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A New Strategy to Tackle the Backlog

In 2008 when our journal was first launched, 20 papers were published. Ten years later, 60 papers will be published, producing a threefold increase. Also we started a new sister journal ADAM, with a somewhat broader scope, featuring 20 papers in 2018. And on top of this, there are still about 100 papers in the editorial process for AMC. Many of these papers will be eventually published in 2019, some of them even in 2020.

This year we adopted a strict policy of controlling the number of papers filtered from the initial quick assessment into the actual refereeing phase. If we want to reduce the backlog, then only about five papers per month can be processed, and otherwise we will have no choice but to decline the rest, or divert them to ADAM. This policy should ensure that there will be no further increase in the backlog.

But actually we would like to significantly reduce the backlog, and so we decided to increase the number of published papers from 15 to 20 per issue (and 60 to 80 in total) for the year 2019. In the meantime we expect that ADAM will also gain in reputation and popularity, and this should help us reduce the large influx of papers to the editorial system of AMC.

Klavdija Kutnar, Dragan Marušič and Tomaž Pisanski

Editors In Chief

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Linking rings structures and semisymmetric graphs: Combinatorial constructions

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Abstract

This paper considers combinatorial methods of constructing LR structures: two isolated constructions, RC and SoP, two closely related constructions, $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ and $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$ using cycle decompositions of tetravalent graphs, a generalization of those, $\mathrm{CS}(\Gamma, \mathcal{B}, k)$ for $k > 2$, and finally a construction LDCS relating to cycle decompositions of graphs of higher even valence. This last construction is used to classify all LR structures of types $\{3, *\}$ or $\{4, *\}$.

Keywords: Graph, automorphism group, symmetry, locally arc-transitive graph, semisymmetric graph, cycle structure, linking ring structure.

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1 Introduction

1.1 History

An *LR structure* is a finite, simple, connected, tetravalent vertex-transitive graph together with a decomposition of its edge-set into cycles that satisfies certain symmetry conditions

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(see Section 1.2 for details). It can also be seen as one of the possible symmetry types of the tetravalent vertex-transitive graphs, namely the one in which the stabiliser of a vertex in some vertex-transitive group of symmetries acts on the neighbourhood as the Klein 4-group in its intransitive action on four points (see [10, Section 1] for more on this topic).

This paper is the third in a "trilogy" developing the theory of LR structures. In the first paper [10], we introduced LR structures and we explained their importance in the search for semisymetric graphs via the function $\mathbb{P}$ which creates a semisymmetric graph from such a structure. In that paper, we also introduced two related families of LR structures and discussed certain double covers of LR structures. All definitions from [10] appear in the next section; please consult [10] for more details.

In the second paper, [9], we showed several purely algebraic constructions for such structures. We first gave a quite general approach to constructing an LR structure from a group having certain automorphisms. We then applied these techniques to several families of abelian groups and to dihedral groups.

We noted in [9] that several of these constructions gave semisymmetric graphs having large vertex-stabilizers. It has been known for decades that the size of a vertex-stabilizer in a *cubic* edge-transitive graph is at most 48 in the dart transitive case (see [12]) and at most 384 in the semisymmetric case (see [4]). No such absolute bounds exist for tetravalent edge-transitive graphs. However, recently Spiga, Verret and the first mentioned author of this paper have discovered efficient bounds on the size of the vertex-stabiliser in terms of the number of the vertices for the case of the tetravalent dart-transitive graphs [7] and for the case of the tetravalent half-arc transitive graphs [11]. Semisymmetric graphs are thus the last remaining case of tetravalent edge-transitive graphs for which no good bounding behavior on the size of the vertex stabilizer is known. Perhaps the examples in [9] and especially the characterisation of the LR structures of type $\{4, q\}$ in this paper will yield some insight into the phenomenon of the large vertex stabilizer in LR structures, and consequently, the tetravalent semisymmetric graphs of girth 4 (see Section 5).

1.2 Definitions

Unless explicitly stated otherwise, all the graphs in this paper are finite, simple and connected. Let Λ be a regular tetravalent graph and $\mathcal{C}$ a partition of its edge-set $E(\Lambda)$ into cycles. We shall call such a pair $(\Lambda, \mathcal{C})$ a *cycle decomposition*.

Two edges of Λ will be called *opposite at vertex* v, if they are both incident with v and belong to the same element of $\mathcal{C}$. The *partial line graph* of a cycle decomposition $(\Lambda, \mathcal{C})$ is the graph $\mathbb{P}(\Lambda, \mathcal{C})$ whose vertices are edges of Λ, and two edges of Λ are adjacent as vertices in $\mathbb{P}(\Lambda, \mathcal{C})$ whenever they share a vertex in Λ and are not opposite at that vertex. A *symmetry* of $(\Lambda, \mathcal{C})$ is any permutation of the vertices of Λ which preserves $\mathcal{C}$. The set of all such is called $\mathrm{Aut}(\Lambda, \mathcal{C})$.

Because the two edges at v that belong to one cycle are connected to both of the edges in the other cycle containing v, the edges at each vertex of Λ form a 4-cycle in $\mathbb{P}(\Lambda, \mathcal{C})$. Thus, the girth of $\mathbb{P}(\Lambda, \mathcal{C})$ is usually 4 and never any larger.

A cycle decomposition $(\Lambda, \mathcal{C})$ is said to be *flexible* provided that for every vertex v and each edge e containing v, there is a symmetry which fixes pointwise the cycle $D \in \mathcal{C}$ containing e and interchanges the other two neighbors of v. The edges joining v to those neighbors are in some other cycle C of $\mathcal{C}$. The symmetry then reverses the cycle C and is called a *C-swapper at v*.

A cycle decomposition $(\Lambda, \mathcal{C})$ is called *bipartite* if $\mathcal{C}$ can be partitioned into two subsets

$\mathcal{G}$ and $\mathcal{R}$ so that each vertex of Λ meets one cycle from $\mathcal{G}$ and one from $\mathcal{R}$. Especially in constructions, we will refer to the edges of the cycles in $\mathcal{G}$ and those in $\mathcal{R}$ as *green* and *red*, respectively. The largest subgroup of $\mathrm{Aut}(\Lambda, \mathcal{C})$ preserving each of the sets $\mathcal{G}$ and $\mathcal{R}$ will be denoted by $\mathrm{Aut}^+(\Lambda, \mathcal{C})$, and we will think of it as the color-preserving group of $(\Lambda, \mathcal{C})$.

Definition 1.1. A cycle decomposition $(\Lambda, \mathcal{C})$ is called a *linking rings structure* (or briefly, an *LR structure*) provided that it is bipartite, flexible and $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ is transitive on the vertices of Λ.

Note that $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ acts transitively on the darts of each color class, and that (since Λ is assumed to be connected) its index in $\mathrm{Aut}(\Lambda, \mathcal{C})$ is at most 2. If there is a symmetry of Λ which preserves $\mathcal{C}$ but interchanges the edge color sets $\mathcal{G}$ and $\mathcal{R}$ (that is, if $\mathrm{Aut}^+(\Lambda, \mathcal{C}) \neq \mathrm{Aut}(\Lambda, \mathcal{C})$), then we say that $(\Lambda, \mathcal{C})$ is *self-dual*.

Since the color preserving group $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ of an LR structure $(\Lambda, \mathcal{C})$ is transitive on $\mathcal{R}$ and on $\mathcal{G}$, all cycles in $\mathcal{R}$ must have the same length, say p, and all cycles in $\mathcal{G}$ must be of the same length, say q. We then say that the LR structure $(\Lambda, \mathcal{C})$ is of *type* $\{p, q\}$. For a self-dual structure, of course, $p = q$.

Two LR structures $(\Lambda_1, \mathcal{C}_1)$ and $(\Lambda_2, \mathcal{C}_2)$ are isomorphic whenever there is a graph isomorphism from Λ_1 to Λ_2 which maps cycles in $\mathcal{C}_1$ to cycles in $\mathcal{C}_2$.

We define the *joining sequences* of an LR structure to be $J_r = [s_r, d_r, w_r]$ and $J_g = [s_g, d_g, w_g]$ where: s_r is the least s such that some two red cycles are joined by two green paths of length s. The number d_r is the least d such that two such green paths have starting points that are d apart on one of the red cycles. If the two paths are j apart on the other red cycle, a symmetry argument shows that d must divide j, and then we set $w_r = \frac{j}{d}$; see Figure 1. In the case that no two red cycles are joined by two green paths of the same length, we declare J_r to be $[0, 0, 0]$. The numbers s_g, d_g, w_g are defined similarly, with colors reversed. If $(\Lambda, \mathcal{C})$ is self-dual, $J_r = J_g$. More usefully, if $J_r \neq J_g$, then the structure is not self-dual.

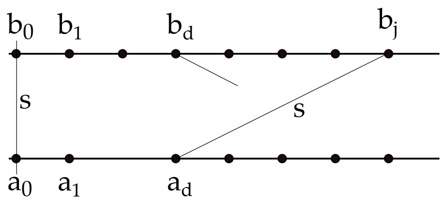


Figure 1: Green paths of length s joining two red cycles.

If $(\Lambda, \mathcal{C})$ is a cycle decomposition, then a cycle C in Λ is said to be $\mathcal{C}$*-alternating* if no two consecutive edges of C belong to the same element of $\mathcal{C}$. If a $\mathcal{C}$-alternating 4-cycle exists, then $J_r = J_g = [1, 1, 1]$, and $(\Lambda, \mathcal{C})$ is the partition of the edges of a toroidal map of type $\{4, 4\}$ into horizontal and vertical cycles.

Definition 1.2. An LR structure $(\Lambda, \mathcal{C})$ is called *suitable* provided that

(1) $(\Lambda, \mathcal{C})$ is not self-dual, and

(2) Λ has no $\mathcal{C}$-alternating 4-cycles.

The primary result of [10] is that:

Theorem 1.3. *The partial line graph construction $\mathbb{P}$ induces a bijective correspondence between the set of suitable LR structures and the set of worthy tetravalent semisymmetric graphs of girth* 4*.*

The word *worthy* in this statement means that no two vertices of the graph have exactly the same neighbors. Every new suitable LR structure gives a new semisymmetric graph, and so we are interested in finding and creating LR structures. In the remainder of this paper, we show how varied examples can be, concentrating on combinatorial constructions.

We first present two simple but non-trivial constructions to show some of the variety possible and to illustrate how the properties of LR structures enter into proofs.

2 Two constructions

2.1 Rows and columns

We construct an LR structure $\mathrm{RC}(n, k)$ in the following way: the vertices are to be all ordered pairs $(i,(r,j))$ and $((i,r),j)$, where i and j are in $\mathbb{Z}_n$, and r is in $\mathbb{Z}_k$. Green edges join $(i,(r,j))$ to $(i\pm1,(r,j))$ and $((i,r),j)$ to $((i,r),j\pm1)$, while red edges join $(i,(r,j))$ to $((i,r\pm1),j)$ and so $((i,r),j)$ to $(i,(r\pm1,j))$.

The function $(i,(r,j)) \mapsto (i+1,(r,j))$ and $((i,r),j) \mapsto ((i+1,r),j)$ is a symmetry of the graph; we abbreviate it by saying $i \mapsto i+1$. Similarly, each of the functions, $j \mapsto j+1$, $r \mapsto r+1$, $i \mapsto -i$, $j \mapsto -j$, $r \mapsto -r$ acts as a symmetry of $\mathrm{RC}(n,k)$. These dihedral symmetries act transitively on the vertices of each kind, and the correspondance $(i,(r,j)) \leftrightarrow ((j,r),i)$ is a symmetry and interchanges the two sets. The green neighbors of $(0,(0,0))$ are $(1,(0,0))$ and $(-1,(0,0))$, while the red neighbors are $((0,1),0)$ and $((0,-1),0)$.

Swappers at $(0,(0,0))$, then, are $i \mapsto -i$ and $r \mapsto -r$. So $\mathrm{RC}(n,k)$ with the given coloring is an LR structure. It has $2n^2k$ vertices, and its group has order at least $8n^2k$. The structure is of type $\{n, \mathrm{LCM}(2,k)\}$. If k is even, the graph described above is disconnected; in this case, re-assign the name $\mathrm{RC}(n,k)$ to the component containing $(0,(0,0))$. Then the graph has only n^2k vertices.

It is easy to check that $J_r = [1,2,1]$, while $J_g = [2,1,1]$ and so this structure is always suitable. This LR structure is also described algebraically in [9].

2.2 SoP

In this section, we describe a family of LR structures whose symmetry groups have arbitrarily large vertex stabilizers. The structure is $\mathrm{SoP}(m,n)$, where m and n are multiples of 4. Let $r = \frac{n}{2}+1$; then we have that $r^2 \equiv 1 \bmod n$. Further, if j is even, then $rj = j$, while if j is odd, $rj = j + \frac{n}{2}$. The vertex-set is $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$. Red edges join (i,j,k) to $(i, j \pm r^k, k)$; for fixed i and j, green edges join the two vertices $(2i,j,0)$ and $(2i,j,1)$ to the two vertices $(2i+1,j,0)$ and $(2i+1,j,1)$ if j is even, to the two vertices $(2i-1,j,0)$ and $(2i-1,j,1)$ if j is odd.

We claim that each of the following mappings $\rho, \mu, \sigma, \tau, \gamma$ and δ is a symmetry of the structure:

$$\begin{aligned}(i,j,k)\rho &= (i,j+2,k)\\ (i,j,k)\mu &= (i,-j,k)\\ (i,j,k)\sigma &= (i+1,j+1,k)\end{aligned}$$

$$\begin{aligned}
(i,j,k)\tau &= (i,rj,1-k)\\
(i,j,k)\gamma &= (-i,j+1,k)\\
(i,j,k)\delta &= \begin{cases}(i,rj,1-k) & \text{if } i\in\{1,2\}\\ (i,j,k) & \text{if } i\notin\{1,2\}\end{cases}
\end{aligned}$$

Together, these symmetries show that the structure is vertex-transitive. The symmetry μ acts as a red swapper at $(0,0,0)$, and δ acts as a green swapper there. Thus $\mathrm{SoP}(m,n)$ is an LR structure of order $2mn$ and type $\{4,n\}$. The conjugates of δ by $\langle\sigma^2\rangle$ commute with each other and so form a subgroup of order $2^{\frac{m}{2}}$. We can see, then, that the order of a vertex-stabilizer is at least $2^{\frac{m-2}{2}}$.

In this case, $J_r = [1,2,1]$, while $J_g = [2,2,1]$ and so this structure is always suitable.

3 LR structures from cycle structures

3.1 Voltage graphs and 2-coverings

We wish to use the mechanism of voltage graphs to describe a family of LR structures. Let us first summarize the voltage construction and some related facts in the special case of 2-coverings: Let Γ be any connected graph or multigraph. A $\mathbb{Z}_2$-voltage assignment on Γ is a function $\zeta\colon E(\Gamma)\to\mathbb{Z}_2$. The corresponding 2-fold covering $\mathrm{Cov}(\Gamma,\zeta)$ has $V(\Gamma)\times\mathbb{Z}_2$ as its vertex-set. The edge-set is $\{\{(u,i),(v,i+\zeta(e))\} \mid e=\{u,v\}\in E(\Gamma), i\in\mathbb{Z}_2\}$. Two $\mathbb{Z}_2$-voltage assignments ζ and ζ' are *equivalent* provided there is an isomorphism between $\mathrm{Cov}(\Gamma,\zeta)$ and $\mathrm{Cov}(\Gamma,\zeta')$ which acts trivially on first coordinates. For any vertex v, define the function μ_v on the set of all $\mathbb{Z}_2$-voltage assignments on Γ by letting $\zeta\mu_v$ be the assignment defined by

$$(\zeta\mu_v)(e) = \begin{cases}1+\zeta(e) & v \text{ is an endvertex of } e\\ \zeta(e) & v \text{ is not an endvertex of } e\end{cases}$$

We call such a function a "local reversal". Then ζ is equivalent to $\zeta\mu_v$, and any two equivalent assignments are related by a series of local reversals. It follows that if ζ and ζ' are equivalent, then there is a set $U\subseteq V(\Gamma)$ such that $\zeta(e)=\zeta'(e)$ exactly when both ends of e are in U or both not in U.

Two voltage assignments ζ and ζ' are *isomorphic* provided that some isomorphism γ of $\mathrm{Cov}(\Gamma,\zeta)$ onto $\mathrm{Cov}(\Gamma,\zeta')$ has the property that for each vertex v, $(v,0)\gamma$ and $(v,1)\gamma$ have the same first coordinate. Certainly ζ and ζ' are isomorphic if there is a symmetry β of Γ such that $\zeta'(e\beta)=\zeta(e)$ for every e. We write $\zeta'=\zeta\beta$ in this case, and then the function which sends (v,i) to $(v\beta,i)$ is an isomorphism of $\mathrm{Cov}(\Gamma,\zeta)$ onto $\mathrm{Cov}(\Gamma,\zeta')$. Finally, we say that a symmetry α of Γ *lifts* to a symmetry $\bar\alpha$ of $\mathrm{Cov}(\Gamma,\zeta)$ provided that for each vertex of $\mathrm{Cov}(\Gamma,\zeta)$, $(v,i)\bar\alpha=(v\alpha,j)$ for some j. Then, clearly, α lifts if and only if $\zeta\alpha$ is equivalent to ζ.

3.2 Voltage description of $\mathbf{CS}(\Gamma,\mathcal{B},i)$

The construction we wish to present here has to do with a kind of highly symmetric cycle decomposition called a *cycle structure*:

Definition 3.1. A *cycle structure* in a tetravalent graph Γ is a cycle decomposition $\mathcal{B}$ of Γ such that $\mathrm{Aut}(\Gamma, \mathcal{B})$ acts transitively on the darts of Γ.

Consider, for example, the graph of the octahedron $\mathcal{O}$, shown in Figure 2. The set $\mathcal{B}$ of

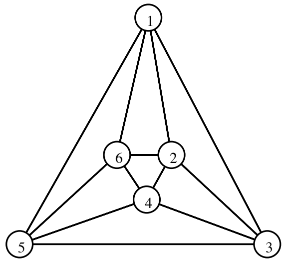


Figure 2: The octahedron.

triangles induced by the triples

$$\{\{1,5,6\},\{1,2,3\},\{2,4,6\},\{3,4,5\}\}$$

forms a cycle structure in $\mathcal{O}$. (In what follows, we will refer to each of these triangles by naming the vertex-triple which induces it rather than specifying its edges.) The group $\mathrm{Aut}(\mathcal{O}, \mathcal{B})$ is isomorphic to the symmetric group S_4, is dart-transitive and acts as S_4 on the four triangles. In particular, $(\mathcal{O}, \mathcal{B})$ is a cycle structure.

Cycle structures were introduced in [5], where it was shown that a vast majority of dart-transitive 4-valent graphs admit a cycle structures—many have more than one. At the end of this section we will show all small cycle structures and see how they contribute semisymmetric graphs.

3.3 The multigraph Γ' and its symmetries

We construct an LR structure from a cycle structure $(\Gamma, \mathcal{B})$ in two steps: we form a multigraph and then 2-cover it. First form the multigraph Γ' from Γ by separating each vertex into a pair of vertices, so that the cycles from $\mathcal{B}$ remain cycles but become disjoint. Then connect the two vertices in each pair with two parallel edges. We will refer to these as "bridge" edges. In our example of the octahedron with cycles $\{1,5,6\}$, $\{1,2,3\}$, $\{2,4,6\}$, $\{3,4,5\}$; Figure 3 shows the result.

To be more specific, the vertices of Γ' are all (C, v) where $C \in \mathcal{B}$, and $v \in C$. "Ordinary" edges join (C, u) to (C, v) where $\{u, v\}$ is an edge in the cycle C. If v belongs to cycles C and D, the corresponding "bridge" edges $e_{v,0}$ and $e_{v,1}$ join (C, v) to (D, v). Continuing the example and setting $A = \{1,5,6\}$, $B = \{1,2,3\}$, $C = \{2,4,6\}$, $D = \{3,4,5\}$, the corresponding labels of vertices in the split graph are shown in Figure 4.

If α is any symmetry in $G = \mathrm{Aut}(\Gamma, \mathcal{B})$, we choose the *canonical representative* α' of α to be the permutation which sends the vertex (C, v) to $(C\alpha, v\alpha)$, the ordinary edge $\{(C, u), (C, v)\}$ to the ordinary edge $\{(C\alpha, u\alpha), (C\alpha, v\alpha)\}$, and the bridge edge $e_{v,i}$ to $e_{v\alpha,i}$. Then α' is clearly a symmetry of Γ'. If we let

$$G' = \{\alpha' \mid \alpha \in G\},$$

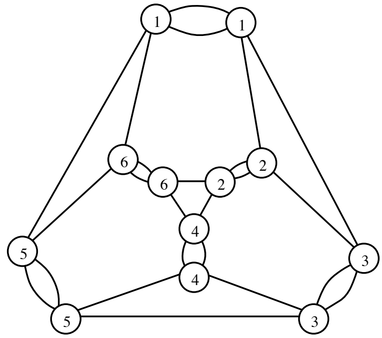


Figure 3: The octahedron, split.

