10 Fermionization in an Arbitrary Number of Dimensions

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Abstract. One purpose of this proceedings-contribution is to show that at least for free massless particles it is possible to construct an explicit boson theory which is exactly equivalent in terms of momenta and energy to a fermion theory. The fermions come as $2^{d/2-1}$ families and the to this whole system of fermions corresponding bosons come as a whole series of the Kalb-Ramond fields, one set of components for each number of indexes on the tensor fields.

Since Kalb-Ramond fields naturally (only) couple to the extended objects or branes, we suspect that inclusion of interaction into such for a bosonization prepared system - except for the lowest dimensions - without including branes or something like that is not likely to be possible.

The need for the families is easily seen just by using the theorem long ago put forward by Aratyn and one of us (H.B.F.N.), which says that to have the statistical mechanics of the fermion system and the boson system to match one needs to have the number of the field components in the ratio $\frac{2^{d-1}-1}{2^{d-1}} = \frac{\#bosons}{\#fermions}$, enforcing that the number of fermion components must be a multiple of 2^{d-1} , where d is the space-time dimension. This "explanation" of the number of dimension is potentially useful for the explanation for the number of dimension put forward by one of us (S.N.M.B.) since long in the Spin-Charge-Family theory, and leads like the latter to typically (a multiple of) 4 families.

And this is the second purpose for our work on the fermionization in an arbitrary number of dimensions - namely to learn how "natural" is the inclusion of the families in the way the Spin-Charge-Family theory does.

Povzetek. Eden od namenov tega prispevka je pokazati, da je za brezmasne bozone mogoče postaviti teorijo, ki je glede na energijo in gibalno količino lahko tudi teorija brezmasnih fermionov v poljubno razsežnih prostorih. Bozoni so v tej teoriji opisani z $2^{d-1} - 1$ realnimi polji Kalb-Ramond-ove vrste, za ekvivalentna fermionska polja, ki so kompleksi Weylovi spinorji, pa Aratyn-Nielsen-ov teorem zahteva, da se pojavijo v sodo razsežnih prostor-čas-ih d v $2^{d/2-1}$ družinah, ker mora biti po tem teoremu razmerje bozonskih in fermionskih polj enako $\frac{2^{d-1}-1}{2^{d-1}} = \frac{\#bozonov}{\#fermionov}$.

Pojav sodega števila družin, ki ga zahteva ta teorija fermionizacije bozonov (ali ekvivalentno bozonizacije fermionov), pritrjuje teoriji spinov-nabojev-družin, ki jo je postavila soavtorica tega prispevka, in ki napoveduje, da je število družin cel mnogokratnik števila 4. Ta prispevek pritrdi teoriji spinov-nabojev-družin, da je pojav družin fermionov v naravi osnovnega pomena.

10.1 Introduction

This is the first draft to the paper, prepared so far only to appear in the Proceedings as the talk of one of the authors (H.B.F.N.). Although many things are not yet strictly proven, the fermionization/bosonization seems, hopefully, to work in any dimensional space-time and also, hopefully, in the presence of a weak background field. We hope, that the fermionization/bosonization procedure might help to better understand why nature has made of choice of spins, charges and families of fermions and of the corresponding gauge and scalar fields, observed in the low energy regime and why the Spin-Charge-Family theory [7,6] might be the right explanations for all the assumptions of the Standard Model.

This talk demonstrates that:

- Bosonization/fermionization is possible in an arbitrary number of dimensions (although the fermions theories are non-local due to the anticommuting nature of fermions, while bosons commute).
- The number of degrees of freedom for fermions versus bosons obeys in our procedure in any d the Aratyn-Nielsen theorem [1].
- The number of families in four dimensional space-time is (a multiple of) four families.

To prove for massless fermions and bosons that the bosonization/fermionization is possible in an arbitrary number of dimensions we use the Jacoby's triple product formula, presented by Leonhard Euler in 1748 [3] and is a special case of Glaisher's theorem [5]

$$\frac{1}{2} \prod_{n=0,1,2,\dots} (1+x^n) = \prod_{m=1,3,5,\dots} \frac{1}{1-x^m}.$$
 (10.1)

Let the reader notices that the product on the left hand side runs over 0 and all positive integers, while on the right hand side it runs only over odd positive integers. One can recognize also that for all positive numbers the number of partitions with odd parts equals the number of partitions with distinct parts. Let us demonstrate this in a special case:

Among the 22 partitions of the number 8 there are 6 that contain only odd parts, namely

(7+1, 5+3, 5+1+1+1, 3+3+1+1, 3+1+1+1+1+1, 1+1+1+1+1).

If we count partitions of 8, in which no number occurs more than once, that is with distinct parts, we obtain again 6 such partitions, namely

(8, 7 + 1, 6 + 2, 5 + 3, 5 + 2 + 1, 4 + 3 + 1).

For every type of restricted partition there is a corresponding function for the number of partitions satisfying the given restriction. An important example is q(n), the number of partitions of n into distinct parts [4]. The generating function for q(n), partitions into distinct parts, is given by

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{k=1}^{\infty} (1+x^k) = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}.$$
 (10.2)

The first few values of q(n) are (starting with q(0)=1):

(1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10,

The pentagonal number theorem can be applied giving a recurrence for q [4]:

$$q(k) = ak + q(k-1) + q(k-2) - q(k-5) - q(k-7) + q(k-12) + q(k-15) - q(k-22) - \dots$$
(10.3)

where ak is $(1)^m$, if $k = (3m^2 - m)$ for some integer m, and is 0 otherwise.



Fig. 10.1. Bosonization Illustrating Formula: the d_{space} space dimensional version (for only a "quadrant") is presented.



