

Proposition of a Hybrid Stochastic Lee-Carter Mortality Model

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Abstract

In the paper, a stochastic hybrid mortality model (EHLC) treated as a solution of stochastic differential equations is introduced. The model is defined analogously to the well-known Lee-Carter mortality model (LC). A parameter estimation procedure including a switching rule are proposed. A comparison of the predictive accuracy of the LC and EHLC models based on the mortality data for Poland has shown that the new model yields better results.

1 Introduction

The analysis of mortality is one of the basic problems faced by mathematical demography. First mortality models appeared in the literature in the 19th century (see Gompertz 1825, Thiele 1872). Since that time, much work has been done to develop the methodology. Complex mortality models have recently been the subject of several research papers authored, for instance, by Booth (2006), Cairns et al. (2008), Plat (2009), Haberman and Renshaw (2011).

Mortality models can be divided into two main groups. The larger group consists of static and stationary models (see, e.g. Cairns et al. 2008, Cairns et al. 2009, Cairns et al. 2011, Renshaw and Haberman 2003a, 2003b, 2006, Haberman and Renshaw 2008, 2009, 2011, Hatzopoulos and Haberman 2011, Pitacco 2004, Luciano et al. 2008, Plat 2009), where the log-odds of death probabilities or mortality rates are often expressed in a parametric form as the functions of age and calendar time.

The second group encompasses dynamic models, where death probabilities or mortality rates are expressed as solutions of stochastic differential equations without jumps (see for instance Russo et al. 2011, Bayraktar et al. 2009, Luciano et al. 2008, Dahl 2004, Schrager 2006, Janssen and Skiadas 1995, Giacometti et al. 2011, Biffis 2005) or with jumps (Bravo and Braumann 2007, Bravo 2009, Coelho et al. 2010, Biffis 2005, Hainaut and Devolder 2008).

When the performance of a mortality model is tested with empirical data, then its parameter estimates are found not to be constant in time – they depend on the period under study. For instance, parameter estimates based on the 1930-1950 time series may

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be significantly different from those obtained for the period 1960-1980 (Giacometti et al. 2011). It therefore seems reasonable to account for the changes the estimated parameter in the dynamic stochastic context by considering a switching rule.

Similar problems have also occurred in other areas of research, including control theory, economics, biology or chemistry. The concept of a switching rule has been considered with respect to the stochastic dynamic hybrid (switched) systems (e.g. Liberzon 2003, Boukas 2005), which are defined as systems composed of several parametric structures described in deterministic or stochastic terms. These structures change depending on the switching rule.

There is a whole range of studies on the estimation methodology applicable to stochastic dynamic hybrid (switched) systems (see, for instance Yin et al. 2002, 2003). Drawing on this methodology, we propose a new concept of stochastic hybrid mortality model.

The paper is organized as follows. In Section 2, the standard Lee-Carter model and basic notations and definitions are introduced. In Section 3, the Lee-Carter model is considered as a solution of stochastic differential equations. Thereafter, in Section 4, its modified representation termed Extended Hybrid Lee-Carter model (EHLIC) is proposed. The parameter estimation procedure and the definition of a switching rule are discussed in Section 5. The parameter estimates calculated for the LC and EHLIC models with Polish mortality data and a comparison between the models are provided in Section 6. The last Section 7 contains concluding remarks.

2 The Lee-Carter model

Before a new mortality model is discussed in the next sections, let us consider a mortality model proposed by Lee and Carter (Lee, Carter 1992). The model allows age-specific mortality rates to be forecasted based on long-term trends.

For the purpose of introducing the Lee-Carter model, let us consider crude age-specific period mortality rates $m_{x,t}$ defined as follows (Cairns et al. 2009)

$$m_{x,t} = \frac{D_{x,t}}{\bar{N}_{x,t}}, \quad x = 0, 1, \dots, \omega, \quad t = 1, 2, \dots, T, \quad (2.1)$$

where

$x = 0, 1, \dots, \omega$ – index of one-year age groups,

$t = 1, 2, \dots, T$ – index of a calendar year under observation,

$D_{x,t}$ – number of deaths in a year t (aged x last birthday),

$\bar{N}_{x,t}$ – death risk exposure (an average population size in a year t , aged x last birthday).

