proof, we may assume that the two triangles defined by $\{x_0, x_4, x_5\}$ and $\{y_0, y_4, y_5\}$ are as shown in Figure 12(a).

(B3): Let ℓ be the line through y_4 which is parallel to x_4x_5 and let M be the interior of the triangle formed by y_0y_5, y_0y_4 and x_4x_5. See Figure 12(a). Since y_4 and y_5 are the only y's in $x_4x_5^-$, then $M \cap Y = \{y_1, y_2, y_3\}$. If we rotate ℓ in clockwise order around y_4 until it reaches y_4y_0, then by Observation 4.7 we have that exactly one of $\{y_1y_4, y_2y_4, y_3y_4\}$ is 8-edge and the other two are 7-edges. The desired assertion follows from (A4).

(B4): Seeking a contradiction, suppose that y_3y_5 does not separate y_4 from the other y's. Then Proposition 4.16 implies that y_3y_5 separates $\{y_0, y_1\}$ from $\{y_2, y_4\}$. On the other hand, since y_3y_4 is an 8-edge and x_4x_5 separates y_3 from y_4, then $y_3y_4^- = X \cup \{y_0, y_2\}$. Thus y_1 must be contained in the triangle R formed by z_0y_3, y_3y_4, y_0y_5. See Figure 12(b). This implies that $y_0^4 = y_1$. Then Proposition 4.11, Observation 4.8(2) and (B3) imply, $e_7^{yy}(P) \geq 6$, a contradiction. □

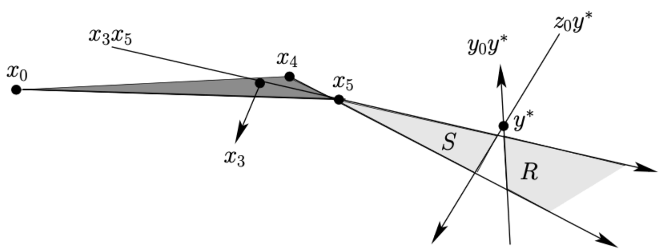

Figure 14: Here x_3x_5 separates y^* from the other y's.

Remark 4.19. From now on, without loss of generality, we assume that the $u \in \{x, y, z\}$ satisfying Proposition 4.17 is x, and that $x_5 \in x_0x_4^+$. Indeed, it is not hard to see that we can get such requirements by rotating and/or reflecting P along x_0x_4, and by appropriately exchanging the labels x, y and z. In particular, we assume that X, Y, Z, x_0, x_4 and x_5 are placed as in Figure 11.

Note that (B4) appears as hypothesis in Propositions 4.17 and 4.18. The following corollary is an immediate consequence of this fact.

Corollary 4.20. *Let $\sigma(x) = y, \sigma(y) = z$ and $\sigma(z) = x$. The following hold for each $u \in \{x, y, z\}$:*

(C1) *u_3u_5 separates $\sigma(u)_5$ from the other $\sigma(u)$'s;*

(C2) *$\sigma(u)_4\sigma(u)_5$ splits the $\sigma(\sigma(u))$'s;*

(C3) $\sigma(u)_1\sigma(u)_4$ *and* $\sigma(u)_2\sigma(u)_4$ *are both* 7*-edges;*

(C4) $\sigma(u)_3\sigma(u)_5$ *separates* $\sigma(u)_4$ *from the other* $\sigma(u)$*'s; and*

(C5) $u_5 \in u_0u_4^+$.

Proof. In view of Remark 4.19, we may assume that x_4x_5 splits the y's, x_3x_5 separates x_4 from the other x's, and X, Y, Z, x_0, x_4 and x_5 are placed as in Figure 11.

First, we show that (C1) – (C4) hold. Proposition 4.18 states exactly (C1) – (C4) for $u = x$ and $v = \sigma(x) = y$. In particular, (C2) and (C4) tell us that y_4y_5 splits the z's and that y_3y_5 separates y_4 from the other y's, respectively. By applying Proposition 4.18 to the last two conclusions on y's we have that (C1) – (C4) also hold for $u = y$ and $v = \sigma(y) = z$. Similarly, we can conclude that (C1) – (C4) also hold for $u = z$ and $v = \sigma(z) = x$.

Now, we show (C5). For $u = x$ the assertion holds by Remark 4.19. We first analyze the case $u = y$. From (A6) and Remark 4.19, we know that $\{x_0, \ldots, x_5\}$ lies on the left side of both y_0y_4 and y_0y_5. Seeking a contradiction, suppose that $y_5 \in y_0y_4^-$. Then, from (A6) and the last two facts, it is easy to verify that $x_0 \in y_3y_5^-$. Similarly, from $y_5 \in y_0y_4^-$ and (C4), we can deduce that $y_2 \in y_3y_5^-$. Thus $x_0, y_2 \in y_3y_5^-$, and so Proposition 4.15 implies that y_4y_5 splits the x's, contradicting (C2). An analogous argument shows that (C5) also holds for $u = z$. □

From Remark 4.19 and Corollary 4.20, we have that the points of P with indices $0, 3, 4$ and 5 are placed as in Figure 15.

