Partial-dual Euler-genus distributions for bouquets with small Euler genus*

Yan Yang [†] ⓘ

School of Mathematics, Tianjin University, Tianjin, China

Xiaoya Zha ⓘ

Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, USA

Abstract

For an arbitrary ribbon graph G, the partial-dual Euler-genus polynomial of G is a generating function that enumerates partial duals of G by Euler genus. When G is an orientable ribbon graph, the partial-dual orientable genus polynomial of G is a generating function that enumerates partial duals of G by orientable genus. Gross, Mansour, and Tucker inaugurated these partial-dual Euler-genus and orientable genus distribution problems in 2020. A bouquet is a one-vertex ribbon graph. Given a ribbon graph G, its partial-dual Euler-genus polynomial is the same as that of some bouquet; this motivates our focus on bouquets. We obtain the partial-dual Euler-genus polynomials for all the bouquets with Euler genus at most two.

Keywords: Ribbon graph, partial dual, Euler-genus polynomial, orientable genus polynomial.

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1 Introduction

Ribbon graphs are well-known to be equivalent to 2-cell embeddings of graphs. Let G be a ribbon graph, $V(G)$ and $E(G)$ denote its vertex-disk set and edge-ribbon (or simply ribbon) set, respectively. In 2009, Chmutov [1] defined the partial dual of G when he was studying signed Bollobás–Riordan polynomials. For any $A \subseteq E(G)$, the partial dual ribbon graph with respect to A is denoted by G^A. As a generalization of the geometric duality, partial

duality has developed into a topic of independent interest for its applications in graph theory and topology. The reader is referred to [1, 3] for more background about ribbon graphs and partial duals.

Given a ribbon graph G, there are $2^{|E(G)|}$ partial duals of G in total, the problem of the enumeration of its partial duals G^A according to their Euler-genus $\varepsilon(G^A)$ or orientable genus $\gamma(G^A)$ was inaugurated by Gross, Mansour, and Tucker [5] in 2020. They defined the partial-dual Euler-genus (and orientable genus) polynomial of a ribbon graph G, which was motivated by the Euler-genus (and orientable genus) polynomial [4] of a graph.

Definition 1.1 ([5])**.** The *partial-dual Euler-genus polynomial* of a ribbon graph G is the generating function

$$^\partial\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)}$$

that enumerates partial duals by Euler genus. The *partial-dual orientable genus polynomial* of a ribbon graph G is the generating function

$$^\partial\gamma_G(z) = \sum_{A \subseteq E(G)} z^{\gamma(G^A)}$$

that enumerates partial duals by orientable genus.

In [5], the authors discussed some properties of each polynomial, gave a recursion for subdivision of an edge, and obtained the partial-dual Euler-genus (or orientable genus) polynomials for some infinite families of ribbon graphs. In [8], Yan and Jin found some bouquets having a non-constant partial-dual orientable genus polynomial with only one non-zero coefficient, which disproved one of conjectures provided in [5], they also obtained the partial-dual Euler-genus polynomials for all bouquets with the number of loops at most 3 and the partial-dual orientable genus polynomials for all bouquets with the number of loops at most 4. They also found two infinite families of counterexamples to another conjecture provided in [5] about the interpolating property of partial-dual Euler-genus polynomials of non-orientable ribbon graphs in [10]. Then Chmutov and Vignes-Tourneret [2] and Yan and Jin [9] proved that the family of counterexamples given in [8] are the only counterexamples for that conjecture, independently.

A *bouquet* is a one-vertex ribbon graph, we denote the bouquet with n loops by B_n. The bouquets with orientable genus 0 and 1 are called *plane bouquets* and *toroidal bouquets*, respectively; the bouquets with nonorientable genus 1 and 2 are called *projective planar bouquets* and *Klein bottle bouquets*, respectively. In this paper we focus on the partial-dual Euler-genus polynomial for these bouquets. The following two lemmas motivate our interest in bouquets.

Lemma 1.2 ([8])**.** *If G is a ribbon graph and $A \subseteq E(G)$, then $^\partial\varepsilon_G(z) = {}^\partial\varepsilon_{G^A}(z)$. When G is orientable, $^\partial\gamma_G(z) = {}^\partial\gamma_{G^A}(z)$.*

Lemma 1.3 ([3, 5])**.** *If A is a spanning tree for G, then the partial dual G^A has only one vertex.*

From Lemmas 1.2 and 1.3, given a ribbon graph G, its partial-dual Euler-genus polynomial and partial-dual orientable genus polynomial when G is orientable will be the same as that of some bouquet. From this point of view, for the partial-dual Euler-genus (and orientable genus) distribution problems, we only need to focus on bouquets.

Given a bouquet B_n, we label each loop by distinct letter. By reading these letters we meet when travelling around the boundary of the vertex-disk of B_n, we obtain a cyclic order of $2n$ letters, call it the *rotation* of B_n. A loop in B_n is called a *twisted loop* if its neighbourhood is homeomorphic to a Möbius band, and *untwisted loop* if its neighbourhood is homeomorphic to an annulus. To distinguish between untwisted loops and twisted loops, we give the same signs $+$ to the two same letters which correspond to a common untwisted loop, and give two different signs (one $+$, the other $-$) to the two same letters which correspond to a common twisted loop. For simplicity, we usually omit the sign $+$. We call the cyclic order of $2n$ signed letters the *signed rotation* of B_n. It is easy to check that there is a 1-to-1 correspondence between the signed rotations and bouquets.