The Lee-Carter model (LC) can be defined as

$$\ln m_{x,t} = a_x + b_x k_t + \xi_{x,t}, \quad (2.2)$$

or equivalently

$$m_{x,t} = \exp \{a_x + b_x k_t + \xi_{x,t}\}, \quad (2.3)$$

where

k_t – time parameters indexed by $t = 1, 2, \dots, T$,

a_x, b_x – age-specific parameters indexed by $x = 0, 1, \dots, \omega$,

$\xi_{x,t}$ – random errors assumed to be iid, $\xi_{xt} \sim N(0, \sigma_\xi^2)$.

To ensure the unique representation of (2.2) or (2.3), some additional constraints are imposed (see Lee, Carter 1992) in the following way

$$\sum_{t=1}^T k_t = 0, \quad \sum_{x=0}^{\omega} b_x = 1, \quad \text{so that} \quad a_x = \frac{1}{T} \sum_{t=1}^T \ln m_{x,t}. \quad (2.4)$$

Parameters a_x describe the general profile of mortality rates. Indices k_t reflect the dominant temporal pattern in the decline of mortality (Tuljapurkar et al. 2000), while the coefficients b_x express the tendency of mortality rates to change when k_t changes. For example, small values of b_x for some x indicate that mortality at x varies a little with changes in the general level of mortality k_t (it is often the case at older ages). Parameters b_x usually have the same sign, but in some cases their movement in the opposite directions is also possible. Consequently, the Lee-Carter model assumes that mortality rates move in tandem, but not in the same direction or by the same amount.

The performance of the Lee-Carter model has been covered in many studies (see, e.g. Lee, Miller 2001). According to the concept underlying this methodology, parameters a_x, b_x, k_t are estimated with the empirical data on the considered population. Lee and Carter have used in their paper the Singular Value Decomposition (SVD) method (Lee, Carter 1992). Two other methods of estimation are also proposed, i.e. Weighted Least Squares (WLS) and the Maximum Likelihood (ML) approach (see Wilmoth 1993, Brouhns et al. 2002a,b).

The estimated values of k_t form a time series, with one value for each year. Because of that, the statistical methods of time series modelling can be employed. Lee and Carter considered a stochastic model of random walk with a (negative) drift. Its discrete representation is the following

$$k_t = k_{t-1} + c + e_t, \quad (2.5)$$

where c is a constant drift, and $e_t, t = 1, \dots, T$ are independent, normally distributed random errors.

The projected k_t , based on (2.5), and the estimates of a_x, b_x are used to forecast mortality rates and any other life-table characteristics, e.g. the expected remaining lifetime.

The estimation of the drift c is based on the following formula

$$\hat{c} = (k_T - k_1)/(T - 1). \quad (2.6)$$

The variance estimator of the error term is defined by

$$\hat{\sigma}_e^2 = \frac{1}{T-1} \sum_{t=2}^T (k_t - k_{t-1} - \hat{c})^2. \quad (2.7)$$

It can be treated as the measure of uncertainty in forecasting of k_t .

3 Lee-Carter model versus stochastic differential equations

We will consider a stochastic process $\mu_x(t)$ representing a hazard rate for a person aged x at time t . The rate will be defined by means of the following stochastic differential equation

$$d\mu_x(t) = \left(\alpha_x(t) + \frac{1}{2}\sigma_x^2 \right) \mu_x(t)dt + \sigma_x \mu_x(t)dW(t), \quad (3.1)$$

with

$$\alpha_x(t) = b_x k'(t), \quad \mu_x(0) = e^{a_x + b_x k(0)}, \quad x = 0, 1, 2, \dots, \omega, \quad (3.2)$$

where a_x, b_x, σ_x are constant model parameters, $k(t)$ is a differentiable deterministic function of time t , and $W(t)$ stands for the Wiener process.