The Idea for the Procedure for a Proof of the Multidimensional Bosonization Formula

- 1. Divide the whole system of all the discretized momentum vectors into "classes" of proportional vector (meaning in practice vectors deviating by a rational factor only), or rays (we might call them the rays of the module).
- 2. For each "class" the proof is given by the 1+1 dimensional case which means by just using the formula by Euler and extending it to both positive and negative integers.

Thinking of the Formulas of Bosonization as Products over Rays/Classes



where c runs over the set rays of the d_{space} -tuples of non-negative integers, that cannot be written as such a tuple multiplied by an over all integer factor.



Fig. 10.2. Splitting the Fock space into Cartesian Product Factors from Each Ray c.

Denoting the Fock space for the theory - it be a boson or a fermion one - as \mathcal{H} for the d_{space}-dimensional theory, and by \mathcal{H}_c the Fock space for the - essentially 1 + 1 dimensional theory associated with the ray/or class c describing the particles with momenta being an integer (though not 0) times the representative for *c*,

namely $(\mathfrak{m}_1(c), \mathfrak{m}_2(c), ..., \mathfrak{m}_{d_{space}}(c))$ - it is suggested that we write the full Fock space as the product

$$\mathcal{H} = \otimes_{c \in RAYS} \mathcal{H}_c. \tag{10.6}$$

Introduction of Creation and Annihilation Operators

We shall introduce for a boson interpretation of the Hilbert space the Fock space \mathcal{H} :

 $a(n_1, n_2, ..., n_{d_{space}})$ annihilates a boson with momentum $(n_1, n_2, ..., n_{d_{space}})$, $a^{\dagger}(n_1, n_2, ..., n_{d_{space}})$ creates a boson with momentum $(n_1, n_2, ..., n_{d_{space}})$,

where the integers can be any, except that they must *not* all d_{space} ones be even. Similarly for fermions:

$$\begin{split} f(n_1,n_2,...,n_{d_{space}}) & \text{annihilates a fermion with momentum } (n_1,n_2,...,n_{d_{space}}) \,, \\ f^{\dagger}(n_1,n_2,...,n_{d_{space}}) & \text{creates} & \text{a fermion with momentum } (n_1,n_2,...,n_{d_{space}}) \,, \end{split}$$

where now the n_i numbers can be any integers.

Boson Operators Dividable into rays or classes c, also Fermions Except for one Type

We can write any "not all even" (discretized) momentum $(n_1, n_2, ..., n_{d_{space}})$ as an *odd* integer n times a representative for a class/ray c

$$\begin{aligned} a(n_1, n_2, ..., n_{d_{space}}) &= a(n_1(c) * n, n_2(c) * n, ..., n_{d_{space}}(c) * n), \\ a^{\dagger}(n_1, n_2, ..., n_{d_{space}}) &= a^{\dagger}(n_1(c) * n, n_2(c) * n, ..., n_{d_{space}}(c) * n) \end{aligned}$$

The boson momentum with a given even/odd combination for its momentum components (say oe...o) goes to a ray/class c with the *same combination of even/odd-ness*.

Similarly one can proceed also for fermions with not all momentum components even; but the fermion momenta that have all components even *get divided into rays/classes with different even/odd combinations*. There are no rays with the even combination *ee...e*, of course, because a tuple of only even numbers could be divided by 2.

10.2 Thoughts on Construction of Fermion Operators

We have made an important step arriving at a model suggesting how it could be possible to match momenta and energies for a system with either fermions or bosons. To completely show the existence of fermionization (or looking the opposite way, bosonization) we should, however, write down the formula for how the fermion (boson) creation and annihilation operators are constructed in terms of the boson (fermion) operators, so that it can become clear (be proven) that the phase conventions and identification of the specific states with a given total momentum and energy for fermions can be identified with specific states for the boson system.

Such a construction is well known for 1+1 dimensions, where it looks like

$$\psi_e(\mathbf{x}) + i\psi_o(\mathbf{x}) = \exp(i\phi_R(\mathbf{x})) \tag{10.7}$$

in the "position" representation, meaning that

$$\psi_e(\mathbf{x}) = \sum_{\mathbf{m} \text{ even}} \exp(i\mathbf{m}\mathbf{x}) \ \mathbf{b}_e(\mathbf{m}) \tag{10.8}$$

$$\psi_{o}(x) = \sum_{m \text{ odd}} \exp(imx) b_{e}(m)$$
(10.9)

$$\phi_{o}(x) = \sum_{m \text{ odd}} \exp(imx) a_{e}(m). \qquad (10.10)$$

Let us think of the case of making the field operators in position space

$$\phi_e(\mathbf{x}), \ \phi_o(\mathbf{x}), \ \psi_o(\mathbf{x})$$

Hermitean by assumming

$$a_o(\mathfrak{m}) = a_o^{\dagger}(-\mathfrak{m}); \text{ for all } \mathfrak{m} \text{ odd},$$
 (10.11)

$$b_o(m) = b_o^{\dagger}(-m); \text{ for all } m \text{ odd},$$
 (10.12)

$$b_e(\mathbf{m}) = b_e^{\dagger}(-\mathbf{m}); \text{ for all } \mathbf{m} \text{ even}.$$
(10.13)

10.2.1 Problem of Extending to Higher Dimensions Even if we Have Bosonization Ray for Ray

At first one might naively think that - since each of our rays (or classes) c functions as the 1+1 dimensional system and we can write the whole fermion, as well as the whole boson, space according to (10.6) - it would be trivial to obtain the bosonization for the whole system and thereby have achieved the bosonization in the arbitrary dimension, which is the major goal of this article.

However, one should notice that constructing in a simple way a system composed from several independent subsystems such it is the whole system \mathcal{H} , composed from the subsystems \mathcal{H}_c (for $c \in rays$), one obtains *commutation* between operators acting solely inside one subsystem *c*, say, and operators acting solely inside another subsystem *c'*, say. But we want for the fermions the *anticommutation* relations rather than the commutation ones, and thus some (little ?) trick is needed to achieve this anticommutation.