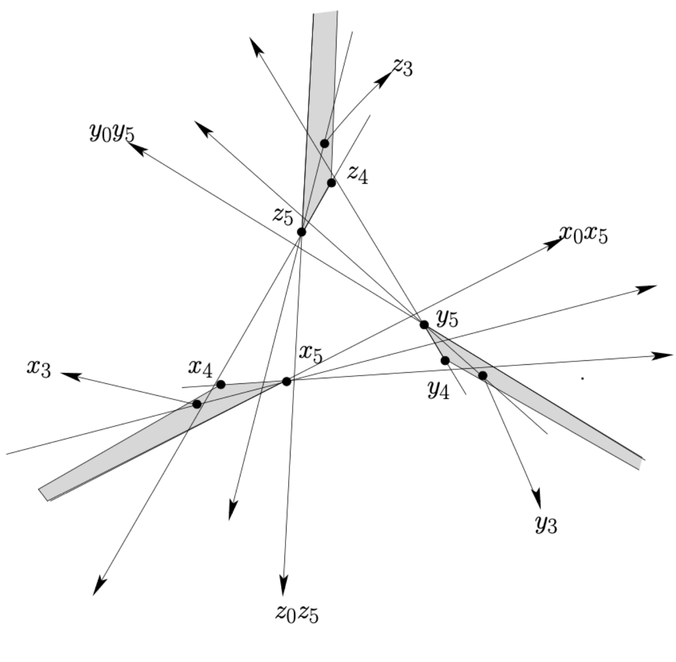

Figure 15: The relative position of the points of P with indices $0, 3, 4$ and 5.

4.4 Determination of pq^+ when pq is a monochromatic edge.

Lemma 3.1 for the case in which pq is a monochromatic edge will follow from Propositions 4.21 and 4.22 below. Regarding the statements of these propositions, we recall from Remark 4.19 and Corollary 4.20 that x_4x_5 splits the ys, y_4y_5 splits the zs, and z_4z_5 splits the xs.

Proposition 4.21. *Let $u \in \{x, y, z\}$, and let v be the element in $\{x, y, z\} \setminus \{u\}$ such that u_4u_5 splits the vs. Then the following hold:*

(D1) $u_0u_5^+ = \{v_0, \dots, v_5\}$;

(D2) $u_0u_4^+ = \{u_1, u_2, u_3, u_5\} \cup \{v_0, \dots, v_5\}$;

(D3) $u_4u_5^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1, v_2, v_3\}$; *and*

(D4) $u_3u_5^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2, v_3, v_4\}$.

Proof. (D1): From (A7), we know that u_0u_5 separates the v's from the w's. Moreover, because u_0u_5 is an edge of the convex hull of U, then u_0u_5 leaves the other u's on the same side. This and (C5) imply that $\{u_1, u_2, u_3, u_4\} \subset u_0u_5^-$. Again, from (C5) and Proposition 4.13 we have that $\{v_0, v_1, v_2, v_3\} \subset u_4u_5^+$, and hence $\{v_0, v_1, v_2, v_3\} \subset u_0u_5^+$. This and the fact that u_0u_5 separates the v's from the w's imply that $\{v_0, \dots, v_5\} \subseteq u_0u_5^+$. We finally note that Observation 4.8(1) implies that $u_0u_5^+ = \{v_0, \dots, v_5\}$, as required.

(D2): From (C5), (D1), and the way in that the points of P were labelled we have that $\{v_0, \dots, v_5\} = u_0u_5^+ \subset u_0u_4^+$. On the other hand, since u_0u_4 is an edge of the convex hull of U, then u_0u_4 leaves the other u's on the same side. This and (C5) imply that $\{u_1, u_2, u_3, u_5\} \subset u_0u_4^+$. Thus $\{u_1, u_2, u_3, u_5\} \cup \{v_0, \dots, v_5\} \subset u_0u_4^+$. Again, Observation 4.8(1) implies that $u_0u_4^+ = \{u_1, u_2, u_3, u_5\} \cup \{v_0, \dots, v_5\}$, as required.

(D3): It follows immediately from (C5) and Proposition 4.13.