One can show that the solution of (3.1)-(3.2) takes the form

$$\ln \mu_x(t) = a_x + b_x k(t) + \sigma_x W(t) \quad (3.3)$$

or in an equivalent representation

$$\mu_x(t) = \exp \{a_x + b_x k(t) + \sigma_x W(t)\} \quad (3.4)$$

4 The extended hybrid Lee-Carter model

In the next step, we shall consider a slightly modified equation (3.1) of the following form

$$d\mu_x(t) = \left(\alpha_x(t) + \frac{1}{2}q_x^2 \right) \mu_x(t)dt + \sigma_x \mu_x(t)dW(t), \quad (4.1)$$

with

$$\alpha_x(t) = b_x k'(t), \quad \mu_x(0) = e^{a_x + b_x k(0)}, \quad (4.2)$$

where q_x^2 is treated as correction term, whereas $\sigma_x^2 > 0$ represent the age-specific volatility parameters.

Using the Ito formula we can express (4.1)-(4.2) in the form (see e.g. Oksendal 1995)

$$\ln \mu_x(t) = a_x + b_x k(t) + \frac{1}{2} (q_x^2 - \sigma_x^2) t + \sigma_x W(t). \quad (4.3)$$

By observing the trends appearing in $k(t)$ (Rossa 2011, p. 124) we have concluded that the estimates of $k(t)$ can be modelled with linear functions of time t over separate time intervals $I_s = (\tau_{s-1}, \tau_s]$, $s = 1, 2, \dots, S$, which will be termed *mortality regimes*. Accordingly, we propose to define $k(t)$ as a piecewise differentiable function

$$k(t) = \begin{cases} k_1(t) & \text{for } t \in I_1, \\ k_2(t) & \text{for } t \in I_2, \\ \vdots & \vdots \\ k_S(t) & \text{for } t \in I_S. \end{cases} \quad (4.4)$$

4.1 Discrete time representation

A discrete representation of (4.3) has the form

$$\ln m_{x,t} = \ln m_{x,t-1} + b_x[k(t) - k(t-1)] + \frac{1}{2}(q_x^2 - \sigma_x^2) + \sigma_x \epsilon_{x,t}, \quad (4.5)$$

where $m_{x,t}$ is defined in (2.1) and $\epsilon_{x,t} = W(t) - W(t-1)$. From the properties of the standard Wiener process, it follows that the expectation of the error term $\epsilon_{x,t}$ is equal to 0, and the variance of $\epsilon_{x,t}$ equals 1.

We will further assume, that $k(t)$ is a piecewise linear function

$$k(t) = \begin{cases} c_1 + \delta_1 t & \text{for } t = 1, 2, \dots, \tau_1, \\ c_2 + \delta_2 t & \text{for } t = \tau_1 + 1, \tau_1 + 2, \dots, \tau_2, \\ \dots & \dots \\ c_S + \delta_S t & \text{for } t = \tau_{S-1} + 1, \tau_{S-1} + 2, \dots, T, \end{cases} \quad (4.6)$$

where τ_s are switching points determining different mortality regimes I_s , and

$$b_x, \sigma_x^2, q_x^2, \delta_s, c_s, \quad x=0, 1, \dots, \omega, s=1, 2, \dots, S, \quad (4.7)$$

form a set of model parameters, under the constraints of (2.4).

The model defined by (4.5)-(4.6) will be termed *Extended Hybrid Lee-Carter model* (EHLC).