First we shall show how this anticommutation can be achieved by means of an ordering of all the rays $c \in$ rays by some ordering inequality being chosen between these rays: >. But this is a very ugly procedure and we shall develop

a slightly more general attempt in which we construct a phase $\delta(c, c')$ for each pair of rays c and c'. Then we shall go on seeking to make the choice of this phase $\delta(c, c')$ in a continuous and more elegant way. Since that shall turn out to be non-trivial, we shall develop the ideas by first seeking for such a construction of the phase for the odd dimensional space of d = 3, meaning d_{space} = 2, to learn the idea, although we are most keen on even space-time dimensions, such it is the experimentally observed number of space-time dimensions, d = 4.

10.2.2 The > Ordered Rays Construction

Let us suppose that we have a formal way of constructing the fermion creation and annihilation operators in terms of the boson operators. We do indeed have such a construction, since we can Fourier transform back and forth the construction in the position representation (10.7) and the 1+1 dimensional bosonization is so well understood. Since for the present problem the details of this 1+1 dimensional bosonization relations are not so important, we shall just assume that we are able to deduce for each ray or class c a series of fermion creation - $b^{\dagger}_{naive o}(m, c)$ and $b^{\dagger}_{naive e}(m, c)$ - and annihilation - $b_{naive o}(m, c)$ and $b_{naive e}(m, c)$ - operators, that function well as fermion operators *inside the ray* c, so to speak. o and e denotes odd and even respectively. The only important thing is that these operators *can be expressed in terms of the bosons annihilation and creation operators belonging to the same ray* c:

$$b^{\dagger}_{naive o}(\mathbf{m}, \mathbf{c}) = b^{\dagger}_{naive o}(\mathbf{m}, \mathbf{c}; \mathbf{a}_{o}(\mathbf{n}, \mathbf{c}), \text{ for } \mathbf{n} \text{ odd}), \quad (10.14)$$

$$b_{naive o}(\mathbf{m}, \mathbf{c}) = b_{naive o}(\mathbf{m}, \mathbf{c}; \mathbf{a}_o(\mathbf{n}, \mathbf{c}), \text{ for } \mathbf{n} \text{ odd}), \quad (10.15)$$

$$b^{\dagger}_{naive e}(\mathbf{m}, \mathbf{c}) = b^{\dagger}_{naive e}(\mathbf{m}, \mathbf{c}; \mathbf{a}_{o}(\mathbf{n}, \mathbf{c}), \text{ for } \mathbf{n} \text{ odd}), \quad (10.16)$$

$$b_{\text{naive }e}(\mathfrak{m}, \mathfrak{c}) = b_{\text{naive }e}(\mathfrak{m}, \mathfrak{c}; \mathfrak{a}_{\mathfrak{o}}(\mathfrak{n}, \mathfrak{c}), \text{ for } \mathfrak{n} \text{ odd}).$$
(10.17)

For these operators we know form the 1+1 dimensional bosonization that we can take them to obey the usual anticommutation rules *provided we keep to only one ray* c:

$$\begin{split} \{b^{\dagger}_{naive o}(m,c;a_{o}(n,c), \text{ for n odd}), b^{\dagger}_{naive o}(p,c;a_{o}(n,c), \text{ for n odd})\}_{-} \\ &= \delta_{n,-p}, \text{ for } m, p \text{ both odd}, \\ \{b_{naive o}(m,c;a_{o}(n,c), \text{ for n odd}), b^{\dagger}_{naive o}(p,c;a_{o}(n,c), \text{ for n odd})\}_{-} \\ &= \delta_{n,p}, \text{ for } m, p \text{ both odd}, \\ \{b_{naive o}(m,c;a_{o}(n,c), \text{ for n odd}), b_{naive o}(p,c;a_{o}(n,c), \text{ for n odd})\}_{-} \\ &= \delta_{n,-p}, \text{ for } m, p \text{ both odd}. \end{split}$$

We have similar anticommutation rules for annihilation and creation operators if exchanging the index o (meaning odd) by the index *e* (meaning even), but now we should take into account that the fermion operators with zero momentum, i.e. m, p = 0, are not constructed from a single ray c. Rather there are - referring to our little problem with the explicit factor $\frac{1}{2}$ in the state counting formulae - not enough degrees of freedom in the 1+1 dimensional boson system to deliver a fermion operator with a zero momentum.

We should therefore imagine that we do not have these zero momentum fermion operators attached to our rays either. This is actually good for our hopes of bosonizing in higher dimensions because the zero momentum fermion operators would have had to be common for the infinitely many rays and we would have had too many candidates for the zero momentum fermion mode. Now instead we totally miss the zero momentum creation and annihilation fermion operators for the many dimensional system. That is, however, not at all so bad as it would have been to get an infinity of them, because we fundamentally can not expect to produce all fermion operators from boson ones because we cannot possibly build up a sector with an odd number of fermions from boson operators acting on say some vacuum with an even number. Therefore one fermion operator must be missing. This becomes the zero momentum one and that is o.k..