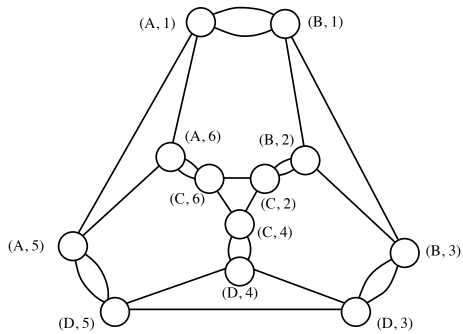


Figure 4: Labels in the octahedron.

then $G' \cong G$. Also, for each $v \in V(\Gamma)$, let σ_v interchange $e_{v,0}$ and $e_{v,1}$ while fixing every vertex of Γ' and every edge other than those two. Clearly, each σ_v is in $\mathrm{Aut}(\Gamma')$. If we let

$$K = \langle \sigma_v : v \in V(\Gamma) \rangle,$$

then $\mathrm{Aut}(\Gamma')$ is the inner semidirect product $K \rtimes G'$. For each $C \in \mathcal{B}$, define σ_C to be the product of all σ_v for $v \in C$, and let

$$L = \langle \sigma_C : C \in \mathcal{B} \rangle \ \leq K.$$

Since the product of all σ_C for $C \in \mathcal{B}$ involves each σ_v twice, the product is trivial. On the other hand, if $\mathcal{D}$ is a proper non-empty subset of $\mathcal{C}$, then an easy connectivity argument shows that the product of all σ_C for $C \in \mathcal{D}$ is non-trivial. Therefore the group L has order $2^{|\mathcal{B}|-1}$.

Now, G' is transitive on the vertices of Γ' and is, in fact transitive on the darts of ordinary edges. So for any (C, v), some $\alpha' \in G'$ acts as a swapper of ordinary edges there. And each non-trivial element of L acts as a swapper of bridge edges.

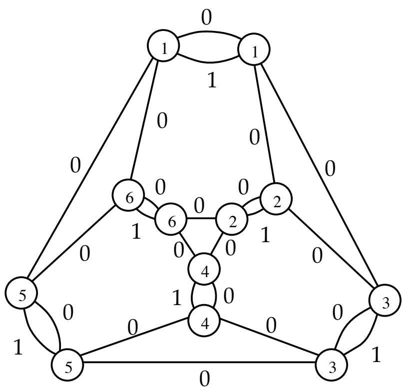
Then $L \rtimes G'$ is transitive on darts in ordinary edges and on darts in bridge edges as well. Thus the partition $\mathcal{C}$ of the edges of Γ' into cycles covering ordinary edges and cycles covering bridge edges is an LR coloring of this multigraph.

In the following sections we will construct two covers of the graph Γ' and show that in both cases, $L \rtimes G'$ is the group of symmetries that lifts.

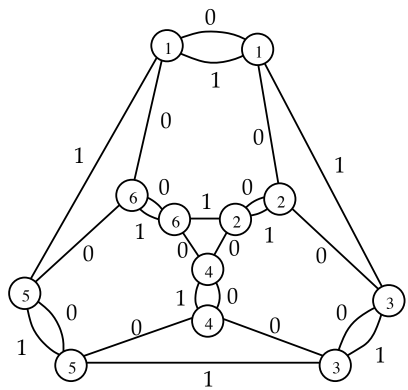
3.4 The coverings of Γ'

We now assign voltages $0, 1$ from $\mathbb{Z}_2$ to the edges of Γ' in two different ways; the assignments will be called ζ_0 and ζ_1. For bridge edges, let $\zeta_i(e_{v,0}) = 0$ and $\zeta_i(e_{v,1}) = 1$ for $i = 0, 1$. We assign $\zeta_0(e) = 0$ for each ordinary edge e. To define ζ_1, we choose one edge in each cycle $C \in \mathcal{B}$ to receive the voltage 1, and assign 0 to the rest of the edges in C. The isomorphism class of the resulting 2-cover is independent of which edge in each cycle is chosen, as we show in the next paragraph. Let $\Lambda(\Gamma, \mathcal{B}, 0)$ and $\Lambda(\Gamma, \mathcal{B}, 1)$ be the 2-covers $\mathrm{Cov}(\Gamma', \zeta_0)$ and $\mathrm{Cov}(\Gamma', \zeta_1)$ of Γ' resulting from ζ_0 and ζ_1, respectively.

Let $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ and $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$ be these graphs together with the decompositions into cycles covering those in $\mathcal{C}$. In Section 3.5 we will show that $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ and $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$ are, in fact, LR structures.



(a) $\zeta_0(\mathcal{O}')$



(b) $\zeta_1(\mathcal{O}')$

Figure 5: Voltage assignments.

To support our claim that the isomorphism class of $\mathrm{Cov}(\Gamma', \zeta_1)$ does not depend on our choice of representatives in each cycle, it will suffice to show that for any $C \in \mathcal{B}$, two $\mathbb{Z}_2$-assignments which are identical except on two consecutive edges of C are isomorphic assignments. So suppose that vertices u, v, w are consecutive in C, and that one of the assignments is ζ such that $\zeta(\{(C, u), (C, v)\}) = 1, \zeta(\{(C, v), (C, w)\}) = 0$, as in Figure 6. Then ζ is isomorphic to $\zeta\sigma_v$, which in turn is equivalent to $\zeta\sigma_v\mu_{(C,v)}$ (where, $\mu_{(C,v)}$ is a local reversal as described in Section 3.1) and this assignment is identical to ζ except on the edges $\{u, v\}, \{v, w\}$, as required.

Thus, by applying products such as $\sigma_v\mu_{(C,v)}$ to the assignment at successive vertices v of C, we can move the edge bearing a 1 from any position in C to any other. By adjusting each cycle in turn, we can show isomorphism of any two such assignments. This in fact shows the following useful fact, which we will refer to later.

Remark 3.2. Let ζ be an assignment on Γ' for which $\zeta(e_{v,0}) = 0$ and $\zeta(e_{v,1}) = 1$ for every vertex v of Γ, and let $C \in \mathcal{B}$. If the sum of all $\zeta(e)$ for $e \in C$ is 0, then ζ is isomorphic to

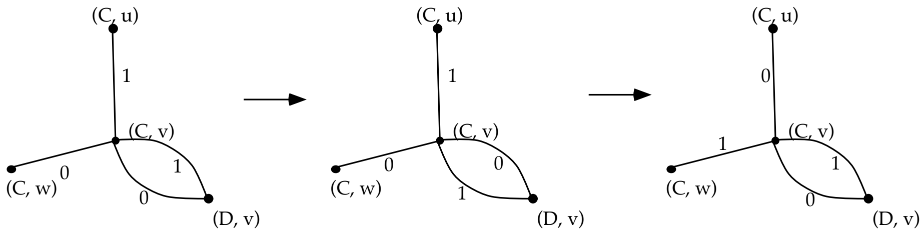


Figure 6: Isomorphic voltage assignments.

some assignment ζ' with $\zeta'(e) = 0$ for all $e \in C$. Similarly, if the above sum is 1, then ζ is isomorphic to some assignment in which every edge of C except one has weight 0, and that one has weight 1. Consequently, every 2-cover of Γ' without multiple edges in which all cycles covering those in $\mathcal{B}$ have the same length must be isomorphic to $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ or $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$. $\Diamond$

3.5 The groups of $\mathbf{CS}(\boldsymbol{\Gamma}, \boldsymbol{\mathcal{B}}, \mathbf{0})$ and $\mathbf{CS}(\boldsymbol{\Gamma}, \boldsymbol{\mathcal{B}}, \mathbf{1})$

We will show in this section that the cycle decompositions $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ and $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$, are LR structures, each admitting a group of order $2^{|\mathcal{B}|}|G|$.

Let $G = \mathrm{Aut}(\Gamma, \mathcal{B})$. Since $(\Gamma, \mathcal{B})$ is a cycle structure, G is transitive on the darts of Γ. Further, let G', K and L be the groups of symmetries of Γ' as defined in Section 3.3, and recall that $\mathrm{Aut}(\Gamma') = K \rtimes G'$.

Observe that, since G' maps a cycle in $\mathcal{B}$ to another cycle in $\mathcal{B}$ and since L is generated by all σ_C, $C \in \mathcal{B}$, L is normalised by G' and hence normal in $\mathrm{Aut}(\Gamma')$. In particular, $\langle L, G' \rangle = L \rtimes G'$.

Fix $i \in \{0, 1\}$ and let T be the group of those symmetries of Γ' that lift to a symmetry of $\mathrm{CS}(\Gamma, \mathcal{B}, i)$. In view of Section 3.1, a symmetry β of Γ' is in T if and only if the voltage assignment $\zeta_i\beta$ is equivalent to ζ_i.

We will prove that $T = L \rtimes G'$ and that the lift of T contains the symmetries needed to show that $\mathrm{CS}(\Gamma, \mathcal{B}, i)$ is an LR structure.

Let us first show that for every $\alpha \in G$ (and thus $\alpha' \in G'$) there exists $\beta \in T$ such that $\beta \in \alpha' K$. In other words, we show that $G' \subseteq TK$, and since $G'K = \mathrm{Aut}(\Gamma')$, that $\mathrm{Aut}(\Gamma') = G'K = TK$.

If $i = 0$, then $\zeta_i\alpha' = \zeta_i$, implying that α' lifts, and we can take β to be α' itself. Suppose now that $i = 1$. Then $\zeta_1\alpha'$ also has one edge in each $C \in \mathcal{B}$ whose voltage is 1. Then as in Section 3.4, there is a (possibly empty) subpath $v_1, v_2, v_3, \ldots, v_r$ of C such that $\zeta_1\alpha'\sigma_{v_1}\mu_{(C,v_1)} \ldots \sigma_{v_k}\mu_{(C,v_r)}$ coincides with ζ_1 on C. Denote

$$\sigma_{(\alpha,C)} = \sigma_{v_1}\sigma_{v_2}\cdots\sigma_{v_r} \quad \text{and} \quad \mu_{(\alpha,C)} = \mu_{(C,v_1)}\mu_{(C,v_2)}\cdots\mu_{(C,v_r)}$$

and observe that μ's and the σ's commute in their action on voltage assignments. Hence, by performing this adjustment for each $C \in \mathcal{B}$ in turn, it follows that

$$\zeta_1 = \zeta_1\alpha' \prod_{j=1}^{k} \sigma_{(\alpha,C_j)} \prod_{j=1}^{k} \mu_{(\alpha,C_j)}$$

and so, letting

$$\beta = \alpha' \prod_{j=1}^{k} \sigma_{(\alpha,C_j)} \in \alpha' K$$

we see that $\zeta_1\beta$ is equivalent to ζ_1 and thus that $\beta \in T$. This completes the proof of the claim that

$$\mathrm{Aut}(\Gamma') = G'K = TK.$$

Let us now show that $T \cap K = L$. This will then imply that $G' \cong G'K/K = TK/K \cong T/(T \cap K) = T/L$ and hence that $T = L \rtimes G'$, as claimed.

First note that each element of L lifts, and hence $L \leq T$. To see this, let $\sigma \in L$ and thus $\sigma = \prod_{C\in\mathcal{D}} \sigma_C$ for some $\mathcal{D} \subseteq \mathcal{B}$. Since $(\zeta_i\sigma)(e) = \zeta_i(e\sigma)$ for every edge e of Γ', we see that ζ_i and $\zeta_i\sigma$ agree on the ordinary edges and differ on exactly those bridge edges $e_{v,0}$ and $e_{v,1}$for which v belongs to exactly one of the cycles in $\mathcal{D}$. In particular,

$$\zeta_i\sigma = \zeta_i \prod_{C\in\mathcal{D}} \prod_{v\in C} \mu_{(C,v)},$$

showing that $\zeta_i\sigma$ is equivalent to ζ_i and thus that $\sigma \in T$.

Suppose now that $\sigma \in K$ and that σ lifts. Then ζ_i is equivalent to $\zeta_i\sigma$, and in view of Section 3.1, there is a collection W of vertices of Γ' such that

$$\zeta_i \prod_{(C,v)\in W} \mu_{(C,v)} = \zeta_i\sigma.$$

Since $\zeta_i\sigma$ agrees with ζ_i on ordinary edges, we see that if a vertex (C, v) is in W, then every vertex (C, u) for all u in C must also be in W. Letting, as before,

$$\mu_C = \prod_{v\in C} \mu_{(C,v)},$$

for each $C \in \mathcal{B}$, we see that there is a $\mathcal{D} \subseteq \mathcal{B}$ such that

$$\zeta_i \prod_{C\in\mathcal{D}} \mu_C = \zeta_i\sigma.$$

Now, because $\sigma \in K$, there is a subset U of $V(\Gamma)$ such that

$$\sigma = \prod_{v\in U} \sigma_v$$

and because $\zeta_i\mu_{(C,v)}$ and $\zeta_i\sigma_v$ agree on bridge edges, U must be the set of those $v \in V(\Gamma)$ that belong to exactly one of the cycles in $\mathcal{D}$. Then σ must be

$$\prod_{C\in\mathcal{D}} \sigma_C,$$

and so $\sigma \in L$. This completes the proof that $K \cap T = L$, and therefore, that $T = L \rtimes G'$ and has order $2^{|\mathcal{B}|-1}|G|$. Then because each $\mathrm{CS}(\Gamma, \mathcal{B}, i)$ has a symmetry which interchanges the two vertices in each fibre, $\mathrm{Aut}(\mathrm{CS}(\Gamma, \mathcal{B}, i))$ has order $2^{|\mathcal{B}|}|G|$.

Finally, we need to show that each $\mathrm{CS}(\Gamma, \mathcal{B}, i)$ has the symmetries required of an LR structure. As $L \rtimes G'$ lifts to a group of color-preserving symmetries and is transitive on vertices, in each case, it suffices to show that $L \rtimes G'$ contains swappers at each vertex of Γ'. Consider a vertex (C, v), as in Figure 6, above. The symmetry σ_C acts as a swapper for the bridge edges at (C, v). Now G contains a symmetry α which sends the dart (u, v) to (w, v); then $C\alpha = C$. As above, there exists a corresponding β that lifts and such that $\beta \in \alpha' K$. This β and $\beta\sigma_C$ both lift, and one of them will swap ordinary edges at (C, v). Thus, $\mathrm{CS}(\Gamma, \mathcal{B}, i)$ is an LR structure of type $\{4, q\}$ (if $i = 0$) or $\{4, 2q\}$ (if $i = 1$), where q is the length of a cycle in $\mathcal{B}$. It has no alternating 4-cycles and cannot be self-dual unless $q = 4$ and $i = 0$.

3.6 Wreath graphs and self-dual LR structures

Define the graph $W(n, k)$ (called a *wreath* graph) to have kn vertices in n groups of k each, the groups arranged in a circular order. The edges join every vertex in one group to every vertex in the groups immediately before and after it. For example, Figure 7 shows the graph $W(5, 2)$.

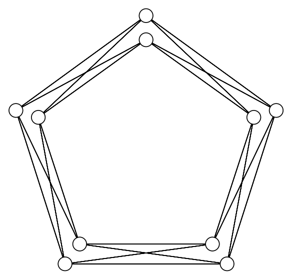


Figure 7: The wreath graph $W(5, 2)$.

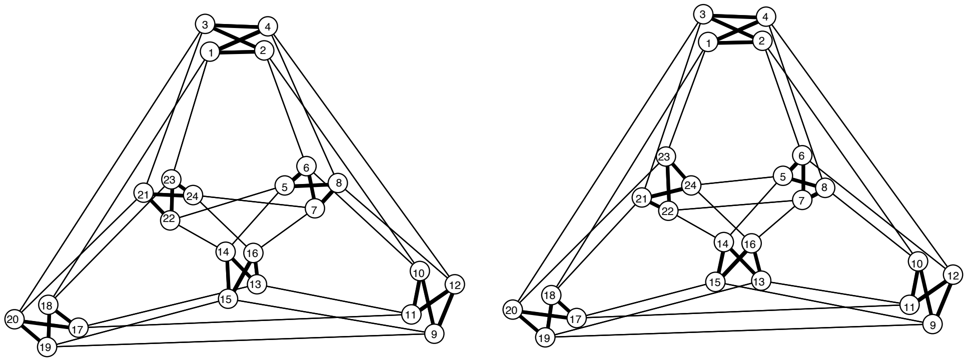
In $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$, let the ordinary edges be colored green and the bridge edges, red. Consideration of joining sequences shows that even if $i = 0$, $p = 4$, then $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ is self-dual if and only if Γ is the graph $W(n, 2)$ for some n and $\mathcal{B}$ is the set of 4-cycles induced by two consecutive groups of vertices. To see this, first check that in $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$, $J_r = [1, 2, 1]$. If $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ is to be self-dual, J_g must be $[1, 2, 1]$ as well. Thus, from two antipodal vertices in each green cycle, red edges must lead to two antipodal vertices in another green cycle. As the red cycles in $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ correspond to vertices in Γ, this implies that each 4-cycle in $\mathcal{B}$ must share two vertices with each other that it meets. Since each 4-cycle meets only two others, they must be arranged in a circle, and the graph must be $W(n, 2)$ for some n.

Thus, in all other cases, the LR structure $\mathrm{CS}(\Gamma, \mathcal{B}, i)$ is suitable.

In our example in which $\Gamma = \mathcal{O}$, the LR structures $\mathrm{CS}(\mathcal{O}, \mathcal{B}, i)$ for $i = 0, 1$ are shown in Figure 8.

We summarize the argument from Sections 3.3, 3.4, 3.5, 3.6 in the following theorem.

Theorem 3.3. *If $\mathcal{B}$ is a cycle structure for the tetravalent graph Γ, then $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ and $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$ are LR stuctures. These LR structures are suitable with the exception that if $\mathcal{B}$ is the decomposition of $W(n, 2)$ into 4-cycles, then $\mathrm{CS}(\mathrm{W}(n, 2), \mathcal{B}, 0)$ is self-dual and so not suitable.*



(a) $\mathrm{CS}(\mathcal{O}, \mathcal{B}, 0)$ (b) $\mathrm{CS}(\mathcal{O}, \mathcal{B}, 1)$

Figure 8: Two LR structures.

3.7 Occurrences of cycle structures

It is a little surprising that nearly every small dart-transitive 4-valent graph has a suitable system of block cycles and most have several. Consider the smallest dart-transitive 4-valent graphs (below, $C_n(a, b)$ denotes the circulant graph on n vertices with connection set $\{\pm a, \pm b\}$ while $R_n(a, r)$ is a *rose window graph*, defined in [13]):

Graph	**Cycle structures**
K_5	two 5-cycles
Octahedron	four 3-cycles, three 4-cycles
$K_{4,4}$	four 4-cycles, two 8-cycles
$C_3 \times C_3$	six 3-cycles, three 6-cycles
$W(5, 2)$	five 4-cycles
$C_{10}(1, 3)$	two 10-cycles
$W(6, 2)$	six 4-cycles, four 6-cycles (two ways), two 12-cycles
$R_6(5, 4)$	eight 3-cycles, six 4-cycles, four 6-cycles

Wreath graphs appear four times in this list: the octahedron is $W(3, 2)$ and $K_{4,4}$ is $W(4, 2)$. The graph $W(n, 2)$ always has a cycle structure $\mathcal{F}$ in which every cycle has length 4. In these cases, as we have shown, the LR structure $\mathrm{CS}(\Gamma, \mathcal{F}, 0)$ is self-dual and thus not suitable, so only $\mathrm{CS}(\Gamma, \mathcal{F}, 1)$ gives a truly semi-symmetric graph. Therefore, applying CS and then $\mathbb{P}$ to these eight graphs and their 16 cycle structures give us 28 semisymmetric graphs, having 40 to 96 vertices.

Suppose $\mathcal{M}$ is a reflexible map. Its medial graph, the graph $\mathrm{MG}(\mathcal{M})$, has one vertex for each edge of $\mathcal{M}$, and two are joined by an edge in $\mathrm{MG}(\mathcal{M})$ if the corresponding edges in $\mathcal{M}$ are consecutive around some face of $\mathcal{M}$. The symmetry group $G = \mathrm{Aut}(\mathcal{M})$ of the map acts on $\mathrm{MG}(\mathcal{M})$ as a group of symmetries. G is transitive on darts, and has at least three block systems of cycles: those corresponding to faces, those corresponding to vertices and those corresponding to Petrie paths. While two of these could be isomorphic (if $\mathcal{M}$ is self-dual, for instance), it will quite often happen that three different cycle structures, and hence as many as six different semisymmetric graphs, will result from one map $\mathcal{M}$.