4.2 Discussion of the model parameters

In the proposed EHLC model we have introduced two parameters q_x^2 and $k(t)$. The introduction of the parameter q_x^2 follows from the definition of the family of stochastic integrals. The Ito stochastic differential equations corresponding to the definitions of the Ito and Stratonovich stochastic integrals are respectively

$$d\mu_x(t) = \alpha_x(t)\mu_x(t)dt + \sigma_x\mu_x(t)dW(t), \quad (4.8)$$

and

$$d\mu_x(t) = \left(\alpha_x(t) + \frac{1}{2}\sigma_x^2 \right) \mu_x(t)dt + \sigma_x\mu_x(t)dW(t), \quad (4.9)$$

respectively.

The term $\frac{1}{2}\sigma_x^2$ is treated as a Stratonovich correction term. In a general case, a correction term corresponding to the definition of the stochastic integral can be treated as $\frac{1}{2}q_x^2$. The interpretation of the proposed term for real physical processes draws on the interpretation of the approximation of the white noise process, which, in fact, is an abstract process used for mathematical modelling purposes as a convenient mathematical tool. The real physical process that approximates the white noise process is a coloured noise (stationary wide-band process) (Willems and Aeyels 1976). This problem has been discussed in the literature (see, for instance Wong 1971 and Socha 2008). However, interpretation of q_x^2 for a demographic model has not been developed yet.

The proposition to represent $k(t)$ as (4.6), where $k_i(t)$ are linear functions of t seems to be the most straightforward one. Functions that are more complex will be considered during future research.

5 Estimation of the EHLC model

5.1 Switching time points estimation

The estimation of the EHLC model starts with the identification of the switching time points τ_s that distinguish successive mortality regimes $I_s = (\tau_{s-1}, \tau_s]$.

Therefore, we first estimated k_t using the SVD estimation of the standard Lee-Carter model and then we determined the unknown switching points τ_s , $s = 1, 2, \dots, S-1$ in the time series $\{k_t\}$ using the statistical adaptive test JL proposed by Janic-Wróblewska and Ledwina (2000).

The underlying concept is as follows. Consider random variables U_t , $t = 1, \dots, T-1$, defined by the differences

$$U_t = k_{t+1} - k_t. \quad (5.1)$$

If for any τ ($1 < \tau < T$) each of the variables U_1, U_2, \dots, U_τ has the same probability distribution as variables $U_{\tau+1}, U_{\tau+2}, \dots, U_{T-1}$, then there is no switching point, otherwise there is at least one switching point τ . The significant switching points in the time series $\{k_t\}$ can be identified with the JL test (see Appendix for details).

5.2 Parameter estimation

The estimation procedure of (4.6)-(4.7) is stepwise. The switching time points allow the mortality regimes I_s , $s = 1, \dots, S$ to be determined. For instance, if there is one switching point τ then two mortality regimes occur. In this case, four parameters $\delta_1, \delta_2, c_1, c_2$ of the model (4.6) can be considered and the model is the following

$$k(t) = \begin{cases} c_1 + \delta_1 t, & t = 1, 2, \dots, \tau, \\ c_2 + \delta_2 t, & t = \tau + 1, \dots, T. \end{cases} \quad (5.2)$$

It is worth noting that the parameters of (5.2) can be easily estimated with the standard Least Squares method.

The parameters σ_x^2 represent the volatility terms. We define the estimator of σ_x^2 as the variance of $y_{x,t} = \ln m_{x,t+1} - \ln m_{x,t}$

$$\hat{\sigma}_x^2 = \frac{1}{T-1} \sum_{t=1}^T (y_{x,t} - \bar{y}_x)^2, \quad (5.3)$$

where \bar{y}_x is a mean of the sequence $\{y_{x,t}\}$ over t .

In order to estimate the remaining parameters, b_x and q_x^2 we use the nonlinear Least Squares method, i.e. we minimize the sum

$$\sum_{t=1}^T \left[\ln m_{x,t} - \left(\ln m_{x,t-1} + b_x [k(t) - k(t-1)] + \frac{1}{2} (q_x^2 - \sigma_x^2) \right) \right]^2, \quad (5.4)$$

using $k(t)$, σ_x^2 obtained in the previous steps.