Two loops a and b in a bouquet are *interlaced* if the rotation of this bouquet is of form $a \cdots b \cdots a \cdots b \cdots$, otherwise *parallel*. A loop e in a bouquet is *trivial* if e is untwisted and not interlaced with any other loops; otherwise *nontrivial*. Notice that, all the twisted loops are nontrivial in this paper, which is different with that in [8] (where a loop in a bouquet is trivial if it is not interlaced with any other loops, so both trivial untwisted loops and trivial twisted loops may exist). An *essential bouquet* is a bouquet whose loops are all nontrivial. A ribbon graph H is a *ribbon subgraph* of G if H can be obtained by deleting vertices and ribbons of G. Given a bouquet B, by deleting all trivial loops, we obtain an essential bouquet B' which is a ribbon subgraph of B, and we call B' the *maximum essential subgraph* of B.

The *join* of two disjoint ribbon graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a ribbon graph which can be obtained by the following way. Firstly we choose an arc p_1 on the boundary of a vertex-disk of G_1 that lies between two consecutive ribbon ends, and choose another such arc p_2 on the boundary of a vertex-disk of G_2. Next by identifying the arcs p_1 and p_2, we merge the two vertex-disks from G_1 and G_2 into one new vertex-disk, the ribbon graph obtained is the graph $G_1 \vee G_2$. For the partial-dual Euler-genus polynomial of the graph $G_1 \vee G_2$, the following result had been obtained by Gross, Mansour, and Tucker.

Lemma 1.4 ([5])**.** *If G_1 and G_2 are disjoint ribbon graphs, then*

$$^{\partial}\varepsilon_{G_1 \vee G_2}(z) = {}^{\partial}\varepsilon_{G_1}(z) \cdot {}^{\partial}\varepsilon_{G_2}(z).$$

For a bouquet, it can be obtained by the join of its maximum essential subgraph with each trivial loops successively. From Lemma 1.4, the following result can be deduced.

Lemma 1.5 ([8])**.** *Let G be a bouquet and G' be the maximum essential subgraph of G. If $|E(G)| - |E(G')| = i$, then*
$$^{\partial}\varepsilon_G(z) = 2^i \cdot {}^{\partial}\varepsilon_{G'}(z).$$

From Lemma 1.5, we can reduce the problem of partial-dual Euler-genus polynomial of a bouquet to that of its maximum essential subgraph. We also note that the Euler genus of a bouquet is equal to that of its maximum essential subgraph, due to the additivity of Euler genus over the join operation.

If $A \subseteq E(G)$, the *induced ribbon subgraph* $G|_A$ is a ribbon subgraph of G whose ribbon set is A and whose the vertex-disk set consists of all ends of ribbons in A. We use $A^c := E(G) \setminus A$ to denote the complement of $A \subseteq E(G)$. By the following lemma, the Euler genus of a partial dual of a bouquet can be computed from the Euler genus of its two ribbon subgraphs.

Lemma 1.6 ([5])**.** *If G is a bouquet and $A \subseteq E(G)$, then*

$$\varepsilon(G^A) = \varepsilon(G|_A) + \varepsilon(G|_{A^c}).$$

We note that $\varepsilon(G) = 2\gamma(G)$ for orientable ribbon graphs, the following two lemmas can be deduced easily.

Lemma 1.7 ([5])**.** *If G is an orientable bouquet and $A \subseteq E(G)$, then*

$$\gamma(G^A) = \gamma(G|_A) + \gamma(G|_{A^c}).$$

Lemma 1.8 ([5])**.** *If G is an orientable ribbon graph, then $^{\partial}\gamma_G(z) = {}^{\partial}\varepsilon_G(z^2)$.*

In this paper we deduce the forms of the signed rotations of maximum essential subgraphs for all projective planar bouquets, Klein bottle bouquets, plane bouquets, and toroidal bouquets, respectively, and obtain the partial-dual Euler-genus polynomials for these bouquets.

2 Partial-dual Euler-genus polynomials of projective planar bouquets and Klein bottle bouquets

Recall that a ribbon graph is equivalent to a 2-cell embedding of a graph. From the Proposition 4.1.5 in [6], we get the following fact easily.

Lemma 2.1 ([6])**.** *If H is a subgraph of a ribbon graph G, then $\varepsilon(H) \leq \varepsilon(G)$.*

Lemma 2.2. *The bouquet B_n is a plane bouquet if and only if all of its loops are trivial.*

Proof. This lemma follows from the definition of trivial loops in B_n. □

Lemma 2.3. *For a nonorientable bouquet B_n,*

  (i) *if there exists a pair of parallel twisted loops, then $\varepsilon(B_n) \geq 2$;*

 (ii) *if there exists a pair of interlaced loops, in which one is twisted and the other one is untwisted, then $\varepsilon(B_n) \geq 2$;*

(iii) *if there exists a pair of interlaced untwisted loops, then $\varepsilon(B_n) \geq 3$.*

Proof. For (i) and (ii), the bouquet B_n has ribbon subgraphs B_2 and B'_2, as shown in Figure 1(a) and (b), respectively. It is easy to check that $\varepsilon(B_2) = \varepsilon(B'_2) = 2$, then $\varepsilon(B_n) \geq 2$ by Lemma 2.1.

