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Low-lying states in the Y-string three-quark potential*

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Abstract. We give a brief summary of our study of low-lying states in the Y-string three-quark potential [1], currently under review. We study the masses of various three-quark SU(6) multiplets in the N=0,1,2 shells, confined by the Y-string three-quark potential, at four levels of approximation with increasing accuracy. We show the general trend of convergence of these four approximations.

1 Introduction

The so-called Y-junction string three-quark potential, defined by

$$V_Y = \sigma \min_{\mathbf{x}_0} \sum_{i=1}^{3} |\mathbf{x}_i - \mathbf{x}_0|. \tag{1}$$

has long been advertised [2–7] as a natural description of the flux tube confinement mechanism, that is allegedly active in QCD. Lattice investigations, Refs. [8–10] contradict each other, however, in their attempts to distinguish between the Y-string, Fig. 1, and the Δ-string potential, see Fig. 2, which is indistinguishable from the sum of three linear two-body potentials. One may therefore view the present lattice results as inconclusive and await the next generation of calculations. Yet, one would wish to resolve this dilemma on a purely theoretical basis: do these two kinds of string potentials predict sufficiently different baryon spectra that can be differentiated by experiment? At this time one must use the quark model in order to try and resolve this dilemma.

$$V_\Delta = \sigma \sum_{i<j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|, \tag{2}$$

Over the past 25 years, the Y-string potential has been used in several studies of baryons in the (constituent) quark model with various hyperfine interactions [4–6,11], and yet some of the most basic predictions of this potential, such as its influence on the splitting of the low-lying three-quark states have remained widely

* Talk presented by V. Dmitrašinović

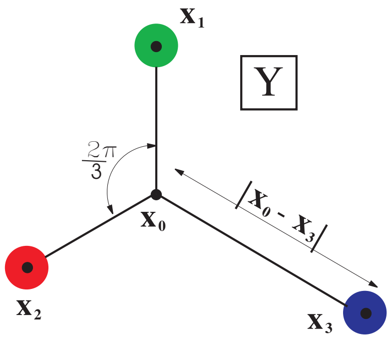

Fig.1. Three-quark Y-junction string potential.

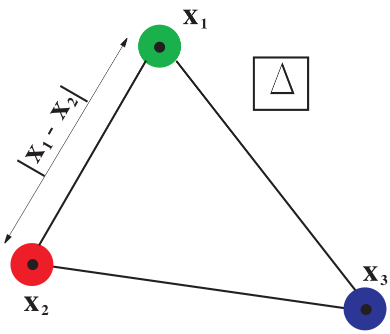

Fig.2. Three-quark Δ-shape string potential.

unknown. That has to do with the technical complications in the implementation of the potential Eq. (1), that can best be seen when expressed in terms of the three-body Jacobi coordinates $\boldsymbol{\rho}, \boldsymbol{\lambda}$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2), \tag{3}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3), \tag{4}$$

as follows. The exact string potential Eq. (1) consists of the so-called Y-string, or three-string term,

$$V_{\text{string}} = V_Y = \sigma\sqrt{\frac{3}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho} \times \boldsymbol{\lambda}|)}, \tag{5}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho} \cdot \boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho} \cdot \boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho} \cdot \boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho} \cdot \boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho} \cdot \boldsymbol{\lambda})^2} \, . \end{cases}$$

and three angle-dependent two-part string, or the so-called V-string, terms,

$$V_{\rm string} = \sigma \left(\sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} \right) \tag{6}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \leq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases}$$

$$V_{\rm string} = \sigma \left(\sqrt{2}\rho + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} \right) \tag{7}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases}$$

$$V_{\rm string} = \sigma \left(\sqrt{2}\rho + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} \right) \tag{8}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases} .$$

Here, the reasons for the lack of use of the exact potential Eq. (1) become apparent: i) it is a genuine three-body operator with a complicated and unusual ("area term") angular dependence under the square-root of the most important term (the Y-junction string potential) that leads to the non-conservation of the individual Jacobi relative coordinates' angular momenta and hugely complicates the equations of motion; ii) the square-roots appearing in all four functional forms of the potential make this task even more difficult; iii) the presence of four different functional forms of the potential depending on the configuration space angles makes the integration of the equations of motion difficult as one cannot easily separate the angular and radial integrals.

