On some generalization of the Möbius configuration

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Abstract

The Möbius (8_4) configuration is generalized in a purely combinatorial approach. We consider $(2n_n)$ configurations $\mathfrak{M}_{(n,\varphi)}$ depending on a permutation φ in the symmetric group S_n. Classes of non-isomorphic configurations of this type are determined. The parametric characterization of $\mathfrak{M}_{(n,\varphi)}$ is given. The uniqueness of the decomposition of $\mathfrak{M}_{(n,\varphi)}$ into two mutually inscribed n-simplices is discussed. The automorphisms of $\mathfrak{M}_{(n,\varphi)}$ are characterized for $n \geq 3$.

Keywords: Möbius configuration, (8_4) configurations, Möbius pair, n-simplex.

Math. Subj. Class.: 51D20, 05B30, 51E30

1 Introduction

The Möbius (8_4) configuration is a certain configuration in a projective 3-dimensional space consisting of two mutually inscribed and circumscribed tetrahedra (cf. [7]). Each vertex of one tetrahedron lies on a face plane of the other tetrahedron and vice versa. Configurations with parameters (n_4) were studied in detail in [4], but this is not the case, since the Möbius (8_4) configuration is not a point-line structure. An important role of the theorem connected with the Möbius configuration (which says, roughly speaking, that the Möbius configuration "closes") in a projective 3-dimensional space was presented in [12]: it is equivalent to the commutativity of the ground division ring.

In this paper we deal with two n-simplices (simplices with n vertices, $n \geq 3$)[1] instead of two tetrahedra (4-simplices). The way how an n-simplex is inscribed into another we define by a permutation φ in the group S_n. The generalization of the Möbius configuration we obtain, is a $(2n_n)$-configuration and it will be referred to as a *Möbius pair of n-simplices*,

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[1]In geometry an n-simplex usually means a simplex having $n + 1$ vertices. Our definition is slightly different.

or shortly a *Möbius n-pair*. Only a combinatorial scheme (an abstract incidence structure, see e.g. [2, 10]) of a Möbius n-pair is investigated and we do not discuss problems regarding embeddability into projective (or other) spaces. Although these problems have been partially solved in [5] (the case with $\varphi = \mathrm{id}$), they are interesting and still open in general.

As we know from [6], in a projective space, up to an isomorphism there are five (8_4) point-plane configurations with the property that at most two planes share two points, and dually at most two points are shared by two planes. These are precisely those configurations with two mutually circumscribed tetrahedra, and thus all of them are sometimes called the Möbius configurations. It is also known (cf. [10]), that these (8_4) configurations correspond to conjugacy classes of the permutation group S_4. We shall prove, that two Möbius n-pairs are isomorphic if and only if the permutations, that determine them, are conjugate. Another important impact of the permutation on the geometry of the Möbius n-pair is that the cycle structure of φ is associated with circuits in the incidence graph of the Möbius n-pair.

As we shall see, the decomposition of the points of the generalized Möbius configuration into two complementary and mutually inscribed simplices is, generally, a unique one. Exceptions appear "near" the classical case $n = 4$. Three of five (8_4) Möbius configurations contain at least two distinct pairs of complementary 4-simplices.

The next problem, which is considered in the paper, involves Möbius subpairs of a Möbius n-pair. We simply delete some number of points and blocks of one n-simplex and the same number of points and blocks of the second n-simplex with a hope to obtain a Möbius pair again. The conditions, under which we get a subpair in the Möbius n-pair, are determined.

In the last part we use most of the established properties to characterize the automorphism group of the Möbius n-pair for $n \geq 3$.

2   Definitions, parameters and basic properties

By a *configuration* we mean any point-block structure $\mathfrak{M} = \langle S, \mathcal{L} \rangle$, where the blocks are subsets of the set of points, i.e. $\mathcal{L} \subseteq 2^S$. The *rank of a point* is the number of blocks containing this point, and dually the *size of a block* is the number of points contained in this block. Let n be a natural number and X be a set. The family of all n-subsets of the set X will be denoted by $\wp_n(X)$. Let $n \geq 3$. We say that a configuration $\mathfrak{M}$ is an n-*simplex* iff $|\mathcal{L}| = n$, there is a subset $V \in \wp_n(S)$ such that for every $V' \in \wp_{n-1}(V)$ there is a unique block $L \in \mathcal{L}$ containing V', and the rank of each point $s \in S \setminus V$ is less than $n - 1$. Elements of V will be called *vertices* of the simplex, and blocks of the simplex are said to be its *faces*. We say that two configurations $\mathfrak{M}_1 = \langle S_1, \mathcal{L}_1 \rangle$, $\mathfrak{M}_2 = \langle S_2, \mathcal{L}_2 \rangle$ are *isomorphic* (and we write $\mathfrak{M}_1 \cong \mathfrak{M}_2$) iff there exists a bijective map $f \colon S_1 \longrightarrow S_2$ such that conditions $k \in \mathcal{L}_1$ and $f(k) \in \mathcal{L}_2$ are equivalent. In case $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}$ the map f will be called an *automorphism of* $\mathfrak{M}$.

Let us consider two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $A \cap B = \emptyset$. Let $\varphi \in S_n$ be a permutation of the set $I = \{1, \ldots, n\}$. Now we introduce the following sets:

$$
\begin{aligned}
\mathcal{L}_A &:= \big\{ A' \cup \{b_i\} : A' \in \wp_{n-1}(A) \text{ and } a_i \notin A' \big\}, \\
\mathcal{L}_B &:= \big\{ B' \cup \{a_{\varphi(i)}\} : B' \in \wp_{n-1}(B) \text{ and } b_i \notin B' \big\}.
\end{aligned}
$$

The configuration

$$
\mathfrak{M}_{(n,\varphi)} := \langle A \cup B, \mathcal{L}_A \cup \mathcal{L}_B \rangle,
$$

will be called a *Möbius n-pair*. The Möbius configurations can be identified with the Möbius 4-pairs, which Levi graphs are Figures 1, 2, 3, 4, 5. All of them are also presented in [10]. In particular, $\mathfrak{M}_{(4,\mathrm{id})}$ is the classical (8_4) Möbius configuration.

Let $\mathcal{M}$ be a Möbius n-pair. We write: A_i, B_i for blocks of $\mathcal{M}$ not containing a_i, b_i, respectively; *a-points*, *b-points*, *A-blocks*, *B-blocks* for points in A, B, and blocks in $\mathcal{L}_A$, $\mathcal{L}_B$, respectively. The configuration $\mathcal{M}$ reflects the main abstract properties of the classical Möbius configuration.

1. The a-points yield a simplex in $\mathcal{M}$: for any $(n-1)$-subset $A \setminus \{a_i\}$ of the a-points there is a unique block of $\mathcal{M}$, which contains this subset (A_i, a face of the simplex in question); the remaining points (b-points) yield another simplex.

2. The simplex with a-points and the simplex with b-points are mutually inscribed: on each face, A_i, of the first simplex there is a unique vertex (b_i) of the second one; on each face, B_i, of the second simplex there is a unique vertex ($a_{\varphi(i)}$) of the first simplex.

Thus, we can decompose $\mathcal{M}$ into two complementary substructures $S_A(\mathcal{M}) = \langle A, \mathcal{L}_A \rangle$ and $S_B(\mathcal{M}) = \langle B, \mathcal{L}_B \rangle$, which we call *simplices of* $\mathcal{M}$ (although, formally, a block of each of them is not a subset of its points; there is one extra point on each of its faces).

In the forthcoming part we will use the notion of the incidence graph (*the Levi graph*) $\mathcal{G}_{\mathcal{M}}$ associated with $\mathcal{M}$. Recall that a Levi graph is a bipartite graph with partition induced by points vs. blocks (cf. [9, 10]). Two of its vertices x, y are said to be *adjacent* (which is written $x \sim y$) if x is a point, y is a block (or vice versa) and $x \in y$ (or $y \in x$). Otherwise x is not adjacent to y, which we write $x \nsim y$. The *rank of a vertex* is the number of vertices adjacent to it. A vertex of $\mathcal{G}_{\mathcal{M}}$ will be called point-vertex, block-vertex, a,b-vertex, A,B-vertex, or simply a_i, b_i, A_i, B_i as it corresponds to the point or to the block of $\mathcal{M}$. The Levi graph associated with $S_A(\mathcal{M}), S_B(\mathcal{M})$ will be denoted by $\mathcal{G}_{S_A(\mathcal{M})}, \mathcal{G}_{S_B(\mathcal{M})}$, respectively.

Remark 2.1. Let $\mathcal{M}$ be a Möbius n-pair. The Levi graph $\mathcal{G}_{\mathcal{M}}$ has the following properties:

(i) for $X = A, B$, every point-vertex from $\mathcal{G}_{S_X(\mathcal{M})}$ is adjacent to all but one block-vertices from $\mathcal{G}_{S_X(\mathcal{M})}$, and vice versa,

(ii) for $X, Y = A, B$ and $X \neq Y$, every point-vertex from $\mathcal{G}_{S_X(\mathcal{M})}$ is adjacent to precisely one block-vertex from $\mathcal{G}_{S_Y(\mathcal{M})}$, and vice versa.