3.8 Larger coverings

We can generalize $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ to k-coverings for $k > 2$. We describe a k-covering of Γ', and we call the covering structure $\mathrm{CS}(\Gamma, \mathcal{B}, k)$. Give the weight 0 to each ordinary edge. Give each pair of bridge edges voltage 1 in opposite directions. Let $\mathrm{CSI}(\Gamma, \mathcal{B}, k))$ be one component of the resulting k-cover. If $(\Gamma, \mathcal{B})$ is bipartite and k is even, then the k-covering has two components, while in all other cases, it has one. Thus if $\mathcal{B}$ has m cycles, each of length n, and so $mn/2$ vertices, then $\mathrm{CSI}(\Gamma, \mathcal{B}, k))$ is of type $\{n, \mathrm{LCM}(2, k)\}$ and has mnk or $mnk/2$ vertices.

It is easy to check that $J_r = [2, 1, 1]$, while $J_g = [1, 2, 1]$ and so this structure is always suitable.

4 Other constructions

4.1 Locally circular cycle structures

Definition 4.1. Suppose $\mathcal{C}$ is a cycle decomposition of a graph Γ of valence $2q$ and let X be the set of all (C, v) such that $C \in \mathcal{C}$, $v \in V(\Gamma)$ and C passes through v. For a vertex v of Γ let X_v be the set of pairs with second coordinate v. If P is a permutation on X such that the orbits of $\langle P \rangle$ are the sets X_v for $v \in V(\Gamma)$, then we will say that $(\Gamma, \mathcal{C}, P)$ is *locally circular*. We will call such a P a *locally circular ordering* on $(\Gamma, \mathcal{C})$.

Definition 4.2. If σ is a symmetry of $(\Gamma, \mathcal{C})$, we will say that σ *respects* P provided that, for each $(C, v) \in X$, $(C, v)\mathcal{P}\sigma$ is either $(C\sigma, v\sigma)P$ or $(C\sigma, v\sigma)P^{-1}$. Let $\mathrm{Aut}(\Gamma, \mathcal{C}, P)$ be the group of all symmetries of $(\Gamma, \mathcal{C})$ which respect P.

Definition 4.3. If $(\Gamma, \mathcal{C}, P)$ is locally circular and $G \leq \mathrm{Aut}(\Gamma)$, we will say it is *G-locally dihedral* provided that the following hold:

(i) G acts transitively on darts,

(ii) every element of G respects P,

(iii) for every $v \in V(\Gamma)$, the stabiliser G_v acts dihedrally on the cycles through v and contains an element which fixes every cycle through v setwise and reverses at least one of them.

A locally circular $(\Gamma, \mathcal{C}, P)$ is *locally dihedral* if it is G-locally dihedral for some $G \leq \mathrm{Aut}(\Gamma)$.

While this definition appears to be very restrictive, notice that a large class of examples arises from reflexible maps: if $\mathcal{M}$ is a reflexible map of type $\{p, 2q\}$, we can consider two edges to be *opposite* at v provided that those edges are q apart in the cycle of edges incident to v. If $\mathcal{M}$ is proper (i.e., no two edges have the same endpoints), the edges fall into cycles in which each edge is joined to the edges opposite it at each end. Then the family of such cycles is a locally dihedral cycle structure.

Construction 4.4. *If* $(\Gamma, \mathcal{C}, P)$ *is locally circular, let* $\mathrm{LDCS}(\Gamma, \mathcal{C}, P)$ *be the bipartite cycle decomposition* $(\Lambda, \mathcal{D})$ *in which vertices of* Λ *are all* (C, v) *such that* v *is a vertex of cycle* $C \in \mathcal{C}$*, green edges are all* $\{(C, u), (C, v)\}$ *such that* $\{u, v\}$ *is an edge of cycle* $C \in \mathcal{C}$*, and red edges are all* $\{(C, v), (C, v)P\}$.

Theorem 4.5. *If* $(\Gamma, \mathcal{C}, P)$ *is a locally dihedral cycle structure, then* $\mathrm{LDCS}(\Gamma, \mathcal{C}, P)$ *is an LR structure which has no alternating* 4*-cycles.*

Proof. Every element of $G = \mathrm{Aut}(\Gamma, \mathcal{C}, P)$ acts on $\mathrm{LDCS}(\Gamma, \mathcal{C}, P)$ as a symmetry. Since G is transitive on darts, $\mathrm{Aut}(\mathrm{LDCS}(\Gamma, \mathcal{C}, P))$ is transitive on vertices. To see that it is flexible, consider a vertex (C, v). Because $(\Gamma, \mathcal{C}, P)$ is locally dihedral, it has a symmetry $\rho \in G$ which fixes v, fixes each cycle at v setwise and reverses C. Then ρ acts as a green swapper at (C, v). Also, because G_v acts dihedrally on the cycles at v, it has a μ which fixes C (setwise) and interchanges the neighboring cycles in the local order. Then μ or $\mu\rho$ is a red swapper at (C, v). If there were an alternating 4-cycle in $\mathrm{LDCS}(\Gamma, \mathcal{C}, P)$, the green edges would correspond to distinct edges in Γ with the same endpoints, which is forbidden in a graph. □

Our last results in this paper show that the constructions CS and LDCS generate all LR structures $(\Lambda, \mathcal{C})$ for which $\mathcal{C}$ contains cycles of length 3 or 4. We begin by showing that LDCS covers all the cases in which $s_r \geq 2$.

Theorem 4.6. *Let* $(\Lambda, \mathcal{C})$ *be an LR structure of type* $\{p, q\}$ *in which no two red cycles (of length* p*) are joined by more than one green edge (that is, if the joining sequence* J_r *is not of the form* $[1, *, *]$*). Then there is a locally dihedral cycle structure* $(\Gamma, \mathcal{D}, P)$*, where* Γ *is a graph of valence* $2p$ *and* $\mathcal{D}$ *is a partition of the edges of* Γ *into* q*-cycles, such that* $(\Lambda, \mathcal{C})$ *and* $\mathrm{LDCS}(\Gamma, \mathcal{D})$ *are isomorphic LR structures.*

Proof. Let Γ be the graph with the vertex set being the set of red cycles in $(\Lambda, \mathcal{C})$ with two red cycles adjacent in Γ whenever they are joined by a green edge in Λ. For each vertex v of Λ, let $\pi(v)$ be the red cycle to which v belongs. We can consider π to be a projection onto Γ of the subgraph of Λ induced by its green edges. Since two red cycles are joined by at most one green edge, this projection π induces a bijection between the green edges in $(\Lambda, \mathcal{C})$ and the edges of Γ.

Let $\mathcal{D}$ be the set of all cycles in Γ of the form $\pi(D)$ where D is a green cycle in $(\Lambda, \mathcal{C})$. Then Γ has valence $2p$ and $\mathcal{D}$ is a cycle decomposition of Γ in which every cycle has length q.

Let X be the set of all $(\pi(D), C)$ such that D is a green cycle in $(\Lambda, \mathcal{C})$ and C is a red cycle in $(\Lambda, \mathcal{C})$ contained (as a vertex of Γ) in the cycle $\pi(D)$.

Let us now define the permutation P on X yielding a locally dihedral $(\Gamma, \mathcal{D}, P)$. For each red cycle C in $(\Lambda, \mathcal{C})$ choose one of the two possible orientations of C. We then let P map a pair $(\pi(D), C) \in X$ to the pair $(\pi(D'), C)$ where D' is the green cycle through the next vertex (with respect to the chosen orientation of C) on C after the unique vertex of Λ that belongs to both C and D. Then all of $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ respects this P. Since $\mathrm{Aut}^+(\Lambda, \mathcal{C})$ is transitive on green darts, it acts dart-transitively on Γ. The set stabilizer of a red cycle acts dihedrally on the set of green cycles meeting it. Any green swapper fixes a red cycle pointwise and so fixes setwise each green cycle meeting it, and reverses at least one cycle. Thus $(\Gamma, \mathcal{D}, P)$ is a locally dihedral cycle structure.

Finally, let φ be the mapping which maps a vertex v of Λ to the vertex $(\pi(D), C)$ of $\mathrm{LDCS}(\Gamma, \mathcal{D}, P)$ where C and D are the red and the green cycle of $(\Lambda, \mathcal{C})$ containing v, respectively. It is a matter of straightforward computation to verify that φ is an isomorphism of the LR structures $(\Lambda, \mathcal{C})$ and $\mathrm{LDCS}(\Gamma, \mathcal{D})$. □

With Theorem 4.6, we can now easily prove that all LR structures of types $\{3, q\}$ or $\{4, q\}$ without alternating 4-cycle arise by constructions in this paper. (We point out that the LR structures with alternating cycles have been characterised in [10, Lemma 6.3].) The first of the two corollaries below follows directly from Theorem 4.6 after observing that an alternating 4-cycle implies that two red cycles are joined by two green edges. The second one requires some additional work.

Corollary 4.7. *If $(\Lambda, \mathcal{C})$ is a LR structure of type $\{3, q\}$ without alternating 4-cycles, then there is a locally dihedral cycle structure $(\Gamma, \mathcal{D}, P)$, where Γ is a graph of valence 6 and $\mathcal{D}$ is a partition of the edges of Γ into q-cycles, such that $(\Lambda, \mathcal{C})$ and $\mathrm{LDCS}(\Gamma, \mathcal{D})$ are isomorphic LR structures.*

Theorem 4.8. *If $(\Lambda, \mathcal{C})$ is an LR structure of type $\{4, q\}$ without alternating 4-cycles, then one of the following happens:*

(1) *there is a locally dihedral cycle structure $(\Gamma, \mathcal{D}, P)$, where Γ is a graph of valence 8 and $\mathcal{D}$ is a partition of the edges of Γ into q-cycles, such that $(\Lambda, \mathcal{C}) \cong \mathrm{LDCS}(\Gamma, \mathcal{D})$; or*

(2) *there is a cycle structure $(\Gamma, \mathcal{B})$, where Γ has valence 4 and $(\Lambda, \mathcal{C}) \cong \mathrm{CS}(\Gamma, \mathcal{B}, i)$ for $i = 0$ or 1.*

Proof. Suppose that $(\Lambda, \mathcal{C})$ is an LR structure, without alternating 4-cycles, in which the red cycles have length 4. Consider a green edge and the red cycles through its endvertices. If no other green edge joins those red cycles then Theorem 4.6 applies, and so (1) holds.

If not, then, because $(\Lambda, \mathcal{C})$ has no alternating 4-cycles, two green edges join two antipodal vertices on one red cycle with two antipodal vertices on the other red cycle. Call two green edges which are arranged in this way, *mated* edges. Then $J_r = [1, 2, 1]$.

Collapsing each red cycle to a single vertex, as in the proof of Theorem 4.6, identifies all pairs of mated green edge to form a tetravalent dart-transitive graph Γ. The green cycles of $(\Lambda, \mathcal{C})$ are projected onto a cycle structure $\mathcal{D}$ in Γ. Since the projection is 2-to-1 on green edges, we see that if mated green edges come from different cycles, those two q-cycles project to a single q-cycle in Γ. If they are from the same cycle, then q must be even and that cycle projects onto a $\frac{q}{2}$-cycle.

Consider now an intermediate projection in which we identify mated green edges, and within a red cycle identify antipodal vertices and opposite edges. This projects $(\Lambda, \mathcal{C})$ onto a multigraph, in which the red "cycles" are actually 2-gons, i.e., each consists of a pair of parallel red edges. This is clearly isomorphic to the graph Γ' formed in the first step of the construction of $\mathrm{CS}(\Gamma, \mathcal{D}, i)$. This presents $(\Lambda, \mathcal{C})$ as a 2-covering of Γ'.

Remark 3.2 shows that in the case where the green cycles of Γ are of length q, the LR structure $(\Lambda, \mathcal{C})$ is isomorphic to $CS(\Gamma, \mathcal{D}, 0)$, and if the green cycles are of length $\frac{q}{2}$, then $(\Lambda, \mathcal{C})$ is isomorphic to $\mathrm{CS}(\Gamma, \mathcal{D}, 1)$. □

5 Conclusion

Though this paper and its predecessors [9, 10] have presented a number of constructions both agebraic and combinatorial, much remains to be done. Every new discovery of an LR structure gives us a new semisymmetric graph of girth and valence 4. Thus, finding LR structures and organizing them into parameterized families is important in the search for semisymmetric graphs. The smallest known LR structure not yet to be seen as part of a family of such has 36 vertices, and there are seven more with 72 vertices. Examples such as SoP, $\mathrm{CS}(\Gamma, \mathcal{B}, 0)$ and $\mathrm{CS}(\Gamma, \mathcal{B}, 1)$, whose vertex-stabilizers can grow without bound add to our growing knowledge about the structure of semisymmetric graphs.

Our ultimate goal of the study of the LR structures is to develop the tools that would enable us to construct a complete list of all "small" LR structures. Such lists exist for both types of edge-transitive cubic graphs (see [1, 3] for the census of cubic edge-transitive graph of order at most 768 and [2] for the extension to order up to $10\,000$ in the case of dart-transitive graphs) and for cubic vertex-transitive graphs [6] for orders up to $1\,280$. Moreover, lists of all dart-transitive and $\frac{1}{2}$-transitive tetravalent graphs of order up to $1\,000$ have recently been compiled (see [6, 8]). The main ingredient of these results was always a theoretical result that bounded the order of the vertex-stabiliser in such a graph. While it has long been known that this order is bounded by a constant in the case of cubic edge-transitive graphs, this is not the case in the cubic vertex-transitive or tetravalent edge-transitive cases. What is more, for these cases, families of graphs where the order of the stabiliser grows exponentially with the order of the graph are known. The crucial point in the enumeration of these graphs was a result that identified the "problematic" families and proved that the order of the vertex-stabiliser in the "non-problematic" graphs is bounded by a tame (possibly sublinear) function of the order of the graph. As it happens, all the problematic graphs contain cycles of girth 4 (and there is a deep group theoretical reason for that). There is strong evidence that a similar result might hold in the case of the LR structures. This leads to the following question (we thank Gabriel Verret for a fruitful discussion on this topic):

Question 5.1. Does there exist a polynomial function f such that for every LR structure $(\Lambda, \mathcal{C})$ of type other than $\{4, q\}$, the symmetry group $\mathrm{Aut}(\Lambda, \mathcal{C})$ has order at most $f(|V(\Lambda)|)$.

This question complements Corollary 4.8 which reduces the classification of the LR structures of type $\{4, q\}$ to the study of 8-valent locally dihedral cycle structures and cycle structures in tetravalent dart-transitive graphs.

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On Wiener inverse interval problem of trees

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Abstract

The Wiener index $W(G)$ of a simple connected graph G is defined as the sum of distances over all pairs of vertices in a graph. We denote by $W[\mathcal{T}_n]$ the set of all values of the Wiener index for a graph from the class $\mathcal{T}_n$ of trees on n vertices. The largest interval of consecutive integers (consecutive even integers in case of odd n) contained in $W[\mathcal{T}_n]$ is denoted by $W^{int}[\mathcal{T}_n]$. In this paper we prove that both sets are of cardinality $\frac{1}{6}n^3 + O(n^{5/2})$ in the case of even n, while in the case of odd n we prove that the cardinality of both sets equals $\frac{1}{12}n^3 + O(n^{5/2})$, which essentially solves two conjectures posed in the literature.

Keywords: Wiener index, Wiener inverse interval problem, Tree.

Math. Subj. Class.: 05C05, 05C90

1 Introduction

The Wiener index of a connected graph G is defined as the sum of distances over all pairs of vertices, i.e.

$$W(G) = \sum_{u,v \in V(G)} d(u,v).$$

It was first introduced in [13] and it was used for predicting the boiling points of paraffins. Since the index was very successful many other topological indices were introduced later which use the distance matrix of a graph. There is a recent survey by Gutman et al. [14] in which finding extremal values and extremal graphs for the Wiener index and several of its variations is nicely presented. Given the class of all simple connected graphs on n vertices it is easy to establish extremal graphs for the Wiener index, those are complete graph K_n and path P_n. The same holds for the class of tree graphs on n vertices in which the minimal

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tree is the star S_n and the maximal tree is the path P_n. Many other bounds on the Wiener index are also established in the literature.

In [4] Gutman and Yeh proposed the inverse Wiener index problem, i.e. for a given value w the problem of finding a graph (or a tree) G for which $W(G) = w$. The first attempt at solving the problem was made in [7] where integers up to 1206 were checked and 49 integers were found that are not Wiener indices of trees. In [1] it was computationally proved that for all integers w between 10^3 and 10^8 there exists a tree with Wiener index w. The problem was finally fully solved in 2006 when two papers were published solving the problem independently. It was proved in [12] that for every integer $w > 108$ there is a caterpillar tree G such that $W(G) = w$. The other proof is from the paper [9] where it was proved that all integers except those 49 are Wiener indices of trees with diameter at most 4. Since the most interesting graphs to be considered are chemical trees (especially those in which maximum vertex degree is at most 3) and hexagon type graphs, in [11] this problem was further considered on classes of such graphs.

A related question is to ask what value of the Wiener index can a graph (or a tree) G on n vertices have? In order to clarify further this problem one may also ask how many such values are there, how are they distributed along the related interval or how many of them are consecutive. In [6] this problem is named the Wiener inverse interval problem (see also a nice recent survey [5] which covers the topic). In that paper the set $W[\mathcal{G}_n]$ is defined as the set of all values of the Wiener index for graphs $G \in \mathcal{G}_n$, where $\mathcal{G}_n$ is the class of simple connected graphs on n vertices. Similarly, $W[\mathcal{T}_n]$ is defined as the set of values $W(T)$ for all trees on n vertices ($\mathcal{T}_n$ denotes the class of trees on n vertices). Also, $W^{int}[\mathcal{G}_n]$ (or analogously $W^{int}[\mathcal{T}_n]$) is defined to be the largest interval of consecutive integers such that $W^{int}[\mathcal{G}_n] \subseteq W[\mathcal{G}_n]$ (or analogously $W^{int}[\mathcal{T}_n] \subseteq W[\mathcal{T}_n]$).

In [6] the Wiener inverse interval problem on the class $\mathcal{G}_n$ was considered. First, the authors noted that obviously $W^{int}[\mathcal{G}_n] \subseteq W[\mathcal{G}_n] \subseteq [W(K_n), W(P_n)]$. Since $W(K_n)$ and $W(P_n)$ are easily computed, the upper bound $\left|W^{int}[\mathcal{G}_n]\right| \leq |W[\mathcal{G}_n]| \leq \frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3} + 1$ easily follows. Introducing dandelion and comet graphs and establishing how the values between the values of the Wiener index for dandelion and comet graph can be obtained, the authors obtain the following lower bound $|W[\mathcal{G}_n]| \geq \left|W^{int}[\mathcal{G}_n]\right| \geq \frac{n^3}{6} - \frac{5}{2}n^2 - \frac{1}{3}n^{3/2} + \frac{19}{3}n + \frac{7}{3}n^{1/2}$. These bounds sandwich the value of $\left|W^{int}[\mathcal{G}_n]\right|$ and $|W[\mathcal{G}_n]|$ in terms of n^3 tightly, therefore the result $\left|W^{int}[\mathcal{G}_n]\right| = \left|W^{int}[\mathcal{G}_n]\right| = \frac{n^3}{6} + O(n^2)$ easily follows. The authors further conjecture that $|W[\mathcal{G}_n]| = \frac{n^3}{6} - \frac{n^2}{2} + \Theta(n)$. Regarding the same problem on the class $\mathcal{T}_n$ the following two conjectures were made.

Conjecture 1.1. *The cardinality of* $W[\mathcal{T}_n]$ *equals* $\frac{1}{6}n^3 + \Theta(n^2)$.

Conjecture 1.2. *The cardinality of* $W^{int}[\mathcal{T}_n]$ *equals* $\Theta(n^3)$.