(D4): Clearly, $\{v_0, \dots, v_5\} \cap u_4u_5^+ \subset \{v_0, \dots, v_5\} \cap u_3u_5^+$. This fact, together with (C1) and (D3), implies that $\{v_0, v_1, v_2, v_3, v_4\} \subset u_3u_5^+$. On the other hand, from (C4) is easy to verify that $\{u_0, \dots, u_5\} \cap u_3u_5^+ = \{u_0, u_1, u_2\}$. Then $\{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2, v_3, v_4\} \subset u_3u_5^+$. We finally note that (A4) implies that $u_3u_5^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2, v_3, v_4\}$, as required. □

Proposition 4.22. *Let $u \in \{x, y, z\}$, and let v be the element in $\{x, y, z\} \setminus \{u\}$ such that u_4u_5 splits the vs. Then the following hold:*

(E1) u_0u_1 *is an* 8*-edge;*

(E2) u_0u_2 *and* u_0u_3 *are* 7*-edges;*

(E3) u_2u_3 *and* u_2u_5 *are* 8*-edges;*

(E4) $u_2u_4^+ = \{u_5\} \cup \{v_0, \dots, v_5\}$;

(E5) $u_1u_4^+ = \{u_2, u_3, u_5\} \cup \{v_0, \dots, v_5\}$ *and* $u_3u_4^+ = \{u_2, u_5\} \cup \{v_0, \dots, v_5\}$;

(E6) $u_0u_2^+ = \{u_5\} \cup \{v_0, \dots, v_5\}$;

(E7) $u_0u_1^+ = \{u_2, u_5\} \cup \{v_0, \dots, v_5\}$ *and* $u_0u_3^+ = \{u_1, u_2, u_5\} \cup \{v_0, \dots, v_5\}$;

(E8) $u_1u_5^+ = \{u_0\} \cup \{v_0, \dots, v_5\}$;

(E9) u_1u_2 *and* u_1u_3 *are* 8*-edges;*

(E10) $u_1u_3^+ = \{u_2, u_5\} \cup \{v_0, \dots, v_5\}$;

(E11) $u_1u_2^+ = \{u_0, u_5\} \cup \{v_0, \dots, v_5\}$;

(E12) $u_2u_5^+ = \{u_0, u_1\} \cup \{v_0, \dots, v_5\}$; *and*

(E13) $u_2u_3^+ = \{u_4, u_5\} \cup \{v_0, \dots, v_5\}$.

Proof. (E1): From Proposition 4.9, Corollary 4.10, and (A5), we know that x_0x_1 is a 7- or an 8-edge. If x_0x_1 is a 7-edge, then $x_1 \in \{x_0^2, x_0^4\}$ and by Proposition 4.11, there are at least three 7-edges of type xx involving x_1 but not x_0. This, Observation 4.8(2), and (C3) imply that $e_7^{xx}(P) \geq 6$, a contradiction. Thus x_0x_1 must be an 8-edge.

(E2): From Observation 4.8(2), we know that there are exactly two 7-edges of the type x_0x. Since (E1) and (A6) imply that none of x_0x_1, x_0x_4, x_0x_5 is a 7-edge, then both x_0x_2 and x_0x_3 are 7-edges, as desired.

(E3): By Corollary 4.10 and (A5), each of x_2x_3 and x_2x_5 is a 7- or an 8-edge. From (C3) and (E2), we know that each of $x_1x_4, x_2x_4, x_0x_2, x_0x_3$ is a 7-edge. Since (A2) guarantees the existence of an additional 7-edge involving x_1 and $e_7^{xx}(P) = 5$, then x_2x_3 and x_2x_5 are 8-edges, as required.

Observation 4.23. Since z_4z_5 separates $\{x_4, x_5\}$ from the other x's (see Figure 15), then x_ix_4 leaves Z on its left for any $i \in \{0, 1, 2, 3\}$.

(E4): From (C3), we know that x_1x_4 and x_2x_4 are 7-edges. This and Observation 4.23 imply that x_2x_4 leaves exactly 1 or exactly 3 points of X on its left.

Suppose first that $x_1 \in x_2x_4^+$. Since $x_0 \in x_2x_4^-$, then $x_2x_4^+ = \{x_1, x_3, x_5\} \cup Y$. Again, Observation 4.23 implies that when we rotate x_2x_4 counterclockwise around x_4, the first two points that such line finds (with the tail) are x_1 and x_3. Since x_1x_4 is a 7-edge and x_3x_4 is an 8-edge, then the first point that such a rotation finds must be x_3, and hence $x_1 \in x_3x_4^+$. This and the way in which the x's were labelled imply that $x_0^4 = x_1$. But then x_0x_1 is a 7-edge, contradicting (E1). Then $x_0, x_1 \in x_2x_4^-$ and hence x_2x_4 leaves exactly three points of X on its left, namely x_0, x_1 and x_3. The desired equality follows from the last conclusion, Observation 4.23, and (C3).

(E5): From (A4), we know that x_3x_4 is an 8-edge, and from (C3) that x_1x_4 is a 7-edge. Since $x_1, x_3 \in x_2x_4^-$, by (E4), then when we rotate x_2x_4 clockwise around x_4, the first two points that such line finds (with the tail) are precisely x_1 and x_3. Since x_1x_4 is a 7-edge and x_3x_4 is an 8-edge, then such a rotation finds first x_3 and then x_1. The desired conclusions are immediate from this fact and (E4).

