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Maximal core size in singular graphs

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Abstract

A graph G is singular of nullity η if the nullspace of its adjacency matrix $\mathbf{G}$ has dimension η. Such a graph contains η cores determined by a basis for the nullspace of $\mathbf{G}$. These are induced subgraphs of singular configurations, the latter occurring as induced subgraphs of G. We show that there exists a set of η distinct vertices representing the singular configurations. We also explore how the nullity controls the size of the singular substructures and characterize those graphs of maximal nullity containing a substructure reaching maximal size.

Keywords: Adjacency matrix, nullity, extremal singular graphs, singular configurations, core width.

Math. Subj. Class.: 05C50, 05C60, 05B20.

1 Introduction

A graph $G = G(\mathcal{V}, \mathcal{E})$ has vertex set $\mathcal{V} = \mathcal{V}_G = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E}$ consisting of pairs of vertices. The *order* $|G|$ of a graph G is the number n of vertices. The graphs we consider are simple, that is, without multiple edges or loops. The complete graph K_n on n vertices has edges between all distinct pairs of vertices.

The graph $G - X$ denotes the graph obtained from G when the set X of vertices and the edges incident to the vertices in X are deleted. The reverse process, starting from H and adding a vertex set X results in $H + X$. Note that $H + X$ is not unique for a particular graph H and set X, since it varies with the choice of edges between X and $\mathcal{V}_H$ and even with the edges among the vertices of X themselves. If $X = \{v\}$, we write $G - v$ and $G + v$ for $G - X$ and $G + X$ respectively.

The adjacency matrix of a graph G, denoted by $\mathbf{G}$, is (a_{ij}), where $a_{ij} = 1$ if $\{ij\}$ is an edge and 0 otherwise. Note that the set of matrices $\{\mathbf{G}\}$ for distinct labellings of the vertices are permutationally similar and therefore a graph G is described completely (up

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to isomorphism) by the corresponding $\mathbf{G}$ for a specific labelling. The *spectrum* $Sp(G)$ of a graph G consists of the collection, with repetitions, of the eigenvalues of $\mathbf{G}$, which are the solutions of the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{G}) = 0$. The *algebraic multiplicity* of an eigenvalue is the number of times it is repeated in $Sp(G)$. The *geometric multiplicity* is the dimension of the corresponding eigenspace. Since $\mathbf{G}$ is real and symmetric, the two multiplicities for an eigenvalue have a common value. The multiplicity of the eigenvalue zero is referred to as the *nullity*[1], $\eta(G)$, of G. The *rank* of G is rank$(\mathbf{G})$, equal to $n-\eta(G)$, a result referred to as *the Dimension Theorem.*

A graph G on n vertices is said to be *singular* if $\eta(G) > 0$; that is, if there exists $\mathbf{x} \neq \mathbf{0}, \mathbf{x} \in \mathbb{R}^n$, such that $\mathbf{G}\mathbf{x} = \mathbf{0}$, where each entry of the vector $\mathbf{0}$ is 0. Since $\mathbf{G}$ satisfies $\mathbf{G}\mathbf{x} = \lambda\mathbf{x}$ for the eigenvalue $\lambda = 0$, we call $\mathbf{x}$ a *kernel eigenvector* of G.

This paper is motivated by the question:
How does the nullity control the size of the singular substructures within a graph?

This we address in section 4. To this end, we survey results on substructures in section 2 and on certain invariants of a graph in section 3, including proofs of theorems that facilitate the reasoning of new results, leading to a clarification of the underlying concepts.

2 Singular graphs

Let $\mathbf{x} \in \mathbb{R}^n$, $n \geq 3$, be a vector in the nullspace of G, which is labelled so that $\mathbf{x} = (\mathbf{x}_F, \mathbf{0})^t$, with each entry of $\mathbf{x}_F$ being non-zero. The vertices corresponding to $\mathbf{x}_F$ induce a subgraph F whose adjacency matrix is the principal $|F| \times |F|$ submatrix $\mathbf{F}$ of $\mathbf{G}$, satisfying $\mathbf{F}\mathbf{x}_\mathbf{F} = \mathbf{0}$. We call $(F, \mathbf{x}_F)$, or simply F, a *core of* G. If $\mathbf{x} = \mathbf{x}_F$, then $G = F$ and G is said to be a *core graph.* Note that a core of G is a core graph in its own right. Linearly independent kernel eigenvectors determine distinct cores of G. The set CV of *core vertices* consists of those vertices that lie on some core of G. If a vertex does not lie on any core of G, then it is said to be *core forbidden.* A core graph without isolated vertices having nullity one must be connected and is said to be a *nut graph.* Nut graphs exist for all orders from seven onwards. There are three nut graphs of order seven and none for smaller order [10].

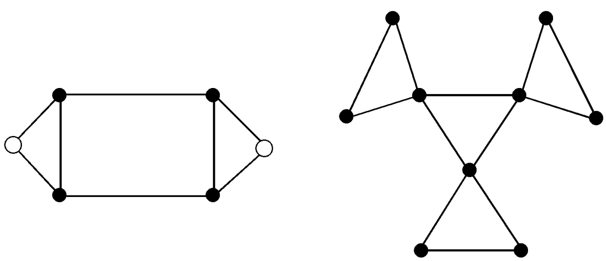

Figure 1: Two singular graphs of nullity one.

Consider the two graphs in Figure 1. The six vertex graph has a nullvector $(1, 1, -1, -1, 0, 0)^t$ and its core is the four cycle C_4 (labelled 1,2,3,4), induced by the solid black vertices, while its core-forbidden vertices (labelled 5,6) are white. The nine vertex graph is a nut graph and therefore has no core-forbidden vertices.