In this paper we will consider these two conjectures. First, we will note that for a tree T on odd number of vertices n the value $W(T)$ can be only an even number. That means that the Wiener inverse interval problem in that case has to be reformulated as the problem of finding the largest interval $W^{int}[\mathcal{T}_n]$ of consecutive even integers such that $W^{int}[\mathcal{T}_n] \subseteq W[\mathcal{T}_n]$. Since $|W[\mathcal{T}_n]| \leq W(P_n) - W(S_n) + 1 = \frac{1}{6}n^3 - n^2 + \frac{11}{6}n$, we now conclude that the cardinality of $W[\mathcal{T}_n]$ in the case of odd n can be at most $\frac{1}{12}n^3 + O(n^2)$. Given that reformulation, we will prove both conjectures to be true in terms of n^3. Even more, we will prove the strongest possible version of Conjecture 1.2 in terms of n^3 by proving that $\left|W^{int}[\mathcal{T}_n]\right|$ also equals $\frac{1}{6}n^3 + O(n^{5/2})$ (i.e. $\frac{1}{12}n^3 + O(n^{5/2})$ in case of odd

n) which is the best possible result given the upper bound on $|W[\mathcal{T}_n]|$ derived from the difference between $W(P_n)$ and $W(S_n)$. These results will yield quite a strong result for the class $\mathcal{T}_n^4$ of chemical trees as a direct corollary.

The present paper is organized as follows. In the next section basic definitions and preliminary results are given. In the third section the problem is solved for trees on even number of vertices, while in the fourth section the problem is solved for trees on odd number of vertices. In the fifth section we conclude the paper with several remarks and possible directions for further research.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple connected graph having $n = |V(G)|$ vertices and $m = |E(G)|$ edges. For a pair of vertices $u, v \in V(G)$ we define the distance $d_G(u, v)$ as the length of the shortest path connecting u and v in G. For a vertex $u \in V(G)$ the degree $d_G(u)$ is defined as the number of neighbors of vertex u in graph G. When it doesn't lead to confusion we will use the abbreviated notation $d(u, v)$ and $d(u)$. Also, for a vertex $u \in V(G)$ and a set of vertices $A \subseteq V(G)$ we will denote $d(u, A) = \sum_{v \in A} d(u, v)$. Similarly, for two sets of vertices $A, B \subseteq V(G)$ we will denote $d(A, B) = \sum_{u \in A, v \in B} d(u, v)$. We say that a vertex $u \in V(G)$ is a leaf if $d_G(u) = 1$, otherwise we will say that u is an interior vertex of a graph G. A graph G which does not contain cycles is called a tree. A tree graph will usually be denoted by T throughout the rest of the paper. We say that a tree T is a caterpillar tree if all its interior vertices induce a path. Such a path will be called the interior path of a caterpillar. Let a and b be positive integers such that $a \leq b$. We say that the interval $[a, b]$ is Wiener $p-$complete if there is a tree T in $\mathcal{T}_n$ such that $W(T) = a + pi$ for every $i = 0, \ldots, \left\lfloor \frac{b-a}{p} \right\rfloor$. We say that the interval $[a, b]$ is Wiener complete if it is Wiener $1-$complete.

Let us now note that the value of the Wiener index for a tree T on odd number of vertices n is an even number. There are various ways to prove this fact, maybe the simplest one is to recall that for a tree T on n vertices it holds that

$$W(T) = \sum_{uv \in E(T)} n_u \cdot n_v$$

where n_u and n_v are the number of vertices in the connected component of $T \backslash \{uv\}$ containing u and v respectively. Obviously, $n_u + n_v = n$ and therefore in the case of odd n the product $n_u \cdot n_v$ must be an even number. I would like here to thank prof. Tomislav Došlić for suggesting this short proof to me and to the anonymous reviewers for referring me to the interesting survey [2] in which this fact is already explained and to several interesting papers ([3], [8] and [10]) in which one can read more on the subject. Before proceeding further, let us state this fact as a formal theorem which we can reference in further text.

Theorem 2.1. *Let T be a tree on odd number of vertices $n \geq 3$. Then $W(T)$ is an even number.*

The main tool for obtaining our results throughout the paper will be a transformation of a tree which increases the value of the Wiener index by exactly four. We will call it Transformation A, but let us introduce its formal definition.

Definition 2.2. Let T be a tree and $u \in V(T)$ a vertex of degree 4 such that neighbors v_1 and v_2 of u are leaves, while neighbors w_1 and w_2 of u are not leaves. We say that a tree

T' is obtained from T by Transformation A if T' is obtained from T by deleting edges uv_1 and uv_2, while adding edges w_1v_1 and w_2v_2.

Theorem 2.3. *Let T be a tree and let T' be a tree obtained from T by Transformation A. Then $W(T') = W(T) + 4$.*

Proof. For simplicity's sake we will use the notation $d'(u, v)$ for $d_{T'}(u, v)$. Let $T_{w_i} = (V_{w_i}, E_{w_i})$ be the connected component of $T\backslash\{u\}$ which contains vertex w_i for $i = 1, 2$. Note that the only distances that change in Transformation A are distances from vertices v_1 and v_2. For every $v \in V_{w_1} \cup V_{w_2}$ we have

$$d'(v_1, v) - d(v_1, v) + d'(v_2, v) - d(v_2, v) = 0.$$

For the vertex u we have

$$d'(v_1, u) - d(v_1, u) + d'(v_2, u) - d(v_2, u) = 2.$$

Finally, we also have $d'(v_1, v_2) - d(v_1, v_2) = 2$. Therefore, $W(T') - W(T) = 4$ which proves the theorem. □

Although Transformation A can be applied on any tree graph, we will mainly apply it on caterpillar trees. Moreover, it is critical to find a kind of caterpillar tree on which Transformation A can be applied repeatedly as many times as possible. For that purpose, let us prove the following theorem.

Theorem 2.4. *Let T be a caterpillar tree and $P = u_1 \ldots u_d$ its interior path. If there is a vertex $u_i \in P$ of degree 4 such that $u_{i\pm j}$ is of degree 3 for every $1 \leq j \leq k-1$, then the interval $[W(T), W(T) + 4k^2]$ is Wiener $4-$complete.*

Proof. Let us denote a caterpillar tree T from the statement of the lemma by T^k (since $1 \leq j \leq k-1$). Also, let us denote $D = \{u_{i\pm j} : j = 0, \ldots, k-1\}$. To obtain the desired result we will systematically apply Transformation A to vertices from D until there is no more vertices in D to which Transformation A can be applied. Let us now explain into greater detail by what system that is done. First, note that in T^k Transformation A can initially be applied only to u_i. By applying transformation A to u_i in T^k we will obtain a caterpillar tree in which Transformation A can be applied to vertices u_{i-1} and u_{i+1}. By applying Transformation A to u_{i-1} and u_{i+1} consecutively we will obtain a caterpillar tree in which Transformation A can be applied to u_{i-2} and u_{i+2} (and u_i but we will not further apply Transformation A to that vertex for the time being). By further applying transformation A to u_{i-2} and u_{i+2} consecutively and repeating this procedure we will reach a caterpillar tree in which Transformation A can be applied to vertices $u_{i-(k-1)}$ and $u_{i+(k-1)}$ and finally apply Transformation A to those two vertices. The caterpillar tree obtained after that last step we can denote by T^{k-1} because of the following: in that tree vertex $u_i \in P$ is of degree 4 and vertices $u_{i\pm j}$ are of degree 3 for every $j = 1, \ldots, k-2$. Note that in the process of transforming T^k to T^{k-1} we will have applied the Transformation A $2k-1$ times. Now, the same process can be repeated on T^{k-1} to obtain T^{k-2}. The procedure stops when we reach T^1 in which u_i is the only vertex in D having degree greater than 2 (to be more precise, the degree of u_i in T^1 equals 4, so Transformation A can be applied to it one more time). Applying Transformation A on u_i in T^1 we finally obtain T^0 in which Transformation A cannot be further applied to vertices from D. Therefore, in transforming T^k to T^0 Transformation A was used $\sum_{j=1}^{k}(2j-1) = k^2$ times and each time the value of the Wiener index increased by 4. □

Note that the Transformation A in Theorem 2.4 is applied k^2 times on a caterpillar in which interior path is of length $d-1$. If we prove that there are $\Theta(n)$ different values of d for which $k=\Theta(n)$, we obtain roughly $\Theta(n^3)$ graphs with different values of the Wiener index which is exactly the result we aim at (of course, here one has to be careful to avoid significant overlapping of the values of the Wiener index for caterpillars with different values of d). So, that is what we are going to do in following sections, but in order to do that with sufficient mathematical precision we will have to construct four different special types of caterpillar trees. To easily construct those four types of caterpillar trees we first introduce two basic types of caterpillars from which those four types will be constructed by adding one or two vertices.

Definition 2.5. Let n, d and x be positive integers such that $n \geq 18$ is even, $\lceil \frac{n-2}{4} \rceil \leq d \leq \frac{n-8}{2}$ and $x \leq \frac{4+4d-n}{2}$. Caterpillar $B_1(n,d,x)$ is a caterpillar on even number of vertices n obtained from path $P = u_{-d} \ldots u_{-1} u_0 u_1 \ldots u_d$ by appending a leaf to vertices u_{-d-1+x} and u_{d+1-x} and by appending a leaf to $2k-1$ consecutive vertices $u_{-(k-1)}, \ldots, u_{k-1}$ where $k = \frac{n-(2d+1)-1}{2}$.

Caterpillar graph $B_1(n,d,x)$ is illustrated by Figure 1 (vertex u_i of the interior path is in the images denoted just by i in order to make labels easier to see).

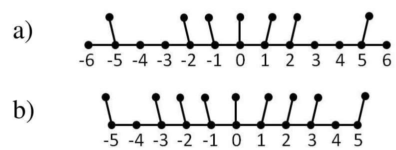

Figure 1: Caterpillar graphs: a) $B_1(20,6,2)$, b) $B_1(20,5,1)$.

Lemma 2.6. *Let n, d and x be integers such that $B_1(n,d,x)$ is defined. Then*

$$W(B_1(n,d,x)) = \frac{n^3}{4} + \left(-\frac{3d}{2} - \frac{5}{4}\right)n^2 + \left(4d^2 + 10d + \frac{13}{2} - 2x\right)n+ \\ + 2x^2 - \frac{8}{3}d^3 - 12d^2 - \frac{46d}{3} - 7.$$

Proof. Let $k = \frac{n-(2d+1)-1}{2}$ and $x' = -d-1+x$. Even though the structure of B_1 is a bit complicated it is still regular enough so that the Wiener index can be computed exactly (as a function in variables n, d and x). Let us divide vertices of B_1 into three sets A, B and C so that set A contains vertices u_i for $i=-d,\ldots,d$, set B contains leaves attached to $2k-1$ consecutive vertices $u_{-(k-1)},\ldots,u_{k-1}$ and set C contains two leaves attached to vertices u_{-d-1+x} and u_{d+1-x}. Note that we have

$$d(A,A)=\sum_{i=-d}^{d}\sum_{j=i+1}^{d}(j-i),\ d(B,B)=\sum_{i=-(k-1)}^{k-1}\sum_{j=i+1}^{k-1}(j-i+2)$$

$$d(C,C)=(2d+2-2(x-1)),\ d(A,B)=\sum_{i=-d}^{d}\sum_{j=-(k-1)}^{k-1}(|i-j|+1)$$

$$d(A,C)=2\sum_{i=-d}^{d}(|i-x'|+1),\ d(B,C)=2\sum_{i=-(k-1)}^{k-1}(|i-x'|+2)$$

Noting that

$$W(B_1(n,d,x))=d(A,A)+d(B,B)+d(C,C)+d(A,B)+d(A,C)+d(B,C)$$

and simplifying the obtained sum yields the formula from the statement of the lemma. □

Note that the caterpillar $B_1(n,d,x)$ is a caterpillar with relatively long interior path. Namely, the value d is roughly half of the length of the interior path and in the definition of $B_1(n,d,x)$ the value of d is relatively large with respect to number of vertices n. We now introduce the formal definition of the second basic caterpillar which will have relatively short interior path.

Definition 2.7. Let n, d and x be positive integers such that $n \geq 18$ is even, $4 \leq d \leq \left\lfloor \frac{n}{4} \right\rfloor$ and $x \leq \frac{n-4d+2}{2}$. Caterpillar $B_2(n,d,x)$ is a caterpillar on even number of vertices n obtained from path $P = u_{-d} \dots u_{-1}u_0u_1 \dots u_d$ by appending a leaf to $2k-1$ consecutive vertices $u_{-(k-1)}, \dots, u_{k-1}$ where $k = d-1$, by appending x leaves to each of the $u_{-(d-1)}$ and $u_{(d-1)}$, and by appending r leaves to each of the u_{-d} and u_d where $r = \frac{n-4d-2x+2}{2}$.

Caterpillar graph $B_2(n,d,x)$ is illustrated by Figure 2 (vertex u_i of the interior path is in the images denoted just by i in order to make labels easier to see).

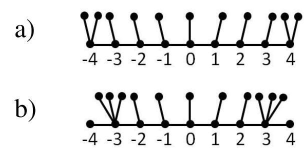

Figure 2: Caterpillar graphs: a) $B_2(20,4,1)$, b) $B_2(20,4,3)$.

Lemma 2.8. *Let n, d and x be integers such that $B_2(n,d,x)$ is defined. Then*

$$W(B_2(n,d,x))=(\frac{d}{2}+1)n^2+(-2d-2)n-\frac{8d^3}{3}+\frac{32d}{3}-5+8x-8dx-2x^2.$$

Proof. Let $k=d-1$ and $r=\frac{n-4d-2x+2}{2}$. To obtain the exact formula for $W(B_2(n,d,x))$ we divide vertices from $B_2(n,d,x)$ into four sets: set A contains vertices u_i for $i =$

$-d, \ldots, d$, set B contains leaves appended to $2k-1$ consecutive vertices $u_{-(k-1)}, \ldots, u_{k-1}$, set C contains x leaves appended to each of the $u_{-(d-1)}$ and $u_{(d-1)}$, while finally set D contains r leaves appended to each of the u_{-d} and u_d. Note that

$$d(A,A) = \sum_{i=-d}^{d} \sum_{j=i+1}^{d} (j-i),\ d(B,B) = \sum_{i=-(k-1)}^{k-1} \sum_{j=i+1}^{k-1} (j-i+2)+$$
$$d(C,C) = 4\binom{x}{2} + x^2(2d),\ d(D,D) = 4\binom{r}{2} + r^2(2d+2)$$

Also, we have

$$d(A,B) = \sum_{i=-d}^{d} \sum_{j=-(k-1)}^{k-1} (|i-j|+1),\ d(A,C) = 2x(3 + \sum_{i=3}^{2d+1}(i-1))$$
$$d(A,D) = 2r\sum_{i=1}^{2d+1} i,\ d(B,C) = 2x \sum_{i=-(k-1)}^{k-1} (i+d+1)$$
$$d(B,D) = 2r \sum_{i=-(k-1)}^{k-1} (i+d+2),\ d(C,D) = 2(3xr + xr(2d+1)).$$

Noting that

$$\begin{aligned} W(B_1(n,d,x)) &= d(A,A) + d(B,B) + d(C,C) + d(D,D)+ \\ &+ d(A,B) + d(A,C) + d(A,D)+ \\ &+ d(B,C) + d(B,D) + d(C,D) \end{aligned}$$

and simplifying the obtained sum yields the formula from the statement of the lemma. □

Finally, let us denote $d_1^{\min} = \lceil \frac{n-2}{4} \rceil$ and $x_1^{\max} = \frac{4+4d_1^{\min}-n}{2}$, while $d_2^{\max} = \lfloor \frac{n}{4} \rfloor$. Note that

$$B_1(n, d_1^{\min}, x_1^{\max}) = B_2(n, d_2^{\max}, 1). \tag{2.1}$$

This equality will provide us with a nice transition from caterpillars based on $B_1(n,d,x)$ to caterpillars based on $B_2(n,d,x)$ in the following sections.

3 Even number of vertices

In this section we will first introduce a special kind of caterpillar based on $B_1(n,d,x)$ which will have a longer interior path, then we will introduce a second special kind of caterpillar based on $B_2(n,d,x)$ which will have a shorter interior path. For each of those two special kinds of caterpillars we will establish a bound on the value of d for which the interval between values of the Wiener index for two consecutive values of x and d is Wiener $4-$complete. The equality (2.1) will then enable us to "glue" all those intervals into one big Wiener $4-$complete interval.

Definition 3.1. Let n, d and x be integers for which $B_1(n-2,d,x)$ is defined. For $s = -1, 0, 1, 2$ caterpillar $T_1(n,d,x,s)$ is a caterpillar on even number of vertices n, obtained from $B_1(n-2,d,x)$ by appending a leaf to the vertex u_s and a leaf to the vertex u_d of the path $P = u_{-d} \ldots u_{-1}u_0u_1 \ldots u_d$ in $B_1(n-2,d,x)$.

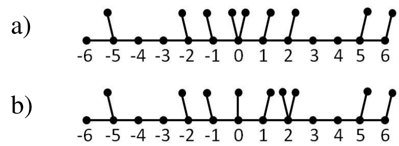

Figure 3: Caterpillar graphs: a) $T_1(22, 6, 2, 0)$, b) $T_1(22, 6, 2, 2)$.

Caterpillar graph $T_1(n, d, x, s)$ is illustrated by Figure 3 (vertex u_i of the path P is in the images denoted just by i in order to make labels easier to see).

Lemma 3.2. *Let n, d, x and s be integers for which $T_1(n, d, x, s)$ is defined. Then*

$$W(T_1(n, d, x, s)) = W(B_1(n-2, d, x)) + \frac{n^2}{4} + \frac{3n}{2} + 2d^2 + 3d + 2s^2 - s - 2x.$$

Proof. Let $k = \frac{(n-2)-(2d+1)-1}{2}$, $x' = -d - 1 + x$. We define a function

$$f(v) = \sum_{i=-d}^{d} (|v-i|+1) + \sum_{i=-(k-1)}^{k-1} (|v-i|+2) + (|x'-v| + 2 + |-x'-v| + 2)$$

Now, the definition of $T_1(n, d, x, s)$ implies

$$W(T_1(n, d, x, s)) = W(B_1(n-2, d, x)) + f(s) + f(d) + d - s + 2.$$

Plugging s and d into the formula for f and simplifying the obtained expression yields the result. □

As a direct consequence of Lemma 3.2 we obtain the following corollary.

Corollary 3.3. *It holds that*

$$\begin{aligned} W(T_1(n, d, x, 1)) &= W(T_1(n, d, x, 0)) + 1, \\ W(T_1(n, d, x, 2)) &= W(T_1(n, d, x, 0)) + 6, \\ W(T_1(n, d, x, -1)) &= W(T_1(n, d, x, 0)) + 3. \end{aligned}$$

The main tool in proving the results will be Transformation A of the graph, which, for a given graph, finds another graph whose value of the Wiener index is greater by 4. Therefore, it is critical to find a graph on which Transformation A can be applied consecutively as many times as possible. That was the basic idea behind constructing graph $T_1(n, d, x, s)$ as we did, so that we can use Theorem 2.4 in filling the interval between values $W(T_1(n, d, x, s))$ for consecutive values of x and d. So, let us first apply Theorem 2.4 (i.e. find the corresponding value of k) to the graph $T_1(n, d, x, s)$.

Lemma 3.4. *Let n, d, x and s be integers for which $T_1(n, d, x, s)$ is defined. For $k = \frac{1}{2}n - d - 4$ the interval $[W(T_1(n, d, x, s)), W(T_1(n, d, x, s)) + 4k^2]$ is Wiener $4-$complete.*

Proof. Let us denote $k_1 = \frac{(n-2)-(2d+1)-1}{2}$. Note that k_1 is half of the number of leaves appended to the vertices $u_{\pm j}$ of the interior path of $T_1(n,d,x,s)$ for $j = 0, \ldots, k-1$. Since $s \leq 2$, note that the definition of $T_1(n,d,x,s)$ and Theorem 2.4 imply the result for $k = k_1 - 2$. □

So, let us now establish for which values of d the gap between $W(T_1(n,d,x,s))$ and $W(T_1(n,d,x-1,s))$ is smaller than $4k^2$ which is the width of an interval which can be filled by repeatedly applying Transformation A on $T_1(n,d,x,s)$ (i.e. by using Lemma 3.4).

Lemma 3.5. *Let n, d, $x \geq 2$ and s be integers for which $T_1(n,d,x,s)$ is defined. For $d \leq \frac{1}{2}(n - \sqrt{2n-8} - 8)$ the interval*

$$[W(T_1(n,d,x,s)), W(T_1(n,d,x-1,s))]$$

is Wiener $4-$complete.