(E6): From (E4) and the way the x's were labelled, we know that when we rotate x_2x_4 clockwise around x_2, the first point that such line finds (with the tail) is one of x_0 or x_1. Since x_0x_2 is a 7-edge, by (E2), then such a point must be x_0 and the desired result follows from (E4).

(E7): From (E6), we know that the first two points that we find when we rotate x_0x_2 counterclockwise around x_0 are x_1 and x_3. Since x_0x_1 is an 8-edge, by (E1), then we have that such a rotation finds x_1 and then x_3. These together with (E6) imply the desired results.

(E8): (D4) implies that $x_3 \in x_1x_5^-$. If $x_2 \in x_1x_5^+$, then $\{x_1, x_3, x_4\} \cup Z \subset x_2x_5^-$. This would imply that x_2x_5 is not an 8-edge, contradicting (E3). Then we can assume that $x_2 \in x_1x_5^-$. Thus, x_0 is the only x on the right side of x_1x_5 and hence x_1x_5 is a (≤ 7)-edge. From Proposition 4.9 and Corollary 4.10, we have that x_1x_5 must be a 7-edge. This implies that $Y \subset x_1x_5^+$, and hence (E8) holds.

(E9): (E4), (E5), (E6), (E7), and (E8) imply, respectively, that $x_2x_4, x_1x_4, x_0x_2, x_0x_3$ and x_1x_5 are 7-edges. Then Proposition 4.9 implies that these five are all the monochromatic edges of type xx. This and Corollary 4.10 imply that x_1x_2 and x_1x_3 must be 8-edges.

(E10): The first assertion of (E7) implies that $x_3, x_4 \in x_0x_1^-$ and $x_2, x_5 \in x_0x_1^+$. From the first assertion of (E5) we know that $x_2, x_3 \in x_1x_5^+$. Then when we rotate x_0x_1 counterclockwise around x_1, the first point that such line finds must be x_3, and so the desired result follows immediately from this and the first assertion of (E5).

(E11): From the first assertion of (E7), we have that $x_3, x_4 \in x_0x_1^-$ and $x_2, x_5 \in x_0x_1^+$. From (E8), we know that $x_2, x_3 \in x_1x_4^-$. Then when we rotate x_0x_1 clockwise around x_1, the first point that we find is x_2, and so the desired result follows from the first assertion of (E7).

(E12): (E8) implies $\{x_2, x_3, x_4\} \cup Z = x_1x_5^-$. From (D4) and the second assertion of (E3), we know that when we rotate x_1x_5 clockwise around x_5, the first point that we find must be x_2, and so the desired result follows from (E8).

(E13): (E4) implies that $Z \cup \{x_0, x_1, x_3\} = x_2x_4^-$. From (E10), we know that $x_2 \in x_1x_3^+$. This and the first assertion of (E3) imply that when we rotate x_2x_4 counterclockwise around x_2, the first point that we find must be x_3, and so the desired result follows from (E4). □

4.5 Determination of pq^+ when pq is a bichromatic (>5)-edge.

We are finally ready to prove Lemma 3.1 for the remaining case, namely when pq is a bichromatic (> 5)-edge. This is achieved in the next statement. We recall from Remark 4.19 and Corollary 4.20 that x_4x_5 splits the ys, y_4y_5 splits the zs, and z_4z_5 splits the xs.

Proposition 4.24. *Let $u \in \{x, y, z\}$, and let v be the element in $\{x, y, z\} \setminus \{u\}$ such that u_4u_5 splits the vs. Then the following hold:*

(F1) $u_5v_4^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1, v_2, v_3\}$;

(F2) $u_5v_5^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2, v_3, v_4\}$;

(F3) $u_5v_3^+ = \{u_0, u_1, u_2, u_3, u_4\} \cup \{v_0, v_1, v_2\}$;

(F4) $u_5v_2^+ = \{u_0, u_1, u_2, u_3, u_4\} \cup \{v_0, v_1\}$;

(F5) $u_5v_1^+ = \{u_0, u_1, u_2, u_3, u_4\} \cup \{v_0\}$;

(F6) $u_4v_4^+ = \{u_0, u_1, u_2, u_3, u_5\} \cup \{v_0, v_1, v_2, v_3\}$;

(F7) $u_4v_5^+ = \{u_0, u_1, u_2, u_3, u_5\} \cup \{v_0, v_1, v_2, v_3, v_4\}$;

(F8) $u_4v_3^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1, v_2\}$;

(F9) $u_4v_2^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1\}$;

(F10) $u_3v_5^+ = \{u_0, u_1, u_2, u_5\} \cup \{v_0, v_1, v_2, v_3, v_4\}$;

(F11) $u_3v_4^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2, v_3\}$;

(F12) $u_3v_3^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2\}$;

(F13) $u_2v_5^+ = \{u_0, u_1\} \cup \{v_0, v_1, v_2, v_3, v_4\}$;

(F14) $u_2v_4^+ = \{u_0, u_1\} \cup \{v_0, v_1, v_2, v_3\}$;

(F15) $u_1v_5^+ = \{u_0\} \cup \{v_0, v_1, v_2, v_3, v_4\}$.