[1]The term $\mathbf{corank}(\mathbf{G})$ is also used for nullity$(\mathbf{G})$ in the literature.

Since it is the existence of $\mathbf{x}_F \in \mathbb{R}^{|F|}$ that determines that a graph is singular, we classify singular graphs according to core-order $|F|$. As shown in Figure 2, there are eight possible cores of order six, three of order five, two of order four and one each of orders three and two.

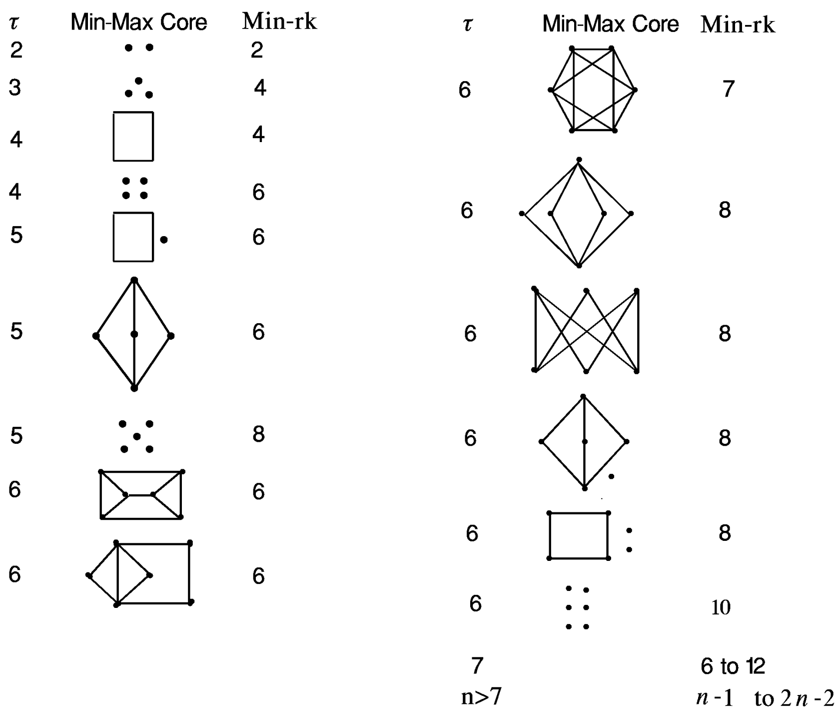

Figure 2: Minimum rank of graphs with core-width τ.

2.1 Singular configurations

Cauchy's inequalities for a Hermitian matrix M, (also known as the *Interlacing Theorem* [3]) control the multiplicity of the eigenvalues of principal submatrices relative to those of M. Applied to graphs we have:

Theorem 2.1. Interlacing Theorem: *Let G be an n-vertex graph and $v \in \mathcal{V}$. If the eigenvalues of G are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and those of $G-v$ are $\mu_1, \mu_2, \ldots, \mu_{n-1}$, both in non-increasing order of magnitude, then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n$.*

Thus when a vertex is added to a graph, the nullity, or multiplicity of the eigenvalue zero, may change by at most one. Let us first consider a graph of nullity one. Such a graph G with nullspace generator $\mathbf{x}$ has a unique core F as an induced subgraph determined by the vertices corresponding to the non-zero restriction $\mathbf{x}_F$ of $\mathbf{x}$. We say the core is $(F, \mathbf{x}_F)$ when $\mathbf{x}_F$ needs to be emphasized. By interlacing, to obtain a graph G of nullity one from a core graph F of nullity η, at least $\eta - 1$ vertices are added. Thus a lower bound for the order of a graph G of nullity one, with core $(F, \mathbf{x}_F)$, is $|F| + \eta(F) - 1$.

Definition 2.2. A graph G, $|G| \geq 3$, is a **singular configuration** (SC), with core $(F, \mathbf{x}_F)$, if it is a singular graph, of nullity one, having $|F| + \eta(F) - 1$ vertices, with F as an induced subgraph, satisfying $|F| \geq 2$, $\mathbf{F}\mathbf{x}_F = \mathbf{0}$ and $\mathbf{G}\begin{pmatrix} \mathbf{x}_F \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$. The vector $\mathbf{x}_F$ is said to be the **non-zero part** of the kernel eigenvector $\begin{pmatrix} \mathbf{x}_F \\ \mathbf{0} \end{pmatrix}$ of G.

Note that by interlacing, among singular graphs of nullity one, a singular configuration G has the least number of vertices for its core $(F, \mathbf{x}_F)$. A core graph may be connected as in the cycles C_{4k}, $k \in \mathbb{Z}^+$ on $4k$ vertices or disconnected as in the empty graph rK_1 consisting of r isolated vertices. An important combinatorial property of core graphs is that they have no pendant edges.

Lemma 2.3. *A singular configuration is a connected graph.*

Proof. Suppose, for contradiction that a singular configuration S is disconnected. Without loss of generality, S has a connected component S_1, $|S_1| \geq 3$, having a non-zero kernel eigenvector $\mathbf{x}_1$. If the vertices of S_1 are labelled first, then $\mathbf{S}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ and $\mathbf{S}_1\mathbf{x}_1 = \mathbf{0}$. Note that S cannot have isolated vertices, since these would contribute to the nullity, so that $\eta(S)$ would be more than one. Since $\mathbf{x}_1 \neq \mathbf{0}$, then $\mathbf{x}_2 = \mathbf{0}$, otherwise $\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{pmatrix}$ would be linearly independent kernel eigenvectors of S, again contradicting that the nullity of S is one. Thus the unique core of S lies in S_1. Suppose that $\mathbf{x}_2$ corresponds to vertices in the non-singular components. But then S does not have a minimum number of vertices for core F, a contradiction. Thus $S = S_1$. □

We now show the relevance of singular configurations. We prove that for nullity–one graphs, of order larger than minimal with respect to core F, some singular configuration is an induced subgraph.