Proof. First note that

$$\begin{aligned} W(T_1(n,d,x-1,s)) - W(T_1(n,d,x,s)) &\leq W(T_1(n,d,2,s)) - W(T_1(n,d,1,s)) = \\ &= 2(n-5) + 2. \end{aligned}$$

Therefore, Lemma 3.4 implies it is sufficient to find integers n and d for which it holds that $4k^2 \geq 2(n-5)+2$ where $k = \frac{1}{2}n - d - 4$. By a simple calculation it is easy to establish that the inequality holds for $d \leq \frac{1}{2}(n - \sqrt{2n-8} - 8)$ so the lemma is proved. □

It is easy to show, using Lemma 3.2, that $W(T_1(n,d,x-1,s)) - W(T_1(n,d,x,s)) = 2n - 4x$ which is divisible by 4 since n is even. Therefore, Lemma 3.5 enables us to "glue" together Wiener $4-$complete intervals

$$[W(T_1(n,d,x,s)), W(T_1(n,d,x-1,s))]$$

into one bigger Wiener $4-$complete interval

$$[W(T_1(n,d,x_1^{\max},s)), W(T_1(n,d,1,s))]$$

where $x_1^{\max} = \frac{4+4d-(n-2)}{2}$. Corollary 3.3 then implies that roughly the same interval will be Wiener complete when we take values for every $s = -1, 0, 1, 2$. We say "roughly" because the difference $W(T_1(n,d,x,2)) = W(T_1(n,d,x,0)) + 6$ makes one point gap at $W(T_1(n,d,x_1^{\max},0)) + 2$. We now want to "glue" together such bigger intervals into one interval on the border between d and $d-1$. The problem is that

$$T_1(n,d,x_1^{\max},s) \neq T_1(n,d-1,1,s),$$

so we have to cover the gap in between. Moreover, it holds that

$$W(T_1(n,d,x_1^{\max},s)) - W(T_1(n,d-1,1,s)) = n-3$$

which is not divisible by 4. Therefore, we have to find enough graphs whose values of the Wiener index will cover the gap of $n-3$ plus the gap of 6 which arises from the "rough" edge of the interval for a given d.

Lemma 3.6. *Let n, d, $x_1^{\max} = \frac{4+4d-(n-2)}{2}$ and s be integers for which $T_1(n, d, x_1^{\max}, s)$ and $T_1(n, d-1, 1, s)$ are defined. For $d \leq \frac{1}{2}(n - \sqrt{n+3} - 6)$ the interval*

$$[W(T_1(n, d-1, 1, s)), W(T_1(n, d, x_1^{\max}, s)) + 6]$$

is Wiener $4-$complete.

Proof. Since

$$W(T_1(n, d, x_1^{\max}, s)) + 6 - W(T_1(n, d-1, 1, s)) = n - 3 + 6 = n + 3,$$

Lemma 3.4 implies that it is sufficient to find for which d it holds that $4k^2 \geq n + 3$ where $k = \frac{1}{2}n - (d-1) - 4$. By a simple calculation one obtains that inequality holds for $d \leq \frac{1}{2}(n - \sqrt{n+3} - 6)$ which proves the theorem. □

Note that the restriction on the maximum value of d is stricter in Lemma 3.5 then in Lemma 3.6 for every $n > 4$.

Now we have taken out all we could from graph T_1, but that covers only caterpillars with relatively large d. We can further expand the Wiener complete interval to the left side, i.e. to caterpillars with smaller d, using graph T_2 which we will construct from the basic graph B_2.

Definition 3.7. Let n, d and x be integers for which $B_2(n-2, d, x)$ is defined. For $s = -1, 0, 1, 2$ caterpillar $T_2(n, d, x, s)$ is a caterpillar on even number of vertices n, obtained from $B_2(n-2, d, x)$ by appending a leaf to the vertex u_s and a leaf to the vertex u_d of the path $P = u_{-d} \ldots u_{-1} u_0 u_1 \ldots u_d$ in $B_2(n-2, d, x)$.

Caterpillar graph $T_2(n, d, x, s)$ is illustrated by Figure 4 (vertex u_i of the path P is in the images denoted just by i in order to make labels easier to see).

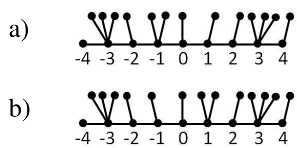

Figure 4: Caterpillar graphs: a) $T_2(22, 4, 3, -1)$, b) $T_2(22, 4, 3, 1)$.

Lemma 3.8. *Let n, d, x and s be integers for which $T_2(n, d, x, s)$ is defined. Then*

$$W(T_2(n, d, x, s)) = W(B_2(n-2, d, x)) + (2d+4)n - 2d^2 - 7d - 6 - 2x + 2s^2 - s.$$

Proof. Let $k = d - 1$ and $r = \frac{n-4d-2x}{2}$. We define a function

$$\begin{aligned} f(v) = & \sum_{i=-d}^{d} (|v-i| + 1) + \sum_{i=-(k-1)}^{k-1} (|v-i| + 2) + \\ & + x(|v + (d-1)| + 2) + x(|v - (d-1)| + 2) + \\ & + r(|v + d| + 2) + r(|v - d| + 2). \end{aligned}$$

Now, the definition of $T_2(n,d,x,s)$ implies

$$W(T_2(n,d,x,s)) = W(B_2(n-2,d,x)) + f(s) + f(d) + d - s + 2.$$

Plugging s and d into the formula for f and simplifying the obtained expression yields the result. $\square$

Again, as a direct consequence of Lemma 3.8 we obtain the following corollary.

Corollary 3.9. *It holds that*

$$\begin{aligned} W(T_2(n,d,x,1)) &= W(T_2(n,d,x,0)) + 1, \\ W(T_2(n,d,x,2)) &= W(T_2(n,d,x,0)) + 6, \\ W(T_2(n,d,x,-1)) &= W(T_2(n,d,x,0)) + 3. \end{aligned}$$

As in the case of large d, the main tool in obtaining the results will be the following lemma.

Lemma 3.10. *Let n, d, x and s be integers for which $T_2(n,d,x,s)$ is defined. For $k = d-3$ the interval $[W(T_2(n,d,x,s)), W(T_2(n,d,x,s)) + 4k^2]$ is Wiener $4-$complete.*

Proof. Let us denote $k_1 = d-1$. Note that k_1 is half of the number of leaves appended to the vertices $u_{\pm j}$ of the interior path of $T_2(n,d,x,s)$ for $j = 0,\ldots,k-1$. Since $s \leq 2$, note that the definition of $T_2(n,d,x,s)$ and Theorem 2.4 imply the result for $k = k_1 - 2$. $\square$

We will first use Lemma 3.10 to cover the interval between $W(T_2(n,d,x,s))$ and $W(T_2(n,d,x-1,s))$, after that we will use it to cover the gap between $W(T_2(n,d-1,1,s))$ and $W(T_2(n,d,x_2^{\max},s))$.

Lemma 3.11. *Let $n, d,$ $x \geq 2$ and s be integers for which $T_2(n,d,x,s)$ is defined. For $d \geq \frac{1}{2}(\sqrt{2n-8}+6)$ the interval*

$$[W(T_2(n,d,x,s)), W(T_2(n,d,x-1,s))]$$

is Wiener $4-$complete.

Proof. First note that for $x_2^{\max} = \frac{(n-2)-4d+2}{2}$ it holds that

$$\begin{aligned} & W(T_2(n,d,x-1,s)) - W(T_2(n,d,x,s)) \leq \\ & \leq W(T_2(n,d,x_2^{\max}-1,s)) - W(T_2(n,d,x_2^{\max},s)) = \\ & = 2(n-5)+2. \end{aligned}$$

Therefore, Lemma 3.10 implies it is sufficient to find for which n and d it holds that $4k^2 \geq 2(n-5)+2$ where $k = d-3$. By a simple calculation it is easy to establish that the inequality holds for $d \geq \frac{1}{2}(\sqrt{2n-8}+6)$ so the theorem is proved. $\square$

Again, it is easy to show that $W(T_2(n,d,x-1,s)) - W(T_2(n,d,x,s)) = 4d+4x-8$ which is divisible by 4. Therefore, using Lemma 3.11 we can again "glue" the interval for different values of x into one bigger interval which will be "roughly" Wiener complete when taking values of $W(T_2(n,d,x,s))$ for every $s = -1,0,1,2$. The next thing is to cover the gap between $W(T_2(n,d-1,1,s))$ and $W(T_2(n,d,x_2^{\max},s))$ which equals $n-3$ plus the gap of 6 which arises from the "rough" ends of the Wiener complete interval for given n and d.

Lemma 3.12. *Let n, d, $x_2^{\max} = \frac{(n-2)-4d+2}{2}$ and s be integers for which $T_2(n,d,x_2^{\max},s)$ and $T_2(n,d-1,1,s)$ is defined. For $d \geq \frac{1}{2}(\sqrt{n+3}+8)$ the interval*

$$[W(T_2(n,d-1,1,s)), W(T_2(n,d,x_2^{\max},s))+6]$$

is Wiener $4-$complete.

Proof. Since

$$W(T_2(n,d,x_2^{\max},s))+6-W(T_2(n,d-1,1,s)) = n+3,$$

Lemma 3.10 implies it is sufficient to find n and d for which it holds that $4k^2 \geq n+3$ where $k = (d-1)-3$. By a simple calculation one obtains that the inequality holds for $d \geq \frac{1}{2}(\sqrt{n+3}+8)$ which proves the theorem. □

Therefore, using graphs $T_1(n,d,x,s)$ and $T_2(n,d,x,s)$ we obtained two big Wiener complete intervals, which it would be nice if we could "glue" together into one big Wiener complete interval. In order to do that, note that the equality (2.1) implies

$$T_2(n,d_2^{\max},1,s) = T_1(n,d_1^{\min},x_1^{\max},s)$$

for $d_2^{\max} = \left\lfloor \frac{n-2}{4} \right\rfloor$, $d_1^{\min} = \left\lceil \frac{n-4}{4} \right\rceil$ and $x_1^{\max} = \frac{4+4d_1^{\min}-(n-2)}{2}$. Now we can state the theorem which gives us the largest Wiener complete interval we have managed to obtain.

Theorem 3.13. *Let $n \geq 30$, $d_2^{\min} = \left\lceil \frac{1}{2}(\sqrt{2n-8}+6) \right\rceil$, $x_2^{\max} = \frac{(n-2)-4d_2^{\min}+2}{2}$ and $d_1^{\max} = \left\lfloor \frac{1}{2}(n-\sqrt{2n-8}-8) \right\rfloor$. The interval*

$$[W(T_2(n,d_2^{\min},x_2^{\max},2)), W(T_1(n,d_1^{\max},1,0))]$$

is Wiener complete.

Now that we have obtained very large Wiener complete interval, we can finally prove the following theorem which is our main result and which proves Conjectures 1.1 and 1.2 in terms of n^3.

Theorem 3.14. *For even $n \geq 30$ it holds that $\left|W^{int}[\mathcal{T}_n]\right| = |W[\mathcal{T}_n]| = \frac{1}{6}n^3 + O(n^{5/2})$.*

Proof. Theorem 3.13 implies

$$|W[\mathcal{T}_n]| \geq \left|W^{int}[\mathcal{T}_n]\right| \geq W(T_1(n,d_1^{\max},1,0)) - W(T_2(n,d_2^{\min},x_2^{\max},2))$$

where $d_2^{\min} = \frac{1}{2}(\sqrt{2n-8}+6)+p$, $x_2^{\max} = \frac{(n-2)-4d_2^{\min}+2}{2}$ and $d_1^{\max} = \frac{1}{2}(n-\sqrt{2n-8}-8)-1+q$ for $p \in [0,1\rangle$ and $q \in \langle 0,1]$. From Lemmas 3.2 and 3.8 we further obtain that

$$\left|W^{int}[\mathcal{T}_n]\right| \geq \frac{1}{6}n^3 - \frac{1}{2}\sqrt{2n^5-8n^4} - 4n^2 + \frac{10}{3}\sqrt{2n^3-8n^2} + \frac{143}{6}n + 21\sqrt{2n-8} - 51.$$

On the other hand, recall that $\left|W^{int}[\mathcal{T}_n]\right| \leq |W[\mathcal{T}_n]| \leq W(P_n) - W(S_n) + 1 = \frac{1}{6}n^3 - n^2 + \frac{11}{6}n$, which proves the theorem. □

Note that caterpillar trees $T_1(n,d,x,s)$ are chemical trees (i.e. trees in which the degree of every vertex is at most 4) for all possible values of its parameters and they remain chemical after repeated application of Transformation A. Therefore, half of these results hold for chemical trees and we obtain the following corollary.

Corollary 3.15. *Let $\mathcal{T}_n^4$ be a class of chemical trees on n vertices where $n \geq 30$ is even. Then $\left|W^{int}[\mathcal{T}_n^4]\right| = \left|W[\mathcal{T}_n^4]\right| = \Theta(n^3)$.*

Proof. Note that Lemmas 3.5 and 3.6 imply that

$$\left|W[\mathcal{T}_n^4]\right| \geq \left|W^{int}[\mathcal{T}_n^4]\right| \geq W(T_1(n,d_1^{\max},1,0)) - W(T_1(n,d_1^{\min},x_1^{\max},2))$$

where $d_1^{\max} = \frac{1}{2}(n-\sqrt{2n-8}-8)-1+p$, $d_1^{\min} = \frac{n-2}{4}+q$ and $x_1^{\max} = \frac{6+4d_1^{\min}-n}{2}$ for $p \in \langle 0,1]$ and $q \in [0,1\rangle$. From Lemma 3.2 we obtain

$$\left|W^{int}[\mathcal{T}_n^4]\right| \geq \frac{1}{12}n^3 - \frac{1}{4}\sqrt{2n^5-8n^4} - \frac{15}{8}n^2 + \frac{5}{3}\sqrt{2n^3-8n^2} + \frac{101}{12}n + \frac{9}{2}\sqrt{2n-8} - \frac{134}{3}.$$

□

Note that this result for chemical trees is obtained using only chemical trees with relatively large diameter and the result is still the best possible with regard to the highest power n^3 (just the power, not the coefficient). This means that this result is something that probably can be significantly improved by considering chemical trees with shorter diameter, but we leave that as an open problem for future research.

4 Odd number of vertices

The strategy to prove the result in the case of odd number of vertices is the same as in the case of even number of vertices. The only difference is that in this case the value of the Wiener index can be only even number so we are aiming at the largest possible interval of consecutive even numbers which are values of the Wiener index for a tree.

Definition 4.1. Let n, d and x be integers for which $B_1(n-1,d,x)$ is defined. For $s = 0,1$ caterpillar $T_3(n,d,x,s)$ is a caterpillar on odd number of vertices n, obtained from $B_1(n-1,d,x)$ by appending a leaf to the vertex u_s of the path $P = u_{-d}\ldots u_{-1}u_0u_1\ldots u_d$ in $B_1(n-1,d,x)$.

Caterpillar tree $T_3(n,d,x,s)$ is illustrated by Figure 5 (vertex u_i of the path P is in the images denoted just by i in order to make labels easier to see).

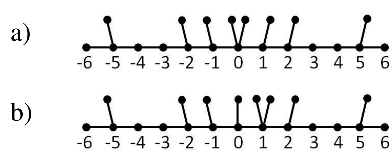

Figure 5: Caterpillar graphs: a) $T_3(21,6,2,0)$, b) $T_3(21,6,2,1)$.

Lemma 4.2. *Let n, d, x and s be integers for which $T_3(n,d,x,s)$ is defined. Then*

$$W(T_3(n,d,x,s)) = W(B_1(n-1,d,x)) + \frac{n^2}{4} - dn + 2d^2 + 5d + \frac{11}{4} - 2x + 2s^2.$$

Proof. Let $k = \frac{(n-1)-(2d+1)-1}{2}$, $x' = -d-1+x$. The definition of $T_3(n,d,x,s)$ implies

$$\begin{aligned} W(T_3(n,d,x,s)) = W(B_1(n-1,d,x)) + \sum_{i=-d}^{d}(|s-i|+1) + \\ + \sum_{i=-(k-1)}^{k-1}(|s-i|+2) + (s-x'+2) + \\ + (-x'-s+2). \end{aligned}$$

Simplifying this expression yields the result. □

As a direct consequence of Lemma 4.2 we obtain the following corollary.

Corollary 4.3. *It holds that $W(T_3(n,d,x,1)) = W(T_3(n,d,x,0)) + 2$.*

We now want to apply Theorem 2.4 to $T_3(n,d,x,s)$, i.e. we want to establish the value of k in the case of this special graph.

Lemma 4.4. *Let n, d, x and s be integers for which $T_3(n,d,x,s)$ is defined. For $k = \frac{1}{2}n - d - \frac{5}{2}$ the interval $[W(T_3(n,d,x,s)), W(T_3(n,d,x,s)) + 4k^2]$ is Wiener $4-$complete.*

Proof. Let us denote $k_1 = \frac{(n-1)-(2d+1)-1}{2}$. Note that k_1 is half of the number of leaves appended to the vertices $u_{\pm j}$ of the interior path of $T_3(n,d,x,s)$ for $j = 0,\ldots,k-1$. Since $s \leq 1$, note that the definition of $T_3(n,d,x,s)$ and Theorem 2.4 imply the result for $k = k_1 - 1$. □

So, let us now establish for which values of d the gap between $W(T_3(n,d,x,s))$ and $W(T_3(n,d,x-1,s))$ is smaller than $4k^2$ where $k = \frac{1}{2}n - d - \frac{5}{2}$.

Lemma 4.5. *Let n, d, $x \geq 2$ and s be integers for which $T_3(n,d,x,s)$ is defined. For $d \leq \frac{1}{2}(n - \sqrt{2n-6} - 5)$ the interval*

$$[W(T_3(n,d,x,s)), W(T_3(n,d,x-1,s))]$$

is Wiener $4-$complete.

Proof. First note that

$$\begin{aligned} T_3(n,d,x-1,s) - T_3(n,d,x,s) \leq T_3(n,d,2,s) - T_3(n,d,1,s) = \\ = 2(n-4)+2. \end{aligned}$$

Therefore, Lemma 4.4 implies it is sufficient to find integers n and d for which it holds that $4k^2 \geq 2(n-4)+2$ where $k = \frac{1}{2}n - d - \frac{5}{2}$. By a simple calculation it is easy to establish that the inequality holds for $d \leq \frac{1}{2}(n - \sqrt{2n-6} - 5)$ so the lemma is proved. □

Using Lemma 4.2 it is easy to establish that

$$W(T_3(n,d,x-1,s)) - W(T_3(n,d,x,s)) = 2(n-2x+1)$$

which is divisible by 4 since n is odd. Moreover, note that for $x_3^{\max} = \frac{4+4d-(n-1)}{2}$ it holds that

$$T_3(n,d,x_3^{\max},s) = T_3(n,d-1,1,s).$$

Therefore we can use Lemma 4.5 and "glue" together intervals both on the border between x and $x-1$ and on the border of d and $d-1$, so we will obtain one large interval which is Wiener 2−complete (because of Corollary 4.3).

Again, here we have used $T_3(n,d,x,s)$ to the maximum, but we have covered thus only caterpillars with large d. Let us now use graph $B_2(n,d,x)$ to create the fourth special kind of caterpillars which we will use to widen our interval to caterpillars with small d.

Definition 4.6. Let n,d and x be integers for which $B_2(n-1,d,x)$ is defined. For $s = 0,1$ caterpillar $T_4(n,d,x,s)$ is a caterpillar on odd number of vertices n, obtained from $B_2(n-1,d,x)$ by appending a leaf to the vertex u_s of the path $P = u_{-d}\ldots u_{-1}u_0u_1\ldots u_d$ in $B_2(n-1,d,x)$.

Caterpillar graph $T_3(n,d,x,s)$ is illustrated by Figure 6 (vertex u_i of the path P is in the images denoted just by i in order to make labels easier to see).

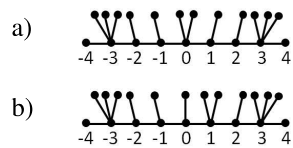

Figure 6: Caterpillar graphs: a) $T_4(21,4,3,0)$, b) $T_4(21,4,3,1)$.

Lemma 4.7. *Let n, d, x and s be integers for which $T_4(n,d,x,s)$ is defined. Then*

$$W(T_4(n,d,x,s)) = W(B_2(n-1,d,x)) + (2+d)n - 2d^2 - 3d - 1 + 2s^2 - 2x.$$

Proof. Let $k = d-1$ and $r = \frac{(n-1)-4d-2x+2}{2}$. The definition of $T_4(n,d,x,s)$ implies

$$\begin{aligned} W(T_4(n,d,x,s)) = W(B_2(n-1,d,x)) + \sum_{i=-d}^{d}(|s-i|+1) + \\ + \sum_{i=-(k-1)}^{k-1}(|s-i|+2) + (s-x'+2) + \\ + 2x(d+1) + 2r(d+2). \end{aligned}$$

Simplifying this expression yields the result. □

Corollary 4.8. *It holds that $W(T_4(n,d,x,1)) = W(T_4(n,d,x,0)) + 2$.*

Let us now apply Theorem 2.4 to $T_4(n, d, x, s)$.