Our real problem remains that these naive fermion operators taken for two different rays c and c' will commute

$$\{b_{naive o}^{\dagger}(m,c;a_{o}(n,c), \text{ for } n \text{ odd}), b_{naive o}^{\dagger}(p,c';a_{o}(n,c'), \text{ for } n \text{ odd})\}_{-} = 0 \quad \text{ for } m, p \text{ both odd}$$

etc.. (10.18)

We could define an $(-1)^{F}$ -operator, where F is the fermion number operator. It sounds at first very easy just to write

$$F_{c} = \sum_{m} b^{\dagger}_{naive o}(m,c) b_{naive o}(m,c) + \sum_{m} b^{\dagger}_{naive e}(m,c) b_{naive e}(m,c),$$
(10.19)

where the sums run over respectively the odd and the even positive values for m for the o and the *e* components. But now this fermion number operator- as taken as a function of the *naive* operators - ends necessarily up being an expression in purely boson operators (from the ray c), and thus it looks at first as being valid except when the expression $(-1)^{F_c}$, which we are interested in, is equal to 1 on all states that can truly be constructed from boson operators. If it were indeed so, our idea of using $(-1)^{F_c}$ to construct the multidimensional fermion operators, would not be so good. However, there is a little detail that we did not have enough bosonic degrees of freedom to construct the zero momentum fermion operator in 1+1 dimensions. Therefore we can not really include in the definition of the "fermion number operator for the ray c", F_c , the term coming from m = 0. This term would formally have been $b_{naive e}^{\dagger}(m = 0, c) b_{naive e}(m = 0, c)$, but we decided to leave it out. This then means that the fermion number operator, for which we decide to use F_c as the number of fermions operator in the ray c is *not* the full fermion number operator for the corresponding 1+1 dimensional theory, but rather only for those fermions, that avoid the zero momentum state. To require this avoidance of the zero momentum is actually very attractive for defining a fermion number operator for the ray c as far as the momentum states included in such a ray really must exclude the zero momentum state in a similar way as a ray in a vector space is determined from the set of vectors in the ray not being zero.

But this precise definition avoiding the zero-momentum fermion operator contribution to the fermion number operator F_c leads to the avoidance of the just

mentioned problem that this fermion number F_c looked as always having to be even when constructed in terms of boson operators.

Now there should namely be enough boson degrees of freedom that one should be able to construct by boson operators all the different possible combinations for fermion states being filled or unfilled (still not the zero momentum included). Thus one does by pure bosons construct both - the even F_c and the odd F_c - states and thus the F_c with the zero momentum fermion state not counted can indeed be a function of the boson operators and can *take on both even and odd values* for momentum, depending on the boson system state. So, we can have - using this leaving out the zero momentum fermion state in the rays - an operator

$$F_c = F_{naive c}(a_e(n), \text{ for } n \text{ odd})$$
(10.20)

The operator $(-1)^{F_c}$ for each ray c counts if the number of fermions in the 1+1 dimensional system is even, then $(-1)^{F_c} = 1$, or odd, then $(-1)^{F_c} = -1$. We construct the following improved fermion operator (annihilation or creation),

$$b_e(m,c) = b_{naive e}(m,c) \prod_{c' < c} (-1)^{F_{c'}}.$$
 (10.21)

The inclusion of this extra operator factor helps to convert the commutation relations between the fermion annihilation and creation operators for different rays into anticommutation relations, as it can easily be seen

$$\begin{split} b_{e}(m,c) \ b_{e}(p,c') &= \\ b_{naive \ e}(m,c) \cdot \prod_{c'' < c} (-1)^{F_{c''}} \ b_{naive \ e}(p,c') \cdot \prod_{c'' < c'} (-1)^{F_{c'''}} \\ b_{naive \ e}(m,c) \cdot \prod_{c' \leq c'' < c} (-1)^{F_{c''}} \ b_{naive \ e}(p,c') = \\ -b_{naive \ e}(m,c) \ b_{naive \ e}(p,c') \cdot \prod_{c' < c'' < c} (-1)^{F_{c''}} \\ -b_{naive \ e}(p,c') \ b_{naive \ e}(m,c) \cdot \prod_{c' < c'' < c} (-1)^{F_{c''}} = \\ -b_{naive \ e}(p,c') \cdot \prod_{c''' < c'} (-1)^{F_{c'''}} \ b_{naive \ e}(m,c) \cdot \prod_{c'' < c'} (-1)^{F_{c''}} \\ = -b_{naive \ e}(p,c') \cdot b_{e}(m,c), \ still \ for \ c > c'. \end{split}$$
(10.22)

Thus we deduced, for c > c' in our in fact at first just chosen ordering of <, that the fermion operators do anticommute. It is not difficult to show similarly also in the case c' > c, that the fermion operators anticommute. The crux of the matter is that when e.g. c' > c there is the factor $(-1)^{F_c}$ contained in the product $\prod_{c'' < c'} (-1)^{F_{c''}}$, which is attached to $b_{naive o} (m, c')$ in order to correct it into $b_o (m, c')$, while there is no analogous factor $(-1)^{F_c'}$ contained in the factor $\prod_{c'' < c} (-1)^{f_{c''}}$ attached at $b_{naive o} (m, c)$ in order to bring it into $b_o (m, c)$. In this way one gets just the one extra minus sign in the product of the fermion operators that makes them anticommute.

10.2.3 Slight Generalization to have a Phase Factor

It is not difficult to see that the idea of using such an ordering < could be slightly generalized to have instead of the factors only minus or plus phase factors of the form $\exp(\delta(c, c'))$

$$b_{e}^{\dagger}(\mathbf{m}, \mathbf{c}) = b_{naive \ e}^{\dagger}(\mathbf{m}, \mathbf{c}) \prod_{c' \neq c, \text{ but } c' \in rays} e^{(i\delta(c, c')F_{c'})}.$$
(10.23)

It is also not difficult to see that, in order to obtain the anticommutation relations instead of the commutation ones (which we have for $b^{\dagger}_{naive\ e}(m,c)$), the phases must obey the rule

$$\delta(c, c') - \delta(c', c) = \pi(\text{mod } 2\pi).$$
(10.24)