Proposition 2.4. *A graph G without isolated vertices, of nullity one, with core $(F, \mathbf{x}_F)$, has (at least) one induced subgraph which is a singular configuration with the same core $(F, \mathbf{x}_F)$.*

Proof. Let the vertices of F be labelled first. Then the first $|F|$ rows of $\mathbf{G}$ may be partitioned as $(\mathbf{F}|\mathbf{C})$ and $(\mathbf{F}|\mathbf{C})^t(\mathbf{x}_F) = 0$. Moreover the rank of $(\mathbf{F}|\mathbf{C})$ is $|F| - 1$. If F is not itself a SC, its nullity $\eta(F)$ is more than one. There exist $\eta(F) - 1$ column vectors of the supplementary matrix $\mathbf{C}$ which are mutually linearly independent and also independent of the columns of F. These form the matrix $\mathbf{C}'$ such that rank$((\mathbf{F}|\mathbf{C}'))$ =rank$((\mathbf{F}|\mathbf{C}))$ $= |F| - 1$. The principal submatrix of $\mathbf{A}(G)$ determined by $(\mathbf{A}(F)|\mathbf{C}')$ is of the form $\mathbf{A}' = \begin{pmatrix} \mathbf{A}(F) & \mathbf{C}' \\ (\mathbf{C}')^{\mathbf{t}} & \mathbf{Q} \end{pmatrix}$, where $\mathbf{Q}$ is a square matrix. The adjacency matrix $\mathbf{A}'$ defines an induced subgraph of G, satisfying Definition 2.2. Therefore G is a singular configuration $S(F, \mathbf{x}_F)$. □

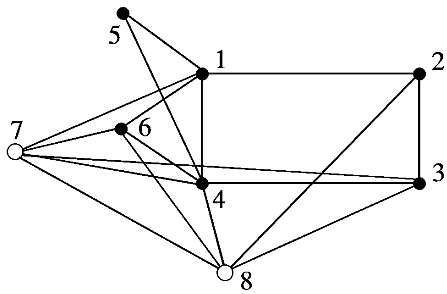

Figure 3: Verices 7 and 8 are core–forbidden.

Example 2.5. A singular configuration S has a unique core $(F, \mathbf{x}_F)$ whose vertices form the full set CV in S. If the set $\mathcal{P}$ of core-forbidden[2] vertices form an independent subset of the vertices of S, then S is said to be a *minimal configuration*. In fact there are $2^{|\mathcal{P}|}$ singular configurations $\{S\}$ obtained from a particular minimal configuration by adding edges between some pairs of distinct vertices of $\mathcal{P}$ and they all satisfy $\mathbf{S}\begin{pmatrix}\mathbf{x}_F\\ \mathbf{0}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}$.

The complete list of minimal configurations of core order up to five may be found in [5].

The singular graph G, of nullity one, shown in Figure 3, has only one core F which is the subgraph (of order 6 and nullity 2) induced by the solid black vertices. By Definition 2.2, a singular configuration with core F is expected to have 7 vertices. There are two non-isomorphic singular configurations, $G - 7$ and $G - 8$, having the same core as G. These two distinct induced subgraphs of G are in fact minimal configurations. Moreover, G and its two singular configurations share the same non-zero part $\mathbf{x}_F = (1, 1, -1, -1, -1, 1)^t$ of a kernel eigenvector generating their nullspace.

3 Some new invariants of singular graphs

The study of singular graphs reveals invariants of a graph. One is the partition of the vertex set $\mathcal{V}(G)$ into the set CV of vertices lying on some core of G and the set of core-forbidden vertices, $\mathcal{V}(G)\backslash\mathrm{CV}$. We show that this partition does not depend on the choice of basis for the nullspace of $\mathbf{G}$. Therefore CV is well defined.

Proposition 3.1. *The set* CV *of core vertices is an invariant of a graph G.*

Proof. A basis B for the nullspace can be transformed into another, B', by linear combinations of the vectors of B. However, the union of the collections of the positions of the non-zero entries in the basis vectors is the same for all bases. Thus the partition of the vertex set $\mathcal{V}(G)$ into CV and core-forbidden vertices, $\mathcal{V}(G)\backslash\mathrm{CV}$, is independent of the basis used for the nullspace. $\square$

[2] For a graph G of nullity more than one, the *periphery* with respect to F is the subset of vertices *not* on the core F. For graphs of nullity more than one, the set of core-forbidden vertices is then the set–intersecton of the peripheries over all cores F of G.

From proposition 3.1, it follows that the set, $\mathcal{V}(G)\backslash$CV, of core-forbidden vertices is also an invariant of G.

3.1 Fundamental system of cores

We now point out properties of subspaces in general that shed light on the structure of a singular graph. For any vector space W, let $wt(\mathbf{x})$ denote the **weight** or number of non-zero entries of the vector $\mathbf{x}$. We adopt the convention to write a basis for a vector space in which the vectors are *ordered* according to the monotonic non-decreasing *sequence* of the weights of its vectors.