For (iii), there exists a ribbon subgraph with three ribbons e_1, e_2 and e_3 in which e_1 and e_2 are interlaced untwisted loops and e_3 is a twisted loop (for any nonorientable bouquet, a twisted loop must exist). The loop e_3 can be interlaced with the other i ($0 \leq i \leq 2$) loops in this ribbon subgraph, by a case analysis, there are three nonequivalent such ribbon subgraphs B_3, B'_3 and B''_3, as shown in Figure 1(c)-(e). One can check that $\varepsilon(B_3) = \varepsilon(B'_3) = \varepsilon(B''_3) = 3$, then $\varepsilon(B_n) \geq 3$ follows from Lemma 2.1. □

The following is easy to prove by induction and simple facial walk counting. We present it as a technical lemma for the convenience of our later proof. It in fact reveals some important differences between twisted and untwisted loops in a bouquet.
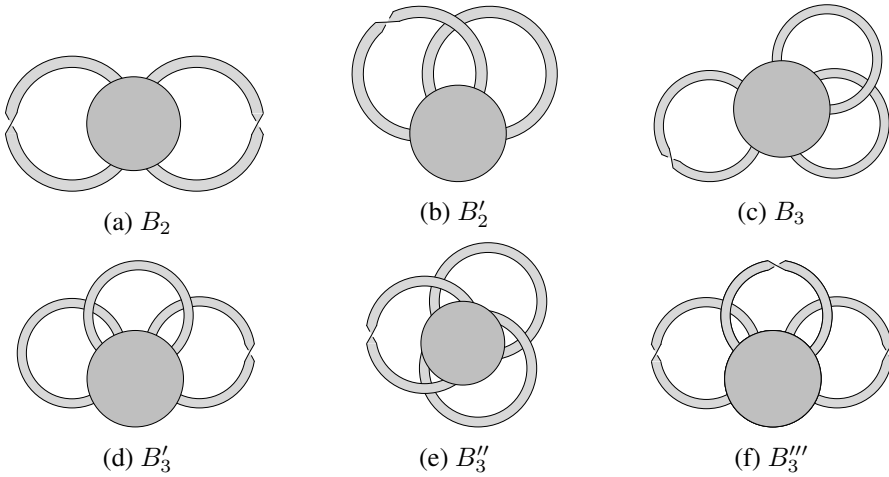
Figure 1: Some bouquets.

Lemma 2.4. *If G is a bouquet and $A \subseteq E(G)$, then*

(i) *If A (A^C, respectively) consists of k untwisted and pairwise parallel loops, then $\varepsilon(G|_A) = 0$ ($\varepsilon(G|_{A^C}) = 0$, respectively);*

(ii) *If A (A^C, respectively) consists of k twisted and pairwise parallel loops, then $\varepsilon(G|_A) = k$ ($\varepsilon(G|_{A^C}) = k$, respectively);*

(iii) *If A (A^C, respectively) consists of k twisted and pairwise interlaced loops, then $\varepsilon(G|_A) = 1$ ($\varepsilon(G|_{A^C}) = 1$, respectively).*

We first consider the partial-dual Euler-genus polynomials for projective planar bouquets.

Lemma 2.5. *If B_n is a projective planar bouquet and B_k is the maximum essential subgraph of B_n, then the signed rotation of B_k has the form*

$$a_1 \ldots a_k(-a_1) \ldots (-a_k), (1 \le k \le n). \tag{1}$$

Proof. Firstly a bouquet with signed rotation (1) has Euler genus 1, from Lemma 2.4(iii). From items (ii) and (iii) in Lemma 2.3, all the untwisted loops in B_n are trivial loops, so all the loops in B_k are twisted loops. From item (i) in Lemma 2.3, any pair of twisted loops in B_k cannot be paralleled to each other, so its signed rotation must have the form (1). The lemma follows. $\qquad\square$

Lemma 2.6. *If G is a bouquet with the signed rotation $a_1 \ldots a_t(-a_1) \ldots (-a_t)$, then*

$$^\partial\varepsilon_G(z) = 2z + (2^t - 2)z^2.$$

Proof. Let $X \subseteq E(G)$ and $x = |X|$. When $x = 0$ or $x = t$, $\varepsilon(G^X) = \varepsilon(G) = 1$. When $0 < x < t$, the signed rotations of both $G|_X$ and $G|_{X^C}$ have the form (1), then

$\varepsilon(G|_X) = \varepsilon(G|_{X^C}) = 1$. By Lemma 1.6, $\varepsilon(G^X) = 2$. Since there are $\binom{t}{x}$ ways to choose the x edges,

$$^\partial \varepsilon_G(z) = 2z + \sum_{x=1}^{t-1} \binom{t}{x} z^2 = 2z + (2^t - 2)z^2. \qquad \square$$