Lemma 4.9. *Let* $n, d,$ x *and* s *be integers for which* $T_4(n, d, x, s)$ *is defined. For* $k = d-2$ *the interval* $[W(T_4(n, d, x, s)), W(T_4(n, d, x, s)) + 4k^2]$ *is Wiener* $4-$*complete.*

Proof. Let us denote $k_1 = d - 1$. Note that k_1 is half of the number of leaves appended to the vertices $u_{\pm j}$ of the interior path of $T_3(n, d, x, s)$ for $j = 0, \ldots, k-1$. Since $s \leq 1$, note that the definition of $T_4(n, d, x, s)$ and Theorem 2.4 imply the result for $k = k_1 - 1$. □

Now we can establish the minimum value of d for which the difference between Wiener index of $T_4(n, d, x, s)$ and $T_4(n, d, x-1, s)$ can be "covered" by Transformation A.

Lemma 4.10. *Let* $n, d,$ $x \geq 2$ *and* s *be integers for which* $T_4(n, d, x, s)$ *is defined. For* $d \geq \frac{1}{2}(\sqrt{2n-6}+4)$ *the interval*

$$[W(T_4(n, d, x, s)), W(T_4(n, d, x-1, s))]$$

is Wiener $4-$*complete.*

Proof. First note that for $x_4^{\max} = \frac{(n-1)-4d+2}{2}$ it holds that

$$\begin{aligned}
&W(T_4(n, d, x-1, s)) - W(T_4(n, d, x, s)) \\
&\leq W(T_4(n, d, x_4^{\max}-1, s)) - W(T_4(n, d, x_4^{\max}, s)) = \\
&= 2(n-4)+2.
\end{aligned}$$

Therefore, Lemma 4.9 implies it is sufficient to find integers n and d for which it holds that $4k^2 \geq 2(n-4)+2$ where $k = d-2$. By a simple calculation it is easy to establish that the inequality holds for $d \geq \frac{1}{2}(\sqrt{2n-6}+4)$ so the lemma is proved. □

Using Lemma 4.7 it is easy to establish that

$$W(T_4(n, d, x-1, s)) - W(T_4(n, d, x, s)) = 4(x+2d-2)$$

which is divisible by 4. Moreover, note that for $x_4^{\max} = \frac{(n-1)-4d+2}{2}$ it holds that

$$T_4(n, d, x_4^{\max}, s) = T_4(n, d-1, 1, s).$$

Therefore we can use Lemma 4.10 and "glue" together intervals both on the border between x and $x-1$ and on the border of d and $d-1$, so we will obtain one large interval which is Wiener $2-$complete (because of Corollary 4.8).

Finally, noting that for $d_3^{\min} = \lceil \frac{n-3}{4} \rceil$, $x_3^{\max} = \frac{4+4d_3^{\min}-(n-1)}{2}$ and $d_4^{\max} = \lfloor \frac{n-1}{4} \rfloor$ it holds that

$$T_3(n, d_3^{\min}, x_3^{\max}, s) = T_4(n, d_4^{\max}, 1, s),$$

we conclude that we can "glue" together two large Wiener $2-$complete intervals we obtained (one for large values of d and the other for small values of d), and thus we obtain the following theorem which gives us the largest Wiener $2-$complete interval we manage to obtain.

Theorem 4.11. *Let* $n \geq 21$, $d_3^{\max} = \left\lfloor \frac{1}{2}(n - \sqrt{2n-6} - 5) \right\rfloor$, $d_4^{\min} = \left\lceil \frac{1}{2}(\sqrt{2n-6} + 4) \right\rceil$ *and* $x_4^{\max} = \frac{(n-1)-4d_4^{\min}+2}{2}$. *The interval*

$$[W(T_4(n, d_4^{\min}, x_4^{\max}, 1)), W(T_3(n, d_3^{\max}, 1, 0))]$$

is Wiener $2-$*complete.*

Having the largest Wiener $2-$complete interval from the previous theorem, we can now finally prove the following theorem which is our main result and which proves Conjectures 1.1 and 1.2 in terms of n^3.

Theorem 4.12. *For odd* $n \geq 21$ *it holds that* $\left|W^{int}[\mathcal{T}_n]\right| = |W[\mathcal{T}_n]| = \frac{1}{12}n^3 + O(n^{5/2})$.

Proof. Using Theorem 3.13 we obtain

$$|W[\mathcal{T}_n]| \geq \left|W^{int}[\mathcal{T}_n]\right| \geq (W(T_3(n, d_3^{\max}, 1, 0)) - W(T_4(n, d_4^{\min}, x_4^{\max}, 1)))/2$$

where $d_3^{\max} = \frac{1}{2}(n - \sqrt{2n-6} - 5) - 1 + p$, $d_4^{\min} = \frac{1}{2}(\sqrt{2n-6} + 4) + q$ and $x_4^{\max} = \frac{(n-1)-4d_4^{\min}+2}{2}$ for $p \in \langle 0, 1]$ and $q \in [0, 1\rangle$. Now, using Lemmas 4.2 and 4.7 we further obtain

$$\begin{aligned} |W[\mathcal{T}_n]| \quad &\geq \quad \left|W^{int}[\mathcal{T}_n]\right| \geq \frac{1}{12}n^3 - \frac{1}{4}\sqrt{2n^5 - 6n^4} - \frac{3}{2}n^2 + \frac{5}{3}\sqrt{2n^3 - 6n^2} + \\ &+ \frac{83}{12}n - \frac{13}{12}\sqrt{2n-6} - \frac{253}{12}. \end{aligned}$$

On the other hand, Theorem 2.1 implies $\left|W^{int}[\mathcal{T}_n]\right| \leq |W[\mathcal{T}_n]| \leq (W(P_n) - W(S_n) + 1)/2 = \frac{1}{12}n^3 - \frac{1}{2}n^2 + \frac{11}{12}n$. □

Again, since caterpillars $T_3(n, d, x, s)$ are chemical trees for all possible values of parameters, and remain chemical after applying repeatedly Transformation A, half of these results hold for chemical trees, i.e. we obtain the following corollary.

Corollary 4.13. *Let* $\mathcal{T}_n^4$ *be a class of chemical trees on* n *vertices where* $n \geq 21$ *is odd. Then* $\left|W^{int}[\mathcal{T}_n^4]\right| = \left|W[\mathcal{T}_n^4]\right| = \Theta(n^3)$.

Proof. Using Lemma 4.5 we obtain

$$\left|W[\mathcal{T}_n^4]\right| \geq \left|W^{int}[\mathcal{T}_n^4]\right| \geq (W(T_3(n, d_3^{\max}, 1, 0)) - W(T_3(n, d_3^{\min}, x_3^{\max}, 1)))/2$$

where $d_3^{\min} = \frac{n-3}{4} + p$, $x_3^{\max} = \frac{4+4d_3^{\min}-(n-1)}{2}$ and $d_3^{\max} = \frac{1}{2}(n - \sqrt{2n-6} - 5) - 1 + q$ for $p \in [0, 1\rangle$ and $q \in \langle 0, 1]$. Using Lemma 4.2 we further obtain

$$\left|W^{int}[\mathcal{T}_n^4]\right| \geq \frac{1}{24}n^3 - \frac{1}{8}\sqrt{2n^5 - 6n^4} - \frac{11}{16}n^2 + \frac{5}{6}\sqrt{2n^3 - 6n^2} + \frac{19}{12}n - \frac{61}{24}\sqrt{2n-6} - \frac{685}{48}.$$

□

Again, this result for chemical trees is obtained by considering only chemical trees with large diameter so it probably can be significantly improved, but we leave that as an open problem for future research.

5 Conclusion

In this paper we have proved that in the case of even n the cardinality of the largest interval $W^{int}[\mathcal{T}_n]$ of consecutive integers which are values of the Wiener index for a tree graph on n vertices equals $\frac{1}{6}n^3 + O(n^{5/2})$. In the case of odd n the value of the Wiener index for a tree on n vertices can be only even number, therefore the cardinality of the largest interval $W^{int}[\mathcal{T}_n]$ of consecutive even integers which are values of the Wiener index for a tree graph on n vertices equals $\frac{1}{12}n^3 + O(n^{5/2})$. Since the set $W^{int}[\mathcal{T}_n]$ is a subset of the set $W[\mathcal{T}_n]$ of all values of the Wiener index for trees on n vertices, this immediately yields the same result on the cardinality of $W[\mathcal{T}_n]$. The upper bound $|W[\mathcal{T}_n]| \leq \frac{1}{6}n^3 - n^2 + \frac{11}{6}n$ (i.e. $|W[\mathcal{T}_n]| \leq \frac{1}{12}n^3 - \frac{1}{2}n^2 + \frac{11}{12}n$ for odd n) is easily established by calculating the difference of the value of the Wiener index for maximal and minimal tree graphs (the path P_n and the star S_n respectively). Comparing this bound with our results it is readily seen that our results are best possible with respect to n^3. Yet, with respect to n^2 the results are not so good, because we obtained $|W[\mathcal{T}_n]| = \left|W^{int}[\mathcal{T}_n]\right| = \frac{1}{6}n^3 + O(n^{5/2})$ (i.e. $\frac{1}{12}n^3 + O(n^{5/2})$ in the case of odd n). This may be due to the fact that in the paper we aimed at the bound for $\left|W^{int}[\mathcal{T}_n]\right|$ and we stopped with our search when the interval was interrupted (when the diameter of a tree became too small or too large). There is the possibility that the same approach extended to the caterpillars of all diameters would yield sufficient improvement on $|W[\mathcal{T}_n]|$ to reduce $O(n^{5/2})$ to $O(n^2)$. But we leave that for future research.

Furthermore, in our research we focused only on caterpillar trees, so the obvious corollary is that the same results hold in the narrower class of caterpillar trees. Half of the caterpillars we used are chemical trees, which yields relatively strong result for the class of chemical trees as a direct corollary. Also, we researched the caterpillars grouped by the length of the interior path (which is nearly the diameter), so the results for trees with given diameter would also follow easily though it is questionable how strong those results would be. Researching the same question in the classes of trees with other given parameters might also be interesting direction of future research.

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Groups in which every non-nilpotent subgroup is self-normalizing

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Abstract

We study the class of groups having the property that every non-nilpotent subgroup is equal to its normalizer. These groups are either soluble or perfect. We describe soluble groups and finite perfect groups with the above property. Furthermore, we give some structural information in the infinite perfect case.

Keywords: Normalizer, non-nilpotent subgroup, self-normalizing subgroup.

Math. Subj. Class.: 20E34, 20D15, 20E32

1 Introduction

A long standing problem posed by Y. Berkovich [3, Problem 9] is to study the finite p-groups in which every non-abelian subgroup contains its centralizer.

In [6], the finite p-groups which have maximal class or exponent p and satisfy Berkovich's condition are characterized. Furthermore, the infinite supersoluble groups with the same condition are completely classified. Although it seems unlikely to be able to get a full classification of finite p-groups in which every non-abelian subgroup contains its centralizer, Berkovich's problem has been the starting point for a series of papers investigating finite and infinite groups in which every subgroup belongs to a certain family or it contains

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its centralizer. For instance, in [7] and [9] locally finite or infinite supersoluble groups in which every non-cyclic subgroup contains its centralizer are described.

A more accessible version of Berkovich's problem has been proposed by P. Zalesskii, who asked to classify the finite groups in which every non-abelian subgroup equals its normalizer. This problem has been solved in [8].

In this paper we deal with the wider class $\mathcal{S}$ of groups in which every non-nilpotent subgroup equals to its normalizer. All nilpotent groups (and hence all finite p-groups) are in $\mathcal{S}$. It is also easy to see that groups in $\mathcal{S}$ are either soluble or perfect. Further obvious examples of groups in $\mathcal{S}$ include the minimal non-nilpotent groups (that is, non-nilpotent groups in which every proper subgroup is nilpotent) and groups in which every subgroup is self-normalizing. Finite minimal non-nilpotent groups are soluble, and their structure is well known (see [18, 9.1.9]). Infinite minimal non-nilpotent groups have been first studied in [14] (see also [4] for more recent results). These groups are either finitely generated or locally finite p-groups (Černikov groups or Heineken-Mohamed groups). Ol'shanskii and Rips (see [15]) showed that there exist finitely generated infinite simple groups all of whose proper non-trivial subgroups are cyclic of the same order (the so-called Tarski monsters). On the other side, groups whose non-trivial subgroups are self-normalizing are periodic and simple. Furthermore, in the locally finite case they are trivial or of prime order. Again, infinite examples are the Tarski p-groups.

We describe soluble groups lying in the class $\mathcal{S}$. It turns out that an infinite polycyclic group lies in the class $\mathcal{S}$ if and only if it is nilpotent (Proposition 3.3). We also prove that a non-periodic soluble group belongs to the class $\mathcal{S}$ if and only if it is nilpotent (Theorem 3.4). Moreover, a periodic soluble group which is not locally nilpotent lies in the class $\mathcal{S}$ if and only if it is a split extension of a nilpotent p'-group by a cyclic p-group whose structure is described in Theorem 3.5. In particular, this result characterizes non-nilpotent soluble finite groups in the class $\mathcal{S}$. Furthermore, a locally nilpotent soluble group belongs to the class $\mathcal{S}$ if and only if it is either nilpotent or minimal non-nilpotent (Theorem 3.7).

In the last part of the paper we prove that a finite perfect group lies in the class $\mathcal{S}$ if and only if it is either isomorphic to the group $\mathrm{PSL}_2(2^n)$ where $2^n - 1$ is a prime number, or to the group $\mathrm{SL}_2(5)$ (Theorem 4.8). Finally, we give some information on the structure of infinite perfect groups lying in the class $\mathcal{S}$.

Our notation is mostly standard (see for instance [3] and [18]). In particular, given any group G, we will denote by $Z(G)$ the center of G, by $Z^\infty(G)$ the hypercenter of G, by $\Phi(G)$ the Frattini subgroup of G, by G' the commutator subgroup of G, and, for all integers $i \geq 1$, by $\gamma_i(G)$ the i-th term of the lower central series of G.

2 General properties of groups in $\mathcal{S}$

It is very easy to prove that the class $\mathcal{S}$ is subgroup and quotient closed. Furthermore, non-nilpotent groups in $\mathcal{S}$ are not products of two proper normal subgroups.

Recall that a group G is said to be perfect if it equals its commutator subgroup G'. Clearly, if $G \in \mathcal{S}$ then G is perfect or G' is nilpotent. Hence the groups in $\mathcal{S}$ are either perfect or soluble.

Suppose now that a cyclic group $\langle x\rangle$ acts on a group H by means of an automorphism x. If a subgroup L of H is invariant with respect to $\langle x\rangle$, we will write $L \leq_x H$. Consider the induced map

$$\rho_x \colon H \to H, \quad \rho_x(h) = [x, h] = h^{-x}h.$$

Clearly, if H is abelian then ρ_x is a homomorphism. We will describe groups belonging to the class $\mathcal{S}$ based on the following property of ρ_x:

$$\forall K \leq_x H,\ (\exists n \geq 1\colon\ \rho_x^n(K) = 1\ \vee\ \langle \rho_x(K)\rangle = K). \tag{$\star$}$$

Lemma 2.1. *Let x act on H by means of an automorphism. Then for every $K \leq_x H$ we have $\langle \rho_x(K)\rangle K' = \rho_x(K)K'$.*

Proof. Let $h_1, h_2 \in K$. Then $[x, h_1h_2] = [x, h_2][x, h_1][x, h_1, h_2]$. It follows that $\rho_x(h_1h_2) \equiv \rho_x(h_1)\,\rho_x(h_2) \pmod{K'}$. □

The following easy observations are used in the sequel.

Lemma 2.2. *Let x act on H by means of an automorphism.*

1. *The action of x is fixed point free if and only if ρ_x is injective.*
2. *If ρ_x is injective and H is abelian, then $(\star)$ implies that ρ_x is an isomorphism.*
3. *If ρ_x is injective (or surjective) and H is finite, then ρ_x satisfies $(\star)$.*

Proof. (i) Note that ρ_x is injective if and only if whenever $[x, h] = 1$ it follows that $h = 1$. This is precisely the same as x acting fixed point freely on H.

(ii) Of course we can assume that H is non-trivial. If ρ_x is injective then there is no positive integer n with the property that $\rho_x^n(K) = 1$, and so $(\star)$ implies that $\langle \rho_x(H)\rangle = H$. If in addition H is abelian then ρ_x is a homomorphism, and hence $\langle \rho_x(H)\rangle = \rho_x(H)$. Therefore ρ_x is an isomorphism.

(iii) If H is assumed to be finite, then ρ_x is injective if and only if it is surjective. In this case ρ_x is bijective, and we have that $\rho_x(K) = K$ for all $K \leq_x H$. Thus ρ_x satisfies $(\star)$. □

Lemma 2.3. *Let $G = \langle x\rangle H$, where H is a nilpotent normal subgroup of G generated by a set Y. Suppose that there exists $n \geq 1$ such that $\rho_x^n(y) = 1$ for every $y \in Y$. Then G is nilpotent.*

Proof. By a theorem of Hall (see for instance [16, Theorem 2.27]) it suffices to show that G/H' is nilpotent. The group H/H' is generated by all yH' with $y \in Y$ and ρ_x induces an endomorphism τ of H/H' such that $\tau^n(H/H') = 1$. Now G/H' is nilpotent of class at most n since $H/H' \subseteq Z_n(G/H')$. □

Lemma 2.4. *Let $G = \langle x\rangle \ltimes H$ be a non-nilpotent group where x has prime order p and H is nilpotent. Assume that ρ_x has property $(\star)$ and suppose that there exists a subgroup $1 \neq K \leq_x H$ such that $\rho_x^n(K) = 1$. Then $Z(G) \neq 1$.*

Proof. As $\langle x\rangle \ltimes K$ is nilpotent by Lemma 2.3, it has a non-trivial center. Thus there exists an element $1 \neq h \in C_K(x)$. Now consider the group $\langle x\rangle \ltimes Z(H)$. By property $(\star)$, we either have $\rho_x(Z(H)) = Z(H)$ or there is a positive integer n such that $\rho_x^n(Z(H)) = 1$. In the latter case, we certainly have an element that belongs to $Z(H)$ and commutes with x, so that $Z(G) \neq 1$. Suppose now that $\rho_x(Z(H)) = Z(H)$ holds. By property $(\star)$ we have

$$\langle Z(H), h\rangle = \rho_x(\langle Z(H), h\rangle) = \rho_x(Z(H)) = Z(H),$$

and hence $h \in Z(H)$. Thus we again have $Z(G) \neq 1$. □

The following proposition shows how property $(\star)$ is tightly related to the class $\mathcal{S}$.

Proposition 2.5. *Let $G = \langle x \rangle \ltimes H$ be a group in $\mathcal{S}$ with x^p acting trivially on a nilpotent subgroup H for some prime p. Then ρ_x has property $(\star)$.*

Proof. Let $K \leq_x H$, and suppose $\langle \rho_x(K) \rangle \subsetneq K$. Consider the subgroup

$$L = \langle x \rangle \ltimes \langle \rho_x(K) \rangle K'$$

of G. As K is nilpotent, it follows that $\langle \rho_x(K) \rangle K' \subsetneq K$ (see for instance [16, Lemma 2.22]). Therefore L is a proper normal subgroup of $\langle x \rangle \ltimes K$. Now, since $\langle x \rangle \ltimes K$ belongs to $\mathcal{S}$, it follows that its normal subgroup L must be nilpotent, and so $\rho_x^n(K) = 1$ for some positive integer n. Therefore ρ_x has property $(\star)$. □

Let p be any prime number. An abelian group A is said to be p-divisible if $A = pA$.

Lemma 2.6. *Let x be an automorphism of order p of an abelian group A. If ρ_x is surjective, then A is p-divisible.*

Proof. Consider A as a $\mathbb{Z}[\langle x \rangle]$-module. In this sense, the operator ρ_x corresponds to the element $1 - x \in \mathbb{Z}[\langle x \rangle]$. We have $(1-x)^p \equiv 0$ modulo $p\mathbb{Z}[\langle x \rangle]$, and so the image of $(\rho_x)^p$ is a subgroup of $p\mathbb{Z}[\langle x \rangle]A = pA$. As ρ_x is assumed to be surjective, it follows that $A = pA$. □

Lemma 2.7. *Let $G = \langle x \rangle \ltimes H$ be a periodic non-nilpotent group with $x^p = 1$ for some prime p and H a nilpotent p'-group. Assume that ρ_x has property $(\star)$. Then every non-nilpotent subgroup $L \leq G$ is conjugate to a subgroup of the form $\langle x \rangle \ltimes K$ for some $K \leq_x H$.*

Proof. Since H is nilpotent, L is not contained in H. It easily follows that L contains an element of the form xh for some $h \in H$. As G is non-nilpotent, it follows from property $(\star)$ and Lemma 2.1 that

$$h \in H = \langle \rho_x(H) \rangle \subseteq \rho_x(H)H'.$$

Hence we can write $h = \rho_x(h_1)h'$ with $h_1 \in H$ and $h' \in H'$. Thus $xh = x^{h_1}h'$. After possibly replacing L by $L^{h_1^{-1}}$, we can assume that $xh' \in L$ for some $h' \in H'$. As ρ_x has property $(\star)$, we have that either $\rho_x^n(H') = 1$ for some positive integer n, or $\langle \rho_x(H') \rangle = H'$.