We may in fact seek to evaluate the product of two fermion creation operators with the ansatz (10.23)

$$\begin{split} b_{e}^{\dagger}(\mathbf{m},\mathbf{c}) & b_{e}^{\dagger}(\mathbf{m}',\mathbf{c}') = \\ &= b_{naive\ e}^{\dagger}(\mathbf{m},\mathbf{c}) \prod_{\substack{c''\neq c, \\ c''\in rays}} e^{i\delta(c,c'')\ F_{c''}} & b_{naive\ e}^{\dagger}(\mathbf{m}',\mathbf{c}') \prod_{\substack{c''\neq c', \\ but \\ c'''\in rays}} e^{i\delta(c,c'')\ F_{c''}} & p_{naive\ e}^{\dagger}(\mathbf{m}',\mathbf{c}') \\ &+ b_{naive\ e}^{\dagger}(\mathbf{m},\mathbf{c}) e^{i\delta(c,c')} \prod_{\substack{c''\neq c \\ nor\ c', but\ c''\in rays}} e^{i(\delta(c,c'')+\delta(c',c''))\ F_{c''}} & b_{naive\ e}^{\dagger}(\mathbf{m}',\mathbf{c}') \\ &= b_{naive\ e}^{\dagger}(\mathbf{m},\mathbf{c}) e^{i\delta(c,c')\ F_{c'}} & b_{naive\ e}^{\dagger}(\mathbf{m}',\mathbf{c}') e^{i\delta(c',c)\ F_{c}} \\ &\cdot \prod_{\substack{c''\neq c \\ nor\ c', but\ c''\in rays}} e^{i(\delta(c,c'')+\delta(c',c''))\ F_{c''}} & e^{i\delta(c,c')\ F_{c'}} \\ &= e^{i(\delta(c,c')-\delta(c',c))} & b_{naive\ e}^{\dagger}(\mathbf{m}',\mathbf{c}') e^{i\delta(c',c)\ F_{c}} & b_{naive\ e}^{\dagger}(\mathbf{m},\mathbf{c}) e^{i\delta(c,c')\ F_{c'}} \\ &\cdot \prod_{\substack{c''\neq c \\ nor\ c', but\ c''\in rays}} e^{i(\delta(c,c'')+\delta(c',c'')\ F_{c''})\ F_{c''}} \\ &= e^{i(\delta(c,c')-\delta(c',c))} & b_{naive\ e}^{\dagger}(\mathbf{m}',\mathbf{c}') e^{i\delta(c,c')-\delta(c',c))} & b_{e}^{\dagger}(\mathbf{m},\mathbf{c}) = \\ &= -b_{e}^{\dagger}(\mathbf{m}',\mathbf{c}') & b_{e}^{\dagger}(\mathbf{m},\mathbf{c}), \end{split}$$

$$(10.25)$$

where in the last step we used (10.24). Thus we see that in this way we can get really in infinitely many ways - some algebraicly defined fermion operators that do indeed anticommute as they should. But it should be had in mind that both these procedure, by choosing $\delta(c, c')$ and the forgoing proposal with the ordering <, are a priori discontinuous and arbitrary.

We expect, however, that the latter method with $\delta(c, c')$ can be lead to a smooth and attractive scheme in the case of $d_{space} = 2$ or equivalently d = 3.

10.2.4 Exercise with Next to Simplest Case d_{space} = 2

In the case of $d_{space} = 2$ we can say that the set of our rays rays form a kind of a set of "rational angles" in the sense that each ray specifies modulo π (rather

than modulo 2π as for an oriented arrow it would specify) an angle, but that one only obtains those angles which rationalize tangenses. But the fact that they are after all implemented as angles - although only modulo π , means that they are at least locally ordered as numbers along a real or rather rational axis. So apart from troubles at the end and beginning we have an ordering and we could attempt to use it even for the implementation of the ordered set of rays method by proposing a "nice" < ordering. However, we think we get a better chance by using the $\delta(c, c')$ method in this d = 3 and thus d_{space} = 2 case.

We have to think about what topological properties we shall and can achieve for the function $\delta(c, c')$ depending on a pair of rays c and c'.

Since the classes or rays are "a kind of rational" directions, though without orientation, the topological space of the rays is like the sphere $S^{d_{space}-1}$ with opposite points identified. This topological space obtained by the identification of the opposite point on the $S^{d_{space}-1}$ sphere is actually topologically identical to the projective space of $d_{space} - 1$ dimensions. For the case d = 3 or $d_{space} = 2$ the topological space rays thus becomes simply the projective line (using real numbers), but that is topologically just the S^1 circle. Had this topological space been naturally orderable we could have used the ordering as the < above. However, it is a circle S^1 and not a simple line with plus and minus infinity; the infinities have so to speak been identified to only one point in the projective line. This means that using the method to define the fermion fields/operators by means of <-method would be very non-elegant, and would probably violate almost everything wanted.

Let us now think about a slight generalization by using the $\delta(c, c')$. We need to make a choice of a function $\delta(\cdot, \cdot)$ defined on the cross product of two projective spaces of dimension $d_{space} - 1$ each. Since it shall obey the condition (10.24), it cannot at all be a smooth or continuous function at the points where c = c'. Let us, for a while, take care that this method works well for d = 3 only.

In this d = 3 case the cross product of the two projective lines becomes topologically simply a two-dimensional torus. So we face topologically to define $\delta(c, c')$ on a two-dimensional torus. However, we are forced to give up having continuity along the "diagonal" - meaning the set of points on this torus with c = c' - and it is thus rather a $\delta(c, c')$ defined as a continuous function on the torus minus its "diagonal", which we must choose/find.

This two dimensional torus minus its "diagonal" is rather like a belt. I.e., it is topologically like the outer surface of a finite piece of a tube. It has two separate edges, each being topologically an S¹ circle, namely two images of the "diagonal" seen from the two sides. In between there is then the two-dimensional bulk area of the topological shape of the surface of the finite piece of a tube. It is inside this bulk region that we shall attempt to construct $\delta(c, c')$ to be smooth and "nice". Choosing

$$\delta(\mathbf{c}, \mathbf{c}') = 2^{\prime\prime} \text{clock average angle}^{\prime\prime}(\mathbf{c}, \mathbf{c}') \tag{10.26}$$

might be a good choice. directions c and c' forms with some coordinate axis (in momentum space). The precise way of defining this "clock average angle"(c, c') is illustrated on the figure 10.3 and consists in the following (let us remind the reader that we are still in the d = 3; $d_{space} = 2$ case):



Fig. 10.3. Examples of clock average directions.