Immediately from the definition of $\mathfrak{M}_{(n,\varphi)}$, the number of its points coincides with the number of its blocks and equals $2n$, and the rank of every point coincides with the size of every block and equals n. Thus the structures we investigate are $(2n_n)$-configurations. A standard parametric question related to configurations is: what is the number of points that are contained in two distinct blocks, and dually: what is the number of blocks containing two distinct points.

Proposition 2.2. *Let k, l be two different blocks of the structure $\mathfrak{M}_{(n,\varphi)}$. Then $|k \cap l| \in \{0, 1, 2, n-2\}$. If both k, l are A-blocks, or both k, l are B-blocks then $|k \cap l| = n - 2$. Otherwise, $k = A_i$ and $l = B_j$ for some $i, j \in I$, and the following equivalences hold*

(i) $|A_i \cap B_j| = 0$ *iff* $\varphi(j) = i = j$,

(ii) $|A_i \cap B_j| = 1$ *iff* $\varphi(j) = i \neq j$ *or* $\varphi(j) \neq i = j$,

(iii) $|A_i \cap B_j| = 2$ *iff* $\varphi(j) \neq i \neq j$.

Proof. It is straightforward from the definition that if k, l are both A-blocks or B-blocks then $k \cap l$ has $n - 2$ elements. Let $k = A_i \in \mathcal{L}_A$ and $l = B_j \in \mathcal{L}_B$ for some $i, j \in I$. Let $i \neq j$. If $\varphi(j) \neq i$ then $A_i \cap B_j = \{b_i, a_{\varphi(j)}\}$. Otherwise, for $\varphi(j) = i$, we get $A_i \cap B_j = \{b_i\}$. Let $i = j$. If $\varphi(i) \neq i$ we obtain $A_i \cap B_i = \{a_{\varphi(i)}\}$. In case $\varphi(i) = i$ it holds $A_i \cap B_i = \emptyset$. $\qquad \square$

Each conjugacy class of S_n corresponds to exactly one decomposition of a permutation $\varphi \in S_n$ into cycles, up to a permutation of the elements of I. Now we describe how the cycle structure of φ is reflected in block paths of $\mathfrak{M}_{(n,\varphi)}$.

Fact 2.3. *A permutation φ contains a cycle of length $k \leq n$ iff there is a closed path of length $2k$ consisting of blocks of $\mathfrak{M}_{(n,\varphi)}$ such that every two consecutive blocks intersect in precisely one point of $\mathfrak{M}_{(n,\varphi)}$.*

Proof. Assume that φ contains the cycle $(i_1 i_2 \ldots i_k)$. Then $a_{i_{j+1}} \in A_{i_j} \cap B_{i_j}$ and $b_{i_{j+1}} \in B_{i_j}, A_{i_{j+1}}$ for each $j \leq k$. Thus, the closed path in question is the following: $A_{i_1}, B_{i_1}, A_{i_2}, B_{i_2}, \ldots, A_{i_k}, B_{i_k}$.

Now assume that there exists a closed path $l_1, l_1', \ldots, l_k, l_k'$ of blocks of $\mathfrak{M}_{(n,\varphi)}$ such that every two consecutive blocks intersect in a point. By Proposition 2.2(ii) every two consecutive blocks of the path are $A_i \in \mathcal{L}_A$, $B_j \in \mathcal{L}_B$ with $\varphi(j) = i \neq j$ or $\varphi(j) \neq i = j$. Suppose $\varphi(j) \neq i = j$ holds for the first two blocks of our path, namely $l_1 = A_i$, $l_1' = B_i$ and $\varphi(i) \neq i$ for some $i \in I$. To obtain $|l_1' \cap l_2| = 1$ we must have $l_2 = A_j$ with $\varphi(i) = j$. Thus the next two blocks are $l_2 = A_{\varphi(i)}$, $l_2' = B_{\varphi(i)}$ and $\varphi(\varphi(i)) \neq \varphi(i)$. In general we obtain $l_j = A_{\varphi^{j-1}(i)}$, $l_j' = B_{\varphi^{j-1}(i)}$ and $\varphi^{j-1}(i) \neq \varphi^{j-2}(i)$ for every $j = 2, \ldots, k$. To close the path we need $\varphi^k(i) = i$. Let us put $i = i_0$. Then the cycle $(i_0, i_1, \ldots, i_{k-1})$, where $i_j = \varphi^j(i)$ for $j = 0, \ldots, k-1$, is one of the cycles in the cycle decomposition of φ. $\qquad \square$

As the configuration $\mathfrak{M}_{(n,\varphi)}$ is symmetric, it makes sense to consider the dual configuration $\mathfrak{M}^\circ_{(n,\varphi)}$.

Fact 2.4. *The configuration $\mathfrak{M}^\circ_{(n,\varphi)}$ is isomorphic to $\mathfrak{M}_{(n,\varphi)}$.*

Proof. It is easy to note that $\mathfrak{M}^\circ_{(n,\varphi)} \cong \mathfrak{M}_{(n,\varphi^{-1})}$. Consider $\alpha \in S_n$ such that $\alpha(1) = 1$ and $\alpha(m) = n - m + 2$ for $m \in I \setminus \{1\}$. Let $x \in \{a, b, A, B\}$, $i \in I$. Then $F : x_i \mapsto x_{\alpha(i)}$ is an isomorphism mapping $\mathfrak{M}_{(n,\varphi^{-1})}$ onto $\mathfrak{M}_{(n,\varphi)}$. $\qquad \square$

The problem of two isomorphic Möbius n-pairs will be considered in general in the last section of the paper. Another parametric characterization is now a simple consequence of Proposition 2.2 and Fact 2.4.

Proposition 2.5. *Let x, y be two different points of $\mathfrak{M}_{(n,\varphi)}$. There exist 0, 1, 2, or $n - 2$ blocks of $\mathfrak{M}_{(n,\varphi)}$ containing x and y.*

3   Hidden Möbius pairs

The goal of this section is to characterize $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$ that can be transformed into Möbius pair with simplices distinct from $S_A(\mathcal{M})$, $S_B(\mathcal{M})$ by a decomposition of the points or by a deletion of some points and blocks. Informally, we say that these Möbius pairs are *hidden* in $\mathcal{M}$.
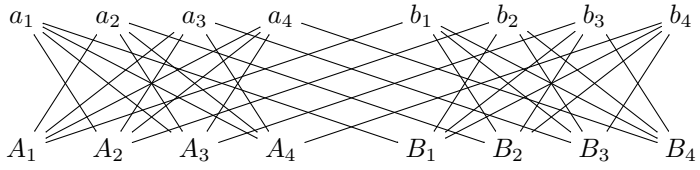
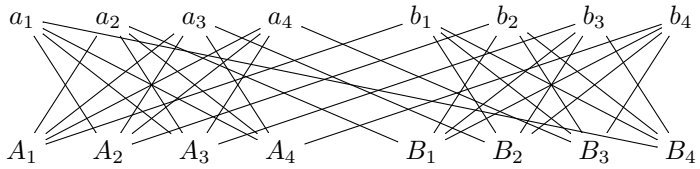Figure 1: The Levi graph of $\mathfrak{M}_{(4,\mathrm{id})}$ (isomorphic to the hypercube graph Q_4).

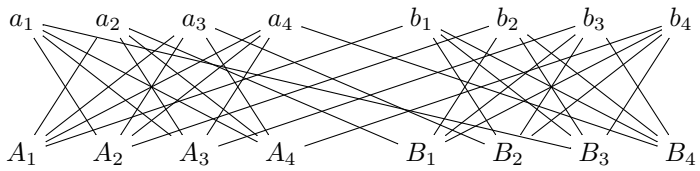Figure 2: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (1234)$.

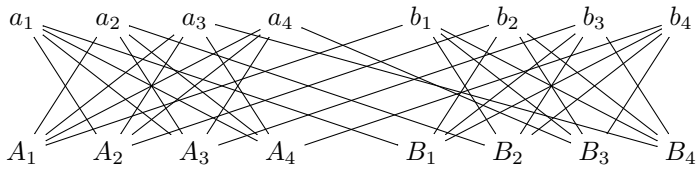Figure 3: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (123)(4)$.

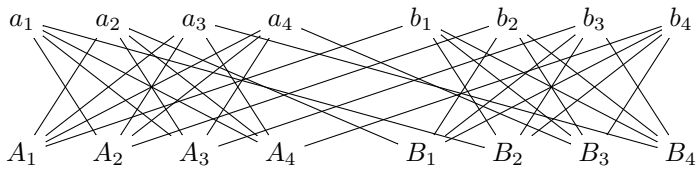Figure 4: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (1)(2)(34)$.

Figure 5: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (12)(34)$.

3.1 Möbius n-pairs with the special decompositions

Let us start with the following combinatorial observation:

Remark 3.1. The Möbius configuration $\mathcal{M} = \mathfrak{M}_{(4,\mathrm{id})}$ can be presented in 3 distinct ways as two mutually circumscribed simplices such that each of them is distinct from $S_A(\mathcal{M}), S_B(\mathcal{M})$.

One could say that there are four Möbius 4-pairs hidden in $\mathfrak{M}_{(4,\mathrm{id})}$. Let $n \geq 4$, $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$, and assume that it is possible to decompose the points of $\mathcal{M}$ into two complementary and mutually inscribed simplices $S_1(\mathcal{M}), S_2(\mathcal{M})$ such that $S_t(\mathcal{M}) \neq S_X(\mathcal{M})$ for each $t = 1, 2$, $X = A, B$. Such a decomposition will be called a *special decomposition*.