Proof. In view of Remark 4.19 and Corollary 4.20, we may assume that the points of P are placed as in Figure 15. Moreover, by symmetry, we only need to verify the case $u = x$ and $v = y$. For brevity, for $m \in \{0, \dots, 5\}$, we let $X_m := \{x_i | i \leq m\}$ and $Y_m := \{y_i | i \leq m\}$.

(F1): From (D3), we know that $x_4x_5^+ = X_3 \cup Y_3$. We claim that the first point $p \in P$ that x_4x_5 finds when it is rotated counterclockwise around x_5 is y_4. From (D4), we know that $x_3x_5^+ = X_2 \cup Y_4$. Then $p = y_4$, and so $x_5y_4^+ = X_3 \cup Y_3$, as required.

(F2): We know that $x_3x_5^+ = X_2 \cup Y_4$ by (D4). We claim that the first point $p \in P$ that x_3x_5 finds when it is rotated counterclockwise around x_5 is y_5. Indeed, from (D1), (E8), and (E12), we know that $y_5 \in x_jx_5^+$ for $j = 0, 1, 2$, respectively. These imply that $p \notin X_2$, and so $p = y_5$. Then $x_5y_5^+ = X_2 \cup Y_4$, as required.

(F3): We know that $x_4x_5^+ = X_3 \cup Y_3$ by (D3). We claim that the first point $p \in P$ that x_4x_5 finds when it is rotated clockwise around x_5 is y_3. Indeed, by applying (E7), (E10), and (E13) to $j = 0, 1$ and $j = 2$ (with $u = y$ and $v = z$), respectively, we have that $X \subset y_jy_3^-$, and so $x_5 \in y_jy_3^-$. These imply that $p \notin Y_2$. This and the fact that $x_5y_0^+ = X_4$ imply that $p = y_3$. Then $x_5y_3^+ = X_4 \cup Y_2$, as required.

(F4): We know that $x_5y_3^+ = X_4 \cup Y_2$ by (F3). We claim that the first point $p \in P$ that x_5y_3 finds when it is rotated clockwise around x_5 is y_2. Indeed, by applying (E6) and (E11) to $j = 0$ and $j = 1$ (with $u = y$ and $v = z$), respectively, we have that $X \subset y_jy_2^-$, and so $x_5 \in y_jy_2^-$. These imply that $p \notin Y_1$. This and the fact that $x_5y_0^+ = X_4$ imply that $p = y_2$. Then $x_5y_2^+ = X_4 \cup Y_1$, as required.

(F5): We know that $x_5y_2^+ = X_4 \cup Y_1$ by (F4). We claim that the first point $p \in P$ that x_5y_2 finds when it is rotated clockwise around x_5 is y_1. Indeed, by taking $u = y$ in (E7), we have that $x_5 \in y_0y_1^-$, and so $p \neq y_0$. This and the fact that $x_5y_0^+ = X_4$ imply that $p = y_1$. Then $x_5y_1^+ = X_4 \cup Y_0$, as required.

(F6): We know that $x_4x_5^+ = X_3 \cup Y_3$ by (D3). We claim that the first point $p \in P$ that x_4x_5 finds when it is rotated counterclockwise around x_4 is y_4. Indeed, by taking $u = y$ and $v = z$ in D3) we get $x_4 \in y_4y_5^-$, and so $p \neq y_5$. This and the fact that $x_0x_4^+ = \{x_1, x_2, x_3, x_5\} \cup Y$ imply that $p = y_4$. Then $x_4y_4^+ = X_3 \cup Y_3 \cup \{x_5\}$, as required.

(F7): We know that $x_4y_4^+ = X_3 \cup Y_3 \cup \{x_5\}$ by (F6). We claim that the first point $p \in P$ that x_4y_4 finds when it is rotated counterclockwise around x_4 is y_5. Indeed, from (D2), we know that $x_0x_4^+ = \{x_1, x_2, x_3, x_5\} \cup Y$. These imply that $p \notin X$, and so $p = y_5$. Then $x_4y_5^+ = X_3 \cup Y_4 \cup \{x_5\}$, as required.

(F8): We know that $x_4x_5^+ = X_3 \cup Y_3$ by (D3). We claim that the first point $p \in P$ that x_4x_5 finds when it is rotated clockwise around x_4 is y_3. Indeed, by applying E7), E10), and E13) to $j = 0, 1$ and $j = 2$ (with $u = y$ and $v = z$), respectively, we have that $X \subset y_jy_3^-$, and so $x_4 \in y_jy_3^-$. These imply that $p \notin Y_2$. From Proposition 4.3(2), we know that $X_3 \subset x_4y_1^+$, and so $p \notin X_3$. All these facts imply that $p = y_3$, and so $x_4y_3^+ = X_3 \cup Y_2$, as required.