A maximal set of linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell$, with the smallest *weight sum* $\sum_{i=1}^{\ell} wt(\mathbf{x}_i)$, are said to form a *minimal basis* B_{min} for W. We say a basis B for W is *reducible* if linear combinations of the vectors of B can produce another basis for W with lower weight sum. Thus for any vector space, a basis is not reducible if and only if it is minimal. The monotonic non-decreasing sequence of weights of vectors (weight–sequence) in a minimal basis provides an invariant of the vector space, a result that is significant because of its generality to bases of any vector space:

Theorem 3.2. [8] *Let W be a $q-$dimensional subspace of $\mathbb{R}^n$. Let t_1 be the least weight of a non-zero vector in W and let $B_1 = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q)$, with weight–sequence $t_1, t_2, \ldots, t_q$, be a basis with minimum weight-sum. If $B_2 = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q)$ is another ordered basis for W with weight–sequence $s_1, s_2, \ldots, s_q$, then $\forall i,\ \ t_i \leq s_i$.*

Although there may be various possible minimal bases for a vector space, by Proposition 3.2, the sequence of weights of their members is unique for the vector space. The following results are immediate.

Corollary 3.3. *A basis for a vector space is not reducible if and only if the monotonic non-decreasing sequence of the weights of its vectors is lexicographically minimal.*

Corollary 3.4. *The weight-sequence of a minimal basis for a vector space is an invariant of the vector space.*

The vertex space for a n-vertex graph G is considered to be $\mathbb{R}^n$ and the nullspace ker($\mathbf{G}$), is a subspace of dimension η. Note that a minimal basis B_{min} for ker($\mathbf{G}$) determines a *fundamental system of cores* of G. We now apply Corollary 3.4 to singular graphs to obtain another invariant associated with the nullspace.

Proposition 3.5. *The monotonic non–decreasing sequence of core-orders in a fundamental system of cores is a graph invariant.*

Remark 3.6. The following result proved in [7], provides a necessary condition, in terms of admissible subgraphs, for a graph to be of a specific nullity η.

Proposition 3.7. *Let H be a singular graph, without isolated vertices, having nullity η. There exist η* SC*s as induced subgraphs of H whose cores form a fundamental system of cores of H.*

3.2 Increasing the nullity of a graph

Proposition 3.7 provides a necessary condition for a graph to be of nullity η. The following proposition, which we shall use repeatedly in what follows, gives a necessary and sufficient condition to increase the number of core vertices in a graph.

Proposition 3.8. *The nullity increases with the addition of a vertex v to a graph G forming a connected graph $G+v$ if and only if v is a core vertex of $(G+v)$.*

Proof. Suppose, for contradiction, that v does not lie on any core of the resulting graph $G+v$ but $\eta(G) < \eta(G+v)$. Then all the cores of $G+v$ lie in G. But then, interlacing demands that $\eta(G) \geq \eta(G+v)$, a contradiction.

Conversely, let v be a core vertex of $(G+v)$. Then there exists a kernel eigenvector $\mathbf{x}$ with a non-zero entry corresponding to v. Let $\mathbf{M}$ be the $k \times n$ matrix whose rows are the k vectors of a basis for the nullspace of $G+v$, labelled so that the first row of $\mathbf{M}$ is $\mathbf{x}$ and the first column corresponds to v. By row-reducing $\mathbf{M}$ to echelon form, with all entries in the columns above and below a leading 1 (or pivot) being zero, a matrix $\mathbf{M}'$ is produced, whose rows give a new basis $B_{\mathcal{Y}}$ for the nullspace of $G+v$, determining a system $\mathcal{Y}$ of cores. The first row determines the only core F in this system with vertex v. Deleting v has the effect of destroying F while retaining all the other cores in $\mathcal{Y}$. Thus $\eta(G) \geq \eta(G+v) - 1$.

Moreover, if $\eta(G) > \eta(G+v) - 1$, then there exists a kernel eigenvector $\mathbf{z}$ of G which is linearly independent of those determined by $\mathcal{Y}$. This kernel eigenvector becomes an additional kernel eigenvector of $G+v$ by adding a zero as a first entry; but then the $k+1$ vectors in $B_{\mathcal{Y}} \cup \{\mathbf{z}\}$ are linearly independent in the k–dimensional nullspace of $G+v$, a contradiction. Hence $\eta(G) = \eta(G+v) - 1$. □

We now show that if a graph G of rank r is labelled so that the first r rows of $\mathbf{G}$ are linearly independent, then $\mathbf{G}$ has a simple block form.

Proposition 3.9. *Let the graph G be of order n and rank r, with adjacency matrix $\mathbf{G}$. Let the first r rows of $\mathbf{G}$ be linearly independent vectors. Then there exist a non–singular $r \times r$ matrix B and a $r \times (n-r)$ matrix $\mathbf{Y}$ such that*

$$\mathbf{A}(G) = \begin{pmatrix} \mathbf{B} & \mathbf{BY} \\ \mathbf{Y}^t\mathbf{B} & \mathbf{Y}^t\mathbf{BY} \end{pmatrix}.$$

Proof. The first r rows of $\mathbf{G}$ form a submatrix $\mathbf{T}$ of the adjacency matrix $\mathbf{G}$ and represent a maximal set of linearly independent row vectors of $\mathbf{G}$. Each of the $\eta(G)$ row vectors $R_j,\ j > r$, is linearly dependent on a subset of the first r row vectors of $\mathbf{G}$. The adjacency matrix $\mathbf{G}$, of rank r and nullity η, can be expressed as $\begin{pmatrix} \mathbf{T} \\ \mathbf{Y}^t\mathbf{T} \end{pmatrix}$ for a $r \times \eta$ matrix $\mathbf{Y}$. To see this, note that each linear relation between $R_j,\ j > r$ and the rows of $\mathbf{T}$ corresponds to a kernel eigenvector in the nullspace of $\mathbf{G}$. Since each of these $\eta(G)$ kernel eigenvectors corresponds to a core with a unique vertex v_j (described by row vector $R_j,\ j > r$), these $\eta(G)$ kernel eigenvectors are linearly independent and so their first r entries form the columns of $\mathbf{Y}$. Because of the symmetry of $\mathbf{G}$, the block $\mathbf{T}$ can be expressed as $(\mathbf{B}\ \ \mathbf{BY})$, where $\mathbf{B}$ is a $r \times r$ non-singular matrix. The result now follows immediately from the symmetry of $\mathbf{G}$. □

