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The 2018 Petra Šparl Award

Dr Petra Šparl was a talented woman mathematician with a promising future who worked in graph theory and combinatorics, but died mid-career in 2016 after a battle with cancer. In her memory, the Petra Šparl Award was established recently to recognise in each even-numbered year the best paper published in the previous five years by a young woman mathematician in one of the two journals *Ars Mathematica Contemporanea* (AMC) and *The Art of Discrete and Applied Mathematics* (ADAM).

Nominations for the inaugural award were invited in AMC in 2017, and cases were considered by a committee (consisting of the three of us) appointed by Dragan Marušič and Tomaž Pisanski as editors of AMC and ADAM.

As judges we were impressed by the large number of papers in AMC over the five years 2013–2017 having a woman author or co-author: almost 60 in total, with well over half of those being women in the early stages of their career. With helpful commentaries from co-authors (in some cases) we drew up a long list of candidates for the 2018 award, sought reports from referees on those, and also considered the papers themselves, before making a decision, which was unanimous.

The winner of the Petra Šparl Award for 2018 is *Dr Monika Pilśniak* (Department of Discrete Mathematics, AGH University, Kraków, Poland), for her paper 'Improving upper bounds for the distinguishing index', in *Ars Mathematica Contemporanea* **13** (2017), 259–274.

Dr Monika Pilśniak

Monika Pilśniak published four papers in AMC in 2016 and 2017, but one stands out: a single-author paper in 2017 on the distinguishing index of a graph. This is the smallest number of classes in a partition of the edge-set such that the only class-preserving automorphism of the graph is the identity automorphism. Monika helped introduce this concept in 2015, and in her 2017 paper in AMC, she classified all graphs with distinguishing index being at least equal to the maximum vertex degree. The main theorem is impressive, and difficult to prove, and it improves on the analogous theorem from 2005 on the distinguishing number (for partitions of vertices).

In summary, and quoting a referee: “Monika richly deserves the Petra Šparl Award: four papers in ACM pioneering a new concept, and one solo paper with an outstanding theorem, worthy of an award by itself.”

We would also like to make special mention of other high quality papers, by Sophie Decelle, María del Río Francos, Klavdija Kutnar, Klara Stokes and Aleksandra Tepeh.

Monika Pilśniak will be awarded a certificate and invited to give a lecture in the Mathematics Colloquium at the University of Primorska, and to give lectures at the University of Maribor and the University of Ljubljana.

Finally, we encourage nominations for the next Petra Šparl Award in 2020, and submissions of high quality new papers that will be worthy of consideration for future awards.

Marston Conder, Asia Ivić Weiss and Aleksander Malnič
Members of the 2018 Petra Šparl Award Committee

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Groups of Ree type in characteristic 3 acting on polytopes*

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Abstract

Every Ree group $\mathrm{R}(q)$, with $q \neq 3$ an odd power of 3, is the automorphism group of an abstract regular polytope, and any such polytope is necessarily a regular polyhedron (a map on a surface). However, an almost simple group G with $\mathrm{R}(q) < G \leq \mathsf{Aut}(\mathrm{R}(q))$ is not a C-group and therefore not the automorphism group of an abstract regular polytope of any rank.

Keywords: Abstract regular polytopes, string C-groups, small Ree groups, permutation groups.

Math. Subj. Class.: 52B11, 20D05

1 Introduction

Abstract polytopes are certain ranked partially ordered sets. A polytope is called "regular" if its automorphism group acts (simply) transitively on (maximal) flags. It is a natural question to try to classify all pairs $(\mathcal{P}, G)$, where $\mathcal{P}$ is a regular polytope and G is an automorphism group acting transitively on the flags of $\mathcal{P}$. An interesting subclass is constituted

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by the pairs $(\mathcal{P}, G)$ with G almost simple, as then a lot of information is available about the maximal subgroups, centralizers of involutions, etc., of these groups, making classification possible for some of these families of groups. Potentially this could also lead to new presentations for these groups, as well as a better understanding of some families of such groups using geometry.

The study of polytopes arising from families of almost simple groups has received a lot of attention in recent years and has been very successful. Mazurov [23] and Nuzhin [25, 26, 27, 28] established that most finite simple groups are generated by three involutions, two of which commute. These are precisely the groups that are automorphism groups of rank three regular polytopes. The exceptions are $PSL_3(q)$, $PSU_3(q)$, $PSL_4(2^n)$, $PSU_4(2^n)$, A_6, A_7, M_{11}, M_{22}, M_{23}, McL, $PSU_4(3)$, $PSU_5(2)$. The two latter, although mentioned by Nuzhin as being generated by three involutions, two of which commute, have been found to be exceptions recently by Martin Mačaj and Gareth Jones (personal communication). We refer to [18] for almost simple groups of Suzuki type (see also [16]); [4, 20, 21] for groups $\mathrm{PSL}_2(q) \leq G \leq \mathrm{P\Gamma L}_2(q)$; [2] for groups $\mathrm{PSL}_3(q)$ and $\mathrm{PGL}_3(q)$; [1] for groups $\mathrm{PSL}(4, q)$; [9] for symmetric groups; [3, 10, 11] for alternating groups; and [13, 19, 22] for the sporadic groups up to, and including, the third Conway group Co_3, but not the O'Nan group. Recently, Connor and Leemans have studied the rank 3 polytopes of the O'Nan group using character theory [5], and Connor, Leemans and Mixer have classified all polytopes of rank at least 4 of the O'Nan group [6].

Several attractive results were obtained in this vein, including, for instance, the proof that Coxeter's 57-cell and Grünbaum's 11-cell are the only regular rank 4 polytopes with a full automorphism group isomorphic to a group $\mathrm{PSL}_2(q)$ (see [20]). Another striking result is the discovery of the universal locally projective 4-polytope of type $\{\{5,3\}_5, \{3,5\}_{10}\}$, whose full automorphism group is $J_1 \times \mathrm{PSL}_2(19)$ (see [14]); this is based on the classification of all regular polytopes with an automorphism group given by the first Janko group J_1.

The existing results seem to suggest that polytopes of arbitrary high rank are difficult to obtain from a family of almost simple groups. Only the alternating and symmetric groups are currently known to act on abstract regular polytopes of arbitrary rank. For the sporadic groups the highest known rank is 5.

The Ree groups $\mathrm{R}(q)$, with $q = 3^{2e+1}$ and $e > 0$, were discovered by Rimhak Ree [29] in 1960. In the literature they are also denoted by ${}^2\mathrm{G}_2(q)$. These groups have a subgroup structure quite similar to that of the Suzuki simple groups $\mathrm{Sz}(q)$, with $q = 2^{2e+1}$ and $e > 0$. Suzuki and Ree groups play a somewhat special role in the theory of finite simple groups, since they exist because of a Frobenius twist, and hence have no counterpart in characteristic zero. Also, as groups of Lie-type, they have rank 1, which means that they act doubly transitively on sets of points without further apparent structure. However, the rank 2 groups which are used to define them, do impose some structure on these sets. For instance, the Suzuki groups act on "inversive planes". For the Ree groups, one can define a geometry known as a "unital". However, these unitals, called *Ree unitals*, have a very complicated and little accessible geometric structure (for instance, there is no geometric proof of the fact that the automorphism group of a Ree unital is an almost simple group of Ree type; one needs the classification of doubly transitive groups to prove this). Also, Ree groups seem to be misfits in a lot of general theories about Chevalley groups and their twisted analogues. For instance, there are no applications yet of the Curtis-Tits-Phan theory for Ree groups; all finite quasisimple groups of Lie type are known to be presented by two elements and 51 relations, except the Ree groups in characteristic 3 [12]. Hence it may be clear that the Ree groups $\mathrm{R}(q)$, with q a power of 3, deserve a separate treatment when investigating group actions on polytopes.

Now, the regular polytopes associated with Suzuki groups are quite well understood (see [16, 18]). But the techniques used for the Suzuki groups are not sufficient for the Ree groups. In the present paper, we carry out the analysis for the groups $\mathrm{R}(q)$. In particular, we ask for the possible ranks of regular polytopes whose automorphism group is such a group, and we prove the following theorem.

Theorem 1.1. *Among the almost simple groups G with $\mathrm{R}(q) \leq G \leq \mathsf{Aut}(\mathrm{R}(q))$ and $q = 3^{2e+1} \neq 3$, only the Ree group $\mathrm{R}(q)$ itself is a C-group. In particular, $\mathrm{R}(q)$ admits a representation as a string C-group of rank 3, but not of higher rank. Moreover, the non-simple Ree group $\mathrm{R}(3)$ is not a C-group.*

In other words, the groups $\mathrm{R}(q)$ behave just like the Suzuki groups: they allow representations as string C-groups, but only of rank 3. Although Nuzhin proved in [27] that these groups allow representations as string C-groups of rank 3 for every q, we will describe a string C-group representation for $\mathrm{R}(q)$, $q \neq 3$, for each value of q to make the paper self-contained. Also, almost simple groups $\mathrm{R}(q) < G \leq \mathrm{Aut}(\mathrm{R}(q))$ can never be C-groups (in characteristic 3).

Rephrased in terms of polytopes, Theorem 1.1 says that among the almost simple groups $\mathrm{R}(q) \leq G \leq \mathrm{Aut}(\mathrm{R}(q))$, only the groups $G := \mathrm{R}(q)$ are automorphism groups of regular polytopes, and that these polytopes must necessarily have rank 3.

Ree groups can also be the automorphism groups of abstract chiral polytopes. In fact, Sah [30] showed that every Ree group $\mathrm{R}(3^{2e+1})$, with $2e+1$ an odd prime, is a Hurwitz group; and Jones [15] later extended this result to arbitrary simple Ree groups $\mathrm{R}(q)$, proving in particular that the corresponding presentations give chiral maps on surfaces. Hence the groups $\mathrm{R}(q)$ are also automorphism groups of abstract chiral polyhedra.

It is an interesting open problem to explore whether or not almost simple groups of Ree type also occur as automorphism groups of chiral polytopes of higher rank.

Note that the Ree groups in characteristic 2 are also very special: they are the only (finite) groups of Lie type arising from a Frobenius twist and having rank at least 2. This makes them special, in a way rather different from the way the Ree groups in characteristic 3 are special. We think that in characteristic 2, quite different geometric methods will have to be used in the study of polytopes related to Ree groups.

2 Basic notions

2.1 Abstract polytopes and string C-groups

For general background on (abstract) regular polytopes and C-groups we refer to McMullen & Schulte [24, Chapter 2].

A polytope $\mathcal{P}$ is a ranked partially ordered set whose elements are called *faces*. A polytope $\mathcal{P}$ of rank n has faces of ranks $-1, 0, \ldots, n$; the faces of ranks 0, 1 or $n-1$ are also called *vertices*, *edges* or *facets*, respectively. In particular, $\mathcal{P}$ has a smallest and a largest face, of ranks -1 and n, respectively. Each flag of $\mathcal{P}$ contains $n+2$ faces, one for each rank. In addition to being locally and globally connected (in a well-defined sense), $\mathcal{P}$ is *thin*; that is, for every flag and every $j = 0, \ldots, n-1$, there is precisely *one* other (j-*adjacent*) flag with the same faces except the j-face. A polytope of rank 3 is a *polyhedron*.

A polytope $\mathcal{P}$ is *regular* if its (automorphism) group $\Gamma(\mathcal{P})$ is transitive on the flags. If $\Gamma(\mathcal{P})$ has exactly two orbits on the flags such that adjacent flags are in distinct orbits, then $\mathcal{P}$ is said to be *chiral*.

The groups of regular polytopes are string C-groups, and vice versa. A *C-group* of *rank* n is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ satisfying the following *intersection property*:

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle \quad (J, K \subseteq \{0, \ldots, n-1\}).$$

Moreover, G, or rather $(G, \{\rho_0, \ldots, \rho_{n-1}\})$, is a *string* C-group (of rank n) if the underlying Coxeter diagram is a string diagram; that is, if the generators satisfy the relations

$$(\rho_j \rho_k)^2 = 1 \qquad (0 \leq j < k-1 \leq n-2).$$

Let $G_i := \langle \rho_j \mid j \neq i \rangle$ for each $i = 0, 1, \ldots, n-1$, and let $G_{ij} := \langle \rho_k \mid k \neq i, j \rangle$ for each $i, j = 0, 1, \ldots, n-1$ with $i \neq j$.

Each string C-group G (uniquely) determines a regular n-polytope $\mathcal{P}$ with automorphism group G. The i-*faces* of $\mathcal{P}$ are the right cosets of the distinguished subgroup G_i for each $i = 0, 1, \ldots, n-1$, and two faces are incident just when they intersect as cosets; formally we must adjoin two copies of G itself, as the (unique) (-1)- and n-faces of $\mathcal{P}$. Conversely, the group $\Gamma(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is a string C-group, whose generators ρ_j map a fixed, or *base*, flag Φ of $\mathcal{P}$ to the j-adjacent flag Φ^j (differing from Φ in the j-face).

2.2 The Ree groups in characteristic 3

We let C_k denote a cyclic group of order k and D_{2k} a dihedral group of order $2k$.

The Ree group $G := \mathrm{R}(q)$, with $q = 3^{2e+1}$ and $e \geq 0$, is a group of order $q^3(q-1)(q^3+1)$. It has a faithful permutation representation on a Steiner system $\mathcal{S} := (\Omega, \mathcal{B}) = S(2, q+1, q^3+1)$ consisting of a set Ω of q^3+1 elements, the *points*, and a family of $(q+1)$-subsets $\mathcal{B}$ of Ω, the *blocks*, such that any two points of Ω lie in exactly one block. This Steiner system is also called a *Ree unital*. In particular, G acts 2-transitively on the points and transitively on the incident pairs of points and blocks of $\mathcal{S}$.

The group G has a unique conjugacy class of involutions (see [29]). Every involution ρ of G has a block B of $\mathcal{S}$ as its set of fixed points, and B is invariant under the centralizer $C_G(\rho)$ of ρ in G. Moreover, $C_G(\rho) \cong C_2 \times \mathrm{PSL}_2(q)$, where $C_2 = \langle \rho \rangle$ and the $\mathrm{PSL}_2(q)$-factor acts on the $q+1$ points in B as it does on the points of the projective line $PG(1, q)$.

The Ree groups $\mathrm{R}(q)$ are simple except when $q = 3$. In particular, $\mathrm{R}(3) \cong P\Gamma L_2(8) \cong \mathrm{PSL}_2(8) : C_3$ and the commutator subgroup $\mathrm{R}(3)'$ of $\mathrm{R}(3)$ is isomorphic to $\mathrm{PSL}_2(8)$.

A list of the maximal subgroups of G is available, for instance, in [32, p. 349] and [17]. Here we briefly review the list for $\mathrm{R}(q)$, with $q \neq 3$, as the maximal subgroups are required in the proof of Theorem 1.1; in parentheses we also note their characteristic properties relative to the Steiner system $\mathcal{S}$.

- $N_G(A) \cong A : C_{q-1}$ (stabilizer of a point), where A is a 3-Sylow subgroup of G;
- $C_G(\rho) \cong C_2 \times \mathrm{PSL}_2(q)$ (stabilizer of a block), where $C_2 = \langle \rho \rangle$ and ρ is an involution of G;
- $\mathrm{R}(q_0)$ (stabilizer of a sub-unital of $\mathcal{S}$), where $(q_0)^p = q$ and p is a prime;

- $N_G(A_i)$, for $i = 1, 2, 3$, where A_i is a cyclic subgroup of G of one of the following kinds:

  – $A_1 = C_{\frac{q+1}{4}}$, with $N_G(A_1) \cong (C_2^2 \times D_{\frac{q+1}{2}}) : C_3$;
  – $A_2 = C_{q+1-3^{e+1}}$, with $N_G(A_2) \cong A_2 : C_6$;
  – $A_3 = C_{q+1+3^{e+1}}$, with $N_G(A_3) \cong A_3 : C_6$.

Note here that $q \equiv 3 \bmod 8$, so $(q-1)/2$ is odd and $(q+1)/2$ is even. Moreover, since p is odd, $q_0 - 1$ and $q_0 + 1$ divide $q - 1$ and $q + 1$, respectively. Finally, $q + 1$ is divisible by 4 but not by 8.

The automorphism group $\mathsf{Aut}(\mathrm{R}(q))$ of $\mathrm{R}(q)$ is given by

$$\mathsf{Aut}(\mathrm{R}(q)) \cong \mathrm{R}(q){:}C_{2e+1},$$

so in particular $\mathsf{Aut}(\mathrm{R}(3)) \cong \mathrm{R}(3)$.

In the proof of our theorem we need the following lemma about normalizers of dihedral subgroups of dihedral groups. The proof is straightforward.

Lemma 2.1. *Let $m, n > 1$ be integers such that $m \mid n$. The normalizer $N_{D_{2n}}(D_{2m})$ of any subgroup D_{2m} of D_{2n} coincides with D_{2m} if n/m is odd, or is isomorphic to a subgroup D_{4m} of D_{2n} if n/m is even.*

3 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on a sequence of lemmas. We begin in Lemma 3.1 by showing that if $\mathrm{R}(q) < G \leq \mathrm{Aut}(\mathrm{R}(q))$ then G can not be a C-group (with any underlying Coxeter diagram). Thus only the Ree groups $\mathrm{R}(q)$ themselves need further consideration. Then we prove in Lemma 3.3 that $\mathrm{R}(q)$ does not admit a representation as a string C-group of rank at least 5. In the subsequent Lemmas 3.9, 3.11 and 3.12 we then extend this to rank 4 and show that $\mathrm{R}(q)$ can also not be represented as a string C-group of rank 4. Finally, in Lemma 3.15 we construct each group $\mathrm{R}(q)$ as a rank 3 string C-group.

All information that we use about the groups $\mathrm{R}(q)$ can found in [17].

We repeatedly make use of the following simple observation. If $A : B$ is a semi-direct product of finite groups A, B such that B has odd order, then each involution in $A : B$ must lie in A. In fact, if $\rho = \alpha\beta$ is an involution, with $\alpha \in A$, $\beta \in B$, then $1 = \rho^2 = \alpha(\beta\alpha\beta^{-1})\beta^2$, where $\alpha(\beta\alpha\beta^{-1}) \in A$ and $\beta^2 \in B$; hence $\beta^2 = 1$, so $\beta = 1$ and $\rho = \alpha \in A$.

3.1 Reduction to simple groups $\mathrm{R}(q)$

We begin by eliminating the almost simple groups of Ree type that are not simple.

Lemma 3.1. *Let $\mathrm{R}(q) < G \leq \mathsf{Aut}(\mathrm{R}(q))$, where $q = 3^{2e+1}$. Then G is not a C-group.*

Proof. Since $\mathsf{Aut}(\mathrm{R}(q)) \cong \mathrm{R}(q){:}C_{2e+1}$ and $2e+1$ is odd, every involution in $\mathsf{Aut}(\mathrm{R}(q))$ lies in $\mathrm{R}(q)$ (by the previous observation), and hence any subgroup of $\mathsf{Aut}(\mathrm{R}(q))$ generated by involutions must be a subgroup of $\mathrm{R}(q)$. Thus no subgroup G of $\mathsf{Aut}(\mathrm{R}(q))$ strictly above $\mathrm{R}(q)$ can be a C-group. (When $e = 0$ we have $\mathsf{Aut}(\mathrm{R}(3)) \cong \mathrm{R}(3)$, so the statement holds trivially.) □

3.2 String C-groups of rank at least five

By Lemma 3.1 we may restrict ourselves to Ree groups $G = \mathrm{R}(q)$. We first rule out the possibility that the rank is 5 or larger.

Lemma 3.2. *Let G be a simple group. Suppose G has a generating set $S := \{\rho_0, \ldots, \rho_{n-1}\}$ of n involutions such that (G, S) is a string C-group. Then $|\rho_i\rho_{i+1}| \geq 3$ for all $i = 0, \ldots, n-2$.*

Proof. This is due to the fact that, as G is simple, G is not directly decomposable, that is, G cannot be written as the direct product of two nontrivial normal subgroups of G. □

Lemma 3.3. *Let $G = \mathrm{R}(q)$, where $q = 3^{2e+1} \neq 3$. Suppose G has a generating set S of n involutions such that (G, S) is a string C-group. Then $n \leq 4$.*

Proof. Let $S = \{\rho_0, \ldots, \rho_{n-1}\}$, so in particular, $G = \langle\rho_0, \ldots, \rho_{n-1}\rangle$. Then ρ_0 commutes with $\rho_2, \ldots, \rho_{n-1}$, since the underlying Coxeter diagram is a string. However, by Lemma 3.2, ρ_0 does not commute with ρ_1 and ρ_{n-1} does not commute with ρ_{n-2}. Now suppose $n \geq 5$ and consider the subgroup $H := \langle\rho_0, \rho_1, \rho_{n-2}, \rho_{n-1}\rangle$ of G. Then H must be isomorphic to $D_{2c} \times D_{2d}$ for some integers $c, d \geq 3$. Inspection of the list of maximal subgroups of $\mathrm{R}(q)$ described above shows that direct products of (non-abelian) dihedral groups never occur as subgroups in G. So n is at most 4. □

3.3 String C-groups of rank four

Next we eliminate the possibility that the rank is 4. We begin with a general lemma about string C-groups that are simple.

Lemma 3.4. *Let (G, S) be a string C-group of rank n, and let G be simple. Then*

$$N_G(G_{01}) \backslash N_G(G_0)$$

must contain an involution (namely ρ_0).

Proof. The involution ρ_0 centralizes G_{01} and hence must lie in $N_G(G_{01})$. On the other hand, ρ_0 cannot also lie in $N_G(G_0)$ for otherwise G_0 would have to be a nontrivial normal subgroup in the simple group G. □

The next two lemmas will be applied to dihedral subgroups in subgroups of type $\mathrm{PSL}_2(q)$ or $C_2 \times \mathrm{PSL}_2(q)$ of $\mathrm{R}(q)$, respectively.

Lemma 3.5. *Let $q = 3^{2e+1}$ and $e \geq 0$. Then the order $2d$ of a non-abelian dihedral subgroup of $\mathrm{PSL}_2(q)$ must divide $q-1$ or $q+1$. Moreover, $d \not\equiv 0 \bmod 4$, and d is even only if $2d$ divides $q+1$.*

Proof. Suppose D_{2d} is a non-abelian dihedral subgroup of $\mathrm{PSL}_2(q)$, so $d \geq 3$. We claim that $2d$ must divide $q+1$ or $q-1$. Recall that under the assumptions on q, the order $2d$ must either be 6 or must divide $q-1$ or $q+1$. It remains to eliminate 6 as a possible order. In fact, since q is an odd power of 3, the only maximal subgroups of $\mathrm{PSL}_2(q)$ with an order divisible by 6 are subgroups $\mathrm{PSL}_2(q_0)$ with q_0 a smaller odd power of 3 (see [8] for a list of the subgroups of $\mathrm{PSL}(2, q)$). If we apply this argument over and over again with smaller odd powers of 3, we eventually are left with a subgroup $\mathrm{PSL}_2(3)$. However,

$\mathrm{PSL}_2(3) \cong A_4$ and hence cannot have a subgroup of order 6. Thus $2d$ must divide $q+1$ or $q-1$. This proves the first statement of the lemma. The second statement follows from the fact that $q \equiv 3 \bmod 8$. $\square$

Lemma 3.6. *Let* $q = 3^{2e+1}$ *and* $e \geq 0$*, let* $2d$ *divide* $q-1$ *or* $q+1$*, and let* D_{2d} *be a non-abelian dihedral subgroup of a group* $C := C_2 \times \mathrm{PSL}_2(q)$*.*

(a) Then there exists a dihedral subgroup D *in* $\mathrm{PSL}_2(q)$ *such that* D_{2d} *is a subgroup of* $C_2 \times D$ *of index* 1 *or* 2*, and* $N_C(D_{2d}) = N_C(C_2 \times D) = C_2 \times N_{\mathrm{PSL}_2(q)}(D)$*. Here the normalizer* $N_{\mathrm{PSL}_2(q)}(D)$ *must lie in a maximal subgroup* D_{q+1} *or* D_{q-1} *of* $\mathrm{PSL}_2(q)$*, and coincide with* $N_{D_{q+1}}(D)$ *or* $N_{D_{q-1}}(D)$*, according as* $2d$ *divides* $q-1$ *or* $q+1$*.*

(b) Let $D_{2d} \cong D$ *(that is, the index is* 2*). If* $2d \mid (q-1)$ *or if* $2d \mid (q+1)$ *and* $(q+1)/2d$ *is odd, then* $N_{\mathrm{PSL}_2(q)}(D) = D$ *and* $N_C(D_{2d}) \cong C_2 \times D_{2d}$*. If* $2d \mid (q+1)$ *and* $(q+1)/2d$ *is even, then* $N_{\mathrm{PSL}_2(q)}(D) \cong D_{4d}$ *and* $N_C(D_{2d}) \cong C_2 \times D_{4d}$*.*

(c) If $D_{2d} = C_2 \times D$ *(that is,* d *is even,* $d/2$ *is odd,* $D \cong D_d$*, and the index is* 1*), then* $N_{\mathrm{PSL}_2(q)}(D) \cong D_{2d}$ *and* $N_C(D_{2d}) \cong C_2 \times D_{2d}$ *(regardless of whether* $2d \mid (q-1)$ *or* $2d \mid (q+1)$*).*

(d) The structure of $N_C(D_{2d})$ *only depends on* d *and* q*, not on the way in which* D_{2d} *is embedded in* C*.*

Proof. For the first part, suppose $C_2 = \langle \rho \rangle$ and $D_{2d} = \langle \sigma_0, \sigma_1 \rangle$ where σ_0, σ_1 are standard involutory generators for D_{2d}. Write $\sigma_0 = (\rho^i, \sigma_0')$ and $\sigma_1 = (\rho^j, \sigma_1')$ for some $i, j = 0, 1$ and involutions σ_0', σ_1' in $\mathrm{PSL}_2(p)$. Then $D := \langle \sigma_0', \sigma_1' \rangle$ is a dihedral subgroup of $\mathrm{PSL}_2(p)$, and D_{2d} lies in $C_2 \times D$. Since the period of $\sigma_0'\sigma_1'$ divides that of $\sigma_0\sigma_1$, the order of D is at most $2d$ and D_{2d} has index 1 or 2 in $C_2 \times D$. If this index is 1 then $D_{2d} = C_2 \times D$ (and d is even and $D \cong D_d$). If the index of D_{2d} in $C_2 \times D$ is 2, then $D \cong D_{2d}$ and $D_{2d} \cap \{1\} \times D$ must have index 1 or 2 in $\{1\} \times D$. If the index of $D_{2d} \cap \{1\} \times D$ in $\{1\} \times D$ is 1 then clearly $D_{2d} = \{1\} \times D$ and D_{2d} can be viewed as a subgroup of $\mathrm{PSL}_2(q)$. If the index of $D_{2d} \cap \{1\} \times D$ is 2, then $D_{2d} \cap \{1\} \times D$ is of the form $\{1\} \times E$ where E is either the cyclic subgroup C_d of D, or d is even and E is one of the two dihedral subgroups of D of order d. (Note here that D_{2d} cannot itself be a direct product in which one factor is generated by ρ, since ρ cannot lie in D_{2d}.)

Next we investigate normalizers. First note that the normalizer of a direct subproduct in a direct product of groups is the direct product of the normalizers of the component groups. Thus $N_C(C_2 \times D) = C_2 \times N_{\mathrm{PSL}_2(q)}(D)$.

We now show that the normalizers in C of the subgroups D_{2d} and $C_2 \times D$ coincide. There is nothing to prove if $D_{2d} = C_2 \times D$ or $D_{2d} = \{1\} \times D$. Now suppose that D_{2d} has index 2 in $C_2 \times D$ and E is as above. Then it is convenient to write D_{2d} in the form

$$D_{2d} \;=\; (\{1\} \times E) \;\cup\; (\{\rho\} \times (D \backslash E)). \tag{3.1}$$

If $(\alpha, \beta) \in C$ then

$$(\alpha, \beta) D_{2d} (\alpha, \beta)^{-1} = (\{1\} \times \beta E \beta^{-1}) \;\cup\; (\{\rho\} \times \beta(D \backslash E)\beta^{-1}). \tag{3.2}$$

Now if $(\alpha, \beta) \in N_C(D_{2d})$ then the group on the left in (3.2) is just D_{2d} itself and therefore $\beta E \beta^{-1} = E$ and $\beta(D \backslash E)\beta^{-1} = D \backslash E$. It follows that β normalizes both E and D, so in particular $(\alpha, \beta) \in N_C(C_2 \times D)$. Hence $N_C(D_{2d}) \leq N_C(C_2 \times D)$.

Now suppose that $(\alpha, \beta) \in N_C(C_2 \times D)$. Then β normalizes D. But $\beta E \beta^{-1}$ must be a subgroup of D of index 2 isomorphic to E, and hence $\beta E \beta^{-1}$ and E are either both cyclic or both are dihedral. Clearly, if both subgroups are cyclic then $\beta E\beta^{-1} = E$. However, the case when both subgroups are dihedral is more complicated. First recall that then d must be even. Now the normalizer $N_{\mathrm{PSL}_2(q)}(D)$ of the dihedral subgroup D of $\mathrm{PSL}_2(q)$ in $\mathrm{PSL}_2(q)$ either coincides with D (that is, D is self-normalized), or is a dihedral subgroup containing D as a subgroup of index 2. We claim that under the assumptions on q, the second possibility cannot occur. In fact, in this case the normalizer would have to be a group of order $4d$, and since d is even, its order would have to be divisible by 8; however, the order of $\mathrm{PSL}_2(q)$ is not divisible by 8 when q is an odd power of 3, so $\mathrm{PSL}_2(q)$ certainly cannot contain a subgroup with an order divisible by 8. Thus $N_{\mathrm{PSL}_2(q)}(D) = D$. But β belongs to the normalizer of D in $\mathrm{PSL}_2(q)$, so then β must lie in D. In particular, $\beta E\beta^{-1} = E$ since E is normal in D. Thus, in either case we have $\beta E\beta^{-1} = E$, and since $\beta D\beta^{-1} = D$, also $\beta(D\backslash E)\beta^{-1} = D\backslash E$. Hence, (3.2) shows that $(\alpha, \beta) \in N_C(D_{2d})$. Hence also $N_C(C_2 \times D) \leq N_C(D_{2d})$.

To complete the proof of the first part, note that D must lie in a maximal subgroup $D_{q\pm1}$ of $\mathrm{PSL}_2(q)$ and $N_{\mathrm{PSL}_2(q)}(D) = N_{D_{q\pm1}}(D)$.

The second and third part of the lemma follow from Lemma 3.2 applied to the dihedral subgroup D of $D_{q\pm1}$. In particular, D is self-normalized in $D_{q\pm1}$ if $(q\pm1)/|D|$ is odd, and $N_{\mathrm{PSL}_2(q)}(D)$ is a dihedral subgroup of $D_{q\pm1}$ of order $2|D|$ if $(q \pm 1)/|D|$ is even. Bear in mind that $(q - 1)/2$ is odd, and $(q + 1)/2$ is even but not divisible by 4.

To establish the last part of the lemma, note that $N_C(D_{2d}) \cong C_2 \times D_{2d}$, except when $D \cong D_{2d}$, $2d \mid (q+1)$ and $(q+1)/2d$ is even. However, since $q \equiv 3 \bmod 8$, if $2d \mid (q+1)$ and $(q + 1)/2d$ is even then d must be odd. In other words, the situation described in the third part of the lemma cannot occur as this would require d to be even. Thus, if $2d \mid (q+1)$ and $(q + 1)/2d$ is even, then we are necessarily in the situation described in second part of the lemma, and so necessarily $N_C(D_{2d}) \cong C_2 \times D_{4d}$. □

Our next lemma investigates possible C-subgroups of $G = \mathrm{R}(q)$ of rank 3. The vertex-figure of a putative regular 4-polytope with automorphism group G would have to be a regular polyhedron with a group of this kind.

Lemma 3.7. *The only proper subgroups of* $\mathrm{R}(q)$ *that could have the structure of a C-group of rank* 3 *are Ree subgroups* $\mathrm{R}(q_0)$ *with* $q_0 \neq 3$ *or subgroups of the form* $\mathrm{PSL}_2(q_0)$*,* $C_2 \times \mathrm{PSL}_2(q_0)$*, or* $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$*.*

Proof. It is straightforward (sometimes by applying Lemma 3.2) to verify that only subgroups of maximal subgroups of $\mathrm{R}(q)$ of the second and third type can have the structure of a rank 3 C-group. Therefore we are left with Ree subgroups $\mathrm{R}(q_0)$ and subgroups of groups $C_2 \times \mathrm{PSL}_2(q_0)$, with q_0 an odd power of 3 dividing q, as well as subgroups of type $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$ inside a subgroup $\mathrm{R}(3)$. A forward appeal to Lemma 3.15 shows that Ree groups $R(q_0)$ with $q_0 \neq 3$ do in fact act flag-transitively on polyhedra, and by [31], so does $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$. The complete list of subgroups of $\mathrm{PSL}_2(q_0)$ is available, for instance, in [20]. As q_0 is an odd power of 3, the group $\mathrm{PSL}_2(q_0)$ does not have subgroups isomorphic to A_5, S_4, or $\mathrm{PGL}_2(q_1)$ for some q_1. Hence, none of the subgroups of $\mathrm{PSL}_2(q_0)$, except for those isomorphic to a group $\mathrm{PSL}_2(q_1)$, with q_1 an odd power of 3 dividing q_0 (and hence q), admits flag-transitive actions on polyhedra. Now the maximal subgroups of $C_2 \times \mathrm{PSL}_2(q_0)$ consist of the factor $\mathrm{PSL}_2(q_0)$, as well as all subgroups of the form $C_2 \times H$ where H is a maximal subgroup of $\mathrm{PSL}_2(q_0)$ from the following list:

$$E_{q_0} : C_{\frac{q_0-1}{2}},\ D_{q_0-1},\ D_{q_0+1},\ \mathrm{PSL}_2(q_1).$$

A subgroup of $C_2 \times \mathrm{PSL}_2(q_0)$ of the form $C_2 \times D_{q_0-1}$ is isomorphic to $D_{2(q_0-1)}$ (since $q_0 \equiv 3 \bmod 8$), so none of its subgroups (including the full subgroup itself) can act regularly on a non-degenerate polyhedron (that is a polyhedron with no 2 in the Schläfli symbol). Similarly, a subgroup of $C_2 \times \mathrm{PSL}_2(q_0)$ of the form $C_2 \times D_{q_0+1}$ is isomorphic to $C_2 \times C_2 \times D_{(q_0+1)/2}$, so again none of its subgroups (including the full subgroup itself) can act regularly on a non-degenerate polyhedron. Finally, a subgroup of $C_2 \times \mathrm{PSL}_2(q_0)$ of the forms $C_2 \times (E_{q_0} : C_{(q_0-1)/2})$ has an order not divisible by 4. Hence, as in the two other cases, none of its subgroups (including the full subgroup itself) can act regularly on a non-degenerate polyhedron.

In summary, the only possible candidates for rank 3 subgroups of $\mathrm{R}(q)$ are of the form $\mathrm{R}(q_0)$, $\mathrm{PSL}_2(q_0)$, $C_2 \times \mathrm{PSL}_2(q_0)$, and $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$. We can further rule out a subgroup of type $\mathrm{R}(3)$, since $\mathrm{R}(3) \cong P\Gamma L_2(8)$ is not generated by involutions. $\square$

For a subgroup B of A we define $N_A^0(B) := \langle a \mid a \in N_A(B), a^2 = 1\rangle$. If B is generated by involutions then $B \leq N_A^0(B) \leq N_A(B)$. We first state a lemma that will be useful in several places.

Lemma 3.8. *Let $H := \mathrm{R}(3) = P\Gamma L_2(8)$, and let $D := D_{2d}$ be a dihedral subgroup of H of order at least 6. Then $d = 3$, 7 or 9, and in all cases $N_H^0(D) = D$.*

Proof. Straightforward. $\square$

The following lemma considerably limits the ways in which Ree groups $\mathrm{R}(q)$ might be representable as C-groups of rank 4.

Lemma 3.9. *If the group $G := \mathrm{R}(q)$ can be represented as a string C-group of rank 4, then*

$$N_G^0(G_{01}) = N_{C_G(\rho_0)}^0(G_{01}). \tag{3.3}$$

Proof. Suppose that G admits a representation as a string C-group of rank 4. Thus

$$G = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle.$$

Since $\mathrm{R}(3)$ is not generated by involutions, we must have $q \neq 3$.

The subgroup $G_{01} = \langle \rho_2, \rho_3 \rangle$ is a dihedral subgroup D_{2d} (say) of the centralizer $C_G(\rho_0)$ of ρ_0, and $C_G(\rho_0) \cong \langle \rho_0 \rangle \times \mathrm{PSL}_2(q)$. Here $d \geq 3$, by arguments similar to those used in the proof of Lemma 3.3. Thus

$$D_{2d} \cong \langle \rho_2, \rho_3 \rangle = G_{01} \leq G_1 = \langle \rho_0, \rho_2, \rho_3 \rangle \leq C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q). \tag{3.4}$$

By Lemma 3.6 applied to G_{01} and $C_G(\rho_0)$, there exists a dihedral subgroup D in the $\mathrm{PSL}_2(q)$-factor of $C_G(\rho_0)$ such that G_{01} is a subgroup of $\langle \rho_0 \rangle \times D = C_2 \times D$ of index at most 2 and

$$N_{C_G(\rho_0)}(G_{01}) = N_{C_G(\rho_0)}(C_2 \times D) = C_2 \times N_{PSL_2(q)}(D).$$

In fact, the proof of Lemma 3.6 shows that this subgroup $C_2 \times D$ is just given by G_1. But $\rho_0 \notin G_{01}$, so G_{01} has index 2 in $C_2 \times D = G_1$, and $D \cong G_{01} \cong D_{2d}$. Then Lemma 3.5, applied to D, shows that $2d$ must divide either $q + 1$ or $q - 1$.

The structure of the normalizer $N_{C_G(\rho_0)}(G_{01})$ can be obtained from Lemma 3.6. In fact, $N_{C_G(\rho_0)}(G_{01}) \cong C_2 \times D_{2d}$, unless $2d \mid (q + 1)$ and $(q + 1)/2d$ is even; in the latter case $N_{C_G(\rho_0)}(G_{01}) \cong C_2 \times D_{4d}$. In particular, $N_{C_G(\rho_0)}(G_{01})$ is generated by involutions and its order is divisible by 4. We will show that the normalizer of G_{01} in $C_G(\rho_0)$ captures all the information about the full normalizer $N_G(G_{01})$ of G_{01} in G that is relevant for us. A key step in the proof is the invariance of the structure of the normalizer of G_{01} in arbitrary subgroups of G of type $C_2 \times \mathrm{PSL}_2(q)$; more precisely, the structure only depends on d and q, not on the way in which G_{01} is embedded in a subgroup $C_2 \times \mathrm{PSL}_2(q)$ (see Lemma 3.6).

The full normalizer $N_G(G_{01})$ of G_{01} in G must certainly contain $N_{C_G(\rho_0)}(G_{01})$ and also have an order divisible by 8. We claim that all involutions of the full normalizer $N_G(G_{01})$ must already lie in $C_G(\rho_0)$ and hence in $N_{C_G(\rho_0)}(G_{01})$.

First note that $N_G(G_{01})$ must certainly lie in a maximal subgroup M of G and then coincide with $N_M(G_{01})$. (Since G is simple, the normalizer of a proper subgroup of G cannot coincide with G.) Inspection of the list of maximal subgroups of G shows that only maximal subgroups M of type $R(q_0)$, $C_2 \times \mathrm{PSL}_2(q)$ or $N_G(A_1)$ have an order divisible by 4. Only those maximal subgroups could perhaps contain $N_{C_G(\rho_0)}(G_{01})$ and hence $N_G(G_{01})$. We investigate the three possibilities for M separately.

Suppose M is a group of type $C_2 \times \mathrm{PSL}_2(q)$. Then the invariance of the structure of the normalizer of G_{01} shows that $N_M(G_{01}) \cong N_{C_G(\rho_0)}(G_{01})$. However, $N_{C_G(\rho_0)}(G_{01}) \leq N_G(G_{01})$ and $N_G(G_{01}) = N_M(G_{01})$, so this gives $N_G(G_{01}) = N_{C_G(\rho_0)}(G_{01})$. But $N_{C_G(\rho_0)}(G_{01})$ is generated by involutions, so $N_{C_G(\rho_0)}(G_{01}) = N^0_{C_G(\rho_0)}(G_{01})$ and (3.3) must hold as well.

Let M be a group of type $N_G(A_1) \cong (C_2^2 \times D_{(q+1)/2}) : C_3$ where A_1 is a group $C_{(q+1)/4}$ (recall that $(q+1)/4$ is odd). Then all involutions of M must lie in its subgroup $K := C_2^2 \times D_{(q+1)/2} = C_2 \times D_{q+1}$. In particular, all involutions of $N_M(G_{01})$ must lie in K and hence in $N_K(G_{01})$; that is, $N^0_M(G_{01}) \leq N_K(G_{01})$. Also, G_{01} itself must lie in K and its order $2d$ must divide $q+1$. The subgroup K lies in the centralizer C of the involution generating the C_2-factor in the direct product factorization $C_2 \times D_{q+1}$ for K, and $N_K(G_{01}) \leq N_C(G_{01})$. This subgroup C is of type $C_2 \times \mathrm{PSL}_2(q)$, and so again the invariance of the structure of the normalizers implies that $N_C(G_{01}) \cong N_{C_G(\rho_0)}(G_{01})$. But $N_G(G_{01}) = N_M(G_{01})$ and therefore

$$N_{C_G(\rho_0)}(G_{01}) = N^0_{C_G(\rho_0)}(G_{01}) \leq N^0_G(G_{01}) = N^0_M(G_{01}) \leq N_K(G_{01}) \leq N_C(G_{01}).$$

Thus $N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01})$, as required.

Now let M be a Ree group $\mathrm{R}(q_0)$ where $(q_0)^p = q$ and p is a prime. We first cover the case when M is a Ree group $\mathrm{R}(3) = \mathrm{PSL}_2(8) : C_3$, that is, $q = 3^p$ where p is a prime. In that case, by Lemma 3.8, $N^0_G(G_{01}) = N^0_M(G_{01}) = G_{01}$. Hence, since also $G_{01} \leq N^0_{C_G(\rho_0)}(G_{01}) \leq N^0_G(G_{01})$, we must have $N^0_G(G_{01}) \leq N^0_{C_G(\rho_0)}(G_{01})$.

Now suppose $q_0 \neq 3$, so in particular M is simple. Then $2d$ must divide $q_0 \pm 1$, since $N_{C_G(\rho_0)}(G_{01})$ lies in M and therefore $\rho_0 \in M$, giving $G_{01} \leq N_{C_M(\rho_0)}(G_{01}) \cong C_2 \times \mathrm{PSL}_2(q_0)$. Since the subgroup $N_G(G_{01})$ of M must have an order divisible by 4, it must lie in a maximal subgroup M' of M of type $\mathrm{R}(q_1)$, $C_2 \times \mathrm{PSL}_2(q_1)$, or $N_{\mathrm{R}(q_0)}(A'_1)$ with $A'_1 \cong C_{(q_0+1)/4}$. The maximal subgroups M' of $M = \mathrm{R}(q_0)$ of types $C_2 \times \mathrm{PSL}_2(q_0)$ and $N_{\mathrm{R}(q_0)}(A'_1)$, respectively, lie in maximal subgroups of G of type $C_2 \times \mathrm{PSL}_2(q)$ or $N_G(A_1)$, so they are subsumed under the previous discussion. (Alternatively we could dispose of these cases for M' directly, using arguments very similar to those in the two previous cases for M.) Then this leaves the possibility that M' is of type $R(q_1)$, in which case we are back at a Ree group. Now continuing in this fashion to smaller and smaller Ree subgroups that could perhaps contain $N_G(G_{01})$, we eventually arrive at either a Ree subgroup $M^{(k)}$

(say) whose parameter $q^{(k)} \pm 1$ (say) is no longer divisible by $2d$, or a Ree group $\mathrm{R}(3)$. In the first case, $\mathrm{R}(q_0)$ does not contribute anything new to $N^0_G(G_{01})$, and the normalizer $N_G(G_{01})$ must already lie in one of the maximal subgroups of type $C_2 \times \mathrm{PSL}_2(q)$ or $N_G(A_1)$ discussed earlier; in particular, $N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01})$, as required. In the second case, the normalizer $N_G(G_{01})$ lies in a Ree subgroup $\mathrm{R}(3) \cong \mathrm{PSL}_2(8) : C_3$, and its involutory part $N^0_G(G_{01})$ must lie in the $\mathrm{PSL}_2(8)$ subgroup. Again, by Lemma 3.8, we have $N^0_G(G_{01}) = N^0_M(G_{01}) = G_{01}$, hence (3.3) must also hold in this case. □

Lemma 3.10. *If the group $G := \mathrm{R}(q)$ can be represented as a string C-group of rank 4, then $q \neq 3$ and both the facet stabilizer G_3 and vertex stabilizer G_0 have to be isomorphic to* $\mathrm{PSL}_2(8) = \mathrm{R}(3)'$ *(i.e. the commutator subgroup of* $\mathrm{R}(3)$*) or a simple Ree group* $\mathrm{R}(q_0)$ *with $q = q_0^m$ for some odd integer m.*

Proof. We consider the possible choices for G_0 in the given C-group representation of G of rank 4. Our goal is to use Lemma 3.4 to limit the choices for G_0 to just $\mathrm{R}(3)'$ or $R(q_0)$. First recall from Lemma 3.7 that the only possible candidates for G_0 are either Ree subgroups $\mathrm{R}(q_0)$ with $q_0 \neq 3$ or subgroups of the form $\mathrm{PSL}_2(q_0)$, $C_2 \times \mathrm{PSL}_2(q_0)$, or $\mathrm{R}(3)' \cong \mathrm{PSL}_2(8)$. To complete the proof we must eliminate the second and third types of candidates. This is accomplished by means of Lemmas 3.4 and 3.6, proving in each case that $N_G(G_{01})\backslash N_G(G_0)$ cannot contain an involution, or equivalently $N^0_G(G_{01}) \leq N_G(G_0)$. Bear in mind that $G_{01} \leq G_0$.

First observe that all subgroups of G of the form $C_2 \times \mathrm{PSL}_2(q_0)$ are self-normalized in G; and the normalizer of a subgroup of G of the form $\mathrm{PSL}_2(q_0)$ is isomorphic to $C_2 \times \mathrm{PSL}_2(q_0)$. In other words, $N_G(G_0) = G_0$ if G_0 is of type $C_2 \times \mathrm{PSL}_2(q_0)$, and $N_G(G_0) = C_2 \times G_0$ if G_0 is of type $\mathrm{PSL}_2(q_0)$. We show that $N^0_G(G_{01}) \leq N_G(G_0)$ for each of these two choices of G_0.

Suppose that $G_0 \cong C_2 \times \mathrm{PSL}_2(q_0)$. We first claim that then $2d \mid q_0 \pm 1$ (where $2d = |G_{01}|$). To see this, note that the intersection of G_{01} with the $\mathrm{PSL}_2(q_0)$-factor of G_0 is a subgroup of index 1 or 2 in G_{01}. If the index is 1, the statement is clear by Lemma 3.5, since then G_{01} lies in the $\mathrm{PSL}_2(q_0)$-factor; and if the index is 2 and the intersection is a cyclic group C_d, the statement follows by inspection of the possible orders of cyclic subgroups of $\mathrm{PSL}_2(q_0)$. Now if the index is 2 and the intersection is a dihedral group D_d, then Lemma 3.6 shows that d must be even, $2d \mid q + 1$, and $d/2$ must be odd; moreover, $d \mid q_0 + 1$ since D_d lies in $\mathrm{PSL}_2(q_0)$, and hence $2d \mid q_0 + 1$ since $q_0 + 1$ is divisible by 4. Thus $2d \mid q_0 \pm 1$, as claimed.

Now, since $G_0 \cong C_2 \times \mathrm{PSL}_2(q_0)$, the normalizer $N_{C_G(\rho_0)}(G_{01})$ coincides with the normalizer $N_H(G_{01})$ of G_{01} taken in a suitable subgroup H of $C_G(\rho_0)$ of type $C_2 \times \mathrm{PSL}_2(q_0)$. In fact, from Lemma 3.6 we know that

$$N_{C_G(\rho_0)}(G_{01}) \leq C_2 \times D_{q\pm1} \leq C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q).$$

But $2d \mid q_0 \pm 1$, so we must have $N_{C_G(\rho_0)}(G_{01}) \leq C_2 \times D_{q_0\pm1}$. However, $C_2 \times D_{q_0\pm1}$ lies in a subgroup H of $C_G(\rho_0)$ isomorphic to $C_2 \times \mathrm{PSL}_2(q_0)$.

To complete the argument (for any given type of group G_0) we show that $N^0_G(G_{01})$ must lie in $N_{G_0}(G_{01})$ and therefore also in G_0 and $N_G(G_0)$. When G_0 is a group of type $C_2 \times \mathrm{PSL}_2(q_0)$, the normalizer $N_{G_0}(G_{01})$ can be determined using Lemma 3.6 (with q replaced by q_0). In fact, by the invariance of the normalizers of G_{01} we know that $N_{G_0}(G_{01})$

and $N_H(G_{01})$ are isomorphic and that both subgroups are generated by involutions. However, then by Lemma 3.9,

$$N_{G_0}(G_{01}) = N^0_{G_0}(G_{01}) \leq N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01}) = N^0_H(G_{01}) = N_H(G_{01}),$$

so clearly $N_{G_0}(G_{01}) = N_H(G_{01})$. Thus $N^0_G(G_{01}) = N_{G_0}(G_{01}) \leq G_0 \leq N_G(G_0)$.

Now let G_0 be of type $\mathrm{PSL}_2(q_0)$. Then $C := N_G(G_0)$ is a group of type $C_2 \times \mathrm{PSL}_2(q_0)$ containing G_0, so we can replace G_0 by C and argue as before. In fact, using the same subgroup H, we see that the normalizers $N_C(G_{01})$ and $N_H(G_{01})$ are isomorphic subgroups generated by involutions. In particular,

$$N_C(G_{01}) = N^0_C(G_{01}) \leq N^0_G(G_{01}) = N^0_{C_G(\rho_0)}(G_{01}) = N^0_H(G_{01}) = N_H(G_{01}),$$

and therefore $N_C(G_{01}) = N_H(G_{01})$. Hence $N^0_G(G_{01}) = N_C(G_{01}) \leq C = N_G(G_0)$. □

Let us now show that $G_0 \not\cong R(3)'$.

Lemma 3.11. *If* $\mathrm{R}(q)$ *has a representation as a string C-group of rank 4 with* $G_0 \cong R(3)'$, *then* $q = 27$.

Proof. Suppose $G := \mathrm{R}(q)$ is represented as a string C-group of rank 4 with generators $\rho_0, \ldots, \rho_3$. Then we know that $G_{01} \leq G_1 \leq C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q)$.

The abstract regular polyhedra with automorphism group $R(3)' = \mathrm{PSL}_2(8)$ are all known and are available, for instance, in [22]. There are seven examples, up to isomorphism, but not all can occur in the present context. In fact, the dihedral subgroup G_{01} of G_0 must also lie $C_G(\rho_0) \cong C_2 \times \mathrm{PSL}_2(q)$ and hence cannot be a subgroup D_{18}. It follows that the polyhedron associated with G_0 (that is, the vertex-figure of the polytope for G) must have Schläfli symbol $\{3, 7\}$, $\{7, 3\}$, or $\{7, 7\}$. We can further rule out the possibility that $G_{01} \cong D_6$ by Lemmas 3.5 and 3.6, giving that $C_2 \times \mathrm{PSL}_2(q)$ has no dihedral subgroup of order 6. Hence $G_{01} \cong D_{14}$.

The fixed point set of every involution in G is a block of the corresponding Steiner system $S(2, q + 1, q^3 + 1)$, and vice versa, every block is the fixed point set of a unique involution. Hence, two involutions with two common fixed points must coincide, since their blocks of fixed points must coincide. Suppose B_0 denotes the block of fixed points of ρ_0. As ρ_2 and ρ_3 centralize ρ_0, they stabilize B_0 globally but not pointwise. However, ρ_2 cannot have a fixed point among the $q + 1$ points in B_0, since otherwise two points of B_0 would have to be fixed by ρ_2 since $q + 1$ is even. Thus ρ_2, and similarly ρ_3, does not fix any point in B_0. Moreover, in order for $G_{01} \cong D_{14}$ to lie in a subgroup of G of type $C_2 \times \mathrm{PSL}_2(q)$, we must have $7 \mid q + 1$ or $7 \mid q - 1$. Using $q = 3^{2e+1}$ and working modulo 7 the latter possibility is easily seen to be impossible. On the other hand, the former possibility occurs precisely when $e \equiv 1 \bmod 3$, and then $3 \mid 2e + 1$. Hence G must have subgroups isomorphic to $\mathrm{R}(27) = \mathrm{R}(3^3)$.

We claim that G itself is isomorphic to $\mathrm{R}(27)$, that is, $q = 27$. Now the subgroup $G_0 \cong R(3)'$ lies in a unique subgroup $K \cong \mathrm{R}(3)$ of G, namely its normalizer $N_G(G_0)$. Indeed, Figure 1 tells us that $G_0 \cong R(3)'$ is in a unique subgroup isomorphic to $R(27)$ (because of the lower 1's on the edges joining the boxes). This subgroup K, in turn, lies in a unique subgroup $H \cong \mathrm{R}(27)$ of G. All Ree subgroups of G are self-normalized in G, so in particular K and H are self-normalized. Relative to the Ree subgroup H, the normalizer $N_H(C_7)$ in H of the cyclic subgroup C_7 of G_{01} is a maximal subgroup of type

$N_H(A_1) = (C_2^2 \times D_{14}) : C_3$ in H, which also contains G_{01} (see Section 2.2 or [7, p. 123]). Note here that this subgroup C_7 is a 7-Sylow subgroup of both K and H, and is normalized by G_{01}. Thus, $N_H(C_7) = (C_2^2 \times D_{14}) : C_3$. We claim that $N_H(G_{01}) = N_H(C_7)$. Clearly, $N_H(G_{01}) \leq N_H(C_7)$. For the opposite inclusion observe that $(C_2^2 \times D_{14}) : C_3$ has four subgroups isomorphic to D_{14}, including G_{01}. The subgroup G_{01} is normalized by the C_3-factor, and the three others are permuted under conjugation by C_3. This is due to the fact that if it were otherwise, the number of subgroups $R(3)'$ containing G_{01} would not be an integer but 4/3. Hence, among these four subgroups only G_{01} is normal and can be thought of as the subgroup D_{14} occurring in the factorization of the semi-direct product. It follows that the subgroups C_2^2 and C_3 normalize G_{01}. Thus

$$N_H(G_{01}) = N_H(C_7) = (C_2^2 \times D_{14}) : C_3.$$

Figure 1 shows the sublattice of the subgroup lattice of G that is relevant to the current situation. Each box contains two pieces of information: a group that describes the abstract structure of the groups in the conjugacy class of subgroups of G depicted by the box, and a number in the lower left corner that gives the number of subgroups in the conjugacy class. This number is the order of G divided by the order of the normalizer in G of a representative subgroup of the conjugacy class. Two boxes are joined by an edge provided that the subgroups represented by the lower box are subgroups of some subgroups represented by the upper box. There are also two numbers on each edge. The number at the top gives the number of subgroups in the conjugacy class for the lower box that are contained in a *given* subgroup in the conjugacy class for the upper box. The number at the bottom similarly is the number of subgroups in the conjugacy class for the upper box that contain a *given* subgroup in the conjugacy class for the lower box. If we know the lengths of the conjugacy classes for the upper box and lower box, then knowing one of these two numbers on the connecting edge gives us the other. For instance, in Figure 1, if we know that there are 36 (conjugate) subgroups D_{14} in a given subgroup $R(3)'$, then there are

$$\frac{|G|}{|R(3)|} \,.\, 36 \,/\, \frac{|G|}{|2^2 \,.\, 3 \,.\, 14|} = 4$$

(conjugate) subgroups $R(3)'$ containing a given subgroup D_{14}.

Returning to our line of argument, as already pointed out above, Figure 1 tells us that $G_0 \cong R(3)'$ is in a unique subgroup isomorphic to $R(27)$, namely H (because of the lower 1's on the edges joining the boxes). It also shows that G_{01} is contained in a unique subgroup $(C_2^2 \times D_{14}) : C_3$, which, in turn, is contained in a unique $R(27)$, namely H. As we saw above, this subgroup $(C_2^2 \times D_{14}) : C_3$ is necessarily the normalizer $N_H(G_{01})$ of G_{01} in H. Moreover, ρ_0 has to lie in this unique subgroup $(C_2^2 \times D_{14}) : C_3$, which itself is a subgroup of H, and therefore $\langle \rho_0, G_0 \rangle \leq H$. This holds because $N_G(G_{01}) = N_H(G_{01})$. That these normalizers coincide can be seen as follows. Clearly, $N_H(G_{01}) \leq N_G(G_{01})$. Now for the opposite inclusion observe that for $g \in N_G(G_{01})$ we have $G_{01} = gG_{01}g^{-1} \leq gN_H(G_{01})g^{-1}$ and (trivially) $G_{01} \leq N_H(G_{01})$. But then Figure 1 shows that a subgroup D_{14} of H must lie in a unique conjugate of $(C_2^2 \times D_{14}) : C_3 = N_H(G_{01})$, so necessarily $gN_H(G_{01})g^{-1} = N_H(G_{01})$. Similarly, since $N_H(G_{01}) \leq H$ and hence $N_H(G_{01}) = gN_H(G_{01})g^{-1} \leq gHg^{-1}$, Figure 1 (at box $R(27)$) gives $gHg^{-1} = H$, so $g \in H$ since H is self-normalized. Thus $G = \langle \rho_0, G_0 \rangle = H \cong R(27)$. □

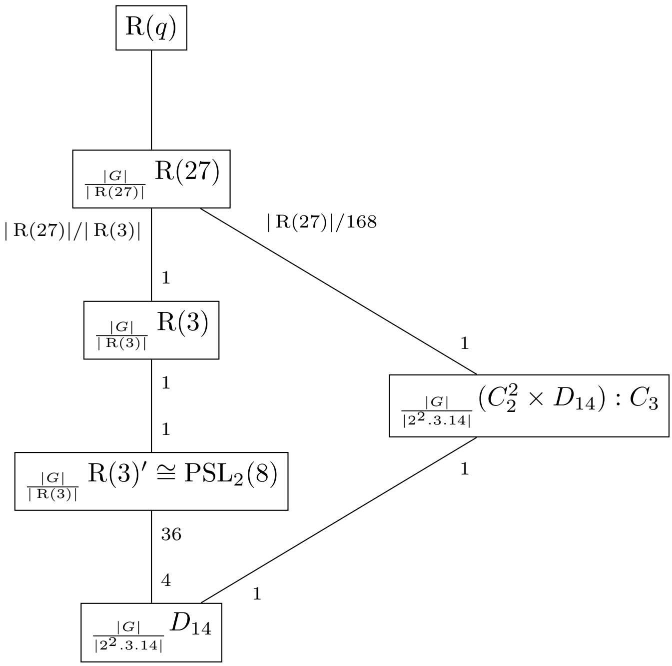

Figure 1: A sublattice of the subgroup lattice of $\mathrm{R}(q)$.

Lemma 3.12. *The group* $\mathrm{R}(27)$ *cannot be represented as a string C-group of rank 4.*

Proof. Let $G \cong \mathrm{R}(27)$. By the previous lemmas we may assume that $G_0 \cong G_3 \cong \mathrm{PSL}_2(8)$. In all other cases we know that G cannot be represented as a rank 4 string C-group. Moreover, from the proof of the previous lemma we already know that $G_{01} \cong D_{14}$ and $N_G(G_{01}) \cong (C_2^2 \times D_{14}) : C_3$. As there is a unique conjugacy class of subgroups $\mathrm{R}(3)'$ in $\mathrm{R}(27)$, and there is also a unique conjugacy class of subgroups D_{14} in $\mathrm{R}(3)'$, the choice of ρ_2, ρ_3 is therefore unique up to conjugacy in $\mathrm{R}(27)$. Once ρ_2, ρ_3 have been chosen, there are three candidates for ρ_0, namely the elements of the subgroup C_2^2 that centralizes D_{14}, and these are equivalent under conjugacy by C_3. Hence there is a unique choice for $\{\rho_0, \rho_2, \rho_3\}$ up to conjugacy. By similar arguments we also know that $G_3 \cong \mathrm{R}(3)'$ and $G_{23} \cong D_{14}$, and that the pair (G_{23}, G_3) is related to (G_{01}, G_0) by conjugacy in $R(27)$. Hence there must exist an element $g \in \mathrm{R}(27)$ such that

- $\rho_0^g = \rho_3$, $\rho_2^g = \rho_1$, $\rho_3^g = \rho_0$, or
- $\rho_0^g = \rho_3$, $\rho_2^g = \rho_0$, $\rho_3^g = \rho_1$.

The second case can be reduced to the first, as the centraliser of ρ_0 contains an element that swaps ρ_2 and ρ_3 (any two involutions in D_{14} are conjugate). Hence, we may assume without loss of generality that g swaps ρ_0 and ρ_3. In particular, $\langle \rho_0, \rho_3 \rangle$ is an elementary abelian group of order 4 normalized by g. All such subgroups are known to be conjugate and have as normalizer a group $(C_2^2 \times D_{14}) : C_3$. In this group, there is no element that will swap ρ_0 and ρ_3 under conjugation. All elements that will conjugate ρ_0 to ρ_3 will necessarily conjugate ρ_3 to $\rho_0\rho_3$. Hence we have a contradiction. $\square$

We therefore know that if a string C-group representation of rank 4 exists for $\mathrm{R}(q)$, both G_0 and G_3 must be subgroups of Ree type. Thus from now on we can assume $G_0 \cong \mathrm{R}(q_0)$ with $q_0 > 3$.

In a Ree group, the dihedral subgroups D_{2n} are such that n must divide one of

$$9,\ q-1,\ q+1,\ \alpha_q := q+1-3^{e+1},\ \beta_q := q+1+3^{e+1}.$$

Note that

$$\alpha_q\beta_q = q^2 - q + 1,$$

so in particular if H is a Ree subgroup $\mathrm{R}(q_0)$ of G then similarly $\alpha_{q_0}\beta_{q_0} = q_0^2 - q_0 + 1$.

Lemma 3.13. *Let $G \cong \mathrm{R}(q)$ with $q = 3^{2e+1}$ and $\langle \rho_0, \rho_1, \rho_2, \rho_3\rangle$ be a string C-group representation of rank 4 of G. Then*

(1) G_{01} is a dihedral subgroup D_{2d} with d a divisor of $q+1$ and of either α_{q_0} or β_{q_0} for some q_0 such that $q = q_0^m$ with m odd (where q_0 is determined by $G_0 = \mathrm{R}(q_0)$);

(2) $m = 3$, and G_0 and G_3 are conjugate Ree subgroups $\mathrm{R}(q_0)$ with $q = q_0^3$;

(3) G_{03} is a dihedral subgroup D_{2t} with t a divisor of α_{q_0} or β_{q_0}.

Proof. (1) By Lemmas 3.10, 3.11 and 3.12, we may assume that G_0 is a simple Ree subgroup of G. Let G_0 be a Ree subgroup $\mathrm{R}(q_0)$, with $q_0 \neq 3$ such that $q_0^m = q$ with m a positive odd integer and let $G_{01} \cong D_{2d}$. As $G_{01} \leq C_G(\rho_0)$ we have that $2d \mid q \pm 1$ by Lemma 3.5. In order to have involutions in $N_G(G_{01})\backslash N_G(G_0)$, the only possibility is that $N_G(G_{01})$ (of order divisible by 4) lies in a maximal subgroup of type $N_G(A_1)$ but not in a maximal subgroup $N_{G_0}(C_{\frac{q_0+1}{4}})$ of G_0; for otherwise, the same techniques as in Lemma 3.10 show that there is no involution in $N_G(G_{01})\backslash N_G(G_0)$. Hence $2d$ divides $q+1$. Observe that $N_G(A_1) \cong (C_2^2 \times D_{\frac{q+1}{2}}) : C_3$ has exactly four subgroups $D_{\frac{q+1}{2}}$ because of the subgroup C_2^2. These four subgroups are not all normalised by the C_3 because of the semi-direct product. Hence the C_3 must conjugate three of them and normalise the fourth one. Similarly, in $R(q_0)$ there are four subgroups D_{q_0+1} in each $N_{R(q_0)}(A_1')$ and it is obvious that $N_{R(q_0)}(D_{2d}) = N_{R(q)}(D_{2d})$ for every divisor d of q_0+1. Hence, in order to find some involutions in $N_G(G_{01})\backslash N_G(G_0)$, we need to have that $2d$ does not divide q_0+1. Moreover, since q_0-1 divides $q-1$, we have also that $(q_0-1, q+1) = 2$. That forces d not to be a divisor of q_0-1 as $d > 2$. Hence, looking at the list of maximal subgroups of $R(q_0)$ we can conclude that d is a divisor of either α_{q_0} or β_{q_0} in order for D_{2d} to be a dihedral subgroup of $R(q_0)$.

(2) Observe that $q_0^3+1 = (q_0+1)\alpha_{q_0}\beta_{q_0}$ divides q^3+1. Let us first show that $2e+1$ must be divisible by 3 in order for d to satisfy (1). Suppose $(3, 2e+1) = 1$. Then $q_0 = 3^{2f+1}$ with $2e+1 = m(2f+1)$ and $(3, m) = 1$. Let p be an odd prime dividing $(\alpha_{q_0}\beta_{q_0}, q+1)$ but not dividing q_0+1. Then p divides $(q_0^3+1, q+1)$ and hence p divides

$$(q_0^6-1, q_0^{2m}-1) = q_0^{2(3,m)} - 1 = q_0^2 - 1 = (q_0+1)(q_0-1)$$

and hence also q_0-1. As p divides $q+1$, and q_0-1 divides $q-1$, and since $(q-1, q+1) = 2$, we have that $p \mid 2$, a contradiction. Hence m must be divisible by 3 and so does $2e+1$. Suppose $m \neq 3$. Then $m = 3m'$ and given a Ree subgroup $\mathrm{R}(q_0)$ of $\mathrm{R}(q)$ with $q_0^m = q$, there exists a Ree subgroup $\mathrm{R}(q_0^3)$ such that $\mathrm{R}(q_0) < \mathrm{R}(q_0^3) < \mathrm{R}(q)$. Using similar arguments as in the proof of Lemma 3.11, it is easy to show that, since $\alpha_{q_0}\beta_{q_0}$ divides

q_0^3+1, we must have $\langle \rho_0, \rho_1, \rho_2, \rho_3\rangle = \mathrm{R}(q_0^3)$ and therefore $m = 3$. Indeed, as we stated in (1), $N_{R(q_0^3)}(D_{2d}) = N_{R(q)}(D_{2d})$ for every divisor d of q_0^3+1. Hence $\rho_0 \in \mathrm{R}(q_0^3)$. This implies that $m = 3$ and $G_0 \cong \mathrm{R}(q_0)$ with $q_0^3 = q$. Dually, $G_3 \cong \mathrm{R}(q_0)$. As all subgroups $\mathrm{R}(q_0)$ are conjugate in $\mathrm{R}(q)$, we have that G_0 and G_3 are conjugate.

(3) is due to the fact that $G_0 \cap G_3 = G_{03}$ and that, by (2), G_0 and G_3 are conjugate in G. Hence, $N_G(G_{03}) \setminus G_0$ has to be nonempty and G_{03} must not be contained in a subgroup H of G_0 such that $N_G(H) \geq N_G(G_{03})$, for if such a subgroup H exists, then $G_0 \cap G_3 \geq H$. If t divides 9 or one of $q_0 \pm 1$, this does not happen. Hence t divides one of α_{q_0} or β_{q_0}. □

Lemma 3.14. *The small Ree groups have no string C-group representation of rank* 4.

Proof. Suppose G is a Ree group that has a string C-group representation of rank 4. By Lemma 3.10 and part (2) of Lemma 3.13 we may assume that $G := \mathrm{R}(q)$ where $q = q_0^3$ with $q_0 = 3^m$ for an odd integer m. Moreover, G_0 and G_3 are conjugate simple Ree subgroups isomorphic to $\mathrm{R}(q_0)$. By part (3) of Lemma 3.13, if $G_{03} = D_{2t}$ then t must be a divisor of either α_{q_0} or β_{q_0}, and since $q = q_0^3$, we also have

$$q + 1 = (q_0 + 1)(q_0^2 - q_0 + 1) = (q_0 + 1)\alpha_{q_0}\beta_{q_0}.$$

Thus t is also a divisor of $q + 1$. We claim that then $G_0 \cap G_3 > G_{03}$, which gives a contradiction to the intersection property. Indeed, since G_{03} lies in a subgroup $H := C_t : C_6$ of G_0, and the normaliser of G_{03} is not contained in G_0 (for otherwise, D_{2t} would have to lie in a unique subgroup $\mathrm{R}(q_0)$), whereas already G_0 and G_3 give two examples of such subgroups, by the previous lemma), we have $N_G(G_{03}) = (C_2 \times C_2 \times D_{2t}) : C_3$. This group contains $H = C_t : C_6 \cong D_{2t} : C_3$ as a normal subgroup, and G_{03} is normal in H. We also have that $N_G(H) = N_G(G_{03})$. But then, as G_{03} is normal in H, any subgroup $R(q_0)$ containing G_{03} must contain H. In particular this applies to G_3. Thus $G_0 \cap G_3 \geq H > G_{03}$, and the intersection property fails. □

3.4 String C-groups of rank 3

It remains to investigate the possibility of representing $\mathrm{R}(q)$ as a string C-group of rank 3. Nuzhin already showed in [27] that there exist triples of involutions, two of which commute, that generate $\mathrm{R}(q)$ for every q. This completes the proof of Theorem 1.1. However, we decided to give here another way to construct an example of a rank three regular polytope for $\mathrm{R}(q)$ for the paper to be self-contained.

Lemma 3.15. *Let $G = \mathrm{R}(q)$, with $q \neq 3$ an odd power of 3. Then there exists a triple of involutions $S := \{\rho_0, \rho_1, \rho_2\}$ in G such that (G, S) is a string C-group.*

Proof. Recall that the fixed point set of an involution in G is a block of the Steiner system $\mathcal{S} := S(2, q+1, q^3+1)$. Pick two involutions ρ_0, ρ_1 from a maximal subgroup M of G of type $N_G(A_3)$ such that $\rho_0\rho_1$ has order $q + 1 + 3^{e+1}$, and let B_0, B_1, respectively, denote their blocks of fixed points. Obviously, $B_0 \cap B_1 = \emptyset$, for otherwise $\langle \rho_0, \rho_1\rangle$ would lie in the stabilizer of a point in $B_0 \cap B_1$, which is not possible because of the order of $\rho_0\rho_1$. Recall here that the point stabilizers are maximal subgroups of the form $N_G(A) = A : C_{q-1}$, where A is a 3-Sylow subgroup of G. Now choose an involution ρ_2 in $C_G(\rho_0)$ distinct from ρ_0 such that its block of fixed points B_2 meets B_1 in a point. Such a ρ_2 exists as all involutions of $C_G(\rho_0)$ have pairwise disjoint blocks of size $q + 1$ and therefore they cover all of the $q^3 + 1$ points. Then $B_1 \cap B_2$ must consist of a single point p (say), and

$B_0 \cap B_2 = \emptyset$ since the stabilizer of a point does not contain Klein 4-groups. Then $\langle \rho_1, \rho_2 \rangle$ lies in the point stabilizer of p, and hence must a dihedral group D_{2n}, with n a power of 3. As $\langle \rho_0, \rho_1 \rangle$ is a subgroup of index 3 in M, and ρ_0 does not belong to M, we see that $\langle \rho_0, \rho_1, \rho_2 \rangle = G$. Moreover, since the orders of $\rho_0\rho_1$ and $\rho_1\rho_2$ are coprime, the intersection property must hold as well. Thus (G, S), with $S := \{\rho_0, \rho_1, \rho_2\}$, is a string C-group of rank 3. □

We have not attempted to enumerate or classify all representations of $\mathrm{R}(q)$ as a string C-group of rank 3.

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On hypohamiltonian snarks and a theorem of Fiorini*

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In loving memory of Ella.

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Abstract

In 2003, Cavicchioli et al. corrected an omission in the statement and proof of Fiorini's theorem from 1983 on hypohamiltonian snarks. However, their version of this theorem contains an unattainable condition for certain cases. We discuss and extend the results of Fiorini and Cavicchioli et al. and present a version of this theorem which is more general in several ways. Using Fiorini's erroneous result, Steffen had shown that hypohamiltonian snarks exist for some orders $n \geq 10$ and each even $n \geq 92$. We rectify Steffen's proof by providing a correct demonstration of a technical lemma on flower snarks, which might be of separate interest. We then strengthen Steffen's theorem to the strongest possible form by determining all orders for which hypohamiltonian snarks exist. This also strengthens a result of Máčajová and Škoviera. Finally, we verify a conjecture of Steffen on hypohamiltonian snarks up to 36 vertices.

Keywords: Hypohamiltonian, snark, irreducible snark, dot product.

Math. Subj. Class.: 05C10, 05C38, 05C45, 05C85

1 Introduction

A graph G is *hypohamiltonian* if G itself is non-hamiltonian, but for every vertex v in G, the graph $G - v$ is hamiltonian. A *snark* shall be a cubic cyclically 4-edge-connected graph

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with chromatic index 4 (i.e. four colours are required in any proper edge-colouring) and girth at least 5. We refer for notions not defined here to [22] and [7].

Motivated by similarities between the family of all snarks and the family of all cubic hypohamiltonian graphs regarding the orders for which such graphs exist, Fiorini [8] studied the hypohamiltonian properties surrounding Isaacs' so-called "flower snarks" [13] (defined rigorously below). The a priori surprising interplay between snarks and hypohamiltonian graphs has been investigated extensively—we now give an overview. Early contributions include Fiorini's aforementioned paper [8], in which he claims to show that there exist infinitely many hypohamiltonian snarks. (In fact, according to Máčajová and Škoviera [18], it was later discovered that a family of hypohamiltonian graphs constructed by Gutt [12] includes Isaacs' snarks, thus including Fiorini's result.)

Skupień showed that there exist exponentially many hypohamiltonian snarks [20], and Steffen [22] proved that there exist hypohamiltonian snarks of order n for every even $n \geq 92$ (and certain $n < 92$)—we will come back to this result in Section 3. For more references and connections to other problems, see e.g. [3, 18, 20, 23]. Hypohamiltonian snarks have also been studied in connection with the famous Cycle Double Cover Conjecture [3] and Sabidussi's Compatibility Conjecture [9].

The smallest snark, as well as the smallest hypohamiltonian graph, is the famous Petersen graph. Steffen [21] showed that every cubic hypohamiltonian graph with chromatic index 4 is *bicritical*, i.e. the graph itself is not 3-edge-colourable but deleting any two distinct vertices yields a 3-edge-colourable graph. Nedela and Škoviera [19] proved that every cubic bicritical graph is cyclically 4-edge-connected and has girth at least 5. Therefore, every cubic hypohamiltonian graph with chromatic index 4 must be a snark.

This article is organised as follows. In Section 2 we discuss the omission in Fiorini's theorem on hypohamiltonian snarks [8]—first observed by Cavicchioli et al. [5]—and its consequences and state a more general version of this theorem. In Section 3 we first rectify a proof of Steffen on the orders for which hypohamiltonian snarks exist which relied on Fiorini's theorem—this erratum is based on giving a correct proof of a technical lemma concerning flower snarks, which may be of separate interest. We then prove a strengthening of Steffen's theorem, which is best possible, as all orders for which hypohamiltonian snarks exist are determined. Our result is stronger than a theorem of Máčajová and Škoviera [17] in the sense that our result implies theirs, while the converse does not hold. Finally, in Section 4 we comment upon and verify a conjecture of Steffen on hypohamiltonian snarks [23] for small orders.

2 Fiorini's theorem revisited

We call two edges *independent* if they have no common vertices. Let G and H be disjoint connected graphs on at least 6 vertices. Consider $G' = G - \{ab, cd\}$, where ab and cd are independent edges in G, put $H' = H - \{x, y\}$, where x and y are adjacent cubic vertices in H, and let a', b' and c', d' be the other neighbours of x and y in H, respectively. Then the *dot product* $G \cdot H$ is defined as the graph

$$(V(G) \cup V(H'), E(G') \cup E(H') \cup \{aa', bb', cc', dd'\}).$$

Two remarks are in order. (1) Under the above conditions, the dot product may be disconnected. (2) In fact, there are eight ways to form the dot product for a specific ab, cd, xy. For the computational results in Section 4 we indeed applied the dot product in all eight possible ways, but for the theoretical proofs in this paper we will perform the dot product in one way, namely as follows. We always construct the dot product by adding the edges aa', bb', cc', dd' and we will abbreviate this as "a, b, c, d are joined by edges to the neighbours of x and y, respectively".

The dot product was introduced by Adelson-Velsky and Titov [1], and later and independently by Isaacs [13]. Its original purpose was to obtain new snarks by combining known snarks. Fiorini then proved that the dot product can also be used to combine two *hypohamiltonian* snarks into a new one. Unfortunately, Fiorini's argument is incorrect. We shall discuss this omission within this section, and correct the proof of a lemma of Steffen [22] which depended on Fiorini's result in Section 3.

In a graph G, a pair (v, w) of vertices is *good* in G if there exists a hamiltonian path in G with end-vertices v and w. Two pairs of vertices $((v, w), (x, y))$ are *good* in G if there exist two disjoint paths which together span G, and which have end-vertices v and w, and x and y, respectively.

Claim 2.1 (Fiorini, Theorem 3 in [8]). *Let G be a hypohamiltonian snark having two independent edges ab and cd for which*

(i) each of $(a, c), (a, d), (b, c), (b, d), ((a, b), (c, d))$ is good in G;

(ii) for each vertex v, exactly one of $(a, b), (c, d)$ is good in $G - v$.

If H is a hypohamiltonian snark with adjacent vertices x and y, then the dot product $G \cdot H$ is also a hypohamiltonian snark, where ab and cd are deleted from G, x and y are deleted from H, and vertices a, b, c, d are joined by edges to the neighbours of x and y, respectively.

Cavicchioli et al. [5] point out the omissions in Claim 2.1: in order for the proof to work, the given vertex pairs need to be good in $G - \{ab, cd\}$ rather than in G. They give a corrected statement of the theorem envisioned by Fiorini and give a new proof.

Claim 2.2 (Cavicchioli et al., Theorem 3.2 in [5]). *Let G be a hypohamiltonian snark having two independent edges ab and cd for which*

(i) each of $(a, c), (a, d), (b, c), (b, d), ((a, b), (c, d))$ is good in $G - \{ab, cd\}$;

(ii) for each vertex v, each of $(a, b), (c, d)$ is good in $G - \{v, ab, cd\}$.

If H is a hypohamiltonian snark with adjacent vertices x and y, then the dot product $G \cdot H$ is also a hypohamiltonian snark, where ab and cd are deleted from G, x and y are deleted from H, and vertices a, b, c, d are joined by edges to the neighbours of x and y, respectively.

In above statements, the fact that the dot product of snarks is itself a snark had already been shown [1, 13], so indeed only the hypohamiltonicity was to be proven.

We point out that the hypotheses in Claim 2.2 are unattainable for $v \in \{a, b, c, d\}$, since (a, b) and (c, d) cannot both be good in $G - \{v, ab, cd\}$ if $v \in \{a, b, c, d\}$. This is tied to the fact that the requirements in (ii) are stronger than what is needed to prove the statement.

In [11, Theorem 1], we gave the following (second) restatement of Claim 2.1 which we used to solve a problem of McKay. Note that in [11] the graphs are required to be cubic and below we do not state this requirement—we do however need the two vertices which are removed to be cubic. This allows us to use exactly the same proof as in [11, Theorem 1]. Nevertheless, we now give a sketch of the proof: first, we assume that $G \cdot H$ does contain a hamiltonian cycle. This however implies that at least one of the factors is hamiltonian, contradicting their hypohamiltonicity. Second, we prove that every vertex-deleted subgraph of $G \cdot H$ is indeed hamiltonian. This is done with a careful case analysis (depending on where the removed vertex lies) using the goodness of various pairs (and pairs of pairs) of vertices in $G - \{ab, cd\}$ and $G - \{v, ab, cd\}$.

Theorem 2.3. *Let G be a non-hamiltonian graph having two independent edges ab and cd for which*

(i) each of $(a, c), (a, d), (b, c), (b, d), ((a, b), (c, d))$ is good in $G - \{ab, cd\}$;

(ii) for each vertex v, at least one of (a, b) and (c, d) is good in $G - \{v, ab, cd\}$.

If H is a hypohamiltonian graph with cubic adjacent vertices x and y, then the dot product $G \cdot H$ is also a hypohamiltonian graph, where ab and cd are deleted from G, x and y are deleted from H, and vertices a, b, c, d are joined by edges to the neighbours of x and y, respectively.

If G and H are planar, and ab and cd lie on the same facial cycle, then the dot product can be applied such that $G \cdot H$ is planar, as well. If g and h are the girth of G and H, respectively, then the girth of $G \cdot H$ is at least $\min\{g, h\}$. *If G and H are cubic, then so is $G \cdot H$.*

Note that the fact that G is non-hamiltonian together with condition (ii) implies that G must be hypohamiltonian.

In the following, we will call the pair of edges ab, cd from the statement of Theorem 2.3 *suitable*. The Petersen graph is the smallest snark, and the two Blanuša snarks on 18 vertices are the second-smallest snarks. All three graphs are also hypohamiltonian. Due to the huge automorphism group of the Petersen graph, it can be verified by hand that it does not contain a pair of suitable edges. Although both Blanuša snarks are dot products of two Petersen graphs, the Petersen graph does not contain a pair of suitable edges. Thus, in a certain sense, Theorem 2.3 is *not* “if and only if”, i.e. there exist dot products whose factors do not contain suitable edges.

Let us end this section with a remark which may prove to be useful in other applications. Throughout its statement and proof, we use the notation from Theorem 2.3.

Observation 2.4. We have that $G \cdot H + ab$, $G \cdot H + cd$, and $G \cdot H + ab + cd$ are hypohamiltonian, as well.

Proof. Put $N(x) = \{a', b', y\}$ and $N(y) = \{c', d', x\}$ such that the unique neighbour of a' (b', c', d') in G is a (b, c, d). Assume $G \cdot H + ab + cd$ contains a hamiltonian cycle $\mathfrak{h}$. Thus, at least one of ab and cd lies in $\mathfrak{h}$, say ab. We treat $H - \{x, y\}$ as a subgraph of $G \cdot H$. If $aa', bb' \in E(\mathfrak{h})$, then $\mathfrak{h} \cap H \cup a'xb' \cup c'yd'$ gives a hamiltonian cycle in H, a contradiction. If $aa', bb' \notin E(\mathfrak{h})$, then the cycle $\mathfrak{h} \cap G + cd$ yields a contradiction. So w.l.o.g. $aa' \in E(\mathfrak{h})$ and $bb' \notin E(\mathfrak{h})$. This implies the existence of a hamiltonian path in $H - \{x, y\}$ with end-vertices a' and $u \in \{c', d'\}$. But this path together with $uyxa'$ is a hamiltonian cycle in H, a contradiction. It follows that $G \cdot H + ab$ and $G \cdot H + cd$ are non-hamiltonian, as well. □

3 On a theorem of Steffen on hypohamiltonian snarks

3.1 Rectifying Steffen’s proof

A snark is *irreducible* if the removal of every edge-cut which is not the set of all edges incident with a vertex yields a 3-edge-colourable graph. Steffen’s article [22] is motivated by the following problem.

Problem 3.1 (Nedela and Škoviera [19]). For which even number $n \geq 10$ does there exist an irreducible snark of order n? In particular, does there exist an irreducible snark of each sufficiently large order?

Steffen settled the second question of Problem 3.1 by giving the following main result from [22].

Theorem 3.2 (Steffen, Theorem 2.5 in [22]). *There is a hypohamiltonian snark of order* n

(1) for each $n \in \{m : m \geq 64$ *and* $m \equiv 0 \bmod 8\}$*,*

(2) for each $n \in \{10, 18\} \cup \{m : m \geq 98$ *and* $m \equiv 2 \bmod 8\}$*,*

(3) for each $n \in \{m : m \geq 20$ *and* $m \equiv 4 \bmod 8\}$*,*

(4) for each $n \in \{30\} \cup \{m : m \geq 54$ *and* $m \equiv 6 \bmod 8\}$*, and*

(5) for each even $n \geq 92$*.*

Isaacs' *flower snark* J_{2k+1}, see [13], is the graph

$$\left(\{a_i, b_i, c_i, d_i\}_{i=0}^{2k}, \{b_i a_i, b_i c_i, b_i d_i, a_i a_{i+1}, c_i d_{i+1}, d_i c_{i+1}\}_{i=0}^{2k}\right),$$

where addition in the indices is performed modulo $2k + 1$.

However, the proof of [22, Lemma 2.3], which is essential for the proof of the theorem, is erroneous, since it uses Fiorini's erroneous Claim 2.1 (and it does not work with Theorem 2.3). We here give a correct proof of that lemma.

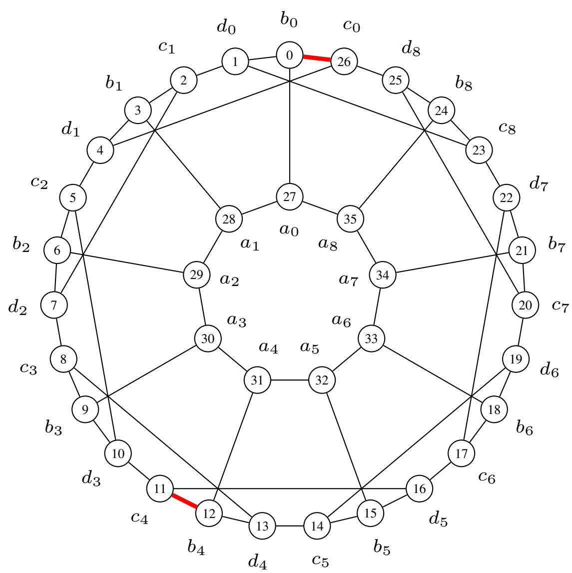

Figure 1: The flower snark J_9. The suitable edges b_0c_0 and b_4c_4 are marked in bold red.

Lemma 3.3 (Steffen, Lemma 2.3 in [22]). *The flower snarks J_9, J_{11}, and J_{13} satisfy the conditions of Theorem 2.3.*

Proof. In [22], in each of the graphs J_9, J_{11}, and J_{13}, the suitable edges were chosen to be b_0c_0 and b_4c_4. However, in [22], for various vertices v, the hamiltonian paths did not satisfy condition (ii) from Theorem 2.3, as the paths used one of the edges b_0c_0 or b_4c_4. This was for instance the case for $v \in \{a_0, a_8, d_0\}$ in J_9, for $v \in \{a_0, a_{10}, d_0\}$ in J_{11}, and for $v \in \{a_0, c_1, c_{12}, d_0\}$ in J_{13}, see Claims 6, 7, and 8 in the Appendix of [22].

We will now prove that b_0c_0 and b_4c_4 are indeed suitable edges for Theorem 2.3 for J_9, J_{11}, and J_{13}. For J_9 the proof is given below (and partially in the Appendix), while the technical details of the proofs for J_{11} and J_{13} can be found in the Appendix. The mapping between the a_i, b_i, c_i, d_i (used by Steffen) and the vertex numbers used in the proof is shown in Figures 1–3. We use numbers as labels in the proof to make it easier to read these graphs using a computer for verifying the results.

Proof that b_0c_0 and b_4c_4 are suitable edges for J_9. Figure 1 shows the flower snark J_9. In J_9, the edges b_0c_0 and b_4c_4 correspond to the edges $(0, 26)$ and $(11, 12)$, respectively.

The pairs $(0, 11)$, $(0, 12)$, $(26, 11)$ and $(26, 12)$ are good in $J_9 - \{(0, 26), (11, 12)\}$ due to the following hamiltonian paths, respectively:

- 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 27, 35, 24, 23, 22, 17, 16, 15, 32, 31, 12, 13, 14, 19, 18, 33, 34, 21, 20, 25, 26, 4, 3, 2, 1, 0
- 12, 13, 14, 15, 16, 11, 10, 5, 4, 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 30, 9, 8, 7, 6, 29, 28, 3, 2, 1, 23, 24, 35, 27, 0
- 11, 10, 5, 4, 3, 2, 1, 0, 27, 28, 29, 6, 7, 8, 9, 30, 31, 12, 13, 14, 19, 18, 17, 16, 15, 32, 33, 34, 35, 24, 23, 22, 21, 20, 25, 26
- 12, 13, 8, 7, 2, 3, 4, 5, 6, 29, 28, 27, 0, 1, 23, 22, 17, 18, 33, 32, 31, 30, 9, 10, 11, 16, 15, 14, 19, 20, 21, 34, 35, 24, 25, 26

Note that $((0, 26), (11, 12))$ is good in $J_9 - \{(0, 26), (11, 12)\}$ due to the following two disjoint paths with end-vertices 0 and 26, and 11 and 12, respectively, which together span J_9:

- 26, 25, 20, 19, 14, 13, 8, 7, 2, 1, 0
- 12, 31, 32, 15, 16, 17, 18, 33, 34, 21, 22, 23, 24, 35, 27, 28, 3, 4, 5, 6, 29, 30, 9, 10, 11

We showed by computer that at least one of $(0, 26)$ or $(11, 12)$ is good in $J_9 - \{v, (0, 26), (11, 12)\}$ for every $v \in V(J_9)$. In each case we verified that the path found by the computer is indeed a valid hamiltonian path in the graph. Below we explicitly show this for $v = 0$. The hamiltonian paths for the other choices of v can be found in the Appendix.

- $v = 0$: 12, 13, 14, 15, 32, 31, 30, 29, 6, 5, 10, 9, 8, 7, 2, 1, 23, 24, 25, 26, 4, 3, 28, 27, 35, 34, 33, 18, 19, 20, 21, 22, 17, 16, 11 □

Since Steffen's statement of Lemma 3.3 remains intact, the proof and statement of his main result, reproduced above as Theorem 3.2, are correct as given in [22]. Even though we prove a stronger version of Steffen's theorem in the next section, we think it is important to fix the proof of Lemma 3.3 as there may be others who rely on this lemma, or might want to rely on it in the future.

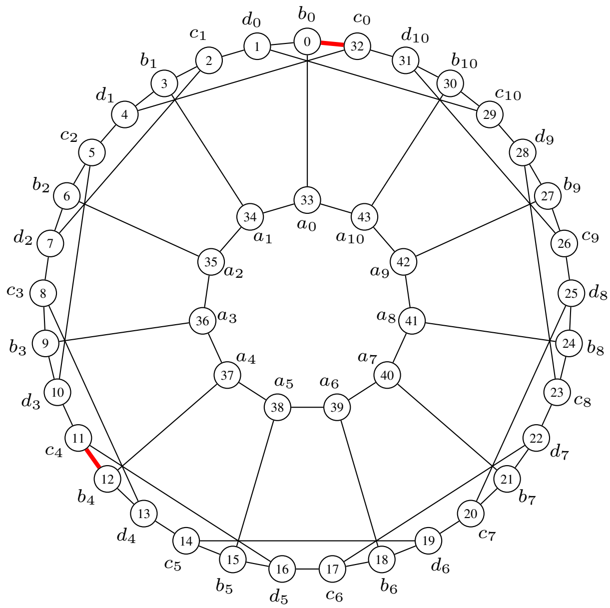

Figure 2: The flower snark J_{11}. The suitable edges b_0c_0 and b_4c_4 are marked in bold red.