- a. We introduce a "clockwise rotation orientation" in the spatial momentum plane.
- b. We draw a circle arrow from one of the two "ends" (half lines) of which the line c (the ray c is basically just a line) in this clockwise direction, and note the angle between this end of c and the first "end" (=half line) of c' (met in the following the circular arrow), which measures less than 180°.
- c. We draw a line, that divide this under b. noted angular region into halves. This line through the (momentum space) origo is denoted "clock average" (as marked on the figure).
- d. Such an unoriented line as the "clock average" defines relative to a coordinate system in spatial momentum space an angle-value modulo π. We call this angle-value "clock average angle"(c, c') and it is as just said defined modulo π (but only modulo π, because the line "clock average" is unoriented).
- e. Multiplying this angle "clock average angle"(c, c') by 2 its ambiguity to be only defined modulo π becomes instead an ambiguity modulo 2π . Thus our proposed expression (10.26) for $\delta(c, c')$ is defined modulo 2π , and that is what we need, since in our construction we exponentiate $\delta(c, c')$ after multiplication by i and an operator $F_{c'}$ that has only integer eigenvalues. Thus the expression, which we use, $\exp(i\delta(c, c')F_{c'})$ becomes well defined even though 2"clock average angle" (c, c') makes sense only modulo 2π .

Let us see whether this proposal is indeed is a good one. To see that our proposal (10.26) is a good one we must first of all check that it obeys (10.24). That is we must see what happens to the expression when we permute the two independent variables c and c'. Since by definition the circular arrow constructed in step b. goes out from the c-line, the first of the two arguments in $\delta(c, c')$, we must draw this circle-arrow after the permutation from c' instead. Therefore the half-angle noted under point b. above will after the permutation differ from the one before the permutation. This means that the line (through the origo) "clock average" gets after the permutation perpendicular to its direction before the permutation of

c and c'. Therefore "clock average angle" (c', c) = "clock average angle" (c, c') + $\pi/2 \pmod{\pi}$, which means that this angle gets shifted modulo π with $\pi/2$. After the multiplication by 2 (point e.) it means that $\delta(c', c) = \delta(c, c') + \pi \pmod{2\pi}$, which is just (10.24). Thus we got indeed by proposal (10.26) the condition (10.24) fulfilled.

We can now remark that quite obviously our proposal (10.26) is continuous as function of the directions c and c' *except where* c *and* c' *just coincide - what means that it is zero* (*mod* π) *angle between them*.

Let us note that had we not chosen the clock-wise rule, but instead taken, say, the smallest angle between c and c' and just found the halfening line between those "ends", we would have got a discontinuity when c and c' were perpendicular to each other. But by our precise choice we avoided that singularity. (For a point close to the diagonal the two arguments, c and c', are approximately the same ray. Permuting them will for a continuous function $\delta(c, c')$ make almost no difference, and thus it cannot possibly change by π , while crossing the "diagonal" the function δ would ask to jump by π .)

10.3 A Guess for Arbitrary Dimension

We propose the generalization of Eq. (10.7) to an arbitrary dimension, due to our experience with the Clifford objects (apart from some modifications due to whether we choose Weyl or Majorana fermions for family or for geometrical components), by using the relation

$$(\psi + \psi_{\mu}\gamma^{\mu} + \psi_{\mu\nu}\gamma^{\mu}\gamma^{\nu} + ... + \psi_{1235...d}\Gamma^{(d-1)} = e^{\phi_{\mu}\gamma^{\mu} + \phi_{\mu\nu}\gamma^{\mu}\gamma^{\nu} + ... + \phi_{1235...d}\Gamma^{(d-1)}}.$$
 (10.27)

10.4 Outlook on Supporting the Spin-Charge-Family theory [7,6]

We started with massless noninteracting bosons or fermions. But we like to work with the interacting fields. There are many Kalb-Ramond fields appearing in our type of fermionizable boson model in higher dimensions and correspondingly it is not easy to see how to make an interacting theory.

There are many ways to come from noninteracting bosonisable (fermionizable) fermion (boson) fields, which might lead to the fermion fields interacting with the boson fields as it is in the spin-charge-family theory.

But on the level of our fermionizable (bosonizable) boson (fermion) model with many Kalb-Ramond fields we must keep in mind that the conserved charges in the Kalb-Ramond theories are vectorial and thus one gets very many vectorial conserved quantities. This makes scattering processes (unless all the scattering particles are without these vectorial charges) very non-trivial.

One chance would be to let either fermion or boson fields to interact with gravity. Crudely speaking gravity couples to energy and momentum, and since

in the free bosonization procedure we have at least sought to get the total dmomentum be the same in the corresponding states of fermions and bosons there might be a chance that we fermionize a theory with both - the bosons of the Kalb-Ramond type *and* gravity through the vierbein formulation - and correspondingly obtain a theory with both fermions *and* bosons, the later would be the gravity degrees of freedom. This might lead to exactly the theory [6,7] that one of us (N.S.M.B.) has postulated as the true model for Nature beyond the standard model (the spin-charge-family theory).

Since our scheme a priori looks to require the Majorana fermions to have real fields like the bosons - at least in the simplest version - we only expect to get chiral fermions in those dimensions wherein Majorana fermions can simultaneously be Weyl (=chiral)as in d=2,6,10,14,... It is therefore even a slight support for the spin-charge-family theory that its phenomenologically favoured dimension is just 13+1=14.