(F9): We know that $x_4y_3^+ = X_3 \cup Y_2$ by (F8). We claim that the first point $p \in P$ that x_4y_3 finds when it is rotated clockwise around x_4 is y_2. Indeed, by applying (E6) and (E11) to $j = 0$ and $j = 1$ (with $u = y$ and $v = z$), respectively, we have that $X \subset y_jy_2^-$, and so $x_4 \in y_jy_2^-$. These imply that $p \notin Y_1$. From this and (D3) it follows that $p = y_2$, and so $x_4y_2^+ = X_3 \cup Y_1$, as required.

(F10): We know that $x_3x_5^+ = X_2 \cup Y_4$ by (D4). We claim that the first point $p \in P$ that x_3x_5 finds when it is rotated counterclockwise around x_3 is y_5. Indeed, by applying (E7), (E10), and (E13) to $j = 0, 1$ and $j = 2$ (with $u = x$ and $v = y$), respectively, we have that $Y \subset x_jx_3^+$, and so $y_5 \in x_jx_3^+$. These imply that $p \notin X_2$. From this and E5) it follows that $p = y_5$, and so $x_3y_5^+ = X_2 \cup Y_4 \cup \{x_5\}$, as required.

(F11): We know that $x_3x_5^+ = X_2 \cup Y_4$ by D4). We claim that the first point $p \in P$ that x_3x_5 finds when it is rotated clockwise around x_3 is y_4. Indeed, by applying (D2), (E5), (E4), and (E5) to $j = 0, 1, 2$ and $j = 3$ (with $u = y$ and $v = z$), respectively, we have that $X \subset y_jy_4^-$, and so $x_3 \in y_jy_4^-$. These imply that $p \notin Y_3$. From Proposition 4.3(2), we know that $x_4 \in x_3y_2^-$, and so $p \neq x_4$. All these facts imply that $p = y_4$, and so $x_3y_4^+ = X_2 \cup Y_3$, as required.

(F12): We know that $x_3y_4^+ = X_2 \cup Y_3$ by (F11). We claim that the first point $p \in P$ that x_3y_4 finds when it is rotated clockwise around x_3 is y_3. Indeed, by applying (E7), (E10), and (E13) to $j = 0, 1$ and $j = 2$ (with $u = y$ and $v = z$), respectively, we have that $X \subset y_jy_3^-$, and so $x_3 \in y_jy_3^-$. These imply that $p \notin Y_2$. By taking $u = x$ in (E5) and (D4), we have that $y_4 \in x_3x_4^+$ and $y_4 \in x_3x_5^+$, respectively. These imply that $p \notin \{x_4, x_5\}$. All these facts imply that $p = y_3$, and so $x_3y_3^+ = X_2 \cup Y_2$, as required.

(F13): From Proposition 4.3(2), we know that $x_2y_3^+ = X_1 \cup Y_2$. We claim that the first point $p \in P$ that x_2y_3 finds when it is rotated counterclockwise around x_2 is y_4. Indeed, by applying (E6) and (E11) to $j = 0$ and $j = 1$ (with $u = x$ and $v = y$), respectively, we have that $Y \subset x_jx_2^+$, and so $p \notin \{x_0, x_1\}$. Finally, by applying (E4), (E12), and (E13) to $j = 4, 5$ and $j = 3$ (with $u = x$ and $v = y$), we have that $p \neq x_4, p \neq x_5$ and $p \neq x_3$, respectively. All these facts imply that $p = y_4$, and so $x_2y_4^+ = X_1 \cup Y_3$, as required.

(F14): We know that $x_2y_4^+ = X_1 \cup Y_3$ by (F13). We claim that the first point $p \in P$ that x_2y_4 finds when it is rotated counterclockwise around x_2 is y_5. As in (F13) we can deduce from (E6) and (E11) that $p \notin \{x_0, x_1\}$. Again, as in (F13) we can deduce from (E4), (E12), and (E13) that $p \neq x_4, p \neq x_5$ and $p \neq x_3$, respectively. All these facts imply that $p = y_5$, and so $x_2y_5^+ = X_1 \cup Y_4$, as required.