6 The results of model estimation

6.1 Estimation of LC and EHLC models for Poland

To illustrate the model estimation results, we estimated the LC and EHLC models using the 1958-2000 mortality data on Poland based on the period age-specific death rates available from www.mortality.org (Human Mortality Data Base).

Figure 1 presents the estimates of a_x 's in the LC model defined in (2.4). Estimates of b_x 's in the LC and EHLC models are very close to each other, thus they have been plotted together in Figure 2. Figure 3 presents the estimates of k_t derived for the LC model and the forecasts to 2012 based on (2.5). The mortality regimes I_s , $s = 1, 2, 3$ in years, 1958-1966, 1967-1990, and 1991-2000 (males), and 1958-1966, 1967-1988, 1989-2000 (females) have been also determined, and the parameter estimates of the function (4.6) derived. The estimated functions $k(t)$ are plotted in Figures 4 and 5. Figures 6 and 7 show the estimates of σ_x^2 and q_x^2 .

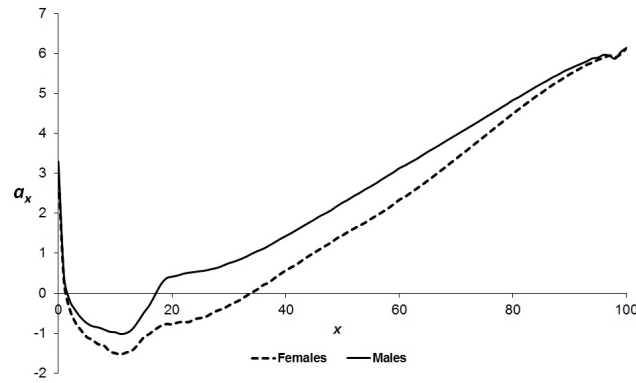


Figure 1: Estimates of a_x (males and females).

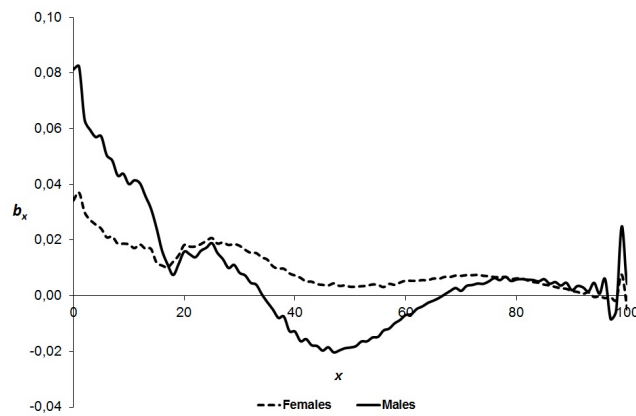


Figure 2: Estimates of b_x (males and females).

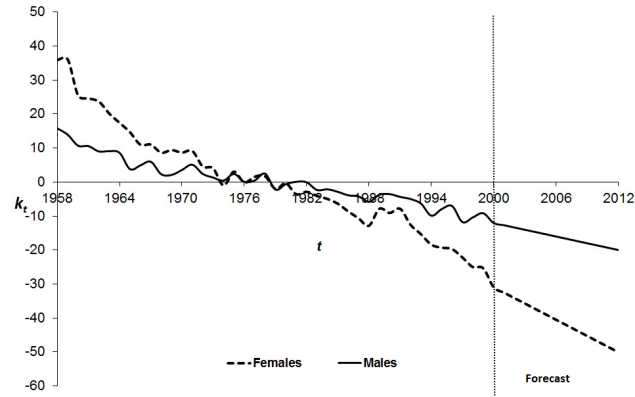


Figure 3: Estimates of k_t in the LC model and their forecasts to 2012 (males and females).



Figure 4: Estimates of k_t in the LC model and $k(t)$ in the EHLC model (males).