Theorem 2.7. *If B_n is a projective planar bouquet with k $(1 \le k \le n)$ twisted loops, then*

$$^\partial \varepsilon_{B_n}(z) = 2^{n-k}(2z + (2^k - 2)z^2).$$

Proof. From Lemma 2.5, the k twisted loops induce the maximum essential subgraph of B_n whose signed rotation has form (1). Combining it with Lemmas 1.5 and 2.6, the theorem follows. $\qquad \square$

We now consider the partial-dual Euler-genus polynomials for Klein bottle bouquets. Firstly, we deduce the form of the signed rotation of maximum essential subgraphs for Klein bottle bouquets. We use the model of disk sum of two projective planes for the Klein bottle. The fundamental group of Klein bottle is generated by two topologically disjoint orientation reversing loops a and b (called *twisted* in this paper). There are three non-separating loops a, b and d up to homeomorphism, where $d = ab$ is orientation preserving (called *untwisted* in this paper).

Lemma 2.8. *If B_n is a Klein bottle bouquet and B_k is the maximum essential subgraph of B_n, then the signed rotation of B_k has the form*

$$a_1 \ldots a_i d_1 \ldots d_q (-a_1) \ldots (-a_i) b_1 \ldots b_j d_q \ldots d_1 (-b_1) \ldots (-b_j) \qquad (2)$$

in which $k = i + j + q$; and

(i) *$i, j, q \ge 1$, or*

(ii) *$i, j \ge 1$ and $q = 0$, or*

(iii) *$i, q \ge 1$ and $j = 0$, or*

(iv) *$j, q \ge 1$ and $i = 0$.*

Proof. It is routine check that the bouquet whose signed rotation with form (2) has Euler genus two. Notice that Case (iii) and Case (iv) are symmetric and we only need to consider Case (iii). We have the following observations:

(α): From Lemma 2.4(ii), there are no three twisted loops that are paralleled to each other, otherwise the Euler genus of B_k is at least three. And there is no subgraph B_3''' as shown in Figure 1(f), because $\varepsilon(B_3''') = 3$. Let B_k' be the ribbon subgraph induced by all the twisted loops in B_k, the signed rotation of B_k' is of form

$$a_1 \ldots a_i(-a_1) \ldots (-a_i) b_1 \ldots b_j(-b_1) \ldots (-b_j),$$

in which $i + j \ge 1$.

(β): From Lemma 2.3(iii), there exist no pairs of interlaced untwisted loops in a Klein bottle bouquet. Each untwisted loop must be interlaced with some twisted loops, otherwise the untwisted loop is trivial.

(γ): Each nontrivial and untwisted loop must be interlaced with all twisted loops, otherwise there exits a ribbon subgraph $\hat{B}$ whose signed rotation is $dad(-a)b(-b)$, and $\varepsilon(\hat{B}) = 3$.

Combining observations (α), (β), and (γ), the lemma follows. $\qquad \square$

According to the four cases listed in Lemma 2.8, the signed rotation of B_k (the maximum essential subgraph of a Klein bottle bouquet) has one of the following three forms,

$$a_1 \ldots a_i(-a_1) \ldots (-a_i)b_1 \ldots b_j(-b_1) \ldots (-b_j), \tag{3}$$
$$a_1 \ldots a_i d_1 \ldots d_q(-a_1) \ldots (-a_i)d_q \ldots d_1, \tag{4}$$
$$a_1 \ldots a_i d_1 \ldots d_q(-a_1) \ldots (-a_i)b_1 \ldots b_j d_q \ldots d_1(-b_1) \ldots (-b_j) \tag{5}$$

in which $k = i + j$, $i, j \geq 1$ in form (3); $k = i + q$, $i, q \geq 1$ in form (4); and $k = i + j + q$, $i, j, q \geq 1$ in form (5).

Next we compute the partial-dual Euler-genus polynomials for bouquets whose signed rotation has form (3), (4) and (5) respectively. Let $A = \{a_1, \ldots, a_t\}, B = \{b_1, \ldots, b_m\}, D = \{d_1, \ldots, d_s\}, X \subseteq E(G)$ and $x = |X|$.

Lemma 2.9. *If G is a bouquet with the signed rotation*

$$a_1 \ldots a_t(-a_1) \ldots (-a_t)b_1 \ldots b_m(-b_1) \ldots (-b_m), (t, m \geq 1),$$

then

$$^{\partial}\varepsilon_G(z) = 4z^2 + 2(2^t + 2^m - 4)z^3 + (2^t - 2)(2^m - 2)z^4.$$

Proof. Let B_t and B_m be projective planar bouquets whose signed rotation have forms $a_1 \ldots a_t(-a_1) \ldots (-a_t)$ and $b_1 \ldots b_m(-b_1) \ldots (-b_m)$, respectively. Then we have $G = B_t \vee B_m$. From Lemmas 1.4 and 2.6, the result follows. ☐

Lemma 2.10. *If G is a bouquet with the signed rotation*

$$a_1 \ldots a_t d_1 \ldots d_s(-a_1) \ldots (-a_t)d_s \ldots d_1, (t, s \geq 1),$$

then

$$^{\partial}\varepsilon_G(z) = 2z + 2(2^s - 1)z^2 + 2(2^t - 2)z^3 + (2^t - 2)(2^s - 2)z^4.$$

Proof. When $x = 0$ or $t + s$, $\varepsilon(G^X) = \varepsilon(G) = 2$. When $0 < x < t + s$, we have the following three cases.