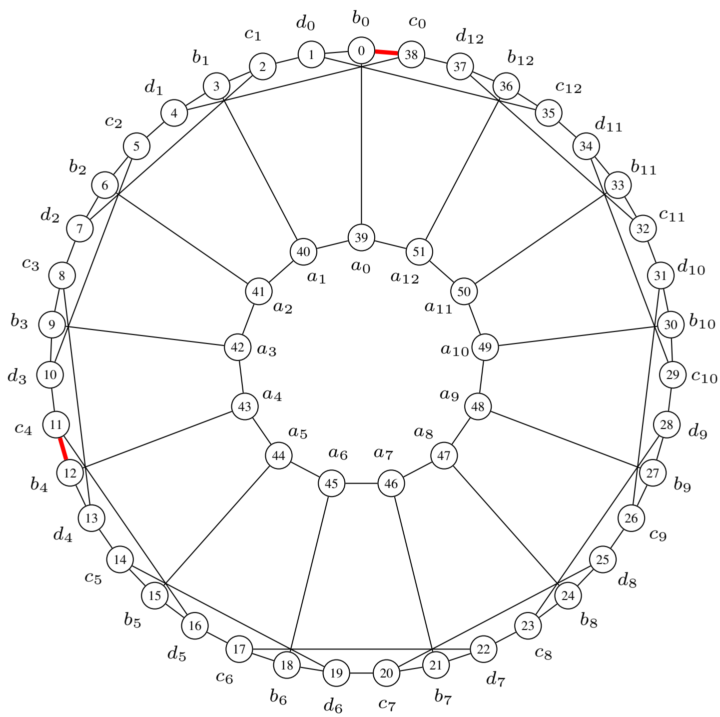

Figure 3: The flower snark J_{13}. The suitable edges b_0c_0 and b_4c_4 are marked in bold red.

3.2 Orders of hypohamiltonian snarks

We shall now prove a strengthening of Steffen's theorem, which in a sense is strongest possible since we will determine all orders for which hypohamiltonian snarks exist. We emphasise that our proof's mechanism contains significantly fewer "moving parts" than Máčajová and Škoviera's [17], and, as mentioned in the introduction, our theorem also strengthens their result. We do need the following two easily verifiable lemmas.

Lemma 3.4. *The second Blanuša snark B_2 shown in Figure 4 has a pair of suitable edges.*

Proof. Figure 4 shows the second Blanuša snark B_2. By computer we determined that B_2 has exactly three pairs of suitable edges: $((6,8),(10,16))$, $((3,9),(12,17))$ and $((4,7),(13,15))$. We will now prove by hand that $((6,8),(10,16))$ is a suitable edge pair.

The pairs $(6,10)$, $(6,16)$, $(8,10)$ and $(8,16)$ are good in $B_2 - \{(6,8),(10,16)\}$ due to the following hamiltonian paths, respectively:

- 10, 11, 12, 17, 16, 15, 13, 14, 0, 1, 5, 4, 7, 8, 9, 3, 2, 6
- 16, 15, 9, 8, 7, 17, 12, 13, 14, 10, 11, 1, 0, 2, 3, 4, 5, 6
- 10, 11, 1, 0, 14, 13, 12, 17, 16, 15, 9, 3, 2, 6, 5, 4, 7, 8
- 16, 15, 9, 3, 4, 5, 6, 2, 0, 1, 11, 10, 14, 13, 12, 17, 7, 8

Note that $((6,8),(10,16))$ is good in $B_2 - \{(6,8),(10,16)\}$ due to the following two disjoint paths with end-vertices 6 and 8, and 10 and 16, respectively, which together span B_2:

- 8, 7, 4, 5, 6
- 16, 17, 12, 11, 1, 0, 2, 3, 9, 15, 13, 14, 10

We now prove that at least one of $(6,8)$ or $(10,16)$ is good in $B_2 - \{v,(6,8),(10,16)\}$ for every $v \in V(B_2)$. By symmetry we only need to prove this for $v = 0,2,4,6,7,8$.

- $v = 0$: 8, 7, 4, 5, 1, 11, 10, 14, 13, 12, 17, 16, 15, 9, 3, 2, 6
- $v = 2$: 8, 9, 3, 4, 7, 17, 16, 15, 13, 12, 11, 10, 14, 0, 1, 5, 6
- $v = 4$: 16, 15, 13, 14, 0, 1, 5, 6, 2, 3, 9, 8, 7, 17, 12, 11, 10
- $v = 6$: 16, 15, 9, 8, 7, 17, 12, 13, 14, 0, 2, 3, 4, 5, 1, 11, 10
- $v = 7$: 8, 9, 15, 16, 17, 12, 13, 14, 10, 11, 1, 0, 2, 3, 4, 5, 6
- $v = 8$: 16, 15, 9, 3, 2, 6, 5, 4, 7, 17, 12, 13, 14, 0, 1, 11, 10 □

Lemma 3.5. *The first Loupekine snark L_1 shown in Figure 5 has a pair of suitable edges.*

Proof. Figure 5 shows the first Loupekine snark L_1. By computer we determined that L_1 has exactly six pairs of suitable edges: $((0,1),(17,20))$, $((0,2),(8,17))$, $((1,5),(14,20))$, $((2,3),(8,10))$, $((3,4),(10,12))$ and $((4,5),(12,14))$. We will now prove by hand that $((3,4),(10,12))$ is a suitable edge pair.

The pairs $(3,10)$, $(3,12)$, $(4,10)$ and $(4,12)$ are good in $L_1 - \{(3,4),(10,12)\}$ due to the following hamiltonian paths, respectively:

- 10, 7, 9, 6, 8, 17, 19, 21, 16, 13, 11, 0, 1, 5, 4, 18, 20, 14, 12, 15, 2, 3

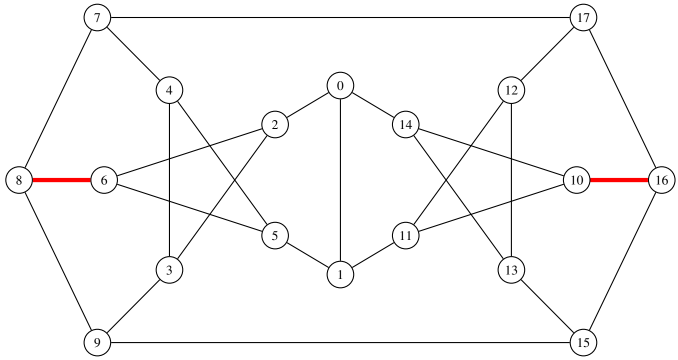

Figure 4: The second Blanuša snark. It has 18 vertices. The suitable edges $(6, 8)$ and $(10, 16)$ are marked in bold red.

- 12, 14, 20, 18, 4, 5, 7, 10, 8, 17, 19, 21, 16, 9, 6, 1, 0, 11, 13, 15, 2, 3
- 10, 7, 5, 1, 0, 11, 13, 16, 9, 6, 8, 17, 20, 14, 12, 15, 2, 3, 19, 21, 18, 4
- 12, 14, 11, 0, 1, 6, 9, 16, 13, 15, 2, 3, 19, 21, 18, 20, 17, 8, 10, 7, 5, 4

Note that $((3, 4), (10, 12))$ is good in $L_1 - \{(3, 4), (10, 12)\}$ due to the following two disjoint paths with end-vertices 3 and 4, and 10 and 12, respectively, which together span L_1:

- 4, 5, 1, 6, 8, 17, 20, 18, 21, 19, 3
- 12, 14, 11, 0, 2, 15, 13, 16, 9, 7, 10

We now prove that at least one of $(3, 4)$ or $(10, 12)$ is good in $L_1 - \{v, (3, 4), (10, 12)\}$ for every $v \in V(L_1)$. By symmetry we only need to prove this for $v = 1, 4, 5, 6, 7, 8, 9, 10, 16, 17, 18, 21$.

- $v = 1$: 12, 15, 13, 16, 9, 6, 8, 17, 20, 14, 11, 0, 2, 3, 19, 21, 18, 4, 5, 7, 10
- $v = 4$: 12, 14, 20, 18, 21, 16, 9, 6, 8, 17, 19, 3, 2, 15, 13, 11, 0, 1, 5, 7, 10
- $v = 5$: 4, 18, 20, 17, 8, 10, 7, 9, 6, 1, 0, 2, 15, 12, 14, 11, 13, 16, 21, 19, 3
- $v = 6$: 4, 5, 1, 0, 2, 15, 12, 14, 11, 13, 16, 9, 7, 10, 8, 17, 20, 18, 21, 19, 3
- $v = 7$: 12, 14, 20, 17, 19, 3, 2, 15, 13, 11, 0, 1, 5, 4, 18, 21, 16, 9, 6, 8, 10
- $v = 8$: 12, 14, 20, 17, 19, 3, 2, 15, 13, 11, 0, 1, 6, 9, 16, 21, 18, 4, 5, 7, 10
- $v = 9$: 4, 5, 7, 10, 8, 6, 1, 0, 2, 15, 12, 14, 11, 13, 16, 21, 18, 20, 17, 19, 3
- $v = 10$: 4, 5, 7, 9, 16, 13, 11, 0, 1, 6, 8, 17, 19, 21, 18, 20, 14, 12, 15, 2, 3
- $v = 16$: 4, 5, 1, 6, 9, 7, 10, 8, 17, 19, 21, 18, 20, 14, 12, 15, 13, 11, 0, 2, 3
- $v = 17$: 4, 5, 1, 6, 8, 10, 7, 9, 16, 13, 11, 0, 2, 15, 12, 14, 20, 18, 21, 19, 3
- $v = 18$: 4, 5, 1, 6, 8, 10, 7, 9, 16, 21, 19, 17, 20, 14, 12, 15, 13, 11, 0, 2, 3
- $v = 21$: 4, 18, 20, 14, 12, 15, 2, 0, 11, 13, 16, 9, 6, 1, 5, 7, 10, 8, 17, 19, 3 □

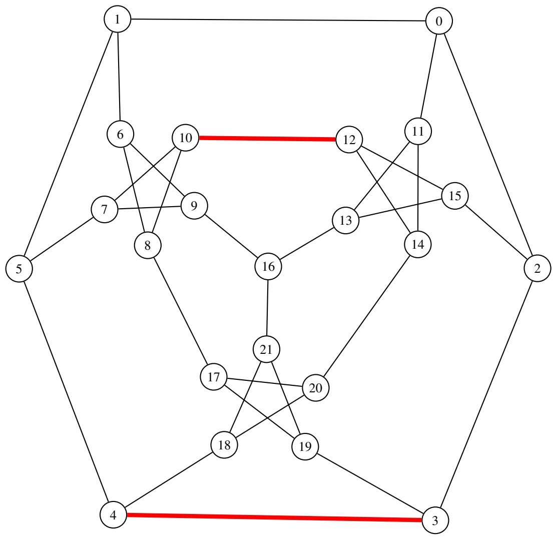

Figure 5: The first Loupekine snark L_1. It has 22 vertices. The suitable edges $(3, 4)$ and $(10, 12)$ are marked in bold red.

The following generalisation of Steffen's Theorem 3.2 is strongest possible.

Theorem 3.6. *A hypohamiltonian snark of order n exists if and only if $n \in \{10, 18, 20, 22\}$ or n is even and $n \geq 26$.*

Proof. For $n = 10$, it is well-known that the Petersen graph is hypohamiltonian and it is also well-known that no snarks exist of order 12, 14 or 16.

In Lemma 3.4 we showed that the second Blanuša snark B_2 (which has order 18) contains a pair of suitable edges. In [3] it was proven that hypohamiltonian snarks exist for all even orders from 18 to 36 with the exception of 24 (see Table 1). Let S_n denote a hypohamiltonian snark of order n. Using Theorem 2.3, we form the dot product $B_2 \cdot S_n$ for $n \in \{18, 20, 22, 26, 28, 30, 32\}$ and obtain hypohamiltonian snarks of all even orders between 34 and 48 with the exception of 40 (recall that the dot product of two snarks is a snark).

By Lemma 3.5 we know that the first Loupekine snark L_1 (which has order 22) contains a pair of suitable edges. Applying Theorem 2.3 to this snark and a 22-vertex hypohamiltonian snark, we obtain a hypohamiltonian snark of order 40.

We form the dot product $B_2 \cdot S_n$ for all even $n \in \{34, \ldots, 48\}$ and obtain hypohamiltonian snarks of all even orders from 50 to 64. This may now be iterated ad infinitum, and the proof is complete. □

3.3 Hypohamiltonian and irreducible snarks

In [17] Máčajová and Škoviera proved the following theorem (which fully settles Problem 3.1).

Theorem 3.7 (Máčajová and Škoviera, Theorems A and B in [17])**.** *There exists an irreducible snark of order n if and only if $n \in \{10, 18, 20, 22\}$ or n is even and $n \geq 26$.*

Nedela and Škoviera [19] proved that a snark is irreducible if and only if it is bicritical, and Steffen [21] showed that every hypohamiltonian snark is bicritical—while the converse is not true, as will be shown in Table 1.

A graph G without k-flow is *k-vertex-critical* if, for every pair of vertices (u, v) of G, identifying u and v yields a graph that has a k-flow; see [4] for more details. In [4] Carneiro, da Silva, and McKay determined all 4-vertex-critical snarks up to 36 vertices and Škoviera [24] showed that a snark is 4-vertex-critical if and only if it is irreducible.

Cavicchioli et al. [5] determined all hypohamiltonian and irreducible snarks on $n \leq 28$ vertices. Later, Brinkmann et al. [3] determined all hypohamiltonian snarks on $n \leq 36$ vertices. These counts can be found in Table 1 together with the number of irreducible snarks from [4]. These graphs can also be downloaded from the *House of Graphs* [2] at `http://hog.grinvin.org/Snarks`.

The number of hypohamiltonian cubic graphs on $n \leq 32$ vertices can be found in [10]. As can be seen from Table 1, there is a significant number of irreducible snarks which are not hypohamiltonian. The smallest such snarks have order 26. So Theorem 3.6 implies Theorem 3.7, while the converse is not true.

Table 1: Number of irreducible and hypohamiltonian snarks (see [4, Table 1] and [3, Table 2]). λ_c stands for cyclic edge-connectivity. The counts of cases indicated with a '$\geq$' are possibly incomplete; all other cases are complete.

Order	irreducible	hypo.	hypo. and $\lambda_c = 4$	hypo. and $\lambda_c \geq 5$
10	1	1	0	1
18	2	2	2	0
20	1	1	0	1
22	2	2	0	2
24	0	0	0	0
26	111	95	87	8
28	33	31	30	1
30	115	104	93	11
32	13	13	0	13
34	40 328	31 198	29 701	1 497
36	13 720	10 838	10 374	464
38	?	?	$\geq$ 51 431	?
40	?	?	$\geq$ 8 820	?
42	?	?	$\geq$ 20 575 458	?
44	?	?	$\geq$ 8 242 146	?

The hypohamiltonian snarks on $n \geq 34$ vertices constructed by the dot product in the proof of Theorem 3.6 clearly all have cyclic edge-connectivity 4. By combining this with Table 1 we obtain:

Corollary 3.8. *Hypohamiltonian snarks of order n and cyclic edge-connectivity 4 exist if and only if $n \in \{18, 26, 28, 30\}$ or n is even and $n \geq 34$.*

As already mentioned, every hypohamiltonian snark is irreducible, thus Corollary 3.8 implies [17, Theorem A]. For higher cyclic edge-connectivity, the following is known.

Theorem 3.9 (Máčajová and Škoviera [18]). *There exists a hypohamiltonian snark of order n and cyclic connectivity 5 for each even $n \geq 140$, and there exists a hypohamiltonian snark of order n and cyclic connectivity 6 for each even $n \geq 166$.*

If we relax the requirements from hypohamiltonicity to irreducibility, more is known:

Theorem 3.10 (Máčajová and Škoviera [17]). *There exists an irreducible snark of order n and cyclic connectivity 5 if and only if $n \in \{10, 20, 22, 26\}$ or n is even and $n \geq 30$, and there exists an irreducible snark of order n and cyclic connectivity 6 for each $n \equiv 4 \pmod 8$ with $n \geq 28$, and for each even $n \geq 210$.*

Note that as every hypohamiltonian snark is irreducible, Theorem 3.9 also implies that $n \geq 210$ can be improved to $n \geq 166$ in Theorem 3.10.

The smallest hypohamiltonian snark of cyclic edge-connectivity 5 has order 10 and is the Petersen graph, and the second-smallest such graph has order 20. The flower snark J_7 of order 28 is the smallest cyclically 6-edge-connected hypohamiltonian snark. We conclude this section with the following two problems motivated by Theorem 3.10 and results of Kochol [14, 16].

Problem 3.11 (Máčajová and Škoviera [17]). Construct a cyclically 6-edge-connected snark (irreducible or not) of order smaller than 118 and different from any of Isaacs' snarks.

Problem 3.12. Determine all orders for which cyclically 6-edge-connected snarks exist.

4 On a conjecture of Steffen on hypohamiltonian snarks

Consider a cubic graph G. We denote with $\mu_k(G)$ the minimum number of edges not contained in the union of k 1-factors of G, for every possible combination of k 1-factors. If $\mu_3(G) = 0$, then G is 3-edge-colourable. In [23], Steffen made the following conjecture on hypohamiltonian snarks.

Conjecture 4.1 (Steffen, Conjecture 4.1 in [23]). *If G is a hypohamiltonian snark, then $\mu_3(G) = 3$.*

If true, this conjecture would have significant consequences, e.g. by Theorem 2.14 from [23], it would imply that every hypohamiltonian snark has a Berge-cover (a bridgeless cubic graph G has a *Berge-cover* if $\mu_5(G) = 0$).

We wrote a computer program for computing $\mu_3(G)$ and tested Conjecture 4.1 on the complete lists of hypohamiltonian snarks up to 36 vertices. This leads to the following observation.

Observation 4.2. There are no counterexamples to Conjecture 4.1 among the hypohamiltonian snarks with at most 36 vertices.

The authors of [3] noted a huge increase (from 13 to 31 198) in the number of hypohamiltonian snarks from order 32 to 34, see Table 1. Using a computer, we were able to determine that 29 365 out of the 29 701 hypohamiltonian snarks with cyclic edge-connectivity 4 on 34 vertices can be obtained by performing a dot product on a hypohamiltonian snark on 26 vertices and the Petersen graph. We also determined that the remaining hypohamiltonian snarks with cyclic edge-connectivity 4 on 34 vertices are obtained by performing a dot product on the Blanuša snarks. Intriguingly, our computations show that some hypohamiltonian snarks can be obtained by performing a dot product on a hypohamiltonian snark on 26 vertices and the Petersen graph, as well as by performing a dot product on the Blanuša snarks.

There is also a (slightly less dramatic) increase in the cyclically 5-edge-connected case—these are obviously not dot products—and we believe it to be due to more general graph products, for instance "superposition" introduced by Kochol [15]. It would be interesting to further explore these transformations in order to fully understand these sudden increases and decreases in numbers.

Using a computer, we determined that all hypohamiltonian snarks with cyclic edge-connectivity 4 up to 36 vertices can be obtained by performing a dot product on two hypohamiltonian snarks. This leads us to pose the following question.

Problem 4.3. Is every hypohamiltonian snark with cyclic edge-connectivity 4 a dot product of two hypohamiltonian snarks?

In [6] Chladný and Škoviera proved that every bicritical snark with cyclic edge-connectivity 4 is a dot product of two bicritical snarks. Since every hypohamiltonian snark is bicritical, this implies that every hypohamiltonian snark with cyclic edge-connectivity 4 is a dot product of two bicritical snarks.

Recall that in Theorem 2.3 the graphs G and H are hypohamiltonian, but the theorem is *not* "if and only if", since although the Petersen graph does not contain a pair of suitable edges, the Blanuša snarks (which are dot products of two Petersen graphs) are hypohamiltonian. Despite the previous paragraph, we believe the answer to Problem 4.3 to be "no" due to the following observation. In order to cover all cases, we would need to add to condition (ii) of Theorem 2.3 the possibility of $((a,b),(c,d))$ being good in $G - \{v, ab, cd\}$. However, we would then need to require from H that it contains a 2-factor containing exactly two (necessarily odd) cycles. Although we were unable to find a counterexample, we believe that there exist hypohamiltonian snarks which do not possess such a 2-factor.

We also determined all hypohamiltonian snarks up to 44 vertices which can be obtained by performing a dot product on two hypohamiltonian snarks. The counts of these snarks can be found in the fourth column of Table 1. We also verified Conjecture 4.1 on these snarks.

Observation 4.4. There are no counterexamples to Conjecture 4.1 among the hypohamiltonian snarks with at most 44 vertices which are a dot product of two hypohamiltonian snarks.

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Appendix

Below we give the technical details which were left out in the proof of Lemma 3.3.

Proof that b_0c_0 and b_4c_4 are suitable edges for J_9 (continued)
We will now prove that at least one of $(0, 26)$ or $(11, 12)$ is good in $J_9 - \{v, (0, 26), (11, 12)\}$ for every $v \in V(J_9)$ except for $v = 0$, which we have already shown above in the proof of Lemma 3.3.

- $v = 1$: 26, 4, 5, 6, 7, 2, 3, 28, 29, 30, 31, 12, 13, 8, 9, 10, 11, 16, 17, 18, 33, 32, 15, 14, 19, 20, 25, 24, 23, 22, 21, 34, 35, 27, 0
- $v = 2$: 12, 31, 32, 15, 14, 13, 8, 7, 6, 5, 10, 9, 30, 29, 28, 3, 4, 26, 25, 24, 23, 1, 0, 27, 35, 34, 33, 18, 19, 20, 21, 22, 17, 16, 11
- $v = 3$: 26, 4, 5, 6, 7, 2, 1, 23, 22, 21, 20, 25, 24, 35, 34, 33, 32, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 31, 30, 29, 28, 27, 0
- $v = 4$: 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 3, 2, 1, 23, 24, 35, 27, 0
- $v = 5$: 26, 4, 3, 2, 1, 23, 22, 21, 20, 25, 24, 35, 34, 33, 32, 15, 14, 19, 18, 17, 16, 11, 10, 9, 30, 31, 12, 13, 8, 7, 6, 29, 28, 27, 0
- $v = 6$: 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 5, 4, 3, 2, 7, 8, 9, 30, 29, 28, 27, 35, 24, 23, 1, 0
- $v = 7$: 26, 4, 5, 6, 29, 30, 31, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 32, 33, 34, 35, 24, 25, 20, 21, 22, 23, 1, 2, 3, 28, 27, 0
- $v = 8$: 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 9, 30, 29, 28, 3, 4, 5, 6, 7, 2, 1, 23, 24, 35, 27, 0
- $v = 9$: 26, 4, 3, 2, 1, 23, 22, 21, 20, 25, 24, 35, 34, 33, 32, 15, 14, 19, 18, 17, 16, 11, 10, 5, 6, 7, 8, 13, 12, 31, 30, 29, 28, 27, 0
- $v = 10$: 12, 13, 14, 15, 32, 31, 30, 9, 8, 7, 2, 3, 28, 29, 6, 5, 4, 26, 25, 24, 23, 1, 0, 27, 35, 34, 33, 18, 19, 20, 21, 22, 17, 16, 11
- $v = 11$: 26, 4, 3, 2, 1, 23, 22, 21, 34, 35, 24, 25, 20, 19, 14, 15, 16, 17, 18, 33, 32, 31, 12, 13, 8, 7, 6, 5, 10, 9, 30, 29, 28, 27, 0
- $v = 12$: 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 30, 9, 8, 13, 14, 15, 16, 11, 10, 5, 4, 3, 2, 7, 6, 29, 28, 27, 35, 24, 23, 1, 0
- $v = 13$: 12, 31, 30, 29, 6, 5, 10, 9, 8, 7, 2, 1, 0, 27, 28, 3, 4, 26, 25, 20, 21, 22, 23, 24, 35, 34, 33, 32, 15, 14, 19, 18, 17, 16, 11
- $v = 14$: 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 15, 16, 11, 10, 9, 8, 13, 12, 31, 30, 29, 28, 3, 4, 5, 6, 7, 2, 1, 23, 24, 35, 27, 0
- $v = 15$: 26, 4, 3, 2, 1, 23, 22, 21, 20, 25, 24, 35, 34, 33, 32, 31, 12, 13, 14, 19, 18, 17, 16, 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 27, 0
- $v = 16$: 12, 13, 8, 7, 2, 3, 28, 29, 6, 5, 4, 26, 25, 24, 23, 1, 0, 27, 35, 34, 33, 18, 17, 22, 21, 20, 19, 14, 15, 32, 31, 30, 9, 10, 11
- $v = 17$: 26, 4, 3, 2, 1, 23, 22, 21, 20, 25, 24, 35, 34, 33, 18, 19, 14, 13, 12, 31, 32, 15, 16, 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 27, 0

- $v = 18$: 26, 25, 20, 19, 14, 15, 32, 33, 34, 21, 22, 17, 16, 11, 10, 9, 8, 13, 12, 31, 30, 29, 28, 3, 4, 5, 6, 7, 2, 1, 23, 24, 35, 27, 0
- $v = 19$: 26, 4, 3, 28, 27, 35, 34, 21, 20, 25, 24, 23, 22, 17, 18, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 5, 6, 29, 30, 9, 8, 7, 2, 1, 0
- $v = 20$: 26, 25, 24, 35, 27, 28, 29, 6, 5, 4, 3, 2, 7, 8, 13, 12, 31, 30, 9, 10, 11, 16, 17, 18, 19, 14, 15, 32, 33, 34, 21, 22, 23, 1, 0
- $v = 21$: 26, 4, 3, 2, 1, 23, 22, 17, 18, 19, 20, 25, 24, 35, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 27, 0
- $v = 22$: 26, 25, 20, 21, 34, 33, 32, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 31, 30, 29, 28, 3, 4, 5, 6, 7, 2, 1, 23, 24, 35, 27, 0
- $v = 23$: 26, 4, 3, 28, 27, 35, 24, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 5, 6, 29, 30, 9, 8, 7, 2, 1, 0
- $v = 24$: 26, 25, 20, 19, 14, 15, 32, 33, 18, 17, 16, 11, 10, 9, 8, 13, 12, 31, 30, 29, 28, 3, 4, 5, 6, 7, 2, 1, 23, 22, 21, 34, 35, 27, 0
- $v = 25$: 26, 4, 3, 2, 1, 23, 24, 35, 34, 33, 18, 17, 22, 21, 20, 19, 14, 13, 12, 31, 32, 15, 16, 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 27, 0
- $v = 26$: 12, 13, 8, 7, 2, 3, 4, 5, 6, 29, 28, 27, 0, 1, 23, 22, 21, 34, 35, 24, 25, 20, 19, 14, 15, 16, 17, 18, 33, 32, 31, 30, 9, 10, 11
- $v = 27$: 26, 4, 5, 6, 7, 2, 3, 28, 29, 30, 31, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 32, 33, 34, 35, 24, 25, 20, 21, 22, 23, 1, 0
- $v = 28$: 12, 13, 14, 15, 32, 31, 30, 29, 6, 5, 10, 9, 8, 7, 2, 3, 4, 26, 25, 24, 23, 1, 0, 27, 35, 34, 33, 18, 19, 20, 21, 22, 17, 16, 11
- $v = 29$: 26, 4, 5, 6, 7, 8, 13, 12, 31, 30, 9, 10, 11, 16, 17, 18, 19, 14, 15, 32, 33, 34, 35, 24, 25, 20, 21, 22, 23, 1, 2, 3, 28, 27, 0
- $v = 30$: 26, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 9, 8, 7, 2, 3, 4, 5, 6, 29, 28, 27, 35, 24, 23, 1, 0
- $v = 31$: 12, 13, 8, 7, 6, 5, 10, 9, 30, 29, 28, 27, 0, 1, 2, 3, 4, 26, 25, 20, 21, 22, 23, 24, 35, 34, 33, 32, 15, 14, 19, 18, 17, 16, 11
- $v = 32$: 26, 25, 24, 23, 1, 2, 7, 6, 5, 4, 3, 28, 29, 30, 31, 12, 13, 8, 9, 10, 11, 16, 15, 14, 19, 20, 21, 22, 17, 18, 33, 34, 35, 27, 0
- $v = 33$: 26, 4, 3, 28, 27, 35, 34, 21, 20, 25, 24, 23, 22, 17, 18, 19, 14, 13, 12, 31, 32, 15, 16, 11, 10, 5, 6, 29, 30, 9, 8, 7, 2, 1, 0
- $v = 34$: 26, 25, 24, 35, 27, 28, 29, 6, 5, 4, 3, 2, 7, 8, 13, 12, 31, 30, 9, 10, 11, 16, 17, 18, 33, 32, 15, 14, 19, 20, 21, 22, 23, 1, 0
- $v = 35$: 26, 4, 3, 2, 1, 23, 24, 25, 20, 19, 18, 17, 22, 21, 34, 33, 32, 31, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 30, 29, 28, 27, 0

Proof that b_0c_0 and b_4c_4 are suitable edges for J_{11}

Figure 2 shows the flower snark J_{11} and here b_0c_0 and b_4c_4 correspond to the edges $(0, 32)$ and $(11, 12)$, respectively.

The pairs $(0, 11), (0, 12), (32, 11)$, and $(32, 12)$ are good in $J_{11} - \{(0, 32), (11, 12)\}$ due to the following hamiltonian paths, respectively:

- 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 43, 30, 29, 28, 23, 22, 17, 16, 15, 38, 37, 12, 13, 14, 19, 18, 39, 40, 21, 20, 25, 24, 41, 42, 27, 26, 31, 32, 4, 3, 2, 1, 0
- 12, 13, 14, 15, 16, 11, 10, 5, 4, 32, 31, 26, 25, 24, 41, 42, 27, 28, 23, 22, 17, 18, 19, 20, 21, 40, 39, 38, 37, 36, 9, 8, 7, 6, 35, 34, 3, 2, 1, 29, 30, 43, 33, 0
- 11, 10, 5, 4, 3, 2, 1, 0, 33, 34, 35, 6, 7, 8, 9, 36, 37, 12, 13, 14, 19, 18, 17, 16, 15, 38, 39, 40, 41, 24, 25, 20, 21, 22, 23, 28, 29, 30, 43, 42, 27, 26, 31, 32
- 12, 13, 8, 7, 2, 3, 4, 5, 6, 35, 34, 33, 0, 1, 29, 28, 23, 22, 17, 18, 39, 38, 37, 36, 9, 10, 11, 16, 15, 14, 19, 20, 21, 40, 41, 24, 25, 26, 27, 42, 43, 30, 31, 32

We have that $((0, 32), (11, 12))$ is good in $J_{11} - \{(0, 32), (11, 12)\}$ due to the following two disjoint paths with end-vertices 0 and 32, and 11 and 12, respectively, which together span J_{11}.

- 32, 31, 26, 25, 20, 19, 14, 13, 8, 7, 2, 1, 0
- 12, 37, 38, 15, 16, 17, 18, 39, 40, 21, 22, 23, 24, 41, 42, 27, 28, 29, 30, 43, 33, 34, 3, 4, 5, 6, 35, 36, 9, 10, 11

The following hamiltonian paths show that at least one of $(0, 32)$ or $(11, 12)$ is good in $J_{11} - \{v, (0, 32), (11, 12)\}$ for every $v \in V(J_{11})$.

- $v = 0$: 12, 13, 14, 15, 38, 37, 36, 35, 6, 5, 10, 9, 8, 7, 2, 1, 29, 30, 31, 32, 4, 3, 34, 33, 43, 42, 41, 24, 23, 28, 27, 26, 25, 20, 19, 18, 39, 40, 21, 22, 17, 16, 11
- $v = 1$: 32, 4, 5, 6, 7, 2, 3, 34, 35, 36, 37, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 38, 39, 40, 41, 24, 23, 22, 21, 20, 25, 26, 31, 30, 29, 28, 27, 42, 43, 33, 0
- $v = 2$: 12, 37, 38, 15, 14, 13, 8, 7, 6, 5, 10, 9, 36, 35, 34, 3, 4, 32, 31, 30, 29, 1, 0, 33, 43, 42, 41, 24, 23, 28, 27, 26, 25, 20, 19, 18, 39, 40, 21, 22, 17, 16, 11
- $v = 3$: 32, 4, 5, 6, 7, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 33, 0
- $v = 4$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 3, 2, 1, 29, 30, 43, 33, 0
- $v = 5$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11, 10, 9, 36, 37, 12, 13, 8, 7, 6, 35, 34, 33, 0
- $v = 6$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 4, 3, 2, 7, 8, 9, 36, 35, 34, 33, 43, 30, 29, 1, 0
- $v = 7$: 32, 4, 5, 6, 35, 36, 37, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 38, 39, 40, 41, 24, 23, 22, 21, 20, 25, 26, 31, 30, 29, 28, 27, 42, 43, 33, 34, 3, 2, 1, 0
- $v = 8$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 9, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 9$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11, 10, 5, 6, 7, 8, 13, 12, 37, 36, 35, 34, 33, 0
- $v = 10$: 12, 13, 14, 15, 38, 37, 36, 9, 8, 7, 2, 3, 34, 35, 6, 5, 4, 32, 31, 30, 29, 1, 0, 33, 43, 42, 41, 24, 23, 28, 27, 26, 25, 20, 19, 18, 39, 40, 21, 22, 17, 16, 11
- $v = 11$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 20, 25, 24, 23, 22, 17, 16, 15, 14, 19, 18, 39, 38, 37, 12, 13, 8, 7, 6, 5, 10, 9, 36, 35, 34, 33, 0

- $v = 12$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 8, 9, 10, 11, 16, 15, 38, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 13$: 12, 37, 36, 35, 6, 5, 10, 9, 8, 7, 2, 1, 0, 33, 34, 3, 4, 32, 31, 26, 27, 28, 29, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11
- $v = 14$: 32, 31, 26, 25, 24, 41, 42, 27, 28, 23, 22, 17, 18, 19, 20, 21, 40, 39, 38, 15, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 15$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 13, 12, 37, 38, 39, 18, 17, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 16$: 12, 13, 8, 7, 2, 3, 34, 35, 6, 5, 4, 32, 31, 30, 29, 1, 0, 33, 43, 42, 41, 24, 25, 26, 27, 28, 23, 22, 17, 18, 39, 40, 21, 20, 19, 14, 15, 38, 37, 36, 9, 10, 11
- $v = 17$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 18, 39, 38, 37, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 18$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 38, 15, 14, 19, 20, 21, 22, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 19$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 20, 25, 24, 23, 22, 17, 18, 39, 38, 37, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 20$: 32, 31, 26, 25, 24, 41, 42, 27, 28, 23, 22, 21, 40, 39, 38, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 21$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 39, 18, 17, 22, 23, 24, 25, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 22$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 21, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 23$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 24, 25, 20, 19, 18, 17, 22, 21, 40, 39, 38, 37, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 24$: 32, 31, 26, 25, 20, 21, 22, 23, 28, 27, 42, 41, 40, 39, 38, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 25$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 24, 23, 22, 17, 18, 19, 20, 21, 40, 39, 38, 37, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 26$: 32, 31, 30, 29, 1, 2, 7, 6, 5, 4, 3, 34, 35, 36, 37, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 38, 39, 40, 41, 24, 25, 20, 21, 22, 23, 28, 27, 42, 43, 33, 0
- $v = 27$: 32, 4, 3, 2, 1, 29, 28, 23, 24, 25, 26, 31, 30, 43, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 28$: 32, 31, 26, 27, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 29$: 32, 4, 3, 34, 33, 43, 30, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 6, 35, 36, 9, 8, 7, 2, 1, 0
- $v = 30$: 32, 31, 26, 25, 20, 21, 22, 23, 24, 41, 40, 39, 38, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 28, 27, 42, 43, 33, 0
- $v = 31$: 32, 4, 3, 2, 1, 29, 30, 43, 42, 41, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 40, 39, 38, 37, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0

- $v = 32$: 12, 13, 8, 7, 2, 3, 4, 5, 6, 35, 34, 33, 0, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 20, 25, 24, 23, 22, 17, 16, 15, 14, 19, 18, 39, 38, 37, 36, 9, 10, 11
- $v = 33$: 32, 4, 5, 6, 7, 2, 3, 34, 35, 36, 37, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 38, 39, 40, 41, 24, 23, 22, 21, 20, 25, 26, 31, 30, 43, 42, 27, 28, 29, 1, 0
- $v = 34$: 12, 13, 14, 15, 38, 37, 36, 35, 6, 5, 10, 9, 8, 7, 2, 3, 4, 32, 31, 30, 29, 1, 0, 33, 43, 42, 41, 24, 23, 28, 27, 26, 25, 20, 19, 18, 39, 40, 21, 22, 17, 16, 11
- $v = 35$: 32, 4, 5, 6, 7, 8, 13, 12, 37, 36, 9, 10, 11, 16, 17, 18, 19, 14, 15, 38, 39, 40, 41, 24, 23, 22, 21, 20, 25, 26, 31, 30, 29, 28, 27, 42, 43, 33, 34, 3, 2, 1, 0
- $v = 36$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 9, 8, 7, 2, 3, 4, 5, 6, 35, 34, 33, 43, 30, 29, 1, 0
- $v = 37$: 12, 13, 8, 7, 6, 5, 10, 9, 36, 35, 34, 33, 0, 1, 2, 3, 4, 32, 31, 26, 27, 28, 29, 30, 43, 42, 41, 40, 21, 22, 23, 24, 25, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11
- $v = 38$: 32, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 15, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 39$: 32, 4, 3, 2, 1, 29, 28, 27, 26, 31, 30, 43, 42, 41, 40, 21, 20, 25, 24, 23, 22, 17, 18, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 40$: 32, 31, 26, 25, 24, 41, 42, 27, 28, 23, 22, 21, 20, 19, 14, 15, 38, 39, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 41$: 32, 4, 3, 2, 1, 29, 28, 27, 42, 43, 30, 31, 26, 25, 24, 23, 22, 17, 18, 19, 20, 21, 40, 39, 38, 37, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0
- $v = 42$: 32, 31, 26, 27, 28, 23, 22, 21, 20, 25, 24, 41, 40, 39, 38, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 37, 36, 35, 34, 3, 4, 5, 6, 7, 2, 1, 29, 30, 43, 33, 0
- $v = 43$: 32, 4, 3, 2, 1, 29, 30, 31, 26, 25, 24, 23, 28, 27, 42, 41, 40, 39, 18, 17, 22, 21, 20, 19, 14, 13, 12, 37, 38, 15, 16, 11, 10, 5, 6, 7, 8, 9, 36, 35, 34, 33, 0

Proof that b_0c_0 and b_4c_4 are suitable edges for J_{13}

Figure 3 shows the flower snark J_{13} and here b_0c_0 and b_4c_4 correspond to the edges $(0, 38)$ and $(11, 12)$, respectively.

The pairs $(0, 11), (0, 12), (38, 11)$, and $(38, 12)$ are good in $J_{13} - \{(0, 38), (11, 12)\}$ due to the following hamiltonian paths, respectively:

- 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 51, 36, 35, 34, 29, 28, 23, 22, 17, 16, 15, 44, 43, 12, 13, 14, 19, 18, 45, 46, 21, 20, 25, 24, 47, 48, 27, 26, 31, 30, 49, 50, 33, 32, 37, 38, 4, 3, 2, 1, 0
- 12, 13, 14, 15, 16, 11, 10, 5, 4, 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 43, 42, 9, 8, 7, 6, 41, 40, 3, 2, 1, 35, 36, 51, 39, 0
- 11, 10, 5, 4, 3, 2, 1, 0, 39, 40, 41, 6, 7, 8, 9, 42, 43, 12, 13, 14, 19, 18, 17, 16, 15, 44, 45, 46, 47, 24, 23, 22, 21, 20, 25, 26, 31, 30, 29, 28, 27, 48, 49, 50, 51, 36, 35, 34, 33, 32, 37, 38
- 12, 13, 8, 7, 2, 3, 4, 5, 6, 41, 40, 39, 0, 1, 35, 34, 29, 28, 23, 22, 17, 18, 45, 44, 43, 42, 9, 10, 11, 16, 15, 14, 19, 20, 21, 46, 47, 24, 25, 26, 27, 48, 49, 30, 31, 32, 33, 50, 51, 36, 37, 38

The pair of pairs $((0, 38), (11, 12))$ is good in $J_{13} - \{(0, 38), (11, 12)\}$ due to the following two disjoint paths with end-vertices 0 and 38, and 11 and 12, respectively, which together span J_{13}.

- 38, 37, 32, 31, 26, 25, 20, 19, 14, 13, 8, 7, 2, 1, 0
- 12, 43, 44, 15, 16, 17, 18, 45, 46, 21, 22, 23, 24, 47, 48, 27, 28, 29, 30, 49, 50, 33, 34, 35, 36, 51, 39, 40, 3, 4, 5, 6, 41, 42, 9, 10, 11

The following hamiltonian paths show that at least one of $(0, 38)$ or $(11, 12)$ is good in $J_{13} - \{v, (0, 38), (11, 12)\}$ for every $v \in V(J_{13})$.

- $v = 0$: 12, 13, 14, 15, 44, 43, 42, 41, 6, 5, 10, 9, 8, 7, 2, 1, 35, 36, 37, 38, 4, 3, 40, 39, 51, 50, 49, 30, 29, 34, 33, 32, 31, 26, 25, 24, 23, 28, 27, 48, 47, 46, 45, 18, 19, 20, 21, 22, 17, 16, 11
- $v = 1$: 38, 4, 5, 6, 7, 2, 3, 40, 41, 42, 43, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 44, 45, 46, 47, 24, 23, 22, 21, 20, 25, 26, 31, 30, 49, 48, 27, 28, 29, 34, 35, 36, 37, 32, 33, 50, 51, 39, 0
- $v = 2$: 12, 43, 44, 15, 14, 13, 8, 7, 6, 5, 10, 9, 42, 41, 40, 3, 4, 38, 37, 36, 35, 1, 0, 39, 51, 50, 49, 30, 29, 34, 33, 32, 31, 26, 25, 24, 23, 28, 27, 48, 47, 46, 45, 18, 19, 20, 21, 22, 17, 16, 11
- $v = 3$: 38, 4, 5, 6, 7, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 39, 0
- $v = 4$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 43, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 3, 2, 1, 35, 36, 51, 39, 0
- $v = 5$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11, 10, 9, 42, 43, 12, 13, 8, 7, 6, 41, 40, 39, 0
- $v = 6$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 43, 12, 13, 14, 15, 16, 11, 10, 5, 4, 3, 2, 7, 8, 9, 42, 41, 40, 39, 51, 36, 35, 1, 0
- $v = 7$: 38, 4, 5, 6, 41, 42, 43, 12, 13, 8, 9, 10, 11, 16, 17, 18, 19, 14, 15, 44, 45, 46, 47, 24, 23, 22, 21, 20, 25, 26, 31, 30, 29, 28, 27, 48, 49, 50, 51, 36, 37, 32, 33, 34, 35, 1, 2, 3, 40, 39, 0
- $v = 8$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 43, 12, 13, 14, 15, 16, 11, 10, 9, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0
- $v = 9$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11, 10, 5, 6, 7, 8, 13, 12, 43, 42, 41, 40, 39, 0
- $v = 10$: 12, 13, 14, 15, 44, 43, 42, 9, 8, 7, 2, 3, 40, 41, 6, 5, 4, 38, 37, 36, 35, 1, 0, 39, 51, 50, 49, 30, 29, 34, 33, 32, 31, 26, 25, 24, 23, 28, 27, 48, 47, 46, 45, 18, 19, 20, 21, 22, 17, 16, 11

- $v = 11$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 46, 47, 24, 25, 20, 19, 14, 15, 16, 17, 18, 45, 44, 43, 12, 13, 8, 7, 6, 5, 10, 9, 42, 41, 40, 39, 0
- $v = 12$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 43, 42, 9, 8, 13, 14, 15, 16, 11, 10, 5, 4, 3, 2, 7, 6, 41, 40, 39, 51, 36, 35, 1, 0
- $v = 13$: 12, 43, 42, 41, 6, 5, 10, 9, 8, 7, 2, 1, 0, 39, 40, 3, 4, 38, 37, 32, 33, 34, 35, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11
- $v = 14$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 15, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0
- $v = 15$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 44, 43, 12, 13, 14, 19, 18, 17, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
- $v = 16$: 12, 13, 8, 7, 2, 3, 40, 41, 6, 5, 4, 38, 37, 36, 35, 1, 0, 39, 51, 50, 49, 30, 29, 34, 33, 32, 31, 26, 25, 24, 23, 28, 27, 48, 47, 46, 45, 18, 17, 22, 21, 20, 19, 14, 15, 44, 43, 42, 9, 10, 11
- $v = 17$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 18, 19, 14, 13, 12, 43, 44, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
- $v = 18$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 19, 14, 15, 44, 45, 46, 21, 22, 17, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0
- $v = 19$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 28, 29, 30, 31, 26, 25, 20, 21, 46, 47, 24, 23, 22, 17, 18, 45, 44, 43, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
- $v = 20$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 25, 26, 27, 28, 23, 22, 21, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0
- $v = 21$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 22, 17, 18, 19, 20, 25, 24, 47, 46, 45, 44, 43, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
- $v = 22$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 24, 23, 28, 27, 26, 25, 20, 21, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0
- $v = 23$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 28, 29, 30, 31, 26, 25, 24, 47, 46, 45, 18, 17, 22, 21, 20, 19, 14, 13, 12, 43, 44, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
- $v = 24$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 47, 46, 21, 22, 23, 28, 27, 26, 25, 20, 19, 14, 15, 44, 45, 18, 17, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0

- $v = 25$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 27, 26, 31, 30, 29, 28, 23, 24, 47, 46, 45, 18, 17, 22, 21, 20, 19, 14, 13, 12, 43, 44, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
- $v = 26$: 38, 37, 32, 31, 30, 29, 34, 33, 50, 49, 48, 27, 28, 23, 22, 21, 20, 25, 24, 47, 46, 45, 44, 15, 14, 19, 18, 17, 16, 11, 10, 9, 8, 13, 12, 43, 42, 41, 40, 3, 4, 5, 6, 7, 2, 1, 35, 36, 51, 39, 0
- $v = 27$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 48, 47, 24, 23, 28, 29, 30, 31, 26, 25, 20, 19, 18, 17, 22, 21, 46, 45, 44, 43, 12, 13, 14, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
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- $v = 29$: 38, 4, 3, 2, 1, 35, 34, 33, 32, 37, 36, 51, 50, 49, 30, 31, 26, 25, 24, 23, 28, 27, 48, 47, 46, 45, 18, 17, 22, 21, 20, 19, 14, 13, 12, 43, 44, 15, 16, 11, 10, 5, 6, 7, 8, 9, 42, 41, 40, 39, 0
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Inherited unitals in Moulton planes*

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Abstract

We prove that every Moulton plane of odd order—by duality every generalised André plane—contains a unital. We conjecture that such unitals are non-classical, that is, they are not isomorphic, as designs, to the Hermitian unital. We prove our conjecture for Moulton planes which differ from $\mathrm{PG}(2, q^2)$ by a relatively small number of point-line incidences. Up to duality, our results extend previous analogous results—due to Barwick and Grüning—concerning inherited unitals in Hall planes.

Keywords: Unitals, Moulton planes.

Math. Subj. Class.: 51E20, 05B25

1 Introduction

A unital is a set of q^3+1 points together with a family of subsets, each of size $q+1$, such that every pair of distinct points are contained in exactly one subset of the family. Such subsets are usually called blocks so that unitals are block-designs $2-(q^3+1, q+1, 1)$. The classical example of a unital arises from the unitary polarity in the Desarguesian projective plane $\mathrm{PG}(2, q^2)$ where the points are the absolute points, and the blocks are the non-absolute lines of the unitary polarity. The name of "Hermitian unital" is commonly used for the

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classical example since the absolute points of the unitary polarity are the points of the Hermitian curve defined over $\mathrm{GF}(q^2)$.

A unital $\mathcal{U}$ is *embedded* in a projective plane Π of order q^2, if its points are points of Π and its blocks are intersections with lines. As usual, we adopt the term "chord" to indicate a block of $\mathcal{U}$. A line ℓ of Π is either a tangent or a $(q+1)$-secant to $\mathcal{U}$ according as $|\ell \cap \mathcal{U}| = 1$ or $|\ell \cap \mathcal{U}| = q+1$, and in the latter case $\ell \cap \mathcal{U}$ is a chord. Examples of unitals embedded in $\mathrm{PG}(2, q^2)$ other than the Hermitian ones are known to exist.

A unital is *classical* if it is isomorphic, as a block-design, to a Hermitian unital. Classical unitals contain no O'Nan configurations, and it has been conjectured that any non-classical unital embedded in $\mathrm{PG}(2, q^2)$ must contain a O'Nan configuration.

In several families of non-desarguesian planes, the problem of constructing and characterizing unitals has also been investigated; see [1, 2, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28]. Apart from the examples of unitals arising from a unitary polarity in a commutative semifield plane, the known examples are inherited unitals from the Hermitian unital. In a non-desarguesian plane Π of order q^2 arising from $\mathrm{PG}(2, q^2)$ by altering some of the point-line incidences, the adjective "inherited" is used for those pointsets of $\mathrm{PG}(2, q^2)$ which keep their intersection properties with lines when moving from $\mathrm{PG}(2, q^2)$ to Π.

In this paper we construct inherited unitals in Moulton planes of odd order q^2, and, by duality, in generalised André planes of the same order; see Theorem 3.1. We also investigate the problem whether these unitals are classical; see Theorems 3.5 and 3.6. We show that if such a plane differs from $\mathrm{PG}(2, q^2)$ by a relatively small number of incidences only, then the inherited unital is non-classical. Also, we exhibit non-classical inherited unitals in case of many point-line incidence alterations. Such unitals appear to be of interest in coding theory; see [25].

What emerges from our work leads us to conjecture that the inherited unitals constructed in our paper are all non-classical. It should be noticed that our results extend previous analogous results due to Barwick and Grüning concerning inherited unitals in Hall planes which are very special André planes; see [8, 16] and Remark 3.4. The methods used in [8] are mostly geometric and involve Baer subplanes and blocking sets. In this paper, we adopt a more algebraic approach that allows us to exploit results on the number of solutions of systems of polynomial equations over a finite field.

2 Two new results on the Hermitian unital

We establish and prove two theorems on Hermitian unitals that will play a role in our study on unitals in Moulton planes.

Up to a change of the homogeneous coordinate system (X_1, X_2, X_3) in $\mathrm{PG}(2, q^2)$, the points of the classical unital $\mathcal{U}$ are those satisfying the equation

$$X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0. \tag{2.1}$$

In the affine plane $\mathrm{AG}(2, q^2)$ arising from $\mathrm{PG}(2, q^2)$ with respect to the line $X_3 = 0$, we use the coordinates (X, Y) where $X = X_1/X_3$ and $Y = X_2/X_3$). Then the points of $\mathcal{U}$ in $\mathrm{AG}(2, q^2)$ are the solutions of the equation

$$X^{q+1} + Y^{q+1} + 1 = 0. \tag{2.2}$$

Since $\mathrm{GF}(q^2)$ is the quadratic extension of $\mathrm{GF}(q)$ by adjunction of a root i of the polynomial $X^2 - s$ with a non-square element s of $\mathrm{GF}(q)$, every element u of $\mathrm{GF}(q^2)$ can uniquely be written as $u = u_1 + iu_2$ with $u_1, u_2 \in \mathrm{GF}(q)$. Then $u^q = u_1 - iu_2$ and $\|u\| = u^{q+1} = u_1^2 - su_2^2$. Therefore, the points $P(x, y) \in \mathcal{U}$ lying in $\mathrm{AG}(2, q^2)$ are those satisfying the equation

$$x_1^2 - sx_2^2 + y_1^2 - sy_2^2 + 1 = 0. \tag{2.3}$$

For a subset $T \subseteq \mathrm{GF}(q) \setminus \{0\}$, let $\mathcal{S}_t$ denote the set of points $\{ (x, y) \mid \|x\| = t \in T \}$. Hence the pointset $\mathcal{S}_t \cap \mathcal{U}$ comprises all points $P(x, y)$ such that both $x_1^2 - sx_2^2 = t$ and (2.3) hold. Therefore, a point $P(x, y) \in \mathrm{AG}(2, q^2)$ is in $\mathcal{S}_t \cap \mathcal{U}$ if and only if $P_1(x_1, x_2) \in \mathrm{AG}(2, q)$ lies on the non-degenerate conic $\mathcal{C}_1 : X^2 - sY^2 - t = 0$ while $P_2(y_1, y_2) \in \mathrm{AG}(2, q)$ does lie on the conic $\mathcal{C}_2 : X^2 - sY^2 + 1 + t = 0$. This shows that $\mathcal{S}_t \cap \mathcal{U}$ has size $(q + 1)^2$ apart from the case $t = -1$ when it consists of the $q + 1$ points of $\mathcal{U}$ lying on the X-axis.

Lemma 2.1. *Let ℓ be a non-vertical line in $\mathrm{AG}(2, q^2)$. Then $|\ell \cap \mathcal{U} \cap \mathcal{S}_t| \in \{0, 1, 2, q+1\}$ for every $t \in T$. If $q + 1$ occurs then ℓ is either a horizontal line, or it passes through the origin.*

Proof. The points $P(x, 0)$ with $\|x\| = t$ form a Baer subline. As $\mathcal{U}$ is classical, $\ell \cap \mathcal{U}$ is a Baer subline of ℓ, and hence the projection of $\ell \cap \mathcal{U}$ on the X-axis from Y_∞ is a Baer-subline, as well. Since two distinct Baer sublines have at most two common points, the first assertion follows. To prove the second one, we need some computation. If ℓ has equation $Y = Xm + b$, we have to count the roots x of the polynomial $f(X) = X^{q+1} + (Xm + b)^{q+1} + 1$ whose norm $\|x\|$ is equal to t. If $\|x\| = t$, then $f(x) = bm^q x^q + b^q mx + t(1 + m^{q+1}) + b^{q+1} + 1$ and hence

$$xf(x) = b^q mx^2 + (t(1 + m^{q+1}) + b^{q+1} + 1)x + bm^q t.$$

If we have at least three such roots x then either $m = 0$ and $t + 1 = -b^{q+1}$, or $b = 0$ and $t(1 + m^{q+1}) = -1$. $\square$

Take any two distinct non-tangent lines ℓ_1 and ℓ_2 of $\mathcal{U}$. We are interested in the intersection of the projection of $\ell_1 \cap \mathcal{U}$ from P on ℓ_2 with $\ell_2 \cap \mathcal{U}$. For any point P outside both ℓ_1 and ℓ_2, the projection of ℓ_1 to ℓ_2 from P takes the chord $\ell_1 \cap \mathcal{U}$ to a Baer subline of ℓ_2. Since two Baer sublines of ℓ_2 intersect in $0, 1, 2$ or $q + 1$ points, one may want to determine the size of the sets Σ_i $(i = 0, 1, 2, q+1)$ consisting of all points P for which this intersection number is equal to i. The points in Σ_i are called *elliptic*, *parabolic*, *hyperbolic*, or *full* with respect to the pair (ℓ_1, ℓ_2), according as $i = 0$, $i = 1$, $i = 2$, or $i = q + 1$, respectively; see [21].

We go on to compute the size of $\Sigma_i \cap \mathcal{U}$. Since the linear collineation group $G \cong \mathrm{PGU}(3, q)$ of $\mathrm{PG}(2, q^2)$ preserving $\mathcal{U}$ acts transitively on the points outside $\mathcal{U}$, we may assume that $Y_\infty = \ell_1 \cap \ell_2$. The stabiliser of Y_∞ in G acts on the pencil with center in Y_∞ as the general projective group $\mathrm{PGL}(2, q)$ on the projective line $\mathrm{PG}(1, q^2)$. Therefore, it has two orbits, one consisting of all tangents the other of all chords to $\mathcal{U}$ through Y_∞. This allows us to assume without loss of generality that ℓ_1 is the line at infinity. Since ℓ_2 is not a tangent to $\mathcal{U}$, its equation is of the form $X = c$ with $c^{q+1} + 1 \neq 0$. Therefore, $c^{q+1} + 1$ is either a non-zero square or a non-square element of $\mathrm{GF}(q)$. These two cases occur depending upon whether a linear collineation $\gamma \in \mathrm{PGL}(2, q)$ taking ℓ_1 to ℓ_2 is in the subgroup isomorphic to the special projective group $\mathrm{PSL}(2, q)$ or not. Accordingly, $\{\ell_1, \ell_2\}$ is called a *special* pair or a *general* pair. Further, since P is a point outside ℓ_1 and ℓ_2, it is an affine point $P = (a, b)$ with $a \neq c$.

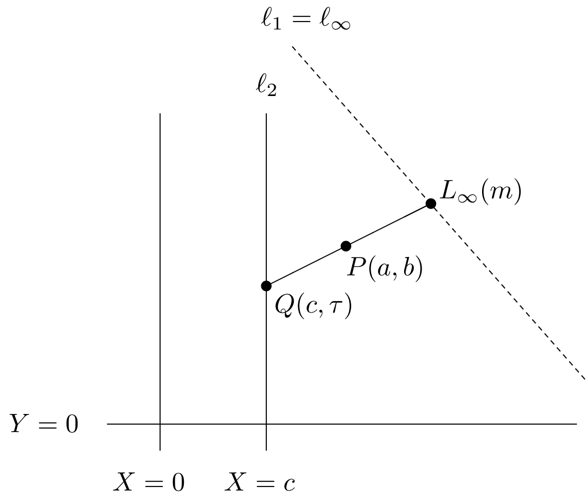

Figure 1: The initial configuration.

Let $P = (a, b)$ denote a point of $\mathcal{U}$, that is,

$$a^{q+1} + b^{q+1} + 1 = 0. \tag{2.4}$$

Take a line r of equation $Y = m(X-a)+b$ through $P = (a, b)$. A necessary and sufficient condition for r to meet both ℓ_1 and ℓ_2 in $\mathcal{U}$ is the existence of a solution $\tau \in \mathrm{GF}(q^2)$ of the system consisting of (2.4) together with

$$c^{q+1} + \tau^{q+1} + 1 = 0, \tag{2.5}$$

$$m^{q+1} + 1 = 0. \tag{2.6}$$

In fact, $Q(c, \tau)$ with $\tau = m(c - a) + b$ is the point of r on ℓ_2. Then (2.5) holds if and only if $Q \in \mathcal{U}$. Furthermore, (2.6) is the necessary and sufficient condition for the infinite point of r to be in $\mathcal{U}$; see Figure 1.

The above discussion also shows how to count lines through P meeting both $\ell_1 \cap \mathcal{U}$ and $\ell_2 \cap \mathcal{U}$. Essentially, one has to find the number of solutions in the indeterminate τ of the system consisting of the equations (2.4), (2.5), and (2.6). Observe that (2.4), (2.5), (2.6) are equivalent to

$$a_1^2 - sa_2^2 + b_1^2 - sb_2^2 + 1 = 0, \tag{2.7}$$

$$c_1^2 - sc_2^2 + \tau_1^2 - s\tau_2^2 + 1 = 0, \tag{2.8}$$

$$b_1\tau_1 - sb_2\tau_2 + a_1c_1 - sa_2c_2 + 1 = 0. \tag{2.9}$$

From this the following result is obtained.

Proposition 2.2. *The number of lines through P meeting both $\ell_1 \cap \mathcal{U}$ and $\ell_2 \cap \mathcal{U}$ equals the number of solutions (τ_1, τ_2), with $\tau_1, \tau_2 \in \mathrm{GF}(q)$, of the system consisting of* (2.7), (2.8), (2.9).

In investigating the above system, two cases are distinguished according as (b_1, b_2) is $(0, 0)$ or not.

In the former case, Equations (2.7) and (2.9) read $a_1^2 - sa_2^2 + 1 = 0$ and $a_1c_1 - sa_2c_2 + 1 = 0$. Geometrically in $\mathrm{AG}(2, q)$, the point $U = (a_1, a_2)$ is the intersection of the ellipse $\mathcal{E}$, with equation $X^2 - sY^2 + 1 = 0$, and the line v with equation $c_1X - sc_2Y + 1 = 0$. Since $c^{q+1} + 1 = c_1^2 - sc_2^2 + 1$ is a non-zero element of $\mathrm{GF}(q)$, v must be either a secant, or an external line to $\mathcal{E}$ and this occurs according as $c_1^2 - sc_2^2 + 1$ is a non-zero square or non-square element in $\mathrm{GF}(q)$. In fact, from (2.7) and (2.9),

$$a_1 = \frac{sc_2a_2 - 1}{c_1}, \quad a_2 = \frac{-sc_2 \pm ic_1\sqrt{c_1^2 - sc_2^2 + 1}}{s(c_1^2 - sc_2^2)}.$$

Therefore, if P is on the X-axis, then P is elliptic in general, apart from the case where $c^{q+1} + 1 = c_1^2 - sc_2^2 + 1$ is a non-square element in $\mathrm{GF}(q)$ and P is one of the two common points of $\mathcal{C}$ and v, namely $P = P(a, 0)$ where

$$a = a_1 + ia_2 = \frac{-1 \pm \sqrt{1 + c^{q+1}}}{c^q}.$$

Further, in the exceptional case, P is a full point as for any $c_1, c_2 \in \mathrm{GF}(q)$ with $c_1^2 - sc_2^2 + 1 \neq 0$, Equation (2.8) always has $q + 1$ solutions (τ_1, τ_2) with $\tau_1, \tau_2 \in \mathrm{GF}(q)$.

In the latter case, either b_1 or b_2 is not zero. If $b_1 \neq 0$, retrieving τ_1 from (2.9) and putting it in (2.8) gives a quadratic equation in the indeterminate τ_2, namely

$$(s^2b_2^2 - sb_1^2)\tau_2^2 - 2sb_2(a_1c_1 - sa_2c_2 + 1)\tau_2 + \\ (a_1c_1 - sa_2c_2 + 1)^2 + b_1^2(1 + c_1^2 - sc_2^2) = 0, \quad (2.10)$$

whose discriminant is $\Delta_1 = sb_1^2\Delta$ with

$$\Delta = (b_1^2 - sb_2^2)(1 + c_1^2 - sc_2^2) + (a_1c_1 - sa_2c_2 + 1)^2$$

which can also be written by (2.7) as

$$\Delta = -(1 + c_1^2 - sc_2^2)(a_1^2 - sa_2^2 + 1) + (a_1c_1 - sa_2c_2 + 1)^2.$$

For $b_2 \neq 0$, retrieving τ_2 from (2.9) and putting it in (2.8) gives the following quadratic equation in the indeterminate τ_1:

$$(-b_1^2 + b_2^2)\tau_1^2 + 2a_1b_1c_1\tau_1 - \\ a_1^2 - s^2a_2^2c_2^2 - b_2^2c_1^2 + sb_2^2c_2^2 + 2sa_2c_2 - b_2^2 - 1 = 0 \quad (2.11)$$

with discriminant $\Delta_2 = s^3b_2^2\Delta$. Since Δ_1 and Δ_2 are simultaneously zero, or a square, or a non-square in $\mathrm{GF}(q)$, each of the equations (2.10) and (2.11) has 2, 1 or zero solutions in $\mathrm{GF}(q)$, depending upon whether Δ is a square element, zero, or a non-square element of $\mathrm{GF}(q)$, respectively. This leads to the study of the zeroes of the polynomial

$$F(X, Y, Z) = -(1 + c_1^2 - sc_2^2)(X^2 - sY^2 + 1) + (c_1X - sc_2Y + 1)^2 - Z^2. \quad (2.12)$$

Geometrically, $F(X, Y, Z) = 0$ is the equation of a quadric $\mathcal{Q}$ in $\mathrm{AG}(3, q)$. Actually, $\mathcal{Q}$

Table 1: Elliptic, parabolic, hyperbolic and full points.

	$P(a,0)$		$P(a,b),\ b \neq 0$	
	$1+\lVert c\rVert \in \square$	$1+\lVert c\rVert \in \not\square$	$1+\lVert c\rVert \in \square$	$1+\lVert c\rVert \in \not\square$
$N_{\mathcal{E}}$	$q+1$	$q-1$	$\dfrac{-3-9q+q^2+q^3}{2}$	$\dfrac{-3-5q-q^2+q^3}{2}$
$N_{\mathcal{P}}$	0	0	$2q-1$	0
$N_{\mathcal{H}}$	0	0	$\dfrac{(q-1)^2}{2}(q+1)$	$\dfrac{(q+1)^2}{2}(q-1)$
$N_{\mathcal{F}}$	0	2	0	0

is a cone. In fact, the system $F_X = F_Y = F_Z = 0$ has a (unique) solution $(c_1, c_2, 0)$ and hence the point $V(c_1, c_2, 0)$ is the vertex of $\mathscr{Q}$. In particular, the intersection of $\mathscr{Q}$ with the plane $Z = 0$ splits into two lines over $\mathrm{GF}(q)$ or its quadratic extension $\mathrm{GF}(q^2)$, and this occurs according as the infinite points of the conic with equation

$$-(1 + c_1^2 - sc_2^2)(X^2 - sY^2) + (c_1X - sc_2Y)^2 = 0$$

lie in $\mathrm{PG}(2, q)$ or in $\mathrm{PG}(2, q^2) \setminus \mathrm{PG}(2, q)$. By a direct computation, this condition only depends on c^{q+1}, namely whether $1 + c^{q+1}$ is a square or a non-square element of $\mathrm{GF}(q)$. Therefore, $\mathscr{Q}$ contains either $2q-1$ or 1 points in the plane $Z = 0$, and this occurs according as the pair $\{\ell_1, \ell_2\}$ is special or general. Also, in the former case there are exactly $2q - 1$ parabolic points P but in the latter case no point P is parabolic. Therefore, the following result holds.

Theorem 2.3. *Let ℓ_1, ℓ_2 be any two distinct non-tangent lines of the classical unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$ whose common point is off $\mathcal{U}$. The number $N_{\mathcal{E}}, N_{\mathcal{P}}, N_{\mathcal{H}}, N_{\mathcal{S}}$, of elliptic, parabolic, hyperbolic and full points of $\mathcal{U}$ with respect to the pair $\{\ell_1, \ell_2\}$ is given in Table 1.*

We state a corollary of Theorem 2.3 that will be used in Section 3. For $i = 1, 2$ let Λ_i be a subset of $\ell_i \cap \mathcal{U}$ such that $|\Lambda_1| = |\Lambda_2| = \lambda$.

Theorem 2.4. *If*

$$\lambda > \sqrt{\frac{(q+1)(q+3)}{2}} \tag{2.13}$$

there exists a non-degenerate quadrangle $A_1B_1A_2B_2$ with vertices $A_i, B_i \in \Lambda_i$ for $i = 1, 2$ such that its diagonal point $A_1B_2 \cap B_1A_2$ lies in $\mathcal{U}$.

Proof. We prove the existence of a hyperbolic point D in $\mathcal{U}$ such that the projection of Λ_1 from D on ℓ_2 share two points with Λ_2. From Theorem 2.3, we have at least $\frac{1}{2}(q-1)^2(q+1)$ hyperbolic points in $\mathcal{U}$. We omit those hyperbolic points projecting $\overline{\Lambda}_1 = (\ell_1 \cap \mathcal{U}) \setminus \Lambda_1$ to a pointset of ℓ_2 meeting $\ell_2 \cap \mathcal{U}$ nontrivially. The number of such hyperbolic points is

$\bar{\lambda}(q-1)(q+1)$ with $\bar{\lambda} = q+1-\lambda$. Similarly we omit all hyperbolic points projecting $\overline{\Lambda}_2 = (\ell_2 \cap \mathcal{U}) \setminus \Lambda_2$ to a pointset of ℓ_1 meeting $\ell_1 \cap \mathcal{U}$ nontrivially. Therefore, the total number of omitted hyperbolic points is $2\bar{\lambda}(q^2-1)-\bar{\lambda}^2(q-1) = (q-1)\bar{\lambda}(2q+2-\bar{\lambda}(q-1))$. From Theorem 2.3, this number is smaller than the total number of hyperbolic points as far as (2.13) holds. □

To state the other new result on the classical unital a couple of *ad hoc* notation in $\mathrm{AG}(2, q^2)$ will be useful: For a non-vertical line ℓ with equation $Y = Xm + b$, $\bar{\ell}$ denotes the non-vertical line with equation $Y = Xm^q + b$. Given a point $P(a, b)$ outside $\mathcal{U}$, two lines ℓ_1 and ℓ_2 are said to be a *good line-pair* whenever the lines $\bar{\ell_1}$ and $\bar{\ell_2}$ meet in a point of $\mathcal{U}$. Our goal is to show that if $a \neq 0$ then there exist many good pairs.

For $i = 1, 2$, write the equations of ℓ_i in the form $Y = (X-a)m_i + b$. Then $\bar{\ell_i}$ has equation $Y = Xm_i^q - am_i + b$. Hence $\bar{P}(x, y) = \bar{\ell_1} \cap \bar{\ell_2}$ where

$$x = \frac{a(m_1 - m_2)}{m_1^q - m_2^q},$$

and hence

$$y = \frac{a(m_1 - m_2)}{m_1^q - m_2^q} m_1^q - am_1 + b.$$

Note that

$$\|x\| = x^{q+1} = a^{q+1}\left(\frac{1}{(m_1 - m_2)^{q-1}}\right)^{q+1} = \frac{\|a\|}{(m_1 - m_2)^{q^2-1}} = \|a\| \neq 0.$$

The condition for $\overline{P}(x, y)$ to lie in $\mathcal{U}$ is

$$x^{q+1} + y^{q+1} + 1 = a^{q+1} + a^{q+1}\left(\frac{(m_1 - m_2)}{(m_1 - m_2)^q} m_1^q - m_1 + \frac{b}{a}\right)^{q+1} + 1 = 0.$$

Let

$$\xi = -\frac{a^{q+1} + 1}{a^{q+1}} \in \mathrm{GF}(q).$$

Then the last equation reads

$$\left(\frac{(m_1 - m_2)}{(m_1 - m_2)^q} m_1^q - m_1 + \frac{b}{a}\right)^{q+1} = \xi. \tag{2.14}$$

Henceforth we assume that

$$\|a\| \neq -1.$$

With

$$m_1 = \alpha + i\beta, \quad m_2 = \gamma + i\delta, \quad \frac{b}{a} = u + iv,$$

(2.14) reads

$$\left(\frac{(\alpha-\gamma) + i(\beta-\delta)}{(\alpha-\gamma) - i(\beta-\delta)}(\alpha - i\beta) - (\alpha + i\beta) + u + iv\right)^{q+1} = \xi,$$

whence

$$(u\alpha - u\gamma - sv\beta + sv\delta)^2 - s(2\beta\gamma - 2\alpha\delta - u(\beta-\delta) + v(\alpha-\gamma))^2 \\ - \xi((\alpha-\gamma)^2 - s(\beta-\delta)^2) = 0,$$

that is,

$$(u(\alpha-\gamma) - sv(\beta-\delta))^2 - s(2\beta\gamma - 2\alpha\delta - u(\beta-\delta) + v(\alpha-\gamma))^2 \\ - \xi((\alpha-\gamma)^2 - s(\beta-\delta)^2) = 0. \quad (2.15)$$

With

$$\gamma = \alpha - \overline{\gamma}, \qquad \delta = \beta - \overline{\delta},$$

Equation (2.15) becomes

$$(u\overline{\gamma} - sv\overline{\delta})^2 - s(-2\beta\overline{\gamma} - 2\alpha\overline{\delta} - u\overline{\delta} + v\overline{\gamma})^2 - \xi(\overline{\gamma}^2 - s\overline{\delta}^2) = 0, \quad (2.16)$$

which can be viewed as a quadratic form in $\overline{\gamma}$ and $\overline{\delta}$:

$$F(\overline{\gamma}, \overline{\delta}) = (u^2 - v^2 s + 4v\beta s - 4\beta^2 s - \xi)\,\overline{\gamma}^2 + 2(-2u\beta s + 2v\alpha s - 4\alpha\beta s)\,\overline{\gamma}\overline{\delta} \\ + (-u^2 s - 4u\alpha s + v^2 s^2 - 4\alpha^2 s + s\xi)\,\overline{\delta}^2 \quad (2.17)$$

with discriminant

$$\Delta = -u^4 s + 2u^2v^2s^2 + 2u^2 s\xi - v^4 s^3 - 2v^2s^2\xi - s\xi^2 \\ + (-4u^3 s + 4uv^2s^2 + 4us\xi)\,\alpha + (-4u^2vs^2 + 4v^3s^3 + 4vs^2\xi)\,\beta \\ - 8uvs^2\,\alpha\beta + (-4u^2 s + 4s\xi)\,\alpha^2 + (-4v^2s^3 - 4s^2\xi)\,\beta^2.$$

Note that $\overline{P}(x, y) \in \mathcal{U}$ if and only if $\Delta = \lambda^2$ for some $\lambda \in \mathrm{GF}(q)$. This leads us to consider the quadric $\mathcal{Q}$ in $\mathrm{AG}(3, q)$ of equation

$$a_{00} + a_{01}X + a_{02}Y + a_{12}XY + a_{11}X^2 + a_{22}Y^2 - Z^2 = 0,$$

where

$$\begin{aligned} a_{00} &= -u^4 s + 2u^2v^2s^2 + 2u^2s\xi - v^4s^3 - 2v^2s^2\xi - s\xi^2, \\ a_{01} &= -4u^3 s + 4uv^2s^2 + 4us\xi, \\ a_{02} &= -4u^2vs^2 + 4v^3s^3 + 4vs^2\xi, \\ a_{12} &= -8uvs^2, \\ a_{11} &= -4u^2 s + 4s\xi, \\ a_{22} &= -4v^2s^3 - 4s^2\xi. \end{aligned}$$

The above coefficients are related by the following equations:

(i) $a_{00} - \frac{1}{2}(\frac{1}{2}a_{01}u - \frac{1}{2}a_{02}v) = s\xi(u^2 - sv^2 - \xi)$;

(ii) $\frac{1}{2}a_{01} - \frac{1}{2}(a_{11}u - \frac{1}{2}a_{12}v) = 0$;

(iii) $\frac{1}{2}a_{02} - \frac{1}{2}(\frac{1}{2}a_{12}u - a_{22}v) = 0$.

Therefore, the determinant D of the 4×4 matrix associated with $\mathcal{Q}$ is equal to $-s\xi(u^2 - sv^2 - \xi)$ multiplied by the determinant of the cofactor of a_{00}. The latter determinant $a_{11}a_{22} - \frac{1}{4}a_{12}^2$ is equal to

$$D_0 = s^3\xi(u^2 - sv^2 - \xi) = s^3(a^{q+1} + b^{q+1} + 1)(a^{q+1} + 1). \tag{2.18}$$

It turns out that

$$D = -(s^2\xi(u^2 - sv^2 - \xi))^2.$$

Observe that $\xi = 0$ if and only if $a^{q+1} = -1$, while

$$u^2 - sv^2 - \xi = \frac{b^{q+1}}{a^{q+1}} + \frac{a^{q+1}+1}{a^{q+1}} = \frac{a^{q+1} + b^{q+1} + 1}{a^{q+1}}$$

vanishes only for $P(a,b) \in \mathcal{U}$. Therefore, $\mathcal{Q}$ is non-degenerate. More precisely, the quadric $\mathcal{Q}$ is either elliptic or hyperbolic according as $q \equiv -1 \pmod 4$ or $q \equiv 1 \pmod 4$. The plane at infinity cuts out from $\mathcal{Q}$ a conic $\mathcal{C}$ with homogeneous equation $a_{11}X^2 + a_{12}XY + a_{22}Y^2 - Z^2 = 0$. Observe that $\mathcal{C}$ is non-degenerate by $D_0 \neq 0$. Thus, the number of points of $\mathcal{Q}$ in $AG(3,q)$ is $q^2 \pm q$ with $q \equiv \pm 1 \pmod 4$. Furthermore, the point at infinity Z_∞ on the Z-axis does not lie on $\mathcal{Q}$, and it is an external point or an internal point to $\mathcal{C}$ according as $-D_0$ is a non-zero square or a non-square in $\mathrm{GF}(q)$. Therefore, the number of tangents to $\mathcal{Q}$ through Z_∞ in $AG(3,q)$ is equal to $q-1$ or $q+1$ according as $-D_0$ is a (non-zero) square or a non-square in $\mathrm{GF}(q)$. From the above discussion, the numbers N_s and N_t of secants and tangents to $\mathcal{Q}$ through Z_∞ are those given in the following lemma:

Lemma 2.5. *For $q \equiv -1 \pmod 4$, either $N_t = q+1$, $N_s = \frac{1}{2}(q-1)^2$, or $N_t = q-1$, $N_s = \frac{1}{2}(q^2-2q-1)$, according as D_0 is a (non-zero) square or a non-square in* $\mathrm{GF}(q)$. *For $q \equiv 1 \pmod 4$, either $N_t = q-1$, $N_s = \frac{1}{2}(q^2+1)$, or $N_t = q+1$, $N_s = \frac{1}{2}(q^2-1)$, according as D_0 is a (non-zero) square or a non-square in* $\mathrm{GF}(q)$.

Going back to the discriminant Δ, we see that Δ vanishes for $N_s + N_t$ ordered pairs (α, β), that is, $N_s + N_t$ is the number of lines ℓ_1 through $P(a,b)$ for which there exists a line ℓ_2 such that (ℓ_1, ℓ_2) is a good line-pair. For each ℓ_1 counted in N_t (resp. N_s), we have $q-1$ (resp. $2(q-1)$) such lines ℓ_2, since if (2.17) has a non-trivial solution $(\bar{\gamma}, \bar{\delta})$ in $\mathrm{GF}(q) \times \mathrm{GF}(q)$ then it has exactly $q-1$ solutions, the multiples of $(\bar{\gamma}, \bar{\delta})$ by the non-zero elements of $\mathrm{GF}(q)$.

If we do not count the $q+1$ tangents to $\mathcal{U}$ through $P(a,b)$, each of the lines through $P(a,b)$ counted in N_s is in at least $2(q-1) - (q+1) = q-3$ good line-pairs. Therefore Lemma 2.5 has the following corollary.

Theorem 2.6. *Let $P(a,b)$ be a point of* $\mathrm{AG}(2,q^2)$ *outside $\mathcal{U}$. If $a \neq 0$, $\|a\| \neq -1$ and $q > 3$, then there exist at least two non-tangent lines ℓ_1, ℓ_2 of $\mathcal{U}$ through P, such that the non-tangent lines $\bar{\ell}_1$ and $\bar{\ell}_2$ meet in a point of $\mathcal{U}$. Further, if $q > 5$ then ℓ_1 and ℓ_2 may be chosen among the lines through $P(a,b)$ other than the horizontal lines and those passing through the origin.*

3 Unitals in Moulton planes

Let T be a non-empty subset of the multiplicative group of $\mathrm{GF}(q)$. The (affine) *Moulton plane* $\mathfrak{M}_T(q^2)$ which is considered in our paper is the affine plane coordinatized by the left quasifield $\mathrm{GF}(q^2)(+,\circ)$ where

$$x \circ y = \begin{cases} xy & \text{if } \|x\| \notin T, \\ xy^q & \text{if } \|x\| \in T, \end{cases}$$

with $\|x\| = x^{q+1}$ being the norm of $x \in \mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$. Geometrically, $\mathfrak{M}_T(q^2)$ is constructed on $\mathrm{AG}(2, q^2)$ by replacing the non-vertical lines with the graphs of the functions

$$Y = X \circ m + b. \tag{3.1}$$

This also shows that to the non-vertical line ℓ of equation $Y = Xm + b$ there corresponds the line of equation $\tilde{\ell}$ of equation $Y = X \circ m + b$ in $\mathfrak{M}_T(q^2)$, and viceversa. It is useful to look at the partition of the points outside the Y-axis into $q-1$ subsets $\mathcal{S}_i$, called stripes, where $P(x, y) \in \mathcal{S}_i$ if and only if $\|x\| = \omega^i$ with ω a fixed primitive element of $\mathrm{GF}(q)$. Such stripes were already defined in Section 2; here we just abbreviate the subscript ω^i by i. In fact, moving to $\mathfrak{M}_T(q^2)$ the point-line incidences $P \in \ell$ in $\mathrm{AG}(2, q^2)$ do not alter as long as $P \in \mathcal{S}_i$ with $\omega^i \notin T$. The projective Moulton plane is the projective closure of $\mathfrak{M}_T(q^2)$ and it has the same points at infinity as $\mathrm{AG}(2, q^2)$. For a similar description of Moulton planes see also [3, 4, 26].

The dual of the Moulton plane is the André plane $\mathfrak{A}_T(q^2)$ coordinatized by the right quasifield $\mathrm{GF}(q^2)(+,*)$ where

$$x * y = \begin{cases} xy & \text{if } \|x\| \notin T, \\ x^q y & \text{if } \|x\| \in T. \end{cases}$$

In this duality, the correspondence occurs between the point (u, v) of $\mathfrak{M}_T(q^2)$ and the line of equation $Y = u * X - v$, as well as between the line of equation $Y = X \circ m + b$ and the point $(m, -b)$ of $\mathfrak{A}_T(q^2)$. The correspondence between points at infinity and lines through Y_∞, and viceversa, is the same as the canonical duality between $\mathrm{PG}(2, q^2)$ and its dual plane $\mathrm{PG}^*(2, q^2)$. If T consists of just one element, then the arising André planes are pairwise isomorphic and they are also known as Hall planes.

Let $\mathcal{U}$ be the classical unital in $\mathrm{PG}(2, q^2)$ given in its canonical form (2.1). We prove that $\mathcal{U}$ is an inherited unital in the Moulton plane, that is, the point-set of $\mathcal{U}$ is a unital in $\mathfrak{M}_T(q^2)$ as well.

Theorem 3.1. *Let $\mathcal{U}$ be the classical unital in* $\mathrm{PG}(2, q^2)$ *given in its canonical form* (2.1). *Then, for any T, $\mathcal{U}$ is a unital in the projective Moulton plane $\mathfrak{M}_T(q^2)$ as well.*

Proof. In the very special case $T = \{-1\}$, the proof is straightforward. It is enough to show that if a non-vertical line ℓ of equation $Y = Xm + b$ meets $\mathcal{U}$ in a point $P(x, y)$ with $\|x\| = -1$ then $y = 0$ and $x = -b/m$ with $(-b/m)^{q+1} = 1$. In fact, the corresponding line $\tilde{\ell}$ in $\mathfrak{M}_T(q^2)$ has the same property: if $P(x, y) \in \tilde{\ell} \cap \mathcal{U}$ then $y = 0$ and $x = (-b/m^q)^{q+1}$. Since $(-b/m)^{q+1} = (-b/m^q)^{q+1}$, the assertion follows for $T = \{-1\}$.

In the general case, it suffices to exhibit a bijective map from $\ell \cap \mathcal{U}$ to $\tilde{\ell} \cap \mathcal{U}$ for every line ℓ of $\mathrm{AG}(2, q^2)$. We may limit ourselves to non-vertical lines with non zero slopes. Let

$Y = Xm + b$ be the equation of such a line ℓ and take any point $P(x, y)$ lying in $\ell \cap \mathcal{U}$. Then $m \neq 0$ and $x = (y-b)m^{-1}$. Define the map $\varphi\colon \ell \mapsto \tilde{\ell}$ by

$$\varphi(P) = \begin{cases} \overline{P}((y-b)m^{-1}, y) & \text{for } \|x\| \notin T, \\ \overline{P}((y-b)m^{-q}, y) & \text{for } \|x\| \in T. \end{cases}$$

Obviously, $\varphi(P) = P$ whenever $\|x\| \notin T$.

Since φ is bijective, it suffices to show that $P \in \mathcal{U}$ yields $\varphi(P) \in \mathcal{U}$, and the converse also holds. $P(x, y) = ((y-b)m^{-1}, y) \in \mathcal{U}$ if and only if

$$((y-b)m^{-1})^{q+1} + y^{q+1} - 1 = (y-b)^{q+1}(m^{-1})^{q+1} + y^{q+1} - 1 = 0.$$

By $(m^q)^{q+1} = m^{q+1}$, the latter equation is equivalent to

$$((y-b)^{q+1}(m^{-q})^{q+1} + y^{q+1} - 1 = ((y-b)m^{-q})^{q+1} + y^{q+1} - 1 = 0,$$

whence the claim follows. □

Theorem 3.1 and its proof also show that if ℓ is a tangent to $\mathcal{U}$ in $\mathrm{AG}(2, q^2)$ then the corresponding line $\tilde{\ell}$ is a tangent to $\mathcal{U}$ in the projective Moulton plane, and the converse also holds. In particular, the tangent to $\mathcal{U}$ at a point outside the X-axis is the line ℓ of equation $Y = X(-cd^{-1})^q - d^{-q}$ with tangency point $P(c, d)$. Therefore, the corresponding line $\tilde{\ell}$ of equation $Y = X \circ (-cd^{-1})^q - d^{-q}$ is a tangent to $\mathcal{U}$ at the point $\varphi(P) = \bar{P}(\bar{c}, d)$ with $\bar{c} = c$ or $\bar{c} = c(cd^{-1})^{q-1}$ according as $\|c\| \notin T$ or $\|c\| \in T$. Since $\|\bar{c}\| = \|c\|$, the tangency points of ℓ and $\tilde{\ell}$ lie in the same stripe. The tangents of $\mathcal{U}$ with tangency point at infinity contain the origin and each of them has equation $Y = Xm$ with $m^{q+1} + 1 = 0$. By the proof of Theorem 3.1, the corresponding lines $Y = X \circ m$ are the tangents of $\mathcal{U}$ in the projective Moulton plane.

Now look at dual plane of the projective Moulton plane $\mathfrak{M}_T(q^2)$ which is the projective André plane $\mathfrak{A}_T(q^2)$. In this duality, the tangent line $\tilde{\ell}$ of $\mathcal{U}$ with equation $Y = X \circ (-cd^{-1})^q - d^{-q}$ corresponds to the point $P^*(u^*, v^*) \in \mathfrak{A}_T(q^2)$ where $u^* = -(-cd^{-1})^q$ and $v^* = d^{-q}$. Since $((-cd^{-1})^q)^{q+1} + (d^{-q})^{q+1} + 1 = 0$, we have $u^{*q+1} + v^{*q+1} + 1 = 0$. Similarly, the tangent line $\tilde{\ell}$ of $\mathcal{U}$ with equation $Y = X \circ m$, $m^{q+1} + 1 = 0$, corresponds to the point $P^*(u^*, v^*) \in \mathfrak{A}_T(q^2)$ where $u^* = u$ and $v^* = 0$. Therefore $u^{*q+1} + v^{*q+1} + 1 = 0$. In terms of $\mathrm{PG}^*(2, q^2)$, the Desarguesian plane which gives rise to the projective André plane $\mathfrak{A}_T(q^2)$, the points $P^*(u^*, v^*)$ lie on the classical unital $\mathcal{U}^*$ given in its canonical form. This shows that $\mathcal{U}^*$ can be viewed as an inherited unital in the projective André plane $\mathfrak{A}_T(q^2)$.

Remark 3.2. If $T = \{-1\}$ then the unique stripe where incidence are altered meets $\mathcal{U}$ in $q+1$ points lying on the X-axis. The unital $\mathcal{U}^*$ in the Hall plane is the Grüning unital [16] while for $T = \{i\}$ with $\omega^i \neq -1$, $\mathcal{U}^*$ in the Hall plane is the Barwick unital [7].

A O'Nan configuration of a unital consists of four blocks b_1, b_2, b_3 and b_4 intersecting in six points P_1, P_2, P_3, P_4, P_5 and P_6 as in Figure 2. As mentioned in the introduction, the Hermitian unital contains no O'Nan configuration. This fundamental result due to O'Nan dates back to 1972, see [22] and [9, Section 4.2].

Lemma 3.3. *If $T = \{-1\}$ then the unital $\mathcal{U}$ of $\mathfrak{M}_T(q^2)$ is non-classical.*

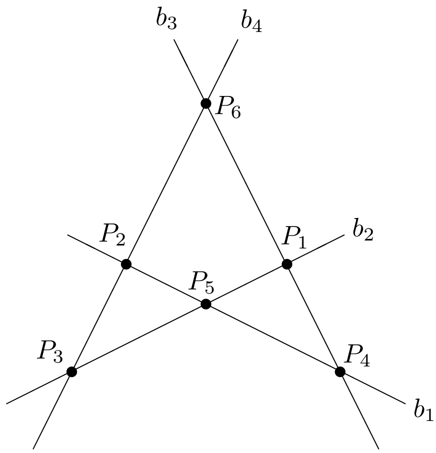

Figure 2: O'Nan configuration of four blocks and six points.

Proof. We show that the unital $\mathcal{U}$ in $\mathfrak{M}_T(q^2)$ with $T = \{-1\}$ contains a O'Nan configuration. Take $\alpha \in \mathrm{GF}(q^2)$ such that $\|\alpha\| = -1$. The line ℓ_1 of equation $Y = X - \alpha$ meets $\mathcal{U}$ in $Q(\alpha, 0)$ and q more points. Take $m \in \mathrm{GF}(q^2)$ such that $m^{q-1} = -1$. The line ℓ_2 of equation $Y = Xm + \alpha m$ meets $\mathcal{U}$ in $R\,(-\alpha, 0)$ and q more points. Further, the common point of ℓ_1 and ℓ_2 is

$$S = \left(\frac{-\alpha(m+1)}{(m-1)}, \frac{-2\alpha m}{(m-1)}\right).$$

Since

$$\left\|\frac{-\alpha(m+1)}{(m-1)}\right\| = -\alpha^{q+1}\frac{(m+1)^{q+1}}{(m-1)^{q+1}} = \\ -\frac{m^{q+1}+m^q+m+1}{m^{q+1}-m^q-m+1} = -\frac{-m^2-m+m+1}{-m^2+m-m+1} = -1,$$

the point S is outside $\mathcal{U}$. Further, in the Moulton plane $\mathfrak{M}_T(q^2)$ with $T = \{-1\}$, the corresponding lines $\tilde{\ell}_1$ and $\tilde{\ell}_2$ meet in $Q(\alpha, 0)$ which is a point of $\mathcal{U}$.

To show that $\mathcal{U}$ is not a classical unital in our Moulton plane $\mathfrak{M}_T(q^2)$, it suffices to exhibit a O'Nan configuration $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ lying in $\mathcal{U}$. The idea is to start off with $P_0 = Q(\alpha, 0)$, and to find four more affine points $P_1, P_2 \in \tilde{\ell}_1$ and $P_3, P_4 \in \tilde{\ell}_2$ each lying in $\mathcal{U}$, so that $\mathcal{U}$ also contains one of the two diagonal points P_5 of the quadrangle $P_1P_2P_3P_4$ that are different from P_0. First we show that $P_1 \in \ell_1$. Let $P_1 = P_1(x_1, y_1)$. Then, $\|x_1\| \neq -1$. In fact, otherwise, we would have $y_1^{q+1} = 0$ and hence $y_1 = 0$, contradicting $P_0 \neq P_1$. Similarly, $P_2 \in \ell_1$ and $P_3, P_4 \in \ell_2$. Now we use a counting argument in $\mathrm{PG}(2, q^2)$ to show that the quadrangle $P_1P_2P_3P_4$ can be chosen in such a way that $P_5 \in \mathcal{U}$. Since $S = \ell_1 \cap \ell_2$ is outside $\mathcal{U}$, the lines of $\mathcal{U}$ joining a point of $\bar{\ell}_1$ with a point of $\bar{\ell}_2$ cover $(q+1)^2(q-1)$ points of $\mathcal{U}$ other than those lying in $\bar{\ell}_1 \cup \bar{\ell}_2$. From $(q+1)^2(q-1) > q^3 + 1 - 2q$, there exists a quadrangle $P_1P_2P_3P_4$ in $\mathrm{PG}(2, q^2)$ such that

$$P_1, P_2 \in \ell_1 \cap \mathcal{U},\; P_3, P_4 \in \ell_2 \cap \mathcal{U},\; P_5 = P_1P_3 \cap P_2P_4 \in \mathcal{U}.$$

Since $(q+1)^2(q-1) > q^3+1-2q+(q+1)$ we may also assume that either $P_5 \in \ell_\infty \cap \mathcal{U}$, or $P_5 = (x_5, y_5)$ with $\|x_5\| \neq -1$. In particular, P_5 is not on the X-axis.