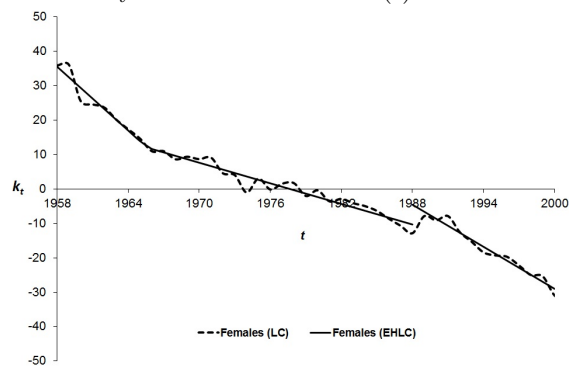


Figure 5: Estimates of k_t in the LC model and $k(t)$ in the EHLC model (females).

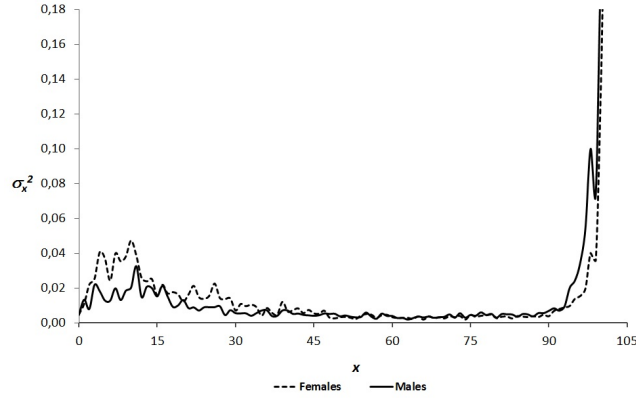


Figure 6: Estimates of σ_x^2 (males and females).

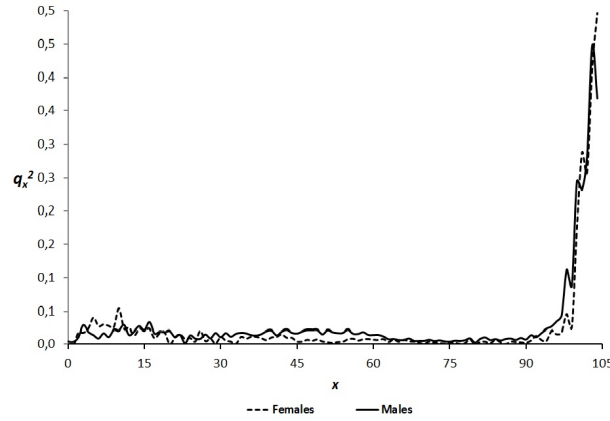


Figure 7: Estimates of q_x^2 (males and females).

6.2 Goodness-of-fit measures for the LC and EHLC models

To compare the quality of the LC and EHLC models, two well-known measures of goodness-of-fit were applied, i.e. the Mean Squared Error and the Mean Absolute Deviation, and the ex-post errors were assessed for the period 2001-2009. The obtained results are summarized in tables 1 and 2.

The Mean Squared Error and the Mean Absolute Deviation were estimated separately to each year t in both models. The measures were denoted $MSE_t^{(LC)}$, $MSE_t^{(EHLC)}$ and $MAD_t^{(LC)}$, $MAD_t^{(EHLC)}$, respectively. Hence we have

$$MSE_t^{(LC)} = \sqrt{\frac{1}{105} \sum_{x=0}^{104} [\ln m_{x,t} - (a_x + b_x k_t)]^2},$$

$$MAD_t^{(LC)} = \frac{1}{105} \sum_{x=0}^{104} |\ln m_{x,t} - (a_x + b_x k_t)|,$$

$$MSE_t^{(EHLC)} = \sqrt{\frac{1}{105} \sum_{x=0}^{104} \left[\ln m_{x,t} - \left(\ln m_{x,t-1} + b_x[k(t) - k(t-1)] + \frac{1}{2} (q_x^2 - \sigma_x^2) \right) \right]^2},$$

$$MAD_t^{(EHLC)} = \frac{1}{105} \sum_{x=0}^{104} \left| \ln m_{x,t} - \left(\ln m_{x,t-1} + b_x[k(t) - k(t-1)] + \frac{1}{2} (q_x^2 - \sigma_x^2) \right) \right|.$$