Case 1: Assume $X \subseteq A$. If $X = A$, then X consists of t twisted and pairwise interlaced loops and X^C consists of s untwisted and pairwise parallel loops. From Lemma 2.4, we have $\varepsilon(G|_X) = 1$ and $\varepsilon(G|_{X^C}) = 0$. Then from Lemma 1.6, we have $\varepsilon(G^X) = 1$.

If $X \subset A$, then the signed rotations of the induced ribbon subgraphs $G|_X$ and $G|_{X^C}$ have form (1) and (4) respectively, so we have $\varepsilon(G|_X) = 1$, $\varepsilon(G|_{X^C}) = 2$ and $\varepsilon(G^X) = 3$. There are $\sum_{x=1}^{t-1} \binom{t}{x} = 2^t - 2$ ways to choose the x edges. Hence in this case, we have one partial dual with Euler genus one, and $2^t - 2$ partial duals with Euler genus three.

Case 2: Assume $X \subseteq D$. The argument is similar to that for Case 1, by using Lemmas 2.4, 2.8 and 1.6, we get $\varepsilon(G|_X)$, $\varepsilon(G|_{X^C})$, and $\varepsilon(G^X)$. Then we calculate the number of ways to choose the x edges, as shown in Table 1.

Table 1: Euler genus distribution in Case 2.

X	$\varepsilon(G|_X)$	$\varepsilon(G|_{X^C})$	$\varepsilon(G^X)$	number
$X = D$	0	1	1	1
$X \subset D$	0	2	2	$2^s - 2$

Table 2: Euler genus distribution in Case 3.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$A \subset X$ and $D \not\subset X$	2	0	2	$2^s - 2$
$D \subset X$ and $A \not\subset X$	2	1	3	$2^t - 2$
$A \not\subset X$ and $D \not\subset X$	2	2	4	$(2^t - 2)(2^s - 2)$

Case 3: Assume $X \cap A \neq \emptyset$ and $X \cap D \neq \emptyset$. We discuss three subcases as shown in Table 2, in which we use Lemmas 2.4, 2.5 and 2.8 to compute $\varepsilon(G\|_X)$ and $\varepsilon(G\|_{X^C})$.

Summarizing the above, we have $^\partial\varepsilon_G(z) = 2z + 2(2^s - 1)z^2 + 2(2^t - 2)z^3 + (2^t - 2)(2^s - 2)z^4$. □

Lemma 2.11. *If G is a bouquet with the signed rotation*

$$a_1 \ldots a_t d_1 \ldots d_s (-a_1) \ldots (-a_t) b_1 \ldots b_m d_s \ldots d_1 (-b_1) \ldots (-b_m), (t, m, s \geq 1),$$

then

$$^\partial\varepsilon_G(z) = 2^{s+1}z^2 + (2^{t+1} + 2^{m+1} - 4)z^3 + (2^{t+m+s} - 2^{t+1} - 2^{m+1} - 2^{s+1} + 4)z^4.$$

Proof. We consider the Euler genus of G^X with the following three cases.

Case 1: Ribbons in X come from one of the three ribbon sets A, B and D. We discuss two subcases as shown in Table 3. And notice that, subcase $X \subseteq D$ includes $X = \emptyset$.

Table 3: Euler genus distribution in Case 1.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$X \subseteq D$	0	2	2	2^s
$X \neq \emptyset$; and $X \subseteq A$ or $X \subseteq B$	1	2	3	$2^t + 2^m - 2$

Case 2: Ribbons in X come from two of the three ribbon sets A, B and D. We discuss three subcases: $X \subseteq A \cup B$, $X \subseteq A \cup D$ and $X \subseteq B \cup D$ respectively, as shown in Table 4.

Table 4: Euler genus distribution in Case 2.

X	$\varepsilon(G\|_X)$	$\varepsilon(G\|_{X^C})$	$\varepsilon(G^X)$	number
$X = A \cup B$	2	0	2	1
$X \subset A \cup B$	2	2	4	$(2^t - 1)(2^m - 1) - 1$
$X = A \cup D$ or $X = B \cup D$	2	1	3	2
$X \subset A \cup D$ or $X \subset B \cup D$	2	2	4	$(2^t - 1)(2^s - 1) + (2^m - 1)(2^s - 1) - 2$