If $P_1, P_2 \neq Q$ and $P_3, P_4 \neq R$ then P_5 remains a diagonal point of the quadrangle $P_1P_2P_3P_4$ in $\mathfrak{M}_T(q^2)$, and we are done.

Otherwise, take the cyclic subgroup G of $PGU(3, q)$ of order $q + 1$ fixing the point S and preserving each line through S. Since $|G| \geq 4$, G contains an element g such that $Q \notin \{g(P_1), g(P_2)\}$ and $R \notin \{g(P_3), g(P_4)\}$. Then g takes the quadrangle $P_1P_2P_3P_4$ to another one, whose vertices are different from both Q and R. The image $g(P_5)$ is on the line r through S and P_5. Since $r \cap \mathcal{U}$ has at most one point on the X-axis, there exists at most one $g \in G$ such that $g(P_5)$ lies on the X-axis. Therefore, if $|G| \geq 5$, some $g \in G$ also takes P_5 either to a point of infinity or a point (x'_5, y'_5) with $\|x'_5\| \neq -1$. In the Moulton plane $\mathfrak{M}_T(q^2)$, the O'Nan configuration $P_0, g(P_1), g(P_2), g(P_3), g(P_4), g(P_5)$ arising from the quadrangle $g(P_1)g(P_2)g(P_3)g(P_4)$ lying in $\mathcal{U}$ has also two diagonal points, namely P_0 and $g(P_5)$, belonging to $\mathcal{U}$. □

Remark 3.4. Lemma 3.3 can also be obtained from Grüning's work. In fact, if $T = \{-1\}$ then $\mathcal{U}$ is isomorphic to its dual, see [16, Theorem 4.2], and the dual of $\mathcal{U}$ contains some O'Nan configuration, see [16, Lemma 5.4c].

We conjecture that Lemma 3.3 holds true for any T. Theorem 3.5 proves this as long as T is small enough. On the other end, Theorem 3.6 provides Moulton planes with large T for which the conjecture holds.

Theorem 3.5. *If $q > 5$ and*

$$|T| < \tfrac{1}{2}\left((q+1) - \sqrt{\tfrac{1}{2}(q+1)(q+3)}\right), \tag{3.2}$$

then $\mathcal{U}$ in the Moulton plane $\mathfrak{M}_T(q^2)$ is a non-classical unital.

Proof. As in the proof of Lemma 3.3, we show the existence of a O'Nan-configuration $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ lying in $\mathcal{U}$. For a point $P(a, b) \in \mathrm{AG}(2, q^2)$ with $a \neq 0$ and $\|a\| \in T \setminus \{-1\}$, Theorem 2.6 ensures the existence of two non-vertical lines ℓ_1 and ℓ_2 through P such that

(i) neither ℓ_1 nor ℓ_2 is horizontal or passes through the origin,

(ii) $P_0 = \bar{\ell}_1 \cap \bar{\ell}_2 \in \mathcal{U}$.

From Lemma 2.1, there exist at least $q + 1 - 2|T|$ points $P(x, y)$ lying on $\ell_1 \cap \mathcal{U}$ such that $\|x\| \notin T$, and the same holds for $\ell_2 \cap \mathcal{U}$. Therefore, Theorem 2.4 applies with $\lambda = q+1-2|T|$ showing that if (3.2) is assumed, then the unital $\mathcal{U}$ in $\mathfrak{M}_T(q^2)$ contains a O'Nan configuration. □

Theorem 3.6. *If $q > 5$, then there exists a T with $|T| > q-4$ such that $\mathcal{U}$ is a non-classical unital in $\mathfrak{M}_T(q^2)$.*

Proof. From the proof of Theorem 3.5, some Moulton plane $\mathfrak{M}_T(q^2)$ contains O'Nan configurations lying in $\mathcal{U}$. If $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ one of them, add each non-zero element $s \in \mathrm{GF}(q)$ to T which satisfies the condition $s \neq \|x_i\|$ for $P_i = P_i(x_i, y_i)$ with $1 \leq i \leq 5$. Then T expands and its size becomes at least $q-4$. In the resulting Moulton plane $\mathfrak{M}_T(q^2)$, the above hexagon $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ is still a O'Nan configuration lying in the unital $\mathcal{U}$. □

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Trilateral matroids induced by n_3-configurations

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Abstract

We define a new class of a rank-3 matroid called a trilateral matroid. When defined, the ground set of such a matroid consists of the points of an n_3-configuration, and its bases are the point triples corresponding to non-trilaterals within the configuration. We characterize which n_3-configurations induce trilateral matroids and provide several examples.

Keywords: Configurations, trilaterals, matroids.

Math. Subj. Class.: 05B30, 51E30, 05C38, 05B35

1 Introduction

A *(combinatorial) n_3-configuration* $\mathcal{C}$ is an incidence structure consisting of n distinct points and n distinct blocks for which each point is incident with three blocks, each block is incident with three points, and any two points are incident with at most one common block. If $\mathcal{C}$ may be depicted in the real projective plane using points and having (straight) lines as its blocks, then it is said to be *geometric*. As observed in [6] (pg. 17–18), it is evident that every geometric n_3-configuration is combinatorial, but the converse of this statement does not hold.

A *trilateral* in a configuration is a cyclically ordered set $\{p_0, b_0, p_1, b_1, p_2, b_2\}$ of pairwise distinct points p_i and pairwise distinct blocks b_i such that p_i is incident with b_{i-1} and b_i for each $i \in \mathbb{Z}_3$ [2]. We may without ambiguity shorten this notation by listing only the points of the trilateral as $\{p_0, p_1, p_2\}$, or more simply as $p_0p_1p_2$. A configuration is *trilateral-free* if no trilateral exists within the configuration. Unless stated otherwise, the n_3-configurations we shall examine are point-line configurations, so that the blocks are lines. But we shall investigate an example of a point-plane configuration in Section 3.

Following the terminology of [7], we define a *matroid* M to be an ordered pair $(E, \mathcal{B})$ consisting of a finite ground set E and a nonempty collection $\mathcal{B}$ of subsets of E called *bases* which satisfy the *basis exchange property:*

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Definition 1.1. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there exists $y \in B_2 - B_1$ such that $B_1 - x \cup y \in \mathcal{B}$.

It is a consequence of this definition that any two bases of M share the same cardinality; this common cardinality is called the *rank* of the matroid. See [7], pg. 16–18 for the details.

It is a standard result that any n_3-configuration $\mathcal{C}$ defines a rank-3 *linear matroid*, or *vector matroid*, $M(\mathcal{C}) = (E, \mathcal{B})$ whose ground set E consists of the points $\{p_1, p_2, \ldots, p_n\}$ of $\mathcal{C}$ and whose set of bases $\mathcal{B}$ consists of the point triples $\{p_a, p_b, p_c\}$ which are *not* collinear in $\mathcal{C}$. Hence the cardinality of $\mathcal{B}$ is $\binom{n}{3} - n$ for the linear matroid $M(\mathcal{C})$ induced by $\mathcal{C}$.

In this work we pose the following associated question: under what conditions do the *trilaterals* of an n_3-configuration $\mathcal{C}$ induce a rank-3 matroid $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$ whose ground set E again consists of the points of $\mathcal{C}$, but now whose bases are the point triples corresponding to non-trilaterals? This question, to our knowledge, has not previously been considered in the literature on configurations and matroids.

Definition 1.2. A *trilateral matroid* $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$, when it exists, is a matroid defined on the set E of points of an n_3-configuration $\mathcal{C}$ whose set of bases $\mathcal{B}$ consists of all of the non-trilaterals of $\mathcal{C}$. When $M_{tri}(\mathcal{C})$ exists, we say that $\mathcal{C}$ *induces* $M_{tri}(\mathcal{C})$.

We shall see that, in contrast to the linear matroid setting, seldom is it the case that an n_3-configuration $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C})$. But thankfully such matroids do exist; for instance, any trilateral-free configuration induces a trilateral matroid, since in this setting *every* point triple forms a base of the matroid. In other words, if $\mathcal{C}$ is a trilateral-free n_3-configuration, then $M_{tri}(\mathcal{C})$ exists and furthermore $M_{tri}(\mathcal{C}) \cong U_{3,n}$, the uniform matroid of rank 3 on n points. Thus our initial motivation to define this new class of matroids stems from the desire to enlarge the class of trilateral-free configurations.

For purposes of instruction, we regard an example of a 15_3-configuration which induces a trilateral matroid on its points. Here is a combinatorial description of this configuration.

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}	l_{15}
1	1	1	2	2	3	3	4	5	5	7	7	9	10	13
2	4	6	4	6	8	11	6	8	9	8	9	11	11	14
3	5	7	14	10	12	13	12	10	13	14	15	12	15	15

This configuration has 10 trilaterals:

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
1	1	1	2	3	7	9	9	9	11
2	2	4	4	11	14	11	11	13	13
4	6	6	6	12	15	13	15	15	15

In Figure 1 we see both a diagram of this 15_3-configuration and a geometric representation of its trilateral matroid. In the geometric representation, each trilateral (that is, each non-basis element) is collinear.

Note that the configuration contains two *complete quadrangles*. The first complete quadrangle is determined by the point set $\{1, 2, 4, 6\}$, and the second by $\{9, 11, 13, 15\}$. This means, for example, that no three points in $\{1, 2, 4, 6\}$ are collinear, and each pair of points is incident to a line of the configuration. So all four point triples present within $\{1, 2, 4, 6\}$ give trilaterals, and hence are not bases of the matroid. Thus every 2-element subset of $\{1, 2, 4, 6\}$ is independent, but no 3-element subset of $\{1, 2, 4, 6\}$ is. Therefore

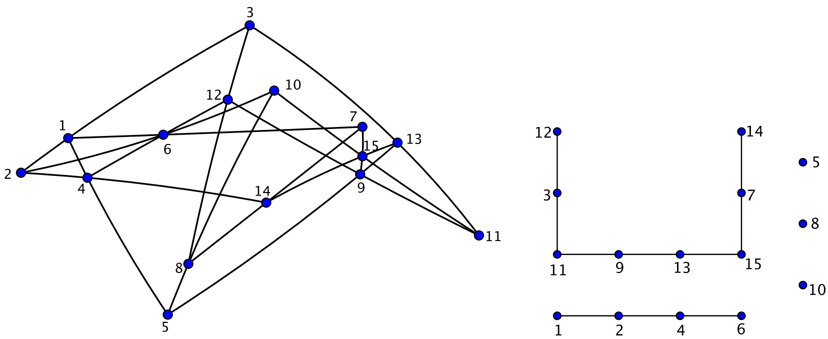


Figure 1: A 15_3-configuration with 10 trilaterals, and a geometric representation of the matroid induced by these trilaterals.

the four-point line that represents the level of dependency of $\{1, 2, 4, 6\}$ in the geometric representation is appropriate. This minor is isomorphic to $U_{2,4}$, which is the unique excluded minor for the class of binary matroids ([7], pg. 501).

We must note that there is a fundamental difference between trilateral matroids and linear matroids. Admittedly a finite set of points and lines in the plane gives a (linear) matroid if and only if any pair of lines meet in at most one point. For suppose there exist two points a and b which are met by two lines, so that points a, b, c are collinear, points a, b, d are collinear, but a, b, c, d are not all on one line. Pick a new point e so that c, d, and e are not collinear, and so that a, b, and e are not collinear. Let $B_1 = abe$ and $B_2 = cde \in \mathcal{B}$; both are bases of the linear matroid. We have $B_1 - B_2 = ab$ and $B_2 - B_1 = cd$. Let $x = e \in B_1 - B_2$, so $B_1 - x = cd$. But if $y \in B_2 - B_1 = ab$, then $B_1 - x \cup y$ equals either abc or abd, neither of which is a base.

Hence a linear matroid cannot have two points common to more than one line. But a trilateral matroid can; if both abc and abd are trilaterals, then the configuration has a chance to induce a trilateral matroid if trilaterals acd and bcd are also present, meaning that points c and d are incident to a particular line of the configuration. In other words, points $\{a, b, c, d\}$ form a *complete quadrangle* within the configuration. We shall explore this necessity further in Theorem 1.7.

Any point of an n_3-configuration is incident to three lines; these three lines are then incident to six points which are distinct from the original point and from each other. Consequently, the maximum number of trilaterals incident to a given point is $\binom{6}{3} - 3 = 12$, since lines are not trilaterals. This maximum is achieved by *every* point of the Fano 7_3-configuration (the smallest n_3-configuration) given in Figure 2.

Proposition 1.3. *Suppose an n_3-configuration $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$. Then each point of the configuration is incident to at most six trilaterals.*

Proof. Let a be a point in $\mathcal{C}$, and let abc, ade, and afg be the lines in $\mathcal{C}$ incident to a. Each of these lines belongs to $\mathcal{B}$, and hence there are at most $\binom{6}{3} - 3 = 12$ trilaterals incident to a, namely

$$abd, abe, abf, abg, acd, ace, acf, acg, adf, adg, aef, \text{ and } aeg.$$

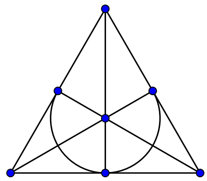


Figure 2: The Fano 7_3-configuration.

Since $B_1 = abc$ and $B_2 = ade$ are bases of $M_{tri}(\mathcal{C})$, the basis exchange property applies to them. This means that if $x \in B_1 - B_2 = bc$, there must exist some $y \in B_2 - B_1 = de$ such that $B_1 - x \cup y \in \mathcal{B}$. Consequently, letting $x = b$, we find at least one of acd and ace must be a base, hence not a trilateral. Likewise, letting $x = c$, it follows that at least one of abd and abe is not a trilateral.

Applying a similar analysis to the pair of bases $B_1 = abc, B_2 = afg$, we find that at least one of acf and acg is not a trilateral, and at least one of abf and abg is not a trilateral. Finally, given $B_1 = ade, B_2 = afg$, we find that at least one of aef and aeg is not a trilateral, and at least one of adf and adg is not a trilateral. Hence at least six of the 12 possible non-collinear triples are not trilaterals, so at most six are trilaterals. $\square$

Corollary 1.4. *Suppose an n_3-configuration $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C}) = (E, \mathcal{B})$. Then $\mathcal{C}$ contains at most $2n$ trilaterals.*

Although Corollary 1.4 admittedly serves as a crude necessary condition for an n_3-configuration to induce a trilateral matroid, it does permit us to eliminate some of the smallest n_3-configurations from consideration, such as the Fano 7_3-configuration (which contains 28 trilaterals) and also the Möbius-Kantor 8_3-configuration (which contains 24 trilaterals). Additionally, two of the three non-isomorphic 9_3-configurations may be dismissed from consideration by this criterion, although the Pappus 9_3-configuration, which contains 18 trilaterals, is still a possibility. We shall soon see, though, that the Pappus configuration does not induce a trilateral matroid on its points.

The upper bound indicated by Proposition 1.3 is sharp, for it turns out that the Desargues 10_3-configuration induces a trilateral matroid. Each of the points of the Desargues configuration is incident to six trilaterals.

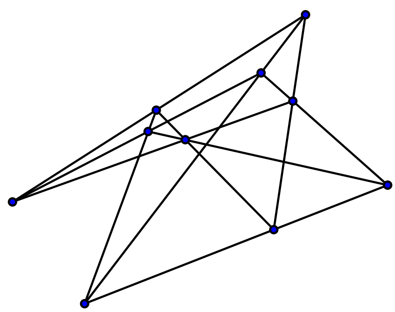


Figure 3: The Desargues 10_3-configuration.

We now establish our main result. This will require the introduction of two types of geometric obstructions (near-complete quadrangles and near-pencils) that, when present within an n_3-configuration $\mathcal{C}$, individually preclude the existence of $M_{tri}(\mathcal{C})$.

Definition 1.5. A *near-complete quadrangle* $[ab:cd]$ consists of four points a, b, c, and d of the configuration, no three of which are collinear, for which five of the six possible lines connecting each pair of points exist within the configuration, except for the pair cd.

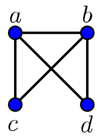


Figure 4: Near-complete quadrangle $[ab:cd]$.

For example, we note the presence of the near-complete quadrangle $[ab : cd]$ in the Pappus configuration in Figure 5.

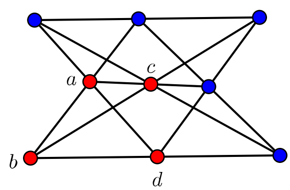


Figure 5: The Pappus 9_3-configuration.

It is important to note that, by our conventions, a complete quadrangle determined by points $\{a, b, c, d\}$ does *not* contain a near-complete quadrangle $[ab: cd]$, since there exists a line in the configuration incident to both c and d. So the Desargues configuration, for example, possesses five complete quadrangles but no near-complete quadrangle.

As we shall witness in greater detail, n_3-configurations which induce trilateral matroids may contain complete quadrangles. Indeed, in a linear matroid, given any two points, at most one line passes between them. But, two trilaterals (call them abc and abd) may share the points a, b provided that acd and bcd are also trilaterals, that is, that line cd is also present within the configuration.

Definition 1.6. A *near-pencil* $[a:bcd]$ consists of four points a, b, c, and d of the configuration, with a incident to each of b, c, and d, and with bcd a line of the configuration.

We regard the near-pencil $[a : bcd]$ in the Möbius-Kantor 8_3-configuration given in Figure 7.

The notations $[ab:cd]$ and $[a:bcd]$ for a near-complete quadrangle and a near-pencil, respectively, are similar in that the points incident to three of the lines which determine the object appear to the left of the colon, and those points incident to two lines appear to the right of the colon.

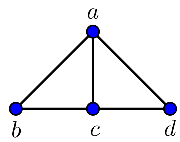


Figure 6: Near-pencil $[a\!:\!bcd]$.

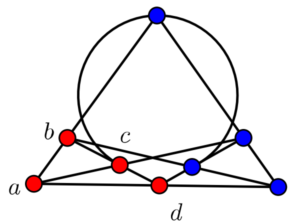
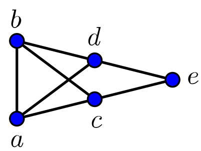


Figure 7: The Möbius-Kantor 8_3-configuration.

Theorem 1.7. *Let $\mathcal{C}$ be an n_3-configuration, and let $\mathcal{B}$ be the set of the non-trilaterals of $\mathcal{C}$. Then $\mathcal{C}$ induces a trilateral matroid $M_{tri}(\mathcal{C})$ if and only if no four points of $\mathcal{C}$ determine either a near-complete quadrangle or a near-pencil.*

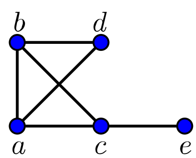
Proof. $(\Rightarrow)$ First suppose that $\mathcal{C}$ contains a near-complete quadrangle $[ab\!:\!cd]$. Let e be the third point on line ace.

Case 1: bde is a line in $\mathcal{C}$. Then the following subfiguration is present inside $\mathcal{C}$.



Let $B_1 = ace$ and $B_2 = bde$; both $B_1, B_2 \in \mathcal{B}$. Then $B_1 - B_2 = ac$ and $B_2 - B_1 = bd$. Let $x = c \in B_1 - B_2$; then $B_1 - x = ae$. But both abe and ade are trilaterals, so $B_1 - x \cup y \notin \mathcal{B}$ for all $y \in B_2 - B_1$. Hence $\mathcal{B}$ cannot be the set of bases of a matroid.

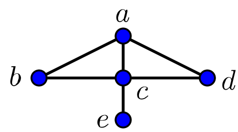
Case 2: bde is not a line in $\mathcal{C}$. Then inside of $\mathcal{C}$ we have



Note that edge be cannot be present, for if so point b would have four lines incident to it, but every point in an n_3-configuration is incident to three lines.

Let $B_1 = abe, B_2 = acd \in \mathcal{B}$. Take $e \in B_1 - B_2$; we have $B_1 - e = ab$. But $B_2 - B_1 = cd$, and both abc and abd are trilaterals. Hence $\mathcal{B}$ cannot be the set of bases of a matroid.

Now suppose $\mathcal{C}$ contains a near-pencil $[a\!:\!bcd]$ as indicated in the diagram. Let e be the third point on line ace.



We have $B_1 = ace, B_2 = bcd \in \mathcal{B}$. Choose $e \in B_1 - B_2$. Then $B_1 - e = ac$. But $B_2 - B_1 = bd$, and both abc and acd are trilaterals. Hence $\mathcal{B}$ cannot be the set of bases of a matroid.

($\Leftarrow$) Suppose that $\mathcal{C}$ does not induce a trilateral matroid $M_{tri}(C)$. Since $\mathcal{B}$ cannot be the set of bases of a matroid, there must exist a pair B_1, B_2 in $\mathcal{B}$ for which the basis exchange property is violated. So there must exist $x \in B_1 - B_2$ such that for all $y \in B_2 - B_1$, $B_1 - x \cup y$ is a trilateral.

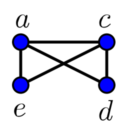
There are several cases to consider, some of which are vacuous.

Case 1: $B_1 = B_2$. Then $B_1 - B_2 = \emptyset$, so a violation of the basis exchange property cannot occur in this circumstance.

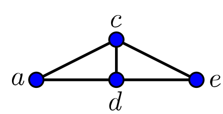
Case 2: $B_1 = abc$, $B_2 = abd$ (distinct letters label distinct points in $\mathcal{C}$.) Then $B_1 - B_2 = c$ and $B_2 - B_1 = d$. For a violation to occur, we require that $B_1 - c \cup d$ be a trilateral. But $B_1 - c \cup d = B_2 \in \mathcal{B}$. Hence no violation can occur in this case as well.

Case 3: $B_1 = abc$, $B_2 = ade$. Then $B_1 - B_2 = bc$ and $B_2 - B_1 = de$. Without loss of generality we assume that $x = b$. For a violation of the basis exchange property to occur, both acd and ace must be trilaterals.

Subcase 3.1: ade is a non-collinear non-trilateral. Then $[ac : de]$ is a near-complete quadrangle.



Subcase 3.2: ade is a line. Then $[c\!:\!ade]$ is a near-pencil.



Case 4: $B_1 = abc$, $B_2 = def$, so $B_1 \cap B_2 = \emptyset$. We may let $x = a$ without loss of generality. So for a violation of the basis exchange property to occur, all three of bcd, bce and bcf must be trilaterals.

Subcase 4.1: Two of d, e, f are collinear with b. Without loss of generality, we assert that bde is a line. Then $[c\!:\!bde]$ is a near-pencil.

Subcase 4.2: No two of d, e, f are collinear with b. Then b must be incident to four lines, a contradiction. $\square$

2 Examples

We have already observed, by Corollary 1.4, that the Fano 7_3-configuration, the Möbius-Kantor 8_3-configuration, and two of the three 9_3-configurations cannot induce trilateral matroids. It is worth noting that the Fano configuration contains no near-complete quadrangle, but many near-pencils; given any line abc of the Fano configuration, and any fourth point d not on this line, then $[d : abc]$ is a near-pencil. Since by Figure 5 we see that the Pappus 9_3-configuration contains a near-complete quadrangle, by Theorem 1.7 it also cannot induce a trilateral matroid.

It is worth noting that there is a matroid associated with the Fano configuration in the sense that no three-element subset of the point set can be an independent set, since every point triple determines a trilateral. But this is really a degenerate case; the matroid is $U_{2,7}$, so every 2-element subset of the point set is independent, but no 3-element subset is. Since $U_{2,7}$ is a rank-2 matroid, and not rank-3, we will not deem it to be a trilateral matroid.

The smallest configuration which does generate a rank-3 trilateral matroid is the Desargues 10_3-configuration provided in Figure 3. There we may readily observe that the configuration contains neither a near-complete quadrangle nor a near-pencil. Since the Desargues configuration contains 20 trilaterals, there are $\binom{10}{3} - 20 = 100$ bases in the associated matroid. Each of the other nine 10_3-configurations contains at least one near-complete quadrangle, and therefore the Desargues configuration is the smallest configuration which induces a trilateral matroid.

Figure 8 depicts a geometric representation of the the trilateral matroid induced by the Desargues configuration in the following fashion. If three points happen to be collinear in the geometric representation, then these points describe a trilateral in the original configuration. Each of the five four-point lines in this representation thus describes four point triples which determine trilaterals; these four points consequently are associated with a complete quadrangle in the Desargues configuration. The Desargues configuration contains five such complete quadrangles, and each point of the configuration is involved in two quadrangles. So we arrive at the star in Figure 8, which is itself a $(10_2, 5_4)$-configuration. This means that there are ten points, with each point incident to two lines, and five lines, with each line incident to four points.

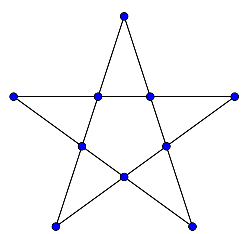


Figure 8: A geometric representation of the trilateral matroid associated with the Desargues configuration.

Interestingly, there is no 11_3-configuration which induces a trilateral matroid. In fact, each of the 31 11_3-configurations contains at least one near-complete quadrangle.

Among the 229 12_3-configurations, there is only one which does not contain a near-complete quadrangle. This configuration also happens not to contain a near-pencil, and hence induces a trilateral matroid on its points. This configuration is the Coxeter 12_3-configuration shown in Figure 9.

12A:

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}
1	1	1	2	2	3	3	4	5	5	6	7
2	4	6	6	8	4	9	8	7	9	9	8
3	5	7	10	11	12	11	10	11	10	12	12

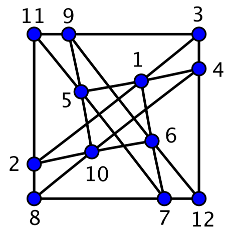


Figure 9: The Coxeter 12_3-configuration.

The automorphism group of this configuration has order 72. This configuration is listed as D88 in Daublebsky von Sterneck's enumeration of the first 228 12_3-configurations in 1895 [4]; the last of the 229 12_3-configurations was found much later in 1990 by Gropp [5]. All 229 12_3-configurations have been recently re-examined in [1], and the provided geometric realization of D88 in Figure 9 stems from this work. Again, by inspection, we see that no near-complete quadrangle is present, as well as no near-pencil.

This configuration contains 12 trilaterals. Each point of the configuration is incident to three of them, with no pair of points belonging to the same trilateral. So these trilaterals are blocks of another 12_3-configuration defined on the same set of points, namely

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
1	1	1	2	2	3	4	4	5	6	6	7
2	3	5	3	8	9	5	8	9	7	9	8
6	4	7	11	10	12	10	12	11	12	10	11

It is not hard to see that this configuration is isomorphic to the previous one. In fact, this is the first instance of a more general phenomenon.

Theorem 2.1. *Suppose that an n_3-configuration $\mathcal{C}$ has n trilaterals, with every point incident to three trilaterals and no pair of points incident to more than one trilateral. Let $\mathcal{C}_{tri}$ be the n_3-configuration formed by these n trilaterals. Then $\mathcal{C}_{tri} \cong \mathcal{C}$.*

Proof. It suffices to show that the dual of $\mathcal{C}$ and the dual of $\mathcal{C}_{tri}$ are isomorphic. Regard one of the lines of the respective duals; call this line p. This is a point of each of the original configurations. The local structure is indicated by the diagram in Figure 10.

We associate the line a with the trilateral t_a as follows: of the three trilaterals incident to p, t_a is chosen so that a is *not* involved in determining this trilateral. In a similar manner, line b is identified with trilateral t_b and line c is identified with trilateral t_c. Our hypotheses allow us to carry this correspondence across the respective dual configurations, with the resulting correspondence between the points of $\mathcal{C}$ and of $\mathcal{C}_{tri}$ (the blocks of the duals) the identity map. Therefore $\mathcal{C}_{dual} \cong (\mathcal{C}_{tri})_{dual}$, whence $\mathcal{C} \cong \mathcal{C}_{tri}$. □

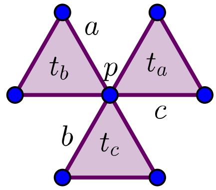


Figure 10: Lines and trilaterals incident to point p.

Next, among the 2036 13_3-configurations, there are four which do not contain a near-complete quadrangle. And among these four, there is only one which does not contain a near-pencil. This is Configuration 13A, given in Figure 11. The automorphism group of

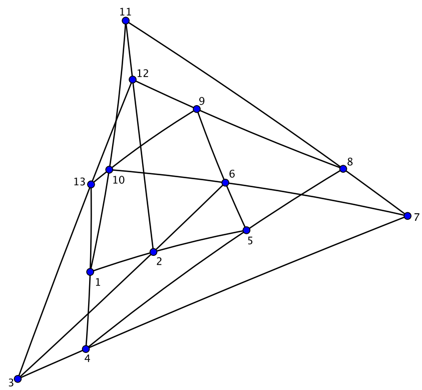


Figure 11: A 13_3-configuration which induces a trilateral matroid.

this configuration has order 39. The configuration contains 13 trilaterals, and each point is incident to three trilaterals, with no pair of points incident to more than one trilateral. So we may derive an associated 13_3-configuration by listing these trilaterals:

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}
1	1	1	2	2	3	4	4	5	6	6	7	9
2	3	5	3	8	11	5	8	10	7	9	8	11
4	7	6	9	10	12	11	13	12	13	11	12	13

By Theorem 2.1 this configuration is isomorphic to Configuration 13A. Configuration 13A is also isomorphic to the cyclic configuration $C_3(13, 1, 4)$, given combinatorially by regarding the lines $\{j, j+1, j+4\}$ mod 13 for $0 \leq j \leq 12$:

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}
0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
4	5	6	7	8	9	10	11	12	0	1	2	3

One may employ these point labels to construct the *Paley graph* of order 13 as follows. Draw an edge between labels a and b if and only if $a - b$ is a perfect square mod 13. This means that $a-b$ can be ± 1, ± 3, or ± 4 mod 13. We thus obtain the following graph where each edge is contained in exactly one triangle, and each triangle in the graph corresponds to a trilateral of the 13_3-configuration.

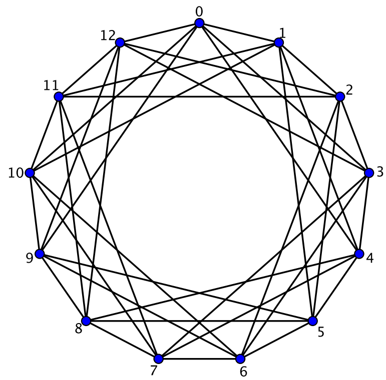


Figure 12: Paley graph associated with Configuration 13A.

More generally, the cyclic n_3-configuration $C_3(n,k,m)$ is given by the lines $\{j, j+k, j+m\}$ mod n for $0 \le j \le n-1$.

Proposition 2.2. *For $n \ge 13$, the cyclic configuration $C_3(n,1,4)$ induces a trilateral matroid on n trilaterals which equals the linear matroid on $C_3(n,3,4)$. In other words, $M_{tri}(C_3(n,1,4)) = M(C_3(n,3,4))$. Moreover, $C_3(n,1,4) \cong C_3(n,3,4)$.*

Proof. In order to determine the trilaterals of $C_3(n,1,4)$, it suffices to ascertain the trilaterals which involve 0, and then extend from this via a cyclic pattern. The trilaterals involving 0 are:

- $0\ 3\ 4$ (using the lines $\{0,1,4\}$, $\{3,4,7\}$, and $\{n-1,0,3\}$)
- $n-4\, n-3\, 0$ (using the lines $\{n-4,n-3,0\}$, $\{n-1,0,3\}$, and $\{n-5,n-4,n-1\}$)
- $n-3\ 0\ 1$ (using the lines $\{n-4,n-3,0\}$, $\{n-3,n-2,1\}$, and $\{0,1,4\}$)

Since $n \ge 13$, no extra trilateral involving 0 is formed (for example, if $n = 12$, then $0\,4\,8$ would be a trilateral.) Hence we see, after extending cyclically, that the trilaterals of $C_3(n,1,4)$ form their own configuration, namely $C_3(n,3,4)$, and thus $M_{tri}(C_3(n,1,4))$ is the linear matroid corresponding to $C_3(n,3,4)$. Finally we may recognize that $C_3(n,1,4)$ is isomorphic to $C_3(n,3,4)$ either by utilizing Theorem 2.1 or by applying the correspondence $t \to (4-t) \mod n$. $\square$

It turns out that $C_3(16,1,4)$ and $C_3(16,1,7)$ are the smallest examples of non-isomorphic cyclic $C_3(n,k,m)$ configurations having n trilaterals each, and hence their corresponding trilateral matroids (which are isomorphic to the linear matroids associated with the respective original configurations) are non-isomorphic to each other as well.

It is possible, however, for a non-cyclic n_3-configuration to induce a trilateral matroid on its n trilaterals, with the trilaterals capable of determining an n_3-configuration in their own right, without the original configuration needing to be cyclic. We have already seen

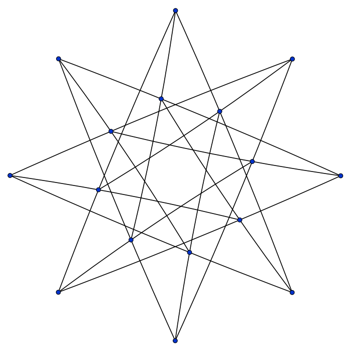


Figure 13: A non-cyclic 16_3-configuration whose trilateral matroid is isomorphic to the linear matroid associated with the configuration.

one example of this with the Coxeter 12_3-configuration given in Figure 9. Another example is the 16_3-configuration provided in Figure 13 whose automorphism group has order 32.

It is additionally possible for an n_3-configuration possessing n trilaterals to induce a trilateral matroid that is not isomorphic to the linear matroid associated with the original configuration. Figure 14 gives a diagram of such a configuration, a 20_3-configuration containing 20 trilaterals. It contains two points which are involved in six trilaterals and four points involved in four trilaterals. A geometric representation of the matroid is also provided.

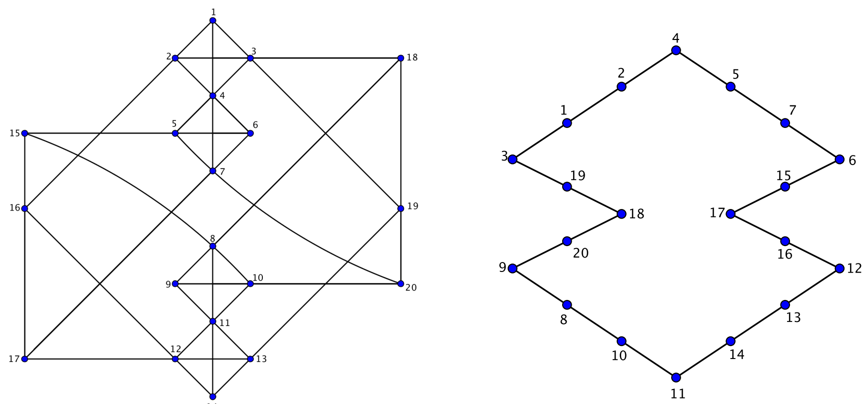


Figure 14: A 20_3-configuration with 20 trilaterals whose trilateral matroid is not isomorphic to the linear matroid of the configuration, and a geometric representation of its trilateral matroid.

We next offer an example of of an 18_3-configuration possessing 20 trilaterals which induces a trilateral matroid. In Figure 15 we provide a picture of this configuration (with several pseudolines) and the accompanying geometric representation of its trilateral matroid. This example presents another instance, in addition to the Desargues 10_3-configuration, of an n_3-configuration containing more than n trilaterals which induces a trilateral matroid.

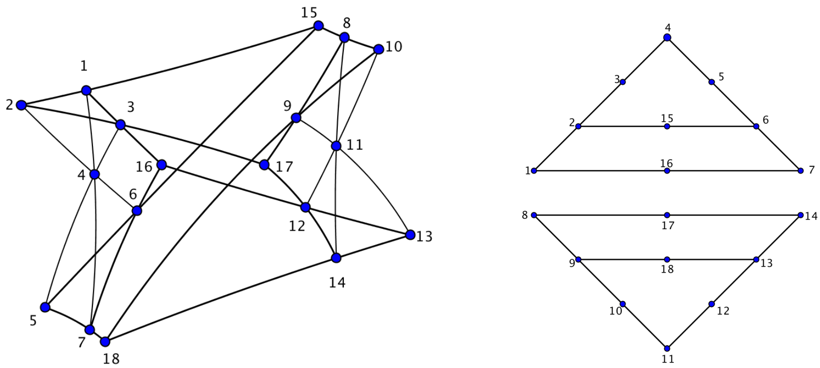


Figure 15: An 18_3-configuration with 20 trilaterals, and a geometric representation of its trilateral matroid.

Note that this configuration contains four complete quadrangles.

We now return to the enumeration of the smallest n_3-configurations which induce trilateral matroids. There are four 14_3-configurations which do so. We label these configurations as 14A, 14B, 14C and 14D, and provide combinatorial depictions of them.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	7	8	9
14A:	2	4	6	4	8	7	8	11	6	12	9	10	13	11
	3	5	7	9	10	12	11	12	13	14	10	14	14	13

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	6	7	8
14B:	2	4	6	4	9	7	10	11	10	12	8	9	9	11
	3	5	7	8	12	11	12	13	14	13	10	13	14	14

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	7	7	10
14C:	2	4	6	4	8	6	13	11	8	12	8	9	10	11
	3	5	7	9	10	11	14	12	13	14	9	14	12	13

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
	1	1	1	2	2	3	3	4	5	5	6	6	7	7
14D:	2	4	6	4	10	8	12	11	8	10	8	10	9	11
	3	5	7	9	13	11	14	12	13	14	9	12	14	13

These configurations contain 14, 10, 10, and 6 trilaterals, respectively. Also, their automorphism groups have orders 14, 1, 4, and 8, respectively.

Figure 16 gives a realization of Configuration 14A, which is isomorphic to the cyclic configuration $C_3(14, 1, 4)$. Hence we know its trilateral matroid is isomorphic to its linear matroid by Proposition 2.2.

Configurations 14B and 14C both contain 10 trilaterals, so it is conceivable that their associated trilateral matroids could be isomorphic. But they are not, for 14B has three points which are each incident to three trilaterals and one point which is incident to only one trilateral, whereas Configuration 14C has two points each incident to three trilaterals

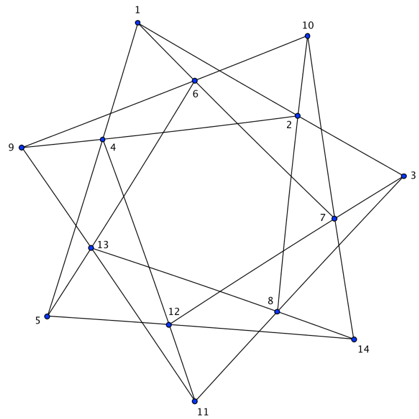


Figure 16: Configuration 14A.

and no point incident to only one trilateral. Figure 17 gives geometric representations of the trilateral matroids associated with Configurations 14B and 14C, respectively.

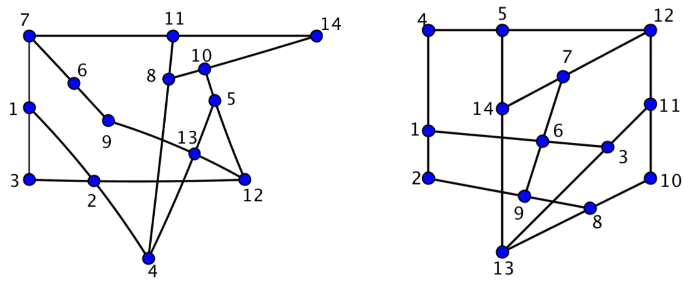


Figure 17: Geometric representations for trilateral matroids for Configurations 14B and 14C.

Figure 18 is a rendering for Configuration 14D with several pseudolines, along with a geometric representation of its associated trilateral matroid.

Proceeding to the $n = 15$ setting, we encounter a substantial increase, to 220, of the number of 15_3-configurations which induce trilateral matroids. One such example is the Cremona-Richmond configuration provided in Figure 19. It is the smallest example of a trilateral-free n_3-configuration. As it is trilateral-free, the trilateral matroid it induces is the uniform matroid on 15 points $U_{3,15}$.

Another example is the cyclic configuration $C_3(15, 1, 4)$, whose induced trilateral matroid (with 15 trilaterals) is isomorphic to the linear matroid on $C_3(15, 1, 4)$ by Proposition 2.2. Its automorphism group has order 30. Each of the other 15_3-configurations which induces a trilateral matroid contains k trilaterals, where $k \in \{4, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

It is clearly not the case that for all n, there exists a one-to-one correspondence between the trilateral matroids themselves and the n_3-configurations which induce them. We know this because there are four non-isomorphic trilateral-free 18_3-configurations [3], so each consequently must induce the same uniform matroid on 18 points.

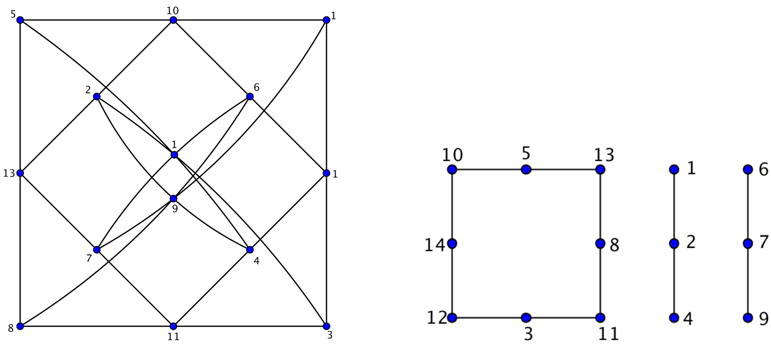


Figure 18: Configuration 14D and its trilateral matroid.

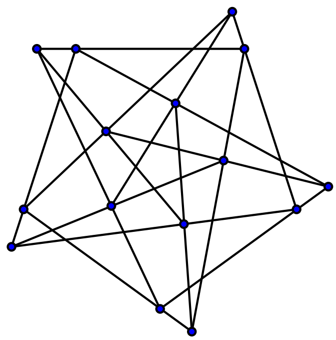


Figure 19: The Cremona-Richmond 15_3-configuration.

It is of interest to contemplate whether smaller non-isomorphic n_3-configurations exist that induce isomorphic trilateral matroids, and indeed this turns out to be true. In fact, this property is satisfied by the following pair of non-isomorphic 15_3-configurations given in Figure 20. Each contains 8 trilaterals and has a symmetry group of order 48. The

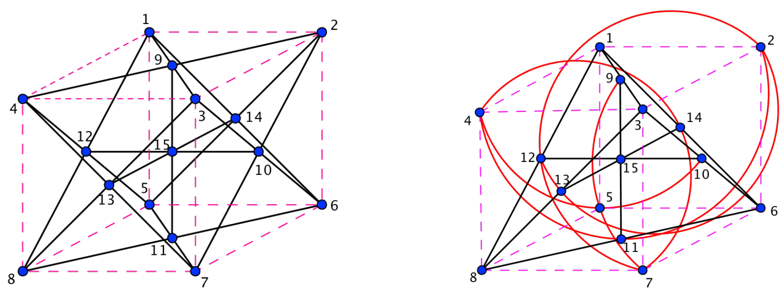


Figure 20: Non-isomorphic 15_3-configurations which induce the same trilateral matroid on 15 points.

set of points for both configurations consists of the eight vertices of a cube, the centers of the six faces of the cube, and the center of the cube itself. In the former configuration the diagonally-opposing points in each face of the cube are incident via a line which passes through the center of the same face, whereas in the latter configuration one pair of diagonally-opposing points in each face are incident via a "line" which passes through the center of the opposite face. The eight trilaterals involved in these respective configurations are identical, and thus their corresponding trilateral matroids are the same. Figure 21 gives this matroid, which is isomorphic to $U_{2,4} \oplus U_{2,4} \oplus U_{3,7}$. Hence the number of

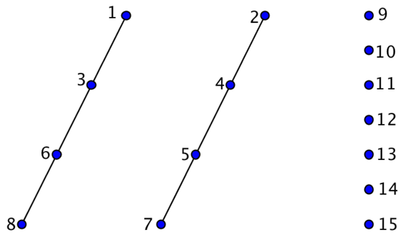


Figure 21: The common trilateral matroid.

trilateral matroids that are induced from 15_3-configurations is smaller than the number of 15_3-configurations which induce trilateral matroids. Our calculations indicate that there are 214 non-isomorphic trilateral matroids that may be found from the 220 15_3-configurations which induce trilateral matroids.

We conclude this section with a table which summarizes the current state of affairs. Here $\#_c(n)$ denotes the number of non-isomorphic n_3-configurations, $\#_{tri}(n)$ denotes the number of these configurations which induce trilateral matroids, and $\#_{mat}(n)$ denotes the number of non-isomorphic trilateral matroids which arise from these configurations.

n	$\#_c(n)$	$\#_{tri}(n)$	$\#_{mat}(n)$
7	1	0	0
8	1	0	0
9	3	0	0
10	10	1	1
11	31	0	0
12	229	1	1
13	2036	1	1
14	21399	4	4
15	245342	220	214

3 A point-plane configuration

A *point-plane n_3-configuration* is an incidence structure consisting of n distinct points and n distinct planes for which each point is incident with three planes, each plane is incident with three points, and any two points are incident with at most one common plane. In such a configuration, we deem a trilateral to be a cyclically ordered set $\{p_0, \pi_0, p_1, \pi_1, p_2, \pi_2\}$ of pairwise distinct points p_i and pairwise distinct planes π_i such that p_i is incident with π_{i-1} and π_i for each $i \in \mathbb{Z}_3$. Once more we may without ambiguity shorten this notation by listing only the points of the trilateral as $\{p_0, p_1, p_2\}$, or more simply as $p_0p_1p_2$.

In Figure 22 we offer an example of a point-plane 12_3-configuration which induces a trilateral matroid on its points. The 12 points are selected from the 20 vertices of the regular dodecahedron so that each of the twelve pentagonal faces contains three points; note that each of the 12 points is the intersection of three faces, so a point-plane 12_3-configuration is achieved. We observe that each of the eight unlabeled red points in the

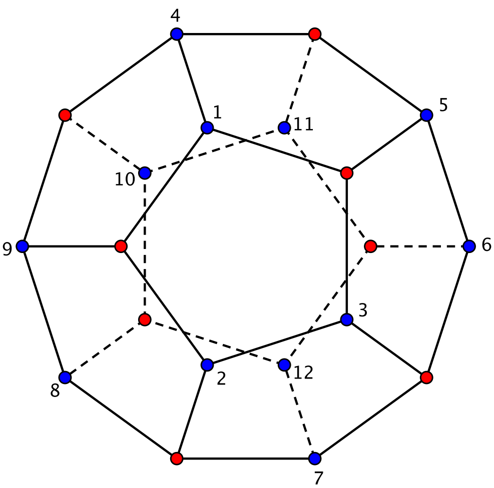


Figure 22: A 12_3 point-plane configuration which induces a trilateral matroid.

diagram corresponds to a trilateral, and that this trilateral may be specified uniquely by cycling through the configuration points that are immediately adjacent to the red point. For example, the triple $\{1, 3, 5\}$ defines a trilateral. We start at 1, then pass through the plane containing both 1 and 3 to 3. We then pass through the plane containing both 3 and 5 to 5, and then finally pass through the plane containing both 5 and 1 back to 1 to complete the cycle. Here are the eight trilaterals.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
1	1	2	3	4	4	6	8
2	3	7	6	5	9	11	10
9	5	8	7	11	10	12	12

Figure 23 gives a geometric representation of the trilateral matroid.

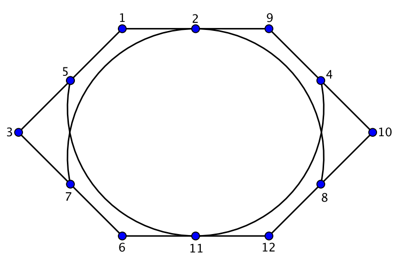


Figure 23: The trilateral matroid of the 12_3 point-plane configuration.

After identifying each trilateral with its corresponding red point in Figure 22, we recognize that the trilateral matroid may also be represented as a point-plane configuration, namely an $(8_3, 12_2)$-configuration. This means the configuration has eight points, with three planes incident to each point, and twelve planes, with two points incident to each plane.

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Growth of face-homogeneous tessellations

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Abstract

A tessellation of the plane is *face-homogeneous* if for some integer $k \geq 3$ there exists a cyclic sequence $\sigma = [p_0, p_1, \ldots, p_{k-1}]$ of integers ≥ 3 such that, for every face f of the tessellation, the valences of the vertices incident with f are given by the terms of σ in either clockwise or counter-clockwise order. When a given cyclic sequence σ is realizable in this way, it may determine a unique tessellation (up to isomorphism), in which case σ is called *monomorphic*, or it may be the valence sequence of two or more non-isomorphic tessellations (*polymorphic*). A tessellation whose faces are uniformly bounded in the hyperbolic plane but not uniformly bounded in the Euclidean plane is called a *hyperbolic tessellation*. Such tessellations are well-known to have exponential growth. We seek the face-homogeneous hyperbolic tessellation(s) of slowest growth rate and show that the least growth rate of such monomorphic tessellations is the "golden mean," $\gamma = (1 + \sqrt{5})/2$, attained by the sequences $[4, 6, 14]$ and $[3, 4, 7, 4]$. A polymorphic sequence may yield non-isomorphic tessellations with different growth rates. However, all such tessellations found thus far grow at rates greater than γ.

Keywords: Face-homogeneous, tessellation, growth rate, valence sequence, exponential growth, transition matrix, Bilinski diagram, hyperbolic plane.

Math. Subj. Class.: 05B45, 05C63, 05C10, 05C12

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1 Introduction

It has long been known that there are finitely many homogeneous tessellations of the Euclidean plane; they all have quadratic growth rate. However, in the hyperbolic plane, for various definitions of "homogeneity," infinitely many homogeneous tessellations are realizable, and their growth rate, if not quadratic, is always exponential. Presently we will give a rigorous definition of growth rate, but for now one should think of this parameter intuitively as the asymptotic rate at which additional tiles (or faces) accrue with respect to some chosen center of a tessellation. In this schema, all Euclidean tessellations have growth rate equal to 1, and hyperbolic tessellations have growth rate strictly greater than 1. The first author has shown by construction in [5] that, given any $\epsilon > 0$, there exists a hyperbolic tessellation with growth rate exactly $1 + \epsilon$. In general, these latter tessellations have few if any combinatorial or geometric symmetries. The question then becomes one of determining the growth rates of hyperbolic tessellations when some sort of homogeneity is imposed. In particular, subject to a homogeneity constraint, how small can the gap be between quadratic and exponential growth?

In a seminal work [8], Grünbaum and Shephard defined a graph to be *edge-homogeneous with edge-symbol* $\langle p, q; k, \ell \rangle$ if every edge is incident with vertices of valences p and q and faces of covalences k and ℓ. They proved that the parameters of an edge-symbol uniquely determine an edge-homogeneous tessellation up to isomorphism.

The notion of homogeneity was extended by Moran [10]. She defined a tessellation to be *face-homogeneous with valence sequence* $[p_0, \ldots, p_{k-1}]$ if every face is a k-gon incident with vertices of valences $p_0, \ldots, p_{k-1}$ in either clockwise or counter-clockwise consecutive order. Unfortunately, no uniqueness property analogous to the Grünbaum-Shephard result holds in general for face-homogeneous tessellations.

Moran's work on growth rates of face-homogeneous tessellations led the authors (together with T. Pisanski) to return to edge-homogeneous tessellations and conclusively determine their growth rates. In [6] they determined the growth rate of any edge-homogeneous tessellation as a function of its edge-symbol and proved that the least growth rate for an exponentially-growing, edge-homogeneous tessellation is $\frac{1}{2}(3 + \sqrt{5}) \approx 2.618$.

The goal of this article is to obtain an analogous result for face-homogeneous tessellations. Our main result is that if a face-homogeneous tessellation with exponential growth rate is determined up to isomorphism by its valence sequence, then its growth rate is at least $\frac{1}{2}(1 + \sqrt{5})$, namely the "golden mean." Moreover, we determine exactly the valence sequences for which this golden mean is realized. A significant by-product of our investigation is an abundance of machinery for computing the growth rates of many classes of face-homogeneous planar tessellations.

Section 2 consists of six subsections. Following some general definitions concerning infinite graphs in the plane, we present (Subsection 2.2) a system for labeling sets of vertices and sets of faces of a tessellation; such a labeling is called a "Bilinski diagram." Subsection 2.3 presents the notion of face-homogeneity and associated notation. Polynomial and exponential growth, defined on the one hand with respect to the standard graph-theoretic metric, and on the other hand with respect to the notion of angle excess, appear in Subsection 2.4. Subsection 2.5 presents a rigorous theoretical treatment of growth rate with respect to regional distance in a Bilinski diagram. Subsection 2.6 concludes the Preliminaries with a review of the completely resolved case of edge-homogeneous tessellations, summarizing results from [8] and [6].

In Subsection 3.1 we lay out our method for filling in the formulas obtained in Subsection 2.5 while introducing the notion of a transition matrix. Analogous to a Markov process, this matrix encodes for given $n \geq 1$ how many faces of each possible configuration are "begotten" at regional distance $n + 1$ from the root of a Bilinski diagram by a face at regional distance n from the root. The maximum modulus of the eigenvalues of the transition matrix are key to the growth rate of T.

Subsection 3.2 applies the machinery of Subsection 3.1 to the significant class of valence sequences that are *monomorphic*, i.e., that are uniquely realizable as a face-homogeneous tessellation and whose Bilinski diagrams are in a certain sense well-behaved, called *uniformly concentric*. It is shown in Theorem 3.7 that for such valence sequences, the partial order defined in Subsection 2.3 is preserved by their growth rates. The six classes of monomorphic sequences of lengths 3, 4, and 5 whose Bilinski diagrams are not uniformly concentric are identified in Subsection 3.3, where it is proved that they are indeed monomorphic. The exhaustive proof that this list is complete is contained in the Appendix [7]. Finally, we present in Subsection 3.4 the main result of the paper, that the least growth rate of a face-homogeneous tessellation with monomorphic valence sequence is the golden mean $\frac{1}{2}(1 + \sqrt{5})$.

Those valence sequences (described as *polymorphic*) which admit multiple non-isomorphic tessellations are alive and well in Subsection 4.1. A general sufficient condition for polymorphism is given. The difficulties posed by polymorphism are illustrated by an example; the polymorphic sequence $[4, 4, 6, 8]$ is considered in some depth in Subsection 4.2. In particular, we see by this example that two different tessellations having the same (polymorphic) valence sequence may well have different growth rates. We conclude the chapter with some conjectures in Subsection 4.3.

The appendix [7] alluded to above is to be found with this article on the arXiv, at `arXiv:1707.03443`. All references therein are to results in the present paper. Due to the considerable length (and tedium!) of the appendix, it will not appear in *Ars Mathematica Contemporanea* with this article.

2 Preliminaries

2.1 Tessellations

For a graph Γ, the symbol $V(\Gamma)$ denotes the vertex set of Γ. If M is a planar embedding of Γ, we call M a *plane map* and denote by $F(M)$ the set of faces of M.

A graph Γ is *infinite* if its vertex set $V(\Gamma)$ is infinite; Γ is *locally finite* if every vertex has finite valence. A graph is 3-*connected* if there is no set of fewer than three vertices whose removal disconnects the graph. It is well known that if the underlying graph Γ of a plane map M is 3-connected (as is generally the case in this work), then every automorphism of Γ induces a permutation of $F(M)$ that preserves face-vertex incidence and can be extended to a homeomorphism of the plane. Thus we tend to abuse language and speak of "the faces of Γ." When a plane map is 3-connected, every edge is incident with exactly two distinct faces. In this case, the number of edges (and hence of vertices) incident with a given face is its *covalence*. A map is *locally cofinite* if the covalence of every face is finite.

An *accumulation point* of an infinite plane map M is a point x in the plane such that every open disk of positive radius (in either the Euclidean or hyperbolic metric) containing x intersects infinitely many map objects, be they faces, edges, or vertices. A map is 1-*ended* when the deletion of any finite submap leaves exactly one infinite component.

Definition 2.1. A *tessellation* is an infinite plane map that is 3-connected, locally finite, locally cofinite, 1-ended, and also admits no accumulation point.

In the terminology of Grünbaum and Shepherd's exhaustive work [9] on tilings of the plane, a tessellation T is *normal* if there is an embedding of T in the plane and radii $0 < r < R$ under a specific metric such that the boundary of each face lies within some annulus with inner radius r and outer radius R. A *Euclidean tessellation* is a tessellation that is normal with respect to the Euclidean metric, and a *hyperbolic tessellation* is one that is normal with respect to the hyperbolic metric but not with respect the Euclidean metric.

2.2 Bilinski diagrams

A very useful tool for computing "growth rate" is what we have called a *Bilinski diagram*, because these diagrams were first used by Stanko Bilinski in his dissertation [1, 2].

Definition 2.2. Let M be a map that is rooted at some vertex x. Define a sequence of sets $\{U_n : n \geq 0\}$ of vertices and a sequence of sets $\{F_n : n \geq 0\}$ of faces of M inductively as follows.

- Let $U_0 = \{x\}$ and let $F_0 = \emptyset$.
- For $n \geq 1$, let F_n denote the set of faces of M not in F_{n-1} that are incident with some vertex in U_{n-1}.
- For $n \geq 1$, let U_n denote the set of vertices of M not in U_{n-1} that are incident with some face in F_n.

The stratification of M determined by the set-sequences $\{U_n\}$ and $\{F_n\}$ is called *the Bilinski diagram of M rooted at x*. In a similar way one can define a Bilinski diagram of M rooted at a face f. In this case $U_0 = \emptyset$ and $F_0 = \{f\}$. Given a Bilinski diagram of T, the induced submap $\langle F_n \rangle$ of T is its n^{th} *corona*.

A Bilinski diagram is *concentric* if each subgraph $\langle U_n \rangle$ induced by U_n $(n \geq 1)$ is a cycle; otherwise the Bilinski diagram is *non-concentric*. If a plane map yields a concentric Bilinski diagram regardless of which vertex or face is designated as its root, then the map is *uniformly concentric*; analogously a map which for every designated root yields a non-concentric Bilinski diagram is *uniformly non-concentric*.

To answer the question as to which tessellations are uniformly concentric we state a sufficient condition and a necessary condition. Let $\mathscr{G}_{a,b}$ denote the class of tessellations all of whose vertices have valence at least a and all of whose faces have covalence at least b. Let $\mathscr{G}_{a+,b}$ be the subclass of $\mathscr{G}_{a,b}$ of tessellations with no adjacent a-valent vertices.

Proposition 2.3 ([3, Corollary 4.2], [11, Theorem 3.2])**.** *Every tessellation $T \in \mathscr{G}_{3,6} \cup \mathscr{G}_{3+,5} \cup \mathscr{G}_{4,4}$ is uniformly concentric, and in every Bilinski diagram of T, for all $n \geq 1$, every face in F_n is incident with at most two edges in $\langle U_{n-1} \rangle$.*

Proposition 2.4 ([3, Theorem 5.1])**.** *If an infinite planar map admits any of the following configurations, then the map is not uniformly concentric:*

1. *a 3-valent vertex incident with a 3-covalent face;*
2. *a 4-valent vertex incident with two nonadjacent 3-covalent faces;*
3. *a 4-covalent face incident with two nonadjacent 3-valent vertices;*
4. *an edge incident with two 3-valent vertices and two 4-covalent faces;*
5. *an edge incident with two 4-valent vertices and two 3-covalent faces.*

2.3 Face-homogeneity and realizability

Let $k \geq 3$ be an integer and let an equivalence relation be defined on the set of ordered k-tuples $(p_0, p_1, \ldots, p_{k-1})$ of positive integers whereby

- $(p_0, p_1, \ldots, p_{k-1}) \equiv (p_1, p_2, \ldots, p_{k-1}, p_0)$, and
- $(p_0, p_1, \ldots, p_{k-1}) \equiv (p_{k-1}, p_{k-2}, \ldots, p_0)$.

The equivalence class of which $(p_0, \ldots, p_{k-1})$ is a member is the *cyclic sequence* $[p_0, \ldots, p_{k-1}]$, and k is its *length*. There is a natural partial order $\leq$ on the set of cyclic sequences:

$$[p_0, \ldots, p_{k-1}] \leq [q_0, \ldots, q_{\ell-1}]$$

if and only if $k \leq \ell$ and there exists a cyclic subsequence $q_{i_0}, q_{i_1}, \ldots, q_{i_{k-1}}$ occurring in either order in $[q_0, q_1, \ldots, q_{\ell-1}]$ such that $p_j \leq q_{i_j}$ for each $j \in \{0, \ldots, k-1\}$. We write $\sigma_1 < \sigma_2$ if $\sigma_1 \leq \sigma_2$ but $\sigma_1 \neq \sigma_2$, where σ_1 and σ_2 are cyclic sequences.

Example 2.5. Let $\sigma_1 = [4, 6, 8, 10]$, $\sigma_2 = [6, 8, 12, 4]$, and $\sigma_3 = [10, 8, 12, 6, 4]$. Then $\sigma_1 < \sigma_2$ and $\sigma_1 < \sigma_3$, but σ_2 and σ_3 are not comparable.

Definition 2.6. Let $\sigma = [p_0, p_1, \ldots, p_{k-1}]$ be a cyclic sequence of integers ≥ 3. Then σ is the *valence sequence* of a k-covalent face f of a tessellation T if the valences of vertices incident with f in clockwise or counter-clockwise order are $p_0, p_1, \ldots, p_{k-1}$. If every face of T has the same valence sequence σ, then T is *face-homogeneous* and σ is the *valence sequence of* T. Thus, to say briefly that a tessellation T has valence sequence σ implies that T is face-homogeneous.

Definition 2.7. Let the cyclic sequnce σ be realizable as the valence sequence of a tessellation. If every tessellation having valence sequence σ is uniformly concentric, then we say that σ is *uniformly concentric*. Otherwise σ is *non-concentric*. If every tessellation having valence sequence σ is non-concentric, then σ is *uniformly non-concentric*.

Notation. By convention, when distinct letters are used to represent terms in a cyclic sequence (e.g. $[p, p, q, r, q]$), the values corresponding to distinct letters are all presumed to be distinct; that is, $p \neq q \neq r \neq p$. Moreover, if some term in the cyclic sequence is given as an integer (usually 3 or 4), then the terms given by letters are presumed to be greater than that integer. For example, if $\sigma = [4, p, q]$, then we understand that $p, q > 4$ and $p \neq q$. When using subscripts in the general form $[p_0, \ldots, p_{k-1}]$, we do not make this assumption.

Remark 2.8. Not all cyclic sequences are realizable as vertex sequences of face-homogeneous tessellations of the plane. For instance, the map with valence sequence $[3, 3, 3]$ (the tetrahedron) is a tessellation of the sphere but not of the plane. More importantly, there are many cyclic sequences for which no face-homogeneous map exists at all. For instance, the valence sequence $[4, 5, 6, p]$ for any $p \geq 3$ is not realizable, because in any such map the valences of the neighbors of a 5-valent vertex in cyclic order would have to alternate between 4 and 6. However, this does not generalize to all cyclic sequences containing a subsequence $[p, q, r]$ where q is odd and $p \neq r$; for instance, $[5, 4, 5, 6, 5, 8]$ is realizable.

Conjecture 2.9. *Suppose σ is the valence sequence of a face-homogeneous tessellation and that σ contains $[p, q, r]$ as a subsequence, with q odd and $p \neq r$. Then σ must contain at least three terms equal to q.*

2.4 Polynomial versus exponential growth

Let x be a vertex of a connected graph Γ. For each nonnegative integer n, the *ball of radius* n *about* x is the set of vertices of Γ at distance $\leq n$ from x, written

$$B_n(x) = \{v \in V(\Gamma) : d(x, v) \leq n\}, \tag{2.1}$$

where $d(-,-)$ is the *standard graph-theoretic metric*, that is, $d(u, v)$ is the length of a shortest path with terminal vertices u and v.

Definition 2.10. An infinite, locally finite, connected graph Γ has *exponential growth* if for some vertex $x \in V(\Gamma)$ there exist real numbers $\alpha > 1$ and $C > 0$ such that, for all $n > 0$, one has $|B_n(x)| > C\alpha^n$; otherwise Γ has *subexponential growth.* We say that Γ has *polynomial growth* of degree $d \in \mathbb{N}$ if there exist positive constants C_1 and C_2 such that $C_1 n^d \leq |B_n(x)| \leq C_2 n^d$ for all but finitely many n.

For example, the graph underlying the square lattice in the plane has quadratic growth ($d = 2$). If x is any vertex, then $|B_n(x)| = 2n^2 + 2n + 1$ for all $n \geq 1$, and one can set $C_1 = 2$ and $C_2 = 3$.

Continuing the notation of Equation (2.1) and Definition 2.10, we consider the generating function

$$\beta_x(z) = \sum_{n=0}^{\infty} |B_n(x)|\, z^n \tag{2.2}$$

We denote the radius of convergence of $\beta_x(z)$ by R_B and define the *ball-growth rate of* Γ *about* x to be the reciprocal of R_B.

If Γ has exponential growth, then we have

$$\beta_x(z) \geq \sum_{n=0}^{\infty} C\alpha^n z^n = \frac{C}{1 - \alpha z}, \tag{2.3}$$

where $\alpha > 1$ is the supremum of values for which the series of Equation (2.2) converges. The convergence is absolute if and only if $|z| < 1/\alpha < 1$. If Γ has polynomial growth of degree d, then

$$C_1 \sum_{n=0}^{\infty} n^d z^n \leq \sum_{n=0}^{\infty} |B_n(x)|\, z^n \leq C_2 \sum_{n=0}^{\infty} n^d z^n.$$

By the "ratio test," the first and third series converge if and only if $|z| < 1$. These computations yield the following.

Proposition 2.11. *Let R_B denote the radius of convergence of the generating function of Equation* (2.2)*. Then $R_B < 1$ if and only if Γ has exponential growth, and $R_B = 1$ if and only if Γ has polynomial growth. Moreover, R_B is independent of the vertex x about which $|B_n(x)|$ is determined.*

It will be seen in the next subsection (see Theorem 2.16) that the value of R_B is independent of the choice of the root vertex x.

It is well known (for example, see [9]) that there exist exactly eleven face-homogeneous Euclidean tessellations, namely the Laves nets. Their valence sequences $[p_0, \dots, p_{k-1}]$

correspond to integer solutions of the equation

$$\sum_{i=0}^{k-1} \frac{1}{p_i} = \frac{k-2}{2}.$$

A necessary condition for the existence of a face-homogeneous hyperbolic tessellation with valence sequence $[p_0, \dots, p_{k-1}]$ is that the inequality

$$\sum_{i=0}^{k-1} \frac{1}{p_i} < \frac{k-2}{2} \tag{2.4}$$

hold. This condition is not sufficient, because as we have seen, not every such integer solution of the inequality (2.4) is realizable as a valence sequence.

Definition 2.12. The *angle excess* of a cyclic sequence $\sigma = [p_0, \dots, p_{k-1}]$ is given by

$$\eta(\sigma) = \left(\sum_{i=0}^{k-1} \frac{p_i - 2}{p_i} \right) - 2.$$

Motivation for this definition comes from Descartes' notion of angular defect in the Euclidean plane. When $\eta(\sigma) > 0$, there are too many faces incident at a vertex for the faces to be regular k-gons in the Euclidean plane.

Proposition 2.13. *For a cyclic sequence $\sigma = [p_0, \dots, p_{k-1}]$, inequality* (2.4) *is equivalent to*

$$\eta(\sigma) > 0 \tag{2.5}$$

and is a necessary condition for σ to be a valence sequence of a face-homogeneous hyperbolic tessellation.

Angle excess provides a quick gauge of the growth behavior of a tessellation with valence sequence σ. If $\eta(\sigma) < 0$, the tessellation is finite. If $\eta(\sigma) = 0$, the tessellation is one of the Laves nets and has polynomial growth of degree 2. If $\eta(\sigma) > 0$, the tessellation has exponential growth. Additionally, we have the following comparison result.

Proposition 2.14. *Let σ_1 and σ_2 be cyclic sequences that are comparable in the partial order. Then $\sigma_1 < \sigma_2$ if and only if $\eta(\sigma_1) < \eta(\sigma_2)$.*

Proof. Suppose that $\sigma_1 < \sigma_2$, where $\sigma_1 = [p_0, \dots, p_{k-1}]$ and $\sigma_2 = [q_0, \dots, q_{\ell-1}]$. By definition there exist $q_{i_0}, \dots, q_{i_{k-1}}$ with $p_j \le q_{i_j}$ for all $j = 0, \dots k-1$. So

$$\eta(\sigma_1) = \sum_{j=0}^{k-1} \frac{p_j - 2}{p_j} \le \sum_{j=0}^{k-1} \frac{q_{i_j} - 2}{q_{i_j}} \le \sum_{i=0}^{\ell-1} \frac{q_i - 2}{q_i} = \eta(\sigma_2). \tag{2.6}$$

If $k = \ell$, then $p_j < q_{i_j}$ for some j and the first inequality in (2.6) is strict. If $k < \ell$, the second inequality in (2.6) is strict. Since $\sigma_1 \neq \sigma_2$, at least one such strict inequality must hold. $\square$

2.5 Growth formulas

In Definition 2.10, the standard graph-theoretical metric was used to define polynomial and exponential growth of a connected graph. However, to measure growth rates of tessellations, it is more convenient to use the notion of "regional distance;" we will count the number of graph objects in the n^{th} corona of a Bilinski diagram centered at a given vertex, and our working definition of "growth rate" will be the following.

Definition 2.15. Let T be a tessellation labeled as a Bilinski diagram rooted at a vertex x. Let R be the radius of convergence of the power series

$$\varphi_x(z) = \sum_{i=1}^{\infty} |F_i| z^i. \tag{2.7}$$

When $0 < R < \infty$, we define the *growth rate* of T (with respect to x) to be $\gamma(T) = 1/R$.

Although it was shown in [6] (see pages 3–4) that, for any connected planar map with bounded covalences, the above definition of growth rate is equivalent to the growth rate with respect to the standard graph-theoretic metric, we need to show that said growth rate is independent of the root of the Bilinski diagram in question.

Theorem 2.16. *The growth rate $\gamma(T)$ of a face-homogeneous tessellation T computed by means of a Bilinski diagram is invariant under the choice of the root of the diagram.*

Proof. Choose an arbitrary vertex x of T and consider a Bilinski diagram rooted at x. Recall that the sequences $\{U_n(x) : 0 \leq n \in \mathbb{Z}\}$ and $\{F_n(x) : 1 \leq n \in \mathbb{Z}\}$ constitute the conventional labeling of T as a Bilinski diagram with root vertex x. As T is face-homogeneous, all faces are k-covalent for some $k \geq 3$. Hence for any $n \geq 1$ and any vertex $v \in U_{n+1}(x)$ there exists a vertex $u \in U_n$ such that $d(u, v) \leq \left\lfloor \frac{k}{2} \right\rfloor$. By induction on n, we obtain $d(x, v) \leq (n + 1) \left\lfloor \frac{k}{2} \right\rfloor$, yielding

$$\bigcup_{i=0}^{n} U_i(x) \subseteq B_{n\lfloor k/2 \rfloor}(x) \tag{2.8}$$

and similarly,

$$B_n(x) \subseteq \bigcup_{i=0}^{n} U_i(x). \tag{2.9}$$

In addition to the power series $\varphi_x(z)$ of Definition 2.15 with radius of convergence R_F, we require the power series $\upsilon_x(z) = \sum |U_n(x)| \, z^n$ with radius of convergence R_U. Writing

$$\Upsilon_x(z) = \frac{\upsilon_x(z)}{1 - z} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} |U_i(x)| \right) z^n = \sum_{n=0}^{\infty} \left| \bigcup_{i=0}^{n} U_i(x) \right| z^n,$$

we have that the radius of convergence of $\Upsilon_x(z)$ equals $\min\{R_U, 1\} \leq R_B$ by Equation (2.8) (where R_B is as in Proposition 2.11). But similarly by Equation (2.9) we have that $R_B \leq \min\{R_U, 1\}$. Hence the radii of convergence of $\Upsilon_x(z)$ and $\beta_x(z)$ are equal, for any choice of root vertex x.

If p is the maximum valence of the vertices in T, each vertex is also incident with at most p faces, while each face is incident with k vertices, giving

$$|U_n(x)| \leq k\,|F_{n+1}(x)| \leq pk\,|U_{n+1}(x)|$$

for each $n \geq 0$, or equivalently,

$$\frac{1}{k}\,|U_n(x)| \leq |F_{n+1}(x)| \leq p\,|U_{n+1}(x)|\,.$$

Hence the radii of convergence of $v_x(z)$ and $\varphi_x(z)$ are equal, and more importantly, $R_F = R_B$; that is, the rate of ball-growth equals the rate of growth when the Bilinski diagram is labeled from a vertex x.

Finally, it follows from Proposition 2.11 that ball-growth rates computed about distinct vertices are asymptotically equal in locally finite, connected, infinite graphs. Hence the radii of convergence of $\varphi_x(z)$, $\beta_x(z)$, $\beta_y(z)$, and $\varphi_y(z)$ are equal for all $x, y \in V$. That is to say, the growth rate of the graph is independent of the choice of root vertex. □

Notation. The subscript on the symbol φ of Definition 2.15 has now been shown to be superfluous and will henceforth be suppressed.

Consider the function $\tau\colon \mathbb{N}_0 \to \mathbb{N}_0$, (where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) given by

$$\tau(n) = \sum_{i=1}^{n} |F_i|.$$

The quantity

$$\lim_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} \tag{2.10}$$

was the definition of the growth rate of a face-homogeneous tessellation used by Moran [10] provided that this limit exists, in which case she called the tessellation *balanced*. Moran's limit fails to converge only when there exist subsequences of the sequence $\left\{\frac{\tau(n+1)}{\tau(n)}\right\}_{n=1}^{\infty}$ with distinct limits.

The following proposition shows that the parameters of a face-homogeneous tessellation determine an upper bound for the limit in Equation (2.10).

Theorem 2.17. *Let T be a face-homogeneous tessellation with valence sequence $[p_0, \ldots, p_{k-1}]$, labeled as a Bilinski diagram. Then*

$$\limsup_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} \leq 1 + \sum_{i=0}^{k-1} p_i - 2k < \infty.$$

Proof. By hypothesis, each face of the tessellation shares an incident vertex with exactly

$$\sum_{i=0}^{k-1} (p_i - 2) = \sum_{i=0}^{k-1} p_i - 2k$$

other faces. So for $n > 0$,

$$|F_{n+1}| \leq |F_n| \left(\sum_{i=0}^{k-1} p_i - 2k \right),$$

which in turn gives that for all $n > 0$,

$$\frac{\tau(n+1)}{\tau(n)} \leq 1 + \frac{|F_n|}{\sum_{i=0}^{n} |F_i|} \left(\sum_{i=0}^{k-1} p_i - 2k \right)$$
$$\leq 1 + \sum_{i=0}^{k-1} p_i - 2k < \infty,$$

since T is locally finite. □

By the "ratio test" of elementary calculus, the above proof implies that in the case of a "balanced" tessellation, Moran's definition of growth rate concurs with Definition 2.15, and

$$\frac{1}{R} = \limsup_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} = \lim_{n\to\infty} \frac{\tau(n+1)}{\tau(n)}.$$

The definition of growth rate in terms of the radius of convergence of a power series also allows us to prove the following result, which is essential in many comparisons of growth rates of various tessellations.

Lemma 2.18 (Comparison Lemma). *Let T_1 and T_2 be tessellations, and for $i = 1, 2$ let $|F_{i,n}|$ be the number of faces in the n^{th} corona of a Bilinski diagram of T_i. Suppose that for some $N \in \mathbb{N}$, we have $|F_{1,n}| \leq |F_{2,n}|$ for all $n \geq N$. Then $\gamma(T_1) \leq \gamma(T_2)$.*

Proof. Let

$$\phi_1(z) = \sum_{n=0}^{\infty} |F_{1,n}| z^n, \quad \phi_2(z) = \sum_{n=0}^{\infty} |F_{2,n}| z^n,$$

and for $i \in \{1, 2\}$, let R_i be the radius of convergence of $\phi_i(z)$ about 0. Then since $|F_{1,n}| \leq |F_{2,n}|$ for sufficiently large n, and

$$\limsup_{n\to\infty} \sqrt[n]{|F_{i,n}|} = \frac{1}{R_i} = \gamma(T_i),$$

we have $\gamma(T_1) \leq \gamma(T_2)$. □

2.6 The edge-homogeneous case

We conclude our presentation of preliminary material with a quick review of what is known about growth rates of edge-homogeneous tessellations, as this case has been completely resolved and its consequences turn out to be useful here and there in attacking the present problem. The point of departure here is the following classification theorem of Grünbaum and Shephard. (Edge-symbols were defined in Section 1.)

Proposition 2.19 ([8, Theorem 1]). *Let $p, q, k, \ell \geq 3$ be integers. There exists an edge-homogeneous, 3-connected, finite or 1-ended map with edge-symbol $\langle p, q; k, \ell \rangle$ if and only if exactly one of the following holds:*

1. *all of p, q, k, ℓ are even;*
2. *$k = \ell$ is even and at least one of p, q is odd;*
3. *$p = q$ is even and at least one of k, ℓ is odd;*

4. *$p = q$, $k = \ell$, and all are odd.*

Such a tessellation is edge-transitive, and the parameters p, q, k, ℓ determine the tessellation uniquely up to homeomorphism of the plane. If $p = q$, then the tessellation is vertex-transitive. If $k = \ell$, then it is face-transitive.

Following up on the Grünbaum-Shephard result, the authors together with T. Pisanski completely determined the growth rates of all edge-homogeneous tessellations. Their main result is the following.

Proposition 2.20 ([6, Theorem 4.1])**.** *Let the function*

$$g\colon \{t \in \mathbb{N} : t \geq 4\} \to [1, \infty)$$

be given by

$$g(t) = \frac{1}{2}\left(t - 2 + \sqrt{(t-2)^2 - 4}\right). \tag{2.11}$$

Let T be an edge-homogeneous tessellation with edge-symbol $\langle p, q; k, \ell\rangle$, and let

$$t = \left(\frac{p+q}{2} - 2\right)\left(\frac{k+\ell}{2} - 2\right). \tag{2.12}$$

Then exactly one of the following holds:

1. *the growth rate of T is $\gamma(T) = g(t)$; or*
2. *the edge-symbol of T or its planar dual is $\langle 3, p; 4, 4\rangle$ with $p \geq 6$, and the growth rate of T is $\gamma(T) = g(t-1)$.*

Observe that each value of $t \geq 4$ corresponds to only finitely many edge-homogeneous tessellations and that pairs of planar duals correspond to the same value of t. As the growth rates of edge-homogeneous tessellations are determined by an increasing function in one variable, the following is immediate.

Corollary 2.21. *The least growth rate of an edge-homogeneous hyperbolic tessellation is $(3 + \sqrt{5})/2$. This value is attained only by the tessellations with edge-symbols $\langle 3, 3; 7, 7\rangle$, $\langle 4, 4; 4, 5\rangle$, $\langle 3, 7; 4, 4\rangle$, and their planar duals.*

Remark 2.22. It is evident from Proposition 2.19 that if a tessellation is both edge- and face-homogeneous, then its edge-symbol and valence sequence have, respectively, either the forms $\langle p, p; k, k\rangle$ and $[p, p, \ldots, p]$ or the forms $\langle p, q; k, k\rangle$ and $[p, q, \ldots, p, q]$, the latter pair being possible only when k is even.

We mention that, by an argument similar to the proof of Theorem 2.17, one easily obtains the following upper bound for the growth rate of an edge-homogeneous tessellation.

Proposition 2.23. *Let T be an edge-homogeneous tessellation with edge-symbol $\langle p, q; k, \ell\rangle$. Then for any labeling of T as a Bilinski diagram, one has*

$$\lim_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} \leq 1 + \max\{pk, qk, p\ell, q\ell\}.$$

3 Accretion and monomorphic valence sequences

3.1 Accretion

Given an arbitrary face-homogeneous tessellation T with valence sequence $\sigma = [p_0, p_1, \ldots, p_{k-1}]$, we wish to apply Definition 2.15 to determine its growth rate. Letting T be labeled as a Bilinski diagram, we require a means to evaluate $|F_n|$ for all $n \in \mathbb{N}$. This is done inductively; each face $f \in F_n$ "begets" a certain number of facial "offspring" in F_{n+1}, and that number is determined by the configuration of f within $\langle F_n \rangle$, that is, what the valences are of the vertices incident with f (in the rotational order of σ) that belong, respectively, to U_{n-1} and more importantly to U_n.

A class of identically configured faces (in any corona) is a *face type*, and is denoted by $\mathbf{f}_i$ for some range of $i = 1, \ldots, r$. The benefit of using face types is that we can define an r-dimensional column vector $\vec{v}_n$, called the n^{th} *distribution vector*, which lists the frequency with which each face type occurs in the n^{th} corona. Thus, if $\vec{j}$ is the r-dimensional vector of 1s, then $|F_n| = \vec{j} \cdot \vec{v}_n$ via the standard dot product.

Figure 1 depicts a face $f \in F_n$ of some tessellation and the faces in F_{n+1} which are determined by the face type of f. These faces are called the *offspring* of f, and the figure is accordingly called the *offspring diagram* for f. As the vertex labeled p_j is incident with

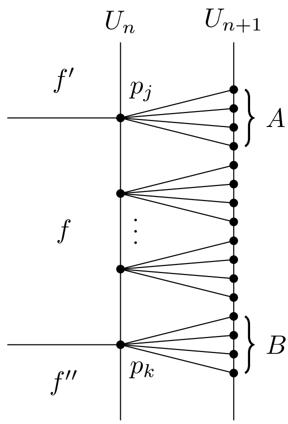

Figure 1: A face f in F_n of a tessellation T, along with the offspring of f in F_{n+1}.

both faces f and $f' \in F_n$, one-half of those faces in F_{n+1} labeled as A in the figure count as offspring of f, and one-half are counted as offspring of f'. Similarly, half of the faces labeled by B count as offspring of f and half as offspring of f''. All those faces between labels A and B in Figure 1 are wholly offspring of f. Those faces which are offspring of f, or offspring of offspring of f, and so on, are called collectively *descendants* of f.

Definition 3.1. With respect to the labeling of a Bilinski diagram, each vertex incident with a face $f \in F_n$ lies in U_{n-1} or U_n. The pattern of valences of vertices in U_{n-1} and in U_n determines the *face type* of f. The three face types occurring most routinely are called *wedges*, *bricks*, and *notched bricks*. A face f in F_n is a *wedge* if it is incident with exactly one vertex in U_{n-1}. The face f is a *brick* if it incident with exactly two adjacent vertices in U_{n-1} and at least two vertices in U_n. Finally, f is a *notched brick* if it is incident with three consecutive vertices of U_{n-1}, of which the middle vertex is 3-valent, and f is incident with two or more vertices in U_n. For a given labeling of a tessellation T as a Bilinski diagram, the face types of T are indexed $\mathbf{f}_1, \dots, \mathbf{f}_r$ for some $r \in \mathbb{N}$; we explain the method by which indices are assigned after the statement of Theorem 3.7.

An algorithm by which one can describe the faces, corona by corona, of a tessellation labeled as a Bilinski diagram is called an *accretion rule*. Often some homogeneous system of recurrence relations determines such an accretion rule. In this case, the n^{th} distribution vector $\vec{v}_n$ defined above has the property that the j^{th} component of $\vec{v}_n$ is the number of faces of type $\mathbf{f}_j$ in the n^{th} corona. We then encode the system of recurrences into a *transition matrix* M such that $\vec{v}_{n+1} = M\vec{v}_n$ holds for all $n \geq 1$. When $M = [m_{i,j}]$ is such a matrix, the entry $m_{i,j}$ is the number of faces of type $\mathbf{f}_j$ that are offspring of a face of type $\mathbf{f}_i$. We require the following result from [6].

Proposition 3.2 ([6, Theorem 3.1]). *Let T be a tessellation labeled as a Bilinski diagram with accretion rule specified by the transition matrix M and first distribution vector $\vec{v}_1$. Then the ordinary generating function for the sequence $\{|F_n|\}_{n=1}^{\infty}$ is*

$$\varphi(z) = |F_0| + z\left(\vec{j} \cdot (I - zM)^{-1}\vec{v}_1\right), \tag{3.1}$$

where I is the identity matrix and $\vec{j}$ is the vector of 1*s.*

By using Definition 2.15, we can prove the following more directly than we did in Theorem 3.4 of [6].

Theorem 3.3. *If M is the transition matrix of a tessellation T and Λ is the maximum modulus of an eigenvalue of M, then $\gamma(T) = \Lambda$.*

Proof. We can write the generating function $\varphi(z)$ of Proposition 3.2 as a rational function $u(z)/v(z)$, with $v(z)$ determined entirely by $(I - zM)^{-1}$. Specifically, using Cramer's rule where r denotes the order of M, we have

$$\begin{aligned} (I - zM)^{-1} &= \frac{1}{\det(I - zM)}\,\mathrm{adj}(I - zM) \\ &= \frac{1}{(-z)^r \det(M - \frac{1}{z}I)}\,\mathrm{adj}(I - zM) \\ &= \frac{1}{(-z)^r \chi(\frac{1}{z})}\,\mathrm{adj}(I - zM) \end{aligned} \tag{3.2}$$

where $\chi(\frac{1}{z})$ is the characteristic polynomial (in $\frac{1}{z}$) of M. Entries of the adjoint $\mathrm{adj}(I - zM)$ are polynomials in z of degree at most $r-1$, and so $v(z) = (-z)^r\chi(\frac{1}{z})$. As $\chi(\frac{1}{z})$ is a polynomial in $\frac{1}{z}$ of degree exactly r, $v(z)$ has a nonzero constant term and the roots of v occur precisely at the roots of $\chi(\frac{1}{z})$. These are precisely the reciprocals of the eigenvalues of M. Thus the minimum modulus of a pole of $\varphi(z)$ is $1/\Lambda$. As this is the definition of the radius of convergence of a power series expanded about 0, we have $\gamma(T) = \Lambda$. $\square$

3.2 Monomorphic, uniformly concentric sequences

As we have already remarked, valence sequences of face-homogeneous tessellations are unlike edge-symbols of edge-homogeneous tessellations in two significant ways: (i) the requirements for realizability of an edge-symbol are simpler and less stringent than the realizability criteria for a cyclic sequence, and (ii) two or more non-isomorphic face-homogeneous tessellations may share a common valence sequence. This latter property motivates the following definition.

Definition 3.4. Let σ be a cyclic sequence. If there exists (up to isomorphism) a unique face-homogeneous tessellation with valence sequence σ, then we say that σ is *monomorphic*. If there exist at least two (non-isomorphic) tessellations with valence sequence σ, then σ is *polymorphic*.

Proposition 3.5 (Moran [10]). *All realizable cyclic sequences of length* 3 *are monomorphic.*

A second property of interest is whether a given valence sequence is uniformly concentric. These two properties thus yield four classes of valence sequences. Not surprisingly, the class most amenable to an elegant and simple accretion rule consists of those that are both monomorphic and uniformly concentric.

One can find in [13] a complete classification of cyclic sequences of length k for $3 \leq k \leq 5$ in terms of Definition 3.4 which will help us to narrow our investigation. (It is actually the equivalent dual problem that is treated in [13], and the term "covalence sequence" is used. In the present work we have opted to follow Moran [10], speaking rather in terms of "valence sequences.")

We now turn to considering the relative growth rates of tessellations with monomorphic valence sequences. The ideal condition would be to have that the partial order on cycic sequences is mirrored by the natural order on growth rates: that is, if T_1 and T_2 are tessellations with valence sequences $\sigma_1 \leq \sigma_2$, then $\gamma(T_1) \leq \gamma(T_2)$. For monomorphic, uniformly concentric valence sequences, this is precisely the case, as stated below in Theorem 3.7. In order to prove the theorem, we now demonstrate the necessary machinery via the following example, which can be readily generalized.

Example 3.6. Consider T_1 and T_2 to be face-homogeneous tessellations with monomorphic valence sequences $\sigma_1 = [4, 5, 4, 5]$ and $\sigma_2 = [4, 6, 6, 4, 5]$, respectively, both labeled as face-rooted Bilinski diagrams. Note that $\sigma_1 < \sigma_2$. We continue the convention that $F_{i,n}$ denotes the set of faces of the n^{th} corona of T_i for $i = 1, 2$. (The reader may follow Figures 2 through 7.) Starting with T_1, we construct by induction a sequence $\{T'_j : j \in \mathbb{N}\}$ of tessellations such that:

1. $T'_0 = T_1$ as a base for the induction,
2. if we denote by $F'_{j,n}$ the set of faces in the n^{th} corona of T'_j, then for each $j \in \mathbb{N}$, the unions of the first n coronas of T'_j satisfy
$$\left\langle \bigcup_{n=1}^{j} F'_{j,n} \right\rangle \cong \left\langle \bigcup_{n=1}^{j} F_{2,n} \right\rangle$$
as induced subgraphs, and
3. $|F_{1,n}| \leq \left|F'_{j,n}\right|$ for all $n \in \mathbb{N}$.

To construct T'_1 from T'_0, the valence sequence of the root face of T'_0 must change from σ_1 to σ_2. To do so, we augment the valence of a 5-valent vertex $v \in U_1$ to 6-valent and then

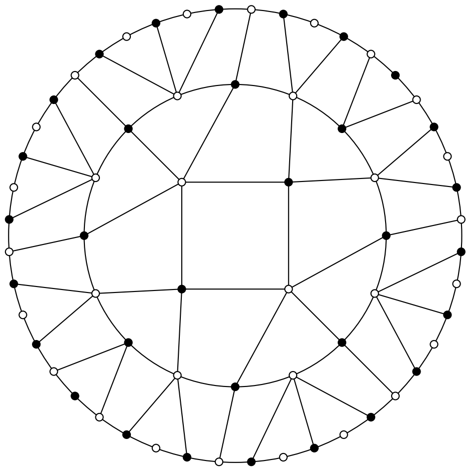

Figure 2: The first three coronas of T_1.

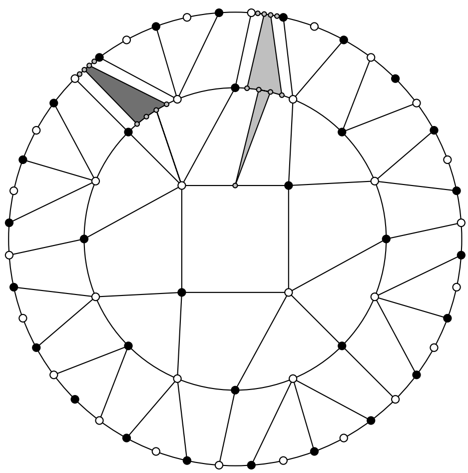

Figure 3: The first three coronas of T_1'. The dark gray region is a subgraph inserted by augmentation of the valence of a vertex from 5 to 6; the light gray region is a subgraph inserted while interpolating a 6-valent vertex along an incident edge. These insertions continue throughout all coronas of T_1'.

subdivide an edge of $\langle U_1 \rangle$ incident with v by inserting a 6-valent vertex. Augmentation and interpolation are both performed via the insertion of an infinite "cone" as follows. We choose a sequence of edges $e_2, e_3, e_4, \ldots$, with $e_i \in \langle U_i \rangle$, such that e_2 and v are incident with a common face in F_1, and for each $i \geq 2$, e_i and e_{i+1} are incident with a common face in F_i. On each of these edges we interpolate vertices, and we insert edges connecting vertices between U_i and U_{i+1} ensuring that every face so created has covalence 5. Furthermore, if a created face is incident only with interpolated vertices, then its valence sequence is σ_2. This insertion is well-defined precisely because σ_2 is monomorphic, i.e., the vertices and edges may be inserted in exactly one way.