In Sect. 2 we give a summary of how we address the above three problems. A summary of our results is shown in Sect. 3. The final Section 4 contains a discussion of our results.

2 Approximations

First we address the above three problems: first, we deal with the angular momentum recoupling algebra necessary to deal with the non-conserved "partial" angular momenta; second we deal with the square root(s) in the Y-string potential, and third we address all four forms of the string potential together.

Perhaps the most common, and the simplest approximation to the exact string potential Eq. (1) is the Y-string, or the three-string potential, Eq. (5), that is used in all geometries, i.e., even when one of the angles exceeds 120°. In that way one avoids the cumbersome transition to the V-string potentials, see problem iii) above. Still, even this simplified approximation suffers from two difficulties mentioned above: i) an unusual ("area term") angular dependence under the square-root that leads to the non-conservation of the individual Jacobi coordinates' angular momenta; ii) the square-root. We shall address these problems in successive steps: i) the area term turns out to be exactly (analytically) integrable in the harmonic oscillator basis, that boils down to some (complicated) angular momenta recoupling algebra and the value of a particular one-dimensional integral. Problem ii), the overall square root, can be treated, at first, by a series expansion, i.e. in perturbation theory, using the angular algebra solution to problem i) and the numerical evaluation of the remaining one-dimensional integral. It turns out that the crucial ingredient for the solution of this problem is the application of the so-called hyper-spherical coordinates/angles [12], more specifically the cosines of the relative angle θ between the Jacobi coordinates $\boldsymbol{\rho}, \boldsymbol{\lambda}$ and of the hyper-angle 2χ defined by the ratio of the moduli ρ, λ of the two Jacobi coordinates. Finally, the third issue iii) (the presence of four different functional forms of the potential) can be tackled as well, by numerically evaluating a two-dimensional integral in a restricted region of the $\chi - \theta$ sub-hyper-space.

3 Summary of Results

In the following we summarize the results of our study of the low-lying parts of the energy spectra of three quarks confined by a pure Y-string potential, without two-quark potentials, in four different approximations: 1) the area-dependent part of the three-string potential as the first order perturbation of the harmonic oscillator; 2) the area-dependent part of the three-string potential as the first order perturbation of the non-harmonic linear potential, i.e. the (approximate) three-string potential expanded up to the first power in hyper-spherical angles; 3) the (approximate) three-string potential to all orders in power expansion in hyper-spherical harmonics, without taking into account the transition(s) to two-string potentials; 4) the exact minimal-length string potential to all orders in power expansion in hyper-spherical harmonics, while taking into account the transition(s) to two-string potentials. Our results are shown in Table 1 and Fig. 3.

An attractive Y-string potential always splits the N=K=2 band states into degenerate $SU_{FS}(6)$ multiplets: $[20, 1^+], [70, 2^+], [56, 2^+], [70, 0^+]$, (in order of descending mass) following approximately the Bowler and Tynemouth (BT) [13] separation rule of 2:2:1. This rule is obeyed by approximations 1) and 2) exactly and by approximation 3) at the level of one per cent corrections. The exact result 4) leads to the 2.25:2.18:1 splitting, i.e. the largest violation of the BT rule is less than 13%.

The mass difference between the first (hyper-) radial excitation of the ground state, that is the "Roper multiplet" $[56', 0^+]$, and the odd-parity K=N=1 $[70, 1^-]$ multiplet is entirely determined by the difference between and the first (hyper-) angular and the first (hyper-) radial excitation eigen-energies in a linearly rising hyper-radial potential, which is always negative.