Lemma 3.2. *Let $S_1(\mathcal{M})$, $S_2(\mathcal{M})$ be two simplices, that arise from a special decomposition of $\mathcal{M}$.*

   (i) *For each $i \in I$, and each $t = 1, 2$, it is impossible to have both B_i, b_i in $S_t(\mathcal{M})$, or both A_i, a_i in $S_t(\mathcal{M})$.*

   (ii) *For each $t = 1, 2$, the blocks of $S_t(\mathcal{M})$ are two B-blocks and two A-blocks.*

Proof. The proof involves only $S_1(\mathcal{M})$, since the reasoning for $S_2(\mathcal{M})$ will be the same.

(i) Assume that $S_1(\mathcal{M})$ contains both of B_i, b_i. Then also some a_j is a point of $S_1(\mathcal{M})$ for $j \in I$. Consider the graph $\mathcal{G}_\mathcal{M}$. The vertices B_i, b_i are not adjacent, so from Remark 2.1(i) $a_j \sim B_i$ and $j = \varphi(i)$. The unique block-vertex not adjacent to a_j in $\mathcal{G}_{S_1(\mathcal{M})}$ is A_j or B_s for some $s \neq \varphi^{-1}(j)$.

Let A_j be this vertex, so from Remark 2.1(ii) $A_j \sim b_i$, and thus $j = i$. Consider in $\mathcal{G}_{S_1(\mathcal{M})}$ another vertex a_t or b_t with $t \neq i$. Since $\varphi(i) = i \neq t$, a contradiction arises: $b_t \not\sim A_i$, and $a_t \not\sim B_i$ (see the scheme presented in Figure 6).
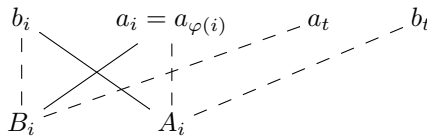

Figure 6: The fragment of $\mathcal{G}_{f(\mathcal{M})}$ containing B_i, b_i and A_i, a_i.
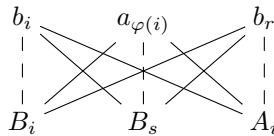
Figure 7: The fragment of $\mathcal{G}_{f(\mathcal{M})}$ containing B_i, b_i and $B_s, a_{\varphi(i)}$.

Assume that $s \neq \varphi^{-1}(j)$ and B_s is a unique vertex not adjacent to a_j in $\mathcal{G}_{S_1(\mathcal{M})}$. We get $s \neq i$, as far as $b_i \sim B_s$. Let us take another vertex: A_t or B_t. For $t \neq i$ there is no B-vertex adjacent to a_j, and $A_t \sim a_j, b_i$ if $t = i$. A vertex, which is not adjacent to A_i, is a_i or b_r with $r \neq i, s$. The vertex a_i is not adjacent to B_i since $\varphi(i) = j \neq i$, and thus a_i cannot be the vertex in question. Consequently, this vertex is $b_r \sim B_i, B_s$. Following the

assumption $n \geq 4$, there exists another block in $S_1(\mathcal{M})$, that is different from B_i, B_s, A_i. We have two b-points in $S_1(\mathcal{M})$ so far, thus this block is a B-block. The B-vertex of $\mathcal{G}_{\mathcal{M}}$, that is associated with this block, must be adjacent to $a_{\varphi(i)}$. So this block is B_i, which is already one of the blocks in $S_1(\mathcal{M})$ (comp. with the scheme presented in Figure 7), a contradiction.

(ii) Let B_i be the unique B-block of $S_1(\mathcal{M})$ for some $i \in I$. Then the remaining blocks of $S_1(\mathcal{M})$ are A-blocks. In view of Lemma 3.2(i), there are $n-1$ b-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$: every A-vertex is associated with the b-vertex, which is not adjacent to it. For $n \geq 4$ a contradiction with Remark 2.1(i) arises: every b-vertex is adjacent to precisely one of A-vertices, and thus it is not adjacent to at least two A-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$.

Let $S_1(\mathcal{M})$ contain at least three B-blocks. Without loss of generality, assume B_1, B_2, B_3 are blocks of $S_1(\mathcal{M})$. From Lemma 3.2(i), b_1, b_2, b_3 are not in $S_1(\mathcal{M})$. Thus, from Remark 2.1(i), $S_1(\mathcal{M})$ contains $a_{i_1}, a_{i_2}, a_{i_3}$ such that $i_j \neq \varphi(j)$ for $j = 1, 2, 3$. Every block-vertex B_j must be adjacent to at least two of the point-vertices $a_{i_{j'}}$ with $j' \neq j$. On the other hand, it is adjacent to at most one of them, what follows from Remark 2.1(ii) applied to $\mathcal{G}_{\mathcal{M}}$. This contradiction actually completes the proof as other cases run dually. $\qquad \square$

By Lemma 3.2 we prove a generalization of Remark 3.1.

Proposition 3.3. *Let* $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$. *The following conditions are equivalent*

(i) *there is a special decomposition of* $\mathcal{M}$,

(ii) $n = 4$ *and there is* $X \subset I$ *such that* $|X| = 2$ *and* $\varphi(X) = X$.

Proof. (i) $\Rightarrow$ (ii): From Lemma 3.2(ii) we get $n = 4$, and two B-vertices and two A-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$. Let (e.g.) B_1, B_2 be the B-vertices of $\mathcal{G}_{S_1(\mathcal{M})}$. In view of Remark 2.1(i), there are vertices x, y in $\mathcal{G}_{S_1(\mathcal{M})}$ such that $x \sim B_1, y \sim B_2$ and $x \nsim B_2, y \nsim B_1$. By Lemma 3.2(i), $x \neq b_2, y \neq b_2$, and thus $x = a_i, y = a_j$ where $\varphi(1) = i, \varphi(2) = j$. Then two A-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$ are A_s, A_t with $s, t \neq i, j$. The remaining two point-vertices must be of the form $b_{s'}, b_{t'}$ with $s', t' \neq 1, 2$, since they must be adjacent to both of B_1, B_2. On the other hand, $b_{s'}, b_{t'}$ need to be adjacent to precisely one of A_s, A_t, so $\{s', t'\} = \{s, t\}$. Thus $s, t \neq 1, 2$, $\{1, 2\} = \{i, j\} = \{\varphi(1), \varphi(2)\}$, and $X = \{1, 2\}$ is the required set.

(ii) $\Rightarrow$ (i): Assume, without loss of generality, $X = \{1, 2\}$ and consider $\mathcal{M} = \mathfrak{M}_{(4,\varphi)}$ with $\varphi(X) = X$. Take blocks B_1, B_2, A_3, A_4 and points $a_{\varphi(1)}, a_{\varphi(2)}, b_3, b_4$ of $\mathcal{M}$, and consider $\mathcal{G}_{\mathcal{M}}$. We have $B_1 \nsim a_{\varphi(2)}, B_2 \nsim a_{\varphi(1)}$, and $B_1, B_2 \sim b_3, b_4$. Similarly $A_3 \nsim b_4$, $A_4 \nsim b_3$, and $A_3, A_4 \sim a_{\varphi(1)}, a_{\varphi(2)}$, since $\varphi(1), \varphi(2) \in \{1, 2\}$. Thus the Levi graph we consider is a Levi graph of a 4-simplex. It is easy to verify that A_1, A_2, B_3, B_4 and b_1, b_2, a_3, a_4 form another 4-simplex. The two obtained simplices are mutually circumscribed. Indeed, $B_1, b_2; B_2, b_1; A_3, a_4; A_4, a_3$, and $A_1, a_{\varphi(1)}$ (or $A_1, a_{\varphi(2)}$); $A_2, a_{\varphi(2)}$ (or $A_2, a_{\varphi(1)}$); $B_3, b_4; B_4, b_3$ are all pairs of adjacent vertices representing blocks (points) of the first simplex and points (blocks) of the second simplex in each pair. In other words, we have found a special decomposition of $\mathcal{M}$. $\qquad \square$

Due to Proposition 3.3 there is a correspondence between the special decompositions of $\mathfrak{M}_{(n,\varphi)}$ and 2-subsets of I preserved by φ. The correspondence is established up to complements, since the special decompositions arise only for $n = 4$, and thus if φ preserves a 2-subset of $\{1, 2, 3, 4\}$ then it preserves its complement as well. So, directly from Proposition 3.3 we get:

Corollary 3.4. *All (up to an isomorphism) Möbius n-pairs with a special decomposition are the following:*

1. $\mathfrak{M}_{(4,\mathrm{id})}$ *with 3 distinct special decompositions associated with*
   $X = \{1,2\}, \{1,3\}, \{1,4\}$,

2. $\mathfrak{M}_{(4,(13)(24))}$ *with the special decomposition associated with $X = \{1,3\}$,*

3. $\mathfrak{M}_{(4,(12)(3)(4))}$ *with the special decomposition associated with $X = \{1,2\}$.*

3.2   Subpairs of Möbius n-pairs

Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$, $n \geq 4$, $k \geq 3$, $k < n$, and $\mathcal{M}'$ be a Möbius k-pair obtained from $\mathcal{M}$ by deleting $2(n-k)$ points and $2(n-k)$ blocks. We call $\mathcal{M}'$ a *k-subpair* of $\mathcal{M}$. The blocks of $\mathcal{M}'$ are *subblocks of $\mathcal{M}$*, that is every block of $\mathcal{M}'$ arises as a block of $\mathcal{M}$ with $n-k$ points removed. The subblocks of the A-blocks, the B-blocks are called the *A-subblocks*, the *B-subblocks*, respectively. Let $S_1(\mathcal{M}')$, $S_2(\mathcal{M}')$ be two simplices of $\mathcal{M}'$. For any $t = 1, 2$, $X = A, B$ we write $S_t(\mathcal{M}') \prec S_X(\mathcal{M})$ if all the points and the blocks of $S_t(\mathcal{M}')$ are points and subblocks of $S_X(\mathcal{M})$. Otherwise we write $S_i(\mathcal{M}') \not\prec S_X(\mathcal{M})$. For $Y \subset I$ by $\varphi \restriction_Y$ we mean the restriction of φ to the set Y.