In the first case $\langle x \rangle \ltimes H'$ is nilpotent by Lemma 2.3. The subgroup $\langle x, h' \rangle$ is finitely generated, periodic and nilpotent, therefore it is finite. Let c be its nilpotency class, and let p^k be the largest power of p that divides $c!$. Set m to be a positive solution to the congruence system

$$\begin{cases} m \equiv 0 & \left(\bmod \dfrac{|h'| c! \exp(\gamma_2(\langle x, h' \rangle))}{p^k}\right) \\ m \equiv 1 & (\bmod p^k). \end{cases}$$

Thus $m(m-1)$ divisible by $c!$, m is divisible by $\exp(\gamma_2(\langle x, h' \rangle))$ and by $|h'|$, and m is coprime to p. In particular, $\binom{m}{i}$ is divisible by $\exp(\gamma_2(\langle x, h' \rangle))$ for all $1 \leq i \leq c$. By the Hall-Petrescu formula (see for instance [3, Appendix 1]) we get

$$(xh')^m = x^m h'^m g_2^{\binom{m}{2}} \ldots g_c^{\binom{m}{c}}$$

with $g_i \in \gamma_i(\langle x, h' \rangle)$. By the choice of m, it follows that $(xh')^m = x^m$. This element belongs to L, and since x has p-power order, we conclude that $x \in L$.

Consider now the case when $\langle \rho_x(H') \rangle = H'$. Thus $\langle x \rangle \ltimes H'$ is non-nilpotent, and we can repeat the argument from above with H replaced by H'. Since H is nilpotent, after finitely many steps we find a conjugate of L that contains x. Replacing L by this conjugate we can thus write $L = \langle x \rangle \ltimes K$ for $K = L \cap H \leq_x H$. □

Proposition 2.8. *Let $G = \langle x \rangle \ltimes H$ be a periodic group with x^p acting trivially on a nilpotent p'-group H for some prime p. Then $G \in \mathcal{S}$ if and only if ρ_x has property $(\star)$.*

Proof. If $G \in \mathcal{S}$, then ρ_x has property $(\star)$ by Lemma 2.5.

Conversely, assume now that ρ_x has property $(\star)$. To prove that G belongs to the class $\mathcal{S}$, take any non-nilpotent subgroup L of G. By Lemma 2.7, we can assume that L is of the form $L = \langle x \rangle \ltimes K$ for some $K \leq_x H$. Let us now show that L is self-normalizing in G. To this end, take an element $x^j c \in N_G(L)$. Then $x^{x^j c} = x^c = x\,\rho_x(c)$, and so we must have $\rho_x(c) \in K$. Furthermore, we also have that $c = x^{-j}(x^j c) \in N_H(K)$. Note that $\langle \rho_x^{-1}(K) \cap N_H(K) \rangle \leq_x H$. It follows that

$$N_G(L) = \langle x \rangle \ltimes \langle \rho_x^{-1}(K) \cap N_H(K) \rangle.$$

Now, for any $h_1, h_2 \in \langle \rho_x^{-1}(K) \cap N_H(K) \rangle$, we have that

$$\rho_x(h_1 h_2) = \rho_x(h_2)\,\rho_x(h_1)[x, h_1, h_2] \in K \cdot K \cdot [K, N_H(K)] \subseteq K.$$

Therefore ρ_x maps $\langle \rho_x^{-1}(K) \cap N_H(K) \rangle$ into K. Since $N_G(L)$ is not nilpotent, it follows from property $(\star)$ that

$$\langle \rho_x^{-1}(K) \cap N_H(K) \rangle = \langle \rho_x(\langle \rho_x^{-1}(K) \cap N_H(K) \rangle) \rangle \subseteq K.$$

This implies $N_G(L) \leq \langle x \rangle \ltimes K = L$, as required. □

Remark 2.9. Assume $G = \langle x \rangle \ltimes H$ with H an abelian finite group and x acting so that x^p acts trivially on H. Then $G \in \mathcal{S}$ if an only if ρ_x has property $(\star)$, which in this case is equivalent by Lemma 2.2 to x acting fixed point freely on H. Confer [8, Theorem 2.13].

Remark 2.10. Let G be a periodic group in $\mathcal{S}$ with a splitting $\langle x \rangle \ltimes H$, and assume that $x^p = 1$. It might not be the case that x acts fixed point freely on H (see Example 2.11). In such a situation, we have $C_G(x) \cap H \neq 1$. Therefore $C_H(x)$ is a subgroup of H with $\rho_x(C_H(x)) = 1$. It follows from Lemma 2.4 that $Z(G) \neq 1$. Now consider the factor group $\langle x \rangle \ltimes H/Z^\infty(G)$. This group is centerless, and so by the above argument x must act fixed point freely on $H/Z^\infty(G)$.

Example 2.11. Let x be the automorphism of the quaternion group Q_8 given by

$$i \mapsto j, \qquad j \mapsto -k.$$

This is an automorphism of order 3. Form the semidirect product $G = \langle x \rangle \ltimes Q_8 \cong \mathrm{SL}_2(3)$. It is readily verified that all proper subgroups of G are nilpotent, and so $G \in \mathcal{S}$. Note that x has a non-trivial fixed point on Q_8, namely $(-1)^x = -1$. So we have $Z^\infty(G) = \langle -1 \rangle$ with $G/\langle -1 \rangle \cong \mathrm{Alt}(4)$, and x acts fixed point freely on $Q_8/\langle -1 \rangle \cong C_2 \times C_2$.

3 Soluble groups in the class $\mathcal{S}$

In this section, we inspect soluble groups in the class $\mathcal{S}$. It turns out that the non-periodic case is only possible if the group is nilpotent, whereas the periodic case is more subtle.

A Fitting group is one which equals its Fitting subgroup. Thus a Fitting group is a product of nilpotent normal subgroups, and therefore it is locally nilpotent. If $G \in \mathcal{S}$ and F denote the Fitting subgroup of G, then clearly G is a Fitting group or F is nilpotent.

Lemma 3.1. *Let $G \in \mathcal{S}$ be a soluble group, and F the Fitting subgroup of G. Then the following hold:*

1. *$G' \leq F$;*
2. *G is a Fitting group or G/F has prime order;*
3. *if G/G' is finitely generated then G is a Fitting group or G/G' is cyclic of prime-power order.*
4. *if G is non-nilpotent then the quotient group G/G' is a locally cyclic p-group for some prime p, and $G' = \gamma_3(G)$.*

Proof. Because G' is normal in G and G is assumed to belong to $\mathcal{S}$, it follows that G' must be nilpotent. Whence it is contained in the Fitting subgroup of G, proving (i).

Suppose now that G is not a Fitting group. Since $G' \leq F$ by (i), the group G/F is abelian. Let N/F be any proper subgroup of G/F. Then N is a proper normal subgroup of G, so N is nilpotent. Thus $N \leq F$. Therefore G/F has no proper non-trivial subgroups, so it has prime order. Hence (ii) is proved.

In order to prove (iii), suppose that G/G' is finitely generated and G is not a Fitting group. Let M_1/G' and M_2/G' be maximal subgroups of G/G'. Since M_1 and M_2 are proper normal subgroups of G, they are nilpotent and hence contained in F. If $M_1 \neq M_2$ it follows that $G = M_1M_2 \leq F$, a contradiction. This means that the finitely generated group G/G' has an unique maximal subgroup. Therefore G/G' is cyclic of prime-power order.

Finally assume G is non-nilpotent, and let H and K be proper normal subgroups of G. Then H and K are nilpotent, hence HK is nilpotent and so $HK \neq G$. Thus (iv) follows by [14, Theorem 2.12]. □

Proposition 3.2. *Let $G = \langle x \rangle \ltimes A$ where A is non-periodic abelian, x acts fixed point freely on A and x^p acts trivially on A for some prime p. Then G does not lie in the class $\mathcal{S}$.*

Proof. Consider $W = \langle x, y \rangle$ with $y \in A$ and of infinite order. The group $W \cap A$ is abelian and finitely generated, since there are only finitely many conjugates of y in W. So also the torsion subgroup T of $W \cap A$ has finite order, m say. Then $(W \cap A)^m$ is finitely generated and torsion-free. By Lemma 2.6, the map ρ_x restriced to $(W \cap A)^m$ is not surjective. Since x acts fixed point freely on A and therefore also on $(W \cap A)^m$, the group $\langle x \rangle \ltimes (W \cap A)^m$ is not nilpotent, and so $\rho_x^n((W \cap A)^m) \neq 1$ for all integers n. Whence ρ_x restricted to $(W \cap A)^m$ does not have property $(\star)$. It follows from Proposition 2.5 that G does not belong to the class $\mathcal{S}$. □

Recall that a group is called just-infinite if it is infinite, but each of its proper quotients is finite. In particular a just-infinite group has no non-trivial finite normal subgroups. The Baer radical of a group G is the subgroup generated by all the cyclic subnormal subgroups of G. It has been proved in [19, Theorem 2] that if G is a just-infinite group with non-trivial Baer radical A then A is free abelian of finite rank and $C_G(A) = A$.

Proposition 3.3. *Every infinite polycyclic group in $\mathcal{S}$ is nilpotent.*

Proof. Assume by a contradiction that the result is false, and let $G \in \mathcal{S}$ be an infinite polycyclic group which is non-nilpotent. Then Lemma 3.1 (iii) ensures that G/G' is cyclic of prime-power order. It easily follows (see for instance [14, Corollary 2.11]) that $G' = \gamma_3(G)$.

Let consider the set $\mathcal{F}$ of all normal subgroups N of G such that G/N is infinite and non-nilpotent. Thus $\mathcal{F}$ is not empty, as it contains the trivial subgroup. Since G satisfies the maximal condition on subgroups, there exists a maximal element $M \in \mathcal{F}$. Hence G/M is a non-nilpotent infinite polycyclic group in $\mathcal{S}$. Moreover, if G/N is any infinite quotient of G/M, then the maximality of M implies that either $N = M$ or G/N is nilpotent. Suppose the latter holds. Then there exists a positive integer t such that $\gamma_t(G) \leq N$. Thus $G' \leq N$, a contradiction since G/G' is finite. Therefore, at expense of replacing G by G/M, we may assume that G is just-infinite.

Let F denote the Fitting subgroup of G. Thus F coincides with the Baer radical of G. Then F is free abelian of finite rank by [19, Theorem 2]. Furthermore, by Lemma 3.1 (ii) we may assume that G/F has prime order p. Hence, for all $x \in G \setminus F$ we can write $G = \langle x \rangle F$ with $x \notin F$ and $x^p \in F$. We claim that $F/C_F(x)$ is torsion-free. Indeed, suppose $a \in F \setminus C_F(x)$ with $a^n \in C_F(x)$ for some integer n. Thus $(a^n)^x = a^n$. Since F is abelian it follows that $(a^x a^{-1})^n = 1$. Hence $n = 0$ since F is normal and torsion-free. Therefore $F/C_F(x)$ is torsion-free, as claimed. Thus $G/C_F(x)$ is infinite, which implies that $C_F(x) = 1$. Hence x acts fixed point freely on F. By Proposition 3.2, the group G does not belong to the class $\mathcal{S}$, our final contradiction. □

Theorem 3.4. *A non-periodic soluble group belongs to the class $\mathcal{S}$ if and only if it is nilpotent.*

Proof. Clearly, if G is nilpotent then $G \in \mathcal{S}$.

Conversely, let $G \in \mathcal{S}$ be a non-periodic soluble group, and assume that G is non-nilpotent. Let us first prove that in this case G is locally nilpotent.

By assumption, there exists a finitely generated infinite subgroup of G, say H. Let K be any finitely generated subgroup of G. Then the subgroup $J = \langle K, H \rangle$ is finitely generated and infinite. Assume that J is non-nilpotent. Then its Fitting subgroup F is nilpotent, and J/F has prime order by Lemma 3.1 (ii). Hence J is polycyclic, a contradiction by Proposition 3.3. Thus J is nilpotent, and so is K. Therefore G is locally nilpotent. Then by Lemma 3.1 (iv) the quotient group G/G' is a p-group for some prime p.

Assume now that G is torsion-free. Since G' is nilpotent and G/G' is periodic, it follows by [17, Lemma 6.33] that G is nilpotent, again a contradiction.

We are left with the case when the torsion subgroup T of G is non-trivial. Since G/T is a torsion-free soluble group belonging to the class $\mathcal{S}$, it is nilpotent by the above. As G/G' is a p-group it follows that the quotient group $(G/T)/(G/T)'$ is a p-group. Hence G/T is a p-group (see for instance [18, 5.2.6]). Therefore G is periodic, our final contradiction. □

Next two results give a complete description of periodic soluble groups in $\mathcal{S}$. In particular, our next theorem characterizes finite soluble non-nilpotent groups in $\mathcal{S}$.

Theorem 3.5. *Let G be a periodic soluble group, and assume that G is not locally nilpotent. Then $G \in \mathcal{S}$ if and only if G splits as $G = \langle x \rangle \ltimes H$, where $\langle x \rangle$ is a p-group for some prime p, H is a nilpotent p'-group, x^p acts trivially on H and ρ_x has property $(\star)$.*

Proof. If G splits according to the above statement, then it follows from Propostion 2.8 that G belongs to the class $\mathcal{S}$.

Assume now that $G \in \mathcal{S}$. Let x be an element of G that does not belong to the Fitting subgroup F. Then $x^p \in F$ for some prime p, by Lemma 3.1. After possibly replacing x by one of its powers, we can assume that the order of x is a power p^m of p. As G is not a Fitting group, we have that F is nilpotent. Hence F is a product of its Sylow subgroups, say $F = \prod_q S_q$. Note that there is at least one prime $q \neq p$ involved: otherwise F is a p-group, hence G is a p-group, but this yields that G is locally nilpotent since it is locally finite, a contradiction. The Sylow subgroups of F are all characteristic in F, so the conjugation action of x preserves them. Therefore x acts component-wise on F.

Note that $x^p \in S_p$. Consider the subgroup $P = \langle x, S_p \rangle$. Clearly, P is a p-group. Assume that $P \neq \langle x \rangle$, and choose an element $y \in P \setminus \langle x \rangle$. Set

$$J = \left\langle x, y, y^x, y^{x^2}, \dots, y^{x^{p^m-1}} \right\rangle.$$

Since J is a finite p-group, and $\langle x \rangle \neq J$, it follows that

$$\langle x \rangle \subsetneq N_J(\langle x \rangle) \subseteq N_P(\langle x \rangle).$$

Hence there exists an element $z \in N_P(\langle x \rangle) \setminus \langle x \rangle$. Thus G contains the subgroup $\langle x \rangle \ltimes \prod_{q \neq p} S_q$ that is not self-normalizing. By assumption, this subgroup must be nilpotent. Since x acts component-wise on F, it follows that G itself should be nilpotent. This is a contradiction, from which it follows that $P = \langle x \rangle$, and so $S_p = \langle x^p \rangle$. This immediately implies that G splits as $G = \langle x \rangle \ltimes H$ with x acting component-wise on $H = \prod_{q \neq p} S_q$.

Since $x^p \in F$, it commutes with all the q-Sylow subgroups of F for $q \neq p$. As the p-Sylow subgroup S_p is cyclic, it follows that $x^p \in Z(G)$.

Finally, let $K \leq_x H$ with $\rho_x^n(K) \neq 1$ for all integers n. Therefore the group $\langle x \rangle \ltimes K$ is non-nilpotent. Consider the group $\langle x \rangle \ltimes \langle \rho_x(K) \rangle K'$. It is a normal subgroup of $\langle x \rangle \ltimes K$, so it must either be equal to $\langle x \rangle \ltimes K$, or else it is nilpotent. The latter case implies that the group $\langle x, \rho_x(K) \rangle$ is nilpotent, which gives that $\rho_x^n(K) = 1$ for some n, a contradiction. Hence we get that $\langle \rho_x(K) \rangle K' = K$, and since K is nilpotent, it follows that $\langle \rho_x(K) \rangle = K$. Thus ρ_x has property $(\star)$. $\square$

Corollary 3.6. *Let $n > 2$. The dihedral group $Dih(n)$ of order $2n$ belongs to $\mathcal{S}$ if and only if either n is a power of 2 or n is odd.*

Theorem 3.7. *A locally nilpotent soluble group lies in the class $\mathcal{S}$ if and only if it is either nilpotent or minimal non-nilpotent.*

Proof. Clearly, nilpotent and minimal non-nilpotent groups belong to the class $\mathcal{S}$.

Let $G \in \mathcal{S}$ be a periodic soluble group which is locally nilpotent, and assume that G is non-nilpotent. We will prove that G is minimal non-nilpotent. For the sake of contradiction, assume that there exists a proper non-nilpotent subgroup H of G. Let B be the last term of the derived series of G which is not contained in H. Then HB has the proper non-nilpotent subgroup H. Hence without loss of generality we may assume that $G = HB$.

Put $L = B \cap H$. Then $B' \leq L$, so L is normal in B. Obviously L is normal in H, thus L is normal in G. The normal series $L < B < G$ can be refined to a (general) principal series of G (see for instance [18, 12.4.1]). Let W/V be any factor of this principal series with $W \leq B$. As G is locally nilpotent, the principal factor W/V is central (see for instance [18, 12.1.6]). Hence $[W, G] \leq V$. This implies that $W \leq N_G(HV) = HV$. Therefore

$$W = W \cap HV = (W \cap H)V \leq LV = V.$$

This means $L = B$, a contradiction, and that proves our result. □

Corollary 3.8. *A locally nilpotent soluble group lying in the class $\mathcal{S}$ is nilpotent or a p-group for some prime p.*

Proof. Let $G \in \mathcal{S}$ be a locally nilpotent soluble periodic group, and assume that G is non-nilpotent. Then by Theorem 3.7 the group G is minimal non-nilpotent, and the result follows by [14, Lemma 4.2]. □

4 Perfect groups in the class $\mathcal{S}$

Lemma 4.1. *Let $G \in \mathcal{S}$ be a finite perfect group, and let F denote its Fitting subgroup. Then G/F is a non-abelian simple group.*

Proof. If there is a proper normal subgroup $F \leq M < G$, then M must be nilpotent since $G \in \mathcal{S}$, and so $M = F$. Thus G/F is simple. As G is also assumed to be perfect, G/F is non-abelian. □

We first classify the finite simple groups in $\mathcal{S}$. This is done with the help of the following lemma.

Lemma 4.2. *Let G be a finite simple group. Then G belongs to $\mathcal{S}$ if and only if all of its maximal subgroups belong to $\mathcal{S}$.*

Proof. Assume that all maximal subgroups of a finite simple group G belong to $\mathcal{S}$, and let H be a non-nilpotent proper subgroup of G. As G is simple, we have $N_G(H) < G$, and so there is a maximal subgroup $M \leq G$ with $N_G(H) \leq M$. Since M belongs to $\mathcal{S}$, it follows that $N_G(H) = N_M(H) = H$, as required. □

Lemma 4.3. *The group $\mathrm{PSL}_2(q)$ belongs to $\mathcal{S}$ if and only if $q = 2^n$ with $q - 1$ a prime, or $q \leq 5$.*

Proof. Suppose that $\mathrm{PSL}_2(q)$ belongs to $\mathcal{S}$ with $q > 5$, and assume first that q is odd. This group contains dihedral subgroups of orders $(q-1)/2$ and $(q+1)/2$ by [10]. Unless $q = 7$, at least one of these does not belong to $\mathcal{S}$ by Corollary 3.6. Note that $\mathrm{PSL}_2(7)$ has a subgroup isomorphic to $\mathrm{Sym}(4)$, so it does not belong to $\mathcal{S}$. Whence we can assume that $q = 2^n$ for some $n \geq 3$. Now $\mathrm{PSL}_2(q)$ contains a diagonal torus of order $q-1$ acting fixed point freely on the unipotent subgroup of order q. It follows from Lemma 3.1 that the torus must be simple, and so $q-1$ is either trivial or a prime, as required. Finally, it follows from [8, Theorem 2.17] that such groups indeed belong to $\mathcal{S}$. □

Proposition 4.4. *A finite non-abelian simple group belongs to $\mathcal{S}$ if and only if it is isomorphic to $\mathrm{PSL}_2(2^n)$, where $2^n - 1$ is a prime.*

Proof. We reduce the situation to the case of Lemma 4.3 by using Lemma 4.2.