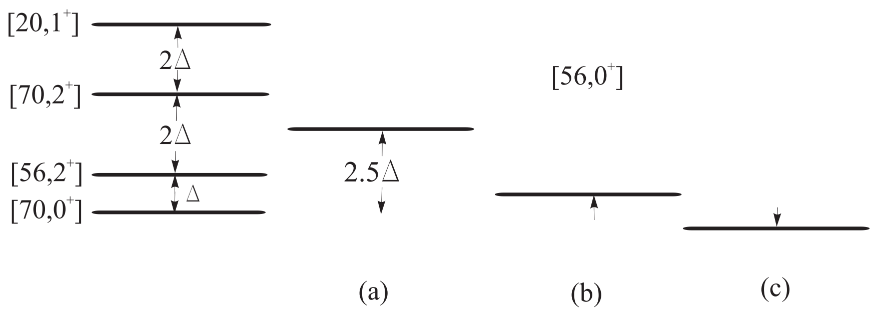

Fig.3. Depiction of the energy splitting of the K = 2 states of the hyper-spherical linear potential spectrum due to attractive three-body potentials: (a) the first-order perturbation approximation to the "harmonic Y-string" potential; (b) the first-order perturbation approximation to the first-power expansion in hyper-spherical coordinates of the "three-string" potential (see text for definition); (c) non-perturbative results of the "three-string" potential solved in hyper-spherical coordinates. The left-hand side of the diagram involving the $[20,1^+]$, $[70,2^+]$, $[56,2^+]$, $[70,0^+]$ multiplets is common to both kinds of potentials, follows the Bowler-Tynemouth rule 2:2:1 to 1%; only the position of the $[56,0^+]$ multiplet (containing the Roper resonance) is variable.

In other words, the Roper resonance cannot be lowered below the odd-parity K=N=1 states, irrespective of the string tension constant and the quark masses, which are the only free parameters in this theory. Consequently, the energy spectrum pattern can not be improved, as compared with the desired/experimental one, by a straightforward application of the Y-string three-body potential.

Table 1. The eigen-values of the unperturbed (solution to the hyper-central approximation, see the text) energy $E^{(0)}_{N_K,K}$, and the two perturbative ($E^{(1)}_{N_K,K,L}$, $E^{(2)}_{N_K,K,L}$), and two non-perturbative ($E^{(3)}_{N_K,K,L}$, $E^{(4)}_{N_K,K,L}$) approximations, where the last one $E^{(4)}_{K}$; is the exact (numerical) result, for the various low-lying K = 0,1,2 states (with all allowed orbital waves L).

K	N_K	$[SU(6), L^P]$	$E^{(0)}_{N_K,K}$	$E^{(1)}_{N_K,K,L}$	$E^{(2)}_{N_K,K,L}$	$E^{(3)}_{N_K,K,L}$	$E^{(4)}_{N_K,K,L}$
0	0	$[56, 0^+]$	3.8175	4.0000	4.6658	4.5182	4.5218
1	1	$[70, 1^-]$	4.6582	5.3333	5.6934	5.5132	5.5176
0	1	$[56, 0^+]$	5.2630	6.6667	6.4326	6.2290	6.2340
2	2	$[70, 0^+]$	5.4290	6.3333	6.3942	6.2493	6.2665
2	2	$[56, 2^+]$	5.4290	6.4667	6.4907	6.3199	6.3279
2	2	$[70, 2^+]$	5.4290	6.7333	6.6837	6.4604	6.4617
2	2	$[20, 1^+]$	5.4290	7.0000	6.8767	6.5993	6.5999

4 Discussion

We have examined the qualitative and quantitative features of the energy spectrum in the Y-string three-quark potential and its differences from the two-body, or the Δ-shaped string one. For this purpose we have studied the low-lying states of the three-quark system (the "baryon") in the Y-string potential (the Δ-string potential had been studied in Ref. [14]). It turns out that the three lowest-lying bands of states that form the (only) set of well-established ("four-star") resonances, do *not* as yet allow a clear distinction to be made between these two types of potentials: there are too few states in these shells, and their wave functions are (tightly) constrained by the permutation symmetry. This is a bit of a surprise, as these two string potentials have (very) different functional forms, which we expected to predict different physics:

So, it turns out that there is only one possible clue to the form of the confining potential in the low-lying baryon resonance spectrum, *viz.* the Roper resonance's (abnormally low) mass, that perhaps could be used to draw conclusions about the existence and/or preponderance of one kind of potential over the other. We have shown, however, that the Y-shaped string always leads to a Roper resonance that is heavier than the lowest-lying odd-parity resonance. This does not mean that the spectra of the Y- and the two-body potentials are identical, rather, it means that one must go to the higher lying bands, and in particular to higher orbital angular momentum states, in order to see the difference between the two.

A detailed study of the possible interference of the two- and three-body potentials on the position of the first hyper-radial excitation (the "Roper resonance") remains a task for the future. Moreover, the behavior of the Y-string in higher orbital angular momentum states remains another place to look for the differences from the Δ-string.

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