The resulting tessellation after the procedure just described is denoted by T_1'. Faces of T_1' fall into three classes: first, there are faces which have valence sequence σ_1 and in T_0' were not incident with any part of the inserted cone; second, there are those faces with valence sequence σ_2 that have been inserted; finally, there are faces which are incident with newly inserted edges but which have neither valence sequence σ_1 nor σ_2. A face f in this third class has covalence equal to the length of σ_2, but some vertices incident with f have valences from σ_1. These faces may occur in all coronas outward from the first corona.

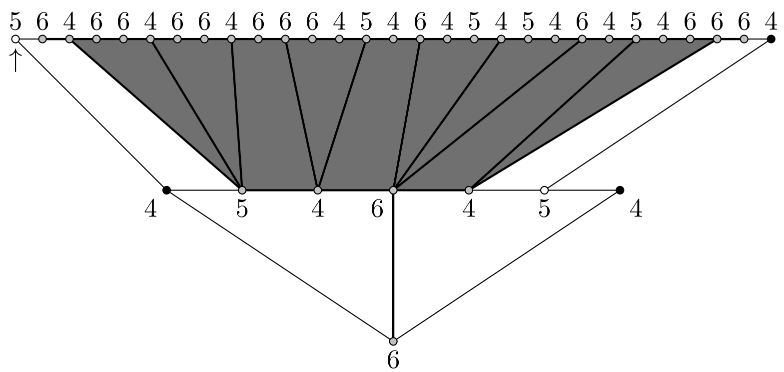

Figure 4: An expanded view of the subgraph inserted when increasing the valence of a 5-valent vertex to 6-valent. Note that the 5-valent vertex in the upper left, marked by the arrow, is disrupting the valence sequence of the white face with which it is incident; if the marked vertex were 6-valent, that face would have valence sequence $\sigma_2 = [4, 6, 6, 4, 5]$.

We compare now the tessellations T_1, T_1', and T_2. In each case, the 0^{th} corona contains only the root face. So from our construction,

$$|F_{1,0}| = \left|F_{1,0}'\right| = |F_{2,0}|\,, \text{ and for all } n \in \mathbb{N}_0,\ |F_{1,n}| \leq \left|F_{1,n}'\right|,$$

as we have inserted faces into every corona outward from the first.

We construct T_2' from T_1' just as we created T_1' from $T_0' = T_1$; there is, however, one additional type of interpolation which may occur. Specifically, a vertex must be interpolated in an edge incident with two adjacent faces in $F_{1,1}'$. In Figure 6, an example of such an edge is marked with an arrow. This obstacle proves to be minor, as the necessary interpolation is shown in Figure 7 – rather than interpolating a vertex on an edge incident with vertices in both U_1 and U_2, the vertex and its two neighbors are interpolated in U_2, replacing a $(5, 4, 5)$-path in $\langle U_2 \rangle$ with a $(5, 4, 6, 4, 5)$-path.

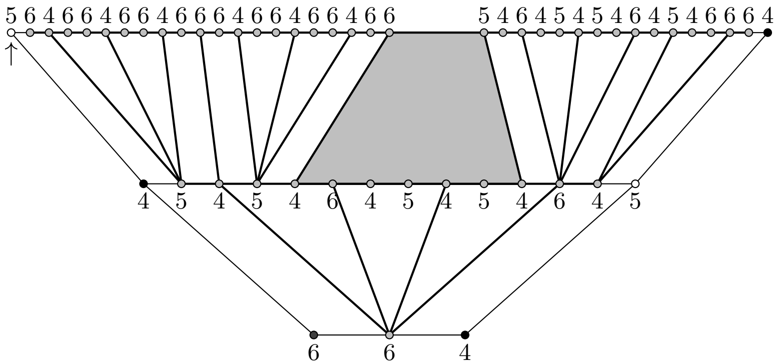

Figure 5: An expanded view of the subgraph inserted when interpolating a 6-valent vertex along an edge incident to the root. Again note the marked 5-valent vertex in the upper left. (The large shaded region represents a number of faces of valence sequence $[4, 6, 6, 4, 5]$ which are too dense to draw nicely in the Euclidean plane.)

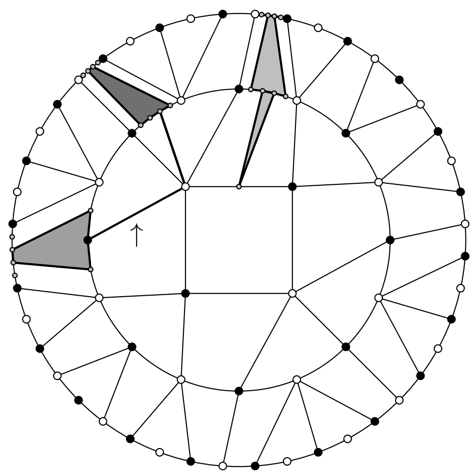

Figure 6: Beginning the construction of T'_2 from T'_1.

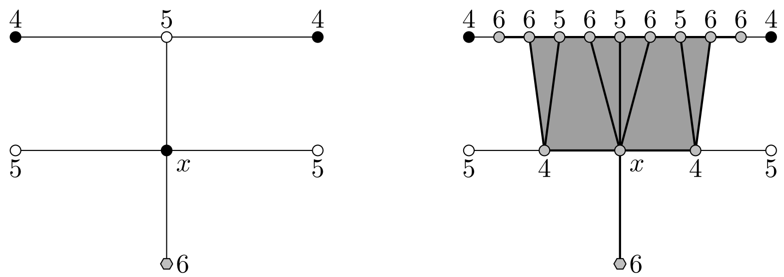

Figure 7: In the diagram to the left, the $(4, 6)$-edge at the bottom must have a 6-valent vertex interpolated, along with the attendant subgraph. However, we wish to avoid non-concentricity; hence the single 4-valent vertex x is expanded to a $(4, 6, 4)$-path as in the diagram on the right.

We continue by induction; suppose a tessellation T'_j has been created by this process. Then in the j^{th} corona, there are finitely many faces which require a finite number of vertices to have their valences increased and a finite number of edges along which we must interpolate a vertex. This creates T'_{j+1} such that

$$|F_{1,n}| \leq \left|F'_{j+1,n}\right| = |F_{2,n}|$$

for $n < j + 1$, as the first j coronas are comprised only of faces with valence sequence σ_2. Furthermore,

$$|F_{1,n}| \leq \left|F'_{j+1,n}\right|$$

for all $n \in \mathbb{N}$. In this manner we can construct an infinite sequence of tessellations, namely $\{T'_j : j \in \mathbb{N}\}$, with the properties that $|F_{1,n}| \leq \left|F'_{j,n}\right|$ for any $j, n \in \mathbb{N}_0$, and $\left|F'_{j,n}\right| = |F_{2,n}|$ whenever $j > n$.

In the previous example, we constructed the sequence in the process of transforming T_1 with valence sequence $[4, 5, 4, 5]$ into T_2 with valence sequence $[4, 6, 6, 4, 5]$; however, the process of creating $\{T'_j : j \in \mathbb{N}_0\}$ is identical in any case where T_1 and T_2 are face-homogeneous and uniformly concentric with monomorphic valence sequences σ_1 and σ_2, respectively, where $\sigma_1 < \sigma_2$. Thus by Lemma 2.18, we obtain the following result.

Theorem 3.7 (Growth Comparison Theorem)**.** *Let σ_1 and σ_2 be monomorphic valence sequences realized by tessellations $T_1, T_2 \in \mathscr{G}_{4,4} \cup \mathscr{G}_{3+,5} \cup \mathscr{G}_{3,6}$, with $\sigma_1 < \sigma_2$. Then $\gamma(T_1) \leq \gamma(T_2)$.*

Our convention is to index the face types ($\mathbf{f}_1, \ldots, \mathbf{f}_r$ for some r) in the following order: first wedges, then bricks, then notched bricks, and finally, other face types if any. A wedge in F_n with face type $\mathbf{f}_i$ is incident with a p_{i-1}-valent vertex in U_{n-1}, for $i = 1, \ldots, k$. Similarly, the indices of face types of bricks begin with a brick in F_n incident with a p_0-valent vertex and a p_{k-1}-valent vertex in U_{n-1}. A new index $\mathbf{f}_j$ is not introduced if there is some $\mathbf{f}_i$ for $i < j$ with the same configuration of vertices in U_{n-1} and U_n, up to orientation. For example, the valence sequence $[4, 6, 8, 8, 6, 4]$ yields seven face types $\mathbf{f}_1, \ldots, \mathbf{f}_7$, of which $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ are wedge types and $\mathbf{f}_4$ through $\mathbf{f}_7$ are brick types.

When a monomorphic sequence $[p_0, \ldots, p_{k-1}]$ is realized by a tessellation in $\mathscr{G}_{4,4} \cup \mathscr{G}_{3+,5} \cup \mathscr{G}_{3,6}$, then every face, with respect to *any* Bilinski diagram, can be only a wedge, a brick, or a notched brick. The indexing of face types when $p_i \neq p_j$ for $i \neq j$ allows a stricter labeling which we can use in several other cases. A face f in F_n is a wedge of type $\mathbf{w}_i$ when the vertex incident with f in U_{n-1} corresponds to valence p_{i-1} in σ. If instead f is a brick with incident vertices in U_{n-1} corresponding to valences p_{i-1} and p_{i-2} (indices here taken modulo k), then f has face type $\mathbf{b}_i$. Finally, if f a notched brick whose incident vertices in U_{n-1} have valences p_i, $p_{i-1} = 3$, and p_{i-2}, then f has face type $\mathbf{n}_i$. It is important to note that if $p_{i-1} \neq 3$, then faces of type $\mathbf{n}_i$ never occur as offspring. This stricter labeling is used explicitly only for the few theorems which follow, by which we determine the number of offspring of each instance of these general face types. Furthermore, we demonstrate a first application of the accretion rules and half-counting of faces that were introduced in Section 3.1.

Notation. Let T be a face-homogeneous tessellation with valence sequence σ, labeled as a Bilinski diagram. We denote by $\Omega(\mathbf{f})$ the number of faces in F_{n+1} that are counted as offspring of a single face of face type $\mathbf{f}$ in F_n, for any $n > 0$. For $T \in \mathscr{G}_{4,4} \cup \mathscr{G}_{3+,5} \cup \mathscr{G}_{3,6}$ we let $\Omega(\mathbf{w}_i)$, $\Omega(\mathbf{b}_i)$, and $\Omega(\mathbf{n}_i)$ denote the number of offspring of a single wedge, brick, or notched brick of, respectively, of the given type.

Lemma 3.8. *For a face-homogeneous tessellation in $\mathscr{G}_{4,4} \cup \mathscr{G}_{3+,5} \cup \mathscr{G}_{3,6}$ with monomorphic valence sequence $\sigma = [p_0, \ldots, p_{k-1}]$, one has for $i \in \{1, \ldots, k\}$,*

$$\Omega(\mathbf{w}_i) = \frac{p_{i-2} + p_i}{2} - 2k + 3 + \sum_{j \notin I_1} p_j, \text{ and} \tag{3.3}$$

$$\Omega(\mathbf{b}_i) = \frac{p_{i-3} + p_i}{2} - 2k + 5 + \sum_{j \notin I_2} p_j, \tag{3.4}$$

where $I_1 = \{i-2, i-1, i\}$ and $I_2 = \{i-3, i-2, i-1, i\}$. Also, when $p_{i-1} = 3$,

$$\Omega(\mathbf{n}_i) = \frac{p_{i-3} + p_{i+1}}{2} - 2k + 7 + \sum_{j \notin I_3} p_j \tag{3.5}$$

with $I_3 = \{i-3, i-2, i-1, i, i+1\}$.

Proof. The reader is referred to the three offspring diagrams shown in Figure 8.

Letting $i \in \{1, \ldots, k\}$, the first diagram applies when $p_{i-1} \geq 4$. If also $p_{i-2}, p_i \geq 4$ as in the diagram, then we have

$$\begin{aligned}\Omega(\mathbf{w}_i) &= \frac{p_{i-2} - 4}{2} + \frac{p_i - 4}{2} + k - 2 + \sum_{j \notin I_1} (p_j - 3) \\ &= \frac{p_{i-2} + p_i}{2} - 2k + 3 + \sum_{j \notin I_1} p_j.\end{aligned}$$

If instead $p_{i-2} = 3$, then the number of wedge offspring of $\mathbf{w}_i$ is

$$\frac{p_i - 4}{2} + \sum_{j \notin I_1} (p_j - 3),$$

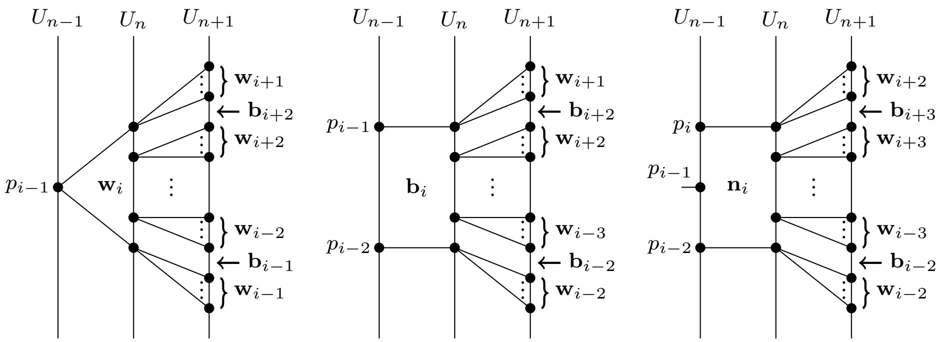

Figure 8: Offspring diagrams for the three general face types (respectively wedges, bricks, and notched bricks) of a tessellation with monomorphic, uniformly concentric valence sequence $[p_0, \ldots, p_{k-1}]$.

the number of brick offspring is $k-3$, and the number of notched brick offspring is $\frac{1}{2}$. Thus when $p_{i-2} = 3$,

$$\begin{aligned}\Omega(\mathbf{w}_i) &= \frac{1}{2} + \frac{p_i - 4}{2} + k - 3 + \sum_{j \notin I_1} (p_j - 3) \\ &= -\frac{1}{2} + \frac{p_i - 4}{2} + k - 2 + \sum_{j \notin I_1} (p_j - 3) \\ &= \frac{p_{i-2} - 4}{2} + \frac{p_i - 4}{2} + k - 2 + \sum_{j \notin I_1} (p_j - 3)\end{aligned}$$

as before; likewise when $p_i = 3$. Analogous arguments hold for the offspring of bricks and notched bricks. □

The process of establishing an accretion rule and accompanying transition matrices is considerably simplified for tessellations in $\mathscr{G}_{4,4}$ by virtue of the absence of notched bricks. By applying the following lemma and Theorem 3.3, one can then compute the growth rate explicitly of any monomorphic valence sequence realizable in $\mathscr{G}_{4,4}$. Recall that by Proposition 2.3, all such valence sequences are uniformly concentric.

Lemma 3.9. *Let $[p_0, \ldots, p_{k-1}]$ be the monomorphic valence sequence for a tessellation $T \in \mathscr{G}_{4,4}$. Then T has an accretion rule which admits the block transition matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$, and $D = (d_{i,j})$ given by

$$a_{i,j} = \begin{cases} 0 & j - i = 0 \\ \frac{1}{2}(p_{i-1} - 4) & j - i \in \{1, k-1\} \pmod{k} \\ p_{i-1} - 3 & \textit{otherwise}, \end{cases} \tag{3.6}$$

$$b_{i,j} = \begin{cases} 0 & j-i \in \{0,1\} \pmod{k} \\ \frac{1}{2}(p_{i-1}-4) & j-i \in \{2,k-1\} \pmod{k} \\ p_{i-1}-3 & \textit{otherwise}, \end{cases} \tag{3.7}$$

$$c_{i,j} = \begin{cases} 0 & j-i \in \{0,1\} \pmod{k} \\ 1 & \textit{otherwise}, \end{cases} \tag{3.8}$$

$$d_{i,j} = \begin{cases} 0 & j-i \in \{0,1,k-1\} \pmod{k} \\ 1 & \textit{otherwise}, \end{cases} \tag{3.9}$$

for $i, j \in \{1, \ldots, k\}$.

Proof. Since all general face types are wedges or bricks, we need demonstrate only that the entries $a_{i,j}$ and $c_{i,j}$ correspond to numbers of offspring of the k face types in wedge configurations and that the entries $b_{i,j}$ and $d_{i,j}$ correspond to numbers of offspring of the k face types in brick configurations.

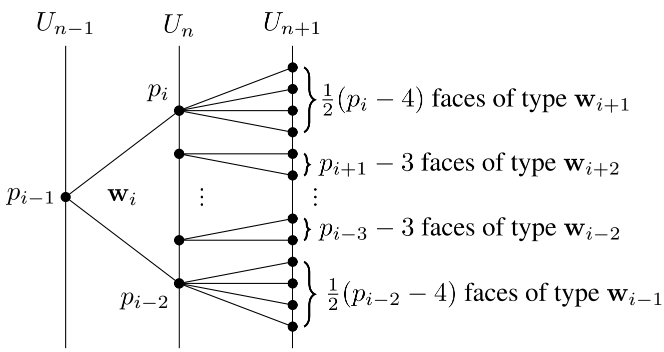

Figure 9: Offspring of a $\mathbf{w}_i$ face in a tessellation $T \in \mathscr{G}_{4,4}$, where $i \in \{1, \ldots, k\}$.

The offspring of wedges of type $\mathbf{w}_i$ are shown in Figure 9, and the offspring of a brick of type $\mathbf{b}_i$ is shown in Figure 10. The ordering of face types is $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{b}_1, \ldots, \mathbf{b}_k$.

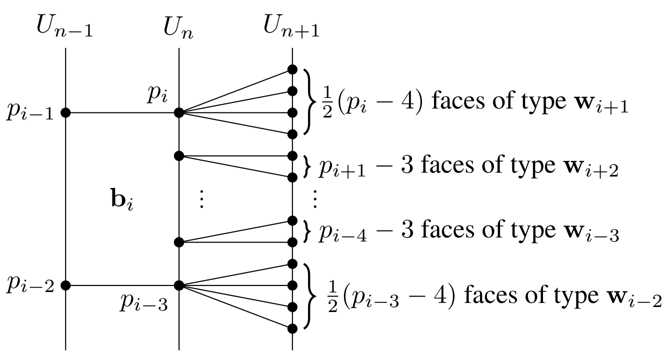

Figure 10: Offspring of a $\mathbf{b}_i$ face in a tessellation $T \in \mathscr{G}_{4,4}$, where $i \in \{1, \ldots, k\}$.

Recalling that the (i,j)-entry of a transition matrix M is the number of faces of the i^{th}

indexed type which are produced in F_{n+1} as offspring of a face of the j^{th} indexed type in F_n, it is straightforward to verify from these two offspring diagrams that the entries of M are correct. $\square$

Remark 3.10. We emphasize the breadth of this class of monomorphic, uniformly concentric valence sequences. In addition to the many monomorphic face-homogeneous tessellations in $\mathscr{G}_{3,6} \cup \mathscr{G}_{3+,5} \cup \mathscr{G}_{4,4}$, there are many with covalence 3 (cf. Proposition 3.5). By Proposition 2.19, all edge-transitive tessellations of constant covalence are included, except for those of the with valence sequence $[3, p, 3, p]$ (edge-symbol $\langle 3, p; 4, 4\rangle$), as they are not uniformly concentric. By Proposition 2.3, a k-covalent tessellation T is uniformly concentric whenever $k \geq 6$. If $k \geq 7$ and if σ is monomorphic, then $\sigma \geq [3, 3, 3, 3, 3, 3, 3]$. In that case, Theorem 3.7 and Proposition 2.20 tell us that σ has growth rate at least $\gamma([3, 3, 3, 3, 3, 3, 3]) = \frac{1}{2}(3 + \sqrt{5})$.

3.3 Monomorphic non-concentric sequences

The purpose of this section is to characterize the six forms of monomorphic, non-concentric valence sequences with positive angle excess. These sequences give rise to face types other than wedges, bricks, and notched bricks, and so the foregoing methods cannot be applied to compute their growth rates.

An interesting situation arises when a tessellation is not *uniformly* concentric but nonetheless, by prudent selection of the root, admits *some* Bilinski diagram that *is* concentric. To illustrate this point, we examine sequences of the form $[4, p, q]$.

Example 3.11. Consider the valence sequence $\sigma = [4, p, q]$ with $4 < p < q$, where $\frac{1}{p} + \frac{1}{q} < \frac{1}{4}$, and let T be a face-homogeneous tessellation with valence sequence σ. For σ to be realizable, clearly p and q must be even. Note as well that the inequality (2.4) is satisfied. While σ is monomorphic and admits a concentric Bilinski diagram, σ is not uniformly concentric (cf. the second case of Proposition 2.4).

When a Bilinski diagram of T admits a 4-valent vertex $v_0 \in U_n$ (for some n) adjacent to the vertices $u_1, u_2 \in U_{n-1}$ and $v_1, v_2 \in U_n$, then the diagram is not concentric; the vertices v_1 and v_2 must also be adjacent, as T is 3-covalent. Hence $\langle\{v_0, v_1, v_2\}\rangle$ is a cycle within $\langle U_n\rangle$, causing the Bilinski diagram to be non-concentric. However, it is possible to avoid this configuration by choosing the root of the Bilinski diagram to be either a p-valent or a q-valent vertex. When so labeled, only four face types occur, as demonstrated by the offspring diagrams in Figure 11.

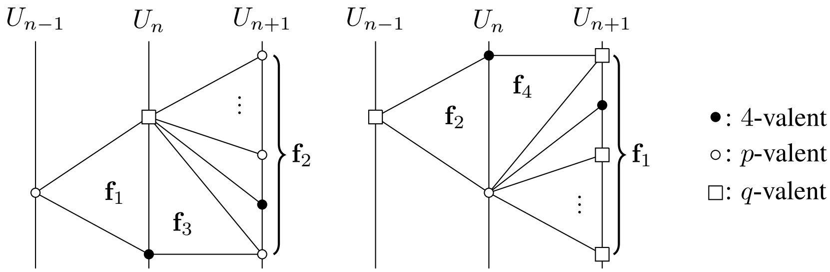

Figure 11: Offspring diagrams for a concentric tessellation with valence sequence $[4, p, q]$.

One sees here that if the root is taken to be a p-valent vertex, the first corona consists entirely of faces of type $\mathbf{f}_1$, which produce in turn only offspring of types $\mathbf{f}_2$ and $\mathbf{f}_3$. Similarly, given a q-valent root, the first corona consists entirely of faces of type $\mathbf{f}_2$, which produce in turn only offspring of types $\mathbf{f}_1$ and $\mathbf{f}_4$. The non-concentric configuration described above can never be produced among the descendants of faces of types $\mathbf{f}_1$ or $\mathbf{f}_2$.

Inspection of Figure 11 gives the first and second columns of the transition matrix M; the third and fourth columns, corresponding to $\mathbf{f}_3$ and $\mathbf{f}_4$, merit further explanation. A face of type $\mathbf{f}_3$ in F_{n+1} has a p-valent vertex in U_{n+1}; this vertex is incident with $p-5$ faces of type $\mathbf{f}_1$ in F_{n+2}. So the behavior of a face of type $\mathbf{f}_3$ is effectively to collapse one of the faces in U_{n+2} of type $\mathbf{f}_1$ begotten by the adjacent face of type $\mathbf{f}_2$. Faces of type $\mathbf{f}_4$ behave similarly, collapsing a face of type $\mathbf{f}_2$. These considerations give us

$$M = \begin{bmatrix} 0 & \frac{1}{2}(p-4) & -1 & 0 \\ \frac{1}{2}(q-4) & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

as the transition matrix M for this accretion rule for T. As the characteristic equation for M is of degree 4, it can be solved to determine that the maximum modulus of an eigenvalue of M is

$$\Lambda = \frac{1}{4}\sqrt{[2(p-4)(q-4)-16] + 2\sqrt{(p-4)^2(q-4)^2 - 16(p-4)(q-4)}}.$$

By Theorem 3.3 and Theorem 2.16, Λ is the growth rate of T. This quantity can be minimized by minimizing pq subject to the initial conditions $\frac{1}{p}+\frac{1}{q}<\frac{1}{4}$ and that p and q be even. We shall see in Section 3.4 the role played by this example.

Growth rate formulas for each of the other five classes are derived in the Appendix.

Theorem 3.12. *Let σ be a valence sequence such that $\eta(\sigma)>0$. Then σ is both monomorphic and non-concentric if and only if σ is of one of the following six forms:*

- *(i)* $[3,p,p]$, *with* $p \geq 14$ *and even;*
- *(ii)* $[4,p,q]$, *with p and q both even,* $4<p<q$, *and* $\frac{1}{p}+\frac{1}{q}<\frac{1}{4}$*;*
- *(iii)* $[3,p,3,p]$, *with* $p\geq 7$*;*
- *(iv)* $[3,p,4,p]$, *with* $p \geq 5$ *and even;*
- *(v)* $[3,3,p,3,p]$, *with* $p\geq 5$*; or*
- *(vi)* $[3,3,p,3,q]$, *with* $p,q\geq 4$ *and* $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$.

Proof. The parity conditions and the inequalities bounding the parameters in each case are minimal such that σ be indeed realizable as a tessellation with $\eta(\sigma)>0$.

As noted in Remark 3.10, all valence sequences of length at least 6 are uniformly concentric. Furthermore, by Proposition 3.5, all valence sequences of length 3 are monomorphic. Valence sequences $[3,p,p]$, $[4,p,q]$, and $[3,p,3,p]$ give rise to tessellations exemplifying cases 1, 2, and 4 respectively of Proposition 2.4, and hence cannot be uniformly concentric. As a face-homogeneous tessellation with valence sequence $[3,p,3,p]$ is also edge-transitive, the sequence must be monomorphic. The proof that the sequence $[3,p,4,p]$

must be monomorphic and uniformly non-concentric is given in the Appendix, where the growth rate of a corresponding tessellation is determined.

We now prove that $[3, 3, p, 3, p]$ is monomorphic for all $p \geq 5$. As a 3-valent vertex is incident with a common face with any two of its neighbors, every 3-valent vertex must be adjacent to at least two p-valent vertices; otherwise some face would be incident with a $(3, 3, 3)$-path. Consider a p-valent vertex v_1. By face-homogeneity, v_1 is adjacent to some 3-valent vertex u_1, with u_1 adjacent in turn to a 3-valent vertex u_2 which is not adjacent to v_0. But then the other vertex adjacent to u_1 must be a p-valent vertex v_2. This forces the pattern of valences at regional distance 1 from v_1 to be $(3, 3, p, \ldots, 3, 3, p)$; as v_1 was arbitrary, this must be the pattern of valences at regional distance 1 from any p-valent vertex. As every vertex is at regional distance 1 from some p-valent vertex, $[3, 3, p, 3, p]$ must be monomorphic; the first two coronas of a tessellation with this valence sequence rooted at a p-valent vertex is depicted in Figure 12. Furthermore, this local configuration to a p-valent vertex forces the local behaviors to a $(3, 3)$-edge and a 3-valent vertex shown in Figure 13. Hence when a 3-valent vertex v_0 is taken as the root of the Bilinski diagram of a tessellation with valence sequence $[3, 3, p, 3, p]$, a pendant vertex occurs in $\langle U_3 \rangle$. This is shown in Figure 14. So $[3, 3, p, 3, p]$ is monomorphic but not uniformly concentric; the argument for $[3, 3, p, 3, q]$ is analogous.

We have shown these six forms to be both monomorphic and non-concentric; that these are the only such valence sequences is proved via the exhaustive examination of cases in the Appendix. □

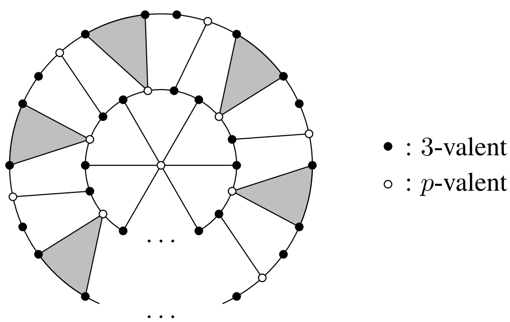

Figure 12: The first two coronas of a tessellation with valence sequence $[3, 3, p, 3, p]$ rooted at a p-valent vertex. Each shaded region indicates $p - 3$ faces in F_2 all having the same face type.

3.4 The main result

The following theorem establishes the so-called "golden mean" as the least rate of exponential growth for face-homogeneous tessellations with monomorphic valence sequences.

Theorem 3.13 (Least Exponential Growth Rate of Monomorphic Valence Sequences)**.** *The least growth rate of a face-homogeneous tessellation with monomorphic valence sequence σ such that $\eta(\sigma) > 0$ is $\frac{1}{2}(1 + \sqrt{5})$ and is attained by exactly the tessellations with valence sequences $[4, 6, 14]$ and $[3, 4, 7, 4]$.*

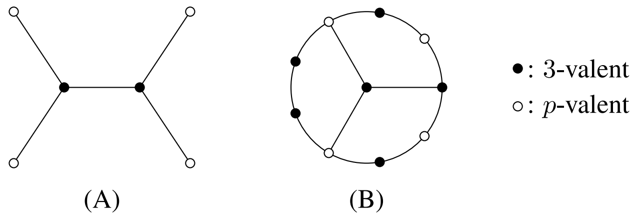

Figure 13: (A) Local configuration along an edge with edge-symbol $\langle 3,3;5,5\rangle$ in a face-homogeneous tessellation with valence sequence $[3,3,p,3,p]$. (B) Local configuration in the same tessellation when rooted at a 3-valent vertex.

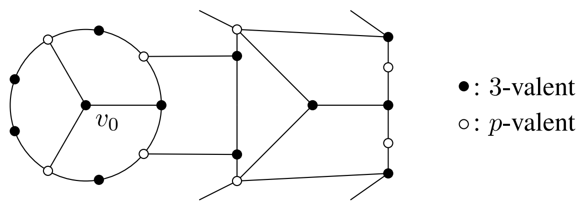

Figure 14: Non-concentricity of $[3,3,p,3,p]$ when rooted at a 3-valent vertex v_0.

Table 1: Table of the least exponential growth rate within each monomorphic class of valence sequences. All rates of growth have been truncated at four decimal places rather than being rounded.

Class	σ	$\gamma(T_\sigma) \approx$	Class	σ	$\gamma(T_\sigma) \approx$
$[p,p,p]$	$[7,7,7]$	2.6180	$[p,p,q,r,q]$	$[4,4,6,5,6]$	6.6650
$[3,p,p]$	$[3,14,14]$	2.6180	$[3,p,q,q,p]$	$[3,4,6,6,4]$	4.9911
$[p,p,q]$	$[6,6,7]$	1.722	$[p,q,r,s,t]$	$[4,6,10,12,8]$	14.5753
$\mathbf{[4,p,q]}$	$\mathbf{[4,6,14]}$	**1.6180**	$[p,p,p,p,p,p]$	$[4,4,4,4,4,4]$	5.8284
$[p,q,r]$	$[6,8,10]$	3.4789	$[p,p,q,p,p,q]$	$[4,4,5,4,4,5]$	7.1347
$[p,p,p,p]$	$[5,5,5,5]$	3.7320	$[p,q,p,q,p,q]$	$[4,5,4,5,4,5]$	7.8729
$[p,p,q,q]$	$[4,4,6,6]$	3.4081	$[p,q,q,p,r,r]$	$[6,4,4,6,8,8]$	13.1291
$[3,p,3,p]$	$[3,7,3,7]$	2.6180	$[p,q,p,r,q,r]$	$[4,5,4,6,5,6]$	9.8115
$[p,q,p,q]$	$[4,5,4,5]$	2.6180	$[p,q,r,p,q,r]$	$[4,6,8,4,6,8]$	13.5612
$[3,p,4,p]$	$[3,6,4,6]$	2.9655	$[p,q,p,r,s,r]$	$[4,5,4,6,7,6]$	10.9033
$\mathbf{[3,p,q,p]}$	$\mathbf{[3,4,7,4]}$	**1.6180**	$[p,q,r,p,s,t]$	$[4,6,8,4,10,12]$	18.1174
$[p,q,p,r]$	$[4,5,4,6]$	3.1462	$[p,q,r,s,t,u]$	$[4,6,10,14,12,8]$	23.9963
$[p,q,r,s]$	$[4,6,10,8]$	7.0367	$[3,p,p,3,p,p]$	$[3,4,4,3,4,4]$	4.3306
$[p,p,p,p,p]$	$[4,4,4,4,4]$	3.7320	$[3,p,3,p,3,p]$	$[3,4,3,4,3,4]$	3.7320
$[3,3,3,3,p]$	$[3,3,3,3,7]$	1.7553	$[3,3,3,p,q,p]$	$[3,3,3,4,5,4]$	4.0265
$[3,3,3,p,p]$	$[3,3,3,6,6]$	3.0217	$[3,p,q,3,q,p]$	$[3,4,6,3,6,4]$	6.8091
$[3,3,p,3,p]$	$[3,3,5,3,5]$	2.6180	$[3,p,3,q,3,r]$	$[3,4,3,5,3,6]$	5.6723
$[3,3,p,3,q]$	$[3,3,4,3,5]$	1.9318	$[3,p,q,r,q,p]$	$[3,4,6,5,6,4]$	8.0601

Proof. With respect to the partial order on valence sequences, if a valence sequence σ has length at least 7, then $[3,3,3,3,3,3,3] \leq \sigma$. A face-homogeneous tessellation T_0 with valence sequence $[3,3,3,3,3,3,3]$ is edge-homogeneous with edge-symbol $\langle 3,3;7,7\rangle$ and so has growth rate $\gamma(T_0) = \frac{1}{2}(3+\sqrt{5})$ by Proposition 2.20. But then if $[3,3,3,3,3,3,3] < \sigma$ and T is a tessellation with monomorphic valence sequence σ, then $\gamma(T_0) \leq \gamma(T)$, by Theorem 3.7. We proceed then by exhaustion: there are only finitely many forms of valence sequences of length at most 6. The Appendix contains an exhaustive classification of realizable valence sequences as monomorphic or polymorphic. For each form of monomorphic valence sequence, the least rate of growth is either determined or bounded below. The minimum growth rate of a minimal representative of each form is listed in Table 1. Of these forms, $[4,6,14]$ and $[3,4,7,4]$ have the least rate of growth, shown to be $\frac{1}{2}(1+\sqrt{5})$ in the Appendix. $\square$

Remark 3.14. It is interesting to observe that the two tessellations realizing the minimum exponential growth rate are closely related. The face-homogeneous tessellation with valence sequence $[4,6,14]$ can be realized by the classical tiling of the hyperbolic plane by triangles with interior angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$. Moreover, a face-homogeneous tessellation with valence sequence $[3,4,7,4]$ is the subgraph of one with valence sequence $[4,6,14]$ obtained by the deletion of all edges joining 6-valent and 14-valent vertices. Many artistic renderings of these tilings exist and can be found on web sites regarding the $(2,3,7)$-triangle group, the Order-7 triangular tiling, or triangular tilings of the hyperbolic plane.

4 Polymorphic valence sequences

4.1 A sufficient condition for polymorphicity

With respect to the ordering of cyclic sequences, the least polymorphic valence sequence with positive angle excess is $[4,4,4,5]$; that is to say, every cyclic sequence σ such that $\sigma < [4,4,4,5]$ is either not realizable as a tessellation, is realizable only by a finite map or a Euclidean tessellation, or is monomorphic. While all valence sequence of length 3 are monomorphic, k-covalent polymorphic valence sequences abound for $k \geq 4$. The following theorem gives a simple sufficient condition under which a realizable valence sequence is polymorphic.

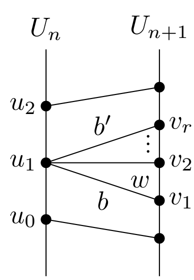

Figure 15: A configuration of faces demonstrating polymorphicity.

Proposition 4.1. *Let $\sigma = [p_0, \ldots, p_{k-1}]$ be the valence sequence of a face-homogeneous tessellation $T \in \mathscr{G}_{4,4} \cup \mathscr{G}_{3+,5}$. If there exist distinct $i, j \in \{0, \ldots, k-1\}$ such that $p_i, p_{i+1} \geq 4$ and either*

1. $p_i = p_j$, $p_{i+1} = p_{j+1}$, *and* $p_{i+2} \neq p_{j+2}$, *or*
2. $p_i = p_j$, $p_{i+1} = p_{j-1}$, *and* $p_{i+2} \neq p_{j-2}$,

then σ *is polymorphic.*

Proof. As the only two forms of valence sequences of length $k = 4$ that satisfy the hypothesis, namely $[p, p, p, q]$ and $[p, p, q, r]$, are polymorphic (see Appendix), we assume that $k \geq 5$. Also, since condition (2) is identical to (1) save for orientation within the cyclic sequence, it suffices to assume that there are distinct i, j such that (1) holds. Furthermore, we may assume $i = 0$ due to the rotational equivalence of valence sequences.

Since $k \geq 5$, there exists for some n a face in F_n incident with three consecutive vertices $u_0, u_1, u_2 \in U_n$ with valence $\rho(u_m) = p_m$ for $m = 0, 1, 2$. Let b be the brick in F_{n+1} incident with the edge u_0u_1, and let b' be the brick (or perhaps notched brick if $p_2 = 3$) in F_{n+1} incident with the edge u_1u_2. Let $v_1, \ldots, v_r$ be the vertices in U_{n+1} incident with u_1 in consecutive order, so that v_1 is incident with b and v_r is incident with b'. Thus $r = p_1 - 2 \geq 2$. If σ contains a subsequence $[q, p_1, p_2]$ with $q \neq p_0$, then $\rho(v_r)$ may equal either p_0 or q, resulting in a choice of face types for b', and we're done. Otherwise we must have $\rho(v_r) = p_0$, which forces the vertex v_{r-2} and subsequent alternate neighbors of u_1 in U_{n+1} also to be p_0-valent.

If p_1 is even, then $\rho(v_1)$ may equal either p_2 or p_{j+2} in which case the wedge $w \in F_{n+1}$ incident with vertices v_1, u_1, v_2 may be of either type $\mathbf{w}_2$ or type $\mathbf{w}_{j+2}$, and T is polymorphic. (See Figure 15.)

If p_1 is odd, then working backward as in the even case forces $\rho(v_1) = p_0$, which implies that either $p_0 = p_2$ or $p_0 = p_{j+2}$, and without loss of generality, we assume the former. Now we may assign $\rho(v_2)$ to be either p_0 or p_{j+2}, and the argument proceeds as in the even case. $\square$

The existence of polymorphic valence sequences considerably complicates the computation of growth rates of face-homogeneous tessellations. The above proof suggests that, unlike in the monomorphic case, polymorphic valence sequences may admit many different accretion rules, as we illustrate in the next section.

4.2 Two non-isomorphic tessellations with the same valence sequence

The minimal polymorphic valence sequence under the partial order on cyclic sequences, namely $[4, 4, 4, 5]$, is unfortunately not amenable to study via our methods. In fact, there is no well-defined transition matrix between coronas, and this problem is shared by all valence sequences of the form $[4, 4, 4, q]$ for $q > 4$. However, $[4, 4, 6, 8]$ provides us with the opportunity to investigate two distinct (but related) accretion rules.

The valence sequence $[4, 4, 6, 8]$ is representative of form $[p, p, q, r]$ discussed in the Appendix. As every face is incident with a pair of adjacent 4-valent vertices, every realization of this valence sequence contains a countable infinity of pairwise-disjoint double rays, each induced exclusively by 4-valent vertices. Figure 16 (A) shows a strip-like patch bordering a double ray of 4-valent vertices. To obtain Figure 16 (B) from this (or vice versa), one can fix pointwise the half-plane on one side of the double ray while translating the half-plane on the other side along one edge of the double ray.

To construct still other such (non-isomorphic) realizations, one can choose to "translate" along any one of these double rays by leaving fixed the half-plane on one side of the double ray but translating the half-plane on the other side by one edge. Since there exists

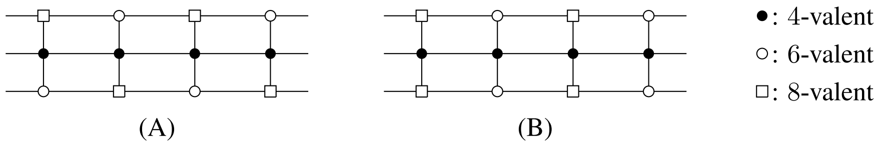

Figure 16: Two non-isomorphic patches of a tessellation with valence sequence $[4, 4, 6, 8]$, showing possible neighborhoods of double rays of 4-valent vertices.

a countable infinity of double rays along which one may choose to translate one or the other or neither of the adjacent half-planes, there exists an uncountable class of pairwise non-isomorphic tessellations that all have the same valence sequence $[4, 4, 6, 8]$.

While one might expect that all tessellations having the same valence sequence always have the same growth rate, we show that this is not so.

We begin by observing that every 4-valent vertex in a face-homogeneous tessellation with valence sequence $[4, 4, 6, 8]$ is adjacent to two other 4-valent vertices and two vertices with valences 6 or 8; thus any given 4-valent vertex either has exactly one 6-valent and one 8-valent neighbor, has two 6-valent neighbors, or has two 8-valent neighbors. Furthermore, every 4-valent vertex lies on a double ray (two-way infinite path) of 4-valent vertices; if one vertex along this path has a 6-valent neighbor and an 8-valent neighbor, then so does every other vertex along the double ray. This is the behavior demonstrated in Figure 16 (A).

If the local configuration specified in Figure 16 (A) is enforced along every double ray of 4-valent vertices, then the tessellation obtained is unique; let this tessellation be T_1. We can then construct offspring diagrams for T_1 as given in Figure 17. It is interesting to note that T_1 is the dual of the Cayley graph of the group with presentation

$$G_1 = \langle a, b, c \mid a^2 = b^2 = c^2 = (bc)^3 = (caba)^4 = 1 \rangle .$$

Encoding the offspring diagrams into a matrix, we obtain the transition matrix M_1 of T_1 given below. The four entries underlined in the matrix are the only entries which change between this example and the next example, T_2, that we construct.

$$M_1 = \begin{bmatrix} 0 & 0 & \underline{0} & \underline{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & \underline{0} & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of M_1 is

$$f_1(z) = (z-1)(z+1)\left(z^2 + 3z + 1\right)\left(z^4 - 3z^3 - 4z^2 - 3z + 1\right),$$

which in turn gives that the eigenvalue of maximum modulus of M_1 is

$$\lambda_1 = \frac{1}{4}\left(3 + \sqrt{33} + 2\sqrt{\frac{13}{2} + \frac{3\sqrt{33}}{2}}\right) \approx 4.13016.$$

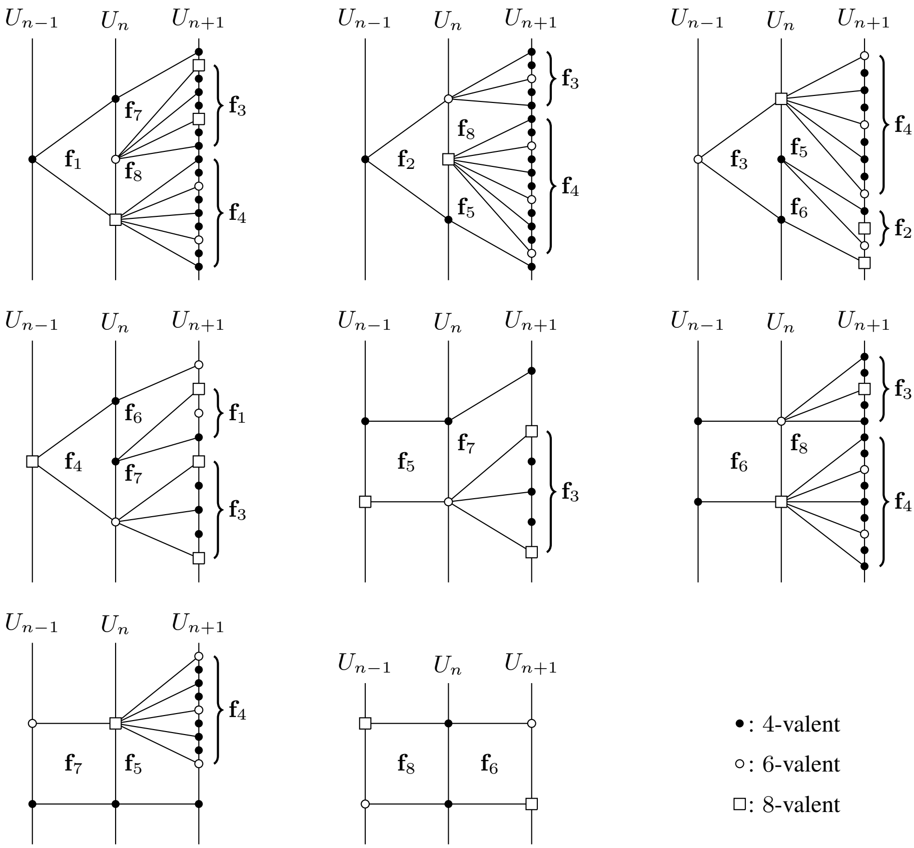

Figure 17: Offspring diagrams for T_1.

Considering again the double-rays of 4-valent vertices, it is trivial to note that if a vertex on such a double ray has two 6-valent neighbors in the tessellation, then both vertices adjacent to it in the double-ray have two 8-valent neighbors. This local behavior is shown in Figure 16 (B).

If this pattern is extended to all such double rays we obtain the tessellation T_2, which is also the dual of a Cayley graph. The underlying group of this Cayley graph is

$$G_2 = \left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (ad)^2 = (cd)^3 = (bc)^4 \right\rangle .$$

The growth behavior of T_2 differs from that of T_1 only in the offspring of faces of types $\mathbf{f}_3$ and $\mathbf{f}_4$, as shown in the offspring diagrams in Figure 18.

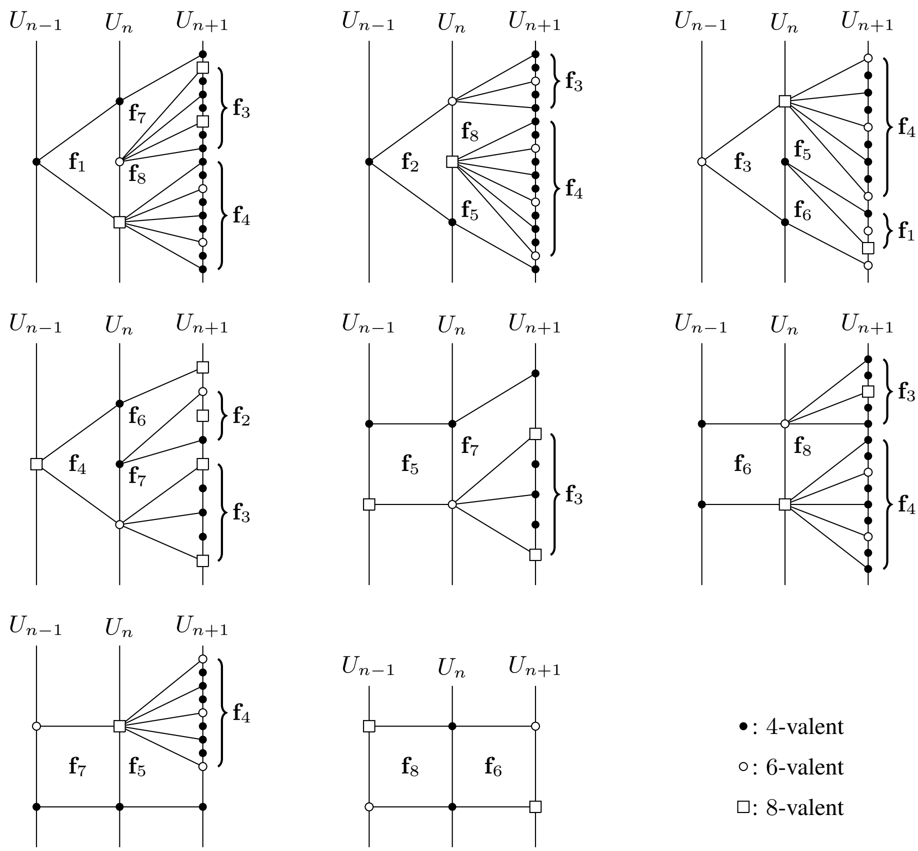

Figure 18: Offspring diagrams for T_2.

The effect of the change of offspring of types $\mathbf{f}_2$ and $\mathbf{f}_3$ in the transition matrix of T_2 lies only in the underlined 2×2 submatrix of M_1, while the remainder of the matrix M_2

remains identical to M_1. Hence we have

$$M_2 = \begin{bmatrix} 0 & 0 & \underline{1} & \underline{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{0} & \underline{1} & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of M_2 is

$$f_2(z) = (z-1)^2 \left(z^6 + 2z^5 - 15z^4 - 40z^3 - 15z^2 + 2z + 1\right).$$

As polynomials of degree 6 are unfortunately not solvable by radicals, we obtain by approximation that the root of maximum modulus is $\lambda_2 \approx 4.14659$.

As these growth rates are nearly the same, there is only a small difference in corona sizes in the first several coronas. However, the size of the coronas and distribution of face types differs greatly farther from the root. To demonstrate this, Table 2 gives corona sizes in Bilinski diagrams of T_1 and of T_2, both rooted at 4-valent vertices. Note that the sizes of the coronas of T_2 dominate those of T_1 only after the 13th corona.

4.3 Some conjectures

Ideally, all tessellations realizing the same polymorphic valence sequence would have the same growth rate. The example of valence sequence $[4, 4, 6, 8]$ illustrates that this is not so. We propose the following definitions.

Definition 4.2. Let σ be some polymorphic valence sequence, and define $\mathscr{T}_\sigma$ to be the set of isomorphism classes of face-homogeneous tessellations with valence sequence σ. Let

$$\underline{\lambda}_\sigma = \inf\{\gamma(T) : T \in \mathscr{T}_\sigma\}, \tag{4.1}$$

$$\overline{\lambda}_\sigma = \sup\{\gamma(T) : T \in \mathscr{T}_\sigma\}, \tag{4.2}$$

$$\mathscr{L}_\sigma = \{T : T \in \mathscr{T}_\sigma \text{ and } \gamma(T) = \underline{\lambda}_\sigma\}, \text{ and} \tag{4.3}$$

$$\mathscr{H}_\sigma = \{T : T \in \mathscr{T}_\sigma \text{ and } \gamma(T) = \overline{\lambda}_\sigma\}. \tag{4.4}$$

We conjecture that the lower and upper bounds $\underline{\lambda}_\sigma$ and $\overline{\lambda}_\sigma$ for any given valence sequence σ are realized.

Conjecture 4.3. *Let σ be a polymorphic valence sequence. Then $\mathscr{L}_\sigma$ and $\mathscr{H}_\sigma$ are nonempty.*

Bearing in mind the polymorphic valence sequence $[4, 4, 6, 8]$ analyzed in Section 4.2, we propose as a conjecture the following sharper version of Theorem 3.7.

Conjecture 4.4. *Let σ_1 and σ_2 be valence sequences such that $\sigma_1 < \sigma_2$. Then*

$$\overline{\lambda}_{\sigma_1} \leq \underline{\lambda}_{\sigma_2}. \tag{4.5}$$

In the spirit of the famous quote of the late George Pólya [12] ("If you can't solve a problem, then there is an easier problem you can solve: find it."), we offer the following (perhaps) easier conjecture.

Table 2: Corona sizes in T_1 and T_2; emphasis on the 14th corona beyond which the coronas of T_2 appear to exceed in size those of T_1.

n	$\lvert F_{1,n}\rvert$	$\lvert F_{2,n}\rvert$	n	$\lvert F_{1,n}\rvert$	$\lvert F_{2,n}\rvert$
1	4	4	29	1.20050×10^{18}	1.27748×10^{18}
2	30	28	30	4.95826×10^{18}	5.29701×10^{18}
3	110	108	31	2.04784×10^{19}	2.19652×10^{19}
4	494	468	32	8.45791×10^{19}	9.10786×10^{19}
5	1938	1900	33	3.49325×10^{20}	3.77673×10^{20}
6	8272	7956	34	1.44277×10^{21}	1.56603×10^{21}
7	33464	32868	35	5.95887×10^{21}	6.49377×10^{21}
8	140046	136380	36	2.46111×10^{22}	2.69268×10^{22}
9	573610	565956	37	1.01648×10^{23}	1.11655×10^{23}
10	2.38167×10^{6}	2.34358×10^{6}	38	4.19821×10^{23}	4.62986×10^{23}
11	9.80378×10^{6}	9.73259×10^{6}	39	1.73393×10^{24}	1.91983×10^{24}
12	4.05773×10^{7}	4.02988×10^{7}	40	7.16140×10^{24}	7.96071×10^{24}
13	1.67365×10^{8}	1.67318×10^{8}	41	2.95777×10^{25}	3.30099×10^{25}
14	$\mathbf{6.91836 \times 10^{8}}$	$\mathbf{6.93034 \times 10^{8}}$	42	1.22161×10^{26}	1.36878×10^{26}
15	2.85585×10^{9}	2.87639×10^{9}	43	5.04544×10^{26}	5.67580×10^{26}
16	1.17992×10^{10}	1.19181×10^{10}	44	2.08385×10^{27}	2.35352×10^{27}
17	4.87218×10^{10}	4.94504×10^{10}	45	8.60662×10^{27}	9.75910×10^{27}
18	2.01257×10^{11}	2.04947×10^{11}	46	3.55467×10^{28}	4.04670×10^{28}
19	8.31149×10^{11}	8.50179×10^{11}	47	1.46814×10^{29}	1.67800×10^{29}
20	3.43297×10^{12}	3.52419×10^{12}	48	6.06363×10^{29}	6.95799×10^{29}
21	1.41782×10^{13}	1.46172×10^{13}	49	2.50438×10^{30}	2.88520×10^{30}
22	5.85596×10^{13}	6.05990×10^{13}	50	1.03435×10^{31}	1.19637×10^{31}
23	2.41857×10^{14}	2.51322×10^{14}	60	1.49395×10^{37}	1.79797×10^{37}
24	9.98918×10^{14}	1.04199×10^{15}	70	2.15777×10^{43}	2.70207×10^{43}
25	4.12567×10^{15}	4.32117×10^{15}	80	3.11654×10^{49}	4.06079×10^{49}
26	1.70397×10^{16}	1.79166×10^{16}	90	4.50134×10^{55}	6.10274×10^{55}
27	7.03766×10^{16}	7.42979×10^{16}	100	6.50145×10^{61}	9.17148×10^{61}
28	2.90667×10^{17}	3.08066×10^{17}	200	2.56861×10^{123}	5.38996×10^{123}

Conjecture 4.5. *Let σ_1 and σ_2 be valence sequences with $\sigma_1 < \sigma_2$. Then*

$$\underline{\lambda}_{\sigma_1} \leq \underline{\lambda}_{\sigma_2}. \tag{4.6}$$

If Conjecture 4.4 holds, then one could delete the condition of monomorphicity from the hypothesis of Theorem 3.7 and therefore from Theorem 3.13 as well. Moreover, the Appendix could be much abbreviated. For example, one could eliminate the exhaustive consideration of the many forms of 6-covalent face-homogeneous tessellations listed and treated there by observing that the least valence sequence σ of length 6 with $\eta(\sigma) > 0$ is $[3, 3, 3, 3, 3, 4]$. Thus, if any tessellation with the polymorphic valence sequence $[3, 3, 3, 3, 3, 4]$ has growth rate greater than $\frac{1}{2}(1 + \sqrt{5})$, then so does every tessellation with valence sequence $\sigma \geq [3, 3, 3, 3, 3, 4]$.

Beyond these conjectures, there are some open questions. Consider the partially ordered set of valence sequences, and in particular, the poset consisting of the polymorphic valence sequences.

Question 4.6. *As one goes up a chain in the poset, do intervals of the form* $[\underline{\lambda}_\sigma, \overline{\lambda}_\sigma]$ *become (asymptotically) longer?*

Question 4.7. *Do the intervals in the complement of*

$$\bigcup_\sigma \{ [\underline{\lambda}_\sigma, \overline{\lambda}_\sigma] : \sigma \text{ is polymorphic} \}$$

become arbitrarily long?

If the answer to Question 4.7 is negative, we pose the following.

Question 4.8. *If x is a sufficiently large real number, is there always some polymorphic valence sequence σ such that $\underline{\lambda}_\sigma \leq x \leq \overline{\lambda}_\sigma$?*

Or, on the other hand,

Question 4.9. *Do there exist polymorphic sequences σ, τ such that*

$$[\underline{\lambda}_\sigma, \overline{\lambda}_\sigma] \cap [\underline{\lambda}_\tau, \overline{\lambda}_\tau] \neq \emptyset?$$

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The 4-girth-thickness of the complete graph

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Abstract

In this paper, we define the 4-girth-thickness $\theta(4, G)$ of a graph G as the minimum number of planar subgraphs of girth at least 4 whose union is G. We prove that the 4-girth-thickness of an arbitrary complete graph K_n, $\theta(4, K_n)$, is $\left\lceil \frac{n+2}{4} \right\rceil$ for $n \neq 6, 10$ and $\theta(4, K_6) = 3$.

Keywords: Thickness, planar decomposition, girth, complete graph.

Math. Subj. Class.: 05C10

1 Introduction

A finite graph G is *planar* if it can be embedded in the plane without any two of its edges crossing. A planar graph of order n and girth g has size at most $\frac{g}{g-2}(n-2)$ (see [6]), and an acyclic graph of order n has size at most $n-1$, in this case, we define its girth as ∞. The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G; i.e. the minimum number of planar subgraphs into which the edges of G can be partitioned.

The thickness was introduced by Tutte [11] in 1963. Since then, exact results have been obtained when G is a complete graph [1, 3, 4], a complete multipartite graph [5, 12, 13] or a hypercube [9]. Also, some generalizations of the thickness for the complete graph K_n have been studied such that the outerthickness θ_o, defined similarly but with outerplanar instead of planar [8], and the S-thickness θ_S, considering the thickness on a surfaces S instead of the plane [2]. See also the survey [10].

We define the *g-girth-thickness* $\theta(g, G)$ of a graph G as the minimum number of planar subgraphs of girth at least g whose union is G. Note that the 3-girth-thickness $\theta(3, G)$ is the usual thickness and the ∞-girth-thickness $\theta(\infty, G)$ is the *arboricity number*, i.e. the minimum number of acyclic subgraphs into which $E(G)$ can be partitioned. In this paper, we obtain the 4-girth-thickness of an arbitrary complete graph of order $n \neq 10$.

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2 The exact value of $\theta(4, K_n)$ for $n \neq 10$

Since the complete graph K_n has size $\binom{n}{2}$ and a planar graph of order n and girth at least 4 has size at most $2(n-2)$ for $n \geq 3$ and $n-1$ for $n \in \{1, 2\}$ then the 4-girth-thickness of K_n is at least

$$\left\lceil \frac{n(n-1)}{2(2n-4)} \right\rceil = \left\lceil \frac{n+1}{4} + \frac{1}{2n-4} \right\rceil = \left\lceil \frac{n+2}{4} \right\rceil$$

for $n \geq 3$ and also $\left\lceil \frac{n+2}{4} \right\rceil$ for $n \in \{1, 2\}$, we have the following theorem.

Theorem 2.1. *The* 4*-girth-thickness* $\theta(4, K_n)$ *of* K_n *equals* $\left\lceil \frac{n+2}{4} \right\rceil$ *for* $n \neq 6, 10$ *and* $\theta(4, K_6) = 3$*.*

Proof. Figure 1 displays equality for $n \leq 5$.

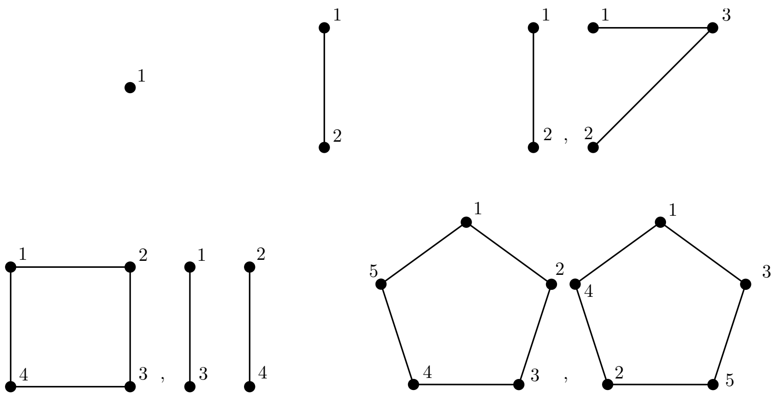

Figure 1: $\theta(4, K_n) = \left\lceil \frac{n+2}{4} \right\rceil$ for $n = 1, 2, 3, 4, 5$.

To prove that $\theta(4, K_6) = 3 > \left\lceil \frac{6+2}{4} \right\rceil = 2$, suppose that $\theta(4, K_6) = 2$. This partition define an edge coloring of K_6 with two colors. By Ramsey's Theorem, some part contains a triangle obtaining a contradiction for the girth 4. Figure 2 shows a partition of K_6 into tree planar subgraphs of girth at least 4.

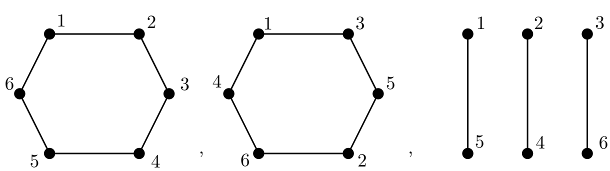

Figure 2: $\theta(4, K_6) = 3$.

For the remainder of this proof, we need to distinguish four cases, namely, when $n = 4k-1$, $n = 4k$, $n = 4k+1$ and $n = 4k+2$ for $k \geq 2$. Note that in each case, the lower bound of the 4-girth thickness require at least $k+1$ elements. To prove our theorem, we exhibit a decomposition of K_{4k} into $k+1$ planar graphs of girth at least 4. The other three cases are based in this decomposition. The case of $n = 4k - 1$ follows because K_{4k-1} is a subgraph of K_{4k}. For the case of $n = 4k + 2$, we add two vertices and some edges to the decomposition obtained in the case of $n = 4k$. The last case follows because K_{4k+1} is a subgraph of K_{4k+2}. In the proof, all sums are taken modulo $2k$.

1. Case $n = 4k$. It is well-known that a complete graph of even order contains a cyclic factorization of Hamiltonian paths, see [7]. Let G be a subgraph of K_{4k} isomorphic to K_{2k}. Label its vertex set $V(G)$ as $\{v_1, v_2, \dots, v_{2k}\}$. Let $\mathcal{F}_1$ be the Hamiltonian path with edges

$$v_1v_2, v_2v_{2k}, v_{2k}v_3, v_3v_{2k-1}, \dots, v_{2+k}v_{1+k}.$$

Let $\mathcal{F}_i$ be the Hamiltonian path with edges

$$v_iv_{i+1}, v_{i+1}v_{i-1}, v_{i-1}v_{i+2}, v_{i+2}v_{i-2}, \dots, v_{i+k+1}v_{i+k},$$

where $i \in \{2, 3, \dots, k\}$.

Such factorization of G is the partition $\{E(\mathcal{F}_1), E(\mathcal{F}_2), \dots, E(\mathcal{F}_k)\}$. We remark that the center of $\mathcal{F}_i$ has the edge $e = v_{i+\lceil \frac{k}{2} \rceil}v_{i+\lceil \frac{3k}{2} \rceil}$, see Figure 3.

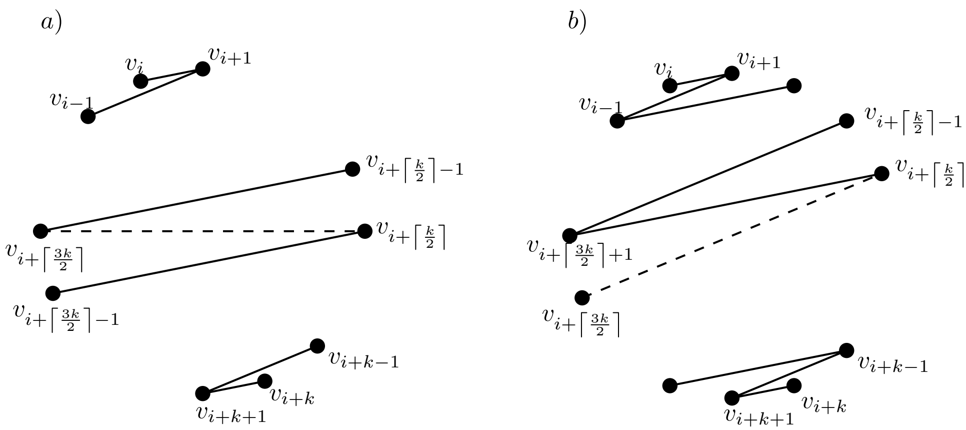

Figure 3: The Hamiltonian path $\mathcal{F}_i$: Left $a)$: The dashed edge e for k odd. Right $b)$ The dashed edge e for k even.

Now, consider the complete subgraph G' of K_{4k} such that $G' = K_{4k} \setminus V(G)$. Label its vertex set $V(G')$ as $\{v'_1, v'_2, \dots, v'_{2k}\}$ and consider the factorization, similarly as before, $\{E(\mathcal{F}'_1), E(\mathcal{F}'_2), \dots, E(\mathcal{F}'_k)\}$ where $\mathcal{F}'_i$ is the Hamiltonian path with edges

$$v'_iv'_{i+1}, v'_{i+1}v'_{i-1}, v'_{i-1}v'_{i+2}, v'_{i+2}v'_{i-2}, \dots, v'_{i+k+1}v'_{i+k},$$

where $i \in \{1, 2, \dots, k\}$.

Next, we construct the planar subgraphs G_1, G_2,...,G_{k-1} and G_k of girth 4, order $4k$ and size $8k - 4$ (observe that $2(4k - 2) = 8k - 4$), and also the matching G_{k+1}, as follows. Let G_i be a spanning subgraph of K_{4k} with edges $E(\mathcal{F}_i) \cup E(\mathcal{F}'_i)$ and

$$v_iv'_{i+1}, v'_iv_{i+1}, v_{i+1}v'_{i-1}, v'_{i+1}v_{i-1}, v_{i-1}v'_{i+2}, v'_{i-1}v_{i+2}, \dots, v_{i+k+1}v'_{i+k}, v'_{i+k+1}v_{i+k}$$

where $i \in \{1, 2, \ldots, k\}$; and let G_{k+1} be a perfect matching with edges $v_j v'_j$ for $j \in \{1, 2, \ldots, 2k\}$. Figure 4 shows G_i is a planar graph of girth at least 4.

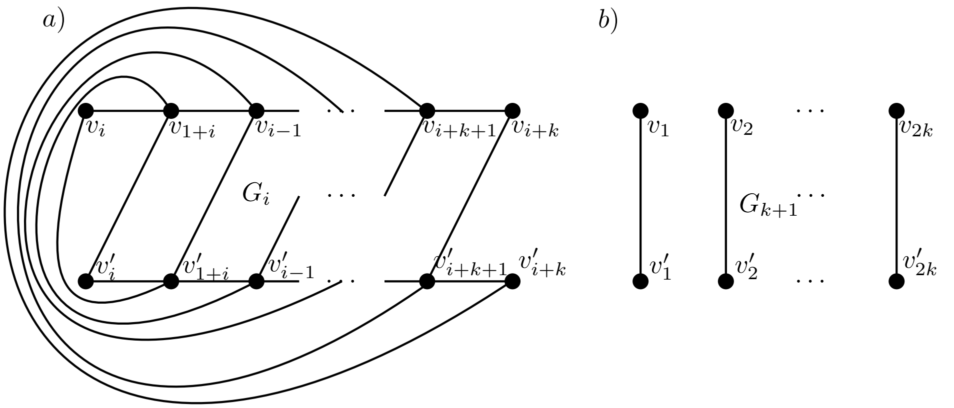

Figure 4: Left $a)$: The graph G_i for any $i \in \{1, 2, \ldots, k\}$. Right $b)$ The graph G_{k+1}.

To verify that $K_{4k} = \bigcup_{i=1}^{k+1} G_i$: 1) If the edge $v_{i_1} v_{i_2}$ of G belongs to the factor $\mathcal{F}_i$ then $v_{i_1} v_{i_2}$ belongs to G_i. If the edge is primed, belongs to G'_i. 2) The edge $v_{i_1} v'_{i_2}$ belongs to G_{k+1} if and only if $i_1 = i_2$, otherwise it belongs to the same graph G_i as $v_{i_1} v_{i_2}$. Similarly in the case of $v'_{i_1} v_{i_2}$ and the result follows.

2. Case $n = 4k - 1$. Since $K_{4k-1} \subset K_{4k}$, we have

$$k + 1 \leq \theta(4, K_{4k-1}) \leq \theta(4, K_{4k}) \leq k + 1.$$

3. Case $n = 4k + 2$ (for $k \neq 2$). Let $\{G_1, \ldots, G_{k+1}\}$ be the planar decomposition of K_{4k} constructed in the Case 1. We will add the two new vertices x and y to every planar subgraph G_i, when $1 \leq i \leq k + 1$, and we will add 4 edges to each G_i, when $1 \leq i \leq k$, and $4k+1$ edges to G_{k+1} such that the resulting new subgraphs of K_{4k+2} will be planar. Note that $\binom{4k}{2} + 4k + 4k + 1 = \binom{4k+2}{2}$.

   To begin with, we define the graph H_{k+1} adding the vertices x and y to the planar subgraph G_{k+1} and the $4k + 1$ edges

$$\{xy, xv_1, xv'_2, xv_3, xv'_4, \ldots, xv_{2k-1}, xv'_{2k}, yv'_1, yv_2, yv'_3, yv_4, \ldots, yv'_{2k-1}, yv_{2k}\}.$$

   The graph H_{k+1} has girth 4, see Figure 5.

   In the following, for $1 \leq i \leq k$, by adding vertices x and y to G_i and adding 4 edges to G_i, we will get a new planar graph H_i such that $\{H_1, \ldots, H_{k+1}\}$ is a planar decomposition of K_{4k+2} such that the girth of every element is 4. To achieve it, the given edges to the graph H_i will be $v'_j x, xv_{j-1}, v_j y, yv'_{j-1}$, for some odd $j \in \{1, 3, \ldots, 2k - 1\}$.

   According to the parity of k, we have two cases:

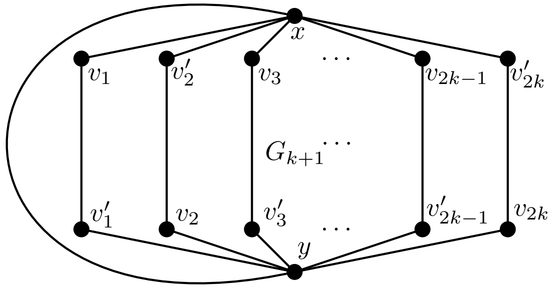

Figure 5: The graph H_{k+1}.

- Suppose k odd. For odd $i \in \{1, 2, \dots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv'_{i+\lceil \frac{3k}{2} \rceil -1}, xv_{i+\lceil \frac{3k}{2} \rceil}, yv_{i+\lceil \frac{3k}{2} \rceil -1}, yv'_{i+\lceil \frac{3k}{2} \rceil}\}$$

  when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv'_{i+\lceil \frac{3k}{2} \rceil -1}, yv_{i+\lceil \frac{3k}{2} \rceil}, xv_{i+\lceil \frac{3k}{2} \rceil -1}, xv'_{i+\lceil \frac{3k}{2} \rceil}\}.$$

  Additionally, for even $i \in \{1, 2, \dots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv'_{i+\lceil \frac{k}{2} \rceil -1}, xv_{i+\lceil \frac{k}{2} \rceil}, yv_{i+\lceil \frac{k}{2} \rceil -1}, yv'_{i+\lceil \frac{k}{2} \rceil}\}$$

  when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv'_{i+\lceil \frac{k}{2} \rceil -1}, yv_{i+\lceil \frac{k}{2} \rceil}, xv_{i+\lceil \frac{k}{2} \rceil -1}, xv'_{i+\lceil \frac{k}{2} \rceil}\}.$$

  Note that the graph H_i has girth 4 for all i, see Figure 6.
- Suppose k even. Similarly that the previous case, for odd $i \in \{1, 2, \dots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv_{i+\lceil \frac{3k}{2} \rceil +1}, xv'_{i+\lceil \frac{3k}{2} \rceil}, yv'_{i+\lceil \frac{3k}{2} \rceil +1}, yv_{i+\lceil \frac{3k}{2} \rceil}\}$$

  when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv_{i+\lceil \frac{3k}{2} \rceil +1}, yv'_{i+\lceil \frac{3k}{2} \rceil}, xv'_{i+\lceil \frac{3k}{2} \rceil +1}, xv_{i+\lceil \frac{3k}{2} \rceil}\}.$$

  On the other hand, for even $i \in \{1, 2, \dots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv_{i+\lceil \frac{k}{2} \rceil}, xv'_{i+\lceil \frac{k}{2} \rceil -1}, yv'_{i+\lceil \frac{k}{2} \rceil}, yv_{i+\lceil \frac{k}{2} \rceil -1}\}$$

  when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv_{i+\lceil \frac{k}{2} \rceil}, yv'_{i+\lceil \frac{k}{2} \rceil -1}, xv'_{i+\lceil \frac{k}{2} \rceil}, xv_{i+\lceil \frac{k}{2} \rceil -1}\}.$$

  Note that the graph H_i has girth 4 for all i, see Figure 7.

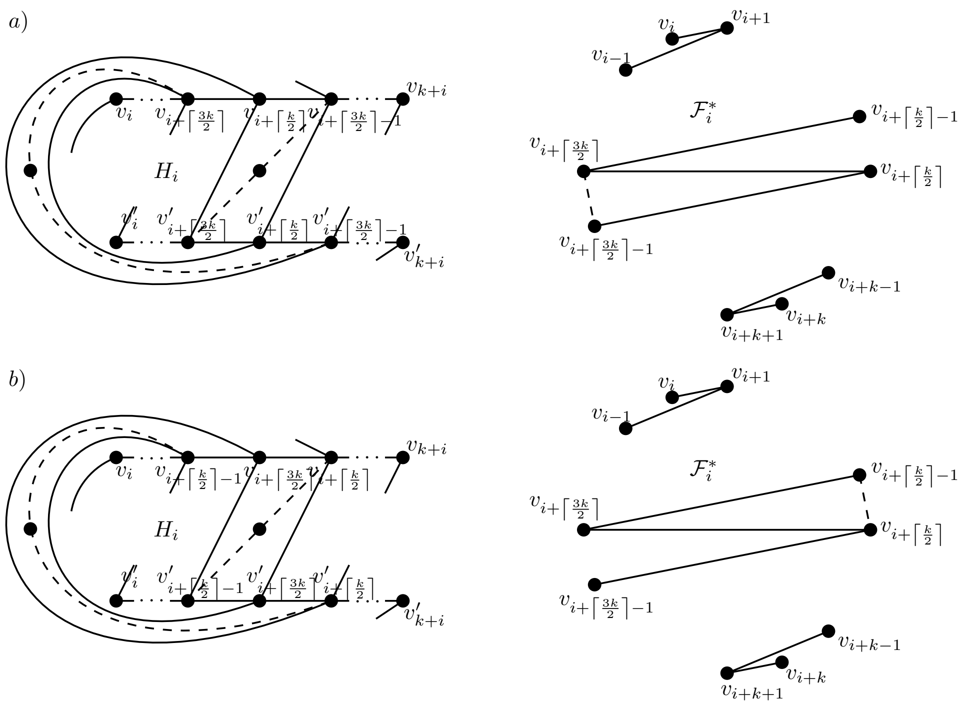

Figure 6: The graph H_i when k is odd and its auxiliary graph $\mathcal{F}_i^*$. Above $a)$ When i is odd. Botton $b)$ When i is even.

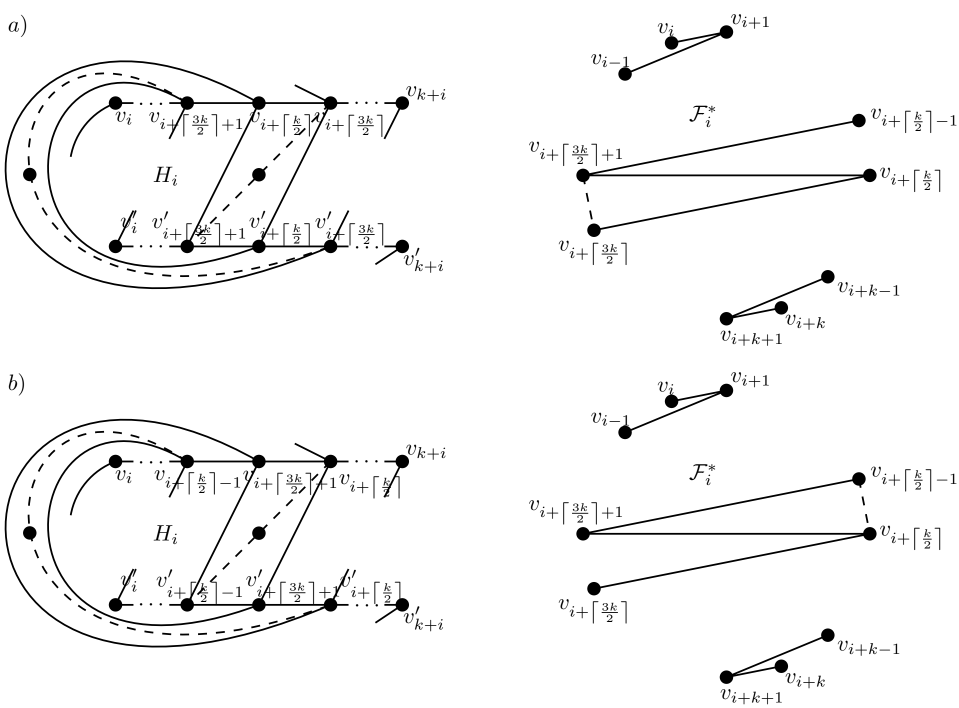

Figure 7: The graph H_i when k is even and its auxiliary graph $\mathcal{F}_i^*$. Above $a)$ When i is odd. Botton $b)$ When i is even.

In order to verify that each edge of the set

$$\{xv'_1, xv_2, xv'_3, xv_3, \ldots, xv'_{2k-1}, xv_{2k}, yv_1, yv'_2, yv_3, yv'_3, \ldots, yv_{2k-1}, yv'_{2k}\}.$$

is in exactly one subgraph H_i, for $i \in \{1, \ldots, k\}$, we obtain the unicyclic graph $\mathcal{F}_i^*$ identifying v_j and v'_j resulting in v_j; identifying x and y resulting in a vertex which is contracted with one of its neighbours. The resulting edge, in dashed, is showed in Figures 6 and 7. The set of those edges are a perfect matching of K_{2k} proving that the added two paths of length 2 in G_i have end vertices v_j and v'_{j-1}, and the other v'_j and v_{j-1}. The election of the label of the center vertex is such that one path is $v_{even}xv'_{odd}$ and $v'_{even}yv_{odd}$ and the result follows.

4. Case $n = 4k + 1$ (for $k \neq 2$). Since $K_{4k+1} \subset K_{4k+2}$, we have

$$k + 1 \leq \theta(4, K_{4k+1}) \leq \theta(4, K_{4k+2}) \leq k + 1.$$

For $k = 2$, Figure 8 displays a decomposition of three planar graphs of girth at least 4 proving that $\theta(4, K_9) = \left\lceil \frac{9+2}{4} \right\rceil = 3$.

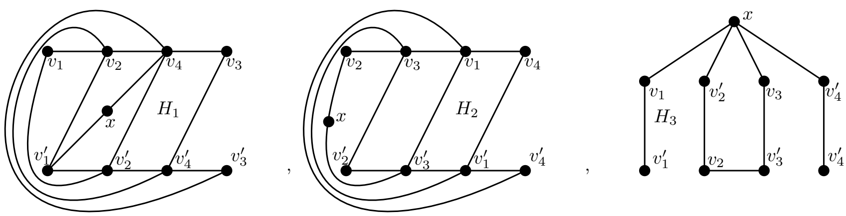

Figure 8: A planar decomposition of K_9 into three subgraphs of girth 4 and 5.

By the four cases, the theorem follows. □

About the case of K_{10}, it follows $3 \leq \theta(4, K_{10}) \leq 4$. We conjecture that $\theta(4, K_{10}) = 4$.

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A note on the thickness of some complete bipartite graphs*

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Abstract

The thickness of a graph is the minimum number of planar subgraphs into which the graph can be decomposed. Determining the thickness for the complete bipartite graph is an unsolved problem in graph theory for over fifty years. Using a new planar decomposition for $K_{4k-4,4k}(k \geq 4)$, we obtain the thickness of the complete bipartite graph $K_{n,n+4}$, for $n \geq 1$.

Keywords: Planar graph, thickness, complete bipartite graph.

Math. Subj. Class.: 05C10

1 Introduction

In this paper, all graphs are simple. A graph G is denoted by $G = (V, E)$ where $V(G)$ is the vertex set and $E(G)$ is the edge set. *A complete graph* is a graph in which any two vertices are adjacent. A complete graph on n vertices is denoted by K_n. A *complete bipartite graph* is a graph whose vertex set can be partitioned into 2 parts, such that every edge has its ends in different parts and every two vertices in different parts are adjacent. We use K_{p_1,p_2} to denote a complete bipartite graph in which the i^{th} part contains p_i vertices, for $i = 1, 2$.

The *thickness* $t(G)$ of a graph G is the minimum number of planar subgraphs into which G can be decomposed [14]. It is a classical topological parameter of a graph and has many applications, for instance, to graph drawing [12] and VLSI design [1]. Since deciding the thickness of a graph is NP-hard [9], it is very difficult to get the exact number of thickness for arbitrary graphs. Battle, Harary and Kodama [3] in 1962 and Tutte [13] in 1963 independently showed that the thickness of K_9 and K_{10} equals 3. Beineke and

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Harary [4] determined the thickness of complete graph K_n for $n \not\equiv 4 \pmod 6$ in 1965, the remaining case was solved in 1976, independently by V.B. Alekseev and V.S. Gonchakov [2] and by J.M. Vasak [15].

For complete bipartite graphs, the problem has not been entirely solved yet. By constructing a planar decomposition of $K_{m,n}$ when m is even, Beineke, Harary and Moon [5] determined the thickness of $K_{m,n}$ for most values of m, n in 1964.

Theorem 1.1. *[5] For $m \leq n$, the thickness of the complete bipartite graph $K_{m,n}$ is*

$$t(K_{m,n}) = \left\lceil \frac{mn}{2(m+n-2)} \right\rceil, \tag{1.1}$$

except possibly when m and n are both odd and there exists an integer k satisfying $n = \left\lfloor \frac{2k(m-2)}{(m-2k)} \right\rfloor$.

We recall that the thickness of $K_{n,n}$ is also obtained in 1968 by Isao and Ozaki [11] independently. The following open problem is adapted from [7] by Gross and Harary.

Problem 1.2. [See Problem 4.1 of [7]] Find the thickness of $K_{m,n}$ for all m, n.

Beineke, Harary and Moon [5] also pointed out that the smallest complete bipartite graph whose thickness is unknown is $K_{17,21}$. From Euler's Formula, the thickness of $K_{17,21}$ is at least 5.

From Theorem 1.1, we need to determine the thickness of $K_{m,n}$ for odd m, n. Since the difference between the two odd numbers is even, we only need to determine the thickness of $K_{n,n+2k}$ for odd n and $k \geq 0$. In this paper, we start to calculate the thickness of $K_{n,n+2k}$ for some small values of k. Indeed, we determine the thickness of $K_{n,n+4}$.

Theorem 1.3. *The thickness of $K_{n,n+4}$ is*

$$t(K_{n,n+4}) = \begin{cases} 1, & \text{if } n \leq 2 \\ \left\lceil \frac{n+3}{4} \right\rceil, & \text{otherwise.} \end{cases}$$

The following corollary follows from Theorem 1.3.

Corollary 1.4. *The thickness of $K_{17,21}$ is 5.*

We may refer the reader to [6, 10, 16] for more background on graph thickness.

2 The thickness of $K_{n,n+4}$

To begin with, we define two special graphs called the pattern graph and the k^{th}-order nest graph. Then, we prove a new planar decomposition of $K_{4k-4,4k}$. Finally, we prove the thickness of $K_{4k-3,4k+1}$ and $K_{n,n+4}$.

2.1 The pattern graph

Let $U = \{u_1, u_2\}$ and X_n be a set of n vertices. A graph is said to be a *pattern graph* of order $n+2$, denoted by $G[u_1 X_n u_2]$, if it can be constructed by the following two steps.

1. Arrange the n vertices in a row, and put vertices u_1, u_2 on the above and below of n vertices, respectively.

2. Join both u_1 and u_2 to the n vertices using straight lines.

From the definition above, the pattern graph is a planar straight-line graph. Figure 1 illustrates the pattern graph $G[u_1 X_n u_2]$.

Remark 2.1. Unless explicitly mentioned, we always join vertices using straight lines in the drawings of the following proofs.

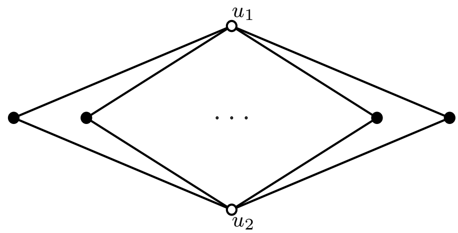

Figure 1: The pattern graph $G[u_1 X_n u_2]$.

2.2 The k^{th}-order nest graph

Let $U_k = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$, $V_k = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$ and $W_{2k+2} = \{w_{l_1}, w_{l_2}, \ldots, w_{l_{2k+2}}\}$, we define a k^{th}-*order nest graph* $G[U_k, V_k, W_{2k+2}]$ as follows:

1. Arrange $2k+2$ vertices $w_{l_1}, w_{l_2}, \ldots, w_{l_{2k+2}}$ in a row.
2. For $1 \leq m \leq k$, place vertices u_{i_m} and v_{j_m} on the above and below of the row, respectively, and join them to $w_{l_1}, w_{l_{2m}}, w_{l_{2m+1}}, w_{l_{2m+2}}$.

Figure 2 illustrates a third-order nest graph $G[U_3, V_3, W_8]$, where $U_3 = \{u_1, u_2, u_3\}$, $V_3 = \{v_1, v_2, v_3\}$ and $W_8 = \{w_1, w_2, \ldots, w_8\}$.

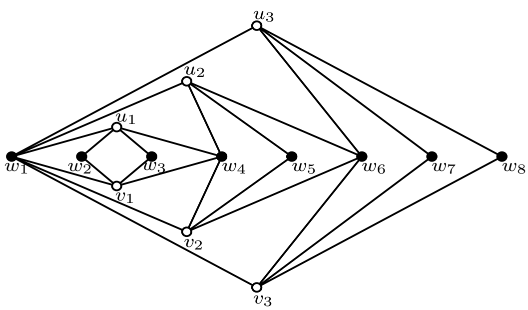

Figure 2: The third-order nest graph $G[U_3, V_3, W_8]$.

2.3 A new planar decomposition of $K_{4k-3,4k+1}$, for $k \geq 4$

In this subsection, we shall construct a planar decomposition for the complete bipartite graph $K_{4k-3,4k+1}$ with k planar subgraphs $G_1, G_2, \ldots, G_k$. Suppose that the vertex partition of $K_{4k-3,4k+1}$ is (X, Y), where $X = \{x_1, x_2, \ldots, x_{4k-3}\}$, $Y = \{y_0, y_1, y_2, \ldots, y_{4k}\}$.

2.3.1 The planar decomposition for $K_{4k-4,4k}$

Let the vertex partition of $K_{4k-4,4k}$ be (X_1, Y_1), where $X_1 = \{x_1, x_2, \dots, x_{4k-4}\}, Y_1 = \{y_0, y_1, \dots, y_{4k-1}\}$. In this subsection, all subscripts in y_j are taken mod $4k$.

1. In the graph G_i $(1 \le i \le k)$, we arrange $4k$ vertices in a row, and divide the $4k$ vertices into two subsets L_{2k} and R_{2k} such that each subset contains $2k$ vertices according to the following steps.

2. In the graph G_i $(1 \le i \le k-1)$, we choose four vertices $x_{4i-3}, x_{4i-2}, x_{4i-1}, x_{4i}$ from X_1 and construct two pattern graphs $G[x_{4i-3}L_{2k}x_{4i-1}]$ and $G[x_{4i-2}R_{2k}\, x_{4i}]$. Then we join both x_{4i-3} and x_{4i-1} to the first vertex and the last vertex in R_{2k}. Finally, we label the vertices in L_{2k} and R_{2k} as $y_1, y_3, y_5, \dots, y_{4k-1}$ and $y_{2i+6}, y_{2i+8}, y_{2i+10}, \dots, y_{2i+4k+4}$ in turn, respectively.

3. In the graph G_k, we label the vertices in L_{2k} and R_{2k} as $y_1, y_3, y_5, \dots y_{4k-1}$ and $y_2, y_4, \dots, y_{4k-2}, y_0$, respectively. First, we construct a $(k-1)^{th}$-order nest graph $G[U_{k-1}, V_{k-1}, W_{2k}]$, where $U_{k-1} = \{x_2, x_6, x_{10}, \dots, x_{4k-6}\}$, $V_{k-1} = \{x_4, x_8, x_{12}, \dots, x_{4k-4}, \}$ and $W_{2k} = \{y_1, y_3, y_5, \dots, y_{4k-1}\}$. We join x_{4i-3} to y_{2i} and y_{2i+2}, for $1 \le i \le k-1$. Second, we construct a union of paths, if k is even, we join x_{4i-1} to y_{2i+2k} and $y_{2i+2+2k}$, for $1 \le i \le k-1$; otherwise k is odd, we join x_{4i-1} to $y_{2i+2k-2}$ and y_{2i+2k}, for $1 \le i \le k-1$.

4. In each graph G_j $(1 \le j \le k-1)$, we put x_{4i-2}, x_{4i} in the quadrangle $x_{4j-3}y_{4j+1}x_{4j-1}y_{4j+3}$, and join them to y_{4j+1} and y_{4j+3}, for $1 \le i < j$. We put the vertices x_{4i-2}, x_{4i} in the quadrangle $x_{4j-3}y_{4j-1}x_{4j-1}y_{4j+1}$, and join both x_{4i-2} and x_{4i} to y_{4j-1} and y_{4j+1}, for $j < i \le k-1$. Next, we put x_{4i-3} in the quadrangle $x_{4j-2}y_{4j-2i+4}x_{4j}y_{4j-2i+6}$, and join x_{4i-3} to $y_{4j-2i+4}, y_{4j-2i+6}$, for $1 \le i < j$. We put x_{4i-3} in the quadrangle $x_{4j-2}y_{4j-2i+4k}x_{4j}y_{4j-2i+4k+2}$, and join x_{4i-3} to $y_{4j-2i+4k}, y_{4j-2i+4k+2}$, for $j < i \le k-1$.

   For each i $(1 \le i \le k-1)$, we define a set $M_i = \{i+1, i+2, \dots, i+k-2\}$. Suppose that $m \in M_i$, if $m \le k-1$, we let $j = m$; otherwise, $j = m-k+1$.

   (i) k is even. If $i+1 \le m \le i+\frac{k-4}{2}$, we put x_{4i-1} in the quadrangle $x_{4j-2}y_{4m-2i+4}x_{4j}y_{4m-2i+6}$, and join x_{4i-1} to $y_{4m-2i+4}, y_{4m-2i+6}$. If $i+\frac{k-4}{2}+1 \le m \le i+k-2$, we put x_{4i-1} in the quadrangle $x_{4j-2}y_{4m-2i+8}x_{4j}y_{4m-2i+10}$, and join x_{4i-1} to $y_{4m-2i+8}, y_{4m-2i+10}$.

   (ii) k is odd. If $i+1 \le m \le i+\frac{k-5}{2}$, we put x_{4i-1} in the quadrangle $x_{4j-2}y_{4m-2i+4}x_{4j}y_{4m-2i+6}$, and join x_{4i-1} to $y_{4m-2i+4}, y_{4m-2i+6}$. If $i+\frac{k-5}{2}+1 \le m \le i+k-2$, we put x_{4i-1} in the quadrangle $x_{4j-2}y_{4m-2i+8}x_{4j}y_{4m-2i+10}$, and join x_{4i-1} to $y_{4m-2i+8}, y_{4m-2i+10}$.