(F15): We know that $x_2y_5^+ = X_1 \cup Y_4$ by F14). We claim that the first point $p \in P$ that x_2y_5 finds when it is rotated counterclockwise around y_5 is x_1. Indeed, from Proposition 4.3(2), we know that $y_5z_0^+ = Y_4$. From the last two equations we have that $p \in X_1 = \{x_0, x_1\}$. Again, from Proposition 4.3(2), we know that $x_0y_5^+ = Y_4$, and so $p = x_1$. Then $x_1y_5^+ = X_0 \cup Y_4$, as required. □

4.6 Conclusion of the proof of Lemma 3.1

In Tables 1 and 2 we give a summary of the results in Propositions 4.3(2)(b), 4.21, 4.22, and 4.24. These tables assume that $u \in \{x, y, z\}$ and $v = \sigma(u)$, where σ is the automorphism of $\{x, y, z\}$ defined in Corollary 4.20, namely $x \overset{\sigma}{\mapsto} y, y \overset{\sigma}{\mapsto} z$, and $z \overset{\sigma}{\mapsto} x$.

In particular, for each $u \in \{x, y, z\}$ and $m, n \in \{0, \dots, 5\}$ with $m < n$, the set $u_m u_n^+$ is given in Table 1. This also determines the set $u_n u_m^+$, since $u_n u_m^+ = u_m u_n^-$, and $u_m u_n^-$ is evidently determined from $u_m u_n^+$. Thus the information in Table 1 suffices to determine pq^+ whenever pq is a monochromatic edge of P.

Now for each $u \in \{x, y, z\}$ and each $m, n \in \{0, \dots, 5\}$, the set $u_m v_n^+$ is given in Table 2. This also determines the set $v_n u_m^+$, since $v_n u_m^+ = u_m v_n^-$, and $u_m v_n^-$ is evidently determined from $u_m v_n^+$. Thus the information in Table 2 suffices to determine pq^+ whenever pq is a bichromatic edge of P. □

$u_i u_j^+$ for each $u_i u_j \in E_k^{\text{mon}}(P)$	Classification of $u_i u_j$	Equality stated in
$u_0 u_5^+ = \{v_0, \dots, v_5\}$	6-edge	(D1)
$u_0 u_4^+ = \{u_1, u_2, u_3, u_5\} \cup \{v_0, \dots, v_5\}$	6-edge	(D2)
$u_0 u_3^+ = \{u_1, u_2, u_5\} \cup \{v_0, \dots, v_5\}$	7-edge	(E7)
$u_0 u_2^+ = \{u_5\} \cup \{v_0, \dots, v_5\}$	7-edge	(E6)
$u_0 u_1^+ = \{u_2, u_5\} \cup \{v_0, \dots, v_5\}$	8-edge	(E7)
$u_1 u_5^+ = \{u_0\} \cup \{v_0, \dots, v_5\}$	7-edge	(E8)
$u_1 u_4^+ = \{u_2, u_3, u_5\} \cup \{v_0, \dots, v_5\}$	7-edge	(E5)
$u_1 u_3^+ = \{u_2, u_5\} \cup \{v_0, \dots, v_5\}$	8-edge	(E10)
$u_1 u_2^+ = \{u_0, u_5\} \cup \{v_0, \dots, v_5\}$	8-edge	(E11)
$u_2 u_5^+ = \{u_0, u_1\} \cup \{v_0, \dots, v_5\}$	8-edge	(E12)
$u_2 u_4^+ = \{u_5\} \cup \{v_0, \dots, v_5\}$	7-edge	(E4)
$u_2 u_3^+ = \{u_4, u_5\} \cup \{v_0, \dots, v_5\}$	8-edge	(E13)
$u_3 u_5^+ = \{u_0, u_1, u_2\} \cup \{v_0, \dots, v_4\}$	8-edge	(D4)
$u_3 u_4^+ = \{u_2, u_5\} \cup \{v_0, \dots, v_5\}$	8-edge	(E5)
$u_4 u_5^+ = \{u_0, \dots, u_3\} \cup \{v_0, \dots, v_3\}$	8-edge	(D3)

Table 1: All the monochromatic edges of P.

5 Concluding remarks

In this work we finally have given the full proof of Theorem 1.2, which was announced at the EuroComb'11 conference [1].

As we mentioned in the Introduction, the exact rectilinear crossing number of K_n is known only for $n \leq 27$ and $n = 30$ [3, 7, 8, 9, 10]. In [2] and [6] we can find non-isomorphic crossing-minimal rectilinear drawings of K_n for both $n = 24$ and $n = 30$. On the other hand, from [6] and the main result of this work, now we know that there is a unique (up to order type isomorphism) crossing-minimal rectilinear drawing of K_n, for $n = 6, 12, 18$. Thus, a plausible conjecture is that K_{6m} has several crossing-minimal rectilinear drawings for each integer $m \geq 4$.