Note that in general there are different choices of the first r rows. Also, each of the last $(n-r)$ vertices in this labelling of G lies on a core and are said to form a **singular configuration vertex- representation** denoted by $\mathcal{R}$. Thus $|\mathcal{R}| = n - r = \eta$, by the Dimension Theorem. The kernel eigenvectors, forming a basis for the nullspace of $\mathbf{G}$, define a system $\mathcal{Y}$ of η distinct cores that correspond to η singular configurations found as induced subgraphs of G. There is a one-one correspondence between $\mathcal{Y}$ and $\mathcal{R}$. The concept of the singular–configuration–vertex–representation has been used in an *ad hoc* manner in the literature and more formally in this paper.

We can also identify a singular–configuration–vertex–representation as the set of vertices corresponding to the pivots in the rows of matrix $\mathbf{M}'$ in the proof of Proposition 3.8. These define a vertex–representation of a fundamental system $\mathcal{Y}$ of $\eta(G)$ distinct cores. By Proposition 3.8, deleting a vertex v in a singular–configuration–vertex–representation reduces the nullity by one and destroys a core in $\mathcal{Y}$ that has vertex v.

Note that any collection of h vectors in B_{min} collectively cover at least h core-vertices of G. So the existence of a singular–configuration–vertex–representation is also guaranteed by *Hall's Theorem* (also known as *the Marriage Problem*).

Retaining the labelling of Proposition 3.9, there exist $\eta(G)$ singular configurations (with distinct spanning minimal configurations) which are induced subgraphs of G such that each singular configuration corresponds to a unique vertex j, $r < j \leq n$. By scaling the corresponding kernel eigenvectors so that the last non-zero entry is 1, the following result is immediate.

Corollary 3.10. *Let G be of order n and rank r, with adjacency matrix $\mathbf{G}$. Let the first r rows of $\mathbf{G}$ be linearly independent vectors. There exists a matrix $\mathbf{M}^t = \begin{pmatrix} \mathbf{K} \\ \mathbf{I} \end{pmatrix}$ whose columns are the vectors of a basis for the nullspace of G, where the order of identity matrix $\mathbf{I}$ is $n - r$.*

Note that if $\mathbf{G} = \begin{pmatrix} \mathbf{B} & \mathbf{E} \\ \mathbf{E^t} & \mathbf{U} \end{pmatrix}$ has rank r and $\mathbf{B}$ is a non–singular $r \times r$ matrix, then $\mathbf{K} = \mathbf{B}^{-1}\mathbf{E} = -\mathbf{Y}$ in Proposition 3.9.

4 Nullity and core width

Remark 4.1. Henceforth we shall refer to a minimal basis B_{min} for ker(G) as $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\eta)$. The integers $wt(\mathbf{u}_1)$ and $wt(\mathbf{u}_\eta)$ are *extremal* values and have been referred to as the graph *singularity* κ in [2, 9] and *core-width* τ in [6], respectively. We focus on the zero, non-zero pattern of the entries of the vectors in ker$(\mathbf{G})$, to show that the nullity controls the core width.

The zero eigenvalue equation, $\mathbf{Gx} = \mathbf{0}$, stipulates that the sum of the entries of $\mathbf{x}$ corresponding to neighbours of any vertex in G is zero. This is also known as the *Zero Sum Rule* [9] which leads to a generic kernel eigenvector $\mathbf{x}_{gen}$ in terms of $\eta(G)$ independent parameters. One way in which to obtain a basis of $\eta(G)$ kernel eigenvectors is to set a parameter in turn equal to one and the remaining parameters equal to zero. If η non-zero entries of $\mathbf{x}_{gen}$, chosen so that they are collectively functions of all the η parameters, are set to zero, then $\mathbf{x}_{gen}$ is forced to vanish. This concept, which has been used in the theory of singular graphs [4], will be expanded upon in this section.

The Lemma that follows appears in [1] for graphs with weighted edges. We give a new proof for the $(0-1)$–adjacency matrix, emphasising the role of the core vertices in a singular graph. The result will enable us to relate the core-width τ to the nullity η, while the proof helps to shed more light on the graph structure.

Lemma 4.2. *If $\eta(G) > k \geq 1$, then there exists $\mathbf{x} \in ker(G)$, such that $\mathbf{x}$ has zero entries in any k specified positions.*

Proof. Let v be a core–forbidden vertex. Then any kernel eigenvector has a zero entry in that position. To prove this lemma, then, we need to show that there exists a kernel eigenvector with zeros in any k positions corresponding to core vertices.

Let the k chosen core vertices be labelled first, followed by the rest of the core vertices and ending with the core-forbidden vertices. Let $\mathbf{M}$ be the matrix whose rows are the vectors of a basis for the nullspace of G, ordered so that the ith entry in the ith row vector is non-zero. The only non-zero entries lie in the first $|\mathrm{CV}|$ columns. Row-reducing $\mathbf{M}$ to echelon form, with all entries in the columns above and below a leading 1 being zero, produces a matrix $\mathbf{M}'$, whose rows give a new basis for the nullspace of G. Thus each row of $\mathbf{M}'$ is non-zero. Furthermore, the pivots in the rows from the $(k+1)$th up to the last row have at least k zero entries preceding them. Thus the kernel eigenvectors represented by each of the last $\eta(G) - k$ rows of $\mathbf{M}'$ satisfy the conditions of the required result. □

Lemma 4.2 guarantees a kernel eigenvector with zero entries in *any* $\eta(G) - 1$ specified positions. In the proof, the rows of $\mathbf{M}'$ form a basis that can determine the minimum weight sequence and one of a possible number of minimal bases. The determination of the minimum rank of a $(0-1)$–adjacency matrix, as τ varies, is regarded as an *extremal* problem. The minimum rank for various values of τ is given in Figure 2. The result that follows holds for minimal bases.