Theorem 2.2. *Let $G_1, G_2, \dots, G_k$ be the planar subgraphs obtained from steps 1, 2, 3 and 4 above, then $\{G_1, G_2, \dots, G_k\}$ is a planar decomposition of $K_{4k-4,4k}$.*

Proof. From the constructions above, we have $E(G_i) \cap E(G_j) = \emptyset$, for $1 \le i \ne j \le k$. In order to prove that $\{G_1, G_2, \dots, G_k\}$ is a planar decomposition of $K_{4k-4,4k}$, we need to show that $E(G_1) \cup E(G_2) \cup \dots \cup E(G_k) = E(K_{4k-4,4k})$. We denote $d_{G_i}(v)$ as the degree of v in G_i, for $1 \le i \le k$.

By the construction above, Step 2 contributes to the degrees of $v_{4i-3}, v_{4i-1}, v_{4i-2}$, and v_{4i} in G_i by terms $2k+2, 2k+2, 2k+1$ and $2k+1$, respectively. In other words, we have $d_{G_i}(v_{4i-3}) = d_{G_i}(v_{4i-1}) = 2k+2$ and $d_{G_i}(v_{4i-2}) = d_{G_i}(v_{4i}) = 2k+1$.

For $1 \leq i \leq k-1$, Step 3 contributes to $d_{G_k}(v_{4i-3}), d_{G_k}(v_{4i-1}), d_{G_k}(v_{4i-2})$ and $d_{G_k}(v_{4i})$ by terms $2, 2, 3$, and 3, respectively.

For $1 \leq j \leq k-1$ and $i \neq j$, Step 4 contributes to each of $d_{G_j}(v_{4i-3}), d_{G_j}(v_{4i-1})$, $d_{G_j}(v_{4i-2})$ and $d_{G_j}(v_{4i})$ a term 2.

In total, for $1 \leq i \leq k-1$, we have
$\sum_{j=1}^{k} d_{G_j}(v_{4i-1}) = \sum_{j=1}^{k} d_{G_j}(v_{4i-3}) = d_{G_i}(v_{4i-3}) + \sum_{1 \leq j \neq i \leq k-1}^{k} d_{G_j}(v_{4i-3}) + d_{G_k}(v_{4i-3})$
$= 2k+2+2(k-2)+2 = 4k$,
and $\sum_{j=1}^{k} d_{G_j}(v_{4i-2}) = \sum_{j=1}^{k} d_{G_j}(v_{4i}) = d_{G_i}(v_{4i}) + \sum_{1 \leq j \neq i \leq k-1}^{k} d_{G_j}(v_{4i}) + d_{G_k}(v_{4i}) =$
$2k+1+2(k-2)+3 = 4k$.

From the discussion above, the result follows. □

2.3.2 Add the vertex x_{4k-3}

1. In the graph $G_i (1 \leq i \leq k-1)$, put the vertex x_{4k-3} in the quadrangle $x_{4i-3}y_{4i-1}$ $x_{4i-1}y_{4i+1}$, and join it to y_{4i-1}, y_{4i+1}.
2. In the graph G_k, place the vertex x_{4k-3} below the row of $2k$ vertices of R_{2k}, and join it to y_1, y_{4k-1} and all the $2k$ vertices of R_{2k}.

2.3.3 Add the vertex y_{4k}

1. In the graph $G_i (1 \leq i \leq k-1)$, put the vertex y_{4k} in the quadrangle $x_{4i-2}y_{4i+8}x_{4i}$ y_{4i+10}, and connect it to x_{4i-2}, x_{4i}.
2. In the graph G_k, place the vertex y_{4k} above the row of vertices of R_{2k}, and join it to $x_1, x_5, \ldots, x_{4k-7}, x_3, x_7, \ldots, x_{4k-3}$.

We have the following theorem.

Theorem 2.3. *The thickness of $K_{4k-3,4k+1}$ is k, for $k \geq 4$.*

Proof. From Theorem 2.2, Subsection 2.3.2 and Subsection 2.3.3, a planar decomposition of $K_{4k-3,4k+1}$ with k planar subgraphs $G_1, G_2, \ldots, G_k$ is obtained. From Euler's formula, we have

$$t(K_{4k-3,4k+1}) \geq \left\lceil \frac{(4k-3)(4k+1)}{2(8k-4)} \right\rceil = k,$$

and so $t(K_{4k-3,4k+1}) = k$. □

Example 2.4. By using the procedure above, the two planar decompositions of $K_{17,21}$ ($k=5$ is odd) and $K_{21,25}$ ($k=6$ is even) are shown in Appendix A (See Figures 3-7) and Appendix B (See Figures 8-13), respectively.

2.4 Proof of Theorem 1.3

From Theorem 1.1, the proof has two cases:
Case 1: $n = 4k-3$ $(k > 0)$. When $1 \leq k \leq 3$, it is routine to check that the theorem is true. For $k \geq 4$, $\left\lfloor \frac{2k(4k-3-2)}{4k-3-2k} \right\rfloor = \left\lfloor 4k+1+\frac{3}{2k-3} \right\rfloor = 4k+1$, thus, the thickness of $K_{4k-3,4k+1}$ can not be determined by Theorem 1.1. By Theorem 2.3, we have $t(K_{4k-3,4k+1}) = k = \left\lceil \frac{n+3}{4} \right\rceil$.
Case 2: $n = 4k-1$ $(k > 0)$. Since $4k-1$ and $4k+3$ are both odd and $4k+3 \neq \left\lfloor \frac{2(k+1)(4k-1-2)}{4k-1-2(k+1)} \right\rfloor$ (See Lemma 1 of [5] for details), the thickness of $K_{4k-1,4k+3}$ can be determined by Theorem 1.1, thus

$$\begin{aligned} t(K_{n,n+4}) = t(k_{4k-1,4k+3}) &= \left\lceil \frac{(4k-1)(4k+3)}{2(4k-1+4k+3-2)} \right\rceil \\ &= \left\lceil k+\frac{1}{2}-\frac{3}{16k} \right\rceil = k+1 = \left\lceil \frac{n+3}{4} \right\rceil. \end{aligned}$$

Summarizing the above, the theorem follows.

3 Conclusion

In this paper, we determine the thickness for $K_{n,n+4}$. The proof replies on a planar decomposition of $K_{4k-3,4k+1}$ and the Theorem 1.1 of Beineke, Harary and Moon. We observe that our approach for the construction of a planar decomposition of $K_{n,n+4}$ is the first step in finding a solution for Problem 1.2. From Theorem 1.1, the next classes of complete bipartite graphs whose thickness is unknown is $K_{4k-1,4k+7}$, for $k \geq 10$. Furthermore, the new smallest complete bipartite graph whose thickness is unknown is $K_{19,29}$. We hope that the construction here helps establish intuition and structure of the Problem 1.2.

Another way of solving the Problem 1.2 is to find a new planar decomposition of $K_{m,n}$, for odd m, n. Actually, using a new planar decomposition of the complete tripartite graph $K_{1,g,n}$ and a recursive construction, we also [8] obtained the thickness of $K_{s,t}$, where s is odd and $t \geq \frac{(s-3)(s-2)}{3}$. Now we split Problem 1.2 into the following two problems.

Problem 3.1. Find the thickness of $K_{n,n+4k}$ for odd n and $k \geq 2$.

Problem 3.2. Find the thickness of $K_{n,n+4k+2}$ for odd n and $k \geq 0$.

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A A planar decomposition $\{G_1, G_2, G_3, G_4, G_5\}$ for $K_{17,21}$

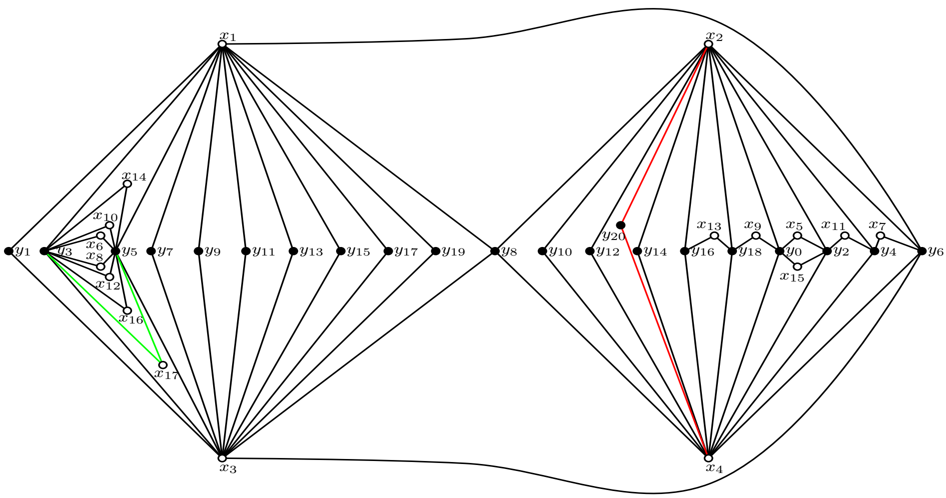

Figure 3: The graph G_1.

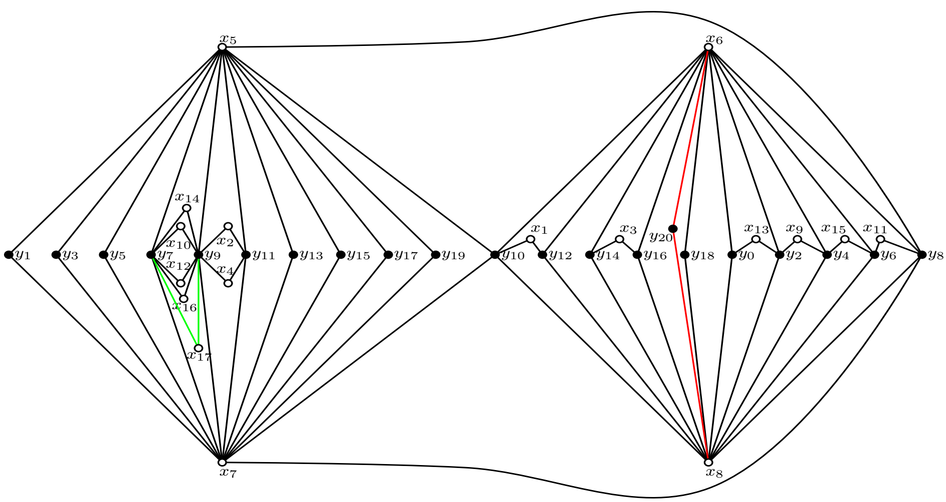

Figure 4: The graph G_2.

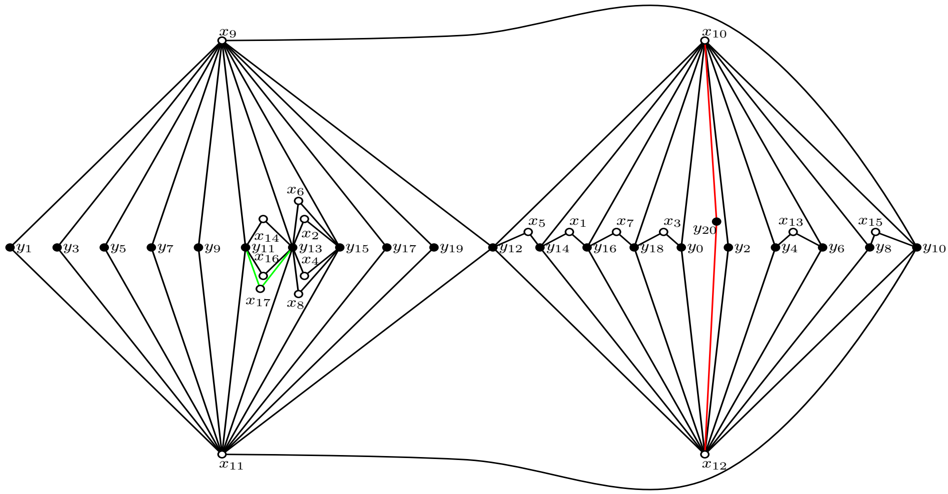

Figure 5: The graph G_3.

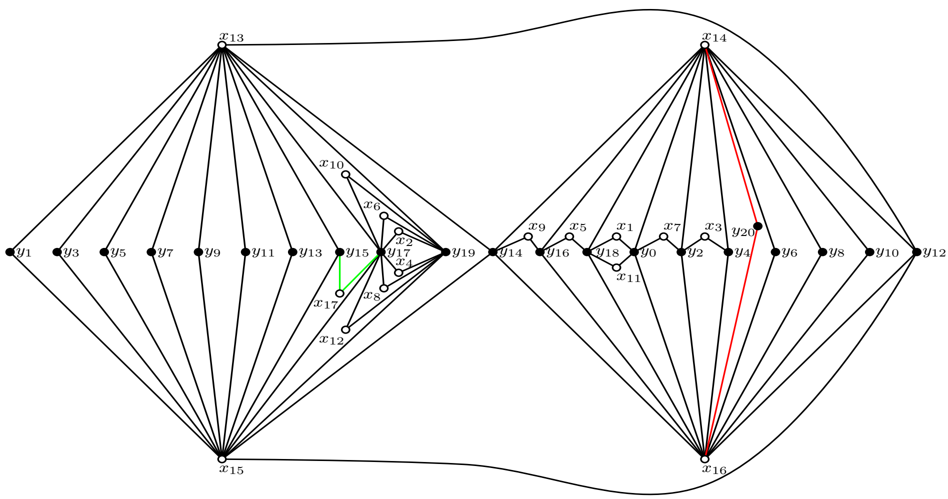

Figure 6: The graph G_4.

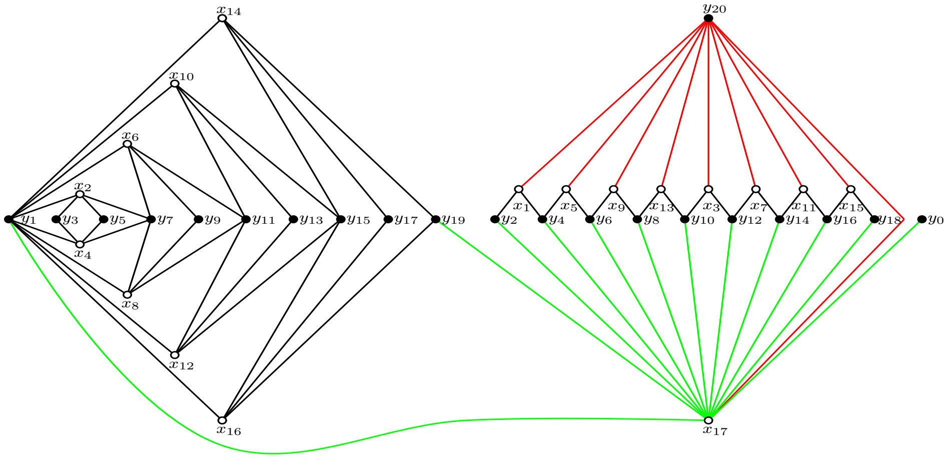

Figure 7: The graph G_5.

B A planar decomposition $\{G_1, G_2, G_3, G_4, G_5, G_6\}$ for $K_{21,25}$

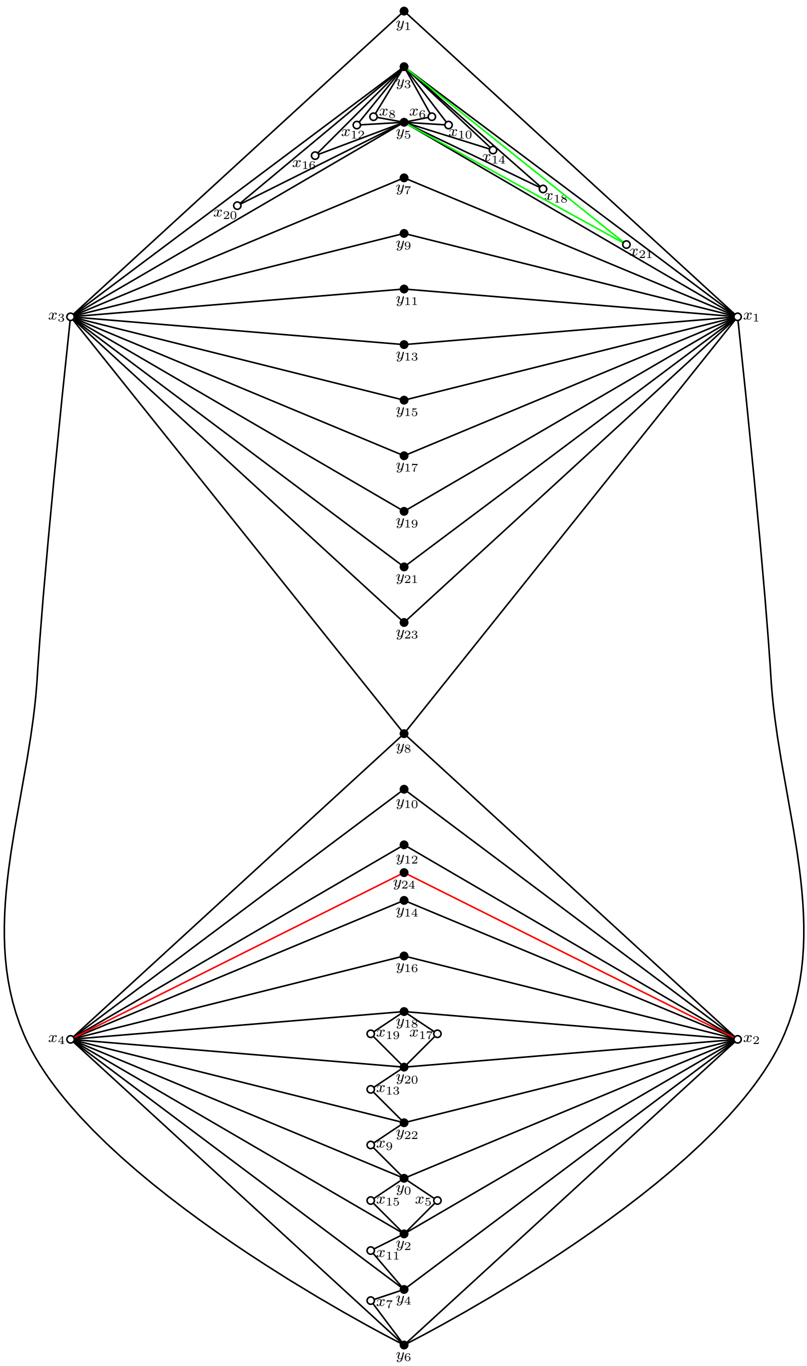

Figure 8: The graph G_1.

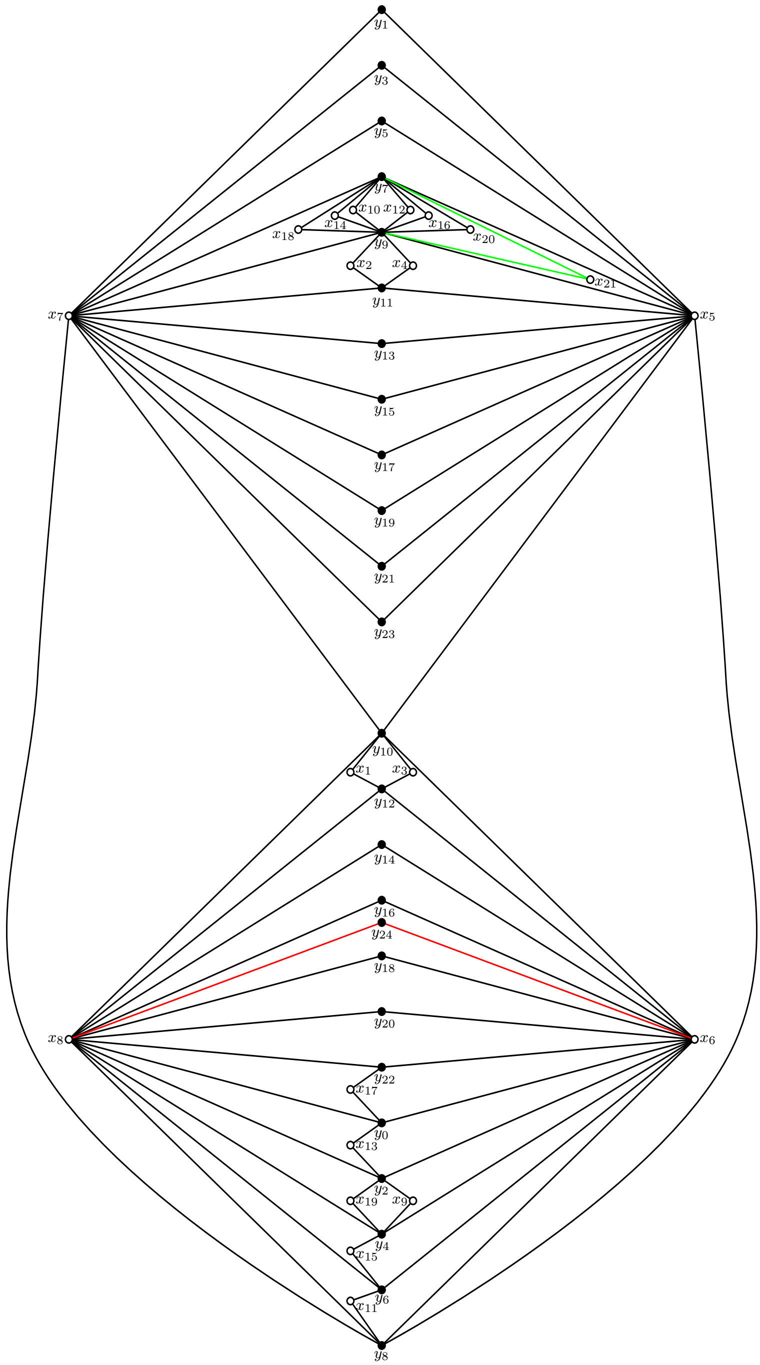

Figure 9: The graph G_2.

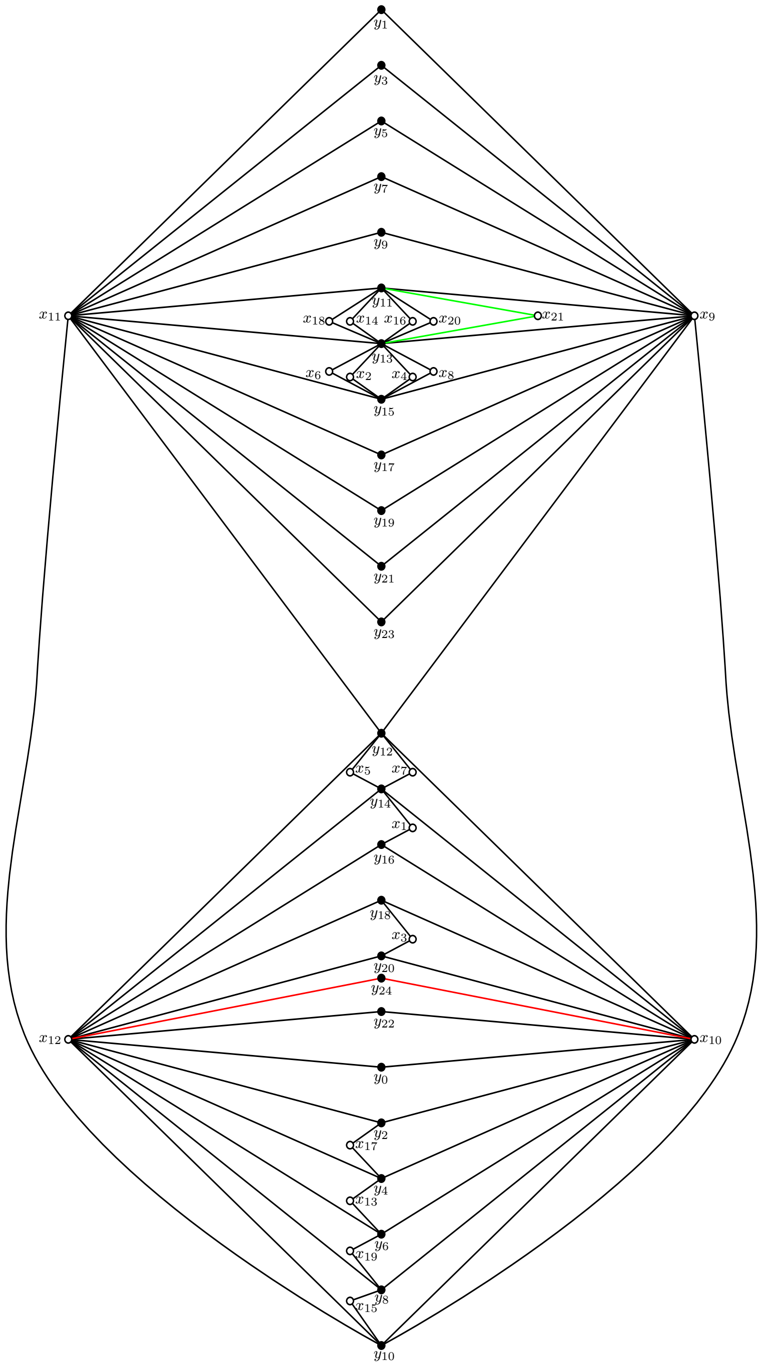

Figure 10: The graph G_3.

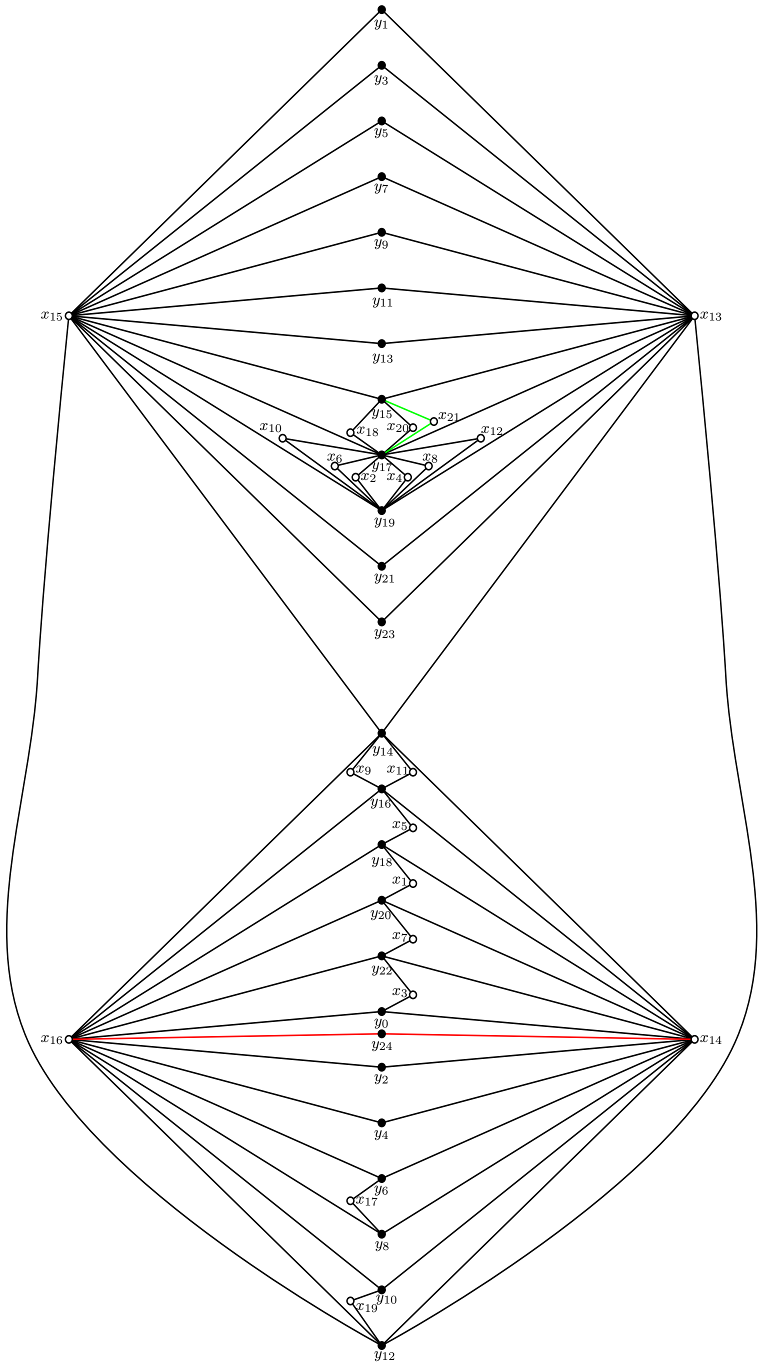

Figure 11: The graph G_4.

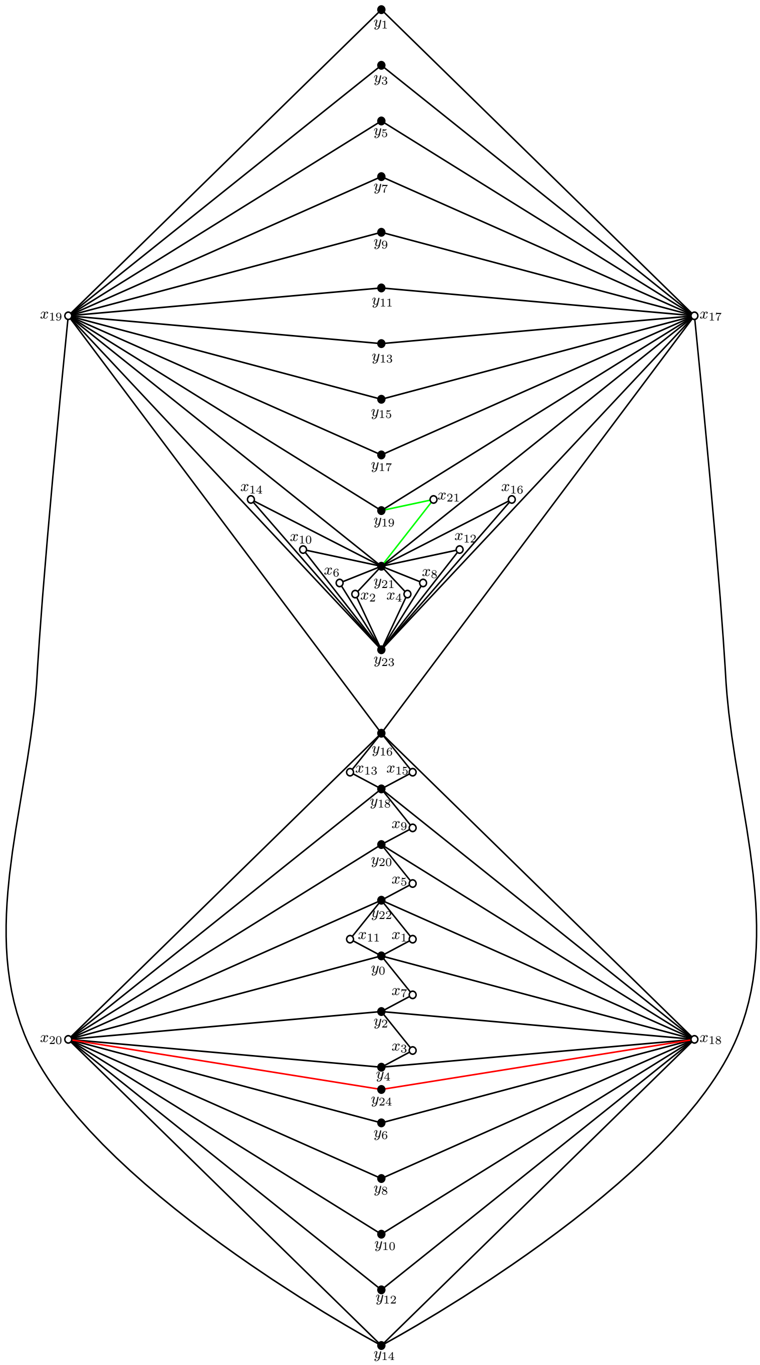

Figure 12: The graph G_5.

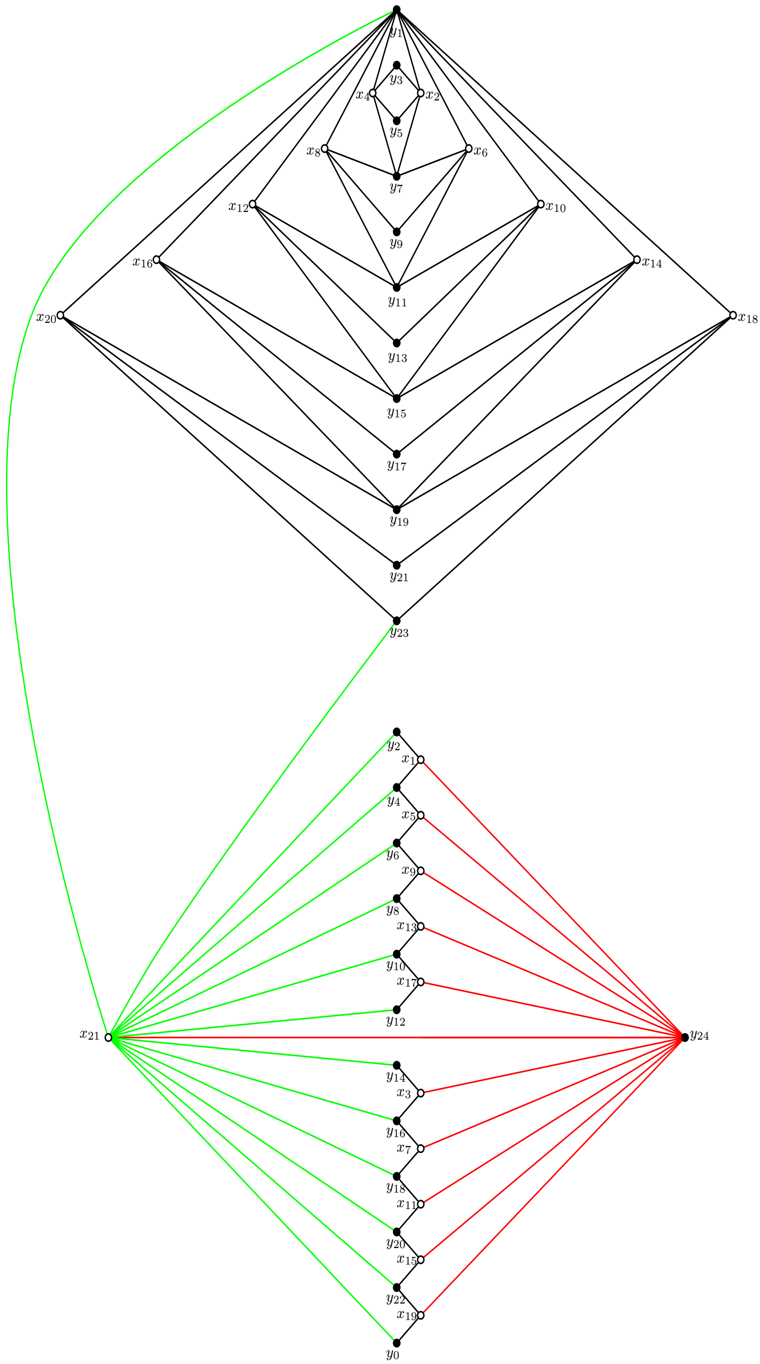

Figure 13: The graph G_6.

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Alphabet-almost-simple 2-neighbour-transitive codes

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Abstract

Let X be a subgroup of the full automorphism group of the Hamming graph $H(m, q)$, and C a subset of the vertices of the Hamming graph. We say that C is an $(X, 2)$-*neighbour-transitive code* if X is transitive on C, as well as C_1 and C_2, the sets of vertices which are distance 1 and 2 from the code. It has been shown that, given an $(X, 2)$-neighbour-transitive code C, there exists a subgroup of X with a 2-transitive action on the alphabet; this action is thus almost-simple or affine. This paper completes the classification of $(X, 2)$-neighbour-transitive codes, with minimum distance at least 5, where the subgroup of X stabilising some entry has an almost-simple action on the alphabet in the stabilised entry. The main result of this paper states that the class of $(X, 2)$ neighbour-transitive codes with an almost-simple action on the alphabet and minimum distance at least 3 consists of one infinite family of well known codes.

Keywords: 2*-neighbour-transitive, alphabet-almost-simple, automorphism groups, Hamming graph, completely transitive.*

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1 Introduction

Ever since Shannon's 1948 paper [18, 19] there has been a great deal of interest around families of error-correcting codes with a high degree of symmetry. The rationale behind this interest is that codes with symmetry should have good error correcting properties. The first families classified were perfect (see [21] or [22]) and nearly-perfect (defined in [12], classified in [15]) codes over prime power alphabets. Note that the classification of nearly-perfect codes follows from the earlier results of [17] on uniformly packed codes, since nearly-perfect codes are uniformly packed codes with maximal packing density. These classifications show that perfect and nearly-perfect codes are rare. In an effort to find further classes of efficient codes, Delsarte [4] introduced *completely regular* codes, a more general class of codes that posses a high degree of combinatorial symmetry. Much effort has been put into classifying particular classes of completely regular codes (see for instance [1, 2]), and new completely regular codes continue to be found [6]. However, completely regular codes have proven to be hard to classify, and this remains an open problem.

Completely transitive codes (first defined in [20], with a generalisation studied in [10]) are a class of codes with a high degree of algebraic symmetry and are a subset of completely regular codes. As such a classification of completely transitive codes would be interesting from the point of view of classifying completely regular codes. This problem also remains open.

Here, the conditions of complete transitivity are relaxed and the family of 2-*neighbour-transitive codes* is studied, a class of codes with a moderate degree of algebraic symmetry. Note that every completely transitive code (see Section 2) is 2-neighbour-transitive. By studying this class of codes we hope to find new codes and gain a better understanding of completely transitive codes. Indeed a classification of 2-neighbour-transitive codes would have as a corollary a classification of completely transitive codes. We also note that codes with 2-transitive actions on the entries of the Hamming graph (which 2-neighbour-transitive codes indeed have), have been of interest lately, where this fact can be used to prove that certain families of codes achieve capacity on erasure channels [14]. The analysis of 2-neighbour-transitive codes is being attacked as three separate problems: *entry-faithful* (see [7]), *alphabet-almost-simple*, and *alphabet-affine*. This paper concerns the alphabet-almost-simple case. The results of this paper do not return any new examples.

However, the results here are of interest from the point of view of perfect codes over an alphabet of non-prime-power size, since in this case a code cannot be alphabet-affine (and also not entry-faithful, by [7]), but may be alphabet-almost-simple. The existence of perfect codes over non-prime-power alphabets with covering radius 1 or 2, is still an open question (see [13]). By Theorem 1.1, if such codes exist, then they cannot be 2-neighbour-transitive (unless they are equivalent to the repetition code of length 3). Note that in the prime power case, for each set of parameters for which a perfect code with covering radius $\rho \geq 2$ exists, a 2-neighbour-transitive code with those parameters exists. That is, the repetition and Golay codes are 2-neighbour-transitive. In fact, the repetition, Hamming and Golay codes are completely transitive (by [11, Example 3.1] for the repetition codes, [20, Proposition 7.3] for the Hamming and binary Golay codes, and [10, Example 3.5.6] for the ternary Golay codes).

1.1 Statement of the main results

Let X be a subgroup of the full automorphism group $S_q^m \rtimes S_m$ of the Hamming graph $\Gamma = H(m,q)$ and let C be a code, that is, a subset of the set of vertices $V\Gamma$. We say that C is an (X,s)*-neighbour-transitive code* if X fixes C setwise and acts transitively on $C = C_0, C_1, \ldots, C_s$ (where C_i are parts of the distance partition, see Section 2). In joint work with Giudici and Praeger [7], the authors classified all $(X,2)$-neighbour-transitive codes for which the group X acts faithfully on the set of entries of the Hamming graph. In this paper, we begin the study of $(X,2)$-neighbour-transitive codes such that the action of X on the entries has a non-trivial kernel.

Let M be the set of entries of the Hamming graph $H(m,q)$ and Q_i be the copy of the alphabet Q in the corresponding entry $i \in M$. Then the vertex set of $H(m,q)$ is:

$$V\Gamma = \prod_{i \in M} Q_i.$$

If C is an $(X,2)$-neighbour-transitive code with minimum distance $\delta \geq 3$, then the subgroup $X_i \leq X$ stabilising the entry $i \in M$ has a 2-transitive action on the alphabet Q_i in that entry (see [7, Proposition 2.7]). Any 2-transitive group G is of affine type ($G \leq \mathrm{AGL}_d(p)$ for some integer d and prime p) or almost-simple type ($S \leq G \leq \mathrm{Aut}(S)$ for some non-abelian simple group S) [5, Theorem 4.1B]. If the action of X on M (see Section 2.1) is transitive with a non-trivial kernel and the action of X_i on the alphabet Q_i is almost-simple then we say C is X*-alphabet-almost-simple*. Our main aim here is to prove the non-existence of codes which are X-alphabet-almost-simple and $(X,2)$-neighbour-transitive with minimum distance $\delta \geq 4$.

Theorem 1.1. *Let C be an X-alphabet-almost-simple and $(X,2)$-neighbour-transitive code in $H(m,q)$ with minimum distance $\delta \geq 3$. Then $\delta = 3$ and C is equivalent to the repetition code in $H(3,q)$, where $q \geq 5$.*

In Section 2 we define the notation used in the paper. In Section 3 we give some results on the structure of codes that are X-alphabet-almost-simple and $(X,2)$-neighbour-transitive, as well as pose some questions about codes for which the action of X_i on the alphabet in the entry $i \in M$ is affine. We present some examples of codes with properties of interest in relation to our results in Section 4. Finally, in Section 5, we give a classification of *diagonally* $(X,2)$*-neighbour-transitive* codes (see Definition 3.1) and prove Theorem 1.1.

2 Preliminaries

Throughout this paper we let $M = \{1, \ldots, m\}$ and $Q = \{1, \ldots, q\}$, with $m, q \geq 2$, though if $q = 2$ we will at times use $Q = \{0, 1\}$. We refer to M as the *set of entries* and Q as the *alphabet*. We use Q_i to denote the disjoint copy of the alphabet Q in the entry $i \in M$. The vertex set $V\Gamma$ of the Hamming graph $\Gamma = H(m,q)$ consists of all m-tuples with entries labeled by the set M, taken from the set Q. An edge exists between two vertices if they differ as m-tuples in exactly one entry. For vertices α, β of $H(m,q)$ the *Hamming distance* $d(\alpha, \beta)$ is the number of entries in which α and β differ, i.e. the usual graph distance in Γ. For $\alpha \in V\Gamma$, we refer to the element of Q appearing in the i-th entry of α as α_i, so that $\alpha = (\alpha_1, \ldots, \alpha_m)$ throughout.

A code C is a subset of the vertex set of the Hamming graph. The *minimum distance* of C is $\delta = \min\{d(\alpha,\beta) \mid \alpha,\beta \in C, \alpha \neq \beta\}$. For a vertex $\alpha \in H(m,q)$, define

$$\varGamma_r(\alpha) = \{\beta \in \varGamma \mid d(\alpha,\beta) = r\}, \quad \text{and} \quad d(\alpha,C) = \min\{d(\alpha,\beta) \mid \beta \in C\}.$$

We then define the *covering radius* to be

$$\rho = \max\{d(\alpha,C) \mid \alpha \in \varGamma\}.$$

For any $r \leq \rho$, define $C_r = \{\alpha \in \varGamma \mid d(\alpha,C) = r\}$. Note that C_i is the disjoint union $\cup_{\alpha\in C}\varGamma_i(\alpha)$ for $i \leq \lfloor \frac{\delta-1}{2} \rfloor$.

2.1 Automorphism groups

The automorphism group $\mathrm{Aut}(\varGamma)$ of the Hamming graph is the semi-direct product $B \rtimes L$, where $B \cong S_q^m$ and $L \cong S_m$ (see [3, Theorem 9.2.1]). We refer to B as the *base group*, and L as the *top group*, of $\mathrm{Aut}(\varGamma)$. Let $g = (g_1,\ldots,g_m) \in B$, $\sigma \in L$ and α be a vertex in $H(m,q)$. Then g and σ act on α as follows:

$$\alpha^g = (\alpha_1^{g_1},\ldots,\alpha_m^{g_m}) \quad \text{and} \quad \alpha^\sigma = (\alpha_{1\sigma^{-1}},\ldots,\alpha_{m\sigma^{-1}}).$$

We define the automorphism group of a code C in $H(m,q)$ to be $\mathrm{Aut}(C) = \mathrm{Aut}(\varGamma)_C$, the setwise stabiliser of C in $\mathrm{Aut}(\varGamma)$. For a subgroup $X \leq \mathrm{Aut}(C)$ we define two other important actions of X which will be useful to us. First, consider the action of X on the set of entries M, which we will write as X^M. In particular $X^M = \mu(X)$, that is, the image of the homomorphism:

$$\begin{array}{rccc} \mu : & X & \longrightarrow & S_m \\ & (h_1,\ldots,h_m)\sigma & \longmapsto & \sigma \end{array}.$$

Note that σ here is not necessarily an automorphism of C, that is, σ is a permutation of M but may not necessarily fix C setwise, though its pre-image $(h_1,\ldots,h_m)\sigma$ is an element of $\mathrm{Aut}(C)$. We define K to be the kernel of the map μ and note that $K = X \cap B$. In this paper we are concerned with $(X,2)$-neighbour-transitive codes where $K \neq 1$.

We also consider the action of the stabiliser $X_i \leq X$ of the entry $i \in M$, on the alphabet Q_i in that entry. We denote this action by $X_i^{Q_i} = \varphi_i(X_i)$, and it is the image of the homomorphism:

$$\begin{array}{rccc} \varphi_i : & X_i & \longrightarrow & S_q \\ & (h_1,\ldots,h_m)\sigma & \longmapsto & h_i \end{array}.$$

Let C be a code in $H(m,q)$ and let X be a subgroup of $\mathrm{Aut}(\varGamma)$. Recall that C is (X,s)*-neighbour-transitive* if each C_i is an X-orbit for $i = 0,\ldots,s$. Note that this implies $X \leq \mathrm{Aut}(C)$ and C is also (X,r)-neighbour-transitive, for $r < s$. If $s = 1$ then C is simply X*-neighbour-transitive* and if $s = \rho$, the covering radius, then C is X*-completely transitive*.

An *almost-simple* group is a group G where $S \leq G \leq \mathrm{Aut}(S)$ for some non-abelian simple group S. The *socle* of a group G, denoted $\mathrm{soc}(G)$, is the product of its minimal normal subgroups. The socle of an almost-simple group G is the non-abelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$. Recall, if C is a code and $X \leq \mathrm{Aut}(C)$ such that $K \neq 1$,

X^M is transitive on M and the $X_i^{Q_i}$ is almost-simple, then we say C is *X-alphabet-almost-simple*. We may sometimes omit the group X from any of the above terms, if the meaning is clear from the context.

We say that two codes, C and C', in $H(m,q)$, are *equivalent* if there exists $x \in \mathrm{Aut}(\Gamma)$ such that $C^x = C'$. Since elements of $\mathrm{Aut}(\Gamma)$ preserve distance, equivalence preserves minimum distance.

2.2 Projections

For a subset $J = \{j_1,\dots,j_k\} \subseteq M$ we define the *projection of α with respect to J* as $\pi_J(\alpha) = (\alpha_{j_1},\dots,\alpha_{j_k})$. For a code C we then define the *projection of C with respect to J* as $\pi_J(C) = \{\pi_J(\alpha) \mid \alpha \in C\}$. So π_J maps a vertex or code from $H(m,q)$ into the smaller Hamming graph $H(k,q)$.

Let X_J be the setwise stabiliser of a subset $J = \{j_1,\dots,j_k\} \subseteq M$. For $x = (h_1,\dots,h_m)\sigma \in X_J$, we define the *projection of x with respect to J* as $\chi_J(x)$ where

$$\pi_J(\alpha)^{\chi_J(x)} = \pi_J(\alpha^x).$$

To be well defined, this requires $x \in X_J$ and it follows that

$$\chi_J(x) = (h_{j_1},\dots,h_{j_k})\hat{\sigma} \in \mathrm{Aut}(H(k,q)),$$

where $\hat{\sigma}$ is the element of $\mathrm{Sym}(J)$ induced by σ. Moreover, we define $\chi_J(X) = \{\chi_J(x) \mid x \in X_J\}$.

3 Structural results

Some results from [8], in which X-alphabet-almost-simple and X-neighbour-transitive codes with $\delta \geq 3$ are characterised, are stated below. This is our starting point when looking at codes that are X-alphabet-almost-simple and $(X,2)$-neighbour-transitive with $\delta \geq 3$, since we then have that C is indeed X-neighbour-transitive. The following definitions are needed first. For a subgroup $T \leq S_q$ define $\mathrm{Diag}_m(T) = \{(h,\dots,h) \in B \mid h \in T\}$.

Definition 3.1. A code C in $H(m,q)$ is *diagonally (X,s)-neighbour-transitive* if C is (X,s)-neighbour-transitive and $X \leq \mathrm{Diag}_m(S_q) \rtimes L$.

Each part of Proposition 3.2 is proved in the relevant citation of [8]. Recall the definitions of: $\pi_J(C)$ and $\chi_J(X)$ (see Section 2.2), the socle $\mathrm{soc}(G)$ and the kernel $K = X \cap B$ for the action of X on M, where $B \cong S_m$ is the base group of $\mathrm{Aut}(\Gamma)$ (see Section 2.1). Note also that G is a *sub-direct* subgroup of a direct product $\prod_{i=1}^n T_i$ of isomorphic groups $T_i \cong T$, where $i \in \{1,\dots,n\}$, if the projection of G in each coordinate is isomorphic to T.

Proposition 3.2. *Suppose C is an X-neighbour-transitive code in $H(m,q)$ with $\delta \geq 3$. Then the following hold:*

i) Let $\mathcal{J}$ be an X-invariant partition of M and $J \in \mathcal{J}$ such that $\pi_J(C)$ is not the complete code. Then $\pi_J(C)$ is $\chi_J(X)$-neighbour-transitive [8, Proposition 3.4]. (Note that the assumption that $\pi_J(C)$ is not the complete code does not appear in [8], but is necessary since the proof assumes that $\pi_J(C)_1$ is non-empty.)

ii) Let $\mathcal{J}$ be an X-invariant partition of M and $J \in \mathcal{J}$ such that $\pi_J(C)$ is not the complete code. Then $\pi_J(C)$ has minimum distance at least 2 [8, Corollary 3.7].

iii) If C is also X-alphabet-almost-simple, then $\operatorname{soc}(K)$ is a sub-direct subgroup of $\prod_{i\in M} \operatorname{soc}\left(X_i^{Q_i}\right)$ [8, Proposition 5.2].

While the next result is not explicitly stated in [8], it is the basis of the characterisation contained within it.

Proposition 3.3. *Let C be an X-alphabet-almost-simple and X-neighbour-transitive code with $\delta \geq 3$. Then there exists an X-invariant partition $\mathcal{J}$ of M such that for all $J \in \mathcal{J}$ the code $\pi_J(C)$ is equivalent to a diagonally $\chi_J(X)$-neighbour-transitive with minimum distance $\delta(\pi_J(C)) \geq 2$.*

Proof. Let T be the non-abelian simple socle of the almost-simple 2-transitive group $X_i^{Q_i}$. By Proposition 3.2-(iii), the group $\operatorname{soc}(K)$ is a sub-direct subgroup of $\prod_{i\in M} \operatorname{soc}\left(X_i^{Q_i}\right)$. Following the discussion after [8, Proposition 5.2], Scott's Lemma [16, p. 328] can be applied to give a partition $\mathcal{J}$ of M such that $\operatorname{soc}(K) = \prod_{J\in\mathcal{J}} D_J$, where each $D_J \cong \operatorname{Diag}_k(T)$ acts on $\pi_J(V\varGamma)$, for all $J \in \mathcal{J}$, where $k = |J|$. Moreover, by [8, Remark 5.5], $\mathcal{J}$ is X-invariant. By examining $\operatorname{soc}(K)$, it can be shown [8, Section 5] that, up to equivalence, two possibilities occur. Either $\chi_J(X) \leq \operatorname{Diag}_k(S_q) \rtimes S_k$, where $k = |J|$, for all $J \in \mathcal{J}$, or $\mathcal{J}$ can be replaced by a more refined X-invariant partition $\hat{\mathcal{J}}$ of M such that $\chi_{\hat{J}}(X) \leq \operatorname{Diag}_{\hat{k}}(S_q) \rtimes S_{\hat{k}}$, where $\hat{k} = |\hat{J}|$, for all $\hat{J} \in \hat{\mathcal{J}}$.

In either case, it follows from Proposition 3.2-(i) and (ii) that, for all $J \in \mathcal{J}$ or $\hat{\mathcal{J}}$ respectively, $\chi_J(X)$ acts transitively on $\pi_J(C)$ and either $\pi_J(C)$ is the complete code or it is $\chi_J(X)$-neighbour-transitive with minimum distance at least 2. Since $\chi_J(X)$ is a diagonal subgroup, we deduce that $\pi_J(C)$ is as in the second case, since no diagonal subgroup acts transitively on the complete code. □

Proposition 3.4. *Let C be an $(X, 2)$-neighbour-transitive code with $\delta \geq 3$ in $H(m, q)$, and suppose $\mathcal{J}$ is an X-invariant partition of M. Then for all $J \in \mathcal{J}$, either;*

i) $\pi_J(C)$ is the complete code, $\delta(\pi_J(C)) = 1$, and $\chi_J(X)$ is transitive on $\pi_J(C)$;

ii) $\pi_J(C)$ has covering radius 1, $\delta(\pi_J(C)) = 2$ or 3, and is $(\chi_J(X), 1)$-neighbour-transitive; or,

iii) $\pi_J(C)$ is $(\chi_J(X), 2)$-neighbour-transitive.

Proof. Let $\bar{C} = \pi_J(C)$. The fact that $\chi_J(X)$ is transitive on $\bar{C}$ and $\bar{C}_1$, if $\bar{C}_1$ is non-empty, follows from Proposition 3.2-(i). From this we deduce (i) and (ii). In particular, suppose the covering radius of $\bar{C}$ is at most 1. If the covering radius is 0 then $\bar{C}$ is the complete code, and if the covering radius is 1 then $\bar{C}$ is not the complete code and the minimum distance is at most 3 so, by Proposition 3.2-(ii), the minimum distance is at least 2. Therefore, we need only show that when $\bar{C}_2$ is non-empty $\chi_J(X)$ is transitive on $\bar{C}_2$.

Suppose $\bar{C}$ has covering radius at least 2. Let $\mu, \nu \in \bar{C}_2$. Then there exists $\alpha, \beta \in C$ such that $d(\mu, \pi_J(\alpha)) = d(\nu, \pi_J(\beta)) = 2$. Let $\hat{\nu} \in H(m, q)$ with $\hat{\nu}_u = \nu_u$ for u in J and $\hat{\nu}_v = \alpha_v$ otherwise. Similarly, let $\hat{\mu} \in H(m, q)$ with $\hat{\mu}_u = \mu_u$ for u in J and $\hat{\mu}_v = \beta_v$ otherwise. We claim that $\hat{\nu}, \hat{\mu} \in C_2$. We show this for $\hat{\nu}$ and note that an identical argument holds for $\hat{\mu}$. First, note that $d(\alpha, \hat{\nu}) = 2$ and $\delta \geq 3$, so $\hat{\nu} \notin C$. Suppose $\hat{\nu} \in C_1$. Then there exists $\alpha' \in C$ such that $d(\hat{\nu}, \alpha') = 1$. We then have $d(\nu, \pi_J(\alpha')) \leq 1$. However, this contradicts $\nu \in \bar{C}_2$. Hence $\hat{\mu}, \hat{\nu} \in C_2$.

As C is $(X, 2)$-neighbour-transitive, there exists an $x = h\sigma \in X$ mapping $\hat{\nu}$ to $\hat{\mu}$. We claim $x \in X_J$. Suppose $x \notin X_J$. Then, since $\mathcal{J}$ is a system of imprimitivity for the action of X on M, there exists $J' \in \mathcal{J}$ such that $J \neq J'$ and $J'^{\sigma} = J$. Since $\pi_{J'}(\hat{\nu}) = \pi_{J'}(\alpha)$, this implies that $\pi_J(\hat{\nu}^x) = \pi_J(\alpha^x) \in \bar{C}$ and hence $\pi_J(\hat{\nu}^x) \neq \mu$, which contradicts the fact that $\hat{\nu}^x = \hat{\mu}$. Thus $x \in X_J$ and

$$\nu^{\chi_J(x)} = \pi_J(\hat{\nu})^{\chi_J(x)} = \pi_J(\hat{\nu}^x) = \pi_J(\hat{\mu}) = \mu.$$

□

Proposition 3.5. *Let C be an $(X, 2)$-neighbour-transitive code in $H(m, q)$ with $\delta \geq 3$, and $\mathcal{J}$ be an X-invariant partition of M. Then, for all $J \in \mathcal{J}$ and $i \in J$,*

1. *$\chi_J(X)_i^{Q_i}$ is 2-transitive on Q; and,*
2. *for $\alpha \in C$, $\chi_J(X)_{\pi_J(\alpha)}$ is transitive on J.*

Proof. As C is X-neighbour-transitive with $\delta \geq 3$, we have that $X_i^{Q_i}$ is 2-transitive, by [7, Proposition 2.7], and X^M is transitive, by [7, Proposition 2.5]. One then deduces that $X_i^{Q_i}$ is 2-transitive for all i. Now, because $\mathcal{J}$ is an X-invariant partition, it follows that $X_i = (X_J)_i$ for all $i \in J$. This in turn implies that $\chi_J(X)_i = \chi_J(X_i)$. It is now straight forward to show that $\chi_J(X_i)^{Q_i} = X_i^{Q_i}$.

Now, since X_α is transitive on M and $\mathcal{J}$ is an X-invariant partition of M, it follows that $(X_\alpha)_J$ is transitive on J. Thus $\chi_J(X_\alpha) \leq \chi_J(X)_{\pi(\alpha)}$ is transitive on J. □

The previous two propositions suggest a study of codes that are $(X, 2)$-neighbour-transitive, have minimum distance $\delta \geq 2$, and where X acts primitively on M. An answer to the following questions would provide us with the building blocks for $(X, 2)$-neighbour-transitive codes with $\delta \geq 3$.

Question 3.6. Can we classify all $(X, 2)$-neighbour-transitive codes with $\delta \geq 2$ such that X^M is primitive and $X_i^{Q_i}$ is 2-transitive?

Question 3.7. Can we classify all $(X, 1)$-neighbour-transitive codes with $\delta = 2$ or 3 and $\rho = 1$ such that X^M is primitive and $X_i^{Q_i}$ is 2-transitive?

Let C be a code and $X \leq \mathrm{Aut}(C)$. If X acts faithfully on M, that is $K = X \cap B = 1$, we say C is *X-entry-faithful*. If $K \neq 1$, X^M is transitive on M and $X_i^{Q_i}$ is affine ($X_i^{Q_i} \leq \mathrm{AGL}_d(p)$ for some integer d and prime p) we say C is *X-alphabet-affine*. Questions 3.6 and 3.7 can be further broken down into X-entry-faithful and non-trivial kernel cases, that is, X-alphabet-affine and X-alphabet-almost-simple (see Section 2.1 for the definition of X-alphabet-almost-simple). By the main result of this paper, the outstanding cases of Question 3.6 are X-alphabet-almost-simple and $(X, 2)$-neighbour-transitive with $\delta = 2$, and X-alphabet-affine and $(X, 2)$-neighbour-transitive, where X^M is primitive and $X_i^{Q_i}$ is 2-transitive.

Given Proposition 3.3, a third question is the following.

Question 3.8. Can we construct $(X, 2)$-neighbour-transitive codes with $\delta \geq 3$ by taking copies of $(X, 1)$-neighbour-transitive codes with $\delta = 2$ or 3 and $\rho = 1$.

4 Examples

We begin this section by considering some examples of codes which have properties relating to the results of the previous section. We first introduce the operators Prod and Rep which allow the construction of new codes from old ones. For an arbitrary code C in $H(m,q)$ we define $\mathrm{Prod}(C,\ell)$ and $\mathrm{Rep}_\ell(C)$ in $H(m\ell,q)$ as

$$\mathrm{Prod}(C,\ell) = \{(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_\ell) \mid \boldsymbol{\alpha}_i \in C\},$$

and

$$\mathrm{Rep}_\ell(C) = \{(\boldsymbol{\alpha},\dots,\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C\}.$$

The *repetition code* $\mathrm{Rep}(m,q)$ in $H(m,q)$ is the set of all vertices $(a,\dots,a)$ consisting of a single element $a \in Q$ repeated m times.

The next two examples present codes which are both X-alphabet-almost-simple and X-completely transitive, though the second example has minimum distance $\delta = 2$.

Example 4.1. Let $C = \mathrm{Rep}(3,q)$, where $q \geq 5$, and $X = \mathrm{Diag}_3(S_q) \rtimes S_3$, as in [11, Example 3.1]. Now,

$$C_1 = \{(a,a,b),(a,b,a),(b,a,a) \mid a,b \in Q; a \neq b\},$$

and

$$C_2 = \{(a,b,c) \mid a,b,c \in Q; a \neq b \neq c \neq a\}.$$

Since S_q acts 3-transitively on Q and S_3 acts transitively on M, it follows that X acts transitively on C, C_1 and C_2. Thus C is $(X,2)$-neighbour-transitive and X-completely transitive, since C has covering radius $\rho = 2$. Also, $X_i^{Q_i} \cong S_q$ is almost-simple, since $q \geq 5$, and $X^M \cong S_3$ is transitive on M. Hence C is X-alphabet-almost-simple and X-completely transitive.

Example 4.2. Let $q \geq 5$, $\ell \geq 2$, $C = \mathrm{Prod}(\mathrm{Rep}(2,q),\ell)$ and $X = (\mathrm{Diag}_2(S_q))^\ell \rtimes U$, where $\mathrm{Diag}_2(S_q)$ is a subgroup of the base group of $\mathrm{Aut}(H(2,q))$ and $U = S_2 \wr S_\ell = S_2^\ell \rtimes S_\ell$ is a subgroup of the top group of $\mathrm{Aut}(H(2\ell,q))$. Let $\mathcal{J} = \{J_1,\dots,J_\ell\}$, with $J_i = \{2i-1,2i\}$, be the partition of M preserved by U. Note that $\delta = 2$. Let $R \subseteq \{1,\dots,\ell\}$ of size s, and $\nu \in H(m,q)$ be such that $\pi_{J_i}(\nu) = (a,b)$, where $a \neq b$ for all $i \in R$, and $a = b$ for all $i \notin R$. Any codeword β is at least distance s from ν, since $d(\pi_{J_i}(\nu),\pi_{J_i}(\beta)) \geq 1$ for each $i \in R$. Also, there exists some codeword α with $\pi_{J_i}(\alpha) = (a,a)$ whenever $\pi_{J_i}(\nu) = (a,b)$ for $i \in \{1,\dots,\ell\}$, and hence $d(\alpha,\nu) = s$. So $\nu \in C_s$. Any vertex ν of $H(2\ell,q)$ can be expressed in this way, for some R, since $\pi_{J_i}(\nu) = (a,b)$ has either $a = b$ or $a \neq b$. Thus, for each s, C_s consists of all such vertices ν where $|R| = s$. It also follows from this that $\rho = \ell$.

Let $\nu \in C_s$, with R as above. Let $x = (h_{J_1},\dots,h_{J_\ell})\sigma \in X$ where $h_{J_i} \in \mathrm{Diag}_2(S_q)$ such that $\pi_{J_i}(\nu)^{h_{J_i}} = (1,2)$, for $i \in R$, and $\pi_{J_i}(\nu)^{h_{J_i}} = (1,1)$, for all $i \notin R$. Moreover, since S_ℓ is ℓ-transitive, there exists $\sigma \in S_\ell \leq S_2 \wr S_\ell$ mapping $\{J_{i_1},\dots,J_{i_s}\}$ to $\{J_1,\dots,J_s\}$ (where $R = \{i_1,\dots,i_s\}$), whilst preserving order within each J_i. Then $\nu^x = \gamma \in C_s$, where $\pi_{J_i}(\gamma) = (1,2)$ for all $i \in \{1,\dots,s\}$ and $\pi_{J_i}(\gamma) = (1,1)$ for all $i \notin \{s+1,\dots,\ell\}$. Since we can map any such ν to γ, X is transitive on C_s for each $s \in \{1,\dots,\ell\}$. Hence C is X-completely transitive, and in particular $(X,2)$-neighbour-transitive for $\ell \geq 2$. Since $X_i^{Q_i} \cong S_q$ and $X^M \cong S_2 \wr S_\ell$ is transitive on M, C is X-alphabet-almost-simple.

Lemma 4.3. *Suppose C is an $(X,2)$-neighbour-transitive code in $H(m,q)$, with $q \geq 3$, and $\mathcal{J}$ is an X-invariant partition of M, such that $\pi_J(C) = \mathrm{Rep}(k,q)$, for all $J \in \mathcal{J}$ where $k = |J|$. Then either $\delta = k = 2$, or $\mathcal{J}$ is a trivial partition.*

Proof. Let $x = (h_1,\ldots,h_m)\sigma \in X$ and $J \in \mathcal{J}$. By the hypothesis it follows that for all $a \in Q$, there exists $\alpha \in C$ such that $\pi_J(\alpha) = (a,\ldots,a)$. Suppose $J^\sigma = J' \in \mathcal{J}$. Then $\pi_{J'}(\alpha^x) = (a^{h_{i_1}},\ldots,a^{h_{i_k}})\sigma = (b,\ldots,b)$ for some $b \in Q$, that is, $a^{h_{i_s}} = a^{h_{i_t}}$ for all $i_s, i_t \in J$. In particular $\chi_J(x\sigma^{-1}) = (h,\ldots,h)$ for some $h \in S_q$, and $X \leq \mathrm{Diag}_k(S_q) \wr U$, where U is the stabiliser of $\mathcal{J}$ in the top group.

Suppose that the partition $\mathcal{J}$ is non-trivial, so that $k, \ell \geq 2$. Since C is a subset of $\mathrm{Prod}(\mathrm{Rep}(k,q),\ell)$, which has minimum distance k, it follows that $\delta \geq k \geq 2$.

Suppose $\delta \geq 3$. As C is a subset of $\mathrm{Prod}(\mathrm{Rep}(k,q),\ell)$ we can replace C by an equivalent code contained in $\mathrm{Prod}(\mathrm{Rep}(k,q),\ell)$ containing $\alpha = (1,\ldots,1)$ and such that

$$\mathcal{J} = \{\{1,\ldots,k\},\{k+1,\ldots,2k\},\cdots,\{m-k+1,\ldots,m\}\}.$$

Consider,

$$\begin{aligned}\mu &= (2,3,1,1,\ldots,1,1,1,1,\ldots,1,\cdots,1,\ldots,1) \quad \text{and}\\ \nu &= (\underbrace{2,1,1,1,\ldots,1}_{k \text{ entries}},\underbrace{2,1,1,\ldots,1}_{k \text{ entries}},\cdots,\underbrace{1,\ldots,1}_{k \text{ entries}}).\end{aligned}$$

If $k=2$, then we claim $\mu \in C_2$. Any vertex $\beta \in \mathrm{Prod}(\mathrm{Rep}(2,q),\ell) \supseteq C$ with $d(\mu,\beta) = 1$ is of the form $\gamma = (a,a,1,\ldots,1)$, where $a = 2$ or 3. However, no such γ is an element of C, since each is distance 2 from α. If $k \geq 3$ then $\mu \in C_2$ since $d(\alpha,\mu) = 2$ and there is no closer codeword as $\pi_{J_1}(\mu) \in \pi_{J_1}(C)_2$. In both cases $\nu \in C_2$ since $d(\alpha,\nu) = 2$ and no codeword is closer, as $\pi_{J_i}(\nu) \in \pi_{J_i}(C)_1$ for $i = 1,2$. Let $x = (h_1,\ldots,h_m)\sigma \in X$ such that $\mu^x = \nu$. We reach a contradiction here, since $h_1 = h_2 = \cdots = h_k = h$ cannot, assuming $k \geq 3$, map the set $\{1,2,3\}$ to either of the sets $\{1,2\}$ or $\{1\}$. In the case $k=2$, in at least one block we must map the set $\{1\}$ to $\{1,2\}$, which is not possible. Hence $2 \geq \delta \geq k \geq 2$. □

Suppose $\mathcal{J}$ is a system of imprimitivity for the action of X on M and C is an X-neighbour-transitive code, with $\delta \geq 3$. The next example shows that it is possible that the projection of C onto each block of $\mathcal{J}$ gives the complete code, though this is not the system of imprimitivity of interest to us in Proposition 3.3.

Example 4.4. Let $\bar{C} = \mathrm{Prod}(C,\ell)$ be a code in $\Gamma = H(m,q)$, where $m = k\ell$ and C is an X-neighbour-transitive code in $H(k,q)$ where $X \cap B$ is transitive on C and $\delta \geq 3$. Let $\bar{X} = \langle (X\cap B)^\ell, \mathrm{Diag}_\ell(X), S_\ell\rangle$ preserve the partition

$$\mathcal{J} = \{\{1,\ldots,k\},\ldots,\{m-k+1,\ldots,m\}\} = \{J_1,\ldots,J_\ell\},$$

of M, where $\chi_J((X\cap B)^\ell) = X \cap B$ and $\chi_J(\mathrm{Diag}_\ell(X)) = X$ for all $J \in \mathcal{J}$, and S_ℓ acts as pure permutations by permuting the blocks of $\mathcal{J}$ whilst preserving the order of entries within a given block. It follows that we preserve two $\bar{X}$-invariant partitions. These being $\mathcal{J}$ and $\mathcal{J}'$, where $\mathcal{J}'$ is attained by taking the corresponding entries, by order, from each copy of C to form each block:

$$\mathcal{J}' = \{\{1,k+1,\ldots,m-k+1\},\ldots,\{\ell,k+\ell,\ldots,m\}\}.$$

Given any $\alpha = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_\ell) \in \bar{C}$, $\boldsymbol{\alpha}_i \in C$, and $\beta = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_\ell) \in \bar{C}$, $\boldsymbol{\beta}_i \in C$ there exists an $x \in (X \cap B)^\ell$ mapping α to β since $X \cap B$ is transitive on C. Hence $\bar{X}$ is transitive on $\bar{C}$. Given any two neighbours $\mu, \nu \in \Gamma_1(\alpha)$, where μ, ν differ from α in the respective blocks J_i and J_j, we can map J_j to J_i via some element $\sigma \in S_\ell$. Then, since $X_{\boldsymbol{\alpha}_i}$ is transitive on $\Gamma_1(\boldsymbol{\alpha}_i)$, there exists an element $x \in \mathrm{Diag}_\ell(X)$ such that $\pi_{J_i}(\nu^{\sigma x}) = \pi_{J_i}(\mu)$. We can then map $\nu^{\sigma x}$ to μ via some element $h \in (X \cap B)^\ell$, where $\chi_{J_i}(h) = 1$, since each $\pi_{J_t}(\nu^{\sigma x})$ and $\pi_{J_t}(\mu)$ are elements of C for $t \neq i$ and $X \cap B$ is transitive on C. Hence $\sigma x h$ maps ν to μ and $\bar{X}$ is transitive on $\bar{C}_1$.

When we consider the projection $\pi_J(\bar{C})$ for any $J \in \mathcal{J}'$ we are left with the complete code. To see this, consider that for $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_\ell) \in \bar{C}$, $\boldsymbol{\alpha}_i \in C$, we may choose an arbitrary element of C as $\boldsymbol{\alpha}_i$ for each i. Since $X_i^{Q_i}$ is 2-transitive on Q_i, each element appears in the first entry for some codeword. Thus, as $\pi_J((\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_\ell))$ when $J = \{1, k+1, \dots, m-k+1\}$ is the first entry of each $\boldsymbol{\alpha}_i$, we have that $\pi_J(\bar{C})$ is the complete code.

5 Alphabet-almost-simple $(X, 2)$-neighbour-transitive codes

Before we prove the final results we define the codes used in this section, which first requires the following definition.

Definition 5.1. Define the *composition* of a vertex $\alpha \in H(m, q)$ to be the set

$$Q(\alpha) = \{(a_1, p_1), \dots, (a_q, p_q)\},$$

where p_i is the number of entries of α which take the value $a_i \in Q$. For $\alpha \in H(m, q)$ define the set

$$\mathrm{Num}(\alpha) = \{(p_1, s_1), \dots, (p_j, s_j)\},$$

where (p_i, s_i) means that s_i distinct elements of Q appear precisely p_i times in α.

Definition 5.2. We define the following codes:

1. $\mathrm{Inj}(m, q)$, where $m < q$, is the set of all vertices $\alpha \in H(m, q)$ such that $\mathrm{Num}(\alpha) = \{(1, m)\}$;
2. for m odd, $W([m/2], 2)$ is the set of vertices in $\alpha \in H(m, 2)$ such that $\mathrm{Num}(\alpha) = \{(m+1)/2, 1), (m-1)/2, 1)\}$; and,
3. $\mathrm{All}(pq, q)$, with $pq = m$, is the set of all vertices $\alpha \in H(m, q)$ such that $\mathrm{Num}(\alpha) = \{(p, q)\}$.

More information on these codes is available in [9, Definition 2]. The following lemma is [9, Lemma 4].

Lemma 5.3. *Let α be a vertex in $H(m, q)$. Then $\mathrm{Num}(\alpha)$ is preserved by $\mathrm{Diag}_m(S_q) \rtimes L$.*

The last result, in combination with the classification of diagonally neighbour-transitive codes [9, Theorem 4.3], allows us to prove the next result.

Proposition 5.4. *Let C be a diagonally $(X, 2)$-neighbour-transitive code in $H(m, q)$. Then one of the following holds:*

1. *$q = 2$ and $C = \{(a, \dots, a)\}$;*
2. *$m = 3$ or $q = 2$, and $C = \mathrm{Rep}(m, q)$;*

3. *$C = \mathrm{Inj}(3, q)$;*
4. *m is odd and $C = W([m/2], 2)$; or,*
5. *$q = 2$ or $q = m = 3$, and there exists some p such that $m = pq$ and C is a subset of $\mathrm{All}(pq, q)$.*

Proof. From [9, Theorem 4.3], we have that a diagonally neighbour-transitive code C is one of: $\{(a, \dots, a)\}$ for some $a \in Q$, $\mathrm{Rep}(m, q)$, $\mathrm{Inj}(m, q)$ with $m < q$, $W([m/2], 2)$ with m odd, or there exists a p such that $m = pq$ and C is a subset of $\mathrm{All}(pq, q)$. Here we consider $m \geq 2$, since if $m = 1$ then C_2 is empty, so C is not $(X, 2)$-neighbour-transitive. Also to prove some C is $(X, 2)$-neighbour-transitive, we need only find some $X \leq \mathrm{Aut}(C)$ such that $X \leq \mathrm{Diag}_m(S_q) \rtimes L$ and X is transitive on C_2, since C is already X-neighbour-transitive, for some X, by [9, Theorem 4.3].

First, if $C = \mathrm{Inj}(2, q)$ then C_2 is empty. Thus, C is not $(X, 2)$-neighbour-transitive. Table 1 lists the remaining cases which are not 2-neighbour-transitive. The second and third columns give a pair $\mu, \nu \in C_2$ such that $\mathrm{Num}(\mu) \neq \mathrm{Num}(\nu)$. Hence, by Lemma 5.3, X is not transitive on C_2. It can be deduced from $\mathrm{Num}(\mu), \mathrm{Num}(\nu)$ that $\mu, \nu \in C_2$, since this makes it clear that we must change μ, ν in at least two entries to get a vertex in C. Note that we let $\alpha = (1, 2, 3, \dots, q) \in H(q, q)$ and in the second last and last rows we assume $\alpha \in C$ and $(\alpha, \dots, \alpha) \in C$, respectively, and observe for the last row $\hat{\mu} = (1, 1, 1, 4, 5, \dots, q)$, $\hat{\nu} = (1, 1, 3, 4, 5, \dots, q)$ are in $\Gamma_2(\alpha)$.

C Conditions	$\mu \in C_2$ $\mathrm{Num}(\mu)$	$\nu \in C_2$ $\mathrm{Num}(\nu)$
$\{(a, \dots, a)\}$ $q \geq 3$	$(b, b, a, \dots, a)$ $\{(m-2, 1), (2, 1)\}$	$(b, c, a, \dots, a)$ $\{(m-2, 1), (1, 2)\}$
$\mathrm{Rep}(m, q)$ $m > q \geq 3$	$(2, 2, 1, \dots, 1)$ $\{(m-2, 1), (2, 1)\}$	$(2, 3, 1, \dots, 1)$ $\{(m-2, 1), (1, 2)\}$
$\mathrm{Inj}(m, q)$ $m \geq 4$	$(1, 1, 1, 4, 5, \dots, m)$ $\{(3, 1), (1, m-3)\}$	$(1, 1, 3, 3, 5, 6, \dots, m)$ $\{(2, 2), (1, m-4)\}$
$\subseteq \mathrm{All}(q, q)$ $q \geq 4$	$(1, 1, 1, 4, 5, \dots, q)$ $\{(3, 1), (1, q-3)\}$	$(1, 1, 3, 3, 5, 6, \dots, q)$ $\{(2, 2), (1, q-4)\}$
$\subseteq \mathrm{All}(pq, q)$ $q > p \geq 2$	$(\hat{\mu}, \alpha, \dots, \alpha)$ $\{(p-1, 2), (p, q-3), (p+2, 1)\}$	$(\hat{\nu}, \hat{\nu}, \alpha, \dots, \alpha)$ $\{(p-2, 1), (p, q-2), (p+2, 1)\}$

Table 1: Diagonally neighbour-transitive codes C which are not diagonally 2-neighbour-transitive, and elements of C_2 which illustrate this. Note: $\hat{\mu} = (1, 1, 1, 4, 5, \dots, q)$, $\hat{\nu} = (1, 1, 3, 4, 5, \dots, q)$ and $\alpha = (1, 2, 3, \dots, q)$.

Now we prove the result for the cases which are 2-neighbour-transitive. Suppose $C = \{(a, \dots, a)\}$ for some $a \in Q$. Let $q = 2$ and $Q = \{0, 1\}$. Then $L = S_m = \mathrm{Aut}(C)$. Without loss of generality, let $a = 0$ so that C_2 is the set of weight two vertices. Since L is transitive on the sets of weight 2 and weight 1 vertices, it follows C is diagonally $(X, 2)$-neighbour-transitive. Let $C = \mathrm{Rep}(m, q)$. It follows from Example 4.1 that $\mathrm{Rep}(3, q)$ is $(\mathrm{Diag}_3(S_q) \rtimes S_3, 2)$-neighbour-transitive. If $q = 2$ then $\mathrm{Aut}(C) \cong \mathrm{Diag}_m(S_2) \rtimes S_m$ and C is completely transitive [11, Example 3.1]. Consider $C = \mathrm{Inj}(m, q)$ with $3 = m < q$ and $q \geq 4$. If $\nu \in C_2$ then $\nu_1 = \nu_2 = \nu_3$, since otherwise $\nu \in C$ or C_1. Since $\mathrm{Diag}_m(S_q) \leq \mathrm{Aut}(C)$, we are transitive on C_2. Suppose $C = W([m/2], 2)$ and m is odd. Then by [9, Corollary 3.4] C is $\mathrm{Diag}(S_2) \rtimes S_m$-completely transitive. Finally, suppose C is a subset of $\mathrm{All}(pq, q)$ for some p such that $m = pq$. Let $p \geq 2$, $q = 2$ and $C = \mathrm{All}(2p, 2)$. Then C_2 is the set of all weight $p \pm 2$ vertices, which $\mathrm{Diag}_2(S_2) \rtimes S_m \leq \mathrm{Aut}(C)$ is transitive on. Let $p = 1$, $q = 3$ and $C = \mathrm{All}(3, 3)$. Then $C_2 = \mathrm{Rep}(3, q)$ and is $\mathrm{Aut}(C)$-completely transitive by Example 4.1. □

With our classification of diagonally $(X, 2)$-neighbour-transitive codes from the previous result, Propositions 3.3 and 3.4 mean we are now in a position to prove the main theorem.

Proof of Theorem 1.1. Suppose C is an X-alphabet-almost-simple and $(X, 2)$-neighbour-transitive code with $\delta \geq 3$ such that $X \cap B \neq 1$. By Proposition 3.3, there exists an X-invariant partition $\mathcal{J} = \{J_1, \ldots, J_\ell\}$, for some ℓ, for the action of X on M. Moreover, $\pi_{J_i}(C)$ has minimum distance at least 2 and is diagonally $\chi_{J_i}(X)$-neighbour-transitive. By Proposition 3.4, either $\pi_{J_i}(C)$ has covering radius $\rho \leq 1$, or $\pi_{J_i}(C)$ is also $(\chi_{J_i}(X), 2)$-neighbour-transitive. Note $\rho \neq 0$, that is, $\pi_{J_i}(C)$ is not the complete code, since $\pi_{J_i}(C)$ has minimum distance at least 2.

Suppose $\pi_{J_i}(C)$ has covering radius $\rho \geq 2$. Since $X_i^{Q_i}$ is almost-simple, it follows that $q \geq 5$. By Proposition 5.4, the only diagonally 2-neighbour-transitive code with $q \geq 5$ and $\delta \geq 2$ is $\mathrm{Rep}(3, q)$ for $q \geq 5$ (note that $\delta = 1$ for $\mathrm{Inj}(3, q)$). Then Lemma 4.3 implies $\mathcal{J}$ is a trivial partition. Since $|J_i| = k = 3 > 1$, it follows that $\ell = 1$, $k = m$, and $C = \mathrm{Rep}(3, q)$.

Suppose $\pi_{J_i}(C)$ has covering radius $\rho = 1$. Now, by [9, Thm. 4 and Cor. 2], the only diagonally neighbour-transitive code with $\delta \geq 2$ and $\rho = 1$ is $\mathrm{Rep}(2, q)$. If $l = 1$ then we have $\delta = 2$, a contradiction. Suppose $l \geq 2$. Then Lemma 4.3 implies $\delta = 2$, a contradiction. □

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Congruent triangles in arrangements of lines

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Abstract

We study the maximum number of congruent triangles in finite arrangements of ℓ lines in the Euclidean plane. Denote this number by $f(\ell)$. We show that $f(5) = 5$ and that the construction realizing this maximum is unique, $f(6) = 8$, and $f(7) = 14$. We also discuss for which integers c there exist arrangements on ℓ lines with exactly c congruent triangles. In parallel, we treat the case when the triangles are faces of the plane graph associated to the arrangement (i.e. the interior of the triangle has empty intersection with every line in the arrangement). Lastly, we formulate four conjectures.

Keywords: Arrangement, congruent triangles.

Math. Subj. Class.: 52C10, 52C30

1 Introduction

A problem from mathematical folklore asks for bounding four congruent triangles with six matchsticks. This is easily done, and left to the reader. Quite naturally, one can ask whether more congruent triangles may be formed by using the same six matchsticks. It seems that this particular problem has not been treated in the literature. Our main focus lies on constructing planar arrangements in which a fixed number ℓ of lines bound as many congruent triangles as possible. For an excellent overview on arrangements and spreads, see Grünbaum's [11]. The results presented in this article are complementary to work of Erdős and Purdy [4, 5] on sets of n points—see also [6].

In this paper, everything happens in $\mathbb{R}^2$. An *arrangement* (of lines) $\mathcal{A}$ shall be a finite family of ℓ lines $L_1, \ldots, L_\ell$. In the following, we will ignore the case when there exists a point common to all lines, and thus assume that $\ell \geq 3$. Denote by $\mathfrak{A}_\ell$ the set of all arrangements of ℓ lines.

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We associate to $\mathcal{A} \in \mathfrak{A}_\ell$ a graph $\Gamma_\mathcal{A}$: the vertices of $\Gamma_\mathcal{A}$ correspond to the intersection points of lines from $\mathcal{A}$ and the edges of $\Gamma_\mathcal{A}$ correspond to the line-segments between these vertices. $\Gamma_\mathcal{A}$ is a plane graph. The vertices, edges, and faces of $\Gamma_\mathcal{A}$ are also said to belong to $\mathcal{A}$.

A *triangle* in $\mathcal{A} \in \mathfrak{A}_\ell$ shall be the convex hull of the set of intersection points of three non-concurrent pairwise non-parallel lines in $\mathcal{A}$. Denote by $F^\mathcal{A}$ the set of all triangles in $\mathcal{A}$. Whenever $\Delta_1, \Delta_2 \in F^\mathcal{A}$ are congruent we write $\Delta_1 \sim \Delta_2$. Let $F_1^\mathcal{A}, \dots, F_p^\mathcal{A}$ be the equivalence classes with respect to $\sim$ such that $\left|F_1^\mathcal{A}\right| \geq \ldots \geq \left|F_p^\mathcal{A}\right|$. Here, $|M|$ denotes the cardinal number of M. We call a triangle $\Delta \in F^\mathcal{A}$ *facial* if it is a face of $\Gamma_\mathcal{A}$, i.e. $L \cap \operatorname{int} \Delta = \emptyset$ for all $L \in \mathcal{A}$. Let $G^\mathcal{A} \subset F^\mathcal{A}$ be the set of all facial triangles in $\mathcal{A}$, and, as before, let $G_1^\mathcal{A}, \dots, G_q^\mathcal{A}$ be the equivalence classes with respect to $\sim$ such that $\left|G_1^\mathcal{A}\right| \geq \ldots \geq \left|G_q^\mathcal{A}\right|$. Put

$$f(\ell) = \max_{\mathcal{A} \in \mathfrak{A}_\ell} \left|F_1^\mathcal{A}\right| \quad \text{and} \quad g(\ell) = \max_{\mathcal{A} \in \mathfrak{A}_\ell} \left|G_1^\mathcal{A}\right|.$$

We shall also be considering restrictions relative to a certain arrangement $\mathcal{A} \in \mathfrak{A}_\ell$, namely, for $k \leq \ell$,

$$f_\mathcal{A}(k) = \max_{\mathcal{B} \subset \mathcal{A},\, \mathcal{B} \in \mathfrak{A}_k} \left|F_1^\mathcal{B}\right| \quad \text{and} \quad g_\mathcal{A}(k) = \max_{\mathcal{B} \subset \mathcal{A},\, \mathcal{B} \in \mathfrak{A}_k} \left|G_1^\mathcal{B}\right|.$$

We call an arrangement $\mathcal{A} \in \mathfrak{A}_\ell$ f*-optimal* (g*-optimal*) if $\left|F_1^\mathcal{A}\right| = f(\ell)$ ($|G_1^\mathcal{A}| = g(\ell)$). If $\mathcal{A}$ is both f-optimal and g-optimal, we simply write *optimal*. A triangle from $F_1^\mathcal{A}$ or $G_1^\mathcal{A}$ is said to be *good*. Note that $F_1^\mathcal{A}$ and $G_1^\mathcal{A}$ need not be unique. In that case, one makes a choice clearly defining $F_1^\mathcal{A}$ and $G_1^\mathcal{A}$. The edges and angles of a good triangle will be called *good*, too.

An arrangement is *simple* if no three lines are concurrent. The lines of an arrangement are in *general position* if no two lines are parallel and no three lines are concurrent. Two arrangements $\mathcal{A}$ and $\mathcal{B}$ are *combinatorially equivalent* if their associated graphs $\Gamma_\mathcal{A}$ and $\Gamma_\mathcal{B}$ are isomorphic. (Note that, as mentioned above, we do not consider line arrangements in which all lines meet in a single point.) $\mathcal{A} \in \mathfrak{A}_\ell$ is *c-unique* if there exists no $\mathcal{B} \in \mathfrak{A}_\ell$ such that $\mathcal{A}$ and $\mathcal{B}$ are (a) not combinatorially equivalent and (b) $|H_1^\mathcal{A}| = |H_1^\mathcal{B}|$, where H is F or G. In the same vein, we say that two arrangements $\mathcal{A} \in \mathfrak{A}_\ell$ and $\mathcal{B} \in \mathfrak{A}_\ell$ are *g-equivalent* if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by translation, rotation, reflection, and scaling. $\mathcal{A} \in \mathfrak{A}_\ell$ is *g-unique* if there exists no $\mathcal{B} \in \mathfrak{A}_\ell$ such that $\mathcal{A}$ and $\mathcal{B}$ are (a) not g-equivalent and (b) $|H_1^\mathcal{A}| = |H_1^\mathcal{B}|$, where H is F or G. A few examples: Three lines in general position yield an arrangement that is c-unique, but not g-unique; any arrangement on four lines that forms exactly two congruent triangles is not c-unique (and thus cannot be g-unique); finally, as we shall see in Theorem 3.5, the f-optimal arrangement from Figure 2 (b) is g-unique (and thus c-unique).

$F(\ell)$ ($G(\ell)$) is defined as the set of all integers u such that there exists an arrangement on ℓ lines having exactly u congruent triangles (congruent facial triangles). We write $[s..t]$ for the set of all integers u with $s \leq u \leq t$, put H for F or G, and

$$h = \begin{cases} f & \text{if } H = F, \\ g & \text{if } H = G. \end{cases}$$

Whenever $H(\ell) = [0..h(\ell)]$, we say that $H(\ell)$ is *complete*. In the following, we will tacitly use the fact that $G(\ell) \subset F(\ell)$. We call an arrangement $\mathcal{A} \in \mathfrak{A}_\ell$ 1*-extendable* if there exists a line L such that $|H_1^{\mathcal{A} \cup L}| = |H_1^\mathcal{A}| + 1$.

2 Preparation

We briefly concern ourselves with the following question, since it will shorten later arguments. What if we drop the condition that the triangles need to be congruent? Kobon Fujimura asked in 1978 in his book "The Tokyo Puzzle" [8]—see also [10, pp. 170–171 and 178]—what the maximum number $K(\ell)$ of facial (not necessarily congruent!) triangles realisable by ℓ lines in the plane is. (Grünbaum treated this problem before Fujimura, but he might have only been interested in arrangements in the projective plane [6].) Recently, Bader and Clément [1], improving upon a result of Tamura, showed the following.

Lemma 2.1 (Bader and Clément)**.**

$$K(\ell) \leq \left\lfloor \frac{\ell(\ell-2)}{3} \right\rfloor - \mathbf{I}_{\{\ell\,:\,(\ell \bmod 6)\in\{0,2\}\}}(\ell),$$

where $\mathbf{I}$ *denotes the indicator function.*

Many arrangements have been constructed in order to find solutions to Fujimura's problem. Fujimura himself gave an example which shows that $K(7) \geq 11$, although it was thought for many years that $K(7) = 10$. In 1996, Grabarchuk and Kabanovitch [13] gave two 10-line, 25-triangle constructions, whereas Lemma 2.1 gives $K(10) \leq 26$. Whether $K(10)$ is 25 or 26 is unknown. Other 10-line, 25-triangle arrangements were found independently by Grünbaum [12, p. 400], Wajnberg, and Honma (see [15] for more details). Good overviews of the best (i.e. the greatest number of triangles for a fixed number of lines) known arrangements can be found in [14] and [15].

Table 1: Bounding $K(\ell)$ for $\ell \leq 12$.

ℓ	3	4	5	6	7	8	9	10	11	12
Bader-Clément bound	1	2	5	7	11	15	21	26	33	39
Best known arrangement	1	2	5	7	11	15	21	25	32	38

Füredi and Palásti [9] construct an arrangement proving $K(\ell) \geq \ell(\ell-3)/3$. See also the article of Forge and Ramírez Alfonsín [7].

We continue with a series of lemmas. Lemma 2.2 is stated without its straightforward proof, but we present the heptagonal case in Figure 1 and its caption.