Table 1: *Ex post* comparison of the LC and EHLC by means of MSE

Years t	Males		Females	
	LC	EHLC	LC	EHLC
2001	0,226	0,145	0,130	0,148
2002	0,220	0,139	0,164	0,158
2003	0,249	0,152	0,149	0,142
2004	0,254	0,152	0,156	0,133
2005	0,262	0,155	0,163	0,173
2006	0,255	0,172	0,166	0,168
2007	0,275	0,189	0,203	0,192
2008	0,308	0,224	0,190	0,171
2009	0,316	0,228	0,205	0,214

Table 2: *Ex post* comparison of the LC and EHLC by means of MAD

Years t	Males		Females	
	LC	EHLC	LC	EHLC
2001	0,179	0,088	0,100	0,095
2002	0,182	0,097	0,126	0,106
2003	0,203	0,105	0,124	0,102
2004	0,214	0,115	0,130	0,098
2005	0,224	0,123	0,133	0,122
2006	0,225	0,141	0,139	0,122
2007	0,230	0,151	0,164	0,143
2008	0,256	0,180	0,160	0,135
2009	0,269	0,187	0,170	0,158

6.3 Comparison of residuals

In this subsection we compare the quality of the LC and EHLC models using the residual analysis. Residuals in the LC model are defined as

$$\epsilon_{x,t} = \ln m_{x,t} - (a_x + b_x k_t), \quad (6.1)$$

whereas residuals in the EHLC model have the form

$$\tilde{\epsilon}_{x,t} = \ln m_{x,t} - \left(\ln m_{x,t-1} + b_x[k(t) - k(t-1)] + \frac{1}{2} (q_x^2 - \sigma_x^2) \right). \quad (6.2)$$

In both models an assumption is embedded that standardized residuals are approximately independent standard normal variables. The simple way of testing this assumption is studying the so-called contour plots. They are plotted in Figures 8-11 for both models. The assumption implies that there should be a random pattern of negative (in white) and

positive (in black) random errors. If the plot reveals some clustering of positive or negative residuals then the assumption becomes inappropriate. Thus, the contour plots are visual tests of the underlying assumption.

We can see strong clustering of positive and negative residuals for the LC model (Figures 8, 10), whereas the contour plots look reasonably random in the case of the EHLC model (Figures 9, 11). However, closer inspection of Figures 9 and 11 reveals horizontal bands, which may suggest some genuine period effects.

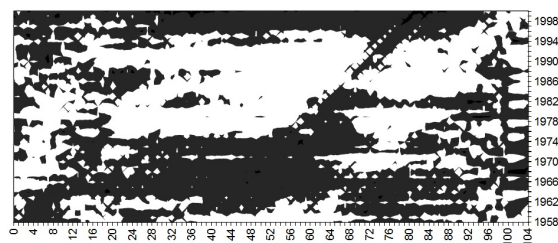


Figure 8: Contour plot of residuals in the LC model (males).

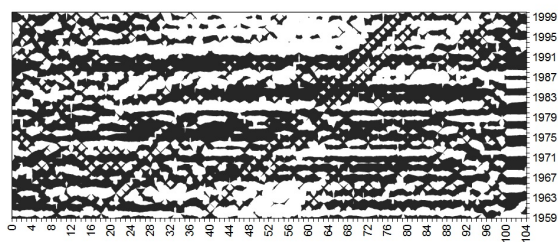


Figure 9: Contour plot of residuals in the EHLC model (males).