Proposition 4.3. *If* $\mathbf{x} \in B_{min}$*, then* $\mathbf{x}$ *has at least* $\eta(G) - 1$ *zero entries.*

Proof. There exists a basis B of kernel eigenvectors which are the rows of the $\eta \times n$ matrix $\mathbf{M}'$, with all entries in the columns above and below a pivot being zero. Recall that the set of vertices corresponding to the pivots represents a singular–configuration–vertex–representation. Thus any row of $\mathbf{M}'$ has at least $|\eta(G)| - 1$ zero entries corresponding to all the other rows of $\mathbf{M}'$. By Theorem 3.2, the weight-sequence of B_{min} is entry-wise less than that of B. Thus if $\mathbf{x} \in B_{min}$, then $\mathbf{x}$ has at least $\eta(G) - 1$ zero entries. □

By Proposition 3.2, among all (ordered) bases for the nullspace of $\mathbf{G}$, a minimal basis has the maximum number of zeros in each vector in turn. The minimum number of zeros in the vectors of a minimal basis is used to establish a method of obtaining an upperbound for the nullity.

Proposition 4.4. *If the number of zero entries in a vector* $\mathbf{x} \in B_{min}$ *is less than* k*, then* $\eta(G) \leq k$*.*

Proof. By Proposition 4.3, all the vectors in B_{min} have $\eta - 1$ or more zero entries so that $k - 1 \geq \eta - 1$, as required. □

Note that Proposition 4.4 can also be proved by using Lemma 4.2 directly. This guarantees a vector $\mathbf{y}$ in the nullspace with at least k zeros if $\eta(G) > k$. So suppose that $\eta(G) > k$ and the number of zero entries in each $\mathbf{x} \in B_{min}$ is $k-1$ or less. Since $\mathbf{y}$ is a linear combination of the vectors in B_{min}, by Theorem 3.2, it cannot have more than $k-1$ zero entries, contradicting the necessary condition of Lemma 4.2. Hence $\eta(G) \leq k$.

To determine the minimum number of zeros over all $\mathbf{x} \in B_{min}$, it suffices to determine the core-width τ, an invariant of the graph. Figure 2 shows all the cores of order up to six and the minimum rank of graphs with a min-max core-order (or core-width) τ.

We now show that τ and η exert mutual control.

Proposition 4.5. *For a graph* G *on* n *vertices of nullity* η *and core width* τ*,* $\tau + \eta \leq n + 1$*.*

Proof. By definition of core-width τ, there exists $\mathbf{y} \in B_{min}$ having τ non-zero entries. If z denotes the number of zero entries in $\mathbf{y}$, then $\tau + z = n$. By Proposition 4.3, $z \geq \eta(G) - 1$. Thus $\tau + \eta(G) - 1 \leq n$, as required. □

We may ask what the threshold order of a singular graph G needs to be for a particular core F to lie in a fundamental system $\mathcal{F}$ of cores.

Corollary 4.6. *A singular graph G of nullity η cannot have a core F_t of order t in $\mathcal{F}$ if $t > n + 1 - \eta$.*

Definition 4.7. Let the order of a singular graph G be n. If the nullity is η and the core width is τ, then G is said to be **extremal singular** if $\eta + \tau$ reaches $n + 1$.

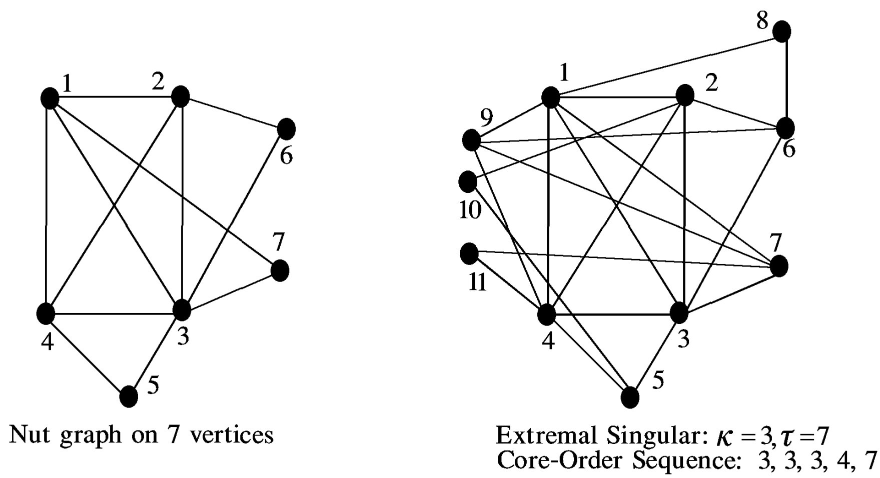

Figure 4: The nut graph 7b and a extremal singular graph of nullity five.

To determine for which graphs $\tau + \eta$ reaches the upper bound $n + 1$, we need to consider as large a core width τ as is possible, for a given core–order n and nullity η. Figure 4 shows a core graph which is extremal singular with the nut graph N_{7b} in a fundamental system of cores.