Lemma 2.2. *The ℓ lines bounding a regular ℓ-gon determine exactly 2ℓ congruent triangles if $\ell \geq 7$, and therefore $2\ell \in F(\ell)$ and $f(\ell) \geq 2\ell$. With the same construction we obtain for $\ell \geq 5$ that $\ell \in G(\ell)$ and $g(\ell) \geq \ell$.*

Lemma 2.3. *Let $\mathcal{A} \in \mathfrak{A}_\ell$ and $h \in \{f, g\}$. Then, for $3 \leq k \leq \ell - 1$,*

$$h_{\mathcal{A}}(\ell) \leq \frac{\ell(\ell-1)(\ell-2)}{k(k-1)(k-2)} \cdot h_{\mathcal{A}}(k).$$

Proof. We observe that every subset of $k < \ell$ lines within the arrangement $\mathcal{A}$ cannot have more than $h_{\mathcal{A}}(k)$ good triangles (good in $\mathcal{A}$). Counting all together, there are at most

$\binom{\ell}{k}h_{\mathcal{A}}(k)$ good triangles, each appearing several times. Since each of them lies in $\binom{\ell-3}{k-3}$ sub-arrangements of k lines, we obtain

$$h_{\mathcal{A}}(\ell) \leq \frac{\binom{\ell}{k}h_{\mathcal{A}}(k)}{\binom{\ell-3}{k-3}}.$$

Thus, for each k we obtain an upper bound for $h_{\mathcal{A}}(\ell)$. □

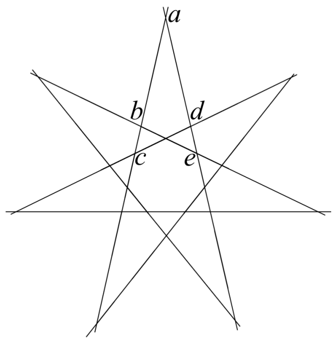


Figure 1: Seven lines bounding a regular heptagon. This arrangement contains fourteen congruent triangles: abe, acd, and their symmetric counterparts obtained by rotating around the barycentre of the heptagon by $2\pi k/7$ for $k = 1, \ldots, 6$. This arrangement proves that $f(7) \geq 14$.

Lemma 2.4 is a direct consequence of Lemma 2.3.

Lemma 2.4. *Let $h \in \{f, g\}$. Then*

$$h(\ell) \leq \min_{3 \leq k \leq \ell-1} \frac{\ell(\ell-1)(\ell-2)}{k(k-1)(k-2)} \cdot h(k).$$

3 Results

3.1 Bounds for the general case

Proposition 3.1. *Let $h \in \{f, g\}$, consider $\mathcal{A} \in \mathfrak{A}_\ell$ and $\mathcal{B} \in \mathfrak{A}_k$, and assume that a good triangle of $\mathcal{A}$ is similar to a good triangle of $\mathcal{B}$. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ each contain two lines intersecting in the boundary of the respective convex hulls of $V(\Gamma_{\mathcal{A}})$ and $V(\Gamma_{\mathcal{B}})$ and forming the same good angle. Then $h(\ell + k - 2) \geq h(\ell) + h(k)$.*

Proof. We scale $\mathcal{B}$ to $\mathcal{B}'$ such that the good triangles of $\mathcal{B}'$ are congruent with the good triangles of $\mathcal{A}$. Consider the convex hull $C_{\mathcal{A}}$ ($C_{\mathcal{B}'}$) of the intersection points of $\mathcal{A}$ ($\mathcal{B}'$). Let $p_{\mathcal{A}}$ ($p_{\mathcal{B}'}$) be an intersection point of $\mathcal{A}$ ($\mathcal{B}'$) lying on the boundary of the convex polygon $C_{\mathcal{A}}$ ($C_{\mathcal{B}'}$) and incident with a good angle $\alpha_{\mathcal{A}}$ ($\alpha_{\mathcal{B}'}$) of $\mathcal{A}$ ($\mathcal{B}'$) such that $\alpha_{\mathcal{A}} = \alpha_{\mathcal{B}'}$. Denote with $L_1^{\mathcal{A}}$ and $L_2^{\mathcal{A}}$ ($L_1^{\mathcal{B}'}$ and $L_2^{\mathcal{B}'}$) two of the lines of $\mathcal{A}$ ($\mathcal{B}'$) intersecting at $p_{\mathcal{A}}$ ($p_{\mathcal{B}'}$) and forming the angle $\alpha_{\mathcal{A}}$ ($\alpha_{\mathcal{B}'}$). We can now identify $L_1^{\mathcal{A}}$ with $L_1^{\mathcal{B}'}$ and $L_2^{\mathcal{A}}$ with $L_2^{\mathcal{B}'}$ such that an arrangement $\mathcal{C}$ is obtained in which, seeing $\mathcal{A}$ and $\mathcal{B}'$ as sub-arrangements of $\mathcal{C}$, no good triangle lies in both $\mathcal{A}$ and $\mathcal{B}'$. (Note that the number of good triangles in $\mathcal{C}$ may be larger than the sum of the number of good triangles in $\mathcal{A}$ and $\mathcal{B}'$, see e.g. Figure 8, in which the arrangements from Figures 1 and 5 (b) are joined: the original arrangements have 14 and 8 good triangles, respectively, but the new arrangement has 26.) □

We write $\tilde{f}(\ell)$ if we consider the values of $f(\ell)$ only for arrangements whose good triangles are not right triangles.

Proposition 3.2. $\tilde{f}(\ell+1) \leq \tilde{f}(\ell) + 3(\ell-1)$ *and* $f(\ell+1) \leq f(\ell) + 4(\ell-1)$.

Proof. Let L be a line being added to $\mathcal{A} \in \mathfrak{A}_\ell$. If $\mathcal{A}$ has the property that $\left|F_1^{\mathcal{A}}\right| = \left|F_2^{\mathcal{A}}\right|$, then consider henceforth only the triangles in $F_1^{\mathcal{A}}$, as well as their edges, to be good. We denote the lengths of good edges with a, b, and c.

There are at most $\ell - 1$ triangles with an edge of length a on L: there are at most $\ell/2$ lines of one of the two lines needed to make a good triangle with an edge on L, each of these lines is part of at most two triangles with an edge of length a on L, and it is impossible for there to be exactly $\ell/2$ of them each of which is part of exactly two triangles. Since this can be applied analogously for edges of length b and c, we have $\tilde{f}(\ell+1) = \tilde{f}(\ell)+3(\ell-1)$.

For good triangles that are right triangles, we argue in the same manner and obtain that for each of the three types of good edge (i.e. of length a, b or c) there are at most $4(\ell-1)/3$ triangles with a good edge of that type on L. □

Proposition 3.3. $\tilde{f}(\ell) \leq \ell(\ell-1)$ *and* $f(\ell) \leq 4\ell(\ell-1)/3$.

Proof. The idea is the same as the one used in the proof of Proposition 3.2. In the case of non-right triangles, we have established that on each line in $\mathcal{A}$ there are at most $3(\ell-1)$ good edges. By multiplying this with ℓ, we obtain an upper bound for the number of good edges in $\mathcal{A}$. Now we divide by three (as there are three edges to each triangle) and have the desired bound. The case of right triangles is settled analogously. □

All angles in $\mathcal{A} \in \mathfrak{A}_\ell$ equal to one of the angles of a good triangle which is not a right triangle will be called *non-right angles*.

Proposition 3.4. $\tilde{f}(\ell) \leq 2\ell(\ell-2)/3$ *for simple arrangements.*

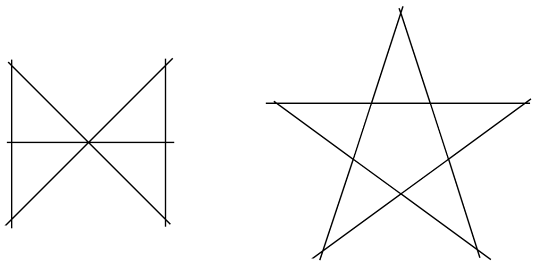
Proof. Consider a simple arrangement $\mathcal{A}$ on ℓ lines admitting a good triangle which is not a right triangle. Let $V(\Gamma_{\mathcal{A}}) = V$, and write V_k for the set of vertices of degree k. As no three lines are concurrent, in $\Gamma_{\mathcal{A}}$ there exist only vertices of degree 2, 3, or 4. Trivially, around a vertex of degree 2 at most one non-right angle resides. Around a vertex of degree 3 likewise (as $\pi/2$ is a right angle, and the sum of two non-right angles must be strictly smaller than π), and around a vertex of degree 4 there may be at most two non-right angles.

Thus, in $\mathcal{A}$, we have as an upper bound for the maximum number of non-right angles $|V_2|+|V_3|+2\cdot|V_4| = |V|+|V_4|$. We have $|V| \leq \ell(\ell-1)/2$. Also $|V_4| \leq |V|-\ell$, because on every line the first and the last vertex belong to $V_2 \cup V_3$, but any such vertex may appear as first or last vertex on two lines of $\mathcal{A}$. Thus, the bound is $\ell^2 - 2\ell$. For odd $\ell \geq 5$, this bound is best possible: for the ℓ lines bounding a regular ℓ-gon we have $|V_2| = \ell$, $|V_3| = 0$, and $|V_4| = \ell(\ell-3)/2$. One non-right angle cannot lie in more than two triangles which are not right triangles, and every triangle requires three angles, whence, the final bound. (In fact, no good angle can lie in more than two good triangles.) □

3.2 $\ell \leq 5$

We have $f(3) = g(3) = 1$ and $f(4) = g(4) = 2$, and $F(3)$, $G(3)$, $F(4)$, and $G(4)$ are complete. We leave the easy proofs to the reader, but mention that in the 4-line case there exist exactly three combinatorially different solutions with two congruent triangles (these coincide for the g-optimal and the f-optimal case): one with three concurrent lines, one with two parallel lines, and one in general position.

We now focus on the first interesting case: $\ell = 5$.



(a) (b)

Figure 2: (a) shows a 5-line arrangement with four congruent triangles constructed as follows. Two lines L_1, L_2 orthogonal to a third line L_3 are considered. Let the intersection points be p_1 and p_2, resp. A fourth and fifth line are considered such that their intersection point is the midpoint of the line-segment p_1p_2 and the angle each forms with L_3 is $\pi/4$. (b) depicts the five lines bounding a regular pentagon. This arrangement contains ten triangles, distributed among two congruence classes of size 5 each.

Theorem 3.5. *(i) We have $f(5) = g(5) = 5$ while $F(5)$ and $G(5)$ are complete. Furthermore, the arrangement from Figure 2 (b) is (ii) optimal, and (iii) g-unique among f-optimal and g-optimal 5-line arrangements.*

Proof. Figure 2 (b) shows that $g(5) \geq 5$ (whence, $f(5) \geq 5$), with which Lemma 2.1 implies $g(5) = 5$. $f(5) = 5$ follows from Lemma 2.4 (with $k = 4$). We now discuss $G(5)$. We have $G(4) = [0..2] \subset G(5)$. Consider the four lines bounding a square and add the two lines containing the square's diagonals. By removing one of the four original lines, we have shown that $3 \in G(5)$. Together with the arrangements from Figure 2, we have that $G(5)$ is complete since $g(5) = 5$. Thus, (i) is proven. (ii) follows directly from (i).

We now prove (iii). First, we show that the arrangement from Figure 2 (b) is c-unique. We use the database provided in Christ's Dissertation [3, Chapter 3.2.5]. (A visualisation of Christ's results is available in [2]. Note that this does not coincide with Grünbaum's isomorphism types of arrangements given in [11, p. 5], since Grünbaum discusses the issue in the projective plane, while here we treat the situation in the Euclidean plane.) Among arrangements of five lines in general position, there are exactly six combinatorially different ones, shown in Figure 3. The arrangement in Figure 2 (b) belongs to the combinatorial class (A).

Only the arrangements in (A) contain five facial triangles, i.e. triangles which are faces in the associated graph. We leave to the reader the straightforward proof that among arrangements with five lines not in general position (i.e. containing two parallel lines or three

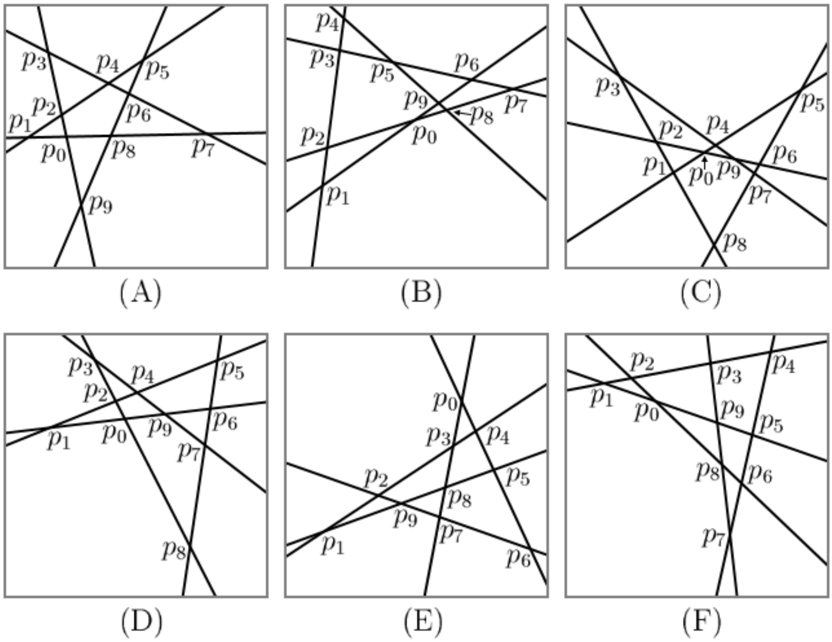


Figure 3: Representations of the six combinatorially different arrangements of five lines in general position.

concurrent lines), there is none featuring five facial triangles. Note that the occurrence of more triangles is impossible due to Lemma 2.1.

We now turn to the case in which triangles are not facial. Let us denote a line-segment between two points x, y with xy and its length with $L(xy)$. We will use the following.

Remark. The sum of the measures α and β of two good angles is π if and only if $\alpha = \beta = \pi/2$.

We write Δ_{ijk} for the triangle with vertices p_i, p_j, p_k. We will tacitly make use of the fact that if in a given arrangement a triangle Δ is strictly contained in a triangle Δ', then $\Delta \nsim \Delta'$ and so Δ and Δ' cannot lie in the same congruence class.

(B) We have $\Delta_{012} \subset \Delta_{149} \cap \Delta_{136}$ (so $\Delta_{149} \nsim \Delta_{012} \nsim \Delta_{136}$), $\Delta_{345} \subset \Delta_{149} \cap \Delta_{248}$, $\Delta_{569} \subset \Delta_{136} \cap \Delta_{237} \cap \Delta_{578}$, $\Delta_{578} \subset \Delta_{237}$, $\Delta_{067} \subset \Delta_{237}$, $\Delta_{089} \subset \Delta_{248} \cap \Delta_{237} \cap \Delta_{067}$. Due to these inclusion relations, only the following set of triangles may form a congruence class of size 5: $\{\Delta_{149}, \Delta_{136}, \Delta_{248}, \Delta_{578}, \Delta_{067}\}$. Assume it is indeed a congruence class. Thus, all angles around p_6 are right. We apply the Remark to the angles surrounding p_6. Combining this with $\Delta_{136} \sim \Delta_{067}$ and $p_0p_6 \subsetneq p_1p_6$, we have $L(p_6p_7) = L(p_1p_6)$, $L(p_0p_6) = L(p_3p_6)$, and $L(p_0p_7) = L(p_1p_3)$. p_1p_3 is the hypothenuse in Δ_{136}, but as p_1p_4 is an edge of Δ_{149} and $\Delta_{149} \sim \Delta_{136}$ we obtain a contradiction, since $L(p_1p_4) > L(p_1p_3)$.

(C) We have $\Delta_{012} \subset \Delta_{134} \cap \Delta_{268} \cap \Delta_{378}$, $\Delta_{134} \cup \Delta_{239} \subset \Delta_{378}$, $\Delta_{457} \cup \Delta_{056} \subset \Delta_{158}$, $\Delta_{049} \subset \Delta_{158} \cap \Delta_{239} \cap \Delta_{378} \cap \Delta_{056}$, $\Delta_{679} \subset \Delta_{158} \cap \Delta_{268} \cap \Delta_{457}$. There is no congruence class of size 5.

(D) We have $\Delta_{012} \subset \Delta_{149} \subset \Delta_{156}$, $\Delta_{234} \subset \Delta_{039} \subset \Delta_{378}$, $\Delta_{457} \cup \Delta_{068} \subset \Delta_{258}$, $\Delta_{679} \subset \Delta_{258} \cap \Delta_{457} \cap \Delta_{068}$. Once more, all congruence classes have size at most 4.

(E) We have $\Delta_{129} \subset \Delta_{138} \subset \Delta_{145}$, $\Delta_{237} \subset \Delta_{246}$, $\Delta_{034} \subset \Delta_{058} \subset \Delta_{067}$, $\Delta_{789} \subset \Delta_{237} \cap \Delta_{246} \cap \Delta_{569}$. As above.

(F) We have $\Delta_{012} \subset \Delta_{139} \subset \Delta_{145}$, $\Delta_{238} \cup \Delta_{056} \subset \Delta_{246}$, $\Delta_{678} \subset \Delta_{579} \subset \Delta_{347}$, $\Delta_{089} \subset \Delta_{238} \cap \Delta_{246} \cap \Delta_{056}$. As above.

Let us show that in a 5-line arrangement $\mathcal{A}$ containing two parallel lines or three concurrent lines, no more than four congruent triangles can be achieved. We first assume that $\mathcal{A}$ contains parallel lines L_1, L_2. If there exists a line L_3 parallel to L_1, we are done, as in $\mathcal{A}$ there are only at most three triples of lines forming triangles. Thus, w.l.o.g. we are in the situation that a third line, L_3, intersects L_1 and L_2. Now assume that a fourth line, L_4, is parallel to L_3. Note that L_1, L_2, L_3, L_4 bound zero triangles. In this situation, a fifth line generates at most four new triangles. Thus, L_4 cannot be parallel to L_3. We have proven that a 5-line arrangement containing three parallel lines or two parallel pairs of parallel lines cannot have more than four congruent triangles.

Denote the open strip bounded by L_1 and L_2 with S, and the complement of its closure by T. Also, let T_1 and T_2 be the connected components of T. In the light of above paragraph, there are three cases (see Figure 4): either (a) L_4 is concurrent with L_1 and L_3 in a point x lying in the closure of T_1, (b) L_4 intersects L_3 in S, or (c) L_4 intersects L_3 in T_2. Denote with L_5 the fifth line of $\mathcal{A}$. We know that L_5 is not parallel to any of the existing four lines. We write Δ_{ijk} for the triangle bounded by the lines L_i, L_j, L_k.

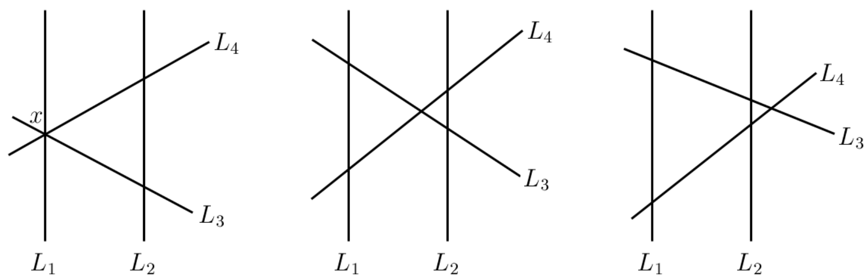


Figure 4: Cases (a)–(c) occurring in the proof of Theorem 3.5.

(a): If $x \in L_5$, then the five lines would bound only three triangles, so we can assume $x \notin L_5$.

Case 1: L_5 intersects both L_3 and L_4 in S. We have six triangles, but $\Delta_{345} = \Delta_{234} \cap \Delta_{145}$ and $\Delta_{245} \subset \Delta_{235}$, so the maximum number of congruent triangles is three.

Case 2: L_5 intersects L_4 in S and L_5 intersects L_3 in $T_1 \cup T_2$. Six triangles appear. Subcase 2.1: L_3 and L_5 intersect in T_1. But then we have $\Delta_{135} \subset \Delta_{345} \subset \Delta_{235}$. Subcase 2.2: L_3 and L_5 intersect in T_2. Here, $\Delta_{235} \subset \Delta_{345} \subset \Delta_{135}$, so once more five congruent triangles cannot occur. (If L_5 intersects L_3 in S and L_4 in T_2, then we are, combinatorially, in the situation treated in Subcase 2.2.)

Case 3: L_5 is concurrent with L_2 and L_3. Subcase 3.1: All intersection points lie in the closure of S. Five triangles appear, but among them one is a subset of another. Subcase 3.2: L_4 and L_5 intersect in T_2. We apply the same argument as before. Subcase 3.3: L_4 and L_5 intersect in T_1. Once more five triangles occur, but one is contained in another.

Case 4: L_5 intersects L_3 and L_4 in points p and p', respectively, which do not lie in S (since this was covered in Cases 1 and 2). Subcase 4.1: If p and p' lie in T_1, six triangles appear, but $\Delta_{345} \subset \Delta_{135} \subset \Delta_{235}$. Subcase 4.2: If p and p' lie in T_2, again six triangles occur, but $\Delta_{245} \subset \Delta_{235} \subset \Delta_{135}$. Subcase 4.3: If $p \in T_1$ and $p' \in T_2$, six triangles are present in the arrangement, but $\Delta_{245} \subset \Delta_{145} \subset \Delta_{345}$, so at most four triangles are congruent.

Case 5: L_5 is concurrent with L_2 and L_4. Subcase 5.1: All intersection points lie in the closure of S. Subcase 5.1 coincides with Subcase 3.1. Subcase 5.2: L_3 and L_5 intersect in T_2. But then we are in the same situation as Subcase 3.2.

(b) Let L_3 and L_4 intersect in y. We know that L_5 is not parallel to L_1. If $y \in L_5$, we obtain six triangles. However, either $\Delta_{135} \cup \Delta_{145} = \Delta_{134}$ and symmetrically $\Delta_{235} \cup \Delta_{245} = \Delta_{234}$ or $\Delta_{134} \cup \Delta_{145} = \Delta_{135}$ and $\Delta_{245} \cup \Delta_{234} = \Delta_{235}$. In either case, the largest congruence class has cardinality at most four. We have treated the cases when L_5 is concurrent with L_1 and L_3, L_1 and L_4, L_2 and L_3, or L_2 and L_4 in (i). We split the remaining cases into four cases according to where the intersection points of L_5 with L_3 and L_4 lie. In each situation, inclusions are given which make the occurrence of a congruence class of cardinality at least five impossible.

Case 1: Both intersection points lie in S. However, we then have $\Delta_{135} \subset \Delta_{145}$, $\Delta_{345} = \Delta_{134} \cap \Delta_{235}$, and $\Delta_{234} \subset \Delta_{245}$.

Case 2: The intersection points of L_5 with L_3 and L_4 lie in T_1 and T_2, respectively. Then $\Delta_{135} \cup \Delta_{245} \subset \Delta_{345}$, $\Delta_{135} \cup \Delta_{234} \subset \Delta_{235}$, and $\Delta_{134} \cup \Delta_{245} \subset \Delta_{145}$.

Case 3: The intersection points of L_5 with L_3 and L_4 lie in T_1 and S, respectively. We have $\Delta_{135} \subset \Delta_{345} \subset \Delta_{235}$, $\Delta_{245} \subset \Delta_{234}$, and $\Delta_{134} \subset \Delta_{145}$.

Case 4: Both intersection points lie in T_1. Then $\Delta_{234} \subset \Delta_{235}$, $\Delta_{135} = \Delta_{235} \cap \Delta_{145}$, and $\Delta_{134} \subset \Delta_{145} \subset \Delta_{245}$.

As situation (c) uses very similar arguments, we skip it.

We have shown that no two lines in $\mathcal{A}$ are parallel. Assume now that three lines L_1, L_2, L_3 of $\mathcal{A}$ intersect at a point q. If L_4 contains q as well, the largest congruence class which may be formed by a fifth line has size 2. So $q \notin L_4$ and L_4 is not parallel to any of L_1, L_2, L_3. W.l.o.g. let the intersection point of L_4 with L_2 lie between the intersection point of L_4 with L_1 and the intersection point of L_4 with L_3. If there are two coincidences (of three lines)—it is easy to see that there cannot be more—we have three combinatorially different cases. W.l.o.g., in each of them L_1, L_4, and L_5 shall be concurrent. We denote this intersection point with a, and the intersection point of L_5 with L_2 and L_3 with b and c, respectively. We differentiate the three cases by the order in which the intersection points occur on L_5.

Case 1: $a - c - b$ (or equivalently $b - c - a$): Eight triangles occur. However, we have $\Delta_{124} \cup \Delta_{234} = \Delta_{134} \subset \Delta_{345}$ and $\Delta_{135} \cup \Delta_{235} = \Delta_{125} \subset \Delta_{245}$. Thus, no five triangles can be congruent.

Case 2: $b - a - c$ (or equivalently $c - a - b$): Again, eight triangles appear, but $\Delta_{245} \cup \Delta_{124} = \Delta_{125}$, $\Delta_{125} \cup \Delta_{135} = \Delta_{235}$, $\Delta_{234} \cup \Delta_{124} = \Delta_{134}$, and $\Delta_{134} \cup \Delta_{135} = \Delta_{345}$.

Case 3: $a - b - c$ (or equivalently $c - b - a$): $\Delta_{124} \cap \Delta_{135} = \Delta_{125}$, $\Delta_{235} \subset \Delta_{234}$, and $\Delta_{245} \subset \Delta_{345}$. Furthermore, every triangle is contained in Δ_{134}.

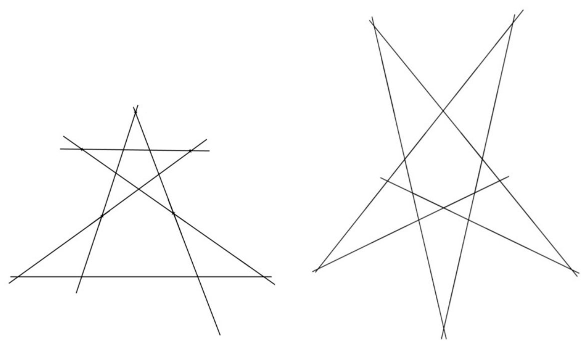
We are left with the case that there is exactly one coincidence of three lines (namely in q). Once more, several cases occur. We leave them to the reader—treating them is a straightforward task in exactly the same spirit as above paragraphs.

Finally, we prove that the construction from Figure 2 (b) is indeed g-unique. Consider five lines bounding a pentagon P such that we obtain an arrangement $\mathcal{A}$ in combinatorial class (A). This implies that no two lines in $\mathcal{A}$ are parallel. Combinatorially, there are two types of triangles in P: those sharing exactly two vertices (and thus an edge) with P, and those sharing exactly one vertex with P. Due to straightforward inclusion arguments, all triangles in a congruence class of size 5 are of the same type.

Consider the first type, and let Δ be one of these five congruent facial triangles. Denote the angles of Δ incident with a vertex of P with α and β. Applying successively the fact that no two lines in $\mathcal{A}$ are parallel, we obtain that $\alpha = \beta$, so Δ is isosceles. This implies that all angles of P must be equal, and since P is a pentagon, the angles of P measure $3\pi/5$ each. Thus $\alpha = 2\pi/5$—in particular, Δ is not equilateral. Hence, the sides of P must have equal length, so P is a regular pentagon.

We treat the second case. We see each triangle of the second type as the union of three faces (of $\Gamma_{\mathcal{A}}$): the pentagon P, which lies in all five triangles, and two facial triangles. Since certain pairs of triangles of the second type share a facial triangle, there are at most two congruence classes C_1 and C_2 of facial triangles. Assume $C_1 \neq C_2$. Thus, there exists a triangle Δ of second type containing a facial triangle in C_1 and a facial triangle in C_2. By considering all five congruent triangles of second type, a contradiction is obtained, since necessarily one of these triangles will contain only triangles from either C_1 or C_2 and thus, it cannot be congruent to Δ. We have proven that all facial triangles are congruent. Now we may argue as in the preceding paragraph. □

3.3 $\ell = 6$



(a) (b)

Figure 5: (a) This arrangement is due to Tudor Zamfirescu and shows that $7 \in F(6)$. To the five lines bounding a regular pentagon a sixth line is added which is parallel to one of the five lines such that seven congruent triangles are present. (b) This arrangement proves that $8 \in F(6)$ and $f(6) \geq 8$. In Theorem 3.6 we show that in fact $f(6) = 8$. The arrangement is obtained by considering six of the seven lines bounding a regular heptagon.

Theorem 3.6. *We have $f(6) = 8$, $6 \leq g(6) \leq 7$, $F(6)$ is complete, and $[0..6] \subset G(6)$.*

Proof. The arrangement from Figure 5 (b) proves that $f(6) \geq 8$. Theorem 3.5 (iii) states that there is exactly one f-optimal arrangement on five lines, shown in Figure 2 (b). We call this arrangement $\mathcal{P}$. We now show that one cannot produce an arrangement on six lines which has $\mathcal{P}$ as a sub-arrangement and features eight (or more) congruent triangles. Assume there exists such an arrangement $\mathcal{A}$. Denote the lines of $\mathcal{P}$ by $L_1, \ldots, L_5$, and the line added to $\mathcal{P}$ in order to obtain $\mathcal{A}$ by L.

First we prove that the addition of L cannot create a "new" congruence class (i.e. a class the triangles of which are non-congruent to every triangle present in $\mathcal{P}$) of congruent triangles of cardinality at least 8. At least one of the angles $\pi/5$, $2\pi/5$, $3\pi/5$, $4\pi/5$ is good in both $\mathcal{P}$ and $\mathcal{A}$, since every triangle bounded by L has at least one angle in $\mathcal{P}$. Among all angles in $\mathcal{A}$, each of the aforementioned four angles appears at least ten times in five pairs of opposite angles, since $\mathcal{P}$ is a sub-arrangement of $\mathcal{A}$. Thus, L forms at least three copies of the angle α with the lines $L_1, \ldots, L_5$, where $\alpha \in \{\pi/5, 2\pi/5, 3\pi/5, 4\pi/5\}$ is fixed. But since the L_i's are pairwise non-parallel, this is only possible if L is parallel to some L_i. But then the addition of L to $\mathcal{P}$ yields at most six new triangles—too few.

Take the two congruence classes $F_1^{\mathcal{P}}$ and $F_2^{\mathcal{P}}$ such that the triangles in $F_1^{\mathcal{P}}$ are facial in $\mathcal{P}$, and notice that $F^{\mathcal{P}} = F_1^{\mathcal{P}} \cup F_2^{\mathcal{P}}$ and $|F_1^{\mathcal{P}}| = |F_2^{\mathcal{P}}| = 5$. Thus, L must add at least three triangles to $F_1^{\mathcal{P}}$ or $F_2^{\mathcal{P}}$. As $\pi/5$ belongs to triangles in $F_1^{\mathcal{P}}$ as well as triangles in $F_2^{\mathcal{P}}$, $\pi/5$ is a good angle in $\mathcal{A}$, so L makes this angle with a line of $\mathcal{P}$, whence, L is parallel to some L_i, say L_1. Among all possible positions of L, only three provide new triangles congruent either to a good triangle in $F_1^{\mathcal{P}}$ or to a good triangle in $F_2^{\mathcal{P}}$, see Figure 6. The number of those new triangles is 1, 1, 2, respectively.

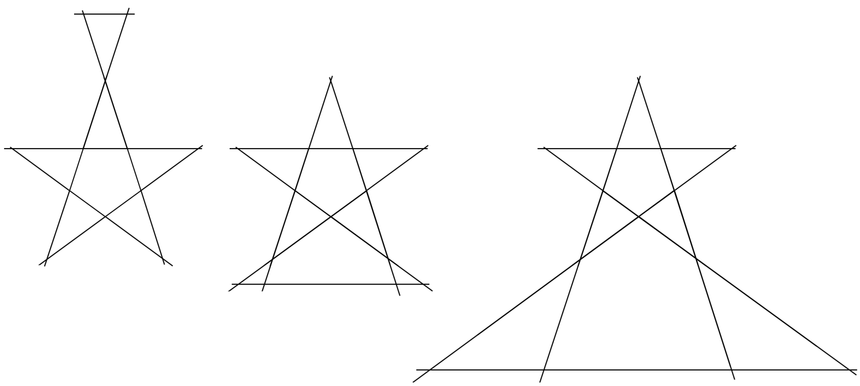


Figure 6: The three essentially different arrangements of five lines bounding a regular pentagon together with a sixth line parallel to one of the five lines forming at least six congruent triangles.

We conclude that in an arrangement on six lines which is f-optimal, every sub-arrangement on five lines contains at most four good triangles. With this in mind, by applying Lemma 2.3 (with $k = 5$), we obtain the desired $f(6) = 8$. Lemmas 2.1 and 2.2 yield the bounds on $g(6)$.

Theorem 3.5 (i) and the Star of David (which proves that $6 \in G(6)$) imply that $[0..6] \subset G(6)$. Together with the arrangements from Figure 5, we are done. $\square$

Among arrangements on six lines bounding exactly six congruent facial triangles, we found three combinatorially non-equivalent ones. (It is unknown whether these are all.)

In the general case, the solution from Figure 5 (b) seems to be unique; see Conjecture 4.1 (which states that $g(6) = 6$) in the final section.

3.4 $\ell = 7$

Theorem 3.7. *We have* $f(7) = 14$, $9 \leq g(7) \leq 11$, $[0..10] \cup \{14\} \subset F(7)$, *and* $[0..9] \subset G(7)$.

Proof. By Theorem 3.6, $f(6) = 8$. Thus, by Lemma 2.2 and Lemma 2.4 (with $k = 6$), $f(7) = 14$. For $g(7)$, the lower bound is given by the construction in Figure 7 (a) (by deleting the line marked h), the upper bound by Lemma 2.1.

Since the Star of David is 1-extendable, we have $7 \in G(7)$. Removing the line marked h in Figures 7 (a) and (b) shows that $9 \in G(7)$ and $8 \in G(7)$, resp. Thus, $[0..9] \subset G(7)$. By considering seven of the eight lines bounding a regular octagon, we obtain $10 \in F(7)$. Together with Lemma 2.2, we have $[0..10] \cup \{14\} \subset F(7)$. $\square$

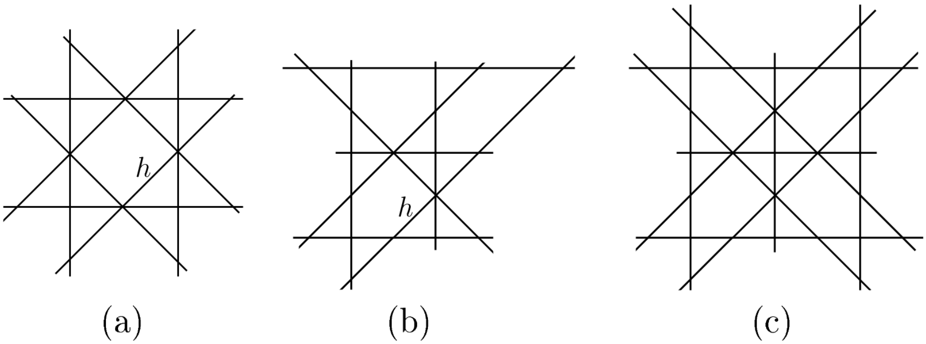


Figure 7: (a) An arrangement proving $12 \in G(8)$. Deleting the line marked h shows that $9 \in G(7)$. (b) An arrangement showing $11 \in G(8)$. Deleting h yields $8 \in G(7)$. (c) This arrangement proves that $g(10) \geq 20$.

3.5 $\ell = 8$

Theorem 3.8. *We have* $16 \leq f(8) \leq 22$, $12 \leq g(8) \leq 15$, $[0..16] \setminus \{13\} \subset F(8)$ *and* $[0..12] \subset G(8)$.

Proof. Lemma 2.2 implies the lower bound for $f(8)$, Theorem 3.7 and Lemma 2.4 (with $k = 7$) the upper bound. For $g(8)$, the lower bound is given by the arrangement from Figure 7 (a), the upper bound by Lemma 2.1.

Figures 7 (a) and (b) show that $\{11, 12\} \subset G(8)$. This, Theorem 3.7 and the fact that the arrangement from Figure 7 (a) minus the line marked h is 1-extendable (which proves that $10 \in G(8)$) yield $[0..12] \subset G(8)$. Applying Lemmas 2.2 and 2.4, we obtain $[0..16] \setminus \{13\} \subset F(8)$. $\square$

3.6 $9 \leq \ell \leq 12$

As the techniques for proving the following results are very similar to what has been shown, we skip them. A notable exception is the construction from Figure 8.

Theorem 3.9. *We have*

$$18 \leq f(9) \leq 33, 15 \leq g(9) \leq 21, [0..18] \subset F(9), [0..15] \subset G(9),$$

$$21 \leq f(10) \leq 48, 20 \leq g(10) \leq 26, [0..21] \subset F(10), [0..20] \subset G(10),$$

$$26 \leq f(11) \leq 66, 23 \leq g(11) \leq 33, [0..26] \subset F(11), [0..23] \subset G(11),$$

and

$$32 \leq f(12) \leq 88, 26 \leq g(12) \leq 39, [0..28] \cup \{32\} \subset F(12), [0..26] \subset G(12).$$

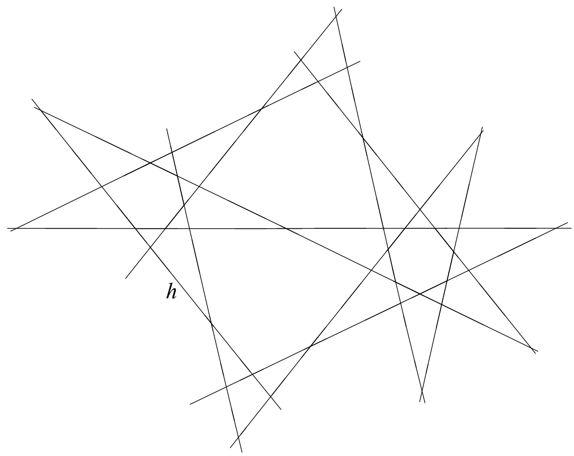


Figure 8: An arrangement proving $f(11) \geq 26$. It is obtained by joining the two arrangements from Figures 1 and 5 (b) with the technique described in the proof of Proposition 3.1, i.e. such that the two arrangements share a pair of lines (forming the same good angle) in the new arrangement. Deleting the line marked h, one obtains $f(10) \geq 21$. By completing the left regular heptagon, we obtain $f(12) \geq 32$.

3.7 Summary

Consider $\ell \leq 12$ lines in the Euclidean plane, and let $f(\ell)$ and $g(\ell)$ be defined as in the Introduction. Then we have the following bounds.

Table 2: Bounding $f(\ell)$ and $g(\ell)$ for $\ell \leq 12$.

ℓ	3	4	5	6	7	8	9	10	11	12
$f(\ell) \geq$	1	2	5	8	14	16	18	21	26	32
$f(\ell) \leq$	1	2	5	8	14	22	33	48	66	88
$g(\ell) \geq$	1	2	5	6	9	12	15	20	23	26
$g(\ell) \leq$	1	2	5	7	11	15	21	26	33	39

We were also able to prove that $f(13) \geq 37, f(14) \geq 44, f(15) \geq 50, f(16) \geq 56$, and $f(17) \geq 61$.

4 Conjectures

Conjecture 4.1. $g(6) = 6$.

If Conjecture 4.1 is true, we would have $g(7) \leq 10$.

Conjecture 4.2. $g(7) = 9$.

If Conjecture 4.2 is true, we would have $g(8) \leq 14$.

Conjecture 4.3. *The f-optimal arrangements on* 6 *and* 7 *lines (consider Figures 5 (b) and 1, resp.) are g-unique.*

Conjecture 4.4. *(a) $F(7)$ is not complete, but (b) for every ℓ, $G(\ell)$ is complete.*

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A note on the directed genus of $K_{n,n,n}$ and K_n

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Abstract

It is proved that a complete graph K_n can have an orientation whose minimum directed genus is $\lceil \frac{1}{12}(n-3)(n-4) \rceil$ if and only if $n \equiv 3, 7 \pmod{12}$. This answers a question of Bonnington et al. by using a method different from current graphs. It is also proved that a complete symmetric tripartite graph $K_{n,n,n}$ has an orientation whose minimum directed genus is $\frac{1}{2}(n-1)(n-2)$.

Keywords: Digraph, complete tripartite graph, directed genus, surfaces.

Math. Subj. Class.: 05C10, 05B05, 05B07

1 Introduction

Throughout this paper, all graphs are assumed to be finite, connected and simple. In a directed graph D, the number of in-arcs at a vertex v is called the *in-degree* of v which is denoted by $d^-(v)$; the number of out-arcs at v is called the *out-degree* of v, denoted by $d^+(v)$. The *degree* of v, denoted by $d(v)$, is the sum of $d^-(v)$ and $d^+(v)$. A digraph D is *Eulerian* if it is connected and every vertex has equal in-degree and out-degree. The underlying graph G of a digraph D is a graph obtained from D by suppressing all directions of the arcs in D. The *orientable surface* of genus h, denoted by S_h, is the sphere with h handles added. A graph is said to be *2-cell embedded* in a surface S, if it is embedded in a surface S such that each component, called a *region,* of $S \setminus D$ is homeomorphic to an open disk. A *2-cell directed embedding* (or *2-cell embedding*) of a digraph D on an orientable surface S means that it is a 2-cell embedding of its underlying graph of D in S such that each region is bounded by a directed cycle. In this paper, all embeddings of graphs

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and digraphs are assumed to be 2-cell embedded on oriented surfaces. Let the genus of a surface S be denoted by $\gamma(S)$. The *directed genus* (or simply say genus) of an embeddable digraph D, denoted by $\gamma(D)$, is the smallest of the numbers $\gamma(S)$ for orientable surfaces S in which D can be directed embedded. Let $|X|$ be the cardinality of a set X.

The study of embeddings of a graph began with Euler. By now, there are many results about the genus ([14, 22, 23, 25, 26, 28, 27, 29]), the maximum genus ([24, 30]), and the genus distribution of a graph ([12, 13, 19, 20]). However, a study of the embeddings of a digraph was started in 2002 by Bonnington et al. in [2]. Bonnington, Hartsfield and Širáň ([3]) gave some obstructions for directed embeddings of digraphs and proved Kuratowski-type theorem for embeddings of digraphs in the plane. This area has remained almost uninvestigated. As we know, genera of only a few kinds of digraphs are known. Hales and Hartsfield calculated the directed genus of the de Bruijn graph in [15]. Hao et al. ([16, 17, 18]) obtained the embedding distributions of some digraphs and maximum embedding properties of digraphs. Chen, Gross and Hu ([4]) derived a splitting theorem for digraph embedding distributions that is analogous to the splitting theorems of [11] and [5] for graph embedding distributions.

Let $\gamma(G)$ denote the genus of a graph G. There are many results on computing genera of undirected graphs. For example, in [25], the genera of the complete graph K_n and the complete tripartite graph $K_{mn,n,n}$ were given as follows: $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$ and $\gamma(K_{mn,n,n}) = \frac{1}{2}(mn-2)(n-1)$. In [28], $\gamma(K_{n,n,n-2}) = \frac{1}{2}(n-2)^2$ for even $n \geq 2$ and $\gamma(K_{2n,2n,n}) = \frac{1}{2}(3n-2)(n-1)$ for $n \geq 1$ were derived. In [26], $\gamma(K_{n,n,n}) = \frac{1}{2}(n-2)(n-1)$ was obtained.

Up to now, the genera of only a few kinds of digraphs are known. For examples, the directed genus of the de Bruijn graph was derived in [15]. In [2], Bonnington et al. determined the genera of the cartesian product $C_n \times C_n$ of two directed cycles, the spoke digraph on $n = 2k+1$ vertices and the directed antiprism DA_k, which are $(n^2-3n+2)/2$, $k-1$ and 0, respectively. Let $\vec{K}_n$ and $\vec{K}_{n,n,n}$ be directed graphs gotten from the complete graph K_n and the complete tripartite $K_{n,n,n}$, respectively, by giving an orientation to each edge. In this paper, we aim to answer the following problem by using a method different from current graphs.

Problem 1.1 ([2]). Which kinds of $\vec{K}_n$ have $\gamma(\vec{G}) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$, the genus of K_n.

A natural question analogue to Problem 1.1 is the following.

Problem 1.2. Which kinds of $\vec{K}_{n,n,n}$ with n vertices in each parts have directed genus $\frac{1}{2}(n-1)(n-2)$, the genus of $K_{n,n,n}$.

In this paper, we solve the Problems 1.1 and 1.2. Problem 1.2 is solved by giving the equivalent conditions for the minimum directed genus embedding of a directed graph $\vec{K}_{n,n,n}$ and a pair of biembeddable Latin squares with order n in an orientable surface. Furthermore, we prove that there is a one to one correspondence between the set of directed embeddings of a digraph D and the set of face-2-colorable embeddings of the underlying graph of D both on orientable surfaces. The result that there exists an orientation on edges of K_n such that the obtained tournament $\vec{K}_n$ has the directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$, when $n \equiv 3, 7 \pmod{12}$ is gotten which answer the Problem 1.1.

2 Alternating rotations, face-2-colorable embeddings, and Latin squares

An *alternating rotation* at a vertex v of D is a cyclic permutation of the arcs incident at v, such that in-arcs alternate with out-arcs. A list of alternating rotations, one for each vertex, is called an *alternating embedding scheme* (also called *alternating rotation system*) for the digraph D. There exists a one to one correspondence between the set of all embeddings (resp. directed embeddings) of a graph G (resp. a digraph D) on orientable surfaces and the set of the embedding schemes (resp. alternating embedding schemes) of G (resp. D). A *color class* is a set of faces with the same color. A *face-2-colorable embedding* of a graph G is an embedding which admits a 2-coloring of regions such that no two distinct regions of the same color shares a common edge. Two colors always mean black and white. Regions in an embedding of a graph are also called *faces*, while regions in a directed embedding of a digraph are partitioned into *faces* which use the arcs in the forward direction and *antifaces* which use arcs traversed against the given orientation.

An embedding is *triangular* if all regions are bounded by 3-cycles. Two face-2-colorable embeddings of K_n are said to be *isomorphic* if there exists a permutation on the n vertices (of the complete graph) such that it maps edges and faces of one embedding to edges and faces of the other one, respectively, see [2]. Equivalently, two face-2-colorable embeddings of K_n are isomorphic if and only if there exists a permutation on the n vertices such that it either preserves the color of the triangles or reverses the color. Let D_1 and D_2 be two digraphs. If D_1 is derived from D_2 by reversing all arcs of D_2, then we say these two digraphs have the *opposite orientation*.

A *transversal design* $TD(3,n)$ is an ordered triple $(V,\mathcal{G},\mathcal{B})$, where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into three disjoint sets (the *groups*) each of which has cardinality n, and $\mathcal{B}$ is a set of three-element subsets of V (the *triples*), such that every unordered pair of elements from V is either contained in precisely one triple or one group, but not both.

Example 2.1. An example of a $TD(3,n)$ of $n=3$. Let

$V=\{1,2,3,\dots,9\}$,
$\mathcal{G}=\{\{4,5,6\},\{7,8,9\},\{1,2,3\}\}$, and
$\mathcal{B}=\{(4,7,3),(4,8,1),(4,9,2),(5,7,1),(5,8,2),(5,9,3),(6,7,2),(6,8,3),(6,9,1)\}$.

Then $(V,\mathcal{G},\mathcal{B})$ is a transversal design $TD(3,3)$.

A *Latin square LS(n) of order* n is an $n\times n$ array filled with n different entries, each occurring exactly once in each row and exactly once in each column.

Example 2.2. A Latin square LS(n) of order n for $n=3$. Let

$$M=\begin{bmatrix}3&1&2\\1&2&3\\2&3&1\end{bmatrix}.$$

Then M is a Latin square LS(3).

There are relations among the face-2-colorable triangular embeddings of $K_{n,n,n}$ on an orientable surface, the transversal design $TD(3,n)$ and the Latin squares as follows.

For a given face 2-colourable triangular embeddings of $K_{n,n,n}$ on an orientable surface, it is proved in [10] that there exists a transversal design which is determined under one of the clockwise and counter-clockwise in each colour class. On the other hand, for a given transversal design $TD(3, n) = (V, \mathcal{G}, \mathcal{B})$, there is a Latin square determined by $TD(3, n)$ by assigning the three groups in $\mathcal{G}$ as labels for the row, columns and entries of the Latin square.

Two color classes $\mathcal{A}$ and $\mathcal{B}$ of a face-2-colorable triangular embedding of $K_{n,n,n}$ on an orientable surface give two Latin squares, corresponding to $\mathcal{A}$ and $\mathcal{B}$ respectively, which is considered as a *biembedding* of these two Latin squares with order n. Two Latin squares A and B are *biembeddable*, denoted by $A \bowtie B$, on an orientable surface S if there is a face-2-colorable (black and white) triangular embedding of $K_{n,n,n}$ in the orientable surface S such that the white face set is $\mathcal{A}$ and the black face set is $\mathcal{B}$. For more details, the readers are referred to [6, 7, 8, 9] and [21].

Example 2.3. Let V_1, V_2 and V_3 be a partition of $V(K_{3,3,3})$, where $V_1 = \{4, 5, 6\}, V_2 = \{7, 8, 9\}$ and $V_3 = \{1, 2, 3\}$. For a given embedding ρ of $K_{3,3,3}$ on an orientable surface, let ρ_v be the rotation at a vertex v. Let

$$\begin{array}{lll} \rho_1 = (7,5,9,6,8,4); & \rho_2 = (7,6,9,4,8,5); & \rho_3 = (7,4,9,5,8,6); \\ \rho_4 = (7,3,9,2,8,1); & \rho_5 = (7,1,9,3,8,2); & \rho_6 = (8,3,7,2,9,1); \\ \rho_7 = (1,5,2,6,3,4); & \rho_8 = (2,5,3,6,1,4); & \rho_9 = (2,4,3,5,1,6). \end{array}$$

Then $\rho = \{\rho_i : i \in \{1, \ldots, 9\}\}$ is a face 2-colourable triangular embedding of $K_{3,3,3}$ on an orientable surface. In fact, a set of faces with the white color is

$$\mathcal{A}_1 = \{(5,7,1), (6,9,1), (4,8,1), (6,7,2), (4,9,2), (5,8,2), (4,7,3), (5,9,3), (6,8,3)\};$$

while a set of faces with the black color is

$$\mathcal{A}_2 = \{(9,5,1), (8,6,1), (7,4,1), (9,6,2), (8,4,2), (7,5,2), (9,4,3), (8,5,3), (7,6,3)\}.$$

There exists a transversal design TD$(3, 3)$, say $(V, \mathcal{G}, \mathcal{B}_1)$, which is determined under the clockwise in white colour class $\mathcal{A}_1$. That is,

$$\begin{aligned} V &= \{1, 2, 3, \ldots, 9\}, \\ \mathcal{G} &= \{\{4,5,6\}, \{7,8,9\}, \{1,2,3\}\}, \text{ and} \\ \mathcal{B}_1 &= \{(5,7,1), (6,9,1), (4,8,1), (6,7,2), (4,9,2), (5,8,2), (4,7,3), (5,9,3), (6,8,3)\}. \end{aligned}$$

There exists another transversal design TD$(3, 3)$, say $(V, \mathcal{G}, \mathcal{B}_2)$, which is determined under the counter-clockwise in black colour class $\mathcal{A}_2$. That is,

$$\begin{aligned} V &= \{1, 2, 3, \ldots, 9\}, \\ \mathcal{G} &= \{\{4,5,6\}, \{7,8,9\}, \{1,2,3\}\}, \text{ and} \\ \mathcal{B}_2 &= \{(5,9,1), (6,8,1), (4,7,1), (6,9,2), (4,8,2), (5,7,2), (4,9,3), (5,8,3), (6,7,3)\}. \end{aligned}$$

Example 2.4. Let $(V, \mathcal{G}, \mathcal{B}_1)$ be a transversal design given in Example 2.3. Assume that $\{4, 5, 6\}$ labels for the row, $\{7, 8, 9\}$ labels for columns and $\{1, 2, 3\}$ labels for entries of the Latin square. Thus

$$\mathcal{B}_1 = \{(5,7,1), (6,9,1), (4,8,1), (6,7,2), (4,9,2), (5,8,2), (4,7,3), (5,9,3), (6,8,3)\}$$

determines the matrix A_1 as

$$\begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \end{array}. \tag{2.1}$$

Thus there is a Latin square A_1 determined by $(V, \mathcal{G}, \mathcal{B}_1)$, where

$$A_1 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Similarly, for a transversal designs $(V, \mathcal{G}, \mathcal{B}_2)$ given in Example 2.3, there is a Latin square A_2 determined by $(V, \mathcal{G}, \mathcal{B}_2)$, where

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

In fact, using $V_1 = \{4, 5, 6\}$ as labels for the row, $V_2 = \{7, 8, 9\}$ as labels for the columns, and $V_3 = \{1, 2, 3\}$ as labels for entries of the Latin square, thus

$$\mathcal{B}_2 = \{(5,9,1), (6,8,1), (4,7,1), (6,9,2), (4,8,2), (5,7,2), (4,9,3), (5,8,3), (6,7,3)\}$$

determines the matrix A_2 as

$$\begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \end{array}. \tag{2.2}$$

As a result, a face-2-colorable triangular embedding ρ of $K_{3,3,3}$ on an orientable surface gives two Latin squares A_1 and A_2, corresponding to two color classes $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. And $A_1 \bowtie A_2$ is a biembedding of these two Latin squares with order 3.

Because an embedding of an embeddable digraph is an embedding of the underlying graph, the following version of Euler's polyhedral formula holds.

Lemma 2.5. *Let $D = (V, A)$ be an embedding digraph, then for any alternating embedding scheme ρ of D, we have*

$$|V| - |A| + |R| = 2 - 2g,$$

where $|R|$ is the number of regions in the embeding scheme ρ and g is the genus of the embedding surface.

Lemma 2.6 ([7]). *There is a unique regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ on an orientable surface for $n \geq 2$.*

Lemma 2.7 ([6]). *For a triangular embedding of $K_{n,n,n}$, it is orientable if and only if it is face-2-colorable embedding.*

The readers are referred to [1] for any undefined notations.

3 The directed genus of $\vec{K}_{n,n,n}$

For an embedding σ of a given digraph $\vec{K}_{n,n,n}$, the alternating embedding scheme is denoted by ρ_σ, the alternating rotation at a vertex $v \in V(D)$ is denoted by $\rho_\sigma(v)$ (or simply ρ_v).

Recall that $K_{n,n,n}$ is a complete tripartite graph. A complete tripartite digraph, denoted by $\vec{K}_{n,n,n}$, obtained from $K_{n,n,n}$ by giving an orientation for each edge in $K_{n,n,n}$. In the following, we find an orientation $\vec{K}_{n,n,n}$ of $K_{n,n,n}$ such that $\vec{K}_{n,n,n}$ has the directed genus $\frac{1}{2}(n-1)(n-2)$, the same as the genus of $K_{n,n,n}$.

Theorem 3.1. *The following two conditions on an orientation $\vec{K}_{n,n,n}$ of the complete tripartite graph $K_{n,n,n}$ are equivalent*:

(1) $\vec{K}_{n,n,n}$ has a directed embedding on the orientable surface of genus $\frac{1}{2}(n-1)(n-2)$, for which we call the sets of faces and antifaces $\mathcal{A}$ and $\mathcal{B}$, respectively.

(2) The sets $\mathcal{A}$ and $\mathcal{B}$ of white faces and black faces for a face-2-colorable triangular embedding of $K_{n,n,n}$ correspond to a pair of biembeddable Latin squares A and B of order n.

Proof. We first show that (1) implies (2).

Assume $\vec{K}_{n,n,n}$ has a directed embedding on an orientable surface of genus $\frac{1}{2}(n-1)(n-2)$ such that the sets of faces and antifaces $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\phi\colon \vec{K}_{n,n,n} \to S$ be this directed embedding of $\vec{K}_{n,n,n}$ and ρ_ϕ be the alternating embedding scheme of ϕ. Note that $\vec{K}_{n,n,n}$ has $3n$ vertices, $3n^2$ arcs and the embedding genus $\frac{1}{2}(n-1)(n-2)$. By Euler's formula of Lemma 2.5, the number of regions in ρ_ϕ is $2n^2$. This implies that each region is bounded by a directed 3-cycle because there are no i-cycles for $i = 1, 2$.

Let the embedding scheme ρ of $K_{n,n,n}$ be the same as ρ_ϕ without considering the directions of arcs, then $\mathcal{A} \cup \mathcal{B}$ is the facial set of the embedding ρ of $K_{n,n,n}$. We color faces in $\mathcal{A}$ with white and antifaces in $\mathcal{B}$ with black. By the definition of a directed embedding, each arc appears once in exactly one facial boundary and exactly one antifacial boundary. That is, no two distinct faces in $\mathcal{A}$ (resp. $\mathcal{B}$) are incident to the same edge. So ρ of $K_{n,n,n}$ is a face-2-colorable triangle embedding with two color classes $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = |\mathcal{B}| = n^2$. Note that two color classes $\mathcal{A}$ and $\mathcal{B}$ of a face-2-colorable triangular embedding of $K_{n,n,n}$ on an orientable surface give two Latin squares, say A and B, corresponding to $\mathcal{A}$ and $\mathcal{B}$ respectively, which is a biembedding of these two Latin squares A and B. The result (2) is obtained.

Secondly, we show that (2) implies (1).

Suppose (2) holds. Note that there exists a face-2-colorable triangular embedding, say ϕ, of $K_{n,n,n}$ on an orientable surface with two facial color classes $\mathcal{A}$ and $\mathcal{B}$ which corresponds a pair of biembeddable Latin squares A and B of order n, respectively. Assume the embedding scheme of the embedding ϕ is ρ_ϕ and the rotation at vertex v in $K_{n,n,n}$ is denoted by $\rho_\phi(v)$. Let $V(K_{n,n,n}) = V_1 \cup V_2 \cup V_3$, where $\{V_1, V_2, V_3\}$ is a partition of $V(K_{n,n,n})$. Suppose $V_1 = \{a_1, a_2, \ldots, a_n\}$, $V_2 = \{b_1, b_2, \ldots, b_n\}$ and $V_3 = \{c_1, c_2, \ldots, c_n\}$.

Note that $\mathcal{A}$ and $\mathcal{B}$ determine transversal designs $(V, \mathcal{G}, \mathcal{A})$ and $(V, \mathcal{G}, \mathcal{B})$ respectively, where $V = V(K_{n,n,n})$, $\mathcal{G} = \{V_1, V_2, V_3\}$ and the faces in each color class form the triples in $\mathcal{A}$ and $\mathcal{B}$ of the transversal designs.

For every edge $uv \in E(K_{n,n,n})$, without loss of generality, let $u = a_i \in V_1$, $v = b_j \in V_2$. By the definition of a transversal design, there is only one triple in $\mathcal{A}$ containing a_i, b_j, say $\{a_i, b_j, c_x\}$ for some $c_x \in V_3$. Thus, vertices b_j and c_x are neighbors of a_i. Without loss of generality, let c_x be the closest successor of b_j in the rotation $\rho_\phi(a_i)$ along the counter-clockwise and the color of the region corresponding to the triple $\{a_i, b_j, c_x\}$ be white. On the other hand, there is exactly one triple in $\mathcal{B}$ containing a_i, b_j, say $\{a_i, b_j, c_y\}$ with $c_y \in V_3$, so b_j is the closest successor of c_y in the rotation $\rho_\phi(a_i)$ along the counter-clockwise and the color of the region corresponding to the triple $\{a_i, b_j, c_y\}$ is black which is illustrated in the left one of Figure 1.

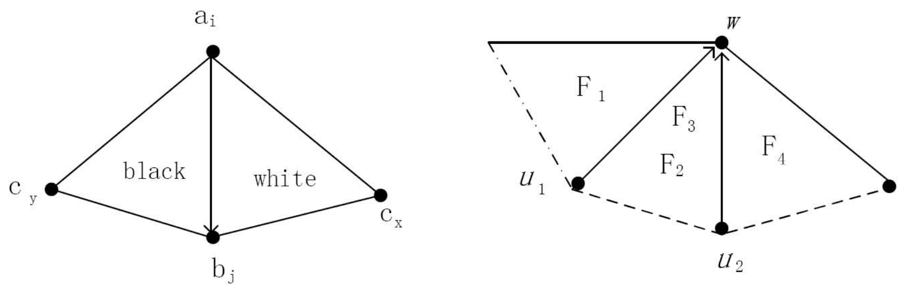

Figure 1: The rotations at vertices a_i and w respectively.

Give the orientation of the edge $uv = a_i b_j$ from $u = a_i$ to $v = b_j$, i.e., the color of the left region of the arc $\overrightarrow{uv}$ is white and the color of the right region is black. By the random choice of uv, all edges in $K_{n,n,n}$ are oriented and the obtained digraph is $\vec{K}_{n,n,n}$.

In the following, we only need to show that this orientation makes the in-arcs and out-arcs alternating at $\rho_\phi(v)$ for any $v \in V(K_{n,n,n})$. By the contrary, suppose there exists a vertex, say $w \in V$, such that in-arcs and out-arcs at w are not alternative. Without loss of generality, suppose two arcs, say $\overrightarrow{u_1w}, \overrightarrow{u_2w}$, are two neighbor in-arcs of w in $\rho_\phi(w)$ and $\rho_\phi(w) = (\ldots, u_1, u_2, \ldots)$ along counter-clockwise. Let the left face and right face of $\overrightarrow{u_1w}$ going from u_1 to w be F_1 and F_2 respectively and the left face and right face of $\overrightarrow{u_2w}$ going along the direction from u_2 to w be F_3 and F_4 respectively. Then $F_2 = F_3$. By the principle of the orientation, F_2 is colored black because of the direction of arc $\overrightarrow{u_1w}$ and F_3 is colored white because of the direction of arc $\overrightarrow{u_2w}$, which is shown in the right graph of Figure 1. It contradicts with face-2-colorable because $F_2 = F_3$. As a result, this orientation makes in-arcs and out-arcs alternating at every vertex $w \in V$ along the rotation $\rho_\phi(w)$.

As a result, $\vec{K}_{n,n,n}$, obtained from $K_{n,n,n}$ by this orientation, has an alternating embedding scheme determined by ϕ such that the sets of faces and antifaces of this directed embedding of $\vec{K}_{n,n,n}$ are $\mathcal{A}$ and $\mathcal{B}$, respectively.

Since each region of this directed embedding of $\vec{K}_{n,n,n}$ is a 3-cycle, the number of regions is $2n^2$. By $|V| = 3n$, the cardinality of arcs in $\vec{K}_{n,n,n}$ being $3n^2$ and Lemma 2.5, it follows $3n - 3n^2 + 2n^2 = 2 - 2g$, where g is the genus of this directed embedding. So $g = \frac{1}{2}(n-1)(n-2)$. Since neither loop nor 2-cycle is in $\vec{K}_{n,n,n}$, the minimum directed genus of $\vec{K}_{n,n,n}$ is $\frac{1}{2}(n-1)(n-2)$. Thus $\vec{K}_{n,n,n}$ has a directed embedding in the orientable surface of genus $\frac{1}{2}(n-1)(n-2)$, for which we call the sets of faces and antifaces $\mathcal{A}$ and

$\mathcal{B}$, respectively. $\square$

Theorem 3.2. *Let $K_{n,n,n}$ be the complete tripartite graph. Then there exists an orientation of $K_{n,n,n}$ such that the obtained digraph $\vec{K}_{n,n,n}$ has the directed genus $\frac{1}{2}(n-1)(n-2)$, the same as the genus of $K_{n,n,n}$.*

Proof. Let $\vec{K}_{n,n,n}$ be the digraph obtained from $K_{n,n,n}$ by giving the orientation to each edge in $K_{n,n,n}$ and g be the directed genus of $\vec{K}_{n,n,n}$.

(1) If $n = 1$, then $K_{n,n,n} = K_{1,1,1}$ is a triangle. Let $\vec{K}_{1,1,1}$ be the digraph obtained by giving an orientation of $K_{1,1,1}$ such that it is a directed 3-cycle. Hence $g = 0$.

(2) If $n \geq 2$, by Lemma 2.6, there is a unique regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ on an orientable surface. By Lemma 2.7, this regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ must be a face-2-colorable embedding and two set of color faces are denoted by $\mathcal{A}$ and $\mathcal{B}$ respectively. By Theorem 3.1 , there is an orientation for $K_{n,n,n}$ such that the resulting digraph $\vec{K}_{n,n,n}$ has a directed embedding in the orientable surface of genus $\frac{1}{2}(n-1)(n-2)$, the set of faces is $\mathcal{A}$ and the set of antifaces is $\mathcal{B}$. Thus the result holds. $\square$

4 The number of different orientations of K_n

Theorem 3.1 for a directed triangular embedding of the directed complete tripartite graph can be generalized to Lemma 4.1 for directed embedding of a general digraph.

Lemma 4.1. *The following two conditions on an orientation $\vec{G}$ of a graph G are equivalent.*

(1) $\vec{G}$ has a directed embedding on an orientable surface of genus g.

(2) G has a face-2-colorable embedding on an orientable surface of genus g.

Proof. We first show that (1) implies (2).

Let $G = (V, E)$ be a graph with n vertices, $\vec{G} = (V, A)$ be a digraph obtained from G by giving an orientation to each edge. So $|V| = n$ and $|E| = |A|$. By (1), $\vec{G}$ has a directed embedding on an orientable surface of genus g. Let ρ be the alternating embedding scheme and $\mathcal{F}_1$ and $\mathcal{F}_2$ be the set of faces and antifaces in $\vec{G}$, respectively. Note that a directed embedding of $\vec{G}$ is an embedding of G and $\mathcal{F}_1 \cup \mathcal{F}_2$ is the set of faces of this embedding of G. We color regions in $\mathcal{F}_1$ with white and rigions in $\mathcal{F}_2$ with black. From the definition of directed embedding, each arc in $\vec{G}$ is incident to exactly one face and exactly one antiface in the directed embedding ρ of $\vec{G}$, so there is no two distinct regions of the same color sharing a common edge in this embedding of G. It implies that this embedding of G is the face-2-colorable embedding on an orientable surface with genus g. So condition (2) holds.

Secondly, we show that (2) implies (1).

Suppose that (2) holds. Let ρ be the embedding scheme of a face-2-colorable embedding of a graph $G = (V, E)$ on an orientable surface S of genus g. And all regions of the embedding ρ can be colored by white and black. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the set of white and black regions, respectively. For each edge $e \in E(G)$, there are exactly two regions sharing the edge e, denoted by F_e^1 and F_e^2. By the definition of the face-2-colorable embedding, F_e^1 and F_e^2 have different colors. Without loss of generality, suppose that $F_e^1 \in \mathcal{F}_1$ and $F_e^2 \in \mathcal{F}_2$. We give the orientation of e such that the left is white region F_e^1 and the right is black region F_e^2 (this is known as *orientational principle*). Since each edge can be oriented, one can obtain a digraph, denoted by $\vec{G}$, from the graph G by this orientational principle. Let the alternating embedding scheme of $\vec{G}$ be the same as ρ. By the orientational principle and face-2-colorability, the in-arcs and out-arcs alternate at each vertex in ρ of $\vec{G}$. Thus this embedding scheme is an alternating embedding scheme of $\vec{G}$ as a directed embedding in the same surface S with genus g, so condition (1) holds. $\square$

Theorem 4.2. *There is a one to one correspondence between the set of directed embeddings of a digraph D on orientable surfaces and the set of face-2-colorable embeddings of the underlying graph of D on orientable surfaces.*

Proof. Let D be a digraph and the underlying graph of D be obtained from D by ignoring the direction of arcs. Theorem 4.2 is obtained directly from Lemma 4.1. $\square$

The following Theorem 4.3 give an answer to the problem in [2].

Theorem 4.3. *If $n \equiv 3, 7 \pmod{12}$, then there exists an orientation on edges of K_n such that the obtained tournament $\vec{K}_n$ has directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$.*

Proof. From Ringel and Youngs' results in [25] and [31], if $n \equiv 3, 7 \pmod{12}$, there exists a face-2-colorable triangular embedding of K_n on an orientable surface. By Lemma 4.1, there exists an orientation on edges of K_n such that the obtained digraph $\vec{K}_n$ has a directed triangular embedding on an orientable surface. By Euler's formula, digraph $\vec{K}_n$ has directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$. $\square$

5 Concluding remarks

In this paper, we show that there is a one to one correspondence between the set of directed embeddings of a digraph D and the set of face-2-colorable embeddings of the underlying graph of D on orientable surfaces. Furthermore, we show that there exist orientations on $K_{n,n,n}$ and K_n such that the obtained graph $\vec{K}_{n,n,n}$ has the directed genus $\frac{1}{2}(n-1)(n-2)$ for $n \geq 1$ and $\vec{K}_n$ has directed genus $\lceil \frac{1}{12}(n-3)(n-4) \rceil$ for $n \equiv 3, 7 \pmod{12}$ which answers the problem about tournaments given in [2] by using a method different from current graphs which were discussed by the same author et al.

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On domination-type invariants of Fibonacci cubes and hypercubes*

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Abstract

The Fibonacci cube Γ_n is the subgraph of the n-dimensional cube Q_n induced by the vertices that contain no two consecutive 1s. Using integer linear programming, exact values are obtained for $\gamma_t(\Gamma_n)$, $n \leq 12$. Consequently, $\gamma_t(\Gamma_n) \leq 2F_{n-10} + 21F_{n-8}$ holds for $n \geq 11$, where F_n are the Fibonacci numbers. It is proved that if $n \geq 9$, then $\gamma_t(\Gamma_n) \geq \lceil (F_{n+2} - 11)/(n-3) \rceil - 1$. Using integer linear programming exact values for the 2-packing number, connected domination number, paired domination number, and signed domination number of small Fibonacci cubes and hypercubes are obtained. A conjecture on the total domination number of hypercubes asserting that $\gamma_t(Q_n) = 2^{n-2}$ holds for $n \geq 6$ is also disproved in several ways.

Keywords: Total domination number, Fibonacci cube, hypercube, integer linear programming, covering codes.

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1 Introduction

Fibonacci cubes were introduced by Hsu [19] because of their appealing properties applicable to interconnection networks. Afterwards they have been extensively studied and found additional applications, see the survey [23]. The interest for Fibonacci cubes continues, recent research of them includes asymptotic properties [24], connectivity issues [7], the structure of their disjoint induced hypercubes [14, 30], the (non)-existence of perfect codes [5], and the q-cube enumerator polynomial [31]. From the algorithmic point of view, Ramras [29] investigated congestion-free routing of linear permutations on Fibonacci cubes, while Vesel [34] designed a linear time recognition algorithm for this class of graphs.

The domination number of Fibonacci cubes was investigated by now in two papers. Pike and Zou [28, Theorem 3.2] proved that $\gamma(\Gamma_n) \geq \lceil (F_{n+2} - 2)/(n-2) \rceil$ for $n \geq 9$, where F_n are the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Exact values of $\gamma(\Gamma_n)$ for $n \leq 8$ were also obtained in [28]. In the second related paper [9] the domination number of Fibonacci cubes was then compared with the domination number of Lucas cubes.

In this note we turn our attention to domination invariants of Fibonacci cubes and of hypercubes with a prime interest on the total domination. We proceed as follows. In the rest of this section we introduce concepts and notation needed. Then, in Section 2, we determine the exact value of the total domination number of Γ_n for $n \leq 12$, and obtain an upper bound and a lower bound on $\gamma_t(\Gamma_n)$. In Section 3 we use integer linear programming to either extend or obtain values for several domination-type invariants on Fibonacci cubes and hypercubes. In the final section we consider the total domination of hypercubes with respect to a recent conjecture from [22]. In particular, using known results from coding theory we show that the conjecture does not hold. It is also observed that for any $c > 0$ there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then $\gamma_t(Q_n) \leq 2^{n-c}$.

The n-dimensional (hyper)cube Q_n, $n \geq 1$, is the graph with $V(Q_n) = \{0,1\}^n$, two vertices being adjacent if they differ in a single coordinate. For convenience we also set $Q_0 = K_1$. The vertices of Q_n will be briefly written as binary strings $b_1 \ldots b_n$. A *Fibonacci string* of length n is a binary string $b_1 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. Fibonacci strings are thus binary strings that contain no consecutive 1s. The *Fibonacci cube* Γ_n, $n \geq 1$, is the subgraph of Q_n induced by the Fibonacci strings of length n. It is well known that $|V(\Gamma_n)| = F_{n+2}$.

If u is a binary string, then the number of its bits equal to 1 is the *weight* of u. If u and v are binary strings, then uv denotes the usual concatenation of the two strings. If u is a binary string and X a set of binary strings, then $uX = \{ux : x \in X\}$.

Let G be a graph. Then $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ is adjacent to some vertex from D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. D is a *total dominating set* if every vertex from $V(G)$ is adjacent to some vertex from D. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. Note that the total domination number is not defined for graphs that contain isolated vertices, hence unless stated otherwise, all graphs in this paper are isolate-free. For more information on the total domination in graphs see the recent book [17] and papers [11, 12].

2 Total domination in Fibonacci cubes

In this section we present exact values of $\gamma_t(\Gamma_n)$ for $n \leq 12$, prove an upper bound on $\gamma_t(\Gamma_n)$, and a lower bound on $\gamma_t(\Gamma_n)$. The exact values were obtained by computer and are collected in Table 1, where the order of the cubes is also given so that the complexity of the problem is emphasized. In particular, $|V(\Gamma_{12})| = 377$.

Table 1: Exact total domination numbers of Fibonacci cubes up to dimension 12.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\|V(\Gamma_n)\|$	2	3	5	8	13	21	34	55	89	144	233	377
$\gamma_t(\Gamma_n)$	2	2	2	3	5	7	10	13	20	30	44	65

More precisely, the results from Table 1 were obtained using integer linear programming as follows. Suppose we associate to each vertex $v \in V(\Gamma_n)$ a binary variable x_v. The problem of determining $\gamma_t(\Gamma_n)$ can then be expressed as a problem of minimizing the objective function

$$\sum_{v \in V(\Gamma_n)} x_v,$$

subject to the condition that for every $v \in V(\Gamma_n)$ we have

$$\sum_{u \sim v} x_u \geq 1.$$

The value of the objective function is then $\gamma_t(\Gamma_n)$.

We have found out that the most efficient solver for the above problem is Gurobi™ Optimizer [15]. For example, it takes less than $9s$ to compute $\gamma_t(\Gamma_{12})$ on a standard desktop machine. On the other hand, we were not able to make the computation for $\gamma_t(\Gamma_{13})$ in real time (note that the order of Γ_{13} is 610), we could only get the estimates

$$97 \leq \gamma_t(\Gamma_{13}) \leq 101\,.$$

Using the above computations, the following result can be derived.

Theorem 2.1. *If $n \geq 11$, then $\gamma_t(\Gamma_n) \leq 2F_{n-10} + 21F_{n-8}$.*

Proof. Consider the so-called fundamental decomposition of Γ_n into the subgraphs induced by the vertices that start with 0 and 10, respectively (cf. [23]). These subgraphs are isomorphic to Γ_{n-1} and Γ_{n-2} respectively, hence we infer that $\gamma_t(\Gamma_n) \leq \gamma_t(\Gamma_{n-1}) + \gamma_t(\Gamma_{n-2})$. From the above computations we know that $\gamma_t(\Gamma_{11}) = 44$ and $\gamma_t(\Gamma_{12}) = 65$. Define the sequence (a_n), $n \geq 11$, with $a_{11} = 44$, $a_{12} = 65$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 13$. Then one can check by a simple induction argument that $a_n = 2F_{n-10} + 21F_{n-8}$ holds for any $n \geq 11$. Since $\gamma_t(\Gamma_n) \leq a_n$ the argument is complete. □

Arnautov [3] and independently Payan [27] proved that

$$\gamma(G) \leq \frac{|V(G)|}{\delta + 1} \sum_{j=1}^{\delta+1} \frac{1}{j} \tag{2.1}$$

holds for any graph G of minimum degree δ. Since $\delta(\Gamma_n) = \lfloor (n+2)/3 \rfloor$, cf. [25, Corollary 3.5], and because $\gamma_t \le 2\gamma$, we get that

$$\gamma_t(\Gamma_n) \le \frac{2F_{n+2}}{\lfloor \frac{n+5}{3} \rfloor} \sum_{j=1}^{\lfloor \frac{n+5}{3} \rfloor} \frac{1}{j} \,. \tag{2.2}$$

Computing the values of the right-hand side of the bound of Theorem 2.1 and of (2.2) we find out that Theorem 2.1 is better than the bound of (2.2) for $n \le 33$.

By using the fact $\gamma_t(\Gamma_{13}) \le 101$ that was obtained by our computations, the bound of Theorem 2.1 can be further improved to give

$$\gamma_t(\Gamma_n) \le 601F_{n-1} - 371F_n,\ n \ge 12\,.$$

We continue by establishing a lower bound on $\gamma_t(\Gamma_n)$.

Theorem 2.2. *If $n \ge 9$, then*

$$\gamma_t(\Gamma_n) \ge \left\lceil \frac{F_{n+2} - 11}{n-3} \right\rceil - 1\,.$$

Proof. The proof mimics the proof of [28, Theorem 3.2] which gives a lower bound on the domination number of Fibonacci cubes, hence we will not give all the details.

For a graph G and its total dominating set D we introduce the *over-total-domination* of D in G as $\mathrm{OD}_G(D) = \sum_{v\in D} \deg(v) - |V(G)|$. Consider now Γ_n, $n \ge 9$, and let D be a total dominating set of Γ_n. In Γ_n, the vertex 0^n is the unique vertex of degree n, vertices 10^{n-1} and $0^{n-1}1$ have degree $n-1$, and all other vertices of weight 1 have degree $n-2$. In addition, the vertices 1010^{n-3}, $10^{n-2}1$, and $0^{n-3}101$ are of degree $n-2$, while all other vertices of Γ_n have degree at most $n-3$, cf. [25].

Let k be the number of vertices of weight 1 from $D \setminus \{10^{n-1}, 0^{n-1}1\}$. In addition, let $\ell = |D \cap \{1010^{n-3}, 10^{n-2}1, 0^{n-3}101\}|$. Note that $k+\ell$ is the number of vertices from D that have degree $n-2$. The proof now proceeds by considering the cases that happen based on the membership of the vertices 0^n, 10^{n-1}, and $0^{n-1}1$ in D. Here we consider only the case when $\{0^n, 10^{n-1}, 0^{n-1}1\} \subseteq D$. We have:

$$\mathrm{OD}_G(D) \le n + 2(n-1) + (k+\ell)(n-2) + (\gamma_t(\Gamma_n) - 3 - k - \ell)(n-3) - F_{n+2}\,.$$

Since clearly $\mathrm{OD}_G(D) \ge 0$, from the above inequality we derive that $\gamma_t(\Gamma_n)(n-3) \ge F_{n+2} - k - \ell - 7$. Because $k + \ell \le n+1$ we get

$$\begin{aligned} \gamma_t(\Gamma_n) \quad &\ge \quad \frac{F_{n+2} - k - \ell - 7}{n-3} \ge \frac{F_{n+2} - (n+1) - 7}{n-3} \\ &= \quad \frac{F_{n+2} - 11 - (n-3)}{n-3} = \frac{F_{n+2} - 11}{n-3} - 1\,, \end{aligned}$$

and the stated inequality holds in this case. All the other cases are treated similarly. □

We conclude the section with Table 2 in which known values and current best bounds on $\gamma_t(\Gamma_n)$ for $n \le 33$ are collected. The values for $n \le 12$ were computed using the linear program explained above. The bounds for $\gamma_t(\Gamma_{13})$ were established by Gurobi, and we conjecture that in fact $\Gamma_t(\Gamma_{13}) = 101$. Finally, the remaining bound in Table 2 were obtained by the bounds given in Theorems 2.1 and 2.2. Recall that $n = 33$ is the last value for which Theorem 2.1 gives a better bound than the bound (2.2).

Table 2: Exact values and current best bounds on $\gamma_t(\Gamma_n)$, $n \leq 33$.

n	$\gamma_t(\Gamma_n)$	n	$\gamma_t(\Gamma_n)$	n	$\gamma_t(\Gamma_n)$
1	2	12	65	23	3749-13276
2	2	13	97-101	24	5779-21481
3	2	14	87-174	25	8926-34757
4	3	15	131-283	26	13816-56238
5	5	16	196-457	27	21424-90995
6	7	17	296-740	28	33280-147233
7	10	18	449-1197	29	51778-238228
8	13	19	682-1937	30	80676-385461
9	20	20	1040-3134	31	125876-623689
10	30	21	1590-5071	32	196649-1009150
11	44	22	2438-8205	33	307580-1632839

3 Additional invariants on small Fibonacci cubes and hypercubes

The integer linear programming approach can be used to compute several additional invariants of Fibonacci cubes (and other graphs). This has recently been done by Ilić and Milošević in [20], where they have computed the domination number, the 2-packing number, and the independent domination number of low dimensional Fibonacci cubes. In particular, they have used integer linear programming to confirm the conjecture from [9] stating that $\gamma(\Gamma_9) = 17$. In addition, an integer linear programming model for the connected domination number has been presented in [13]. In this section we add to the list of integer linear programming models paired domination and signed domination. The concepts mentioned in this paragraph that have not been introduced yet are defined next.

A set $X \subseteq V(G)$ is a 2-*packing* if $d(x, y) \geq 3$ holds for any $x, y \in X$, $x \neq y$. The maximum size of a 2-packing of G is the 2-*packing number* of G denoted $\rho(G)$. The *independence domination number* $i(G)$ of G is the minimum size of a dominating set that induces no edges [26]. The *connected domination number* $\gamma_c(G)$ of G is the order of a smallest dominating set that induces a connected graph [10]. The *paired domination number* $\gamma_p(G)$ is the order of a smallest dominating set $S \subseteq V(G)$ such that the graph induced by S contains a perfect matching [2]. Finally, we say that $f\colon V(G) \to \{-1, 1\}$ is a *signed dominating function* if $\sum_{u \in N[v]} f(u) \geq 1$ holds for every $v \in V(G)$, where $N[v]$ is the closed neighborhood of v, that is, $N[v] = \{v\} \cup \{u : vu \in E(G)\}$. The *signed domination number* $\gamma_s(G)$ is the minimum of $\sum_{v \in V(G)} f(v)$ taken over all signed dominating functions f of G, see [18].

We now present the problems to determine the paired domination number of a graph and the signed domination number of a graph as integer linear programs. To model the paired domination problem for a graph G we introduce a binary variable x_e indicating whether the edge $e \in E(G)$ is present in the graph induced by a paired dominating set of G. Then we can model the problem as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E(G)} x_e \\ \text{subject to} \quad & \sum_{u \sim v} x_{uv} \quad \leq 1, \quad v \in V(G) \\ & \sum_{u \sim v} \sum_{w \sim u} x_{uw} \geq 1, \quad v \in V(G)\,. \end{aligned}$$

Similarly, to model the signed domination number we introduce a binary variable x_v associated with every vertex $v \in V(G)$ indicating whether v is assigned weight 1 or -1, respectively. Then we have the following linear program.

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V(G)} (2x_v - 1) \\ \text{subject to} \quad & \sum_{u \in N[v]} (2x_u - 1) \geq 1, \quad v \in V(G)\,. \end{aligned}$$

Our computational results are collected in Tables 3 and 4. In the rows for $\gamma(\Gamma_n)$, $\rho(\Gamma_n)$, and $i(\Gamma_n)$, the results from [20] are in normal font, while the new values are in bold. We have thus extended the results from [20] for one additional dimension. It is interesting to observe that the gap between the independent domination number and the domination in dimension 9 is equal to 2, but then in dimensions 10 and 11 the difference goes down to 1.

Table 3: Additional invariants for small Fibonacci cubes and hypercubes.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\gamma(\Gamma_n)$	1	1	2	3	4	5	8	12	17	25	**39**	**54-61**
$\rho(\Gamma_n)$	1	1	2	2	3	5	6	9	14	20	29	**42**
$i(\Gamma_n)$	1	1	2	3	4	5	8	12	19	26	**40**	**?-?**

Table 4: Additional invariants for small Fibonacci cubes and hypercubes.

n	1	2	3	4	5	6	7	8	9	10
$\gamma_c(\Gamma_n)$	1	1	2	3	5	7	10	14	22	
$\gamma_c(Q_n)$	1	2	4	6	10	16	28			
$\gamma_p(\Gamma_n)$	2	2	2	4	6	8	10	14	20	30
$\gamma_p(Q_n)$	2	2	4	4	8	14	24	32		
$\gamma_s(\Gamma_n)$	2	3	3	2	5	9	10	17	25	40
$\gamma_s(Q_n)$	2	2	4	6	12	16	32			

4 On total domination in hypercubes

It has recently been conjectured in [22, Conjecture 4.6] that $\gamma_t(Q_n) = 2^{n-2}$ holds for $n \geq 6$. In [4] Arumugam and Kala first observed that $\gamma_t(Q_1) = \gamma_t(Q_2) = 2$ and $\gamma_t(Q_3) = \gamma_t(Q_4) = 4$, and then followed by proving that $\gamma_t(Q_5) = 8$ [4, Theorem 5.1] and

$\gamma_t(Q_6) = 14$ [4, Theorem 5.2]. The last result is then a sporadic counterexample to the conjecture. Actually, at this moment the exact value of $\gamma_t(Q_n)$ is known for $n \leq 10$: $\gamma_t(Q_7) = 24$, $\gamma_t(Q_8) = 32$, $\gamma_t(Q_9) = 64$, and $\gamma_t(Q_{10}) = 124$, see [33, Appendix B, p. 40]. Hence Q_7 and Q_{10} are additional sporadic counterexamples (and so are Q_8 and Q_9 since $\gamma_t(Q_8) = 32 \neq 2^6$ and $\gamma_t(Q_9) = 64 \neq 2^7$).

Total dominating sets of Q_n can be in coding theory equivalently described as *covering codes of empty spheres* (of length n and covering radius 1). The following result was first proved back in [21], see also [35, Theorem 1(b)]. Let us rephrase the result here in graph-theoretical terms and give a corresponding argument.

Proposition 4.1. *If $n = 2^k$, $k \geq 0$, then $\gamma_t(Q_n) = 2^{n-k}$.*

Proof. From [32] we know that if $n = 2^k$, then $\gamma(Q_n) = 2^{n-k}$ and from [16] that if $n = 2^k - 1$, then also $\gamma(Q_n) = 2^{n-k}$. Let $n = 2^k$ and consider Q_n. Let Q^L_{n-1} and Q^R_{n-1} be the subgraphs of Q_n induced by the sets of vertices $X_0 = \{0b_2 \dots b_n : b_i \in \{0,1\}\}$ and $X_1 = \{1b_2 \dots b_n : b_i \in \{0,1\}\}$, respectively. Clearly, $V(Q_n)$ partitions into X_0 and X_1, and in Q_n every vertex of X_0 has a unique neighbor in X_1. Moreover, Q^L_{n-1} and Q^R_{n-1} are both isomorphic to Q_{n-1}. Let C_L be a perfect code of Q^L_{n-1} and let C_R be its copy in Q^R_{n-1}. Then $C_L \cup C_R$ is a total dominating set of Q_n of order 2^{n-k}. Since on the other hand $\gamma_t(Q_n) \geq \gamma(Q_n) = 2^{n-k}$, the conclusion follows. □

It follows from (2.1) that

$$\gamma(G) \leq |V(G)| \left(\frac{1 + \ln(\delta + 1)}{\delta + 1} \right) \tag{4.1}$$

holds for any graph G. Hence, again using the fact that $\gamma_t(G) \leq 2\gamma(G)$, we get for hypercubes that

$$\gamma_t(Q_n) \leq 2^{n+1} \left(\frac{1 + \ln{(n+1)}}{n+1} \right) .$$

Directly from this inequality we infer:

Remark 4.2. For any $c > 0$ there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then

$$\gamma_t(Q_n) \leq 2^{n-c} .$$

Two remarks are in place here. First, (4.1) also follows from a more general result on transversals in hypergraphs due to Alon [1]. Second, the state of the art on the upper bounds on the domination number in terms of the minimum degree and the order of a given graph is given in [8].

It follows from the fact that $\gamma_t(Q_n) \leq 2\gamma(Q_{n-1})$ and from Proposition 4.1 that $\gamma_t(Q_{2^k+1}) \leq 2\gamma(Q_{2^k}) = 2^{2^k-k+1}$. As proved in [32], the equality actually holds here, that is, $\gamma_t(Q_{2^k+1}) = 2^{2^k-k+1}$. More generally, $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$ holds for any n, a result very recently proved in [6].

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On t-fold covers of coherent configurations

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Abstract

We introduce the covering configuration induced by a regular weight defined on a coherent configuration. This construction generalizes the well-known equivalence of regular two-graphs and antipodal double covers of complete graphs. It also recovers, as special cases, the rank 6 association schemes connected with regular 3-graphs, and certain extended Q-bipartite doubles of cometric association schemes. We articulate sufficient conditions on the parameters of a coherent configuration for it to arise as a covering configuration.

Keywords: Association scheme, coherent configuration, regular weight, double cover, two-graph, t-graph.

Math. Subj. Class.: 05C22, 05C50, 05E30

1 Introduction

The Seidel matrix of a graph Γ may be viewed as a weight on the complete graph: edges of Γ are weighted (-1) and non-edges $(+1)$. If Γ is strongly regular with $n = 2(2k - \lambda - \mu)$, it lies in the switching class of a regular two-graph and we call the weight, analogously, *regular* on K_n. This condition on Γ is well known, and dates to 1977, in [25]. The same year, the equivalence of regular two-graphs and antipodal double covers of complete graphs was established in [26].

Martin, Muzychuk and Williford ([18]) defined the *extended Q-bipartite double* of a cometric association scheme, extending the notion of the bipartite double of a distance regular graph. This construction produces, as special cases, the antipodal double covers of complete graphs from the strongly regular graphs affording regular two-graphs.

In recent work, Kalmanovich ([16]) has also generalized the regular two-graph result, working from an unpublished draft of D. G. Higman's ([9]) on *regular 3-graphs*. As defined in [14], a t-graph weights the edges of K_n with elements of the group of roots of unity of

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order t, U_t. The regularity condition ensures that the matrix of edge weights has a quadratic minimal polynomial. The work of Kalmanovich-Higman establishes the equivalence of regular 3-graphs with cyclic antipodal 3-fold covers of K_n ([6]). Regular 3-graphs are shown to give rise to certain rank 6 association schemes, and the necessary conditions under which a rank 6 scheme arises in this way are given.

In this paper there are two main results. First, working with a regular weight with values in U_t, defined on a coherent configuration (CC), we show that there is always a *covering configuration*; that is, a CC constructed using a t-fold cover in a natural way, to convert the weight into a CC of higher rank (by a factor of t). As special cases, we recover the equivalence between regular two-graphs and antipodal double covers of complete graphs; some extended Q-bipartite doubles of cometric schemes; the rank 6 schemes associated with regular 3-graphs, and an extension of these to regular t-graphs.

A CC with a regular weight has two sets of parameters: the structure constants for the weighted adjacency algebra, $\{\beta_{ij}^k\}$, which lie in $\mathbb{C}$ or more specifically in the ring of integers with a primitive t^{th} root of unity adjoined, and the non-negative integers $\{\beta_{ij}^k(\nu)\}$ which count certain triangles with a specified weight. They are related by

$$\beta_{ij}^k = \sum_{\nu \in U_t} \nu \beta_{ij}^k(\nu).$$

The weighted adjacency algebra is in general not a coherent algebra, and may in fact have a coherent closure that is much higher in rank than the original CC. In the regular two-graph case, for instance, it is precisely when the (-1) edges form an SRG that we get a minimal closure: a natural fission of the edge set into $(+1)$ and (-1) edges that yields a (rank 3) association scheme. The covering configuration is the realization of a CC whose structure constants are the $\beta_{ij}^k(\nu)$. Some properties, namely homogeneity and commutativity of a CC carry over to the covering configuration. Symmetry is preserved only if $t = 2$. Metric and cometric properties are not.