In order to determine all Möbius n-pairs with k-subpairs we need to prove some auxiliary facts.

Lemma 3.5. *One of the following conditions holds*

   (i)   *$S_1(\mathcal{M}') \prec S_A(\mathcal{M})$ and $S_2(\mathcal{M}') \prec S_B(\mathcal{M})$,*

  (ii)   *$S_2(\mathcal{M}') \prec S_A(\mathcal{M})$ and $S_1(\mathcal{M}') \prec S_B(\mathcal{M})$,*

 (iii)   *$S_1(\mathcal{M}') \not\prec S_A(\mathcal{M}), S_B(\mathcal{M})$ and $S_2(\mathcal{M}') \not\prec S_A(\mathcal{M}), S_B(\mathcal{M})$.*

Moreover, if $\mathcal{M}'$ satisfies (iii) *then there is a special decomposition of $\mathcal{M}'$.*

Proof. Let $S_1(\mathcal{M}') \prec S_A(\mathcal{M})$ and $S_2(\mathcal{M}') \not\prec S_B(\mathcal{M})$. So there is an a-point or A-subblock in $S_2(\mathcal{M}')$. We consider only the case with an a-point, as the case with an A-subblock is symmetric. From Remark 2.1(ii) applied to $\mathcal{G}_\mathcal{M}$, and Remark 2.1(i) applied to $\mathcal{G}_{\mathcal{M}'}$, there are at most two B-subblocks in $S_2(\mathcal{M}')$. Since $k \geq 3$, there is at least one A-subblock in $S_2(\mathcal{M}')$. Note that a unique A-subblock, which does not contain an a-point of $S_1(\mathcal{M}')$, is a block of $S_1(\mathcal{M}')$. Thus all the points of $S_1(\mathcal{M})$ are in an A-subblock of $S_2(\mathcal{M}')$. This yields a contradiction with Remark 2.1(ii). The proof for each of the remaining cases (i.e. $S_2(\mathcal{M}') \prec S_A(\mathcal{M})$ and $S_1(\mathcal{M}') \not\prec S_B(\mathcal{M})$, $S_1(\mathcal{M}') \prec S_B(\mathcal{M})$ and $S_2(\mathcal{M}') \not\prec S_A(\mathcal{M})$, or $S_2(\mathcal{M}') \prec S_B(\mathcal{M})$ and $S_1(\mathcal{M}') \not\prec S_A(\mathcal{M})$) is analogous.

Let $\mathcal{M}'$ satisfy (iii). The steps of the proof of Lemma 3.2 can be repeated for simplices of $\mathcal{M}'$. As a result we get $k = 4$, and two A-subblocks and two B-subblocks in each of simplices of $\mathcal{M}'$. Let $Y \subset I$ be the set of subscripts of A-subblocks and B-subblocks in one of these simplices. From the reasoning analogous to the first part of the proof of Proposition 3.3 we get that Y is the set of all the subscripts used for labelling the points and the blocks of $\mathcal{M}'$, and there is a two-element set $X \subset Y$ such that $\varphi \restriction_Y (X) = X$. Therefore, in view of Proposition 3.3, there is a special decomposition of $\mathcal{M}'$. $\qquad\square$

Lemma 3.6. *If the number of deleted B-blocks and the number of deleted A-blocks coincide (and equals $n-k$), then there is $X \subset I$ such that $|X| = n-k$ and $\varphi(X) = X$.*

Proof. Assume that $B_{i_1}, \ldots, B_{i_{n-k}}$ and $A_{j_1}, \ldots, A_{j_{n-k}}$ are removed blocks. Consider a vertex $a_{\varphi(i_s)}$ with $s = 1, \ldots, n - k$ of $\mathcal{G}_{\mathcal{M}'}$, and assume $a_{\varphi(i_s)}$ is in $\mathcal{G}_{S_1(\mathcal{M}')}$ (the case with $a_{\varphi(i_s)}$ in $\mathcal{G}_{S_2(\mathcal{M}')}$ will be analogous). Note that $a_{\varphi(i_s)} \sim B_{i_s}$, and from Remark 2.1(ii) B_{i_s} is a unique B-vertex adjacent to $a_{\varphi(i_s)}$. According to Lemma 3.5 two cases arise: (i) or (iii) holds for $\mathcal{M}'$. Let $\mathcal{M}'$ satisfy (i) of Lemma 3.5. Then there is a B-vertex in $\mathcal{G}_{S_2(\mathcal{M}')}$ adjacent to $a_{\varphi(i_s)}$, a contradiction. If $\mathcal{M}'$ satisfies (iii) of Lemma 3.5 then there is a special decomposition of $\mathcal{M}'$. So, by Proposition 3.3, there is a B-vertex in $\mathcal{G}_{S_1(\mathcal{M}')}$ adjacent to $a_{\varphi(i_s)}$, a contradiction again. Therefore all $a_{\varphi(i_1)} \ldots, a_{\varphi(i_{n-k})}$ are removed. Likewise we consider the pairs $a_{j_s}, A_{j_s}, b_{j_s}, A_{j_s}$, and b_{i_s}, B_{i_s}. Each of these reasonings leads us to contradiction. Consequently points $a_{j_1} \ldots, a_{j_{n-k}}, b_{j_1} \ldots, b_{j_{n-k}}$, and $b_{i_1} \ldots, b_{i_{n-k}}$ are deleted as well. Hence

$$\{j_1, \ldots, j_{n-k}\} = \{\varphi(i_1), \ldots, \varphi(i_{n-k})\} \text{ and } \{i_1, \ldots, i_{n-k}\} = \{j_1, \ldots, j_{n-k}\}.$$

Finally we get $\{\varphi(i_1), \ldots, \varphi(i_{n-k})\} = \{i_1, \ldots, i_{n-k}\}$, and $X = \{i_1, \ldots, i_{n-k}\}$ is the set from our claim. $\qquad\square$

Let us present a condition, which is sufficient and necessary to find a k-subpair in $\mathfrak{M}_{(n,\varphi)}$.

Proposition 3.7. *Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$. The following conditions are equivalent*

(i) *there is $\mathcal{M}'$, which is a k-subpair of $\mathcal{M}$,*

(ii) *there is $X \subset I$ such that $|X| = n - k$ and $\varphi(X) = X$.*

Furthermore, if (ii) *holds then $\mathcal{M}' \cong \mathfrak{M}_{(k, \varphi\restriction_{(I \setminus X)})}$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 3.5 $\mathcal{M}'$ satisfies one of (i) – (iii) of Lemma 3.5. In cases (i) and (ii) of Lemma 3.5 the numbers of A-blocks and B-blocks deleted from $\mathcal{M}$ coincide and are equal to $n - k$. The claim follows directly from Lemma 3.6. If case (iii) of Lemma 3.5 holds, then there is a special decomposition of $\mathcal{M}'$, and we get our claim by Proposition 3.3.

(ii) $\Rightarrow$ (i): Without any loss of generality, let $X = \{1, \ldots, n - k\}$. Recall that the rank of every vertex in $\mathcal{G}_{\mathcal{M}}$ is n. Observe the Levi graph obtained from $\mathcal{G}_{\mathcal{M}}$ by removing the vertices a_i, A_i and b_i, B_i for every $i \in X$, and all edges passing through these vertices. We denote this Levi graph by $\mathcal{H}$. Note that $a_{\varphi(i)}$ is not a vertex of $\mathcal{H}$, since $\varphi(i) \in X$. Let $j \notin X$ and take A_j. Clearly A_j is a vertex of $\mathcal{H}$. There are $n - k$ edges joining A_j with all a_i in $\mathcal{G}_{\mathcal{M}}$. Thus, the rank of A_j in $\mathcal{H}$ is $n - (n - k) = k$. Similarly we set ranks of the remaining vertices a_j, b_j, B_j of $\mathcal{H}$. All these ranks are k. From this and the construction of $\mathcal{H}$ we get that $\mathcal{H}$ is the Levi graph of two mutually circumscribed k-simplices, where the way they are inscribed one into another is induced by the action of φ on the set $I \setminus X$. Therefore $\mathcal{H} = \mathcal{G}_{\mathcal{M}'}$ for some $\mathcal{M}'$, which is a k-subpair of $\mathcal{M}$. $\qquad\square$

4 Isomorphisms and automorphisms

4.1 Isomorphic Möbius n-pairs

Recall that the Möbius (8_4) configurations (i.e. Möbius 4-pairs) correspond to conjugacy classes of the permutation group S_4. In this section we generalize this property to all Möbius n-pairs.