Case 3: Ribbons in X come from all of three ribbon sets A, B and D. We discuss three subcases as shown in Table 5. Notice that, subcase $X^C \subset D$ includes $X^C = \emptyset$. And when ribbons in X^C come from at least two of A, B and D, the signed rotations of both $G\|_X$

and $G|_{X^C}$ have form (2), thus $\varepsilon(G^X) = \varepsilon(G|_X) + \varepsilon(G|_{X^C}) = 2 + 2 = 4$. There are

$$\sum_{x=1}^{t-1} \binom{t}{x} \sum_{x=1}^{m-1} \binom{m}{x} \sum_{x=1}^{s-1} \binom{s}{x} + \sum_{x=1}^{m-1} \binom{m}{x} \sum_{x=1}^{s-1} \binom{s}{x}$$

$$+ \sum_{x=1}^{t-1} \binom{t}{x} \sum_{x=1}^{s-1} \binom{s}{x} + \sum_{x=1}^{t-1} \binom{t}{x} \sum_{x=1}^{m-1} \binom{m}{x}$$

$$= (2^t - 2)(2^m - 2)(2^s - 2) + (2^m - 2)(2^s - 2)$$

$$+ (2^t - 2)(2^s - 2) + (2^t - 2)(2^m - 2)$$

$$= 2^{t+m+s} - 2^{t+m} - 2^{t+s} - 2^{m+s} + 4$$

ways to choose X.

Table 5: Euler genus distribution in Case 3.

X	$\varepsilon(G|_X)$	$\varepsilon(G|_{X^C})$	$\varepsilon(G^X)$	number
$X^C \subset D$	2	0	2	$2^s - 1$
$X^C \neq \emptyset$; and $X^C \subseteq A$ or $X^C \subseteq B$	2	1	3	$2^t + 2^m - 4$
Ribbons in X^C come from at least two of A, B and D	2	2	4	$2^{t+m+s} - 2^{t+m} - 2^{t+s} - 2^{m+s} + 4$

Summarizing the above, we have

$$^{\partial}\varepsilon_G(z) = (2^s + 1 + 2^s - 1)z^2 + (2^t + 2^m - 2 + 2 + 2^t + 2^m - 4)z^3$$

$$+ ((2^t - 1)(2^m - 1) + (2^t - 1)(2^s - 1) + (2^m - 1)(2^s - 1) - 3$$

$$+ 2^{t+m+s} - 2^{t+m} - 2^{t+s} - 2^{m+s} + 4)z^4$$

$$= 2^{s+1}z^2 + (2^{t+1} + 2^{m+1} - 4)z^3$$

$$+ (2^{t+m+s} - 2^{t+1} - 2^{m+1} - 2^{s+1} + 4)z^4. \qquad \square$$

From Lemmas 1.5, 2.9, 2.10 and 2.11 the following three theorems can be deduced.

Theorem 2.12. *If B_n is a Klein bottle bouquet and the signed rotation of its maximum essential subgraph is of form $a_1 \ldots a_i(-a_1) \ldots (-a_i)b_1 \ldots b_j(-b_1) \ldots (-b_j)$, $(i, j \geq 1)$, then*

$$^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-j}(4z^2 + 2(2^i + 2^j - 4)z^3 + (2^i - 2)(2^j - 2)z^4).$$

Theorem 2.13. *If B_n is a Klein bottle bouquet and the signed rotation of its maximum essential subgraph is of form $a_1 \ldots a_i d_1 \ldots d_q(-a_1) \ldots (-a_i)d_q \ldots d_1$, $(i, q \geq 1)$, then*

$$^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-q}(2z + 2(2^q - 1)z^2 + 2(2^i - 2)z^3 + (2^i - 2)(2^q - 2)z^4).$$

Theorem 2.14. *If B_n is a Klein bottle bouquet and the signed rotation of its maximum essential subgraph is of form*

$$a_1 \ldots a_i d_1 \ldots d_q(-a_1) \ldots (-a_i)b_1 \ldots b_j d_q \ldots d_1(-b_1) \ldots (-b_j), \quad (i, j, q \geq 1),$$

then

$$^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-j-q}(2^{q+1}z^2 + (2^{i+1} + 2^{j+1} - 4)z^3 + (2^{i+j+q} - 2^{i+1} - 2^{j+1} - 2^{q+1} + 4)z^4).$$

3 Partial-dual orientable genus polynomials of plane bouquets and toroidal bouquets

In this section, we will deduce the partial-dual orientable genus polynomials of plane bouquets and toroidal bouquets, then their partial-dual Euler-genus polynomials follow.

Theorem 3.1. *If B_n is a plane bouquet, then ${}^\partial\gamma_{B_n}(z) = {}^\partial\varepsilon_{B_n}(z) = 2^n$.*

Proof. From Lemma 2.2, the maximum essential subgraph of any plane bouquet is a bouquet with no loop. Combining it with Lemmas 1.5 and 1.8, the theorem follows. □

For toroidal bouquets, in order to obtain the signed rotation of their maximum essential subgraphs, we will use the structure of the maximal nonhomotopic loop system of the torus. Assume x is a point on surface S. A *nonhomotopic loop system*, denoted by $\mathfrak{L} = \{l_i : i = 1, \ldots, t\}$, is a collection of noncontractible loops with base point x such that l_i and l_j only intersect at x and are not homotopic to each other for $1 \leq i < j \leq t$. A nonhomotopic loop system $\mathfrak{L}$ is *maximal* if adding any noncontractible loop l with x as the base point to $\mathfrak{L}$ then l will either be homotopic to some loop l_i in $\mathfrak{L}$ or intersect l_i at some point other than x. Let $\rho(S) = \max\{|\mathfrak{L}|\}$, where $\mathfrak{L}$ is a maximal nonhomotopic loop system of S, $|\mathfrak{L}|$ is the number of loops in $\mathfrak{L}$, and the maximality is taken over all such system of S. For the torus, the following result can be found in both [6] (in Chapter 4) and [7].