The second main result of this paper is the articulation of sufficient conditions for a CC to be the covering configuration of a regular weight.

In the final section, we describe a family of regular weights on the Hamming Scheme $H(n, 2)$ with values in U_4, due to Ada Chan. These weights all fuse to regular 4-graphs, providing an infinite family that may be of interest as complex Hadamard matrices. These regular weights, and their fusions, admit covering configurations of ranks $4(n + 1)$ and 8 respectively, on 2^{n+2} points.

2 Preliminaries

In this section, we give the definitions that are essential to what follows. Much more can be found in [17] and in the original developments of the area by Weisfeiler and Lehman in [28] and by D. G. Higman in [11, 12], and [14].

2.1 Coherent configurations

Definition 2.1. Let $\{A_i\}_{0 \leq i < r}$ be a set of 01-matrices with rows and columns indexed by a finite set X. Let $\mathcal{I} := \{0, 1, \ldots, r\}$. The linear span $\mathcal{A} := \langle A_i \rangle_{\mathbb{C}}$ is a *coherent algebra* if:

(i) $\sum_{i \in \mathcal{I}} A_i = J$, where J is the all-ones matrix,

(ii) $\sum_{i \in \mathcal{L}} A_i = I$, for some subset $\mathcal{L} \subset \mathcal{I}$,

(iii) for each i there exists $i^* \in \mathcal{I}$ such that $A_i^T = A_{i^*}$,

(iv) $A_i A_j = \sum p_{ij}^k A_k, p_{ij}^k \in \mathbb{Z}^+$.

A coherent algebra (CA), is *homogeneous* if $|\mathcal{L}| = 1$; *symmetric* if $i^* = i$ for all i, and *commutative*, clearly, if $p_{ij}^k = p_{ji}^k$ for all i, j, k. The homogeneous CAs are (possibly non-symmetric) association schemes. Commutative schemes which have the *metric* or *P-polynomial* property are synonymous with *distance-regular graphs* (DRGs); those of diameter 2 are the *strongly regular graphs* (SRGs). Some familiarity with these structures is assumed. References for readers lacking this background are [1, 2, 4, 5, 19], and [27]. In the association scheme literature, a rank r scheme is often referred to as an $(r-1)$-class scheme: 'rank' counts the trivial relation, while the number of 'classes' does not.

Every algebra of n by n matrices over $\mathbb{C}$ that is closed under transpose and entry-wise multiplication, and contains both I and J is a coherent algebra, and as such it has a basis of 01-matrices satisfying (i)–(iv). Each A_i in a CA is the adjacency matrix of a digraph Γ_i with vertex set X, which is simple for $i \notin \mathcal{L}$ and a graph when $i^* = i$. Viewing these graphs as relations on X, define a *coherent configuration* (CC) to be a set of binary relations on X, indexed by $\mathcal{I}$, with analogous properties to (i)–(iv) above. Denote it $\mathfrak{A} := (X, \{R_i\}_{i\in\mathcal{I}})$.

The constant p_{ij}^k counts the number of i-j paths from a vertex x to a vertex z, given that $(x, z) \in R_k$ and this number is necessarily independent of the choice of edge in Γ_k. It is convenient to denote each instance of an i-j path by a *triangle* (x, y, z) *of type* (i, j, k). That is, $(x, y, z) \in X^3$ is a triangle of type (i, j, k) if $(x, y) \in R_i$, $(y, z) \in R_j$, and $(x, z) \in R_k$ as indicated in Figure 1.

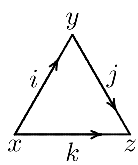

Figure 1: Triangle (x, y, z) of type (i, j, k).

Define the *intersection matrices* M_j of a CC by $M_j := \left(p_{ij}^k\right), 0 \le i, k < r$ thus the map

$$\gamma\colon A_j \mapsto M_j$$

is the right regular representation of $\mathcal{A}$.

We treat CAs and CCs as equivalent structures and move freely between the notations of matrices, relations, and graphs. As $\{A_i\}$ forms the *standard basis* of $\mathcal{A}$, we refer to $\{R_i\}$ and $\{\Gamma_i\}$ as the *basic relations* and *basic graphs* of $\mathfrak{A}$ respectively.

2.2 Fusion and fission

A *fusion* is a merging of relations in a CC according to a partition of $\mathcal{I}$. A fusion will be deemed *coherent* if the resulting configuration is coherent. A coherent *fission* or *refinement* is a partition of each basic relation such that the resulting set of relations forms a CC.

The rank 2 CC represented by K_n is the minimum element in the lattice of all CCs on a given vertex set X of size n ([12, Prop. 3]). The maximum element has rank n^2, with the full matrix algebra $M_X(\mathbb{C})$ as its coherent algebra.

2.3 Regular weights

Let $U = U_t$ be the group of complex t^{th} roots of unity, and fix a primitive root ζ as the generator of U.

Definition 2.2. A *weight* with values in U is a 2-cochain $\omega\colon X^2 \to U$. Viewed as a matrix, a weight is Hermitian with unit diagonal.

The *coboundary* of ω is a function on triangles:

$$\delta\omega(x,y,z) := \omega(y,z)\overline{\omega(x,z)}\omega(x,y)$$

and we refer to this value as the *weight* of the triangle (x,y,z). Analogous to Seidel switching on a graph, switching a weight ω at vertex x_i by a factor of $\alpha \in U$ multiplies the weight on (x_i,y) edges by α and on (y,x_i) edges by $\bar{\alpha}$ for all $y \neq x_i$. In matrix form, this is a similarity transform by the diagonal matrix $\operatorname{diag}(1,1,\ldots,1,\alpha,1,\ldots,1)$ with α in position i. We refer to two weights as *switching equivalent* if one is obtained from the other by some sequence of switches, and observe that $\delta\omega$ is invariant under switching.

Definition 2.3. A *t-graph* is $\delta\omega$ for some weight ω. It is *regular* if

$$|\{y \mid \delta\omega(x,y,z) = \alpha\}|$$

is independent of x and z, for each value $\alpha \in U$.

This is one of a number of natural generalizations of the regular two-graph ([9, 16, 22, 23, 24, 25]). Since a 2-cochain is equivalent to a weight on the edges of a complete graph, the notion of regularity can be extended to weights on CAs.

The entry-wise product $\omega \circ A_i$ gives a matrix with (x,y) entry equal to $\omega(x,y)$ where $(x,y) \in R_i$. Denote this *weighted adjacency matrix* A_i^ω.

Definition 2.4 ([14])**.** A weight ω is *regular* on a CC if for $(x,z) \in R_k$ the number of triangles (x,y,z) of type (i,j,k) and weight α is independent of x and z. In this case, the number of such triangles depends on i, j, k, and α and we denote this parameter $\beta_{ij}^k(\alpha)$.

If ω is regular on $\mathfrak{A}$, then $\sum_\alpha \beta_{ij}^k(\alpha) = p_{ij}^k$. By a straight-forward counting argument,

$$A_i^\omega A_j^\omega = \sum_k \beta_{ij}^k A_k^\omega \text{ where } \beta_{ij}^k := \sum_{\alpha\in U} \alpha\beta_{ij}^k(\alpha)$$

thus $\mathcal{A}^\omega := \langle A_i^\omega \rangle$ is a self-adjoint matrix algebra containing I and we refer to the β_{ij}^k as the *parameters* or *structure constants* of the $\mathcal{A}^\omega$. Note that this *weighted adjacency algebra* is not necessarily closed under the entry-wise product, hence it is not, in general, a coherent algebra. The *weighted intersection matrices* are defined in the obvious way,

$$M_j := \left(\beta_{ij}^k\right), 0 \leq i,k < r.$$

Switching equivalent weights have identical parameters and therefore identical intersection matrices.

2.4 The fission induced by a weight

The weighted CC $(\mathfrak{A}, \omega)$ has a natural fission in which R_i is partitioned according to distinct values of ω. Put

$$(A_i^\alpha)_{xy} := \begin{cases} 1 & \text{if } (A_i^\omega)_{xy} = \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

Some useful properties are:

1. $A_i^\alpha \circ A_j^\beta = \delta_{i,j}\delta_{\alpha,\beta}A_i^\alpha$;
2. $A_i = \sum_\alpha A_i^\alpha$;
3. $A_i^\omega = \sum_\alpha \alpha A_i^\alpha$.

Definition 2.5. $(\mathfrak{A}, \omega)$ has *minimal closure* if the fission $\{A_i^\alpha\}$ forms a CC.

The terminology draws on the notion of the *coherent closure* of a set of matrices as the smallest CA containing them (see [21, 28] for more). The coherent closure of $(\mathfrak{A}, \omega)$ is the CC whose CA is the coherent closure of the matrix algebra $\mathcal{A}^\omega$. Clearly

$$\sum_{i \in \mathcal{L}} A_i^1 = I$$

and the A_i^α sum to J. Furthermore, by the Hermitian property of the weight,

$$(A_i^\alpha)^T = A_{i^*}^{\overline{\alpha}}$$

but the fission is not in general coherent and may in particular generate a matrix algebra of dimension greater than rt.

A weighted CC may be represented in a natural way as a t-fold cover of the configuration. The main goal of this work is to characterize regular weights on CCs in this way, and to describe the construction of a CC of rank rt – the *covering configuration* – derived from the cover.

Let $U = U_t$ with generator ζ, let Γ be a graph or digraph with vertex set X, and ω a weight on Γ. Following [14], we define the t-fold cover of Γ *afforded* by ω as follows. The vertex set is $X \times \{1, 2, \ldots, t\}$. We abuse notation, denoting the t copies of each vertex x by $\{x_1, x_2, \ldots, x_t\}$. Assign adjacencies by $x_i \sim y_j$ whenever $x \sim y$ in Γ and $\omega(x, y) = \zeta^{i-j}$. The induced permutation of indices, $i \mapsto j$ determines a permutation σ of U, namely $\zeta^k \mapsto \zeta^{k+j-i}$ which is simply multiplication by ζ^{j-i}. Let Z_σ be the image of $\sigma \in U$ in the left regular representation of U as a multiplicative group. Then $\{Z_\sigma \mid \sigma \in U\}$ is a cyclic group generated by Z_ζ and the element Z_{ζ^k} corresponds to the k^{th} power of the cycle $(1, 2, \ldots, t)$ on indices. Observe that $\sum_{\sigma \in U} Z_\sigma$ is the all-ones matrix J. Indeed, the Z_σ are the adjacency matrices of a cyclic group scheme on t points.

Example 2.6. We construct a weight with values in U_3 on the cycle C_3, a DRG of diameter 3. The non-trivial basic graphs are shown in Figure 2. Define a weight ω by:

$$A_1^\omega = \begin{bmatrix} & \alpha & & & & \overline{\alpha} \\ \overline{\alpha} & & \alpha & & & \\ & \overline{\alpha} & & \alpha & & \\ & & \overline{\alpha} & & \alpha & \\ & & & \overline{\alpha} & & \alpha \\ \alpha & & & & \overline{\alpha} & \end{bmatrix}, \qquad A_2^\omega = \begin{bmatrix} & & \overline{\alpha} & & \alpha & \\ & & & \overline{\alpha} & & \alpha \\ \alpha & & & & \overline{\alpha} & \\ & \alpha & & & & \overline{\alpha} \\ \overline{\alpha} & & \alpha & & & \\ & \overline{\alpha} & & \alpha & & \end{bmatrix},$$

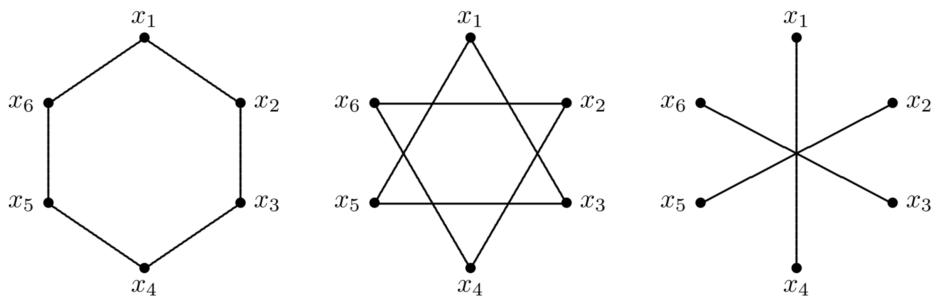

Figure 2: Distance graphs of C_3.

$$A_3^\omega = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{bmatrix}.$$

Working out the products, we see that

$$\begin{aligned} (A_1^\omega)^2 &= 2I + A_2^\omega, & (A_2^\omega)^2 &= 2I + A_2^\omega, \\ A_1^\omega A_2^\omega = A_2^\omega A_1^\omega &= A_1^\omega + 2A_3^\omega, & A_2^\omega A_3^\omega = A_3^\omega A_2^\omega &= A_1^\omega, \\ A_1^\omega A_3^\omega = A_3^\omega A_1^\omega &= A_2^\omega, & (A_3^\omega)^2 &= I, \end{aligned}$$

and therefore the weighted intersection matrices are

$$M_1^\omega = \begin{bmatrix} & 1 & & \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ & & 1 & \end{bmatrix}, \quad M_2^\omega = \begin{bmatrix} & & 1 & \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ & 1 & & \end{bmatrix}, \quad M_3^\omega = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Note. Merging the non-trivial relations or, equivalently, summing $A_i, i \neq 0$, and also the A_i^ω, we see that this weight fuses to a regular 3-graph.

3 Main theorem

Theorem 3.1. *Let* $\mathfrak{A} = (X, \{R_i\}_{i \in \mathcal{I}})$ *be a coherent configuration of rank* r *on* $n := |X|$ *vertices and suppose* ω *is a regular weight on* $\mathfrak{A}$ *with values in* $U = U_t$*. Then* ω *induces a rank* tr *coherent configuration on* tn *vertices with relations given by*

$$\sum_{\alpha \in U} A_i^\alpha \otimes Z_{\sigma\alpha} \quad (i \in \mathcal{I}, \sigma \in U)$$

and parameters $\{\beta_{ij}^k(\alpha)\}$.

Proof. Let $T := \{1, 2, \ldots, t\}$ and let Γ_i be one of the basic graphs in $\mathfrak{A}$. The t-fold cover of Γ_i that is induced by ω has vertex set $Y := X \times T$, and adjacency matrix

$$\sum_{\alpha \in U} A_i^\alpha \otimes Z_\alpha.$$

Motivated by this, and looking to define the matrices of a CA on Y, we put

$$C_{i,\sigma} := \sum_{\alpha \in U} A_i^\alpha \otimes Z_{\sigma\alpha}, \tag{3.1}$$

for $i \in \mathcal{I}$ and $\sigma \in U$, we claim that $\mathcal{C} := \langle C_{i,\sigma} \rangle_{\mathbb{C}}$ is the coherent algebra of a CC $\mathfrak{C}$.

We show that $\mathcal{C}$ satisfies (i)–(iv) of Definition 2.1. We have observed that $\sum_{\sigma \in U} Z_\sigma = J$. Since $\sum_{\alpha \in U} A_i^\alpha = A_i$ for all i, and $\sum_{i \in \mathcal{I}} A_i = J$, we see that

$$\begin{aligned}
\sum_{i \in \mathcal{I}} \sum_{\sigma \in U} C_{i,\sigma} &= \sum_{i \in \mathcal{I}} \sum_{\sigma \in U} \sum_{\alpha \in U} A_i^\alpha \otimes Z_{\sigma\alpha} \\
&= \left(\sum_{i \in \mathcal{I}} \sum_{\alpha \in U} A_i^\alpha \right) \otimes \left(\sum_{\sigma \in U} Z_{\sigma\alpha} \right) \\
&= \left(\sum_{i \in \mathcal{I}} A_i \right) \otimes J_t \\
&= J_n \otimes J_t \\
&= J_{nt}.
\end{aligned} \tag{3.2}$$

Hence $\mathcal{C}$ satisfies (i).

Let $\mathcal{L} \subseteq \mathcal{I}$ be the unique set of indices such that $\sum_{i \in \mathcal{L}} A_i = I_n$. (Assume, without loss of generality, that $\mathcal{L} = \{0\}$ if $\mathcal{A}$ is homogeneous.) We claim $I_{nt} = \sum_{i \in \mathcal{L}} C_{i,1}$. Since $\omega(x,x) = 1$ for all x, $A_i^\alpha = 0$ if $i \in \mathcal{L}$ and $\alpha \neq 1$. Consequently, $i \in \mathcal{L}$ implies $A_i = A_i^1$. Hence,

$$\begin{aligned}
\sum_{i \in \mathcal{L}} C_{i,1} &= \sum_{i \in \mathcal{L}} \sum_{\alpha} A_i^\alpha \otimes Z_\alpha \\
&= \sum_{i \in \mathcal{L}} A_i \otimes Z_0 \\
&= \left(\sum_{i \in \mathcal{L}} A_i \right) \otimes I_t \\
&= I_n \otimes I_t = I_{nt}.
\end{aligned} \tag{3.3}$$

This proves that $\mathcal{C}$ satisfies (ii).

The transpose of $M \otimes N$ is $M^T \otimes N^T$. Since $\omega(y,x) = \overline{\omega(x,y)}$, $(A_i^\alpha)^T = (A_{i^*})^{\overline{\alpha}}$. The transpose of a permutation matrix is its matrix inverse, hence $Z_\sigma^T = Z_\sigma^{-1} = Z_{\sigma^{-1}}$. Therefore,

$$C_{i,\sigma}^T = \sum_{\alpha} A_{i^*}^{\overline{\alpha}} \otimes Z_{(\sigma\alpha)^{-1}} = \sum_{\overline{\alpha}} A_{i^*}^{\overline{\alpha}} \otimes Z_{\overline{\sigma}\,\overline{\alpha}} = C_{i^*,\overline{\sigma}}$$

thus $\mathcal{C}$ satisfies (iii).

Finally, we obtain the structure constants as follows. We claim:

$$\left(\sum_{\alpha \in U} A_i^\alpha \otimes Z_{\sigma\alpha} \right) \left(\sum_{\beta \in U} A_j^\beta \otimes Z_{\tau\beta} \right) = \sum_{k \in \mathcal{I}} \sum_{\nu \in U} \beta_{ij}^k(\nu) \sum_{\gamma \in U} \left(A_k^\gamma \otimes Z_{\sigma\tau\nu\gamma} \right). \tag{3.4}$$

The left hand side of equation (3.4) is equal to

$$\sum_{\alpha,\beta\in U}\left(A_i^\alpha A_j^\beta\right)\otimes(Z_{\sigma\alpha}Z_{\tau\beta}) = \sum_{\alpha,\beta\in U} A_i^\alpha A_j^\beta \otimes Z_{\sigma\tau\alpha\beta}$$
$$= \sum_{\mu\in U}\left(\sum_{\alpha\beta=\mu} A_i^\alpha A_j^\beta\right)\otimes Z_{\sigma\tau\mu},$$

combining terms with the same second tensorand. We now consider the (x,z) entry of each product $A_i^\alpha A_j^\beta$ for a fixed $(x,z)\in R_k$, setting $\gamma := \omega(x,z)$. This equals the number of triangles (x,y,z) of type (i,j,k) with weight $\alpha\beta\bar{\gamma}$. Since we are summing these products over all α and β with $\alpha\beta=\mu$, we account for all such triangles, and the number of these is $\beta_{ij}^k(\alpha\beta\bar{\gamma})$. Thus

$$\begin{aligned}\sum_{\mu\in U}\left(\sum_{\alpha\beta=\mu} A_i^\alpha A_j^\beta\right)\otimes Z_{\sigma\tau\mu} &= \sum_{\mu\in U}\left(\sum_{\gamma\in U}\sum_{k\in\mathcal{I}}\beta_{ij}^k(\mu\bar{\gamma})A_k^\gamma\right)\otimes Z_{\sigma\tau\mu} \\ &= \sum_{\mu\in U}\sum_{\gamma\in U}\sum_{k\in\mathcal{I}}\beta_{ij}^k(\mu\bar{\gamma})\left(A_k^\gamma\otimes Z_{\sigma\tau\mu}\right).\end{aligned} \tag{3.5}$$

Next, observe that $\beta_{ij}^k(\nu)$ occurs exactly t times, once for each γ with $\mu=\nu\gamma$. Factoring gives

$$\sum_{k\in\mathcal{I}}\sum_{\nu\in U}\beta_{ij}^k(\nu)\sum_{\gamma\in U}A_k^\gamma\otimes Z_{\sigma\tau\nu\gamma} \tag{3.6}$$

which proves the claim. Hence $\beta_{ij}^k(\nu)$ is the coefficient of $C_{k,\sigma\tau\nu}$ in the product $C_{i,\sigma}C_{j,\tau}$. □

Remark 3.2. If $\mathfrak{A}$ is an association scheme, then $\mathcal{L}=\{0\}$, and $C_{0,1}=I_{nt}$.

Remark 3.3. The following are clear from the proof of Theorem 3.1.

(i) $\mathfrak{C}$ is homogeneous if and only if $\mathfrak{A}$ is homogeneous.

(ii) $\mathfrak{C}$ is symmetric if and only if $\mathfrak{A}$ is symmetric and ω is real-valued, that is, $t=2$. C_{i1} is always symmetric if A_i is.

(iii) $\mathfrak{C}$ is commutative if and only if $\mathfrak{A}$ is commutative.

(iv) If the A_i^α form a CC, then we are in the case of minimal closure, and $\mathfrak{C}$ is a fusion of a Kronecker product configuration.

(v) The parameter $\beta_{ij}^k(\nu)$ in the proof of (iv) clearly does not depend on σ or τ. This means that each parameter of $\mathfrak{C}$ is duplicated t^2 times:

$$p_{i\sigma,j\tau}^{k\sigma\tau\nu} = p_{i1,j1}^{k\nu} \quad \forall\sigma,\tau\in U_t.$$

4 Discussion and analysis

Example 4.1. This example relates to $\Gamma=\mathrm{SRG}(112,30,2,10)$ which is known by many names in the literature, including the collinearity graph of $\mathrm{GQ}(3,9)$, $O^-(6,3)$, and the first sub-constituent of the McLaughlin graph, McL_1 to name just three. It has a *strongly regular decomposition* into two Gewirtz graphs ($\mathrm{SRG}(56, 10, 0, 2)$) [7].

Let $\mathfrak{A}$ be the rank 3 scheme afforded by Γ. We construct a regular weight on $\mathfrak{A}$ with values in U_2, making use of the decomposition. Let X_1 and X_2 be the two sets of 56 vertices. Define $\omega(x, y)$ for $x \neq y$ to be -1 when x and y are in the same half of this partition, and $+1$ otherwise. Note that ω restricted to either Gewirtz graph is a trivial weight with matrix $2I - J$.

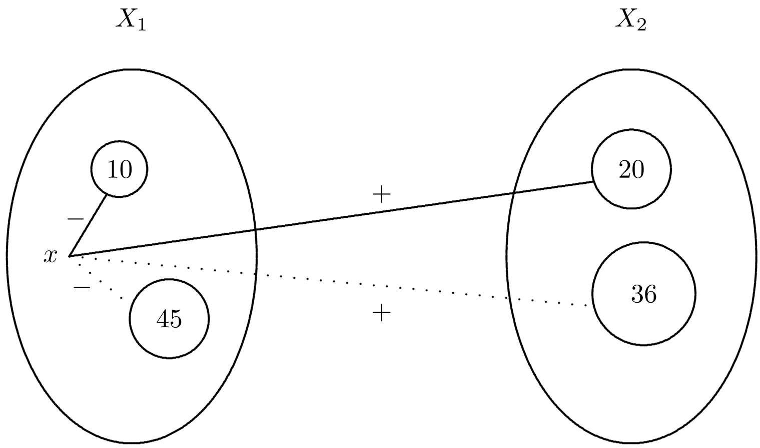

Figure 3: Strongly regular decomposition of $\mathrm{SRG}(112, 30, 2, 10)$.

In Figure 3, a solid line indicates adjacency in Γ, a dotted line non-adjacency. This weighted SRG has minimal closure, since the A_i^α form a rank 5 scheme, in fact a *strongly regular design* or SRD ([13]). Since $\omega(x, y)$ is determined by the parity of $\{x, y\} \cap X_1$, the four non-trivial relations are given by the four combinations of attributes: adjacency/non-adjacency, and this parity. There are many related configurations. For example, another copy of the Gewirtz graph may be adjoined to construct an example of *triality* ([15]). These 112 vertices form the first subconstituent of the McLaughlin graph; the second subconstituent also admits a strongly regular decomposition ([3]).

Some interesting properties of this example:

1. Minimal closure is rare (see [21]).

2. The SRD is cometric, but not metric, which is also rare.

3. The covering configuration $\mathfrak{C}$ is also cometric, but not metric, having rank 6 on 224 points. This example arises as the Q-bipartite double of McL_1 (see [18]).

The Gewirtz graph admits a non-trivial regular weight with values in U_4, constructed via a monomial representation of $2.\mathrm{L}_3(4)$ ([20]). The covering configuration is neither metric nor cometric, has rank 12 on 224 vertices, and contains the doubled Gewirtz graph ($\mathrm{DRG}[10, 9, 8, 2, 1; 1, 2, 8, 9, 10]$) as a quotient.

4.1 Intersection matrices

Lemma 4.2. *The intersection matrices of $\mathcal{C}$ have the form $M_{j\tau} = \sum_{\nu \in U_t} M_j^\nu \otimes Z_{\tau\nu}$ where $\left[M_j^\nu\right]_{ik} := \beta_{ij}^k(\nu)$.*

Proof. We may assume the relations $C_{i\sigma}$ are ordered lexicographically, that is first by i and then by $\sigma \in \{1, \zeta, \ldots, \zeta^{t-1}\}$ so that the intersection matrix $M_{j\tau}$ has (i,k) block given by $\left(p_{i\sigma,j\tau}^{k\sigma\tau\nu}\right)_{\sigma,\sigma\tau\nu}$. By equation (3.6) this block has the value $\beta_{ij}^k(\nu)$ in position $(\sigma, \sigma\tau\nu)$ which means that it has the form $\sum_\nu \beta_{ij}^k(\nu) Z_{\tau\nu}$. Hence $M_{j\tau}$ is the required sum of Kronecker products. □

Lemma 4.3. *Let ω be a regular weight on the cc $\mathfrak{A}$, and let $\widetilde{\omega}$ be an equivalent weight obtained by switching ω by a factor of $\tau = \zeta^l$ at vertex x. Further let $\mathfrak{C} = (Y, \{C_{i\sigma}\})$ and $\widetilde{\mathfrak{C}} = (Y, \{\widetilde{C}_{i\sigma}\})$ be the covering configurations induced by ω and $\widetilde{\omega}$ respectively. Then $\widetilde{\mathfrak{C}}$ is obtained from $\mathfrak{C}$ by permuting $\{x_i\}$ according to the permutation $(1, 2, \ldots, t)^l$ resulting from multiplication by τ on U.*

Proof. Suppose $\omega(x,y) = \alpha$. For some i and j, $(x_1, y_j) \in C_{i1}$ of $\mathcal{C}$, thus $\alpha = \zeta^{j-1}$. Now, $\widetilde{\omega}(x,y) = \tau\alpha$ by assumption, so we have $(x_{1-l}, y_j) \in \widetilde{C}_{i1}$. But this implies that $\widetilde{\mathfrak{C}}$ is obtained from $\mathfrak{C}$ by the permutation $x \mapsto x_{1-l}$, which corresponds to multiplication by τ on U. □

4.2 Special cases

(i) If ω has minimal closure, $\mathfrak{C}$ is a fusion of a tensor product of two CCs.

(ii) If ω is trivial in the sense that $A_i^\alpha = 0$ for all but one value of α, ω has minimal closure, and $\mathcal{C} = \mathcal{A}^\omega \otimes Z$.

(iii) If $\mathfrak{A}$ has rank 2 (ω is regular on K_n), $\mathfrak{C}$ is a t-fold cover of K_n. It is not necessarily distance regular. This case encompasses the regular two-graphs ($t = 2$), and the regular 3-graphs ($t = 3$) of Higman [9] and Kalmanovich [16].

(iv) If $t = 2$, $\mathfrak{A}$ is a (symmetric) scheme, and $\mathcal{A}^\omega$ has minimal closure (say $\mathfrak{B}$, where $\mathfrak{B} = (X, \{B_i\})$), then the covering configuration is isomorphic to the extended Q-bipartite double of $\mathfrak{B}$, when it exists, if the rank of $\mathfrak{B}$ is odd ([18, 3.1]). Existence requires $\mathfrak{B}$ to be cometric with an additional condition on the Krein parameters. For even rank, the covering configuration has a fusion (merging just two classes) that is isomorphic to the extended Q-bipartite double, provided that there is exactly one class of $\mathfrak{A}$ on which ω is constant. Note that a minimal closure of a weight with values in U_2 has even rank only when the weight is constant on an odd number of classes of $\mathfrak{A}$. The isomorphism is $M \otimes N \mapsto N \otimes M$ on the $C_{i\sigma}$ of the cover configuration.

4.2.1 Necessary conditions for a covering configuration

In the case of commutative CCs we extend [16, Prop. 5.4] in a natural way, as follows.

Let $\mathfrak{C} = (X, \{R_i\})$ be a commutative CC of rank tr such that the first t intersection matrices have the form $M_j = I_r \otimes Z_{\zeta^j}$, for $0 \le j < t$, and let $U = \langle\zeta\rangle$ the group of roots of unity of order t. Index the relations according to the r blocks of size t, so that

$$C_{i,\zeta^k} = R_{it+k}$$

and suppose that for any i, j, k and ν:

$$p^{k\sigma\tau\nu}_{i\sigma,j\tau} = p^{k\nu}_{i1,j1}$$

for all σ and τ in U. We intend to show that under these conditions, $\mathfrak{C}$ must arise as the covering configuration of a regular weight on a quotient of $\mathfrak{C}$.

Lemma 4.4. *If $j < t$ and $p^k_{ij} \neq 0$, then $k = i + j \pmod t$; in particular, i and k lie in the same block of M_j.*

Proof. This follows from $M_j = I \otimes Z_{\zeta^j}$. □

Observe that $E := \cup_{j=0}^{t-1} R_j$ is a *parabolic* in the sense of [10]. Indeed, $M_0 = I_{rt}$ implies that R_0 is the identity relation of $\mathfrak{C}$. Further, E is symmetric, since $(x, y) \in R_i$ for $i < t$ implies that $p^0_{i^*i} \neq 0$, so i^* is in the same block of M_i as 0. That is, $(y, x) \in E$. Given $(x, y) \in R_i$ and $(y, z) \in R_j$ with $0 \leq i, j < t$, we see that $(x, z) \in R_k$ for some $k < t$, because k must lie in the same block of M_j as i, since all non-diagonal blocks are zero. Hence, E is a transitive relation.

As a parabolic, E induces an equivalence relation on the indices: If there exist x, x', y, $y' \in X$ such that $(x, x') \in E$, $(y, y') \in E$, $(x, y) \in R_i$ and $(x', y') \in R_j$, then $i \sim j$. Write $[i]$ for the equivalence class of i. In addition, the parabolic affords a quotient (homogeneous) configuration $\mathfrak{A} := (\overline{X}, \{\overline{R}_{[i]}\})$ with an associated partition of the vertex set X into *fibres* of size t. The fibre containing x is

$$[x] = \{y \mid (x, y) \in E\}.$$

We will henceforth suppress the bracket notation for fibres, writing $x = \{x_1, x_2, \ldots x_t\}$.

For $j \in [0]$, Lemma 4.4 implies that $p^k_{kj} = 0$ for $j \neq 0$. But then R_k restricted to $x \times y$ has valency at most 1. We conclude that the number of relations occurring between any two fibres is t. We have: For $k \in \mathcal{I}$ and $x \in X$,

$$|[k]| = |x| = t.$$

Denoting the graph of R_j by Γ_j, we have proved the following:

Lemma 4.5. *For all $j \notin [0]$, Γ_j is a t-fold cover of $\Gamma_{[j]}$.*

Corollary 4.6. *The natural partition of $\mathcal{I}$ according to blocks of M_j, for $0 \leq j < t$ is the same as that determined by the equivalence classes of the parabolic. That is,*

$$[mt] = \{mt, mt + 1, \ldots, mt + t - 1\}.$$

Proof. Suppose $j \in [i]$ so that there exist $x_1, x_2, y_1, y_2 \in X$ with $(x_1, y_1) \in R_i$ and $(x_2, y_2) \in R_j$. Then, by the discussion above, $(x_1, y_3) \in R_j$ for some $y_3 \in y$ and therefore $p^j_{ik} \neq 0$ for some $k < t$.

But then $j = i + k \pmod t$ by Lemma 4.4. □

Recall that $C_{0,\zeta^k} = R_k$ for $k < t$, $C_{0,\sigma}$ has intersection matrix $I_r \otimes Z_\sigma$, and $C_{m,1} = R_{mt}$ for $0 \leq m < r$. Fix a fibre a (from here on), and order it so that $(a_i, a_{i+1}) \in C_{0\zeta}$, for each i, with addition modulo t. This ensures that the perfect matching induced on a corresponds to the permutation $(1, 2, \dots, t)$ on indices, which in turn corresponds to the permutation of U induced by multiplication by ζ.

For each $x \in \overline{X}$, $(a, x) \in \overline{R}_{[mt]}$ for some m. Order x so that $(a_j, x_j) \in C_{m,1}$. In what follows, we mix the notations regarding indexation of the relations of $\mathfrak{C}$. Where two indices are given, we refer to $C_{i,\sigma}$ as above; where one index is given we refer to the original numbering of the relations.

Lemma 4.7. *With notation as above,* $(x_i, x_{i+1}) \in C_{0,\zeta}$ *for all* $x \in \overline{X}$.

Proof. For some σ, $(x_i, x_{i+1}) \in C_{0,\sigma}$; $(a_i, x_{i+1}) \in R_l$ for some l, and $(a_i, x_i) \in R_{m1}$ for some m. Note that $l \in [m]$. Since a_i, a_{i+1}, and x_{i+1} form a triangle of type $(0\zeta, m1, l)$,

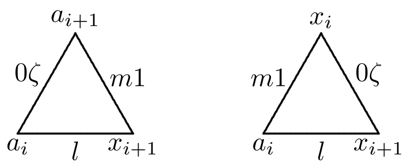

Figure 4: Triangles (a_i, a_{i+1}, x_{i+1}) and (a_i, x_i, x_{i+1}).

we see that $p^l_{0\zeta,m1} \neq 0$. Since $\mathfrak{C}$ is commutative, $R_l = C_{m\zeta}$ by Lemma 4.4. Now observe that a_i, x_i, and x_{i+1} form a triangle of type $(m1, 0\sigma, m\zeta)$, and therefore $\sigma = \zeta$. $\square$

Next, following [16] we show that all matchings are cyclic.

Lemma 4.8. *With notation as above, all matchings between fibres of* $\mathfrak{C}$ *are cyclic.*

Proof. Suppose that $(x_i, y_j) \in R_k$ and $(x_{i+1}, y_l) \in R_k$. We must show that $l = j + 1$. The triangle (x_i, x_{i+1}, y_j) has type $(1, m, k)$ for some m, indicating that $p^k_{1m} \neq 0$. As in the previous lemma, this implies that $k = m + 1$. On the other hand, the triangle (x_{i+1}, y_{l-1}, y_l) has type $(b, 1, k)$ for some b, hence $k = b + 1$. But then $m = b$, and by Lemma 4.5, $y_{l-1} = y_j$ as desired. $\square$

Corollary 4.9. *For all* $x \in X$, $(x_i, x_{i+k}) \in R_k$*, thus* R_k *induces on each fibre the perfect matching corresponding to the* k^{th} *power of the cycle* $(1, 2, \dots, t)$.

Proof. The result follows by Lemma 4.7 and induction (on k) applied to the triangles (x_{i-k}, x_i, x_{i+1}) . $\square$

Lemma 4.10. *For* $x \in X$, $(a_i, x_{i+k}) \in R_{mt+k}$ *for* $0 \leq k < t$.

Proof. The case $k = 0$ holds by choice of ordering of x. Induction applied to the triangles $(a_i, x_{i+k-1}, x_{i+k})$ gives the desired result. $\square$

We now define a weight on $\mathfrak{A}$ by means of C_{i1}. Let $x, y \in \overline{X}$ and suppose $(x, y) \in \overline{R}_{[j]}$. Then $C_{j,1}$ provides a cyclic matching between x and y corresponding to, say, $\alpha \in U$. Set $\omega(x, y) := \alpha$. Observe that $\omega(a, x) = 1$ for all x.

The next lemma shows how to determine the weight of an edge in $\overline{\Gamma}_{[i]}$ from any edge in Γ_i.

Lemma 4.11. *If $(x_i, y_j) \in C_{k\sigma}$, then $\omega(x, y) = \bar{\sigma}\zeta^{j-i}$.*

Proof. Consider $(x_i, y_j) \in C_{k,\sigma}$. Let l be such that $(x_i, y_l) \in C_{k1}$ and note that the triangle (x_i, y_l, y_j) has type $(k1, 0\zeta^{j-l}, k\sigma)$. By Proposition 4.6, $\sigma = \zeta^{j-l}$. This implies that $(x_i, y_{l+m}) \in C_{k,\zeta^m}$. We conclude that the matching between x and y in $C_{k,\sigma}$ is $\alpha\sigma$, where $\alpha = \omega(x, y)$. □

We now prove the second main result which is the extension of [16, Prop. 5.4].

Theorem 4.12. *Let $\mathfrak{C} = (X, \{R_i\})$ be a commutative CC of rank rt with the first t intersection matrices given by*

$$M_j = I_r \otimes Z_{\zeta^j} \quad 0 \le j < t,$$

where $U = U_t = \langle\zeta\rangle$ is the group of roots of unity of order t. Label the relations according to the blocking of M_j:

$$C_{i,\zeta^k} := R_{it+k} \quad 0 \le i < r, 0 \le k < t$$

and suppose that the CC parameters satisfy, for any i, j, k and ν:

$$p^{k\sigma\tau\nu}_{i\sigma,j\tau} = p^{k\nu}_{i1,j1}$$

for all σ and τ in U. Then $\mathfrak{C}$ arises as the covering configuration (in the sense of Theorem 3.1) from a regular weight ω on the quotient scheme $\mathfrak{A} = \mathfrak{C}/E$.

Proof. From the discussion and lemmas above, what remains to be shown is that ω is regular on the quotient configuration $\bar{\mathfrak{C}} = (\bar{X}, \{\bar{R}_{[i]}\})$. Let $(x, z) \in \bar{R}_{[k]}$. We consider all y such that (x, y, z) has type (i, j, k) and weight ν. Let l be such that $(x_1, z_l) \in C_{k\nu}$. If $(x, y) \in \bar{R}_{[i]}$ and $(y, z) \in \bar{R}_{[j]}$, then $(x_1, y_m) \in C_{i1}$ for some m, and this determines (exactly one) τ with $(y_m, z_l) \in C_{j\tau}$. By Lemma 4.11,

$$\begin{aligned} \delta\omega(x, y, z) &= \omega(x, y)\omega(y, z)\overline{\omega(x, z)} \\ &= \zeta^{m-1}\bar{\tau}\zeta^{l-m}\nu\zeta^{1-l} \\ &= \bar{\tau}\nu \end{aligned}$$

from which we see that triangles of weight ν occur exactly when $\tau = 1$. These triangles are counted by the parameter $p^{k\nu}_{i1,j1}$ which is independent of the choice of x_1 and z_l. □

Note that in the proof above we are counting distinct y, and that for each y there is exactly one y_m as indicated. Thus we may use C_{i1} without loss of generality, since $(x_1, y_m) \in C_{i\sigma}$ would yield the same result. In fact triangles of type $(i\sigma, j\tau, k\nu)$ will have weight ν exactly when $\sigma\tau = 1$, which is expected as in that case $p^{k\nu}_{i\sigma,j\tau} = p^{k\nu}_{i1,j1}$

5 Examples

5.1 A rank 12 scheme on 18 points

The covering configuration of Example 2.6 has rank 12 ($= 4 \cdot 3$) on 18 ($= 6 \cdot 3$) points. It is isomorphic to as18[88] on Hanaki and Izumi's list ([8]).

5.2 A family of CCs from regular weights on $H(n, 2)$ with values in U_4

This construction is due to Ada Chan (personal communication). We define a regular weight on the Hamming Scheme $H(n, 2)$ with values in U_4 with generator $\mathbf{i}$. Let $\mathbf{t}$ be an indeterminate, and K the 2 by 2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Form $(I + \mathbf{t}K)^{\otimes n}$, a polynomial in $\mathbf{t}$ with coefficients in the ring of matrices $M_{2,2}(\mathbb{R})^{\otimes n} \simeq M_{2^n,2^n}(\mathbb{R})$. Now let A_k^ω be the coefficient of $\mathbf{t}^k$, scaled by a factor of $\mathbf{i}^k \in U_4$. We claim this is a regular weight on the Hamming scheme. Indeed, replacing $\mathbf{i}$ with 1 and K with $J - I$ in this process yields the adjacency matrices of the Hamming scheme, with the standard P-polynomial ordering. Noting that $K^2 = -I$ it is straight-forward to see that $\mathrm{Span}(A_k^\omega)$ is coherent. For regularity, we note that p_{ij}^k is nonzero only when $i + j + k$ is even, and this implies $\beta_{ij}^k(\pm\mathbf{i}) = 0$ for all i, j, k. Proposition 1 of [21] applies, and we conclude that ω is regular.

The covering configuration induced by this weight is a rank $4(n + 1)$ CC on 2^{n+2} vertices. There is a fusion to regular 4-graph, which is easily seen: replace $\mathbf{t}$ by $\mathbf{i}$, setting

$$\widetilde{\omega} := (I + \mathbf{i}K)^{\otimes n},$$

then verify directly that $\widetilde{\omega}^2 = 2^n\widetilde{\omega}$ thus $\widetilde{\omega}$ is the matrix of a regular 4-graph. The covering configuration of $\widetilde{\omega}$ has rank 8 and is symmetric, but not necessarily distance regular.

For $n = 2$, the weight is given by:

$$A_1^\omega = \begin{bmatrix} 0 & i & i & 0 \\ -i & 0 & 0 & i \\ -i & 0 & 0 & i \\ 0 & -i & -i & 0 \end{bmatrix} \quad \text{and} \quad A_2^\omega = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The rank 12 covering configuration has color matrix $(\sum iA_i)$ below.

$$\begin{bmatrix}
0 & 1 & 2 & 3 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 8 & 9 & 10 & 11 \\
3 & 0 & 1 & 2 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 11 & 8 & 9 & 10 \\
2 & 3 & 0 & 1 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 10 & 11 & 8 & 9 \\
1 & 2 & 3 & 0 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 8 \\
5 & 6 & 7 & 4 & 0 & 1 & 2 & 3 & 10 & 11 & 8 & 9 & 7 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 & 3 & 0 & 1 & 2 & 9 & 10 & 11 & 8 & 6 & 7 & 4 & 5 \\
7 & 4 & 5 & 6 & 2 & 3 & 0 & 1 & 8 & 9 & 10 & 11 & 5 & 6 & 7 & 4 \\
6 & 7 & 4 & 5 & 1 & 2 & 3 & 0 & 11 & 8 & 9 & 10 & 4 & 5 & 6 & 7 \\
5 & 6 & 7 & 4 & 10 & 11 & 8 & 9 & 0 & 1 & 2 & 3 & 7 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 & 9 & 10 & 11 & 8 & 3 & 0 & 1 & 2 & 6 & 7 & 4 & 5 \\
7 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 2 & 3 & 0 & 1 & 5 & 6 & 7 & 4 \\
6 & 7 & 4 & 5 & 11 & 8 & 9 & 10 & 1 & 2 & 3 & 0 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 0 & 1 & 2 & 3 \\
11 & 8 & 9 & 10 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 3 & 0 & 1 & 2 \\
10 & 11 & 8 & 9 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 2 & 3 & 0 & 1 \\
9 & 10 & 11 & 8 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 1 & 2 & 3 & 0
\end{bmatrix}$$

The regular 4-graph $\widetilde{\omega} := I + A_1^\omega + A_2^\omega$ satisfies $\widetilde{\omega}^2 = 4I$. The covering configuration of $\widetilde{\omega}$ has rank 8 and may also be obtained through fusion of the rank 12 above.

In summary, this construction gives regular weights with values in U_4 on the Hamming Schemes H$(n, 2)$. These have rank $n + 1$ on 2^n vertices. The covering configurations thus have rank $4(n+1)$ on 2^{n+2} vertices. These weights fuse to regular 4-graphs always, and the covering configurations of those have rank 8. In examples constructed to date, the covering configurations are not metric, nor are their symmetrizations, and they are not cometric.

5.3 CCs afforded by groups

A CC may have relations determined by the orbitals of a group G acting on a set X, in which the centralizer algebra of the natural permutation representation is the coherent algebra $\mathcal{A}$. In this case, a regular weight may exist such that $\mathcal{A}^\omega$ is the centralizer algebra of a monomial representation of G, induced from a linear representation of a point stabilizer ([14]).

For example, the rank 3 scheme containing the Petersen graph is afforded by the action of A_5 on 2-sets from $\{1,2,3,4,5\}$. The stabilizer of $\{1,2\}$ is a group $H \simeq S_3$, containing $A := \langle 3,4,5\rangle$ as a subgroup of index 2. This index determines that the monomial representation will afford a weight with values in U_2. Defining the linear representation

$$\phi\colon H \to C_2 \quad \text{by} \quad \phi(g) = \begin{cases} 1 & g \in A, \\ -1 & g \notin A, \end{cases}$$

the induced representation $M := \phi\big|_H^G$ is a monomial representation of G. The $M(g)$ for $g \in G$ are signed permutation matrices. The centralizer algebra of M, $\mathcal{A}^\omega$, defines a regular weight on the Petersen graph.

This construction can be done in general when the point stabilizer H has a normal subgroup A of index t, such that $H/A \simeq C_t$. The monomial representation induced may or may not afford a nontrivial regular weight on the underlying CC.

In this example, the covering configuration $\mathfrak{C}$ is a rank 6 scheme on 20 points, in fact the unique (antipodal, non-bipartite) distance-regular graph DRG$\{3,2,1,1,1;1,1,1,2,3\}$, that is the dodecahedron graph. (It is not the bipartite double of the Petersen graph, which is DRG$\{3,2,2,1,1;1,1,2,2,3\}$.)

We obtain a permutation representation from M, via

$$M(g) \mapsto M^+(g) \otimes Z_1 + M^-(g) \otimes Z_2$$

where M^+, M^-, Z_1 and Z_2 are defined as in Section 2. It is natural to ask whether $\mathcal{C}$ is the centralizer algebra of this permutation representation. In fact, $\mathcal{C}$ is properly contained in this centralizer algebra. It affords a CC with valencies 1, 1, 3, 3, 3, 3, 3, 3 which has a fusion to $\mathfrak{C}$. The group affording $\mathfrak{C}$ is $A_5 \times C_2$, an extension of our group G by the cyclic C_2, the latter generated by the even permutation interchanging each x_1 and x_2. This is of course the symmetry group of the dodecahedron and is not isomorphic to S_5.

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On the number of additive permutations and Skolem-type sequences*

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Abstract

Cavenagh and Wanless recently proved that, for sufficiently large odd n, the number of transversals in the Latin square formed from the addition table for integers modulo n is greater than $(3.246)^n$. We adapt their proof to show that for sufficiently large t the number of additive permutations on $[-t, t]$ is greater than $(3.246)^{2t+1}$ and we go on to derive some much improved lower bounds on the numbers of Skolem-type sequences. For example, it is shown that for sufficiently large $t \equiv 0$ or $3 \pmod 4$, the number of split Skolem sequences of order $n = 7t+3$ is greater than $(3.246)^{6t+3}$. This compares with the previous best bound of $2^{\lfloor n/3 \rfloor}$.

Keywords: Additive permutation, Skolem sequence, transversal.

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1 Introduction

This paper is concerned with counting additive permutations and Skolem-type sequences. Additive permutations are related to certain kinds of transversals in Latin squares. A Latin square of order n may be envisaged as an $n \times n$ array having n distinct entries, each of which appears once in any one row and once in any one column. We adopt a slightly wider

*We thank the referees for helpful comments and for drawing to our attention the papers [6, 7, 8].
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than usual definition of a transversal in an $n \times n$ array (not necessarily a Latin square) as a set of n (row, column, entry) triples that cover every row, every column and n distinct entries from the array. Consequently, a transversal in a Latin square will necessarily cover every entry precisely once.

In a recent paper, Cavenagh and Wanless [3] proved that, for all sufficiently large odd n, the number of transversals in the Latin square formed from the addition table for integers modulo n is greater than $(3.246)^n$. This result was a substantial improvement on previous results in this area, although Vardi [10] conjectured that this number exceeds $c^n n!$ for some constant $c \in (0, 1)$. A proof of this conjecture is claimed in the arXiv paper [6]. However, it is important to note that not all transversals are suitable for our purposes.

For integers a and b with $a < b$ we use the notation $[a, b]$ to denote the set of integers i such that $a \leq i \leq b$. An additive permutation π on $[-t, t]$ is a permutation of these integers such that $\{i + \pi(i) : i \in [-t, t]\}$ is also a permutation on the same set of integers. This definition of an additive permutation is the one employed by Abram [1] and others in connection with Skolem sequences, but the reader is cautioned that it differs from that used in [6] where pointwise addition of permutations on $[1, n]$ is taken modulo n. An examination of the proof given in [3] shows that it is possible to adapt the proof to show that the number of additive permutations on $[-t, t]$ is greater than $(3.246)^{2t+1}$ for all sufficiently large t, and this is done in Theorem 2.1 below. There are strong connections between additive permutations and Skolem-type sequences. We investigate some of these connections and obtain much improved lower bounds on the numbers of some Skolem-type sequences.

A *pure Skolem sequence*, sometimes simply called a *Skolem sequence*, of order n is a sequence $(s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the following conditions.

(C1) For each $k \in \{1, 2, \ldots, n\}$ there are precisely two elements of the sequence, say s_i and s_j, such that $s_i = s_j = k$.

(C2) If $s_i = s_j = k$ and $i < j$ then $j - i = k$.

For example, $(4, 1, 1, 5, 4, 2, 3, 2, 5, 3)$ is a pure Skolem sequence of order 5. It is well known that a pure Skolem sequence of order n exists if and only if $n \equiv 0$ or $1 \pmod 4$. For this and other existence results mentioned below see, for example, [4, 5].

An *extended Skolem sequence* of order n is a sequence $(s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers satisfying (C1) and (C2) above and such that precisely one element of the sequence is zero. An extended Skolem sequence of order n exists for every positive integer n. If the zero element of an extended Skolem sequence of order n appears in the $2n$-th position, i.e. $s_{2n} = 0$, then the sequence is called a *hooked Skolem sequence*. A hooked Skolem sequence of order n exists if and only if $n \equiv 2$ or $3 \pmod 4$. If the zero element of an extended Skolem sequence of order n appears in the $(n + 1)$-th position, i.e. $s_{n+1} = 0$, then the sequence is called a *split Skolem sequence* or a *Rosa sequence*. A split Skolem sequence of order n exists if and only if $n \equiv 0$ or $3 \pmod 4$.

A *split-hooked Skolem sequence* (also known as a *hooked Rosa sequence*) of order n is a sequence $(s_1, s_2, \ldots, s_{2n+2})$ of $2n + 2$ integers satisfying (C1) and (C2) above and such that $s_{n+1} = s_{2n+1} = 0$. A split-hooked Skolem sequence of order n exists if and only if $n \equiv 1$ or $2 \pmod 4$ and $n \neq 1$.

The various types of Skolem sequence described above may be used to construct solutions to Heffter's first and second difference problems. These, in turn, may be used to construct cyclic Steiner triple systems. We will refer to the sequences just described, and to some near relatives, somewhat loosely as Skolem-type sequences. It was shown in [1, 2] that for many of these Skolem-type sequences (in particular, pure, hooked, split and split-hooked) the number of them is essentially bounded below by $2^{\lfloor \frac{n}{3} \rfloor}$, where n is the order of the sequence.

2 Additive permutations

Define the Latin square A_n of odd order $n = 2t + 1$ to have its rows and columns indexed by the integers in $[-t, t]$ and the (i, j) entry $k \in [-t, t]$ given by $k \equiv i + j \pmod{n}$. The array A_n gives the addition table on $\mathbb{Z}_n$. A $\mathbb{Z}$-transversal T in A_n is a transversal in which every (row, column, entry) triple $(i, j, k) \in T$ has $k = i + j$ in $\mathbb{Z}$, so that no triples of a $\mathbb{Z}$-transversal have $i + j < -t$ or $i + j > t$. Not all transversals in A_n are $\mathbb{Z}$-transversals, for example the transversal formed by the leading diagonal in A_n is not a $\mathbb{Z}$-transversal. We will only count $\mathbb{Z}$-transversals: in effect the (i, j) cells in A_n where $i + j < -t$ or $i + j > t$ are ignored. The entries in these cells are therefore irrelevant to our discussions and it will sometimes be helpful to take $i + j$ as the entry, rather than $i + j$ reduced modulo n. This has the advantage that the "ignored" cells are easily identified, particularly when considering subarrays of A_n – such cells are then precisely those with entries outside the range $[-t, t]$.

Figure 1 shows the array A_{19} with the "ignored" entries greyed-out, and with a $\mathbb{Z}$-transversal having its entries marked in boxes. For the present, disregard the highlighting of the subarrays.

If the (row, column, entry) triples of a $\mathbb{Z}$-transversal in A_n are listed as a $3 \times n$ array T^* with row numbers of A_n forming the first row of T^*, column numbers the second, and entries the third, then each row of this array contains the integers $[-t, t]$, and the entries in the third row are the sums of the corresponding entries in the other two rows. Taking the first row of T^* as $[-t, t]$, the second row as a permutation π on $[-t, t]$, and the third row as the vector sum of the first two rows, then π is an additive permutation on $[-t, t]$. Conversely, if π is an additive permutation on $[-t, t]$ then the (row, column, entry) triples $(i, \pi(i), i + \pi(i))$ for $i \in [-t, t]$ form a $\mathbb{Z}$-transversal in A_n. If the entries in the third row of such a $3 \times n$ array T^* are multiplied by (-1), then each column of the resulting array sums to zero, and the new array is called a zero-sum array. Thus, there is an equivalence between $\mathbb{Z}$-transversals, additive permutations, and zero-sum arrays. Table 1 (taken from [3]) gives the number of these, here denoted by z_n, for $n = 2t + 1 \leq 23$. These numbers form the sequence A002047 in Sloane's encyclopaedia [9] and have been independently checked by ourselves using our own computer program. The table also gives a *rounded down* value for $(z_n)^{\frac{1}{n}}$ which will be used subsequently. We remark that z_{2t+1} is also the number of extremal Langford sequences with defect $t + 1$ (i.e. starting with $t + 1$) – see [1, 4] for definitions.

Theorem 2.1. *Suppose that b and n are odd and that $n \geq 3b \geq 9$. Then $z_n \geq (z_b)^{2\lfloor \frac{n-b}{2b} \rfloor}$.*

Proof. Our proof is a re-working of that of Cavenagh and Wanless [3], ensuring that for general b and n the subarrays R and S (defined below) have appropriate (sub-)transversals and that the transversals constructed in A_n are indeed $\mathbb{Z}$-transversals. We take $b = 2a + 1$ (so that $a \geq 1$) and $n = 2t + 1$. Put $k = \lfloor \frac{n-b}{2b} \rfloor = \lfloor \frac{t-a}{b} \rfloor$, $s = t - a - bk$, and $r = s + b$. Then $0 \leq s < b$, $b \leq r < 2b$, and one of r, s is odd and the other is even.

Next define the subarray $M_{(i,j),c}$ of A_n to be the $c \times c$ block whose top left entry is in

	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9
−9	1	2	3	4	5	6	7	8	9	$\boxed{-9}$	−8	−7	−6	−5	−4	−3	−2	−1	0
−8	2	3	4	5	6	7	8	9	−9	−8	$\boxed{-7}$	−6	−5	−4	−3	−2	−1	0	1
−7	3	4	5	6	7	8	9	−9	$\boxed{-8}$	−7	−6	−5	−4	−3	−2	−1	0	1	2
−6	4	5	6	7	8	9	−9	−8	−7	−6	−5	−4	−3	$\boxed{-2}$	−1	0	1	2	3
−5	5	6	7	8	9	−9	−8	−7	−6	−5	−4	$\boxed{-3}$	−2	−1	0	1	2	3	4
−4	6	7	8	9	−9	−8	−7	−6	−5	−4	−3	−2	$\boxed{-1}$	0	1	2	3	4	5
−3	7	8	9	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	$\boxed{3}$	4	5	6
−2	8	9	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	$\boxed{6}$	7
−1	9	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	$\boxed{4}$	5	6	7	8
0	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	$\boxed{9}$
1	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	$\boxed{8}$	9	−9
2	−7	$\boxed{-6}$	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	−9	−8
3	−6	−5	$\boxed{-4}$	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	−9	−8	−7
4	$\boxed{-5}$	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	−9	−8	−7	−6
5	−4	−3	−2	−1	0	$\boxed{1}$	2	3	4	5	6	7	8	9	−9	−8	−7	−6	−5
6	−3	−2	−1	$\boxed{0}$	1	2	3	4	5	6	7	8	9	−9	−8	−7	−6	−5	−4
7	−2	−1	0	1	$\boxed{2}$	3	4	5	6	7	8	9	−9	−8	−7	−6	−5	−4	−3
8	−1	0	1	2	3	4	$\boxed{5}$	6	7	8	9	−9	−8	−7	−6	−5	−4	−3	−2
9	0	1	2	3	4	5	6	$\boxed{7}$	8	9	−9	−8	−7	−6	−5	−4	−3	−2	−1

Figure 1: The Latin square A_{19}.

Table 1: The number of $\mathbb{Z}$-transversals in A_n.

n	z_n	$(z_n)^{\frac{1}{n}} >$
3	2	1.259
5	6	1.430
7	28	1.609
9	244	1.841
11	2 544	2.039
13	35 600	2.239
15	659 632	2.443
17	15 106 128	2.644
19	425 802 176	2.845
21	14 409 526 080	3.046
23	577 386 122 880	3.246

position (i, j) of the array A_n. For example, if $n = 19$ (see Figure 1) then

$$M_{(-6,2),3} = \begin{pmatrix} -4 & -3 & -2 \\ -3 & -2 & -1 \\ -2 & -1 & 0 \end{pmatrix}.$$

As previously mentioned, it is convenient to take the entries of subarrays unreduced modulo n. Allowing this, $M_{(i,j),c}$ has entries from $i + j$ to $i + j + 2(c - 1)$ and, if c is odd, there is a central entry $i + j + c - 1$. We use the following subarrays:

Type 1: $D_i = M_{(-t+ib,-a+ib),b}$ for $i = 0, 1, \ldots, (k - 1)$,

Type 2: $E_i = M_{(-t+r+(k+i)b,-t+ib),b}$ for $i = 0, 1, \ldots, (k - 1)$,

Type 3: $R = M_{(-t+kb,-a+kb),r}$ and

Type 4: $S = M_{(t-s+1,-a-s),s}$.

In Figure 1, $n = 19$ and $b = 3$, so that $t = 9$, $a = 1$, $k = 2$, $s = 2$ and $r = 5$. The subarrays of types 1 and 2 are lightly shaded and the subarrays R and S are shaded more heavily.

Altogether there are $2k+2$ subarrays of the four types. No two of these have a common row and the total number of rows covered is $2bk + r + s = n$, so each row of A_n is covered by precisely one of these subarrays. Similarly, each column of A_n is covered by precisely one of these subarrays. Consequently we may attempt to construct $\mathbb{Z}$-transversals in A_n from transversals in the subarrays.

The type 1 subarray D_i has a central entry $2ib - t + a$. If this value is subtracted from every entry in D_i, the resulting array is a copy of A_b. Since A_b has z_b $\mathbb{Z}$-transversals, each D_i has z_b transversals that are symmetric about its central entry, that is to say transversals each covering the entries from $2ib - t$ to $2ib - t + (b - 1)$ inclusive. Note that these transversals avoid "ignored" cells of A_n. Similarly, the type 2 subarray E_i has z_b transversals symmetric about its central entry $2ib - t + 3a + 1$, and each of these covers the entries from $2ib - t + b$ to $2ib - t + 2b - 1$ inclusive. Collectively the type 1 and type 2 subarrays have $(z_b)^{2k}$ transversals covering the entries from $-t$ to $t - r - s$ inclusive. All of these are partial $\mathbb{Z}$-transversals of A_n.

To complete the proof for all $b \geq 3$ and $n \geq 3b$ we must show that the remaining subarrays R and S have appropriate transversals (i.e. avoiding "ignored" cells of A_n) that cover the entry values from $t - r - s + 1$ to t inclusive. It will then follow that A_n has at least $(z_b)^{2k} = (z_b)^{2\lfloor \frac{n-b}{2b} \rfloor}$ $\mathbb{Z}$-transversals.

The subarray R has entries ranging from $t - r - s - a + 1$ to $t + a$ inclusive, and so R always contains some "ignored" cells of A_n, namely those with entries exceeding t. The subarray S has entries ranging from $t - r - s + 2 + a$ to $t - a - 1$ inclusive, so S does not contain any "ignored" cells of A_n. If we subtract $t - s - a$ from all the entries in R and S we obtain equivalent arrays R' and S', where R' has entries ranging from $-(s + 2a)$ to $s+2a$, while S' has entries ranging from $-(s-1)$ to $s-1$, and we are seeking transversals in these arrays that cover the entry values from $-(s+a)$ to $s+a$ inclusive. Cells in R' that contain entries greater than $s + a$ correspond to the "ignored" cells of R. Our proof that such transversals always exist falls into a number of cases, the details of which are lengthy, and so are postponed until Section 4. □

Corollary 2.2. *If n is odd and sufficiently large, then $z_n > (3.246)^n$.*

Proof. Theorem 2.1 gives $z_n \geq (z_b)^{2\lfloor\frac{n-b}{2b}\rfloor} \geq (z_b)^{\frac{n}{b}-3}$ for b odd and all sufficiently large n. From Table 1 we have $(z_{23})^{\frac{1}{23}} > 3.246$, so taking $b = 23$ we obtain $z_n > (3.246)^n$ for all sufficiently large n. □

Putting the corollary into words, the number of additive permutations on $[-t, t]$ is greater than $(3.246)^{2t+1}$ for all sufficiently large t.

3 Skolem-type sequences

A connection between additive permutations and Skolem-type sequences is formed by so-called $(m, 3, c)$-systems. A set $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, where each D_i is a triple of positive integers $(a_i, b_i, a_i + b_i)$ with $a_i < b_i$ and $\bigcup_{i=1}^m D_i = \{c, c+1, \ldots, c+3m-1\}$ is called an $(m, 3, c)$-system. As remarked in [1], such a system exists if and only if

(i) $m \geq 2c - 1$, and

(ii) $m \equiv 0$ or $1 \pmod 4$ if c is odd, or $m \equiv 0$ or $3 \pmod 4$ if c is even.

Given an $(m, 3, c)$-system $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, where $D_i = (a_i, b_i, a_i + b_i)$, and putting $r = c + 3m - 1$, a (Skolem-type) sequence $(x_{-r}, x_{-r+1}, \ldots, x_{r-1}, x_r)$ may be constructed by putting $x_{-(a_i+b_i)} = a_i$, $x_{-b_i} = a_i$, $x_{-a_i} = a_i + b_i$, $x_{a_i} = b_i$, $x_{b_i} = a_i + b_i$, $x_{a_i+b_i} = b_i$ for $i = 1, 2, \ldots, m$, and $x_j = 0$ for $-c < j < c$. For example, if $c = 2$ and $m = 3$, and if $\mathcal{D} = \{D_1, D_2, D_3\}$ where $D_1 = (2, 6, 8)$, $D_2 = (3, 7, 10)$, $D_3 = (4, 5, 9)$ then the constructed sequence is

$$(3, 4, 2, 3, 2, 4, 9, 10, 8, 0, 0, 0, 6, 7, 5, 9, 8, 10, 6, 5, 7).$$

Observe that in such a sequence, for each $k \in \{c, c+1, \ldots, r\}$ the two positions occupied by k are precisely k apart. Further observe that, independently for each $i \in \{1, 2, \ldots, m\}$, we may replace x_j for $j \in \{-a_i - b_i, -b_i, -a_i, a_i, b_i, a_i + b_i\}$ by x'_j where $x'_{-(a_i+b_i)} = b_i$, $x'_{-b_i} = a_i + b_i$, $x'_{-a_i} = b_i$, $x'_{a_i} = a_i + b_i$, $x'_{b_i} = a_i$, $x'_{a_i+b_i} = a_i$. Thus we may obtain 2^m distinct sequences of length $2r + 1$ each of which has the property that for each $k \in \{c, c+1, \ldots, r\}$, the two positions occupied by k are precisely k apart. Each such sequence has zeros in the central $2c - 1$ positions.

If π is an additive permutation on $[-t, t]$ then a $(2t + 1, 3, t + 1)$-system is formed by the set of triples $\{D_i : i \in [-t, t]\}$ where

$$D_i = (i + 2t + 1, \pi(i) + 4t + 2, i + \pi(i) + 6t + 3).$$

Note that the first entries in these triples cover the interval $[t + 1, 3t + 1]$, the second entries cover $[3t + 2, 5t + 2]$, and the third entries cover $[5t + 3, 7t + 3]$. If π_1 and π_2 are two different additive permutations on $[-t, t]$, then the two resulting $(2t + 1, 3, t + 1)$-systems contain different triples. Consequently the number of different $(2t + 1, 3, t + 1)$-systems is bounded below by z_{2t+1}, and hence by $(3.246)^{2t+1}$ for all sufficiently large t. Combining this with the previous observation that each such sequence gives rise to 2^{2t+1} Skolem-type sequences, we obtain the following result.

Theorem 3.1. *For all sufficiently large t, there are more than $(6.492)^{2t+1}$ Skolem-type sequences of length $14t + 7$ having the following properties:*

(a) there are zeros in the central $2t + 1$ positions, and

(b) *for* $k \in \{t+1, t+2, \ldots, 7t+3\}$*, the two positions occupied by* k *are precisely* k *apart.*

If the central $2t+1$ zero entries in such a Skolem-type sequence are replaced by a split Skolem sequence of order t (which exists for $t \equiv 0$ or $3 \pmod 4$) then a split Skolem sequence of order $7t+3$ is obtained. Hence we have the corollary:

Corollary 3.2. *For sufficiently large* $t \equiv 0$ *or* $3 \pmod 4$*, there are more than* $(6.492)^{2t+1}$ *split Skolem sequences of order* $7t+3$*.*

In fact we can achieve slightly better than this because the number of split Skolem sequences of order t is at least $2^{\lfloor \frac{t}{3} \rfloor}$ for all $t \equiv 0$ or $3 \pmod 4$ [1, 2], so we have at least that number of choices for replacing the central zeros. The bound $(6.492)^{2t+1} > 2^{5.3973t}$ is (for large t) substantially better than the previous best bound of $2^{\lfloor \frac{7t+3}{3} \rfloor}$.

Given a split Skolem sequence of order n, we can form a pure Skolem sequence of order $n+1$ by replacing the central zero with $n+1$ and placing a further entry $n+1$ at either the start or the end of the sequence. Hence we obtain:

Corollary 3.3. *For sufficiently large* $t \equiv 0$ *or* $3 \pmod 4$*, there are more than* $(6.492)^{2t+1}$ *pure Skolem sequences of order* $7t+4$*.*

In the next two corollaries, the lower bound of $(6.492)^{2t+1}$ is extended to hooked and split-hooked Skolem sequences of orders $7t+4$ and $7t+5$ respectively. In each case the basic approach is as follows. For given t, choose a small positive integer c such that $t - c \equiv 2$ or $11 \pmod{12}$ if c is odd, or $t - c \equiv 8$ or $11 \pmod{12}$ if c is even. Put $m = (t-c+1)/3$ and assume that t is large enough to ensure that $m \geq 2c-1$ (i.e. $t \geq 7c-4$). Then there exists an $(m, 3, c)$-system from which a Skolem-type sequence T may be constructed that has length $2t+1$, has zeros in the central $2c-1$ positions and, for each $k \in \{c, c+1, \ldots, t\}$, the two positions occupied by k are precisely k apart. The sequence T is used to replace the central $2t+1$ zeros in each sequence S of length $14t+7$ given by Theorem 3.1. We denote the resulting sequence as $T \rightsquigarrow S$ (T into S), and this sequence has zeros in its central $2c-1$ positions. These are then replaced by a sequence Q of length $2c-1$ to form $Q \rightsquigarrow (T \rightsquigarrow S)$, and a short sequence R of further entries is appended at the right-hand end of this sequence to form a sequence $S' = (Q \rightsquigarrow (T \rightsquigarrow S)) \wedge R$ (where $\wedge$ denotes appending). By choosing c, Q and R appropriately, it is possible to form hooked and split-hooked Skolem sequences S'.

To illustrate the procedure, we explain how to convert a Skolem-type sequence S of length 147 of the form described in Theorem 3.1, to a hooked Skolem sequence S' of order 74. Note that for $k \in \{11, 12, \ldots, 73\}$, the two positions in S occupied by k are precisely k apart, and that S has zeros in the central 21 positions. Next take a $(3, 3, 2)$-system (for example, the one previously described) and from it form a Skolem-type sequence T of length 21 that has zeros in the central three positions and, for each $k \in \{2, 3, \ldots, 10\}$, the two positions in T occupied by k are precisely k apart. Replace the central 21 zeros of S by the sequence T to form $T \rightsquigarrow S$. Then $T \rightsquigarrow S$ has length 147, zeros in the central three positions and, for each $k \in \{2, 3, \ldots, 73\}$, the two positions occupied by k are precisely k apart. Finally, replace the central three zeros in $T \rightsquigarrow S$ by the sequence $Q = (1, 1, 74)$, and append the sequence $R = (0, 74)$ to the right-hand end of $Q \rightsquigarrow (T \rightsquigarrow S)$. The resulting sequence S' has length 149, has a zero in the penultimate position and, for each $k \in \{1, , 2, \ldots, 74\}$, the two positions occupied by k are precisely k apart. Hence S' is a hooked Skolem sequence of order 74. Clearly different sequences S give rise to different sequences S'.

We now return to the general cases. For given t it is obvious that different sequences S will result in different sequences $S' = (Q \rightsquigarrow (T \rightsquigarrow S)) \wedge R$. So, to extend the bound, it suffices to specify c (and hence T), Q and R, and to check the parity conditions for $t-c$. In the next two corollaries to Theorem 3.1, we establish the bound by tabulating appropriate c, Q and R. We leave the reader to check the parity conditions for $t-c$ and that the sequence S' is of the required type.

Corollary 3.4. *For sufficiently large* $t \equiv 1$ *or* $2 \pmod 4$, *there are more than* $(6.492)^{2t+1}$ *hooked Skolem sequences of order* $7t+4$.

Proof. Table 2 covers the possible values of t modulo 12.

Table 2: Construction of hooked Skolem sequences.

$t \pmod{12}$	c	Q	R	$t \geq 7c-4$
$1, 10$	2	$(1, 1, 7t+4)$	$(0, 7t+4)$	$t \geq 10$
$2, 5$	3	$(1, 1, 2, 7t+4, 2)$	$(0, 7t+4)$	$t \geq 17$
$6, 9$	7	$(2, 4, 2, 5, 6, 4, 3, 7t+4, 5, 3, 6, 1, 1)$	$(0, 7t+4)$	$t \geq 45$

For each Skolem-type sequence S of the form described in Theorem 3.1, the resulting sequence $S' = (Q \rightsquigarrow (T \rightsquigarrow S)) \wedge R$ is a hooked Skolem sequence of order $7t+4$. □

Corollary 3.5. *For sufficiently large* $t \equiv 0$ *or* $3 \pmod 4$, *there are more than* $(6.492)^{2t+1}$ *split-hooked Skolem sequences of order* $7t+5$.

Proof. Table 3 covers the possible values of t modulo 12.

Table 3: Construction of split-hooked Skolem sequences.

$t \pmod{12}$	c	Q	R	$t \geq 7c-4$
$0, 3$	4	$(1, 1, 7t+5, 3, 7t+4, 0, 3)$	$(7t+5, 7t+4, 2, 0, 2)$	$t \geq 24$
$4, 7$	5	$(2, 3, 2, 4, 3, 7t+5, 0, 4, 7t+4)$	$(1, 1, 7t+5, 0, 7t+4)$	$t \geq 31$
$8, 11$	9	$(4, 5, 6, 8, 4, 7, 5, 7t+5, 6, 7t+4, 0, 8, 7, 3, 1, 1, 3)$	$(7t+5, 7t+4, 2, 0, 2)$	$t \geq 59$

For each Skolem-type sequence S of the form described in Theorem 3.1, the resulting sequence $S' = (Q \rightsquigarrow (T \rightsquigarrow S)) \wedge R$ is a split-hooked Skolem sequence of order $7t+5$. □

The bound obtained in each of the preceding four corollaries only applies to restricted parts of the appropriate residue classes. We believe that it is possible to extend the bound to all possible residue classes in each case. We do not give a proof of this because our argument breaks into a considerable number of subcases. However, to support our contention, we give one example for split Skolem sequences. Corollary 3.2 gives the bound

$(6.492)^{2t+1}$ when $n = 7t+3$ and $t \equiv 0$ or $3 \pmod 4$, thereby dealing with $n \equiv 3$ or $24 \pmod{28}$. The necessary and sufficient conditions on n for the existence of a split Skolem sequence of order n may be written as $n \equiv 0, 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24$ or $27 \pmod{28}$. For our example, we show how the bound may be extended to $n \equiv 0$ or $7 \pmod{28}$.

Put $n = 7t + 7$ where $t \equiv 0$ or $3 \pmod 4$. Take S to be a Skolem-type sequence as described in Theorem 3.1. Depending on the residue of t modulo 12, take c as in Table 4 and put $m = (t-c+1)/3$. For $t \geq 7c-4$, use an $(m, 3, c)$-system to construct a Skolem-type sequence T of length $2t+1$ having zeros in the central $2c-1$ positions and such that for each $k \in \{c, c+1, \ldots, t\}$, the two positions occupied by k are precisely k apart. Take Q and R as specified in the table and form $S' = (Q \rightsquigarrow (T \rightsquigarrow S)) \wedge R$. Then S' is a split Skolem sequence of length $14t+15$ (i.e. of order $n = 7t+7$). Hence, for all sufficiently large t, there are more than $(6.492)^{2t+1}$ split Skolem sequences of order $n = 7t+7$ where $t \equiv 0$ or $3 \pmod 4$, so that $n \equiv 0$ or $7 \pmod{28}$.

Table 4: Further split Skolem sequences.

$t \pmod{12}$	c	Q	R	$t \geq 7c-4$
$4, 7$	5	$(4, 7t+7, 3, 7t+6, 4, 3, 7t+4, 7t+5, 0)$	$(7t+7, 7t+6, 7t+4, 2, 7t+5, 2, 1, 1)$	$t \geq 31$
$8, 11$	9	$(1, 1, 3, 8, 4, 3, 7t+7, 7, 4, 7t+6, 6, 8, 0, 7t+5, 7, 7t+4, 6)$	$(5, 7t+7, 2, 7t+6, 2, 5, 7t+5, 7t+4)$	$t \geq 59$
$0, 3$	16	$(11, 8, 4, 7, 13, 9, 4, 14, 10, 8, 7, 11, 12, 7t+7, 9, 15, 7t+6, 13, 10, 0, 7t+5, 14, 7t+4, 6, 12, 3, 1, 1, 3, 6, 15)$	$(5, 7t+7, 2, 7t+6, 2, 5, 7t+5, 7t+4)$	$t \geq 108$

There are methods other than the one described above for generating Skolem-type sequences from additive permutations. For certain orders the construction technique given below can give improved bounds.

Suppose that $s > \ell > 0$ and that $\mathcal{S}$ is a Skolem-type sequence of length $2s+1$ having zeros in the central $2\ell-1$ positions, and such that for $k \in \{\ell, \ell+1, \ldots, s\}$ the two positions where k appears in $\mathcal{S}$ are precisely k apart. If $\ell = 1$ then $\mathcal{S}$ is a split Skolem sequence of order s (and such a sequence exists if $s \equiv 0$ or $3 \pmod 4$), otherwise the earlier discussion following Corollary 3.3 shows that such a sequence exists when $s - \ell \equiv 2$ or $11 \pmod{12}$ if ℓ is odd, or $s - \ell \equiv 8$ or $11 \pmod{12}$ if ℓ is even, provided that $s \geq 7\ell - 4$. Let $\mathcal{S}$ be indexed by $[-s, s]$.

Construction 3.6.

- *For $j = \ell, \ell+1, \ldots, s$, denote by a_j, b_j (with $a_j < b_j$) the positions in $\mathcal{S}$ occupied by the entry j, so that $b_j - a_j = j$.*
- *For each $j = \ell, \ell+1, \ldots, s$, let π_j be an additive permutation on $[-t, t]$, and denote by Π the ordered $(s-\ell+1)$-tuple $(\pi_\ell, \pi_{\ell+1}, \ldots, \pi_s)$.*

- *Form a new sequence $\mathcal{S}_\Pi$ indexed by $[-((2t+1)s+t), (2t+1)s+t]$ by placing the entry $(2t+1)j+\pi_j(i)$ at positions $(2t+1)a_j+i$ and $(2t+1)b_j+i+\pi_j(i)$ for $j=\ell,\ell+1,\dots,s$ and $i\in[-t,t]$. For each j these entries cover the interval $[(2t+1)j-t,(2t+1)j+t]$, and so they collectively cover $[(2t+1)\ell-t,(2t+1)s+t]$. These entries cover the positions*

$$[-((2t+1)s+t),(2t+1)s+t]\setminus[-(2t+1)\ell+t+1,(2t+1)\ell-t-1].$$

Place zeros in the vacant positions. Then $\mathcal{S}_\Pi$ is a Skolem-type sequence of length $2(2st+s+t)+1$ having zeros in the central $2((2t+1)\ell-t)-1$ positions, and such that for $k\in[(2t+1)\ell-t,(2t+1)s+t]$ the two positions where k appears in $\mathcal{S}_\Pi$ are precisely k apart.

- *Now suppose that $t\equiv 0$ or $3 \pmod 4$ and let $\mathcal{P}$ be a split Skolem sequence of order t, indexed by $[-t,t]$. Apply the previous three steps to $\mathcal{P}$ using additive permutations σ_j on $[-(\ell-1),\ell-1]$ for $j=1,2,\dots,t$ to form a new sequence $\mathcal{P}_\Sigma$, where Σ is the ordered t-tuple $(\sigma_1,\sigma_2,\dots,\sigma_t)$. Then $\mathcal{P}_\Sigma$ is a Skolem-type sequence of length $2(2t(\ell-1)+t+(\ell-1))+1=2((2t+1)\ell-t)-1$ having zeros in the central $2((2(\ell-1)+1)-(\ell-1)))-1=2\ell-1$ positions, and such that for $k\in[\ell,(2t+1)\ell-t-1]$ the two positions where k appears in $\mathcal{P}_\Sigma$ are precisely k apart. Replace the central zeros of $\mathcal{S}_\Pi$ by $\mathcal{P}_\Sigma$ to form the sequence $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$.*

Then $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$ is a Skolem-type sequence of length $2(2st+s+t)+1$ having zeros in the central $2\ell-1$ positions, and such that for $k\in\{\ell,\ell+1,\dots,2st+s+t\}$ the two positions where k appears in $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$ are precisely k apart.

Given a sequence $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$ constructed in this fashion for given ℓ,s and t, the ingredients $\mathcal{S},\Pi,\mathcal{P}$ and Σ may be recovered by considering entries and positions. Suppose that entry $e>0$ occupies positions a and b with $a<b$. If $e\geq(2t+1)\ell-t$ then $e=(2t+1)j+\pi_j(i)$ for some i and j, and $j=\lfloor(e+t)/(2t+1)\rfloor$, while $a=(2t+1)a_j+i$, so $a_j=\lfloor(a+t)/(2t+1)\rfloor$. Similarly, $b_j=\lfloor(b+t)/(2t+1)\rfloor$, while $i,\pi_j(i)\in[-t,t]$ are given by $i\equiv a \pmod{2t+1}$ and $\pi_j(i)\equiv e \pmod{2t+1}$. This process recovers $\mathcal{S}$ and Π. If $e\leq(2t+1)\ell-t-1=(2(\ell-1)+1)t+(\ell-1)$, then $\mathcal{P}$ and Σ may be recovered in the same way from $\lfloor(e+(\ell-1))/(2\ell-1)\rfloor$, etc.

Hence, varying any of $\mathcal{S},\Pi,\mathcal{P}$ and Σ will yield different Skolem-type sequences $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$. Disregarding variation due to selection of $\mathcal{S},\mathcal{P}$, and Σ, the following result is obtained.

Theorem 3.7. *Suppose that there exists a Skolem-type sequence $\mathcal{S}$ of length $2s+1$ having zeros in the central $2\ell-1$ positions, and such that for $k\in\{\ell,\ell+1,\dots,s\}$ the two positions where k appears in $\mathcal{S}$ are precisely k apart. Then for $t\equiv 0$ or $3 \pmod 4$ there are at least $(z_{2t+1})^{s-\ell+1}$ Skolem-type sequences $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$ of length $2(2st+s+t)+1$ having zeros in the central $2\ell-1$ positions, and such that for $k\in\{\ell,\ell+1,\dots,2st+s+t\}$ the two positions where k appears in $\mathcal{P}_\Sigma\rightsquigarrow\mathcal{S}_\Pi$ are precisely k apart.*

This result generates lower bounds for the numbers of pure, split, hooked and split-hooked Skolem sequences of various orders. Recall from Corollary 2.2 that for sufficiently large t, $z_{2t+1}>(3.246)^{2t+1}$.

Corollary 3.8. *If $s\equiv 0$ or $3 \pmod 4$, then for all sufficiently large $t\equiv 0$ or $3 \pmod 4$, the number of split Skolem sequences of order $2st+s+t$ is greater than $(3.246)^{2st+s}$.*

Proof. Invoke the above construction with $\ell = 1$. □

As an example, taking $s = 3$ gives that for all sufficiently large $t \equiv 0$ or $3 \pmod 4$, the number of split Skolem sequences of order $n = 7t+3$ is greater than $(3.246)^{6t+3}$. This is substantially better than the bound given earlier in Corollary 3.2, although this approach does not appear to offer the generalization to $n \equiv 0$ or $3 \pmod 4$ mentioned in connection with the previous method.

If the central zero in a split Skolem sequence of order n is replaced by $n+1$ and an additional entry $n+1$ is placed at either the start or the end of the sequence, then a pure Skolem sequence of order $n+1$ is formed. Applying this adaptation to the split Skolem sequence constructed in Corollary 3.8 gives the following result.

Corollary 3.9. *If $s \equiv 0$ or $3 \pmod 4$, then for all sufficiently large $t \equiv 0$ or $3 \pmod 4$, the number of pure Skolem sequences of order $2st+s+t+1$ is greater than $(3.246)^{2st+s}$.*

To deal with hooked and split-hooked Skolem sequences, we may replace the central $2\ell - 1$ zeros of $\mathcal{P}_\Sigma \rightsquigarrow \mathcal{S}_\Pi$ with an appropriate sequence Q, and append a short sequence R to the right hand end to form $(Q \rightsquigarrow (\mathcal{P}_\Sigma \rightsquigarrow \mathcal{S}_\Pi)) \wedge R$.

Corollary 3.10. *If $s \equiv 1$ or $2 \pmod 4$ and $s \geq 45$, then for all sufficiently large $t \equiv 0$ or $3 \pmod 4$, the number of hooked Skolem sequences of order $2st+s+t+1$ is greater than $(3.246)^{(s-6)(2t+1)}$.*

Proof. Given s and t, put $m = 2st + s + t$. The sequences Q and R for possible values of s modulo 12 are covered by using Table 2 with t replaced by s, c replaced by ℓ and $7t+3$ replaced by m. Note that in Table 2, $c \leq 7$ and so we may assume that $\ell \leq 7$. In each case, the resulting sequence $(Q \rightsquigarrow (\mathcal{P}_\Sigma \rightsquigarrow \mathcal{S}_\Pi)) \wedge R$ is a hooked Skolem sequence of order $m+1 = 2st+s+t+1$, and the result follows. □

Corollary 3.11. *If $s \equiv 0$ or $3 \pmod 4$ and $s \geq 59$, then for all sufficiently large $t \equiv 0$ or $3 \pmod 4$, the number of split-hooked Skolem sequences of order $2st+s+t+2$ is greater than $(3.246)^{(s-8)(2t+1)}$.*

Proof. Given s and t, put $m = 2st + s + t$. The sequences Q and R for possible values of s modulo 12 are covered by using Table 3 with t replaced by s, c replaced by ℓ and $7t+3$ replaced by m. Note that in Table 3, $c \leq 9$ and so we may assume that $\ell \leq 9$. In each case, the resulting sequence $(Q \rightsquigarrow (\mathcal{P}_\Sigma \rightsquigarrow \mathcal{S}_\Pi)) \wedge R$ is a split-hooked Skolem sequence of order $m+2 = 2st+s+t+2$, and the result follows. □

4 Completing the proof of Theorem 2.1

As previously described, the arithmetic is simplified by subtracting $t-s-a$ from all the entries in R and S to obtain equivalent arrays R' and S', where R' has entries ranging from $-(s+2a)$ to $s+2a$, and S' has entries ranging from $-(s-1)$ to $s-1$. Transversals are sought in these arrays that cover the entry values from $-(s+a)$ to $s+a$ inclusive. We renumber the rows and columns so that for R' the row and column numbers run from 0 to $r-1 = s+2a$ and for S' they run from 0 to $s-1$. The entry in cell (i,j) of R' is then $i+j-(s+2a)$, and that in cell (i,j) of S' is $i+j-(s-1)$. The identification of transversals falls into several cases.

Table 5: Case 1 (r even), transversal in S'.

	Range	Row	Column	Entry
(a)	$j = 0, \dots, \frac{s-1}{2}$	j	$\frac{s-1}{2} + j$	$-\frac{s-1}{2} + 2j$
(b) †	$j = 0, \dots, \frac{s-1}{2} - 1$	$\frac{s-1}{2} + 1 + j$	j	$-\frac{s-1}{2} + 1 + 2j$

Case 1: r even. Since r is even, s must be odd. Table 5 identifies a suitable transversal in S'. Line (b) of the table, marked with †, is omitted when $s = 1$. In S', and subject to †, line (a) of the table covers rows 0 to $\frac{s-1}{2}$, and line (b) covers rows $\frac{s-1}{2} + 1$ to $s - 1$. For columns, line (b) covers 0 to $\frac{s-1}{2} - 1$, and line (a) covers $\frac{s-1}{2}$ to $s - 1$. As regards entries, lines (a) and (b) together cover $-\frac{s-1}{2}$ to $\frac{s-1}{2}$.

Subcase 1.1: $r \equiv 0 \pmod 4$. Table 6 identifies a suitable transversal in R'. The line of the table marked with $*$ is omitted when $s = b - 2$, and the line marked † is omitted when $s = 1$. Subject to $*$ and †, rows, columns and entries of R' are covered by lines of the

Table 6: Subcase 1.1 ($r \equiv 0 \pmod 4$), transversal in R'.

	Range	Row	Column	Entry
(a)	$j = 0, \dots, \frac{r}{4} - 1$	j	$a + j$	$-a - s + 2j$
(b)	$j = 0, \dots, a - \frac{r}{4}$	$\frac{r}{4} + j$	$a + \frac{r}{4} + s + j$	$\frac{s-1}{2} + 1 + 2j$
(c)	$j = 0, \dots, \frac{r}{4} - 1$	$a + 1 + j$	j	$-a - s + 1 + 2j$
(d)	$j = 0, \dots, \frac{s-1}{2}$	$a + \frac{r}{4} + 1 + j$	$a + \frac{r}{4} + \frac{s-1}{2} + j$	$a + 1 + 2j$
(e) †	$j = 0, \dots, \frac{s-1}{2} - 1$	$a + \frac{r}{4} + \frac{s-1}{2} + 2 + j$	$a + \frac{r}{4} + j$	$a + 2 + 2j$
(f) $*$	$j = 0, \dots, a - \frac{r}{4} - 1$	$a + \frac{r}{4} + s + 1 + j$	$\frac{r}{4} + j$	$\frac{s-1}{2} + 2 + 2j$

table in the following orders, where notation such as (a&c) means that entries from lines (a) and (c) are interleaved and taken together. Rows 0 to $2a + s$ by lines (a)(b)(c)(d)(e)(f) in that order. Columns 0 to $2a + s$ by lines (c)(f)(a)(e)(d)(b) in that order. Entries $-a - s$ to $-\frac{s-1}{2} - 1$ by lines (a&c), and entries $\frac{s-1}{2} + 1$ to $a + s$ by lines (b&f)(d&e) in that order; the remaining entries required to complete the transversal values from $-(s + a)$ to $(s + a)$ come from the transversal in S'.

Subcase 1.2: $r \equiv 2 \pmod 4$. Table 7 identifies a suitable transversal in R'. The line of the table marked with † is omitted when $s = 1$. Subject to †, rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a + s$ by lines (a)(b)(c)(d)(e)(f) in that order. Columns 0 to $2a + s$ by lines (c)(f)(a)(e)(d)(b) in that order. Entries $-a - s$ to $-\frac{s-1}{2} - 1$ by lines (a&c), and entries $\frac{s-1}{2} + 1$ to $a + s$ by lines (b&f)(d&e) in that order; the remaining entries required to complete the transversal values from $-(s + a)$ to $(s + a)$ come from the transversal in S'.

Case 2: r odd. If r is odd then s must be even. If $s = 0$ then $r = b$ and R is a copy of A_b

Table 7: Subcase 1.2 ($r \equiv 2 \pmod 4$), transversal in R'.

	Range	Row	Column	Entry
(a)	$j = 0, \dots, \frac{r+2}{4} - 1$	j	$a + j$	$-a - s + 2j$
(b)	$j = 0, \dots, a - \frac{r+2}{4}$	$\frac{r+2}{4} + j$	$a + \frac{r+2}{4} + s + j$	$\frac{s-1}{2} + 2 + 2j$
(c)	$j = 0, \dots, \frac{r+2}{4} - 2$	$a + 1 + j$	j	$-a - s + 1 + 2j$
(d)	$j = 0, \dots, \frac{s-1}{2}$	$a + \frac{r+2}{4} + j$	$a + \frac{r+2}{4} + \frac{s-1}{2} + j$	$a + 1 + 2j$
(e) †	$j = 0, \dots, \frac{s-1}{2} - 1$	$a + \frac{r+2}{4} + \frac{s-1}{2} + 1 + j$	$a + \frac{r+2}{4} + j$	$a + 2 + 2j$
(f)	$j = 0, \dots, a - \frac{r+2}{4}$	$a + \frac{r+2}{4} + s + j$	$\frac{r+2}{4} - 1 + j$	$\frac{s-1}{2} + 1 + 2j$

(with rows and columns appropriately renumbered) and any one of the transversals already identified in A_b provides a suitable transversal in R. So throughout Case 2, we may assume that $s > 0$, and then Table 8 identifies a suitable transversal in S'. Line (b) of the table, marked with †, is omitted when $s = 2$. Subject to †, rows, columns and entries of S' are

Table 8: Case 2 (r odd), transversal in S'.

	Range	Row	Column	Entry
(a)	$j = 0, \dots, \frac{s}{2} - 1$	j	$\frac{s}{2} - 1 + j$	$-\frac{s}{2} + 2j$
(b) †	$j = 0, \dots, \frac{s}{2} - 2$	$\frac{s}{2} + j$	j	$-\frac{s}{2} + 1 + 2j$
(c)	single cell	$s - 1$	$s - 1$	$s - 1$

covered by lines of the table in the following orders. Rows 0 to $s - 1$ by lines (a)(b)(c) in that order. Columns 0 to $s - 1$ by lines (b)(a)(c) in that order. Entries $-\frac{s}{2}$ to $\frac{s}{2} - 2$ by lines (a&b), and entry $s - 1$ by line (c).

