On the Terwilliger algebra of a certain family of bipartite distance-regular graphs with $\Delta_2 = 0$

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Abstract

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let X denote the vertex set of Γ, and let A_i $(0 \leq i \leq D)$ denote the distance matrices of Γ. We abbreviate $A := A_1$. For $x \in X$ and for $0 \leq i \leq D$, let $\Gamma_i(x)$ denote the set of vertices in X that are distance i from vertex x.

Fix $x \in X$ and let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where for $0 \leq i \leq D$, E_i^* represents the projection onto the ith subconstituent of Γ with respect to x. We refer to T as the *Terwilliger algebra* of Γ with respect to x. By the *endpoint* of an irreducible T-module W we mean $\min\{i \mid E_i^*W \neq 0\}$.

In this paper we assume Γ has the property that for $2 \leq i \leq D - 1$, there exist complex scalars α_i, β_i such that for all $y, z \in X$ with $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$, we have $\alpha_i + \beta_i|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|$.

We study the structure of irreducible T-modules of endpoint 2. Let W denote an irreducible T-module with endpoint 2, and let v denote a nonzero vector in E_2^*W. We show that $W = \text{span}\big(\{E_i^* A_{i-2} E_2^* v \mid 2 \leq i \leq D\} \cup \{E_i^* A_{i+2} E_2^* v \mid 2 \leq i \leq D - 2\}\big)$.

It turns out that, except for a particular family of bipartite distance-regular graphs with $D = 5$, this result is already known in the literature. Assume now that Γ is a member of this particular family of graphs. We show that if Γ is not almost 2-homogeneous, then up to isomorphism there exists exactly one irreducible T-module with endpoint 2 and it is not thin. We give a basis for this T-module.

Keywords: Distance-regular graphs, Terwilliger algebra, irreducible modules.

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O Terwilligerjevi algebri določene družine dvodelnih razdaljno-regularnih grafov z $\Delta_2 = 0$

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Povzetek

Naj bo Γ dvodelni razdaljno-regularni graf s premerom $D \geq 4$ in valenco $k \geq 3$. Naj bo X množica točk grafa Γ, in naj bodo A_i $(0 \leq i \leq D)$ razdaljne matrike grafa Γ. Uporabljamo okrajšavo $A := A_1$. Za $x \in X$ in za $0 \leq i \leq D$, naj bo $\Gamma_i(x)$ množica tistih točk iz X, katerih razdalja od točke x je i.

Za dani $x \in X$ naj bo $T = T(x)$ podalgebra algebre $\text{Mat}_X(\mathbb{C})$, generirana z A, E_0^*, E_1^*, ..., E_D^*, kjer je E_i^* projekcija na i-to podkonstituento grafa Γ glede na x, in to za vsak i, ki ustreza pogoju $0 \leq i \leq D$. Tedaj je T *Terwilligerjeva algebra* grafa Γ glede na točko x. *Krajišče* ireducibilnega T-modula W definiramo kot $\min\{i \mid E_i^* W \neq 0\}$.

V tem članku privzamemo, da ima Γ lastnost, da za vsak i, ki ustreza pogoju $2 \leq i \leq D - 1$, obstajata kompleksna skalarja α_i, β_i, tako da za vse $y, z \in X$, ki ustrezajo pogoju $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$, velja $\alpha_i + \beta_i |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|$.

Raziskujemo strukturo ireducibilnih T-modulov s krajiščem 2. Naj bo W ireducibilni T-modul s krajiščem 2, in naj bo v neničeln vektor v $E_2^* W$. Dokažemo, da je tedaj $W = \text{span}\left(\{E_i^* A_{i-2} E_2^* v \mid 2 \leq i \leq D\} \cup \{E_i^* A_{i+2} E_2^* v \mid 2 \leq i \leq D - 2\}\right)$.

Izkaže se, da je, razen za določeno družino dvodelnih razdaljno-regularnih grafov s premerom $D = 5$, ta rezultat že znan v literaturi. Privzemimo zdaj, da je Γ član te družine grafov. Dokažemo, da če Γ ni skoraj 2-homogen, potem obstaja, do izomorfizma natančno, en sam ireducibilen T-modul s krajiščem 2, in ta modul ni tanek. Predstavimo bazo tega T-modula.

Ključne besede: Razdaljno-regularni grafi, Terwilligerjeva algebra, ireducibilni moduli.

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