Lemma 3.2 ([6, 7]). *If S_1 is the torus, then $\rho(S_1) = 3$.*

Lemma 3.3. *If B_n is a toroidal bouquet, then all the nontrivial loops can be partitioned into at least two homotopy classes and at most three homotopy classes. Furthermore, any pair of nonhomotopic nontrivial loops interlace mutually.*

Proof. For B_n, viewing it as an embedding of one-vertex graph on torus, all the nontrivial loops are noncontractible and all the trivial loops are contractible. By Lemma 3.2 all the nontrivial loops of B_n can be partitioned into at most three homotopy classes. The fundamental group of torus is $Z \oplus Z$, generated by two elements which are not homotopic and intersect each other transversely, so all the nontrivial loops of B_n can be partitioned into at least two homotopy classes. On the torus, two nontrivial loops are homotopic if and only if they are homotopically disjoint. Therefore any pair of nonhomotopic and nontrivial loops interlace mutually. □

Lemma 3.4. *If B_n is a toroidal bouquet, then the signed rotation of its maximum essential subgraph has the form*

$$a_1 \ldots a_i b_1 \ldots b_j d_1 \ldots d_q a_i \ldots a_1 b_j \ldots b_1 d_q \ldots d_1, \tag{6}$$

in which $i, j \geq 1, q \geq 0$ and $i + j + q \leq n$.

Proof. From Lemma 3.3, loops in the maximum essential subgraph are partitioned into two or three homotopy classes, the loops in the same homotopy class are paralleled with each other, and loops from different homotopy classes are interlaced with each other, the theorem follows. □

In the following, we will discuss the partial dual distributions of toroidal bouquets in two cases depending on whether $q = 0$ or not in Lemma 3.4.

Lemma 3.5. *If G is a bouquet with the signed rotation*

$$a_1 \ldots a_t b_1 \ldots b_m a_t \ldots a_1 b_m \ldots b_1, (t, m \geq 1), \tag{7}$$

then

$$^{\partial}\gamma_G(z) = 2 + 2(2^t + 2^m - 3)z + (2^t - 2)(2^m - 2)z^2.$$

Proof. When $x = 0$ or $t + m$, $\gamma(G^X) = \gamma(G) = 1$. When $0 < x < t + m$, we have the following three cases as shown in Table 6, in which we get $\gamma(G|_X)$, $\gamma(G|_{X^C})$ by Lemmas 2.4 and 3.4, and get $\gamma(G^X)$ by Lemma 1.7.

Table 6: Orientable genus distribution when $0 < x < t + m$.

X	$\gamma(G|_X)$	$\gamma(G|_{X^C})$	$\gamma(G^X)$	number
$X = A$ or $X = B$	0	0	0	2
$X \subset A$, or $X \subset B$, or $X^C \subset A$, or $X^C \subset B$	0	1	1	$2\sum_{x=1}^{t-1}\binom{t}{x} + 2\sum_{x=1}^{m-1}\binom{m}{x}$ $= 2(2^t + 2^m - 4)$
$X \cap A \neq \emptyset$, $X \cap B \neq \emptyset$, $X^C \cap A \neq \emptyset$, and $X^C \cap B \neq \emptyset$	1	1	2	$\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x}$ $= (2^t - 2)(2^m - 2)$

Summarizing the above, we have $^{\partial}\gamma_G(z) = 2 + 2(2^t + 2^m - 3)z + (2^t - 2)(2^m - 2)z^2$. $\qquad\square$

Lemma 3.6. *If G is a bouquet with the signed rotation*

$$a_1 \ldots a_t b_1 \ldots b_m d_1 \ldots d_s a_t \ldots a_1 b_m \ldots b_1 d_s \ldots d_1, (t, m, s \geq 1), \tag{8}$$

then

$$^{\partial}\gamma_G(z) = 2(2^t + 2^m + 2^s - 2)z + (2^{t+m+s} - 2^{t+1} - 2^{m+1} - 2^{s+1} + 4)z^2.$$

Proof. We will consider the orientable genus of G^X into the following three cases.
Case 1: Ribbons in X come from one of the three ribbon sets A, B and D.
Case 2: Ribbons in X come from two of the three ribbon sets A, B and D.
In this case, we could only consider that ribbons in X come from both A and B, by symmetry. Then we have the following two subcases, i.e.,
    Subcase 2a: $X = A \cup B$ and **Subcase 2b**: $X \neq A \cup B$.
Case 3: Ribbons in X come from all of the three ribbon sets A, B and D.
In this case, we have the following two subcase, i.e.,
    Subcase 3a: $X^C \subset A$, or B, or D and **Subcase 3b**: Otherwise.
In each case, we discuss $\gamma(G|_X)$, $\gamma(G|_{X^C})$ and $\gamma(G^X)$, and compute the number of ways to choose X, as shown in Table 7.