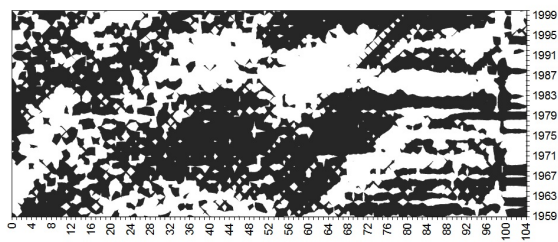


Figure 10: Contour plot of residuals in the LC model (females).

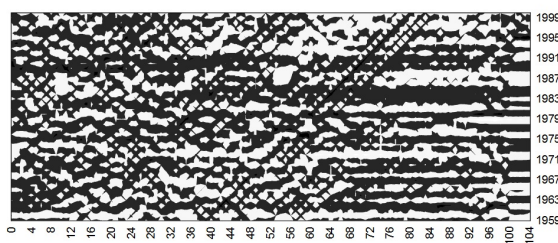


Figure 11: Contour plot of residuals in the EHLC model (females).

7 Concluding remarks

In the paper a mortality model representing the family of stochastic differential equations is proposed. It is based on the Lee-Carter model (LC) treated as a solution of stochastic differential equations. The new stochastic model has been termed an Extended Hybrid Lee-Carter model (EHLC). The switching rule for mortality decline has been derived for the EHLC model from the statistical adaptive test developed by Janic-Wróblewska and Ledwina. Both EHLC and LC were applied to model age-specific mortality rates in Poland. The results have shown the EHLC model to perform better than a standard LC model regarding the ex-post predictive accuracy. Further research will seek to define a more complex switching rule and another representation of a mortality model as a solution of stochastic partial differential equations.

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Appendix: Adaptive Test of Janic-Ledwina (JL)

Let us assume that we observe a sequence of continuous random variables U_1, U_2, \dots, U_N . Each variable U_t is distributed according to a distribution function F_t . We will consider an adaptive statistical test proposed by Janic-Wróblewska and Ledwina (2000) to verify the null hypothesis H_0

$$H_0 : F_1 = F_2 = \dots = F_N,$$

against the alternative hypothesis H_1

$$H_1 : \exists_{\eta \in (0,1)}, \quad F_1 = F_{[N\eta]} \neq F_{[N\eta+1]} = \dots = F_N,$$

where $[N\eta]$ denotes the integer part of the number $N\eta$.

The test statistics M_N is defined as follows

$$M_N(e, p_N) = \max_{[eN] \leq m \leq [(1-e)N]} T(S(m, p_N), m),$$

where

N – sample size,

$e \in (0, \frac{1}{2})$ – fixed value (in the paper it is assumed that $e = 0, 1$),

$p_N = 1, 5 \log N$ – positive value,

$S(m, p_N)$ – statistics defined as

$$S(m, p_N) = \min\{k : 1 \leq k \leq d_N; T(k, m) - k \cdot p_N \geq T(l, m) - l \cdot p_N; l=1, \dots, d_N\},$$

$d_N > 0$ – integer value representing the complexity of the problem (e.g. $d_N = 10$),

$$T(k, m) = \sum_{n=1}^k L^2(m, b_n)$$

$$L(m, b_n) = \sum_{t=1}^N c_{mt} \cdot b_n \left(\frac{R_t - 0,5}{N} \right),$$

R_t – rank of $U_{x,t}$ (for each fixed x),

c_{mt} – weights defined as

$$c_{mt} = \begin{cases} \sqrt{\frac{m(N-m)}{N}} \cdot \frac{1}{m}, & t = 1, 2, \dots, m, \\ -\sqrt{\frac{m(N-m)}{N}} \cdot \frac{1}{N-m}, & t = m+1, \dots, N. \end{cases}$$

$b_n, n = 1, \dots, k$ – the Legendre orthonormal polynomials on the interval $[0, 1]$.

Large values of M_N support rejecting the null hypothesis in favor of the alternative. The authors have derived critical values of the test by using the Monte-Carlo methods. They have also proved that for $k = 1$ the JL test reduces to the well-known Wilcoxon rank test.