We close this paper with a discussion on a question raised by an anonymous reviewer of an earlier version of this paper: to what degree would it be possible to get a computer-assisted proof of Theorem 1.2?

uv^+ for each $uv \in E_k^{\text{bi}}(P)$	Classification of uv	Equality stated in
$u_0v_5^+ = \{v_0, \dots, v_4\}$	5-edge	Proposition 4.3(2)
$u_0v_4^+ = \{v_0, v_1, v_2, v_3\}$	4-edge	Proposition 4.3(2)
$u_0v_3^+ = \{v_0, v_1, v_2\}$	3-edge	Proposition 4.3(2)
$u_0v_2^+ = \{v_0, v_1\}$	2-edge	Proposition 4.3(2)
$u_0v_1^+ = \{v_0\}$	1-edge	Proposition 4.3(2)
$u_0v_0^+ = \emptyset$	0-edge	Proposition 4.3(2)
$u_1v_5^+ = \{u_0\} \cup \{v_0, \dots, v_4\}$	6-edge	(F15)
$u_1v_4^+ = \{u_0\} \cup \{v_0, v_1, v_2, v_3\}$	5-edge	Proposition 4.3(2)
$u_1v_3^+ = \{u_0\} \cup \{v_0, v_1, v_2\}$	4-edge	Proposition 4.3(2)
$u_1v_2^+ = \{u_0\} \cup \{v_0, v_1\}$	3-edge	Proposition 4.3(2)
$u_1v_1^+ = \{u_0\} \cup \{v_0\}$	2-edge	Proposition 4.3(2)
$u_1v_0^+ = \{u_0\}$	1-edge	Proposition 4.3(2)
$u_2v_5^+ = \{u_0, u_1\} \cup \{v_0, \dots, v_4\}$	7-edge	(F14)
$u_2v_4^+ = \{u_0, u_1\} \cup \{v_0, v_1, v_2, v_3\}$	6-edge	(F13)
$u_2v_3^+ = \{u_0, u_1\} \cup \{v_0, v_1, v_2\}$	5-edge	Proposition 4.3(2)
$u_2v_2^+ = \{u_0, u_1\} \cup \{v_0, v_1\}$	4-edge	Proposition 4.3(2)
$u_2v_1^+ = \{u_0, u_1\} \cup \{v_0\}$	3-edge	Proposition 4.3(2)
$u_2v_0^+ = \{u_0, u_1\}$	2-edge	Proposition 4.3(2)
$u_3v_5^+ = \{u_0, u_1, u_2, u_5\} \cup \{v_0, \dots, v_4\}$	7-edge	(F10)
$u_3v_4^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2, v_3\}$	7-edge	(F11)
$u_3v_3^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1, v_2\}$	6-edge	(F12)
$u_3v_2^+ = \{u_0, u_1, u_2\} \cup \{v_0, v_1\}$	5-edge	Proposition 4.3(2)
$u_3v_1^+ = \{u_0, u_1, u_2\} \cup \{v_0\}$	4-edge	Proposition 4.3(2)
$u_3v_0^+ = \{u_0, u_1, u_2\}$	3-edge	Proposition 4.3(2)
$u_4v_5^+ = \{u_0, u_1, u_2, u_3, u_5\} \cup \{v_0, \dots, v_4\}$	6-edge	(F7)
$u_4v_4^+ = \{u_0, u_1, u_2, u_3, u_5\} \cup \{v_0, \dots, v_3\}$	7-edge	(F6)
$u_4v_3^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1, v_2\}$	7-edge	(F8)
$u_4v_2^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1\}$	6-edge	(F9)
$u_4v_1^+ = \{u_0, u_1, u_2, u_3\} \cup \{v_0\}$	5-edge	Proposition 4.3(2)
$u_4v_0^+ = \{u_0, u_1, u_2, u_3\}$	4-edge	Proposition 4.3(2)
$u_5v_5^+ = \{u_0, u_1, u_2\} \cup \{v_0, \dots, v_4\}$	8-edge	(F2)
$u_5v_4^+ = \{u_0, \dots, u_3\} \cup \{v_0, \dots, v_3\}$	8-edge	(F1)
$u_5v_3^+ = \{u_0, \dots, u_4\} \cup \{v_0, v_1, v_2\}$	8-edge	(F3)
$u_5v_2^+ = \{u_0, \dots, u_4\} \cup \{v_0, v_1\}$	7-edge	(F4)
$u_5v_1^+ = \{u_0, \dots, u_4\} \cup \{v_0\}$	6-edge	(F5)
$u_5v_0^+ = \{u_0, u_1, u_2, u_3, u_4\}$	5-edge	Proposition 4.3(2)

Table 2: All the bichromatic edges of P.

At the beginning of this project we asked ourselves the same question, but we are convinced that a traditional proof might be easier to verify. It is worth mentioning that a heavily computer-assisted proof seems to be out of reach, most likely involving several hundred million CPU hours. On the other hand, we believe that a partially computer-assisted proof would be more difficult to follow and perhaps also less reliable. Using computer-assisted proofs needs a very careful preparation and description of what is done, and proofs of the correctness of the results. The code must be explained in full detail, as well as how the program can be executed (including the operating system, compiler versions, etc.). We believe that in this particular case the task of verifying all this information would end up being more taxing on the reader than the current purely theoretical proof.

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