Notice that, in Case 1, because the case $X = \emptyset$ has been counted three times in the three summations, we subtract 2 from the counting. So we have $2^t + 2^m + 2^s - 2$ partial duals with orientable genus one in this case. In Case 2, by symmetry, we have three partial duals with orientable genus one and

$$(2^t - 1)(2^m - 1) - 1 + (2^t - 1)(2^s - 1) - 1 + (2^m - 1)(2^s - 1) - 1$$
$$= 2^{t+m} + 2^{m+s} + 2^{t+s} - 2^{t+1} - 2^{m+1} - 2^{s+1}$$

Table 7: Some orientable genus distribution of G^X.

X	$\gamma(G\|_X)$	$\gamma(G\|_{X^C})$	$\gamma(G^X)$	number
Case 1	0	1	1	$\sum\limits_{x=0}^{t}\binom{t}{x}+\sum\limits_{x=0}^{m}\binom{m}{x}+\sum\limits_{x=0}^{s}\binom{s}{x}-2$ $=2^t+2^m+2^s-2$
Subcase 2a	1	0	1	1
Subcase 2b	1	1	2	$\sum\limits_{x=1}^{t}\binom{t}{x}\sum\limits_{x=1}^{m}\binom{m}{x}-1$ $=(2^t-1)(2^m-1)-1$
Subcase 3a	1	0	1	$\sum\limits_{x=1}^{t-1}\binom{t}{x}+\sum\limits_{x=1}^{m-1}\binom{m}{x}+\sum\limits_{x=1}^{s-1}\binom{s}{x}+1$ $=2^t+2^m+2^s-5$
Subcase 3b	1	1	2	$2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4$

partial duals with orientable genus two. In Subcase 3a, notice that, we haven't include $X^C=\emptyset$ in any of the three summations, so we add one to the counting. In Subcase 3b, there are

$$\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x}\sum_{x=1}^{s-1}\binom{s}{x}+\sum_{x=1}^{m-1}\binom{m}{x}\sum_{x=1}^{s-1}\binom{s}{x}$$

$$+\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{s-1}\binom{s}{x}+\sum_{x=1}^{t-1}\binom{t}{x}\sum_{x=1}^{m-1}\binom{m}{x}$$

$$=(2^t-2)(2^m-2)(2^s-2)+(2^m-2)(2^s-2)$$

$$+(2^t-2)(2^s-2)+(2^t-2)(2^m-2)$$

$$=2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4$$

ways to choose the x edges. Hence, in Case 3, we have $2^t+2^m+2^s-5$ partial duals with orientable genus one and $2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4$ partial duals with orientable genus two.

Summarizing the above, we have

$$^\partial\gamma_G(z)=(2^t+2^m+2^s-2+3+2^t+2^m+2^s-5)z$$

$$+(2^{t+m}+2^{m+s}+2^{t+s}-2^{t+1}-2^{m+1}$$

$$-2^{s+1}+2^{t+m+s}-2^{t+m}-2^{t+s}-2^{m+s}+4)z^2$$

$$=2(2^t+2^m+2^s-2)z+(2^{t+m+s}-2^{t+1}-2^{m+1}-2^{s+1}+4)z^2. \quad \square$$

From Lemmas 1.5, 1.8, 3.5 and 3.6, the following two theorems can be deduced.

Theorem 3.7. *If B_n is a toroidal bouquet and the signed rotation of its maximum essential subgraph has the form $a_1\ldots a_ib_1\ldots b_ja_i\ldots a_1b_j\ldots b_1$ in which $i,j\ge 1$ and $i+j\le n$, then*

$$^\partial\gamma_{B_n}(z)=2^{n-i-j}(2+2(2^i+2^j-3)z+(2^i-2)(2^j-2)z^2),$$

and

$$^\partial\varepsilon_{B_n}(z)=2^{n-i-j}(2+2(2^i+2^j-3)z^2+(2^i-2)(2^j-2)z^4).$$

Theorem 3.8. *If B_n is a toroidal bouquet and the signed rotation of its maximum essential subgraph has the form $a_1 \ldots a_i b_1 \ldots b_j d_1 \ldots d_q a_i \ldots a_1 b_j \ldots b_1 d_q \ldots d_1$, in which $i, j, q \geq 1$ and $i + j + q \leq n$, then*

$$^{\partial}\gamma_{B_n}(z) = 2^{n-i-j-q}(2(2^i + 2^j + 2^q - 2)z + (2^{i+j+q} - 2^{i+1} - 2^{j+1} - 2^{q+1} + 4)z^2),$$

and

$$^{\partial}\varepsilon_{B_n}(z) = 2^{n-i-j-q}(2(2^i + 2^j + 2^q - 2)z^2 + (2^{i+j+q} - 2^{i+1} - 2^{j+1} - 2^{q+1} + 4)z^4).$$

ORCID iDs

Yan Yang ⓘ https://orcid.org/0000-0002-9666-5167
Xiaoya Zha ⓘ https://orcid.org/0000-0003-1940-7428

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