Let us start with a key lemma that gives an account on isomorphisms of configurations $\mathfrak{M}_{(n,\varphi)}$ with the unique decomposition into two n-simplices.

Lemma 4.1. *Let f be an isomorphism mapping $\mathfrak{M}_{(n,\varphi)}$ onto $\mathfrak{M}_{(n,\psi)}$. Assume that either $n = 4$ and both $\varphi, \psi \neq \mathrm{id}$ contain no cycle of length 2, or $n \geq 5$. There is $\alpha \in S_n$ such that $f(B_i) = B_{\alpha(i)}$ for each $i \in I$, or $f(B_i) = A_{\alpha(i)}$ for each $i \in I$.*

 (i) *If $f(B_i) = B_{\alpha(i)}$ then $f(b_i) = b_{\alpha(i)}$, $f(A_i) = A_{\alpha(i)}$, $f(a_i) = a_{\alpha(i)}$ for each $i \in I$.*

 (ii) *If $f(B_i) = A_{\alpha(i)}$ then $f(b_i) = a_{\alpha(i)}$, $f(A_i) = B_{\psi^{-1}(\alpha(i))}$, $f(a_i) = b_{\psi^{-1}(\alpha(i))}$ for each $i \in I$.*

Furthermore, $\alpha\varphi = \psi\alpha$ holds in both cases: (i) *and* (ii).

Proof. Let $\mathcal{M}_1 := \mathfrak{M}_{(n,\varphi)}$ and $\mathcal{M}_2 := \mathfrak{M}_{(n,\psi)}$. Let $i, j \in I$ and B_i be an arbitrary B-block of $\mathcal{M}_1$. Clearly, either $f(B_i) = B_j$ for some B-block B_j of $\mathcal{M}_2$, or $f(B_i) = A_j$ for some A-block A_j of $\mathcal{M}_2$.

 Assume that $f(B_i) = B_j$. In view of Corollary 3.4, both $\mathcal{M}_1$, $\mathcal{M}_2$ are Möbius n-pairs without the special decompositions. Thus all the B-blocks of $\mathcal{M}_1$ are mapped onto B-blocks of $\mathcal{M}_2$. We introduce a map $\alpha \in S_n$ associated with f by the formula

$$\alpha \colon i \mapsto j \text{ iff } f(B_i) = B_j,$$

for all $i, j \in I$. Then $f(B_i) = B_{\alpha(i)}$. Let us analyze graphs $\mathcal{G}_{\mathcal{M}_1}$ and $\mathcal{G}_{\mathcal{M}_2}$: $f(b_i) = b_{\alpha(i)}$ as b_i, $b_{\alpha(i)}$ are unique b-vertices not adjacent to B_i, $B_{\alpha(i)}$ respectively in graphs $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(A_i) = A_{\alpha(i)}$ as A_i, $A_{\alpha(i)}$ are unique A-vertices adjacent to b_i, $b_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(a_i) = a_{\alpha(i)}$ as a_i, $a_{\alpha(i)}$ are unique a-vertices not adjacent to A_i, $A_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. On the other hand, $f(a_{\varphi(i)}) = a_{\psi(\alpha(i))}$ as $a_{\varphi(i)}$, $a_{\psi(\alpha(i))}$ are unique a-vertices adjacent to B_i, $B_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. So $a_{\alpha(\varphi(i))} = a_{\psi(\alpha(i))}$ and thus $\alpha\varphi = \psi\alpha$.

 In case $f(B_i) = A_j$ the map $\alpha \in S_n$ is determined by the condition

$$\alpha \colon i \mapsto j \text{ iff } f(B_i) = A_j,$$

for all $i, j \in I$. Then we proceed in a similar way as in the former case, namely: $f(b_i) = a_{\alpha(i)}$ as b_i, $a_{\alpha(i)}$ are a unique b-vertex and a-vertex not adjacent to B_i, $A_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(A_i) = B_{\psi^{-1}(\alpha(i))}$ as A_i, $B_{\psi^{-1}(\alpha(i))}$ are a unique A-vertex and B-vertex adjacent to b_i, $a_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(a_i) = b_{\psi^{-1}(\alpha(i))}$ as a_i, $b_{\psi^{-1}(\alpha(i))}$ are a unique a-vertex and b-vertex not adjacent to A_i, $B_{\psi^{-1}(\alpha(i))}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. But also $f(a_{\varphi(i)}) = b_{\alpha(i)}$ as $a_{\varphi(i)}$, $b_{\alpha(i)}$ are a unique a-vertex and b-vertex adjacent to B_i, $A_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. Hence $b_{\psi^{-1}(\alpha(\varphi(i)))} = b_{\alpha(i)}$, and consequently $\alpha\varphi = \psi\alpha$. $\square$

 We are ready to characterize two isomorphic Möbius n-pairs.

Theorem 4.2. *Let $n \geq 4$ and $\varphi, \psi, \alpha \in S_n$. The following conditions are equivalent:*

 (i) $\varphi^\alpha = \psi$,

 (ii) $\mathfrak{M}_{(n,\varphi)} \cong \mathfrak{M}_{(n,\psi)}$.

Proof. Let $\mathcal{M}_1 = \mathfrak{M}_{(n,\varphi)}$ and $\mathcal{M}_2 = \mathfrak{M}_{(n,\psi)}$.

 (i) $\Rightarrow$ (ii): Let $i \in I$, a_i, b_i be points and A_i, B_i be blocks of $\mathcal{M}_1$. Consider a map f associated to the permutation α given by the formula

$$f(x_i) = x_{\alpha(i)} \text{ for } x \in \{a, b\},$$

which maps the points of $\mathcal{M}_1$ onto the points of $\mathcal{M}_2$. Then $f(A_i) = A_{\alpha(i)}$ and $f(B_i) = B_{\alpha(i)}$, as the conditions $a_i \notin A_i$, $b_i \notin B_i$ uniquely determine blocks A_i, B_i, respectively. Clearly, conditions $b_i \in A_i$ and $b_{\alpha(i)} \in A_{\alpha(i)}$ are equivalent. Note that $a_{\alpha(\varphi(i))} \in B_{\alpha(i)}$ is equivalent to $a_{\psi(\alpha(i))} \in B_{\alpha(i)}$ as well, since $\alpha\varphi = \psi\alpha$. Thus f is the required isomorphism.

(ii) $\Rightarrow$ (i): We restrict ourselves to $n \geq 5$ since for $n = 4$ this fact is well known, as it was mentioned at the beginning of this section. Let f be an isomorphism mapping $\mathcal{M}_1$ onto $\mathcal{M}_2$. By Lemma 4.1, there is $\alpha \in S_n$ associated with f such that $\alpha\varphi = \psi\alpha$. $\qquad\square$

According to Theorem 4.2, the number of non-isomorphic configurations $\mathfrak{M}_{(n,\varphi)}$ is equal to the number of partitions $p(n)$ of a positive integer n. There is the generating function, recursive formula, asymptotic formula, and direct formula for $p(n)$ (cf. [1]). The increase of n implies quick growth of $p(n)$: $p(5) = 7, p(6) = 11, \ldots, p(100) = 190569292, \ldots, p(1000) = 24061467864032622473692149727991$.

4.2 The automorphism group structure of a Möbius n-pair

For $n = 3$ the structure $\mathfrak{M}_{(n,\varphi)}$ consists of two mutually inscribed triangles. From [8] the automorphism group of $\mathfrak{M}_{(3,\varphi)}$ is isomorphic to $S_3 \ltimes C_2$. From the original paper of Möbius [7] the automorphism group of $\mathfrak{M}_{(4,\mathrm{id})}$ has order 192. The Möbius configuration is also a particular case of the Cox configuration. Recall the definition of the Cox configuration (comp. [3]). Let X be a set with n elements. The incidence structure

$$(\mathbf{Cx})_X = (\mathbf{Cx})_n = \left\langle \bigcup \{ \wp_{2k+1}(X) : 0 \leq k \leq n \}, \{ \wp_{2k}(X) : 0 \leq k \leq n \}, \subset \cup \supset \right\rangle$$

is the $\binom{2^{n-1}}{n}$ configuration, which is called the Cox configuration. Since the automorphism group of $(\mathbf{Cx})_n$ is established in [11] and $\mathfrak{M}_{(4,\mathrm{id})} = (\mathbf{Cx})_4$ (see Figure 8), we get the following:

Fact 4.3. *The automorphism group of $\mathfrak{M}_{(4,\mathrm{id})}$ is isomorphic to $S_4 \ltimes C_2^3$.*

It follows from Theorem 4.2 that the centralizer of φ in S_n consists of automorphisms of $\mathfrak{M}_{(n,\varphi)}$ for any n. Nevertheless, we will give a detailed characterization of automorphism group of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi \neq \mathrm{id}$, and of $\mathfrak{M}_{(n,\varphi)}$ with $n \geq 5$.

Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$ and $1 \leq \nu_1 < \ldots < \nu_r$ be the lengths of the cycles which are contained in the cycle decomposition of $\varphi \in S_n$. Assume that there are m_t cycles of length ν_t, so $n = \sum_{t=1}^{r} m_t \nu_t$. In other words

$$\varphi = \varphi_1^{\nu_1} \varphi_2^{\nu_1} \cdots \varphi_{m_1}^{\nu_1} \varphi_1^{\nu_2} \varphi_2^{\nu_2} \cdots \varphi_{m_2}^{\nu_2} \cdots \varphi_1^{\nu_r} \varphi_2^{\nu_r} \cdots \varphi_{m_r}^{\nu_r},$$

where $\varphi_k^{\nu_t}$ is a cycle of length ν_t for $k \leq m_t, t \leq r$. In view of Theorem 4.2 we can assume, that each cycle consists of consecutive natural numbers. If we set $\mu_k^t := \sum_{i=1}^{t-1} m_i \nu_i + (k-1)\nu_t + 1$ then

$$\varphi_k^{\nu_t} : \mu_k^t \mapsto \mu_k^t + 1 \mapsto \mu_k^t + 2 \mapsto \ldots \mapsto \mu_k^t + (\nu_t - 1) \mapsto \mu_k^t,$$

and the effective domain of $\varphi_k^{\nu_t}$ is the set $X_k^{\nu_t} := \{\mu_k^t, \mu_k^t + 1, \ldots, \mu_k^t + (\nu_t - 1)\} \subseteq I$. Taking all the domains of all the cycles we obtain the family of pairwise disjoint sets $X_1^{\nu_1}, \ldots, X_{m_1}^{\nu_1}, X_1^{\nu_2}, \ldots, X_{m_2}^{\nu_2}, \ldots, X_1^{\nu_r}, \ldots, X_{m_r}^{\nu_r}$ that yields a covering of I. Thus for any cycle $\varphi_k^{\nu_t}$ we have $\varphi_k^{\nu_t}(X_k^{\nu_t}) = X_k^{\nu_t}$ and $\varphi_k^{\nu_t} \restriction_{I \setminus X_k^{\nu_t}} = \mathrm{id}$.
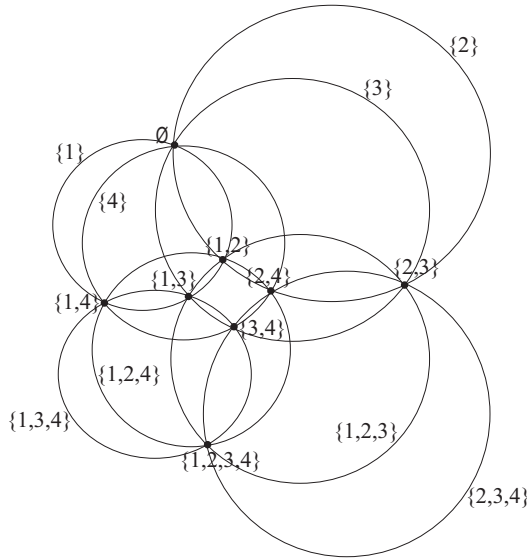
Figure 8: The Möbius configuration as $(\mathbf{Cx})_4$.

The points and the blocks of $\mathcal{M}$ can be identified with the sequences (t, k, i, ε) such that $t \leq r$, $k \leq m_t$, $i = 0, \ldots, \nu_t - 1$, and $\varepsilon \in \{1, 2, -1, -2\}$ according to the formula:

$$(t, k, i, \varepsilon) = \begin{cases} a_{i+\mu_k^t} & \text{for } \varepsilon = 1, \\ b_{i+\mu_k^t} & \text{for } \varepsilon = -1, \\ A_{i+\mu_k^t} & \text{for } \varepsilon = 2, \\ B_{i+\mu_k^t} & \text{for } \varepsilon = -2. \end{cases} \tag{4.1}$$

Let $v_t = (v_1^t, \ldots, v_{m_t}^t) \in C_{\nu_t}^{m_t}$, $\alpha_t \in S_{m_t}$, and $v = (v_1, \ldots, v_r) \in \times_{t=1}^{r} C_{\nu_t}^{m_t}$, $\alpha = (\alpha_1, \ldots, \alpha_r) \in \times_{t=1}^{r} S_{m_t}$. With the pair (v, α) we associate the map $f_{(v,\alpha)}$ as follows:

$$f_{(v,\alpha)}((t, k, i, \varepsilon)) = (t, \alpha_t(k), i + v_k^t \bmod \nu_t, \varepsilon). \tag{4.2}$$

In like manner we define the map $g_{(v,\alpha)}$ by:

$$g_{(v,\alpha)}((t, k, i, \varepsilon)) = \begin{cases} (t, \alpha_t(k), i + v_k^t - 1 \bmod \nu_t, -\varepsilon) & \text{for } \varepsilon = 1, 2, \\ (t, \alpha_t(k), i + v_k^t \bmod \nu_t, -\varepsilon) & \text{for } \varepsilon = -1, -2. \end{cases} \tag{4.3}$$

Lemma 4.4. *The map $f_{(v,\alpha)}$ is an automorphism of $\mathcal{M}$, which preserves each of simplices S_A, S_B.*

Proof. It follows directly from (4.2), that $f_{(v,\alpha)}$ maps S_A onto S_A, and $f_{(v,\alpha)}$ maps S_B onto S_B. Let $i \in X_k^{\nu_t}$ and $j \in I$.

Assume that $b_j \in B_i$. By (4.1), $B_i = (t, k, i_0, -2)$ for some $i_0 \in \{0, \ldots, \nu_t - 1\}$, and $b_j = (t', k', j_0, -1)$ for some $t' \leq r$, $k' \leq m_{t'}$, $j_0 \in \{0, \ldots, \nu_{t'} - 1\}$. Then $f(B_i) = (t, \alpha_t(k), i_0 + v_{\alpha_t(k)}^t \bmod \nu_t, -2)$ and $f(b_j) = (t', \alpha_{t'}(k'), j_0 + v_{\alpha_{t'}(k')}^{t'} \bmod \nu_{t'}, -1)$. Set $i' = (i_0 + v_{\alpha_t(k)}^t \bmod \nu_t) + \mu_{\alpha_t(k)}^t$ and $j' = (j_0 + v_{\alpha_{t'}(k')}^{t'} \bmod \nu_{t'}) + \mu_{\alpha_{t'}(k')}^{t'}$, so

$f(B_i) = B_{i'}$ and $f(b_j) = B_{j'}$. Recall that $b_j \in B_i$ iff $j \neq i$. If $j' \neq i'$ then: firstly $t' = t$, next $\alpha_t(k') = \alpha_t(k)$ and thus $k' = k$, and finally $j_0 = i_0$. It means that $j = i$, which yields a contradiction. Hence $f(b_j) \in f(B_i)$.

Let $a_j \in B_i$. Then $j = \varphi(i)$. We have $a_{\varphi(i)} = (t, k, i_0 + 1 \bmod \nu_t, 1)$, so $f(a_{\varphi(i)}) = (t, k, i_0 + 1 + v_k^t \bmod \nu_t, 1) = a_{\varphi(i')}$. Therefore $f(a_{\varphi(i)}) \in f(B_i)$.

The incidence (membership) relation is preserved by $f_{(v, \alpha)}$ in case $a_j \in A_i$ and in case $b_j \in A_i$ as well, that can be easily proved by similar reasoning. □

Let $\mathbf{v}_t = \underbrace{(v, \ldots, v)}_{t}$ for all $t \leq r$, and $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)$. Let us put $g_0 := g_{(\mathbf{0}, \mathrm{id})}$.

Lemma 4.5. *The map g_0 is an automorphism of $\mathcal{M}$, which interchanges simplices S_A, S_B.*

Proof. Immediately from (4.2), g_0 maps S_A onto S_B, and S_B onto S_A. We restrict our proof to the incidence relation involving the B-blocks, as the case with the A-blocks runs similarly. Let $i \in X_k^{\nu_t}$. From (4.1) B_i is represented by the sequence $(t, k, i_0, -2)$ for some $i_0 \in \{0, \ldots, \nu_t - 1\}$. The points that belongs to B_i are b_j with $j \in I \setminus \{i\}$ and $a_{\varphi(i)}$. Clearly, $g_0(b_j) = a_j \in A_i = g_0(B_i)$. We have $a_{\varphi(i)} = (t, k, i_0 + 1 \bmod \nu_t, 1)$ and thus $g_0(a_{\varphi(i)}) = (t, k, i_0, -1) = b_i$. Then finally $g_0(a_{\varphi(i)}) \in g_0(B_i)$. □

Since $g_{(v, \alpha)} = g_0 f_{(v, \alpha)}$, from Lemma 4.4 and Lemma 4.5 we infer that:

Corollary 4.6. *The map $g_{(v, \alpha)}$ is an automorphism of $\mathcal{M}$, which interchanges simplices S_A, S_B.*

We write $\mathcal{M}_k^{\nu_t}$ for the set of all the points and the blocks of $\mathcal{M}$ labelled by the elements of the set $X_k^{\nu_t}$, and $\mathcal{M}^{\nu_t} = \{\mathcal{M}_k^{\nu_t} : k \leq m_t\}$.

Lemma 4.7. *Let f be an automorphism of $\mathcal{M}$, which*

(1) maps the B-blocks onto the B-blocks, or

(2) maps the B-blocks onto the A-blocks.

There is $v \in \times_{t=1}^{r} C_{\nu_t}^{m_t}$ and $\alpha \in \times_{t=1}^{r} S_{m_t}$ such that
(i) $f = f_{(v, \alpha)}$ in case (1), or

(ii) $f = g_{(v, \alpha)}$ in case (2).
In particular, for each $k \leq m_t$ there is $k' \leq m_t$ such that $f(\mathcal{M}_k^{\nu_t}) = \mathcal{M}_{k'}^{\nu_t}$.