Proposition 4.8. *A graph G is extremal singular of nullity η, if and only if it is a core graph and the largest core in a fundamental system is a nut graph N and there are exactly $\eta - 1$ vertices of G not on N.*

Proof. Let $\tau + \eta = n + 1$ and $\mathbf{x} \in B_{min}$ have τ non-zero entries representing core F_τ. If z denotes the number of zero entries in $\mathbf{x}$, then $\tau + z = n$ and z needs to be $\eta - 1$. As in the proof of Proposition 4.3, there exists a subset L consisting of $\eta - 1$ vertices in a singular–configuration–vertex–representation representing the cores for B_{min} other than F_τ, such that at the vertex positions of L, the entries of $\mathbf{x}$ are zero, but the entries of the other $\eta - 1$ vectors in B_{min} are collectively non-zero. Then corresponding to each vertex of G, there is a kernel eigenvector with a non-zero entry in the associated position. Hence each vertex of G is a core vertex. Now deleting L destroys exactly $\eta - 1$ cores of G leaving a subgraph H of G of order τ with core F_τ whose kernel eigenvector has τ non-zero entries. Thus G is a core graph and H is a nut graph whose kernel eigenvector is the non-zero part of $\mathbf{x}$.

Conversely, if in a core graph G, the core corresponding to the core–width τ is a nut graph N, then $\tau + \eta(N) = |N| + 1$. By Proposition 3.8, as each of the $\eta - 1$ vertices is added to N to produce G, the graph proceeds through a series of core graphs. Both sides of the equation increases by one at each stage. Thus $\tau + \eta(G) = |G| + 1$. □

Proposition 4.9. *A graph G is extremal singular of nullity one if and only if G is a nut graph.*

Proof. Recall that a nut graph is a core graph of nullity one. Thus $\tau = n$ and $\tau + \eta$ reaches the upper bound. Now for a graph with nullity one which is not a nut graph, there exist core-forbidden vertices so that $\tau < n$. □

Thus a nut graph of order n_1 is extremal singular and may be grown into larger extremal singular graphs with core width n_1.

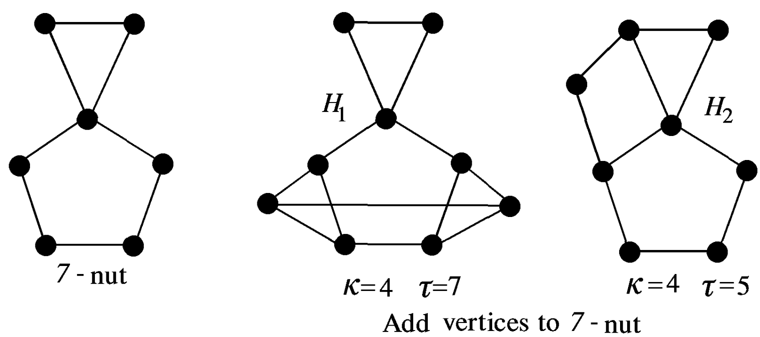

Figure 5: Non–extremal core graphs H_1 and H_2 of nullity two - with the nut graph N_{7a} as an induced subgraph.

When starting from a nut graph $(N, \mathbf{x}_N)$, to construct a singular graph of nullity η, in such a way that the nullity increases with each vertex addition, the vector $\mathbf{x}_N$ need not remain in a minimal basis. Indeed, let us start with a nut graph N and add vertices to produce a core graph. If with each vertex added, a core graph is created while $(N, \mathbf{x}_N)$ is preserved, then the nullity increases by one with each vertex addition and the equality $|N| + \eta(G) = n + 1$ holds. If the core–width $\tau(G) = |N|$, then G is extremal singular. However, there are two other possibilities that might occur in the process of vertex addition until the n-vertex core graph G is produced. With some vertex addition, either a core is not created or the core $(N, \mathbf{x}_N)$ is not preserved in a basis B_{min} for the enlarged graph G. In the former case the nullity does not reach the maximum possible for n and τ, whereas in the latter case $\mathbf{x}_N$ does not remain the non–zero part of a vector in a basis B_{min} for the nullspace of the adjacency matrix of the enlarged graph obtained. If either of these cases occurs, even though the nut graph N remains an induced subgraph and $\mathbf{x}_N$ is still a kernel eigenvector of the enlarged graph, $\tau + \eta < n + 1$, and the core graph obtained is not extremal singular. The graphs in Figure 5 illustrate these two cases. The nine-vertex graph H_1 has $\kappa = 4$ and $\tau = 7$, with F_τ being the nut graph N_{7a} (the smallest nut graph possible [10]). The nullity is two, however, not three as required for $\tau+\eta = n+1$, because the nullity did not increase on adding the first vertex to the nut graph N_{7a}. Note that the eight vertex graph H_2, in Figure 5, has nullity two as well. Although the nut graph N_{7a} is an induced subgraph and a core of H_2, τ is not preserved. Thus the nut graph N_{7a} fails to remain in a Fundamental System of cores (determined by a $B_{min}(H_2)$) after the eighth vertex is added.

Proposition 4.10. *Let G be a singular graph of nullity η, having a fundamental system $\mathcal{F}$ of cores. Then* $\max\limits_{F\in\mathcal{F}}(|F| + \eta(F)) + \eta(G) \leq n + 2$.