One should for appreciating this idea of adding gravity without fermionizing it have in mind that one *does not have to bosonize all degrees of freedom*, but rather can - if one wishes - decide to fermionize some degrees of freedom but not all. Especially, if the motivation were to make all fermions from bosons because one claims that fermions are not properly local and should not be allowed to exist, then of course it is enough that we start with a purely boson theory as the fundamental one - and then we better only fermionize a part of bosons unless we could identify a purely fermionic theory with nature. But of course there seemingly are bosons in nature and we thus must end phenomenologically with a theory with *both* bosons and fermions.

Starting from fundamental bosons only that is only achievable by only a *partial fermionization*.

Hope for the Progress

The hope is, which is evidently from we haveproposed in this contribution, that we shall construct formulas for the higher dimensional cases by generalizing the formulas we already have for the one dimensional case, generalizing as well the "classes" to higher dimensions. In the spirit of seeking to identify the fields characterized by their "odd/even" indices with spin components, we hope to derive from the bosonization formula a scheme formally stating the relation between the boson and the fermion second quantized fields, $2^{d_{space}} - 1$ boson field components, while there will be $2^{d_{space}}$ fermion components.

10.5 Outlook on the Connection to the Spin-Charge-Family Theory

Let us try to clarify how the here discussed fermionization procedure is supposed to be, so to speak, the root for a theory beyond the Spin-Charge-Family theory of Norma Susana Mankoč Borštnik [7,6] (and her collaborators), The (one of) way we see as a very promising hope that one could justify this Spin-Charge-Family theory by the hoped fermionization is as follows: We build up a model with only bosons as the fundamental theory in say - 13 +1 dimensions - in the sense that this 13 +1 dimensional purely bosonic theory with a series of the Kalb-Ramond fields and with usual 13+1 dimensional gravity should be the fundamental choice of nature (not necessarily starting in d=13+1). Then this theory should be partly fermionized in the sense that only the series of Kalb Ramond fields get fermionized, but not the gravity (bosonic) degrees of freedom. The latter remain gravitational degrees of freedom hopefully now functioning as gravity for the fermions that came out of the fermionization. The Spin-Charge-Family theory will show up out of the Kalb-Ramond components.

1. The first assumption of our new scheme, which might be the pre-scheme of the Spin-Charge-Family theory, is that *fermions a priori do not obey proper locality*. The accusation towards all the theories with fermions which are fundamental fermions rather than fermionized bosons is that the axiom of locality in a quantum field theory is for the fermions

$$\{\psi_{\alpha}(x), \psi_{\beta}(y)\}_{+} = 0$$
, for the space like separation of y and x, (10.28)

while true physical locality should have been a *commutation rule* like the one obeyed by the boson fields

$$\{\phi(x), \phi(y)\}_{-} = 0$$
, for the space like separation of x and y. (10.29)

True locality means, one would think, that each little region in space is approximately a completely separate system that only interacts very indirectly with a far away different little region. If so, the physical operators describing the situation in one little region, should commute with those describing the situation in a different little region, and not anticommute as the fermion fields do. One might like to assume that only products of an even number of fermion fields are considered as proper operators describing the little system region, what satisfies the requirement of getting commutation relations between the field variables describing the situation in different regions. But such an assumption must be justified as a physical assumption, discussing seriously also odd products of fermion fields.

The point of view we suggest here is that we admit that we cannot have fermions at all in a truly local way! This then means that the fundamental physics should be a model *without fermions* so that all fermions come from bosons that become fermionized.

2. Since it is not easy to find so terribly many systems of bosons that can be fermionized, and thus if one finds some way of fermionizing, then this way is presumably already likely to be almost the only one possible.

At least we expect that the fermionization of a boson system of fields can only be made provided the number of fermions and the number of bosons agree with the theorem which one of us and Aratyn [1] put forward many years ago. For massless free fermions on the one side and massless free bosons on the other side we obtained that the number of components for the bosons and the fermions counted in the same way with respect to the fields being real or complex, should be in the ratio

$$\frac{\text{#fermion components}}{\text{#boson components}} = \frac{2^{d_{space}}}{2^{d_{space}} - 1},$$
(10.30)

where the dimension is $d = d_{space} + 1$, or the spatial dimension is d_{space} . The number of components - at least the number of real counted components - must of course be positive integer or zero. Thus the minimal number of fermion components must be $2^{d_{space}}$, while the number of boson components must be $2^{d_{space}} - 1$ or the numbers must be an integer multiplum of these numbers.

Alone this theorem of ours [1] makes appreciable restriction for when bosonization/fermionization is at all possible.

3. We are suggesting here the starting point with the bosonic degrees of freedom only, consisting of "series of the Kalb-Ramond fields, all the chain through, except for one (pseudo)scalar". By this we mean that we have as the bosons a series of separate fields $A_{\mu\nu\cdots\rho}$ with all the values of the number of indexes, antisymmetric with respect to all their indexes.

There is a simple way in which one could get the number $2^{d_{space}} - 1$ of boson components, if we arrange to have - by some gauge choice - only spatial values of the indexes μ , ν ,... on the A-fields, removing the A field with zero indexes. The number of components become equal to the number of subsets of d_{space} letters, which is $2^{d_{space}}$. Removing pure scalar, we get this number $2^{d_{space}} - 1$, as we want for the theorem of[1].

- 4. From the $2^{d_{space}} 1$ bosons represented by the Kalb-Ramond fields with the scalar removed, then according to the Aratyn *et al.* theorem [1] theorem there must be the $2^{d_{space}}$ components of fermion fields. This means for the Weyl spinor representation of fermion fields in even $d = d_{space} + 1$, with $2^{d/2-1}$ members that there are $2 \times 2^{d/2-1}$ real fermion fields. To get $2^{d_{space}}$ real Weyl spinor representation fermion fields there must be $\frac{2^{d_{space}} + 1}{2^{(d_{space}+1)/2}} = 2^{d/2-1} = 2^{d_{space}/2-1/2}$ families.
- 5. From the bosonization requirement we obviously get out that there must exist an even number of families as it also comes out from the Spin-Charge-Family theory of one of us [7,6].
- 6. But now there is correction due to the components of the KalbRamond fields with time indices, the 0. This gives very interesting corrections as we may postpone till later.