Proof. (i): Let $i \in X_k^{\nu_t}$. Assume that $f(B_i) = B_j$ for some $j \in I$. According to (4.1) there is $i_0 \in \{0, \ldots, \nu_t - 1\}$ such that $B_i = (t, k, i_0, -2)$, and $j_0 \in \{0, \ldots, \nu_{t'} - 1\}$ such that $B_j = (t', k', j_0, -2)$ for some $t' \leq r$, $k' \leq m_{t'}$. Then, by Lemma 4.1(ii) we get $f((t, k, i_0, \varepsilon)) = (t', k', j_0, \varepsilon)$ for each value of ε. The unique B-block containing $a_i = (t, k, i_0, 1)$ is $B_{\varphi^{-1}(i)} = (t, k, i_0 - 1 \bmod \nu_t, -2)$, and a unique B-block containing a_j is $B_{\varphi^{-1}(j)} = (t', k, j_0 - 1 \bmod \nu_{t'}, -2)$. Hence, f maps $(t, k, i_0 - 1 \bmod \nu_t, -2)$ onto $(t', k', j_0 - 1 \bmod \nu_{t'}, -2)$, and f maps $(t, k, i_0 - 1 \bmod \nu_t, \varepsilon)$ onto $(t', k', j_0 - 1 \bmod \nu_{t'}, \varepsilon)$ generally. By induction we get

$$f : (t, k, i_0 - u \bmod \nu_t, \varepsilon) \mapsto (t', k', j_0 - u \bmod \nu_{t'}, \varepsilon) \text{ for all } u = 0, \ldots, \nu_t - 1.$$

This characterizes the action of f on $\mathcal{M}_k^{\nu_t}$, in particular, $f(\mathcal{M}_k^{\nu_t}) \subseteq \mathcal{M}_{k'}^{\nu_{t'}}$. Conversely, f^{-1} maps B_j onto B_i. By the reasoning, analogous to this, which has already been done, we come to $f^{-1}(\mathcal{M}_{k'}^{\nu_{t'}}) \subseteq \mathcal{M}_k^{\nu_t}$. Consequently, $f(\mathcal{M}_k^{\nu_t}) = \mathcal{M}_{k'}^{\nu_{t'}}$, and therefore $t' = t$ since f is a bijection. It provides that f preserves the set $\mathcal{M}^{\nu_t}$. We define the map $\alpha \in S_{m_t}$ associated with $f \restriction_{\mathcal{M}^{\nu_t}}$ by the formula

$$\alpha \colon k \mapsto k' \text{ iff } f(\mathcal{M}_k^{\nu_t}) = \mathcal{M}_{k'}^{\nu_t},$$

for all $k, k' \leq m_t$. Set $v_k^t = j_0 - i_0 \bmod \nu_t$. Finally the formula for f is the following:

$$f \colon (t, k, i, \varepsilon) \mapsto (t, \alpha(k), i + v_k^t \bmod \nu_t, \varepsilon) \text{ for all } i = 0, \dots, \nu_t - 1.$$

(ii): Based on Lemma 4.5, $g_0 f$ is an automorphism of $\mathcal{M}$, which maps the B-blocks onto the B-blocks. Then, from Lemma 4.7(i), $g_0 f = f_{(v, \alpha)}$ for some $v \in \times_{t=1}^r C_{\nu_t}^{m_t}$ and $\alpha \in \times_{t=1}^r S_{m_t}$, and thus $f = g_0^{-1} f_{(v, \alpha)}$. Note that $g_0^{-1} = g_{(\mathbf{1}, \mathrm{id})}$. Consequently, $f = g_{(\mathbf{1}, \mathrm{id})} f_{(v, \alpha)} = g_0 f_{(v+\mathbf{1}, \alpha)} = g_{(v+\mathbf{1}, \alpha)}$. What is more, f preserves the set $\mathcal{M}^{\nu_t}$, that follows directly from (4.3). $\qquad \square$

Now we characterize automorphisms of $\mathfrak{M}_{(n,\varphi)}$, which can be uniquely decomposed into two mutually inscribed n-simplices.

Theorem 4.8. *Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$ and $1 \leq \nu_1 < \dots < \nu_r$ be the lengths of the cycles in the cycle decomposition of $\varphi \in S_n$. Assume that either $n = 4$ and $\varphi \neq \mathrm{id}$ contains no cycle of length 2, or $n \geq 5$. Then $\mathrm{Aut}(\mathcal{M}) \cong \bigoplus_{i=1}^r \left(C_{2\nu_i}^{m_i} \rtimes S_{m_i} \right)$.*

Proof. Let F be an automorphism of $\mathcal{M}$. By Proposition 3.3, there is no special decomposition of $\mathcal{M}$. Thus, F either interchanges $S_A(\mathcal{M})$ with $S_B(\mathcal{M})$ or preserves each of them. According to Lemma 4.7 there is $v_0 \in \times_{t=1}^r C_{\nu_t}^{m_t}$ and $\alpha_0 \in \times_{t=1}^r S_{m_t}$ such that $F = f_{(v_0, \alpha_0)}$ or $F = g_{(v_0, \alpha_0)} = g_0 f_{(v_0, \alpha_0)}$. Furthermore, every $f_{(v, \alpha)}$, $g_{(v, \alpha)}$ with $v \in \times_{t=1}^r C_{\nu_t}^{m_t}$ and $\alpha \in \times_{t=1}^r S_{m_t}$ is an automorphism of $\mathcal{M}$ by Lemma 4.4 and Corollary 4.6. Since, by Lemma 4.7, F preserves each of the sets $\mathcal{M}^{\nu_t}$, we can restrict the proof to the one fixed set $\mathcal{M}^{\nu_t}$. Thus, we assume that $i = 0, \dots, \nu_t - 1$, $k \leq m_t$. For the simplicity of the notation, we will write $(\alpha(k), i + v_{\alpha(k)}, \varepsilon)$ instead of $(t, \alpha_t(k), i + v_{\alpha_t(k)}^t \bmod \nu_t, \varepsilon)$. Moreover, we identify $f_{(v, \alpha)}$ with $f_{(v, \alpha)} \restriction_{\mathcal{M}^{\nu_t}}$, and $g_{(v, \alpha)}$ with $g_{(v, \alpha)} \restriction_{\mathcal{M}^{\nu_t}}$, so we assume $v \in C_{\nu_t}^{m_t}$, $\alpha \in S_{m_t}$. Let $w \in C_{\nu_t}^{m_t}$, $\beta \in S_{m_t}$ and note that

$$f_{(w, \beta)} f_{(v, \alpha)}((k, i, \varepsilon)) = f_{(w, \beta)}((\alpha(k), i + v_k, \varepsilon)) = (\beta\alpha(k), i + v_k + w_{\alpha(k)}, \varepsilon).$$

Let $\phi^\alpha \colon S_{m_t} \longrightarrow \mathrm{Aut}(C_{\nu_t}^{m_t})$ be the map defined by

$$\phi_\alpha \colon (v_1, \dots, v_{m_t}) \mapsto (v_{\alpha(1)}, \dots, v_{\alpha(m_t)}).$$

Then the formula for the composition of $f_{(v, \alpha)}$ and $f_{(w, \beta)}$ is

$$f_{(w, \beta)} f_{(v, \alpha)} = f_{(v + \phi_\alpha(w), \beta\alpha)}.$$

It is not difficult to check that g_0 and $f_{(v, \alpha)}$ commute. Note also that

$$g_0^z = \begin{cases} f_{(-\frac{z}{2}, \mathrm{id})} & \text{if } z \text{ is even}, \\ g_{(-\frac{z-1}{2}, \mathrm{id})} & \text{if } z \text{ is odd}. \end{cases}$$

Let $k' \leq m_{t'}$. We introduce the family of maps

$$g_{0_k}((k', i, \varepsilon)) = \begin{cases} g_0((k', i, \varepsilon)) & \text{if } k' = k, \\ (k', i, \varepsilon) & \text{otherwise.} \end{cases}$$

Then the following equalities hold

$$\begin{aligned}
\{g_{0_k}^z : z = 0, \ldots, 2\nu_t - 1 \text{ and } z \text{ is even}\} &= \{f_{(v,\mathrm{id})} : v_{k'} = 0 \text{ for } k' \neq k\}, \\
\{g_{0_k}^z : z = 0, \ldots, 2\nu_t - 1 \text{ and } z \text{ is odd}\} &= \{g_{(v,\mathrm{id})} : v_{k'} = 0 \text{ for } k' \neq k\}.
\end{aligned}$$