Proof. Let $F \in \mathcal{F}$ have kernel eigenvector $\mathbf{x}_F$. By Proposition 3.7, there exists a singular configuration H of order $|F| + \eta(F) - 1$ with core $(F, \mathbf{x}_F)$ as an induced subgraph of G. By interlacing, G has at least $|H| + \eta(G) - 1$ vertices. Hence $n \geq |F| + \eta(F) + \eta(G) - 2$ for all $F \in \mathcal{F}$. □

A core in a fundamental system may be a nut graph in which case it is both a core of G and an induced singular configuration of G. If G is extremal singular, then by Proposition 4.8, a core of maximum order τ in $\mathcal{F}$ is a nut graph. Recall that any core F, in a fundamental system $\mathcal{F}$ of cores of a graph G is an induced subgraph of a singular configuration H which is in turn an induced subgraph of G. Now by definition, $|F| \leq \tau$ but is $|H| \leq \tau$? This is extremely pertinent when a core in a fundamental system $\mathcal{F}$ of cores of a graph G is rK_1. In this case, the order, $2|F| - 1$, of H is relatively large when compared to $|F|$, in contrast with the case when F is a nut graph and $|H| = |F|$.

Lemma 4.11. *Let G be an extremal singular graph with core width τ and nullity η. Let F be a core in a fundamental system of cores of G. Then*
$\tau \geq |F| + \eta(F) - 1$.

Proof. Since G is extremal singular, $\eta(G) + \tau = n+1$. By Proposition 4.10, $|F| + \eta(F) - 1 \leq n + 1 - \eta(G) = \tau$. □

Proposition 4.12. *Let G be an extremal singular graph of nullity η and core–width τ. Let F be a core having a fundamental system $\mathcal{F}$ of cores. If H is an induced subgraph of G which is a singular configuration with core F, then $|H| \leq \tau$.*

Proof. Let H be a singular configuration which is an induced subgraph of G corresponding to the core F. By Lemma 4.11, $|H| = |F| + \eta(F) - 1 \ \leq \tau$. □

Example 4.13. The singular configurations of the extremal singular graph of Figure 4, associated with the cores $3K_1, 3K_1, 3K_1, C_4,$ and the nut graph N_{7a} in a fundamental system, have order $5, 5, 5, 5, 7$ respectively, supporting Proposition 4.12.

The graphs H_1 and H_2 of Figure 5 are not extremal singular. The graph H_1 has core–order sequence $\{4, 7\}$ corresponding to cores C_4 and the nut graph N_{7a}, associated to singular configurations of order 5 and 7 respectively. The graph H_2 has core–order sequence $\{4, 5\}$ corresponding to cores $4K_1$ and $C_4\dot{\cup}K_1$, associated to singular configurations of order 5 and 7 respectively.

Corollary 4.14. *If core rK_1 is in fundamental system of cores of an extremal singular graph, then $r \leq \lceil \frac{\tau}{2} \rceil$.*

Proof. A singular configuration H with core rK_1 has $(2r-1)$ vertices. Then $|H| = 2r-1$. By Proposition 4.12, $|H| \leq \tau$. Hence the integer $r \leq \frac{\tau+1}{2}$. $\square$

5 Conclusion

The symmetry of the $(0-1)$–adjacency matrix $\mathbf{G}$ of rank r enables its representation as a block matrix with a maximally non-singular $(r \times r)$–principal submatrix of $\mathbf{G}$. The nullspace of $\mathbf{G}$ can be expressed in terms of submatrices of this representation.

The presence of singular configurations as induced subgraphs in a graph is a necessary condition for a graph to be singular. In a graph G of nullity η, we showed that there is a vertex representation $\mathcal{R}$ consisting of η vertices corresponding to η singular configurations which are induced subgraphs of G. The choice of the vertices in $\mathcal{R}$, which are core vertices, is not unique. A graph may have core-forbidden vertices that do not lie on any core of G. Deletion of a core-forbidden vertex either increases the nullity or leaves it unchanged.

The core-width τ (maximum weight of the vectors in a minimal basis for the nullspace of G) is an invariant of G and was shown to have extremal properties tied with the nullity η in a n-vertex graph. Among all graphs on n vertices, $\tau + \eta$ reaches a maximum in extremal singular graphs. Not only are the core–orders in a fundamental system $\mathcal{F}$ of cores bounded above by τ but in extremal singular graphs, the orders of the singular configurations corresponding to the cores of $\mathcal{F}$ are also bounded above by τ.

References

[1] AIM Minimum Rank – Special Graphs Work Group, Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.* **428** (2008), 1628–1648.

[2] M. Brown, J. W. Kennedy and B. Servatius, Graph singularity, *Graph Theory Notes of New York*, The New York Academy of Sciences **25** (1993), 23–32.

[3] A. J. Schwenk and R. J. Wilson, On the eigenvalues of a graph, in: L. W. Beineke and R. J. Wilson (eds.), *Selected Topics in Graph Theory*, Academic Press, London, 1978, 307–336.

[4] I. Sciriha, On the coefficient of λ in the characteristic polynomial of singular graphs, *Util. Math.* **52** (1997), 97–111.

[5] I. Sciriha, On the construction of graphs of nullity one, *Discrete Math.* **181** (1998), 193–211.

[6] I. Sciriha, On the rank of graphs, in: Y. Alavi, D. R. Lick and A. Schwenk (eds.), *Combinatorics, graph theory and algorithms* **2**, New Issues Press, Kalamazoo, Michigan, 1999, 769–778.

[7] I. Sciriha, A characterization of singular graphs, *Electron. J. Linear Algebra* **16** (2007), 451–462.

[8] I. Sciriha, S. Fiorini and J. Lauri, A minimal basis for a vector space, *Graph Theory Notes of New York*, The New York Academy of Sciences **31** (1996), 21–24.

[9] I. Sciriha and I. Gutman, Graphs with maximum singularity, *Graph Theory Notes of New York*, The New York Academy of Sciences **30** (1996), 17–20.

[10] I. Sciriha and I. Gutman, Nut graphs: maximally extending cores, *Util. Math.* **54** (1998), 257–272.