10.5.1 A Hope for that the Gravity Interaction Can Be Added

There is an interesting hope for that actually our at first free bosons being fermionized to free fermions could be generalized to have an universal coupling to a gravitational field - the bosonic field, which we do not fermionize, keeping it as gravitation, interacting with the fermions - so that we finally arrive at a theory with several families of fermions and gravity.

Above we wrote down a formula for counting the number of states for the fermion and the boson systems having the same number of Fock states with given momentum and energy for the free massless case of our bosonization/fermionization.

We used in reality an infrared cut off that meant that we in fact considered a torus world with for different components different periodicity conditions: Some components of fields had antiperiodicity while the others had periodicity property along various coordinate directions.

We shall note now that we could consider these momentum eigenstates for the single particles with given periodicity restriction as *topologically* specified in the following sense:

The wave functions for the momentum eigenstates are as is well known all along taking on only pure phase factor values, i.e. they obey $|\phi(x)| = 1$ all along. The number of turns around zero, which they perform when one goes around the torus along the different coordinates, is an integer (or a half integer depending on the boundary condition). We can consider this number of turns going around the torus in different ways (along different coordinates) a topological quantity in the sense that it as an integer cannot change under a small deformation.

Our main idea is at this point that we in this way can introduce at least a not too strong gravitational field and still have single particle solutions to the equations of motion characterized by the same system of (topological) quantum numbers.

That should suggest that we have the same set up for making the in this work studied bosonization in a not too strong gravitational field as in the free case. We namely should be able to classify the single particle states as functions of the space-time variables x on the by gravitational fields deformed torus (torus due to infra red cut off) according to a topological classification in terms of the number of times the wave function encircles the value zero in the complex plane when the one follows a closed curve, following, say, the coordinates of the deformed torus. For the massless theory we have scale-invariance for the matter fields - the series of the Kalb-Ramond fields or the fermions - so, as long as we consider the gravitational field itsef - we can scale up the momenta of the single particles by just letting the phase of an eigen-solution be scaled up by a factor. Only the periodicity conditions will enforce such scalings to be by integer factors, just as they must be also in the free flat case.

So we argue that with a background gravitational field, that is with a not too strong field, we have a possible description in terms of a discretized enumeration quite in the correspondence with the one for the flat case.

Remembering that we obtained the bosonization w.r.t. state counting in fact class c for class, meaning that the momentum eigenstates in the classes corresponding to rays went separately from boson to fermion or oppositely, we may have given arguments at least suggesting that a corresponding bosonization correspondence as the one in the free flat case also applies to the case with some (may be not too large though) gravitational field as a background field.

This may require further study but we take it that there is at least a hope for that the bosonization/fermionization procedure *can also be performed in a background gravitational field*.

Since we now with our expansions in power series seek to guarantee that we shall make the bosonization or fermionizations just in such a way that the d-momentum will be the same for the fermion configuration and the boson configuration corresponding to each other, we might hope that we could formulate the exact correspondence and the interpretations in terms of fields with spin indexes so that indeed the momentum densities would be the same for the fermions and for the corresponding bosons. If we succeed in that then the action on the gravitational fields which only feel the matter via the energy momentum tensor $T_{\mu\nu}$ would be the same for the bosons and the fermions in the corresponding states. In that case the development of the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitational fields would be the same for the corresponding to the gravitati

References

- Aratyn, H.B. Nielsen, "Constraints On Bosonization In Higher Dimensions", NBI-HE-83-36, Conference: C83-10-10.2 (Ahrenshoop Sympos.1983:0260), p.0260 Proceedings.
- Jacobi, C. G. J. , Fundamenta nova theoriae functionum ellipticarum (in Latin), Königsberg: Borntraeger (1829), ISBN 978-1-108-05200-9, Reprinted by Cambridge University Press 2012
- 3. Euler, L.: Demonstratio theorematis circa ordinem in summis divisorum observatum, Novi Commentarii Academiae scientiarum Imperialis Petropolitanae 5, 75 (1760), Translated as Bell, J., A demonstration of a theorem on the order observed in the sums of divisors [arXiv:math/0507201]; Euler, L. Evolutio producti infiniti $(1 - x)(1 - xx)(1 - x^3)(1 - x^4)(1 - x^5)$ etc. in seriem simplicem, Acta Academiae Scientarum Imperialis Petropolitinae 1780, pp. 47-55, 1783. Opera Omnia, Series Prima, Vol. 3. pp. 472-479, Translated as Bell, J. The Expansion of the Infinite Product $(1 - x)(1 - xx)(1 - x^3)(1 - x^6)(1 - x^6)$ etc. into a Single Series [arXiv:math.HO/0411454].
- Abramowitz, Milton; Stegun, Irene A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematics Series 55 (10 ed.), New York, USA: United States Department of Commerce, National Bureau of Standards; Dover Publications, page 825.
- 5. Glaisher, J.W.L., Messenger of Mathematics, 12, 158 (1883).
- 6. N.S. Mankoč Borštnik, "The explanation for the origin of the Higgss scalar and for the Yukawa couplings by the *Spin-Charge-Family* Theory", to appear in *J. of Modern Phys.*, [arxiv:1409.4981].
- N.S. Mankoč Borštnik, "Matter-antimatter asymmetry in the *spin-charge-family* theory", *Phys. Rev.* D 91, 065004 (2015) [arxiv:1409.7791].