To deal with R', we consider four subcases depending on the values of r and s modulo 4.

Subcase 2.1: $r \equiv 1,\ s \equiv 0 \pmod 4$. These conditions imply that $b \equiv 1 \pmod 4$ and we may assume that $s \geq 4$. Table 9 identifies a suitable transversal in R'. Lines of the table marked with $*$ are omitted when $s = b - 1$, and lines marked with † are omitted when $s = 4$. Subject to $*$ and †, rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a + s$ by lines (a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)(l)(m) in that order. Columns 0 to $2a + s$ by lines (k)(f)(i)(g)(e)(b)(d)(l)(c)(m)(a)(h)(j) in that order. Entries $-a - s$ to $-\frac{s}{2} - 1$ by lines (b&f)(e)(d&g)(i)(c)(a&k) in that order, entries $\frac{s}{2} - 1$ to $s - 2$ by lines (h&l), and entries s to $a + s$ by lines (j&m); the remaining entries required to complete the transversal values from $-(s + a)$ to $(s + a)$ come from the transversal in S'.

Subcase 2.2: $r \equiv 1,\ s \equiv 2 \pmod 4$. These conditions imply that $b \equiv 3 \pmod 4$ and a is odd. Table 10 identifies a suitable transversal in R' when $s \geq 6$. The line of the table

Table 9: Subcase 2.1 ($r \equiv 1,\ s \equiv 0 \pmod 4$), transversal in R'.

	Range	Row	Column	Entry
(a) $*$	$j = 0, \dots, \frac{a}{2} - \frac{s}{4} - 1$	j	$a + s + j$	$-a + 2j$
(b)	$j = 0, \dots, \frac{s}{4} - 1$	$\frac{a}{2} - \frac{s}{4} + j$	$\frac{a}{2} + \frac{s}{4} + j$	$-a - s + 2j$
(c)	single cell	$\frac{a}{2}$	$\frac{a}{2} + s - 1$	$-a - 1$
(d) †	$j = 0, \dots, \frac{s}{4} - 2$	$\frac{a}{2} + 1 + j$	$\frac{a}{2} + \frac{s}{2} + j$	$-a - \frac{s}{2} + 1 + 2j$
(e)	single cell	$\frac{a}{2} + \frac{s}{4}$	$\frac{a}{2} + \frac{s}{4} - 1$	$-a - \frac{s}{2} - 1$
(f) †	$j = 0, \dots, \frac{s}{4} - 2$	$\frac{a}{2} + \frac{s}{4} + 1 + j$	$\frac{a}{2} - \frac{s}{4} + j$	$-a - s + 1 + 2j$
(g) †	$j = 0, \dots, \frac{s}{4} - 2$	$\frac{a}{2} + \frac{s}{2} + j$	$\frac{a}{2} + j$	$-a - \frac{s}{2} + 2j$
(h)	$j = 0, \dots, \frac{s}{4} - 1$	$\frac{a}{2} + \frac{3s}{4} - 1 + j$	$\frac{3a}{2} + \frac{3s}{4} + j$	$\frac{s}{2} - 1 + 2j$
(i)	single cell	$\frac{a}{2} + s - 1$	$\frac{a}{2} - 1$	$-a - 2$
(j)	$j = 0, \dots, \frac{a}{2}$	$\frac{a}{2} + s + j$	$\frac{3a}{2} + s + j$	$s + 2j$
(k) $*$	$j = 0, \dots, \frac{a}{2} - \frac{s}{4} - 1$	$a + s + 1 + j$	j	$-a + 1 + 2j$
(l)	$j = 0, \dots, \frac{s}{4} - 1$	$\frac{3a}{2} + \frac{3s}{4} + 1 + j$	$\frac{a}{2} + \frac{3s}{4} - 1 + j$	$\frac{s}{2} + 2j$
(m)	$j = 0, \dots, \frac{a}{2} - 1$	$\frac{3a}{2} + s + 1 + j$	$\frac{a}{2} + s + j$	$s + 1 + 2j$

marked with $*$ is omitted when $s = b - 1$, and the line of the table marked with † is omitted when $s = 6$. Subject to $*$ and †, rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a+s$ by lines (a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)(l)(m) in that order. Columns 0 to $2a+s$ by lines (k)(f)(i)(g)(e)(b)(d)(l)(c)(m)(a)(h)(j) in that order. Entries $-a - s$ to $-\frac{s}{2} - 1$ by lines (b&f)(e)(d&g)(c)(i)(a&k) in that order, entries $\frac{s}{2} - 1$ to $s - 2$ by lines (h&l), and entries s to $a + s$ by lines (j&m); the remaining entries required to complete the transversal values from $-(s + a)$ to $(s + a)$ come from the transversal in S'.

The case $s = 2$ may be obtained from the table by omitting lines (b), (d), (e), (f), (g) and (l). Subject to $*$ (i.e. when $b = 3$), rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a + 2$ by lines (a)(c)(h)(i)(j)(k)(m) in that order. Columns 0 to $2a + 2$ by lines (k)(i)(c)(m)(a)(h)(j) in that order. Entries $-a - 2$ to -2 by lines (c)(i)(a&k) in that order, entry 0 by line (h), and entries 2 to $a + 2$ by lines (j&m); the remaining entries required to complete the transversal values from $-(a + 2)$ to $(a + 2)$ come from the transversal in S'.

Subcase 2.3: $r \equiv 3,\ s \equiv 0 \pmod 4$. These conditions imply that $b \equiv 3 \pmod 4$ and a is odd. Table 11 identifies a suitable transversal in R'. The line of the table marked with $*$ is omitted when $s = b - 3$, and the lines of the table marked with † are omitted when $s = 4$. Subject to $*$ and †, rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a + s$ by lines (a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)(l)(m) in that order. Columns 0 to $2a + s$ by lines (k)(f)(i)(g)(e)(b)(d)(l)(c)(m)(a)(h)(j) in that order. Entries $-a - s$ to $-\frac{s}{2} - 1$ by lines (b&f)(e)(d&g)(i)(c)(a&k) in that order, entries $\frac{s}{2} - 1$ to $s - 2$ by lines (h&l), and entries s to $a + s$ by lines (j&m); the remaining entries required to complete the transversal values from $-(s + a)$ to $(s + a)$ come from the transversal in S'.

Table 10: Subcase 2.2 ($r \equiv 1,\ s \equiv 2 \pmod 4$), transversal in R'.

	Range	Row	Column	Entry
(a) $*$	$j = 0, \ldots, \frac{a-1}{2} - \frac{s-2}{4} - 1$	j	$a+s+j$	$-a+2j$
(b)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a-1}{2} - \frac{s-2}{4} + j$	$\frac{a-1}{2} + \frac{s-2}{4} + 1 + j$	$-a-s+2j$
(c)	single cell	$\frac{a-1}{2}$	$\frac{a-1}{2} + s - 1$	$-a-2$
(d)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a-1}{2} + 1 + j$	$\frac{a-1}{2} + \frac{s}{2} + j$	$-a - \frac{s}{2} + 2j$
(e)	single cell	$\frac{a-1}{2} + \frac{s-2}{4} + 1$	$\frac{a-1}{2} + \frac{s-2}{4}$	$-a - \frac{s}{2} - 1$
(f)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a-1}{2} + \frac{s-2}{4} + 2 + j$	$\frac{a-1}{2} - \frac{s-2}{4} + j$	$-a-s+1+2j$
(g) †	$j = 0, \ldots, \frac{s-2}{4} - 2$	$\frac{a-1}{2} + \frac{s}{2} + 1 + j$	$\frac{a-1}{2} + 1 + j$	$-a - \frac{s}{2} + 1 + 2j$
(h)	$j = 0, \ldots, \frac{s-2}{4}$	$\frac{a-1}{2} + \frac{3s-2}{4} + j$	$\frac{3a+1}{2} + \frac{3s-2}{4} + j$	$\frac{s}{2} - 1 + 2j$
(i)	single cell	$\frac{a-1}{2} + s$	$\frac{a-1}{2}$	$-a-1$
(j)	$j = 0, \ldots, \frac{a-1}{2}$	$\frac{a-1}{2} + s + 1 + j$	$\frac{3a+1}{2} + s + j$	$s+1+2j$
(k) $*$	$j = 0, \ldots, \frac{a-1}{2} - \frac{s-2}{4} - 1$	$a+s+1+j$	j	$-a+1+2j$
(l)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{3a+1}{2} + \frac{3s-2}{4} + 1 + j$	$\frac{a-1}{2} + \frac{3s-2}{4} + j$	$\frac{s}{2} + 2j$
(m)	$j = 0, \ldots, \frac{a-1}{2}$	$\frac{3a+1}{2} + s + j$	$\frac{a-1}{2} + s + j$	$s+2j$

Subcase 2.4: $r \equiv 3,\ s \equiv 2 \pmod 4$. These conditions imply that $b \equiv 1 \pmod 4$ and a is even. Table 12 identifies a suitable transversal in R' when $s \geq 6$. The line of the table marked with $*$ is omitted when $s = b-3$, and the line of the table marked with † is omitted when $s = 6$. Subject to $*$ and †, rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a+s$ by lines (a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)(l)(m) in that order. Columns 0 to $2a+s$ by lines (k)(f)(i)(g)(e)(b)(d)(l)(c)(m)(a)(h)(j) in that order. Entries $-a-s$ to $-\frac{s}{2}-1$ by lines (b&f)(e)(d&g)(i)(c)(a&k) in that order, entries $\frac{s}{2}-1$ to $s-2$ by lines (h&l), and entries s to $a+s$ by lines (j&m); the remaining entries required to complete the transversal values from $-(s+a)$ to $(s+a)$ come from the transversal in S'.

The case $s = 2$ may be obtained from the table by omitting lines (b), (d), (e), (f), (g) and (h). Subject to $*$ (i.e. when $b = 5$), rows, columns and entries of R' are covered by lines of the table in the following orders. Rows 0 to $2a+2$ by lines (a)(c)(i)(j)(k)(l)(m) in that order. Columns 0 to $2a+2$ by lines (k)(i)(l)(c)(m)(a)(j) in that order. Entries $-a-2$ to -2 by lines (i)(c)(a&k) in that order, entry 0 by line (l), and entries 2 to $a+2$ by lines (j&m); the remaining entries required to complete the transversal values from $-(a+2)$ to $(a+2)$ come from the transversal in S'.

This concludes the proof of Theorem 2.1.

Table 11: Subcase 2.3 ($r \equiv 3,\ s \equiv 0 \pmod 4$), transversal in R'.

	Range	Row	Column	Entry
(a)	$j = 0, \dots, \frac{a-1}{2} - \frac{s}{4}$	j	$a + s + j$	$-a + 2j$
(b)	$j = 0, \dots, \frac{s}{4} - 1$	$\frac{a+1}{2} - \frac{s}{4} + j$	$\frac{a-1}{2} + \frac{s}{4} + j$	$-a - s + 2j$
(c)	single cell	$\frac{a+1}{2}$	$\frac{a-1}{2} + s - 1$	$-a - 1$
(d) †	$j = 0, \dots, \frac{s}{4} - 2$	$\frac{a+1}{2} + 1 + j$	$\frac{a-1}{2} + \frac{s}{2} + j$	$-a - \frac{s}{2} + 1 + 2j$
(e)	single cell	$\frac{a+1}{2} + \frac{s}{4}$	$\frac{a-1}{2} + \frac{s}{4} - 1$	$-a - \frac{s}{2} - 1$
(f) †	$j = 0, \dots, \frac{s}{4} - 2$	$\frac{a+1}{2} + \frac{s}{4} + 1 + j$	$\frac{a-1}{2} - \frac{s}{4} + j$	$-a - s + 1 + 2j$
(g) †	$j = 0, \dots, \frac{s}{4} - 2$	$\frac{a+1}{2} + \frac{s}{2} + j$	$\frac{a-1}{2} + j$	$-a - \frac{s}{2} + 2j$
(h)	$j = 0, \dots, \frac{s}{4} - 1$	$\frac{a+1}{2} + \frac{3s}{4} - 1 + j$	$\frac{3a+1}{2} + \frac{3s}{4} + j$	$\frac{s}{2} + 2j$
(i)	single cell	$\frac{a+1}{2} + s - 1$	$\frac{a-1}{2} - 1$	$-a - 2$
(j)	$j = 0, \dots, \frac{a-1}{2}$	$\frac{a+1}{2} + s + j$	$\frac{3a+1}{2} + s + j$	$s + 1 + 2j$
(k) $*$	$j = 0, \dots, \frac{a-1}{2} - \frac{s}{4} - 1$	$a + s + 1 + j$	j	$-a + 1 + 2j$
(l)	$j = 0, \dots, \frac{s}{4} - 1$	$\frac{3a+1}{2} + \frac{3s}{4} + j$	$\frac{a-1}{2} + \frac{3s}{4} - 1 + j$	$\frac{s}{2} - 1 + 2j$
(m)	$j = 0, \dots, \frac{a-1}{2}$	$\frac{3a+1}{2} + s + j$	$\frac{a-1}{2} + s + j$	$s + 2j$

5 Concluding remarks

Our results improve the known lower bounds for the number of additive permutations, zero-sum arrays, some Skolem-type sequences, and some extremal Langford sequences. It seems highly likely that the bounds obtained in this paper apply to all pure, split, hooked and split-hooked Skolem sequences of sufficiently large orders. The recent paper [8] combines such bounds with graph labellings to generate Langford sequences. It seems likely that our new bounds can be combined with these techniques to generate improved estimates for the numbers of Langford sequences.

For small orders, the numbers of (pure) Skolem sequences and hooked Skolem sequences (and other related sequences) are tabulated in [4], while Table 8 of [7] gives the numbers of split Skolem (Rosa) sequences of orders $n \leq 12$. These numerical results strongly suggest that further improvements to our lower bounds are possible.

Since Skolem sequences may be used to construct solutions to Heffter's first and second difference problems, the bounds inform the numbers of these and of resulting cyclic Steiner triple systems. If improved bounds for z_{2t+1} are obtained in the future, these methods will lead to improved bounds for many related sequences. From Table 1, it will be seen that the ratio z_{2t+1}/z_{2t-1} appears to increase with t, and to exceed $2t$ for $t \geq 6$, strongly suggesting that $z_{2t+1} > 2^t t!$ for all sufficiently large t. This is a weaker bound than might be suggested by Vardi's conjecture, but it is strongly supported by the computational evidence, and one might expect that most transversals are not $\mathbb{Z}$-transversals.

Table 12: Subcase 2.4 ($r \equiv 3,\ s \equiv 2 \pmod 4$), transversal in R'.

	Range	Row	Column	Entry
(a)	$j = 0, \ldots, \frac{a}{2} - \frac{s-2}{4} - 1$	j	$a + s + j$	$-a + 2j$
(b)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a}{2} - \frac{s-2}{4} + j$	$\frac{a}{2} + \frac{s-2}{4} + j$	$-a - s + 2j$
(c)	single cell	$\frac{a}{2}$	$\frac{a}{2} + s - 1$	$-a - 1$
(d)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a}{2} + 1 + j$	$\frac{a}{2} + \frac{s}{2} - 1 + j$	$-a - \frac{s}{2} + 2j$
(e)	single cell	$\frac{a}{2} + \frac{s-2}{4} + 1$	$\frac{a}{2} + \frac{s-2}{4} - 1$	$-a - \frac{s}{2} - 1$
(f)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a}{2} + \frac{s-2}{4} + 2 + j$	$\frac{a}{2} - \frac{s-2}{4} - 1 + j$	$-a - s + 1 + 2j$
(g) †	$j = 0, \ldots, \frac{s-2}{4} - 2$	$\frac{a}{2} + \frac{s}{2} + 1 + j$	$\frac{a}{2} + j$	$-a - \frac{s}{2} + 1 + 2j$
(h)	$j = 0, \ldots, \frac{s-2}{4} - 1$	$\frac{a}{2} + \frac{3s-2}{4} + j$	$\frac{3a}{2} + \frac{3s+2}{4} + j$	$\frac{s}{2} + 2j$
(i)	single cell	$\frac{a}{2} + s - 1$	$\frac{a}{2} - 1$	$-a - 2$
(j)	$j = 0, \ldots, \frac{a}{2}$	$\frac{a}{2} + s + j$	$\frac{3a}{2} + s + j$	$s + 2j$
(k) $*$	$j = 0, \ldots, \frac{a}{2} - \frac{s-2}{4} - 2$	$a + s + 1 + j$	j	$-a + 1 + 2j$
(l)	$j = 0, \ldots, \frac{s-2}{4}$	$\frac{3a}{2} + \frac{3s+2}{4} + j$	$\frac{a}{2} + \frac{3s-6}{4} + j$	$\frac{s}{2} - 1 + 2j$
(m)	$j = 0, \ldots, \frac{a}{2} - 1$	$\frac{3a}{2} + s + 1 + j$	$\frac{a}{2} + s + j$	$s + 1 + 2j$

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Classification of regular balanced Cayley maps of minimal non-abelian metacyclic groups*

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Abstract

In this paper, we classify the regular balanced Cayley maps of minimal non-abelian metacyclic groups. Besides the quaternion group Q_8, there are two infinite families of such groups which are denoted by $M_{p,q}(m,r)$ and $M_p(n,m)$, respectively. Firstly, we prove that there are regular balanced Cayley maps of $M_{p,q}(m,r)$ if and only if $q = 2$ and we list all of them up to isomorphism. Secondly, we prove that there are regular balanced Cayley maps of $M_p(n,m)$ if and only if $p = 2$ and $n = m$ or $n = m + 1$ and there is exactly one such map up to isomorphism in either case. Finally, as a corollary, we prove that any metacyclic p-group for odd prime number p does not have regular balanced Cayley maps.

Keywords: Regular balanced Cayley map, minimal non-abelian group, metacyclic group.

Math. Subj. Class.: 05C25, 05C30

1 Introduction

A *Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$ is a graph based on a group G and a finite set $X = \{x_1, x_2, \ldots, x_k\}$ of elements in G which does not contain 1_G, contains no repeated elements, is closed under the operation of taking inverses, and generates all of G. In this

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paper, we call X a *Cayley subset* of G. The vertices of the Cayley graph Γ are the elements of G, and two vertices g and h are adjacent if and only if $g = hx_i$ for some $x_i \in X$. The ordered pairs (h, hx) for $h \in G$ and $x \in X$ are called the darts of Γ. Let ρ be any cyclic permutation on X. Then the Cayley map $\mathcal{M} = \mathrm{CM}(G, X, \rho)$ is the 2-cell embedding of the Cayley graph $\mathrm{Cay}(G, X)$ in an orientable surface for which the orientation-induced local ordering of the darts emanating from any vertex $g \in G$ is always the same as the ordering of generators in X induced by ρ; that is, the neighbors of any vertex g are always spread counterclockwise around g in the order $(gx, g\rho(x), g\rho^2(x), \dots, g\rho^{k-1}(x))$.

An (orientation preserving) *automorphism* of a Cayley map $\mathcal{M}$ is a permutation on the dart set of $\mathcal{M}$ which preserves the incidence relation of the vertices, edges, faces, and the orientation of the map. The full automorphism group of $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$, is the group of all such automorphisms of $\mathcal{M}$ under the operation of composition. This group always acts semi-regularly on the set of darts of $\mathcal{M}$, that is, the stabilizer in $\mathrm{Aut}(\mathcal{M})$ of each dart of $\mathcal{M}$ is trivial. If this action is transitive, then we say that the Cayley map $\mathcal{M}$ is a regular Cayley map. As the left regular multiplication action of the underlying group G lifts naturally into the full automorphism group of any Cayley map $\mathrm{CM}(G, X, \rho)$, Cayley maps are proved to be a very good source of regular maps. There are many papers on the topic of regular Cayley maps, we refer the readers to [4, 10] and [11] and the references therein. Furthermore, A Cayley map $\mathrm{CM}(G, X, \rho)$ is called *balanced* if $\rho(x)^{-1} = \rho(x^{-1})$ for every $x \in X$. In [11], Škoviera and Širáň showed that a Cayley map $\mathrm{CM}(G, X, \rho)$ is regular and balanced if and only if there exists a group automorphism σ such that $\sigma|_X = \rho$, where $\sigma|_X$ denotes the restricted action of σ on X. Therefore, to determine all the regular balanced Cayley maps of a group is equivalent to determine all the orbits of its automorphisms that can be Cayley subsets.

In this paper, a non-abelian group G is called minimal if each of its proper subgroups H (that is $H < G$ but $H \neq G$) is abelian. In 1903, Miller and Moreno gave a full classification of minimal non-abelian groups, one may refer to [7] for detailed results. A group G is metacyclic if it has a cyclic normal subgroup N such that the factor group G/N is cyclic. As one can see in [7], there are three classes of minimal non-abelian metacyclic groups:

(1) the quaternion group Q_8;

(2) $M_{p,q}(m, r) = \langle a, b \mid a^p = 1, b^{q^m} = 1, b^{-1}ab = a^r\rangle$, where p and q are distinct prime numbers, m is a positive integer and $r \not\equiv 1 \pmod{p}$ but $r^q \equiv 1 \pmod{p}$;

(3) $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, b^{-1}ab = a^{1+p^{n-1}}, n \geq 2, m \geq 1\rangle$.

One can also cite [3, Theorem 2.1] for reference or [13, pp. 123] for details.

For regular balanced Cayley maps, it has been shown that all odd order abelian groups possess at least one regular balanced Cayley map [4]. Wang and Feng [12] classified all regular balanced Cayley maps for cyclic, dihedral and generalized quaternion groups. In [9], Oh proved the non-existence of regular balanced Cayley maps with semi-dihedral groups. In this paper, we pay our attentions to the regular balanced Cayley maps of minimal non-abelian metacyclic groups. Since the regular balanced Cayley maps of Q_8 have been classified in [12] (Q_8 has exactly one regular balanced Cayley map up to isomorphism), we only consider the groups $M_{p,q}(m, r)$ and $M_p(n, m)$. In Section 3, we show that $M_{p,q}(m, r)$ has regular balanced Cayley maps if and only if q is 2 and we list all of them up to isomorphism. In Section 4, we show that $M_p(n, m)$ has regular balanced Cayley maps if and only if $p = 2$ and $n = m$ or $n = m + 1$. In either case, it has exactly one regular balanced Cayley map up to isomorphism and the map has valency 4. Moreover, as a corollary any metacyclic p-group for odd prime p doesn't have regular balanced Cayley maps.

2 Preliminaries

Lemma 2.1. *Take an element $b^t a^s \in M_{p,q}(m, r)$, where $t \neq 0$, then the order of $b^t a^s$ is q^m if and only if $(t, q) = 1$.*

Proof. The group $M_{p,q}(m, r)$ is the union of one cyclic group of order p and p conjugate cyclic subgroups of order q^m. If $t \neq 0$, then $b^t a^s$ belongs to one of the cyclic subgroups of order q^m. Therefore, the order of $b^t a^s$ is q^m if and only if $(t, q) = 1$. □

Lemma 2.2. *The automorphism group of $M_{p,q}(m, r)$ is*

$$\mathrm{Aut}(M_{p,q}(m, r)) = \{\sigma \mid a^\sigma = a^i,\ b^\sigma = b^j a^k, 1 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant q^m-1, q \mid (j-1)\}.$$

Proof. Assume $\sigma \in \mathrm{Aut}(M_{p,q}(m, r))$. According to Lemma 2.1, $a^\sigma = a^i, b^\sigma = b^j a^k$ for some $1 \leqslant i \leqslant p-1, 1 \leqslant j \leqslant q^m - 1$ and $(j, q) = 1$. If $M_{p,q}(m, r) = \langle a^\sigma, b^\sigma \rangle$, then we can get the relation $q \mid (j-1)$.

In fact, since $(a^r)^\sigma = (b^{-1}ab)^\sigma = (b^{-1})^\sigma a^\sigma b^\sigma = b^{-j} a^i b^j = a^{ir^j} = a^{ir}$, we have $a^{ir(r^{j-1}-1)} = 1$. Moreover, from $(ir, p) = 1$ and $a^p = 1$, we get $(r^{j-1} - 1) \equiv 0 \pmod p$, that is $r^{j-1} \equiv 1 \pmod p$. As $r^q \equiv 1 \pmod p$ and q is prime, we have $q \mid (j-1)$. □

Lemma 2.3 ([5])**.** *The automorphism group of $M_p(n, m)$ is listed as follows:*

(i) *If $n \leq m$, then* $\mathrm{Aut}(M_p(n, m)) = \{\sigma \mid a^\sigma = b^j a^i, b^\sigma = b^s a^r, (i, p) = 1, 1 \leq i \leq p^n, j = kp^{m-n+1}, 0 \leq k < p^{n-1}, 1 \leq r \leq p^n, s \equiv 1 \pmod p, 1 \leq s \leq p^m\}$.

(ii) *If p is odd and $n > m \geq 1$ or $p = 2$ and $n > m > 1$, then* $\mathrm{Aut}(M_p(n, m)) = \{\sigma \mid a^\sigma = b^j a^i, b^\sigma = b^s a^r, (i, p) = 1, 1 \leq i \leq p^n, 1 \leq j \leq p^m, r = kp^{n-m}, 0 \leq k < p^m, s \equiv 1 \pmod p, 1 \leq s \leq p^m\}$.

The following Lemma 2.4 is a basic result in group theory and we omit the proof.

Lemma 2.4. *Let G be a finite group and N be a normal subgroup of G. Take $\alpha \in \mathrm{Aut}(G)$. If $N^\alpha = N$, then $\bar{\alpha} : Ng \mapsto Ng^\alpha$ is an automorphism of G/N which is called the induced automorphism of α.*

Lemma 2.5. *Let G be a finite group and N be a proper characteristic subgroup of G. Take $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. If $X = g^{\langle\alpha\rangle}$ is a Cayley subset of G, then $\overline{X} = \overline{g^{\langle\alpha\rangle}} = \bar{g}^{\langle\bar{\alpha}\rangle}$ is a Cayley subset of $\overline{G} = G/N$. Moreover, if the order of α is a power of 2 and $\bar{g}$ is not an involution, then $|X| = |\overline{X}|$.*

Proof. By Lemma 2.4, $\bar{\alpha}$ is an automorphism of G/N induced by α. Set $X = g^{\langle\alpha\rangle}$, then $\overline{X} = \overline{g^{\langle\alpha\rangle}} = \bar{g}^{\langle\bar{\alpha}\rangle}$. If X is a Cayley subset of G, then the relations $\langle\overline{X}\rangle = \overline{G}$, $\overline{X} = \overline{X}^{-1}$ follow naturally. Since $N < G$, we have $\overline{X} \neq \overline{1}^{\langle\bar{\alpha}\rangle}$ and then $\overline{1} \notin \overline{X}$. So, $\overline{X}$ is a Cayley subset of $\overline{G}$.

If the order of α is 2^s for some positive integer s, then the order of $\overline{\alpha}$ is 2^t for some integer $t \leq s$. From $g^{\alpha^{2^{s-1}}} = g^{-1}$, we have $\overline{g}^{\overline{\alpha}^{2^{s-1}}} = \overline{g}^{-1}$. While $\overline{g}^{\overline{\alpha}^{2^t}} = \overline{g}$, then $t > s - 1$. So, $s = t$ and $|X| = |\overline{X}|$. □

As a direct corollary of Lemmas 2.4 and 2.5, we give the following Corollary 2.6.

Corollary 2.6. *If a group G has regular balanced Cayley maps, then so does the quotient group G/N for any proper characteristic subgroup N of G.*

There are many ways to get proper characteristic subgroups. In the following, we give a method to get such subgroups. These results are exercises for students, so we omit the proof.

Lemma 2.7. *Let G be a finite group, $S \subseteq G$, $\sigma \in \mathrm{End}(G)$, K be a characteristic subgroup of G and n be a positive integer. Then,*

(i) $\langle S \rangle^\sigma = \langle S^\sigma \rangle$;

(ii) $H_1 = \langle x^n \mid x \in K \rangle$ *is a characteristic subgroup of G;*

(iii) $H_2 = \langle y \mid y \in G, y^n \in K \rangle$ *is a characteristic subgroup of G.*

As for isomorphism of regular maps, one may refer to [10] for the following Lemma 2.8.

Lemma 2.8. *Assume $M_1 = \mathrm{CM}(G, X_1, \rho_1)$ and $M_2 = \mathrm{CM}(G, X_2, \rho_2)$ are two regular balanced Cayley maps of the finite group G, where $X_1 = g^{\langle \sigma_1 \rangle}$ and $X_2 = h^{\langle \sigma_2 \rangle}$ are orbits of two group elements g and h under the action of two automorphisms σ_1 and σ_2 of G, respectively. Then M_1 and M_2 are isomorphic if and only if $|X_1| = |X_2| = k$ and there is some $\tau \in \mathrm{Aut}(G)$ such that $h^{\sigma_2^i} = g^{\sigma_1^i \tau}$, $1 \le i \le k$.*

As a special case and an application of Lemma 2.8, we have the following Lemma 2.9.

Lemma 2.9. *Let G be a finite group. Take $\alpha \in \mathrm{Aut}(G)$ and two elements $g, h \in G$. Assume $X = g^{\langle \alpha \rangle}$ is a Cayley subset of G. If there is some $\sigma \in \mathrm{Aut}(G)$ such that $g^\sigma = h$, then $Y = h^{\langle \sigma^{-1} \alpha \sigma \rangle}$ is also a Cayley subset of G and $Y = X^\sigma$. Under this situation, the two regular balanced Cayley maps $\mathrm{CM}(G, X, \alpha|_X)$ and $\mathrm{CM}(G, Y, \sigma^{-1}\alpha\sigma|_Y)$ are isomorphic.*

Proof. Because $Y = h^{\langle \sigma^{-1}\alpha\sigma \rangle} = g^{\sigma \langle \sigma^{-1}\alpha\sigma \rangle} = g^{\sigma\sigma^{-1} \langle \alpha \rangle \sigma} = g^{\langle \alpha \rangle \sigma} = X^\sigma$ and X is a Cayley subset, it follows that Y is also a Cayley subset. The result that $\mathrm{CM}(G, X, \alpha|_X)$ and $\mathrm{CM}(G, Y, \sigma^{-1}\alpha\sigma|_Y)$ are isomorphic follows from Lemma 2.8. □

3 Regular balanced Cayley maps of $M_{p,q}(m, r)$

As we mentioned in the introduction, to determine all the regular balanced Cayley maps of a group is equivalent to determine all the orbits of its automorphisms that can be Cayley subsets. In this section, we divide our discussion into two parts according to the parity of q.

Lemma 3.1. *The center $Z(M_{p,q}(m, r))$ of $M_{p,q}(m, r)$ is generated by b^q and the quotient group $M_{p,q}(m, r)/Z(M_{p,q}(m, r)) \cong M_{p,q}(1, r)$.*

Proof. From the defining relation of $M_{p,q}(m, r)$, we have $b^{-q}ab^q = a^{r^q} = a$. So, $b^q \in Z(M_{p,q}(m, r))$. Since $M_{p,q}(m, r)$ is not abelian and generated by a and b, we have $a, b \notin Z(M_{p,q}(m, r))$, hence $Z(M_{p,q}(m, r)) = \langle b^q \rangle$. The formula $M_{p,q}(m, r)/Z(M_{p,q}(m, r)) \cong M_{p,q}(1, r)$ follows directly from the definition of $M_{p,q}(m, r)$. □

Theorem 3.2. *If q is odd, then the group $M_{p,q}(1, r)$ does not have regular balanced Cayley maps.*

Proof. For brevity, set $H = M_{p,q}(1,r)$. Suppose there exists a $\sigma \in \mathrm{Aut}(H)$ and $b^v a^u \in H$ such that $X = (b^v a^u)^{\langle\sigma\rangle}$ is a Cayley subset of H. The derived subgroup of H is $H' = \langle a \rangle$ which is a characteristic subgroup. Let $\overline{H} = H/H'$ and $\overline{\sigma}$ be induced by σ. By Lemma 2.2, $b^\sigma = ba^k$ for some integer k and as a result $\overline{b}^{\overline{\sigma}} = \overline{b}$. So, $\overline{X} = \overline{b^v a^u}^{\langle\overline{\sigma}\rangle} = \overline{b^v}^{\langle\overline{\sigma}\rangle} = \{\overline{b^v}\}$. While $\overline{X} = \overline{X}^{-1}$, $o(b) = q$ and $o(\overline{b^v}) \mid o(b)$, we have $\overline{b^v} = \overline{1}$ and so $b^v \in H'$. It follows that $\langle X \rangle \leq H' < H$ contradicting to $H = \langle X \rangle$. □

As a corollary of Lemmas 3.1 and 2.5, we have the following Theorem 3.3.

Theorem 3.3. *If q is odd, then $M_{p,q}(m,r)$ does not have regular balanced Cayley maps.*

It is known that $\mathbb{Z}_{2^n}^* \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} = \langle \overline{-1} \rangle \times \langle \overline{5} \rangle$, where $\overline{-1}$ and $\overline{5}$ denote the class of integers equaling to -1 and 5 modular 2^n, respectively. In a p-group G, let $\mho_1(G) = \langle a^p \mid a \in G \rangle$. Then, $\mho_1(\mathbb{Z}_{2^n}^*) = \langle \overline{5}^2 \rangle$ which does not contain $\overline{-1}$.

Lemma 3.4. *For a positive integer $n \geq 2$, the equation $x^k \equiv -1 \pmod{2^n}$ holds if and only if k is odd and $x \equiv -1 \pmod{2^n}$.*

Proof. It is obviously true when $n = 2$. So, we may assume $n \geq 3$. Let u be a solution of the equation $x^k \equiv -1 \pmod{2^n}$, then the integer u should be odd, so $\overline{u} \in \mathbb{Z}_{2^n}^* = \langle \overline{-1} \rangle \times \langle \overline{5} \rangle$. From the discussion preceding to the lemma, suppose k is even, then $\overline{-1} \equiv \overline{u}^k = (\overline{u}^{\frac{k}{2}})^2 \in \mho_1(\mathbb{Z}_{2^n}^*)$, a contradiction. So, k is odd.

Let $\overline{u} = \overline{a}\overline{b}$ for some $\overline{a} \in \langle \overline{-1} \rangle$ and $\overline{b} \in \langle \overline{5} \rangle$ such that $\overline{u^k} = \overline{-1}$. Then, $\overline{u^k} = \overline{a^k b^k} = \overline{-1}$. There are two choices of $\overline{a}$, that is $\overline{1}$ and $\overline{-1}$. But $\overline{a} \neq \overline{1}$, for otherwise $\overline{b^k} = \overline{-1}$, a contradiction. So, $\overline{b^k} = \overline{1}$ and as a result $\overline{b} = \overline{1}$ and $\overline{u} = \overline{-1}$. □

In a group G, for any element $g \in G$, we use $o(g)$ to denote the order of g. Now we look at the group $M_{p,2}(m,r)$. In the definition of $M_{p,2}(m,r)$, one can see that $r \equiv -1 \pmod{p}$. In particular, if $m = 1$, then $M_{p,2}(m,r)$ is a dihedral group of order $2p$. One may refer to [12] for the classification of the regular balanced Cayley maps of dihedral groups. For the sake of completeness, We restate the result in the following theorem.

Theorem 3.5 ([12, Theorem 3.3])**.** *The dihedral group D_{2p} of order $2p$ has $p-1$ non-isomorphic regular balanced Cayley maps, where p is an odd prime number.*

When $m \geq 2$, we have the following Theorem 3.6.

Theorem 3.6. *Let $G = M_{p,2}(m,r)$, where $m \geq 2$, p is an odd prime and $r \equiv -1 \pmod{p}$. If $p - 1 = 2^e s$, where s is odd, then G has s non-isomorphic regular balanced Cayley maps. In particular, if p is a Fermat prime, then G has exactly one regular balanced Cayley map up to isomorphism.*

Proof. If the orbit of $b^v a^u$ under the action of $\sigma \in \mathrm{Aut}(G)$ is a Cayley subset of G, then the integer v must be odd. In fact, both the subgroups $\langle a \rangle$ and $Z(G) = \langle b^2 \rangle$ are characteristic in G, so $\langle (b^v a^u)^{\langle\sigma\rangle} \rangle$ is a proper subgroup of G if $(v,2) \neq 1$. By Lemma 2.2, there is some $\alpha \in \mathrm{Aut}(G)$ such that $(b^v a^u)^\alpha = b$. According to Lemma 2.9, we only need to consider the orbit of b under the action of σ.

For brevity, we denote the automorphism $\sigma \in \mathrm{Aut}(G)$ satisfying $a^\sigma = a^i$ and $b^\sigma = b^j a^k$ by $\sigma_{i,j,k}$ and $X = b^{\langle\sigma_{i,j,k}\rangle}$ by $X_{i,j,k}$. Let $\rho_{i,j,k}$ be the arrangement of the elements in $X_{i,j,k}$ which respects the order of the elements in the orbit. Assume $X_{i,j,k}$ is a Cayley subset of G for some integer i coprime to p and odd integer j. Note that $k \not\equiv 0 \pmod p$ for otherwise contradicting to the Cayley subset assumption of $X_{i,j,k}$.

In the quotient group $\overline{G} = G/\langle a\rangle$, $\overline{X_{i,j,k}} = \overline{b}^{\langle \overline{\sigma_{i,j,k}}\rangle}$ should be a Cayley subset of $\overline{G}$. Therefore, there exists some integer t such that $\overline{b}^{\overline{\sigma_{i,j,k}}^t} = \overline{b}^{-1}$. Clearly, $\overline{b}^{\overline{\sigma_{i,j,k}}^t} = \overline{b}^{j^t} = \overline{b}^{-1}$, so $j^t \equiv -1 \pmod{2^m}$. From Lemma 3.4, t is odd and $j \equiv -1 \pmod{2^m}$. Moreover, as $X_{i,-1,1}^{\sigma_{k,1,0}} = X_{i,-1,k}$, we may assume $k = 1$. Under these conditions, we only need to pay attention to $X_{i,-1,1}$. By direct enumeration one can easily get

$$b^{\sigma_{i,-1,1}^{\ell}} = b^{(-1)^{\ell}} a^{i^{\ell-1}+i^{\ell-2}+\cdots+i+1},$$

for any positive integer ℓ. Since $X_{i,-1,1}$ is a Cayley subset, there exists some positive integer n such that $b^{\sigma_{i,-1,1}^{n}} = b^{-1}$. So, n is odd and

$$i^{n-1} + i^{n-2} + \cdots + i + 1 \equiv 0 \pmod p.$$

If $i \equiv 1 \pmod p$, then $b^{\sigma_{1,-1,1}^{p}} = b^{-1}$ and

$$X_{1,-1,1} = \{b, b^{-1}a, ba^2, \ldots, ba^{p-1}, b^{-1}, (b^{-1}a)^{-1}, \ldots, (ba^{p-1})^{-1}\}$$

is a Cayley subset of G of valency $2p$.

If $1 < i \leq p-1$, then $i^{n-1} + i^{n-2} + \cdots + i + 1 \equiv 0 \pmod p$ if and only if $i^n \equiv 1 \pmod p$. Let $S = \{x \mid x \in \mathbb{Z}_p^*, o(x) \text{ is odd}\}$, then $|S| = s$. Since n is odd, any i satisfying $i^n \equiv 1 \pmod p$ corresponds to $\overline{i} \in S$. And for any $\overline{i} \in S \setminus \{1\}$, if $o(\overline{i}) = n$, then $b^{\sigma_{i,-1,1}^{n}} = b^{-1}$ and

$$X_{i,-1,1} = \{b, b^{-1}a, ba^{i+1}, b^{-1}a^{i^2+i+1}, \ldots, ba^{i^{n-2}+\cdots+i+1}, b^{-1}, \ldots, (ba^{i^{n-2}+\cdots+i+1})^{-1}\}$$

is a Cayley subset of G of valency $2n$. From all the above, when $i > 1$, $X_{i,-1,1}$ is a Cayley subset of G if and only if $\overline{i} \in S$ and $|X_{i,-1,1}|$ is twice of $o(\overline{i})$.

For any two distinct $\overline{i_1}$ and $\overline{i_2}$ in $S \setminus \{1\}$, Cayley maps $\mathrm{CM}(G, X_{i_1,-1,1}, \rho_{i_1,-1,1})$ and $\mathrm{CM}(G, X_{i_2,-1,1}, \rho_{i_2,-1,1})$ are not isomorphic. Otherwise, according to Lemma 2.8, there exists some $\beta \in \mathrm{Aut}(G)$ such that $b^\beta = b$ and for each $\ell \geq 1$,

$$(b^{(-1)^{\ell}} a^{i_1^{\ell-1}+i_1^{\ell-2}+\cdots+i_1+1})^{\beta} = b^{(-1)^{\ell}} a^{i_2^{\ell-1}+i_2^{\ell-2}+\cdots+i_2+1}.$$

In particular, $(b^{-1}a)^\beta = b^{-1}a$ and therefore β is the identical automorphism. Therefore, G has s non-isomorphic regular balanced Cayley maps. When p is a Fermat prime, then $p-1$ is a power of 2, so G has exactly one regular balanced Cayley map up to isomorphism. □

4 Regular balanced Cayley maps of $M_p(n, m)$

For minimal non-abelian p-group, one may refter to [1, 2] or [14] for the following Lemma 4.1.

Lemma 4.1 ([14, Theorem 2.3.6]). *Let G be a finite p-group, $d(G)$ be the number of elements in a minimal generating subset of G. Then, the followings are equivalent.*

(i) The group G is a minimal non-abelian group;

(ii) $d(G) = 2$ and $|G'| = p$;

(iii) $d(G) = 2$ *and* $Z(G) = \Phi(G)$*, where* $\Phi(G)$ *denotes the Frattini subgroup of* G*.*

Lemma 4.2. *Assume* G *is a finite* p*-group for some prime number* p *and* $d(G) = 2$*. Let* $\beta \in \mathrm{Aut}(G)$*,* $g \in G$ *and* $X = g^{\langle\beta\rangle}$*. If* $G = \langle X\rangle$*, then* $G = \langle g,\ g^\beta\rangle$*.*

Proof. Because $d(G) = 2$, it follows that $\overline{G} = G/\Phi(G) \cong Z_p \times Z_p$. Suppose $\langle g,\ g^\beta\rangle < G$, then in the quotient group the subgroup generated by g and g^β has order p, that is $|\overline{\langle g,\ g^\beta\rangle}| = p$. So, $g^\beta \in \langle g\Phi(G)\rangle$. As $\Phi(G)$ is a characteristic subgroup of G, for each $k > 1$ the element $g^{\beta^k} \in \langle g^{\beta^{k-1}}\Phi(G)\rangle$. Therefore, $X \subseteq \langle g\Phi(G)\rangle$ and then $\langle X\rangle \leq \langle g\Phi(G)\rangle < G$, a contradiction. So, $G = \langle g,\ g^\beta\rangle$. □

Remark Lemma 4.2 may not be true for a non-p-group. For example, the symmetry group S_n can be generated by two elements $(1\ 2)$ and $(1\ 2\ldots n)$. Take $g = (1\ 2) \in S_n$ and β the automorphism of S_n induced from the conjugation by the element $(2\ 3\ldots n)$, then $X = g^{\langle\beta\rangle} = \{(1\ 2), (1\ 3), \ldots, (1\ n)\}$ is a Cayley subset of S_n and $g^\beta = (1\ 3)$. But it is obvious that $S_n \neq \langle(1\ 2), (1\ 3)\rangle$ when $n \geq 4$.

Theorem 4.3. *Let* $G = M_p(n, n)$*, where* $n \geq 2$ *and* p *is an odd prime number. Then, the group* G *does not have regular balanced Cayley maps.*

Proof. Let $N = \langle x \in G \mid x^{p^{n-1}} \in G'\rangle$. According to Lemma 2.7, N is a characteristic subgroup of G. One can see from the defining relations of G that $G' = \langle a^{p^{n-1}}\rangle \cong \mathbb{Z}_p$ and $N = \langle a, b^p\rangle$. Take $\sigma \in \mathrm{Aut}(G)$ such that $a^\sigma = b^{kp}a^i$ and $b^\sigma = b^s a^r$, where the integers i, s, r satisfy the conditions in Lemma 2.3 and especially $s \equiv 1 \pmod p$. Suppose $X = (b^u a^v)^{\langle\sigma\rangle}$ is a Cayley subset of G. Then $b^u a^v \notin N$ and therefore $(u, p) = 1$. In the quotient group $\overline{G} = G/N$, $\overline{X} = \overline{(b^u a^v)^{\langle\sigma\rangle}} = \overline{b^u}^{\langle\bar\sigma\rangle}$ is a Cayley subset of $\overline{G}$. So, there exists some integer n such that $\overline{b^{-u}} = \overline{b^{s^n u}}$. As a result, one can get $s^n u \equiv -u \pmod p$. While $(u, p) = 1$, then $s^n \equiv -1 \pmod p$. But this result contradicts to $s \equiv 1 \pmod p$. □

Theorem 4.4. *Let* $G = M_p(n, m)$*, where* $n \geq 2, m \geq 1, m \neq n$ *and* p *is an odd prime number. Then, the group* G *does not have regular balanced Cayley maps.*

Proof. We firstly assume $m > n$. Set $N = \{x^{p^n} \mid x \in G\}$. By Lemma 2.7, $N = \langle b^{p^n}\rangle$ is a characteristic subgroup of G. The quotient group

$$\overline{G} = G/N = \langle \bar a, \bar b \mid \bar a^{p^n} = \bar b^{p^n} = 1, \bar a^{\bar b} = \bar a^{1+p^{n-1}}\rangle \cong M_p(n, n).$$

According to Theorem 4.3 and Lemma 2.5, G does not have regular balanced Cayley maps.

When $m < n$, suppose there exists some $\sigma \in \mathrm{Aut}(G)$ such that $X = (b^u a^v)^{\langle\sigma\rangle}$ is a Cayley subset of G. Because $Z(G) = \langle a^p,\ b^p\rangle$ is characteristic of G, one can assume $u = 0,\ v = 1$ from the results of Lemma 2.3 and Lemma 2.9. That is, $X = a^{\langle\sigma\rangle}$. Assume $a^\sigma = b^j a^i,\ o(\sigma) = 2k$ and $\tau = \sigma^k$, then $a^\tau = a^{-1},\ (b^j a^i)^\tau = a^{-i}b^{-j}$. Recall that $G' = \langle a^{p^{n-1}}\rangle \cong \mathbb{Z}_p$ and $[a, b^j] \in G' < \langle a\rangle$, so $[a, b^j]^\tau = [a, b^j]^{-1}$. While

$$[a, b^j]^\tau = ([a, a^i][a, b^j][a, b^j, a^i])^\tau = [a, b^j a^i]^\tau = [a^\tau, (b^j a^i)^\tau] = [a^{-1}, a^{-i}b^{-j}] = [a^{-1}, b^{-j}],$$

and $[a^{-1}, b^{-j}]$ belongs to the center, the result

$$[a, b^j]^\tau = [a^{-1}, b^{-j}] = b^{-j}a^{-1}[a^{-1}, b^{-j}]ab^j = [a, b^j]$$

follows. Therefore, $[a, b^j]^{-1} = [a, b^j]$, that is $[a, b^j]^2 = 1$. But the order of $[a, b^j]$ is a power of p which is coprime with 2, we get $[a, b^j] = 1$. And from Lemma 4.2, one can get $G = \langle a, a^\sigma \rangle = \langle a, b^j a^i \rangle$. So G is abelian, a contradiction. Thus in either case, G doesn't have regular balanced Cayley maps. □

Remark 4.5. In the paper of Newman and Xu ([8]), they claimed that for odd primes p every metacyclic p-group is isomorphic to one of the groups

$$G = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, b^{-1}ab = a^{1+p^r} \rangle, \tag{4.1}$$

where r, s, t, u are non-negative integers with r positive and $u \leq r$, and these groups are pairwise non-isomorphic. In the following Lemma 4.6, one will see that the metacyclic p-group has an 'intimate' connection with the minimal non-abelian metacyclic p-group.

Lemma 4.6. *Let G be a metacyclic p-group for some odd prime number p and $N < G'$ be a maximal subgroup of the derived subgroup G'. Then N is a characteristic subgroup of G and the quotient group $\overline{G} = G/N$ is a minimal non-abelian metacyclic p-group.*

Proof. Because G' is cyclic and G' is characteristic of G, it follows that N is also characteristic of G. While N is a proper subgroup of G', the quotient group $\overline{G} = G/N$ is non-abelian and metacyclic, generated by two elements because G is generated by two elements. As $\overline{G}' = \overline{G'} \cong \mathbb{Z}_p$ and so $|\overline{G}'| = p$. The quotient group $\overline{G}$ is mininal non-abelian follows from Lemma 4.1. □

From the results of Lemma 2.5 and Theorems 4.3 and 4.4, we get the following Corollary 4.7.

Corollary 4.7. *For any odd prime number p, the metacyclic p-group does not have regular balanced Cayley maps.*

Theorem 4.8. *Let $G = M_2(n, m)$, where m and n are positive integers and $m > n \geq 2$. Then G does not have regular balanced Cayley maps.*

Proof. According to Lemma 2.3, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^j a^i,\ b^\sigma = b^s a^r\}$, where $(is, 2) = 1$, $1 \leq i \leq 2^n$, $1 \leq s \leq 2^m$, $j = 2^{m-n+1}k$, $0 \leq k < 2^{n-1}$, $1 \leq r \leq 2^n$. From the defining relations of G, one can see that both a^2 and b^2 belong to the center of G. Set $N = \langle a^2,\, b^4 \rangle = \{x \in Z(G) \mid x^{2^{m-2}} = 1\}$. By Lemma 2.7, N is a characteristic subgroup of $Z(G)$. Since $Z(G)$ is characteristic in G, N is characteristic in G. Suppose there is some $\sigma \in \mathrm{Aut}(G)$ and $b^u a^v \in G$ such that $X = (b^u a^v)^{\langle \sigma \rangle}$ is a Cayley subset of G. By Lemma 2.9, one may assume $u = 1$ and $v = 0$, that is, $X = b^{\langle \sigma \rangle}$.

Assume $a^\sigma = b^j a^i$ and $b^\sigma = b^s a^r$, then $4 \mid j$, $(s, 2) = 1$ and so $s^2 \equiv 1 \pmod 4$. According to Lemma 4.2, $G = \langle b, b^s a^r \rangle = \langle b, a^r \rangle$ and so $(r, 2) = 1$. In the quotient group $\overline{G} = G/N$, $\overline{X} = \overline{b^{\langle \sigma \rangle}}$ should be a Cayley subset of $\overline{G}$. Noticing that $2 \mid (s + i)$, $4 \mid j$ and $G' \leq N$, we have $\overline{(b^s a^r)^\sigma} = \overline{(b^s a^r)^s (b^j a^i)^r} = \overline{b^{s^2} a^{rs} b^{jr} a^{ir}} = \overline{b^{s^2+jr} a^{r(s+i)}} = \overline{b^{s^2}}$. Since $o(\overline{b}) = 4$ and $s^2 \equiv 1 \pmod 4$, we have $\overline{b^{s^2}} = \overline{b}$. So, $\overline{X} = \{\overline{b}, \overline{b^s a^r}\}$. But $(r, 2) = 1$, $\overline{b}^{-1} \notin \overline{X}$. Then, $\overline{X}$ is not a Cayley subset, a contradiction. □

Theorem 4.9. *Let $G = M_2(n, m)$, where m and n are positive integers, $n > m + 1$ and $m \geq 2$. Then G does not have regular balanced Cayley maps.*

Proof. In this case, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^j a^i,\ b^\sigma = b^s a^r\}$, where $(is, 2) = 1$, $1 \le i \le 2^n$, $1 \le s \le 2^m$, $1 \le j \le 2^m$, $r = k2^{n-m}$, $0 \le k < 2^m$. Let $N = \langle a^4, b^2\rangle = \{x \in Z(G) \mid x^{2^{n-2}} = 1\}$. According to Lemma 2.7, N is characteristic in $Z(G)$. Since $Z(G)$ is characteristic in G, N is characteristic in G. Similar to the proof of Theorem 4.8, we only need to show that $X = a^{\langle\sigma\rangle}$ is not a Cayley subset of G for any $\sigma \in \mathrm{Aut}(G)$.

Assume $a^\sigma = b^j a^i$ and $b^\sigma = b^s a^r$. Then $(s, 2) = 1$, $4 \mid r$, $(i, 2) = 1$ and so $i^2 \equiv 1 \pmod 4$. And from Lemma 4.2, $G = \langle a, b^j a^i\rangle = \langle a, b^j\rangle$ and so $(j, 2) = 1$. If X is a Cayley subset, then $\overline{X} = \overline{a^{\langle\sigma\rangle}}$ is a Cayley subset of $\overline{G} = G/N$. While from $2 \mid (s+i)$, $4 \mid r$ and $G' \le N$, we have $\overline{(b^j a^i)^\sigma} = \overline{(b^s a^r)^j (b^j a^i)^i} = \overline{b^{sj} a^{rj} b^{ji} a^{i^2}} = \overline{b^{j(s+i)} a^{i^2+rj}} = \overline{a^{i^2}}$. And from $o(\overline{a}) = 4$, $i^2 \equiv 1 \pmod 4$, we have $\overline{a^{i^2}} = \overline{a}$. So, $\overline{X} = \{\overline{a}, \overline{b^j a^i}\}$. But $(j, 2) = 1$ implies $\overline{a}^{\,-1} \notin \overline{X}$. So, $\overline{X}$ is not a Cayley subset, a contradiction. □

In Theorem 4.9, if we allow $m = 1$ and so $n > 2$, then the group $M_2(n, 1)$ belongs to one of the p-groups with a cyclic maximal subgroup which had been considered by D. D. Hou, Y. Wang and H. P. Qu in [6]. We list the result in the following theorem.

Theorem 4.10 ([6, Theorem 3.3])**.** *For positive integers $n > 2$, $M_2(n, 1)$ does not have regular balanced Cayley maps.*

Now, there are still two cases about which we have not said anything, that is $M_2(n, n)$ for $n \ge 2$ and $M_2(n+1, n)$ for $n \ge 1$. One may look back at Lemma 2.3 and can easily see that the automorphism groups of both $M_2(n, n)$ and $M_2(n+1, n)$ are 2-groups.

Theorem 4.11. *Let $G = M_2(n, n)$, $n \ge 2$. Then G has exactly one regular balanced Cayley map of valency 4 in the sense of isomorphism.*

Proof. By Lemma 2.3, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^{2k} a^i, b^\sigma = b^s a^r\}$, where $(si, 2) = 1$, $1 \le i, s, r \le 2^n$, $1 \le k \le 2^{n-1}$, and both a^2 and b^2 belong to $Z(G)$.

We firstly show that if for some $g \in G$ and $\sigma \in \mathrm{Aut}(G)$, $X = g^{\langle\sigma\rangle}$ is a Cayley subset of G, then $|X| = 4$. Set $N = \{x \in G \mid x^{2^{n-2}} \in G'\}$. According to Lemma 2.7, N is a characteristic subgroup of G and $N = \langle a^2, b^4\rangle$. Without loss of generality, we assume $g = b$, then in the quotient group $\overline{G} = G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, the order of $\overline{b}$ is 4. While there are exactly four order-4 elements in $\overline{G}$ and $\overline{X} = \overline{b}^{\langle\overline{\sigma}\rangle}$ is a Cayley subset of $\overline{G}$, $\overline{X}$ should contain all these four elements. Because the order of σ is a power of 2 and b is not involution, according to the results in Lemma 2.5, we have $|X| = |\overline{X}| = 4$.

Take $\sigma_1 \in \mathrm{Aut}(G)$ such that $a^{\sigma_1} = b^2 a^{-1}$ and $b^{\sigma_1} = ba^{2^{n-1}-1}$. By a direct calculation, $X_1 = b^{\langle\sigma_1\rangle} = \{b, ba^{2^{n-1}-1}, b^{-1}, (ba^{2^{n-1}-1})^{-1}\}$ is clearly a Cayley subset of G.

For any $\sigma_2 \in \mathrm{Aut}(G)$ such that $a^{\sigma_2} = b^{2k} a^i$, $b^{\sigma_2} = b^s a^r$, where k, i, s, r satisfy the conditions listed in the first paragraph, and $X_2 = b^{\langle\sigma_2\rangle} = \{b, b^s a^r, b^{-1}, (b^s a^r)^{-1}\}$ is a Cayley subset of G, one may take $\tau \in \mathrm{Aut}(G)$ such that $a^\tau = b^{1-s} a^{-r(1+2^{n-1})}$ and $b^\tau = b$. It is easy to check that $(ba^{2^{n-1}-1})^\tau = b^s a^r$.

Therefore, by Lemma 2.8, the two regular balanced Cayley maps $\mathrm{CM}(G, X_1, \sigma_1|_{X_1})$ and $\mathrm{CM}(G, X_2, \sigma_2|_{X_2})$ are isomorphic. So, G has exactly one regular balanced Cayley map of valency 4 in the sense of isomorphism. □

Theorem 4.12. *Let $G = M_2(n+1, n)$, $n > 1$. Then G has exactly one regular balanced Cayley map up to isomorphism and this map is of valency 4.*

Proof. By Lemma 2.3, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^j a^i, b^\sigma = b^s a^{2k}\}$, where $(si, 2) = 1$, $1 \le i \le 2^{n+1}$, $1 \le j, s, k \le 2^n$ and both a^2 and b^2 belong to $Z(G)$.

We firstly show that if $g \in G$, $\sigma \in \mathrm{Aut}(G)$ and $X = g^{\langle\sigma\rangle}$ is a Cayley subset of G, then $|X| = 4$. Set $N = \{x \in G \mid x^{2^{n-1}} = 1\}$. According to Lemma 2.7, N is a characteristic subgroup of G and $N = \langle a^4, b^2\rangle$. In the quotient group $\overline{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, the order of $\overline{a}$ is 4. There are exactly four order-4 elements in $\overline{G}$, similar to the proof of Theorem 4.11, $\overline{X} = \overline{a}^{\langle\overline{\sigma}\rangle}$ is a Cayley subset of $\overline{G}$ of order 4 and $|X| = |\overline{X}| = 4$.

Take $\sigma_1 \in \mathrm{Aut}(G)$ such that $a^{\sigma_1} = b^{-1}a$ and $b^{\sigma_1} = b^{-1}a^2$. Then, $Y_1 = a^{\langle\sigma_1\rangle} = \{a, b^{-1}a, a^{-1}, (b^{-1}a)^{-1}\}$ is a Cayley subset of G.

For any $\sigma_2 \in \mathrm{Aut}(G)$ such that $a^{\sigma_2} = b^j a^i$, $b^{\sigma_2} = b^s a^{2k}$, where j, i, s, k satisfy the conditions listed in the first paragraph, and $Y_2 = a^{\langle\sigma_2\rangle} = \{a, b^j a^i, a^{-1}, (b^j a^i)^{-1}\}$ is a Cayley subset of G, one may take $\tau \in \mathrm{Aut}(G)$ such that $a^\tau = a$ and $b^\tau = b^{-j}a^{1-i}$. It is easy to check that $(b^{-1}a)^\tau = b^j a^i$. Therefore, the two regular balanced Cayley maps $\mathrm{CM}(G, Y_1, \sigma_1|_{Y_1})$ and $\mathrm{CM}(G, Y_2, \sigma_2|_{Y_2})$ are isomorphic and so G has only one regular balanced Cayley map of valency 4 in the sense of isomorphism. □

To be more clear, we list the number of non-isomorphic regular balanced Cayley maps of minimal non-abelian metacyclic groups in Table 1. For brevity, we use $|G|$, N, RBCM and MNAMG to denote the order of group G, the number of regular balanced Cayley maps up to isomorphism, regular balanced Cayley maps and minimal non-abelian metacyclic groups, respectively.

Table 1: Number of RBCM of MNAMG.

	G	$\lvert G\rvert$	N
1	Q_8	8	1
2	$M_{p,2}(1, r) \cong D_{2p}$	$2p$	$p-1$
3	$M_{p,2}(m, r), m \ge 2, p-1 = 2^e s, (s, 2) = 1$	$2^m p$	s
4	$M_{p,q}(m, r), q \ne 2$	pq^m	0
5	$M_2(2, 1) \cong D_8$	8	2
6	$M_2(n, 1), n > 2$	2^{n+1}	0
7	$M_2(n, n), n \ge 2$	2^{2n}	1
8	$M_2(n+1, n), n \ge 2$	2^{2n+1}	1
9	$M_2(n, m)$, $m \ne n$ and $m \ne n-1$	2^{n+m}	0
10	$M_p(n, m), p \ne 2$	p^{n+m}	0

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A note on extremal results on directed acyclic graphs

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Abstract

This paper studies the maximum number of edges of a Directed Acyclic Graph (DAG) with n vertices in terms of it's longest path ℓ. We prove that in general this number is the Turán number $t(n, l+1)$, the maximum number of edges in a graph with n vertices without a clique of size $\ell+2$. Furthermore, we find the maximum number of edges in a DAG which is either reduced, strongly reduced or extremely reduced and we relate this extremal result with the family of intersection graphs of families of boxes with transverse intersection.

Keywords: Directed graphs, Turán numbers, intersection graphs of families of boxes.

Math. Subj. Class.: 05C20, 52C99

1 Introduction

One of the fundamental results in extremal graph theory is the Theorem of Turán (1941) which states that a graph with n vertices that has more than $t(n, k)$ edges, will always contain a complete subgraph of size $k + 1$. The *Turán graph* $T(n, k)$, is a k–partite graph on n vertices whose partite sets are as nearly equal in cardinality, and has the property

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that contains the maximum posible number of edges $t(n,k)$ of any graph not containing a clique of size $k+1$. It is known that $t(n,k) \le (1-\frac{1}{k})\frac{n^2}{2}$, and equality holds if k divides n. In fact, $\lim_{n\to\infty}\frac{t(n,m)}{n^2/2} = 1-\frac{1}{m}$. See [1].

Turán numbers for several families of graphs have been studied in the context of extremal graph theory, see for example [3] and [4]. In ([2, 7]) the authors analyze, among other things, the intersection graphs of boxes in $\mathbb{R}^d$ proving that, if $\mathcal{T}(n,k,d)$ denotes the maximal number of intersection pairs in a family $\mathcal{F}$ of n boxes in $\mathbb{R}^d$ with the property that no $k+1$ boxes in $\mathcal{F}$ have a point in common (with $n \ge k \ge d \ge 1$), then $\mathcal{T}(n,k,d) = \mathcal{T}(n-k+d,d) + \mathcal{T}(n,k-d+1,1)$, with $\mathcal{T}(n,k,1) = \binom{n}{2} - \binom{n-k+1}{2}$ being the precise bound in dimension 1 for the family of interval graphs.

Turán numbers have played and important role for several variants of the Turán Theorem and its relation with the fractional Helly Theorem (see [5, 6]).

The purpose of this paper is to study the maximum number of edges in directed acyclic graphs with n vertices with respect to it's longest path. That turns out to be related with the extremal behavior of the family of intersection graphs for a collection of boxes in $\mathbb{R}^2$ with transverse intersection.

The first result, Theorem 2.10, states that in a directed acyclic graph with n vertices, if the longest path has length ℓ, then the maximal number of edges is the Turán number $t(n,\ell+1)$.

Theorem 3.19 and its Corollaries state that given a directed acyclic graph $\vec{G}$ with n vertices such that the longest path has length ℓ, then if $\vec{G}$ is either reduced, strongly reduced or extremely reduced, $\vec{G}$ has at most $t(n-\ell+1,2) + \mathcal{T}(n,\ell,1)$ edges, where again $\mathcal{T}(n,\ell,1)$ denotes the maximal number of intersecting pairs in a family $\mathcal{F}$ of n intervals in $\mathbb{R}$ with the property that no $\ell+1$ intervals in $\mathcal{F}$ have a point in common.

In fact, this bound is best possible. The bound is reached by the intersection graph of a collection of boxes in $\mathbb{R}^2$ with transverse intersection (see Proposition 4.6). This graph is extremely reduced (and thus is also strongly reduced and reduced, see Proposition 4.4).

2 Directed acyclic graphs

By a *directed acyclic graph*, DAG, we mean a simple directed graph without directed cycles. A DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, with vertex set $\mathcal{V}$ and directed edge set $\vec{\mathcal{E}}$ is *transitive* if for every $x,y,z \in \mathcal{V}$, if $\{x,y\},\{y,z\} \in \vec{\mathcal{E}}$ then $\{x,z\} \in \vec{\mathcal{E}}$.

Definition 2.1. A *topological order* of a directed graph $\vec{G}$ is an ordering $\mathcal{O}$ of its vertices $\{v_1, v_2, \dots, v_n\}$ so that for every edge $\{v_i, v_j\}$ then $i<j$.

The following proposition is a well known result:

Proposition 2.2. *A directed graph $\vec{G}$ is a DAG if and only if $\vec{G}$ has a topological order.*

Given any set X, by $|X|$ we denote the cardinal of X.

Definition 2.3. The *indegree*, $\deg^-(v)$, of a vertex v is the number of directed edges $\{x,v\}$ with $x \in \mathcal{V}$. The *outdegree*, $\deg^+(v)$, of a vertex v is the number of directed edges $\{v,x\}$ with $x \in \mathcal{V}$. Notice that each directed edge $\{v,w\}$ adds one outdegree to the vertex v and one indegree to the vertex w. Therefore, $\sum_{v\in\mathcal{V}}\deg^+(v) = \sum_{v\in\mathcal{V}}\deg^-(v) = |(\vec{\mathcal{E}})|$.

The degree of a vertex is $\deg(v) = \deg^-(v) + \deg^+(v)$.

Definition 2.4. A vertex v such that $\deg^-(v) = 0$ is called *source*. A vertex v such that $\deg^+(v) = 0$ is called *sink*.

It is well known that every DAG $\vec{G}$ has at least one source and one sink.

Definition 2.5. Given a DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, a directed path $\vec{\gamma}$ in G is a sequence of vertices $\{v_0, \ldots, v_n\}$ such that $\{v_{i-1}, v_i\} \in \vec{\mathcal{E}}$ for every $1 \le i \le n$. Here, $\vec{\gamma}$ has *length* n, and *endpoint* v_n.

Observe that since DAG's are acyclic, all the vertices on a directed path are different.

Definition 2.6. Given a DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, let $\Gamma \colon \mathcal{V} \to \mathbb{N}$ be such that $\Gamma(v) = k$ if there exists a directed path $\vec{\gamma}$ in G of length k with endpoint v and there is no directed path $\vec{\gamma}'$ with endpoint v and length greater than k.

Definition 2.7. Given a DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ suppose that $\ell = \max\{k \mid \Gamma(v) = k$ for every $v \in \mathcal{V}\}$. Notice that, since $\vec{G}$ has no directed cycle, $\ell \le |\mathcal{V}|$. Then, let us define a partition $P_\Gamma = \{V_0, \ldots, V_\ell\}$ of $\mathcal{V}$ such that $V_i := \{v \in \mathcal{V} \mid \Gamma(v) = i\}$ for every $0 \le i \le \ell$.

Notice that V_0 is exactly the set of sources in $\vec{G}$ and V_ℓ is contained in the set of sinks in G.

Lemma 2.8. *V_i is nonempty for every $0 \le i \le \ell$.*

Proof. Let $\{v_0, \ldots, v_\ell\}$ be a directed path of maximal length in $\vec{G}$. Clearly, for every $0 \le i \le \ell$, $v_i \notin V_j$ if $j < i$. Suppose $v_i \in V_j$ with $i < j \le \ell$. Then, there is a directed path $\{v'_0, \ldots, v'_j = v_i\}$ with $j > i$ and $\{v'_0, \ldots, v'_j, v_{i+1}, \ldots, v_\ell\}$ is a directed path with length $j + l - i > \ell$ which contradicts the hypothesis. □

Lemma 2.9. *The induced subgraph with vertices V_i, $G[V_i]$, is independent (has no edges) for every i.*

Proof. Let $v_i, v'_i \in V_i$ and suppose $\{v_i, v'_i\} \in \vec{\mathcal{E}}$. Let $\{v_0, \ldots, v_i\}$ be a path of length i with endpoint v_i. Then, $\{v_0, \ldots, v_i, v'_i\}$ defines a directed path of length $i+1$ which contradicts the fact that $v'_i \in V_i$. □

Recall that $T(n, \ell)$ denote the ℓ-partite Turán graph with n vertices and $t(n, \ell)$ denote the number of edges of $T(n, \ell)$.

Theorem 2.10. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices such that the longest directed path has length ℓ. Then, $\vec{G}$ has at most $t(n, \ell + 1)$ edges.*

Proof. Consider the partition $P_\Gamma = \{V_0, \ldots, V_\ell\}$ of $\mathcal{V}$. By Lemma 2.9, this defines an $(\ell+1)$-partite directed graph. Thus, neglecting the orientation we obtain a complete $(\ell+1)$-partite graph with partition sets $V_0, \ldots, V_\ell$. Therefore, the number of edges is at most $t(n, \ell + 1)$. □

Remark 2.11. It is readily seen that the bound in Theorem 2.10 is best possible. Consider the Turán graph $T(n, \ell + 1)$ and any ordering of the $\ell + 1$ independent sets $V_0, \ldots, V_\ell$. Then, for every edge $\{v_i, v_j\}$ in $T(n, \ell)$ with $v_i \in V_i$, $v_j \in V_j$ and $i < j$ let us assume the orientation $\{v_i, v_j\}$. It is trivial to check that the resulting graph is a DAG with $t(n, \ell + 1)$ edges.

3 Reduced, strongly reduced and extremely reduced DAGs

Let $\mathcal{O}$ be a topological ordering in a DAG $\vec{G}$. Given any two vertices v, w, and two directed paths in $\vec{G}$, γ,γ', from v to w, let us define $\gamma \cup_{\mathcal{O}} \gamma'$ as the sequence of vertices defined by the vertices in $\gamma \cup \gamma'$ in the order given by $\mathcal{O}$. Of course, this need not be, in general, a directed path from v to w.

Let $\Gamma(u, v)$ be the set of all directed paths from u to v. Let $\cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\}$ represent the sequence of all the vertices from the paths in $\Gamma(u, v)$ ordered according to $\mathcal{O}$.

Definition 3.1. A finite DAG $\vec{G}$ is *strongly reduced* if for any topological ordering $\mathcal{O}$ of $\vec{G}$, every pair of vertices, v, w, and every pair of directed paths, γ, γ', from v to w, then $\gamma \cup_{\mathcal{O}} \gamma'$ defines a directed path from v to w.

Remark 3.2. Let $\vec{G}$ be DAG. Given any two vertices v, w, and two directed paths in $\vec{G}$, γ,γ', from v to w, let us define $\gamma \leq \gamma'$ if every vertex in γ is also in γ'. Clearly, "$\leq$" is a partial order.

Definition 3.3. A vertex w is *reachable* from a vertex v if there is a directed path from v to w.

Proposition 3.4. *Given a finite DAG $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, the following properties are equivalent:*

i) For every pair of vertices v, w and every pair of paths, γ, γ', from v to w, there exists a directed path from v to w, γ'', such that $\gamma, \gamma' \leq \gamma''$.

ii) For every pair of vertices v, w such that w is reachable from v, there is a directed path from v to w, γ_M, such that for every directed path, γ, from v to w, $\gamma \leq \gamma_M$.

iii) For every topological ordering $\mathcal{O}$ of $\vec{G}$ and any pair of vertices v, w, $\cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\}$ defines a directed path from v to w.

Proof. Since the graph is finite and the relation '$\leq$' is transitive, *i)* and *ii)* are trivially equivalent.

If *ii)* is satisfied, then it is trivial to see that $\cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\} = \gamma_M$ and *iii)* is satisfied. Also, it is readily seen that *iii)* implies *ii)* taking $\gamma_M := \cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\}$. □

Definition 3.5. We say that a finite DAG $\vec{G}$ is *reduced* if it satisfies any of the properties from Proposition 3.4.

Proposition 3.6. *If a finite DAG $\vec{G}$ is strongly reduced, then $\vec{G}$ is reduced.*

Proof. Since the graph is finite, it is immediate to see that being strongly reduced implies *iii)*. □

Remark 3.7. The converse is not true. The graph in the left from Figure 1 is clearly reduced. Notice that the directed path $\gamma_M := \{v_1, v_2, v_3, v_4, v_5\}$ is an upper bound for every directed path from v_1 to v_5. However, if we consider the directed paths $\gamma = \{v_1, v_2, v_5\}$ and $\gamma' = \{v_1, v_4, v_5\}$ with the topological order $\mathcal{O} = \{v_1, v_2, v_3, v_4, v_5\}$, then $\gamma \cup_{\mathcal{O}} \gamma' = \{v_1, v_2, v_4, v_5\}$ which is not a directed path.

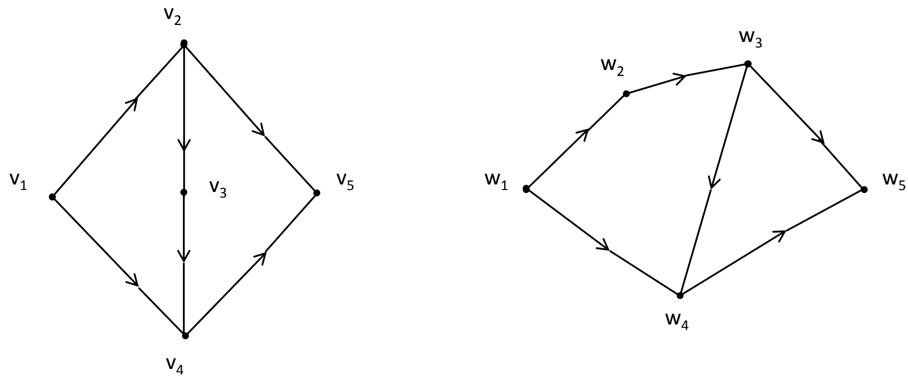

Figure 1: Being reduced does not imply being strongly reduced and being strongly reduced does not imply being extremely reduced.

Definition 3.8. Given a finite DAG $\vec{G}$ and a vertex $v \in \mathcal{V}$ we say that w is an *ancestor* of v if there is a directed path $\{w = v_0, \dots, v_k = v\}$ and w is a *descendant* of v if there is a directed path $\{v = v_0, \dots, v_k = w\}$.

Definition 3.9. We say that a finite DAG $\vec{G}$ is *extremely reduced* if for every pair of non-adjacent vertices x, y, if x, y have a common ancestor, then they do not have a common descendant.

Proposition 3.10. *If a DAG $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ is extremely reduced, then it is strongly reduced.*

Proof. Let $\gamma = \{v, v_1, \dots, v_n, w\}$ and $\gamma' = \{v, w_0, \dots, w_m, w\}$ be two directed paths in $\vec{G}$ from v yo w. Let $\mathcal{O}$ be any topological order in $\vec{G}$ and consider $\gamma \cup_{\mathcal{O}} \gamma' = \{v, z_1, \dots, z_k, w\}$. First, notice that z_1 is either v_1 or w_1. Therefore, $\{v, z_1\} \in \vec{\mathcal{E}}$. Also, z_k is either v_n or w_m, and $\{z_k, w\} \in \vec{\mathcal{E}}$. Now, for every $1 < i \le k$, let us see that $\{z_{i-1}, z_i\} \in \vec{\mathcal{E}}$. If $z_{i-1}, z_i \in \gamma$ or $z_{i-1}, z_i \in \gamma'$, then they are consecutive vertices in a directed path and we are done. Otherwise, since z_{i-1}, z_i have a common ancestor v and a common descendant w, then there is a directed edge joining them and, since z_{i-1}, z_i are sorted by a topological order, $\{z_{i-1}, z_i\} \in \vec{\mathcal{E}}$. □

Remark 3.11. The converse is not true. The graph in the right from Figure 1 is strongly reduced. However, vertices w_2 and w_4 are not adjacent and have a common ancestor and a common descendent.

Proposition 3.12. *If $\vec{G}$ is transitive, then the following properties are equivalent:*

- *$\vec{G}$ is extremely reduced,*
- *$\vec{G}$ is strongly reduced,*
- *$\vec{G}$ is reduced.*

Proof. By Proposition 3.10 if $\vec{G}$ is extremely reduced, then it is strongly reduced. By Proposition 3.6, if $\vec{G}$ is strongly reduced, then it is reduced.

Suppose $\vec{G}$ is reduced and suppose that two vertices x, y have a common ancestor, v, and a common descendant, w. Then, there are two directed paths γ, γ' from v to w such that $x \in \gamma$ and $y \in \gamma'$. By property *i)* in Proposition 3.4, there exists a path γ'' in $\vec{G}$ from v to w such that $\gamma, \gamma' \leq \gamma''$. In particular, $x, y \in \gamma''$. Therefore, either x is reachable from y or y is reachable from x in $\vec{G}$. Since $\vec{G}$ is transitive, this implies that x, y are adjacent. Therefore, $\vec{G}$ is extremely reduced. □

Definition 3.13. Given a DAG $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, the graph with vertex set $\mathcal{V}$ and edge set $\vec{\mathcal{E}}' := \vec{\mathcal{E}} \cup \{\{v, w\} \mid w \text{ is reachable from } v\}$ is called the *transitive closure* of $\vec{G}$, $T[\vec{G}]$.

It is immediate to check the following:

Proposition 3.14. *Given any DAG $\vec{G}$, $T[\vec{G}]$ is transitive.*

Proposition 3.15. *If a DAG $\vec{G}$ is reduced, then the transitive closure $T[\vec{G}]$ is also reduced.*

Proof. Suppose $\vec{G}$ satisfies *i)* in Proposition 3.4 and let $\gamma = \{v = v_0, \dots, v_n = w\}$, $\gamma' = \{v = w_0, \dots, w_m = w\}$ be any pair of paths from v to w in $T[\vec{G}]$. Therefore, v_i is reachable from v_{i-1} in $\vec{G}$ for every $1 \leq i \leq n$ and w_i is reachable from w_{i-1} in $\vec{G}$ for every $1 \leq i \leq m$. Thus, there exist a path γ_0 in $\vec{G}$ such that $\gamma \leq \gamma_0$ and a path γ_0' in $\vec{G}$ such that $\gamma' \leq \gamma_0'$. By property *i)*, there is a directed path from v to w such that $\gamma_0, \gamma_0' \leq \gamma_0''$. Therefore, $\gamma, \gamma' \leq \gamma_0''$ and $T[\vec{G}]$ satisfies *i)*. □

Then, from Propositions 3.6, 3.10, 3.12, 3.14 and 3.15:

Corollary 3.16. *If a DAG $\vec{G}$ is reduced, then the transitive closure $T[\vec{G}]$ is extremely reduced and strongly reduced. In particular, if $\vec{G}$ is extremely reduced or strongly reduced, then $T[\vec{G}]$ is extremely reduced and strongly reduced.*

Let us recall that

$$\mathcal{T}(n, \ell, 1) = \binom{n}{2} - \binom{n - \ell + 1}{2} = (n - \ell + 1)(\ell - 1) + \frac{(\ell - 1)(\ell - 2)}{2} \tag{3.1}$$

As it was proved in [7]:

Lemma 3.17. *For $n \geq \ell$ and $d \geq 1$,*

$$\mathcal{T}(n + d, \ell, 1) - \mathcal{T}(n, \ell, 1) = d(\ell - 1).$$

In particular, $\mathcal{T}(n + 2, \ell, 1) - \mathcal{T}(n, \ell, 1) = 2(\ell - 1)$.
Also, from [7]:

Lemma 3.18. *For $1 \leq d \leq n$,*

$$t(n + d, d) - t(n, d) = (d - 1)n + \binom{d}{2}$$

In particular, $t(n + 2, 2) - t(n, 2) = n + 1$.

Theorem 3.19. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices and such that the longest directed path has length $\ell \geq 1$. If $\vec{G}$ is extremely reduced, then $\vec{G}$ has at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

Proof. Let us prove the result by induction on n. Suppose that the longest directed path has length ℓ.

First, let us see that the result is true for $n = \ell + 1$ and $n = \ell + 2$.

If $n = \ell + 1$ then $\vec{G}$ has at most $\frac{\ell(\ell+1)}{2} = \frac{(\ell-2)(\ell-1)}{2} + 2(\ell - 1) + 1 = \mathcal{T}(n, \ell, 1) + t(n - \ell + 1, 2)$ edges. The last equation follows immediately from (3.1) and the fact that $t(2, 2) = 1$.

If $n = \ell + 2$ then there are $\ell + 1$ vertices which define a directed path $\gamma = \{v_0, \ldots, v_\ell\}$ and one vertex w such that neither $\{w, v_0\}$ nor $\{v_\ell, w\}$ is a directed edge. Then, the partition $P_\Gamma = \{V_0, \ldots, V_\ell\}$ of $\vec{G}$ satisfies that $v_i \in V_i$ for every $0 \le i \le \ell$. Also, $w \in V_j$ for some $0 \le j \le \ell$ and $\{w, v_j\}$, $\{v_j, w\}$ are not directed edges. Hence, $\deg(w) \le \ell$. Therefore, $\vec{G}$ has at most $\frac{\ell(\ell+1)}{2} + \ell = \frac{(\ell-2)(\ell-1)}{2} + 3(\ell - 1) + 2 = \mathcal{T}(n, \ell, 1) + t(n - \ell + 1, 2)$ edges. The last equation follows immediately from (3.1) and the fact that $t(3, 2) = 2$.

Suppose the induction hypothesis holds when the graph has n vertices and let $\#(\mathcal{V}) = n + 2$. Also, by Proposition 3.15 we may assume that the graph is transitive.

Consider the partition $P_\Gamma = \{V_0, \ldots, V_\ell\}$ of $\mathcal{V}$. Let $\#(V_i) = r_i$. Let $v \in V_0$ and w be any sink of $\vec{G}$. Consider any pair of vertices $v_i, v_i' \in V_i$. Since $\vec{G}$ is extremely reduced and every two vertices in V_i are non-adjacent, v_i, v_i' can not be both descendants of v and ancestors of w simultaneously. Hence, the number of edges joining the sets $\{v, w\}$ and V_i are at most $r_i + 1$. Therefore, there are at most $n + \ell - 1$ edges joining $\{v, w\}$ and $G \setminus \{v, w\}$

Since $G \setminus \{v, w\}$ has n vertices, by hypothesis, it contains at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.

Finally, there is at most 1 edge in the subgraph induced by $\{v, w\}$.

Therefore, by Lemmas 3.17 and 3.18, $|\vec{E}(G)| \le t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1) + n + \ell = t(n - \ell + 3, 2) + \mathcal{T}(n + 2, \ell, 1)$. □

By Corollary 3.16 we know that the extremal graph for reduced and strongly reduced graphs is transitive. Thus, from Theorem 3.19 and Proposition 3.12 we obtain the following.

Corollary 3.20. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices and such that the longest directed path has length $\ell \ge 1$. If $\vec{G}$ is reduced, then $\vec{G}$ has at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

Corollary 3.21. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices and such that the longest directed path has length $\ell \ge 1$. If $\vec{G}$ is strongly reduced, then $\vec{G}$ has at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

4 Directed intersection graphs of boxes

Definition 4.1. Let $\mathcal{R}$ be a collection of boxes with parallel axes in $\mathbb{R}^2$. Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a directed graph such that $\mathcal{V} = \mathcal{R}$ and given $R, R' \in \mathcal{R}$ with $R = I \times J$, $R' = I' \times J'$ then $\{R, R'\} \in \vec{\mathcal{E}}$ if and only if $I \subset I'$ and $J' \subset J$ (i.e. there is an edge if and only if the intersection is *transverse* and the order is defined by the subset relation in the first coordinate). Let us call $\vec{G}$ the *directed intersection graph* of $\mathcal{R}$.

Definition 4.2. Let $\mathcal{R}$ be a collection of boxes with parallel axes in $\mathbb{R}^2$. We say that $\mathcal{R}$ is a collection with *transverse intersection* if for every pair of boxes either they are disjoint or their intersection is transverse.

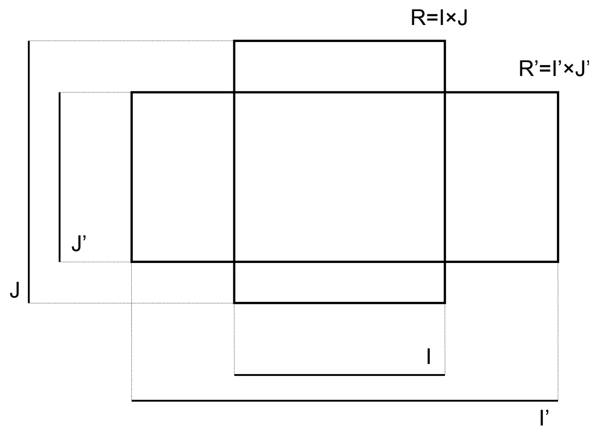

Figure 2: The transverse intersection above induces a directed edge $\{R, R'\}$.

Proposition 4.3. *Let $\mathcal{R}$ be a collection of boxes with parallel axes in $\mathbb{R}^2$ and $\vec{G}$ be the induced directed intersection graph. If two vertices v, w have both a common ancestor and a common descendant in $\vec{G}$, then the corresponding boxes R_v, R_w intersect.*

Proof. Let a be a common ancestor and $R_a = I_a \times J_a$ be the corresponding box. Let b be a common descendant and $R_b = I_b \times J_b$ be the corresponding box. Then if $R_v = I_v \times J_v$, $R_w = I_w \times J_w$ are the boxes corresponding to v and w respectively, it follows by construction that $I_a \subset I_v, I_w$ and $J_b \subset J_v, J_w$. Therefore, $I_a \times J_b \subset R_v, R_w$ and $R_v \cap R_w \neq \emptyset$. □

Proposition 4.4. *If $\mathcal{R}$ is a collection of boxes with parallel axes in $\mathbb{R}^2$ with transverse intersection, then the induced directed intersection graph G is extremely reduced and transitive.*

Proof. First notice that the transitivity holds simply by the transverse intersection property. Let v, w be two vertices such that there is no edge joining them. This means, by construction, that their corresponding boxes do not have a transverse intersection. Since $\mathcal{R}$ has transverse intersection, this implies that these boxes do not intersect. Thus, by Proposition 4.3, if v, w have a common ancestor, then they can not have a common descendant. □

Remark 4.5. Consider the bipartite graph G from Figure 3 with the partition given by $\{\text{letters}, \text{numbers}\}$ and assume all directed edges go from letters into numbers. Note that G is extremely reduced, transitive and acyclic. Notice that the induced subgraphs given by the sets $C_1 := \{1, 2, A, B\}$, $C_2 := \{3, 4, C, D\}$ and $C_3 := \{5, 6, E, F\}$ are three cycles of length 4. Furthermore the induced subgraph given by the set of vertices $\{1, 2, 3, 4, 8, 9, A, B, C, D, H, I\}$ is realizable as boxes in $\mathbb{R}^2$ (see Figure 4) note, that contains C_1 and C_2 and its realization force one of them to be inside the other say C_1 inside C_2. Similarly the induced subgraphs given by the set of vertices $\{1, 2, 5, 6, A, B, E, F, 7, 12, G, L\}$ and the set of vertices $\{3, 4, 5, 6, C, D, E, F, 10, 11, J, K\}$ forces necessarily a system of tree squares one inside the other. However, intervals given by $\{7, 8, 9, 10, 11, 12\}$ and $\{G, H, I, J, K, L\}$ are forced to have more intersections that those given by the graph. In

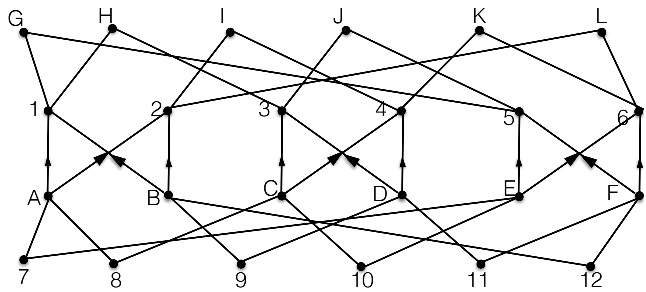

Figure 3: The bipartite, transitive, and extremely reduced DAG, G with partition given by $\{\text{letters}, \text{numbers}\}$ and edges directed from letters into numbers. This graph is not realizable as a family of boxes in $\mathbb{R}^2$.

other words, there is no family of boxes (or intervals) that realizes such a graph or for which it is induced the graph G. Then, the converse of Proposition 4.4 is not true.

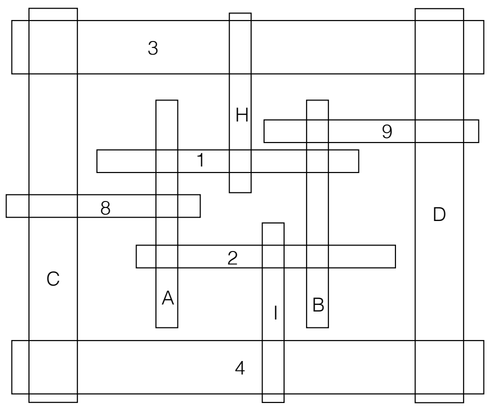

Figure 4: Realization in $\mathbb{R}^2$ of the induced subgraph with vertices $\{1, 2, 3, 4, 8, 9, A, B, C, D, H, I\}$ of the graph shown in Figure 3.

Let $G[r, l, s]$ be the graph, $G(\mathcal{V}, \vec{\mathcal{E}})$, such that:

- $\mathcal{V} = \{x_1, \ldots, x_r, y_1, \ldots, y_{l-1}, z_1, \ldots, z_s\}$
- $\{x_i, x_j\} \notin \vec{\mathcal{E}}$ for any $i \neq j$,
- $\{z_i, z_j\} \notin \vec{\mathcal{E}}$ for any $i \neq j$,
- $\{x_i, y_j\} \in \vec{\mathcal{E}}$ for every i, j,
- $\{y_i, y_j\} \in \vec{\mathcal{E}}$ for every $i < j$,
- $\{y_i, z_j\} \in \vec{\mathcal{E}}$ for every i, j,
- $\{x_i, z_j\} \in \vec{\mathcal{E}}$ for every i, j.

This is the directed intersection graph from the collection of boxes in Figure 5.

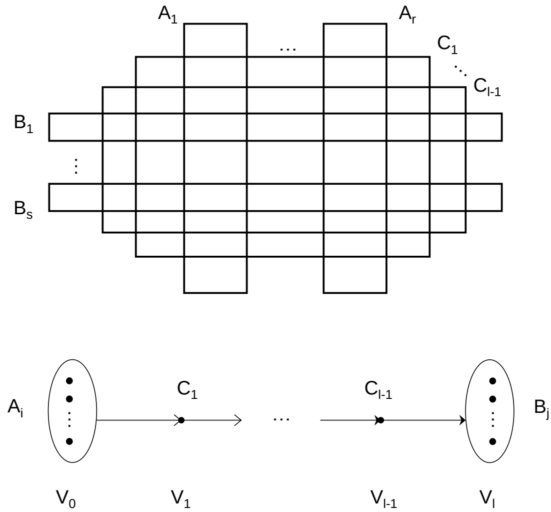

Figure 5: The graph $G[r, l, s]$ corresponds to the directed intersection graph of the collection in the figure where $x_i \sim A_i$, $y_j \sim C_j$ and $z_k \sim B_k$. Notice that the graph is transitive although not every edge is represented in the figure.

By Proposition 4.4, $G[r, l, s]$ is a transitive extremely reduced DAG. In particular, $G[r, l, s]$ is strongly reduced and reduced.

Now, to prove that the bound obtained in Theorem 3.19 and its corollaries is best possible, it is immediate to check the following:

Proposition 4.6. *If $n - \ell$ is even, $G[\frac{n-\ell}{2}, \ell, \frac{n-\ell}{2}]$ has $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges. If $n - \ell$ is odd, $G[\frac{n-\ell+1}{2}, \ell, \frac{n-\ell-1}{2}]$ has $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

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