Therefore, for each $v \in C_{\nu_t}^{m_t}$ we have $f_{(v,\mathrm{id})} = g_{0_1}^{z_1} g_{0_2}^{z_2} \ldots g_{0_{m_t}}^{z_{m_t}}$, where all numbers $z_k = 0, \ldots, 2\nu_t - 1$ are even. Likewise $g_{(v,\mathrm{id})} = g_{0_1}^{z_1} g_{0_2}^{z_2} \ldots g_{0_{m_t}}^{z_{m_t}}$, where all numbers z_k are odd. Hence, for each $F \in \mathrm{Aut}(\mathcal{M})$ there is $v \in C_{\nu_t}^{m_t}$ and $\alpha \in S_{m_t}$ such that $F = f_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)}$ or $F = g_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)}$. To complete the proof it suffices to determine the remaining compositions:

$$\begin{aligned}
f_{(v,\mathrm{id})} f_{(w,\mathrm{id})} &= f_{(w+v,\mathrm{id})}, \\
f_{(\mathbf{0},\alpha)} f_{(\mathbf{0},\beta)} &= f_{(\mathbf{0},\beta\alpha)}, \\
f_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)} &= f_{(\phi^\alpha(v),\alpha)}, \\
g_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)} = g_0 f_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)} &= g_0 f_{(\phi^\alpha(v),\alpha)} = g_{(\phi^\alpha(v),\alpha)}, \\
f_{(v,\mathrm{id})} g_{(w,\mathrm{id})} = f_{(v,\mathrm{id})} g_0 f_{(w,\mathrm{id})} &= g_0 f_{(v,\mathrm{id})} f_{(w,\mathrm{id})} = g_0 f_{(w+v,\mathrm{id})} = g_{(w+v,\mathrm{id})}, \\
g_{(v,\mathrm{id})} g_{(w,\mathrm{id})} = g_0 f_{(v,\mathrm{id})} g_0 f_{(w,\mathrm{id})} &= g_0^2 f_{(w+v,\mathrm{id})} = f_{(-\mathbf{1},\mathrm{id})} f_{(w+v,\mathrm{id})} = f_{(w+v-\mathbf{1},\mathrm{id})}.
\end{aligned}$$

$\square$

The Möbius n-pairs, which automorphism groups are not characterized by Theorem 4.8, admit a special decomposition. We say that an automorphism f of a Möbius n-pair $\mathcal{M}$ *yields a special decomposition* of $\mathcal{M}$ if f maps the pair $\{S_A, S_B\}$ onto a distinct pair of mutually inscribed simplices.

Theorem 4.9. *The automorphism group of $\mathfrak{M}_{(4,\varphi)}$ is isomorphic to*

    (i) $(C_4 \oplus S_2) \rtimes C_2$ *if $\varphi \in S_4$ contains precisely one cycle of length 2,*

    (ii) $(C_4^2 \rtimes S_2) \rtimes C_2$ *if $\varphi \in S_4$ contains two cycles of length 2.*

Proof. In view of Theorem 4.2, without loss of generality we can consider $\mathcal{M}_1 = \mathfrak{M}_{(4,\varphi_1)}$ with $\varphi_1 = (1)(2)(34)$ in case (i), and $\mathcal{M}_2 = \mathfrak{M}_{(4,\varphi_2)}$ with $\varphi_2 = (12)(34)$ in case (ii) (comp. Figures 4, 5). Let $F_s \in \mathrm{Aut}(\mathcal{M}_s)$ for $s = 1, 2$. By Corollary 3.4, there is the special decomposition of each of $\mathcal{M}_s$. Thus, F_s maps the pair $\{S_A, S_B\}$ onto $\{S_A, S_B\}$ or F_s yields the special decomposition of $\mathcal{M}_s$. In case F_s maps the pair $\{S_A, S_B\}$ onto $\{S_A, S_B\}$, by Lemma 4.7, there is $v_0 \in \{0\} \times \{0\} \times C_2$, $\alpha_0 \in S_2 \times \{\mathrm{id}\}$ for $\mathcal{M}_1$, or $v_0 \in C_2 \times C_2$, $\alpha_0 \in S_2$ for $\mathcal{M}_2$ such that $F_s = f_{(v_0,\alpha_0)}$ or $F_s = g_{(v_0,\alpha_0)} = g_0 f_{(v_0,\alpha_0)}$, respectively for $s = 1, 2$. By Lemma 4.4 and Corollary 4.6 all maps $F_s f_{(v,\alpha)}$, $F_s g_{(v,\alpha)}$, where $v \in \{0\} \times \{0\} \times C_2$ and $\alpha \in S_2 \times \{\mathrm{id}\}$ if $s = 1$, or $v \in C_2 \times C_2$, $\alpha \in S_2$ if $s = 2$, are automorphisms of $\mathcal{M}_s$ preserving the pair $\{S_A, S_B\}$. Based on the proof of Theorem 4.8, these maps form the group $C_4 \oplus S_2$ if $s = 1$, and the group $C_4^2 \rtimes S_2$ if $s = 2$.

Consider the following two transformations:

x	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
$\tilde{f}(x)$	b_1	b_2	a_4	a_3	a_1	a_2	b_3	b_4
$\hat{f}(x)$	a_2	a_1	b_3	b_4	b_1	b_2	a_3	a_4

The map $\tilde{f}$ is an automorphism, which yields a special decomposition of $\mathcal{M}_1$; and $\hat{f}$ is an automorphism, which yields a special decomposition of $\mathcal{M}_2$. Assume that F_s yields a special decomposition of $\mathcal{M}_s$. Then $F_1 = \tilde{f}F_1'$ and $F_2 = \hat{f}F_2'$, where F_s' is the automorphism of $\mathcal{M}_s$ given by (4.2) or (4.3).

Let us set the commutativity rules in the automorphism group of $\mathcal{M}_s$. By (4.1), the points of $\mathcal{M}_1$, $\mathcal{M}_2$ correspond to the sequences (t, i, k, ε) with $\varepsilon = 1, -1$. Using the convention introduced at the beginning of this paragraph we get $t = 1, 2$, $\nu_1 = 1$, $\nu_2 = 2$, $m_1 = 2$, $m_2 = 1$ and $X_1^1 = \{1\}$, $X_2^1 = \{2\}$, $X_1^2 = \{3, 4\}$ for $\mathcal{M}_1$; $t = 1$, $\nu_1 = 2$, $m_1 = 2$, and $X_1^2 = \{1, 2\}$, $X_2^2 = \{3, 4\}$ for $\mathcal{M}_2$. To avoid any misunderstanding, in case $\mathcal{M}_2$ we will write Y_1^2, Y_2^2 instead of X_1^2, X_2^2 respectively. Then $\tilde{f}$ maps the points of $\mathcal{M}_1$ by the formula:

$$\tilde{f}((t, k, i, \varepsilon)) = \begin{cases} (t, k, i, -\varepsilon) & \text{for } i + \mu_k^t \in X_1^1, X_2^1, \\ (t, k, i+1 \bmod 2, \varepsilon) & \text{for } \varepsilon = 1,\ i + \mu_k^t \in X_1^2, \\ (t, k, i, \varepsilon) & \text{for } \varepsilon = -1,\ i + \mu_k^t \in X_1^2. \end{cases} \qquad (4.4)$$

The map $\hat{f}$ can be defined on points of $\mathcal{M}_2$ as:

$$\hat{f}((t, k, i, \varepsilon)) = \begin{cases} (t, k, i+1 \bmod 2, \varepsilon) & \text{for } \varepsilon = 1,\ i + \mu_k^t \in Y_1^2, \\ (t, k, i, \varepsilon) & \text{for } \varepsilon = -1,\ i + \mu_k^t \in Y_1^2, \\ (t, k, i, -\varepsilon) & \text{for } i + \mu_k^t \in Y_2^2. \end{cases} \qquad (4.5)$$

Note, that $\tilde{f}^2 = \hat{f}^2 = \text{id}$. Hence the cyclic group generated by $\tilde{f}$ and the cyclic group generated by $\hat{f}$ both coincide with C_2. All the formulas for compositions of $\tilde{f}$ with g_0, and $\tilde{f}$ with $f = f_{(v,\alpha)}$ can be calculated using (4.4) and (4.5) (it is rather technical and thus omitted) and then we get

$$\tilde{f}f = f\tilde{f} \quad \text{and} \quad \tilde{f}g_0 = \tilde{f}g_{(\mathbf{0},\text{id})} = g_{(\tau_{(0,0,1)}(\mathbf{0}),\text{id})}\tilde{f},$$

where $\tau_{(0,0,1)}(v) = \tau_{(0,0,1)}((v_1^1, v_2^1, v_1^2)) = (v_1^1, v_2^1, v_1^2 + 1)$. Analogous calculation can be done for $\hat{f}$. If we set $\tau_{(1,1)}(v) = \tau_{(1,1)}((v_1^1, v_2^1)) = (v_1^1 + 1, v_2^1 + 1)$, then

$$\hat{f}g_0 = \hat{f}g_{(\mathbf{0},\text{id})} = g_{(\tau_{(1,1)}(\mathbf{0}),\text{id})}\hat{f},$$
$$\hat{f}f_{(v,\alpha)} = f_{(v,\alpha)}\hat{f} \qquad \text{if only} \qquad \alpha = \text{id},$$
$$\hat{f}f_{(v,\alpha)} = g_0 f_{(v,\alpha)}\hat{f} = g_{(v,\alpha)}\hat{f} \quad \text{provided that} \quad i + \mu_k^t \in Y_1^2 \text{ and } \alpha = (12),$$
$$\hat{f}f_{(v,\alpha)} = g_0^{-1}f_{(v,\alpha)}\hat{f} = g_{(\tau_{(1,1)}(v),\alpha)}\hat{f} \quad \text{as long as} \quad i + \mu_k^t \in Y_2^2 \text{ and } \alpha = (12).$$

$\square$

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