- *Alternating groups*. It may be verified readily that $\mathrm{Alt}(n)$ belongs to $\mathcal{S}$ if and only $n = 5$, since $\mathrm{Sym}(4)$ is contained in $\mathrm{Alt}(n)$ for every $n \geq 6$.
- *Linear groups* $\mathrm{PSL}_n(q)$. If $n = 2$, this case is covered by Lemma 4.3. If $n \geq 3$, then there is a block embedding of $\mathrm{SL}_2(q)$ into $\mathrm{PSL}_n(q)$. The image of this subgroup is normalized by the class of a diagonal matrix of the form $\mathrm{diag}(\alpha, \beta, \gamma, 1, \ldots, 1)$. As long as $\alpha \neq \beta$, this diagonal matrix does not belong to the image of the embedding of $\mathrm{SL}_2(q)$, and so $\mathrm{PSL}_n(q)$ does not belong to $\mathcal{S}$. The only exceptional case is when $|\mathbb{F}_q^\times| = 1$, i.e., $q = 2$, in which case either $n = 3$ or $\mathrm{PSL}_n(2)$ contains $\mathrm{SL}_3(2)$ via a block diagonal embedding. Both of these groups quotient onto $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, which does not belong to $\mathcal{S}$.
- *Symplectic groups* $\mathrm{PSp}_{2n}(q)$. If $n = 1$, then $\mathrm{PSp}_2(q) \cong \mathrm{PSL}_2(q)$ and this is covered above. Now let $n > 1$. Letting W be a maximal isotropic subspace of the $2n$-dimensional vector space on which $\mathrm{Sp}_{2n}(q)$ acts, the stabilizer of the decomposition $W \oplus W^\perp$ is $\mathrm{GL}_n(q) \rtimes C_2$, and so $\mathrm{PSp}_{2n}(q)$ contains $\mathrm{PGL}_n(q) \rtimes C_2$. Therefore these groups do not belong to $\mathcal{S}$.
- *Unitary groups and orthogonal groups*. Their associated root systems contain a subsystem of type A_2, and so they contain subgroups that are isomorphic to either $\mathrm{SL}_3(q)$ or $\mathrm{PSL}_3(q)$. None of these belong to $\mathcal{S}$ by above. See [2].
- *Exceptional Chevalley groups*. We have an inclusion
$$G_2(q) \subset F_4(q) \subset E_6(q) \subset E_7(q) \subset E_8(q),$$
and the list of maximal subgroups of $G_2(q)$ in [20, p. 127] shows that $G_2(q)$, and hence all of the above groups, does not belong to $\mathcal{S}$.
- *Steinberg groups* ${}^2E_6(q^2)$ *and* ${}^3D_4(q^3)$. By [20, Theorem 4.3], the group ${}^3D_4(q^3)$ has a maximal subgroup which is isomorphic to $G_2(q^3)$, hence it is not in $\mathcal{S}$ by the above. Similarly, $F_4(q^2)$ embeds into ${}^2E_6(q^2)$ by [20, p. 173], hence the latter is not in $\mathcal{S}$.
- *Suzuki groups* $\mathrm{Sz}(q)$. By [20, Theorem 4.1], these contain Frobenius groups $C_{q+\sqrt{2q}+1} \ltimes C_4$ whose Fitting subgroups are of index 4. Such groups do not belong to $\mathcal{S}$ by Lemma 3.1.
- *Ree families*. By [20, Theorem 4.2], $2 \times \mathrm{PSL}_3(2n+1)$ is a maximal subgroup of ${}^2G_2(3^{2n+1})$, and $\mathrm{Sz}(2^{2n+1}) \wr 2$ is a maximal subgroup of ${}^2F_4(2^{2n+1})$ by [20, Theorem 4.5]. For the remaining case, ${}^2F_4(2)'$, we use ATLAS [5] to conclude that this group contains $\mathrm{Sym}(6)$.
- *Sporadic groups*. Inspection of ATLAS reveals that each of 26 sporadic groups has a maximal subgroup which is clearly not in $\mathcal{S}$. $\square$

To deal with perfect finite groups in $\mathcal{S}$, we make use of the theory of Schur covering groups. In particular, we will require the following.

Theorem 4.5 (Hauptsatz 23.5 of [11])**.** *Let G be a finite group and suppose there is an extension*
$$1 \to K \to E \to G \to 1$$
with the property that $K \leq Z(E) \cap E'$. Then K embeds into the Schur multiplier $M(G)$.

Proposition 4.6. *Let $G \in \mathcal{S}$ be a perfect non-simple finite group, and let F denote its Fitting subgroup. Assume that the group G/F contains two elements a and b of distinct prime orders with the additional property that $N_{G/F}(\langle a\rangle) \supsetneq \langle a\rangle$ and $N_{G/F}(\langle b\rangle) \supsetneq \langle b\rangle$. Then the group $S_p/\Phi(S_p)$ embeds into the Schur multiplier $M(G/F)$, for every p-Sylow subgroup S_p of F.*

Remark 4.7. It is easy to find such elements a, b for the simple groups $\mathrm{PSL}_2(2^n)$ that appear in Proposition 4.4. One can take a to be an involution (normalized by the Sylow 2-subgroup of order 2^n) and b a diagonal matrix of order $q-1$ (normalized by the class of the flip $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$).

Proof. The group F is nilpotent, so we can write $F = \prod_q S_q$ where S_q is a q-group. Now fix a prime p and consider $G_1 = G/\prod_{q\neq p} S_q$. The Fitting subgroup of G_1 is isomorphic to S_p. Further, consider the group $G_2 = G_1/\Phi(S_p)$. The Fitting subgroup F_2 of G_2 is an elementary abelian p-group, and G_2 belongs to the class $\mathcal{S}$.

Write $S = G/F$. Then S is simple by Lemma 4.1. The group G_2 acts on its subgroup F_2 by conjugation. There is thus an induced homomorphism $G_2 \to \mathrm{Aut}(F_2)$. This homomorphism factors through F_2, so we get a homomorphism

$$\psi\colon S \cong G_2/F_2 \to \mathrm{Aut}(F_2).$$

As S is a simple group, we have that either ψ is injective or trivial. Let us show that ψ must be trivial.

For the sake of contradiction, assume that $\ker\psi = 1$. Since F_2 is a p-group, at least one of the elements a, b from the statement of the proposition has order coprime to p. Without loss of generality, assume this element is a. Now consider the group $H = \langle a, F_2\rangle \leq G_2$. By our assumption on the element a, the group H is not self-normalized in G_2. But it is also not nilpotent. Indeed, the element a acts nontrivially on F_2 because ψ is an embedding of S into $\mathrm{Aut}(F_2)$. The order of a is coprime to p, so ψ restricted to $\langle a\rangle$ is a completely reducible representation of $\langle a\rangle$ on the $GF(p)$-vector space F_2. This representation splits as a sum of 1-dimensional representations, and so a is a diagonalizable element in the image of ψ. Being non-trivial, we can not have that $\psi(a) - I$ is a nilpotent matrix, and so the group H can not be nilpotent. This leads to a contradiciton with the fact that $G_2 \in \mathcal{S}$.

We therefore have that ψ is trivial, and so S acts trivially on F_2. This means that F_2 is central in G_2. Since G_2 is also assumed to be perfect, the extension $1 \to F_2 \to G_2 \to S \to 1$ has the property that $F_2 \leq Z(G_2) \cap G_2'$. It follows from Theorem 4.5 that $F_2 \cong S_p/\Phi(S_p)$ embeds into $M(S)$. □

Our next result, together with Proposition 4.4, gives a complete classification of all finite perfect groups in $\mathcal{S}$.

Theorem 4.8. *A finite perfect group G belongs to the class $\mathcal{S}$ if and only if it is either isomorphic to $\mathrm{PSL}_2(2^n)$ where $2^n - 1$ is a prime, or to $\mathrm{SL}_2(5)$.*

Proof. The Fitting quotient of G is a finite simple group belonging to $\mathcal{S}$, so it must be one of the PSL's appearing in Proposition 4.4. Note that we have $M(\mathrm{PSL}_2(4)) \cong C_2$ and all the other PSL's have trivial Schur multipliers. Therefore the only possibility for a non-simple perfect group G in $\mathcal{S}$ is a group whose Fitting quotient is $\mathrm{PSL}_2(4)$. Such a group must have F a 2-group with cyclic Frattini quotient, so F itself is cyclic. But now as G/F acts trivially on the Frattini quotient of F, it follows that the image of the homomorphism $G/F \to \mathrm{Aut}(F)$ is a p-group [12, Exercise 4.4]. Since G/F is a non-abelian simple group, this implies that G/F must act trivially even on F. Hence F is central in G. This implies that G is a Schur covering extensions of G/F by F, so it follows that $|F| = |M(\mathrm{PSL}_2(4))| = 2$ and $G \cong \mathrm{SL}_2(5)$. □

Now we deal with infinite perfect groups in the class $\mathcal{S}$.

Lemma 4.9. *Let G be a perfect group lying in the class $\mathcal{S}$. Then G is simple if and only if its Fitting subgroup is trivial.*

Proof. Let F denote the Fitting subgroup of G. First suppose G is simple. If $G = F$ then G is nilpotent, a contradiction since G is perfect. Therefore $F = 1$. Now suppose $F = 1$, and let N be any proper normal subgroup of G. Since $G \in \mathcal{S}$, the subgroup N is nilpotent, so $N \leq F$. Therefore $N = 1$, and G is simple. □

Lemma 4.10. *An infinite perfect group lying in the class $\mathcal{S}$ cannot be a Fitting group.*

Proof. Let $G \in \mathcal{S}$ be an infinite perfect group, and suppose that G is a Fitting group. The group G cannot be minimal non-nilpotent by (see [4, Proposition 144] and [1, Corollary 1.4]), so there exists a proper non-nilpotent subgroup H of G. Choose $x \in G \setminus H$. Since G is generated by its nilpotent normal subgroups, there exists a normal subgroup N of G such that N is nilpotent and $x \in N$. Hence $N \not\subseteq H$. Let B be the last term of the derived series of N which is not contained in H. Put $K = HB$. Then $K \in \mathcal{S}$ is locally nilpotent and non-nilpotent. Put $L = B \cap H$. Thus L is normal in K, and the normal series $L < B < K$ can be refined to a (general) principal series of K. As in the proof of Theorem 3.7, all factors of this principal series which lie between L and B are trivial. This means $L = B$, a contradiction. □

Note that the above shows that the finiteness hypothesis in Lemma 4.1 may be omitted.

Proposition 4.11. *Let $G \in \mathcal{S}$ be a perfect group, and let F denote its Fitting subgroup. Then G/F is a non-abelian simple group.*

Proof. By Lemma 4.1 we may assume that G is infinite. Moreover, by Lemmas 4.9 and 4.10 we may assume that F is a non-trivial proper subgroup of G. Clearly F is infinite and contains all proper normal subgroups of G. □

We leave it as an open problem whether or not there exist infinite perfect groups in $\mathcal{S}$ which are not simple. Note that, if such a group G is locally graded and finitely generated, then G/F is still locally graded (see for instance [13]), and hence it has to be finite. Therefore, by Proposition 4.4, G/F is isomorphic to $\mathrm{PSL}_2(2^n)$, where $2^n - 1$ is a prime.

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On prime-valent symmetric graphs of square-free order*

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Abstract

Symmetric graphs of valencies 3, 4 and 5 and square-free order have been classified in the literature. In this paper, we will present a complete classification of symmetric graphs of square-free order and any prime valency which admit a soluble arc-transitive group, and a complete classification of 7-valent symmetric graphs of square-free order.

Keywords: Symmetric graph, normal quotient graph, automorphism group.

Math. Subj. Class.: 20B15, 20B30, 05C25

1 Introduction

Throughout the paper, graphs considered are assumed to be undirected and simple with valency at least three.

For a graph Γ, denote by $V\Gamma$ and $A\Gamma$ the vertex set and arc set of Γ respectively, denote by $\mathrm{Aut}\Gamma$ the full automorphism group of Γ, and denote by $\Gamma(\alpha)$ the set of neighbors of a

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vertex α in Γ. Then Γ is called *X-vertex-transitive* or *X-arc-transitive*, with $X \leq \mathsf{Aut}\Gamma$, if X is transitive on $V\Gamma$ or $A\Gamma$ respectively. An arc-transitive graph is also called a *symmetric graph*. In particular, Γ is called *arc-regular* if $\mathsf{Aut}\Gamma$ is regular on $A\Gamma$.

For a positive integer s, an *s-arc* of a graph Γ is a sequence $v_0, v_1, \dots, v_s$ of $s+1$ vertices such that v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. If Γ has an s-arc and $X \leq \mathsf{Aut}\Gamma$ is transitive on the set of s-arcs of Γ, then Γ is called *(X, s)-arc-transitive*. If Γ is $(\mathsf{Aut}\Gamma, s)$-arc-transitive but not $(\mathsf{Aut}\Gamma, s+1)$-arc-transitive, then Γ is simply called *s-transitive*.

Characterizing symmetric graphs was initiated by a nice result of Tutte (1949) which says that there exists no s-arc-transitive cubic graph with $s \geq 6$. This result was generalized by Weiss [27] who proved that there is no s-arc-transitive graph with $s \geq 8$ of valency at least 3. Since then, studying transitive graphs has been one of the main topics in algebraic graph theory, and numerous results have been obtained. In particular, transitive graphs of square-free order (not divisible by the square of a prime) have received considerable attention; for example, symmetric graphs of valencies 3, 4 and 5 and square-free order have been classified by [16, 17] and [6] respectively, and arc-regular graphs of square-free order and prime valency have been determined by [9]. The main purpose of this paper is to give a complete classification of symmetric graphs of square-free order and prime valency admitting a soluble arc-transitive group, and a complete classification of 7-valent symmetric graphs of square-free order.

The terminology and notation used in this paper are standard. For example, we denote by J_1 the Janko simple group, by HS the Higman-Sims simple group, and by M_n, with $n = 11, 12, 22, 23, 24$, the five Mathieu simple groups. For a positive integer m, denote by A_m and S_m the alternating group and symmetric group of degree m, and by $\mathbb{Z}_m, \mathsf{F}_m$ and D_m (with m even) the cyclic group, Frobenius group and dihedral group of order m respectively. Given two groups N and H, denote by $N \times H$ the direct product of N and H, by $N.H$ an extension of N by H, and if such an extension is split, then we write $N{:}H$ instead of $N.H$.

A graph Γ is called a *Cayley graph* if there exists a group G and a subset $S \subseteq G \setminus \{1\}$ with $S = S^{-1}{:} = \{s^{-1} \mid s \in S\}$ such that the vertex set $V\Gamma = G$ and a vertex x is adjacent to a vertex y if and only if $yx^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(G, S)$. The following Cayley graphs of dihedral groups give rise to an infinite family of prime-valent symmetric graphs, where the first two letters 'CD' of the name of the graph $\mathrm{CD}_{2m,p,k}$ stand for 'Cayley graph of a dihedral group'.

Example 1.1. Let $G = \langle a, b \mid a^m = b^2 = 1, a^b = a^{-1}\rangle \cong \mathsf{D}_{2m}$ with m a positive integer, and let p be an odd prime and k a solution of the equation

$$x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv 0 \pmod{m}.$$

Set

$$\mathrm{CD}_{2m,p,k} = \mathsf{Cay}(G, \{b, ab, a^{k+1}b, \dots, a^{k^{p-2}+k^{p-3}+\cdots+k+1}b\}).$$

The following theorem determines the prime-valent symmetric graphs of square-free order which admit a soluble arc-transitive automorphism group. We remark that cubic graphs which admit a soluble edge-transitive or arc-transitive automorphism group have been characterized by [20] and [8], respectively.

Theorem 1.2. *Let Γ be a connected p-valent symmetric graph of square-free order n with p an odd prime, and suppose that Γ admits a soluble arc-transitive automorphism group G. Then either*

(1) $\Gamma \cong \mathsf{K}_{p,p}$, and $G \cong (((\mathbb{Z}_p : \mathbb{Z}_{l_1}) \times (\mathbb{Z}_p : \mathbb{Z}_{l_2})).\mathbb{Z}_r).\mathbb{Z}_2 \leq \mathsf{S}_p \wr \mathbb{Z}_2$, where $l_i r \mid p-1$ for $i = 1, 2$; or

(2) $\Gamma \cong \mathrm{CD}_{n,p,k}$, $n = 2 \cdot p^s p_1 p_2 \cdots p_t$ and $G = \mathsf{Aut}\Gamma \cong \mathsf{D}_n{:}\mathbb{Z}_p$, where $0 \leq s \leq 1$, $t \geq 1$, and $p_1, p_2, \ldots, p_t$ are distinct primes such that $p \mid p_i - 1$ for $i = 1, 2, \ldots, t$. Further, there are exactly $(p-1)^{t-1}$ non-isomorphic such graphs of order n.

The next theorem present a complete classification of 7-valent symmetric graphs of square-free order, where the graph $\mathcal{C}_{330}$ in Table 1 is introduced in Example 3.2 for convenience.

Theorem 1.3. *Let Γ be a connected 7-valent symmetric graph of square-free order n. Then one of the following statements holds.*

(1) $\Gamma \cong \mathrm{CD}_{n,7,k}$, and the tuple $(n, \mathsf{Aut}\Gamma)$ is as in part (2) of Theorem 1.2 with $p = 7$.

(2) The triple $(\Gamma, n, \mathsf{Aut}\Gamma)$ lies in Table 1.

(3) $\mathsf{Aut}\Gamma \cong \mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$, where $p \geq 13$ is a prime such that $p(p^2-1) \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7n$.

Table 1: Two 'sporadic' 7-valent symmetric graphs

Row	Γ	n	$\mathsf{Aut}\Gamma$	$(\mathsf{Aut}\Gamma)_\alpha$	Transitivity	Remark
1	$\mathsf{K}_{7,7}$	14	$\mathsf{S}_7 \wr \mathbb{Z}_2$	$\mathsf{S}_7.\mathsf{S}_6$	3-transitive	bipartite
2	$\mathcal{C}_{330}$	330	$\mathsf{M}_{22}.\mathbb{Z}_2$	$\mathbb{Z}_2^4 : \mathsf{SL}(3,2)$	2-transitive	not bipartite

Remark 1.4. Graphs appearing in part (3) of Theorem 1.3 can be expressed as coset graphs of $\mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$ (refer to [10] for the definition of the coset graph). However, it seems infeasible to determine all the possible values of p (and so the corresponding symmetric graphs Γ) for general square-free integer n.

2 Preliminaries

In this section, we introduce some preliminary results that will be used later.

For a group G with a subgroup H, let $C_G(H)$ and $N_G(H)$ denote the centralizer and normalizer of H in G, respectively.

Lemma 2.1 ([14, Ch. I, Lemma 4.5])**.** *Let G be a group and H a subgroup of G. Then $N_G(H)/C_G(H) \lesssim \mathsf{Aut}(H)$.*

For a group G, the largest nilpotent normal subgroup of G is called the *Fitting subgroup* of G. Clearly, the Fitting subgroup is a characteristic subgroup. The next lemma gives a property of the Fitting subgroup of soluble groups.

Lemma 2.2 ([26, P. 30, Corollary])**.** *Let F be the Fitting subgroup of a soluble group G. Then $F \neq 1$ and $C_G(F) \leq F$.*

The maximal subgroups of the simple group $\mathsf{PSL}(2,q)$ are known, see [5, Section 239].

Lemma 2.3. *Let $T = \mathsf{PSL}(2,q)$, where $q = p^n \geq 5$ with p a prime. Then a maximal subgroup of T is isomorphic to one of the following groups, where $d = (2, p-1)$.*

(1) $\mathsf{D}_{\frac{2(q-1)}{d}}$*, where $q \neq 5, 7, 9, 11$;*

(2) $\mathsf{D}_{\frac{2(q+1)}{d}}$*, where $q \neq 7, 9$;*

(3) $\mathbb{Z}_p^n : \mathbb{Z}_{\frac{q-1}{d}}$*;*

(4) A_4*, where $q = 5$, or $q = p \equiv 3, 13, 27, 37 \pmod{40}$;*

(5) S_4*, where $q = p \equiv \pm 1 \pmod 8$;*

(6) A_5*, where $q = p \equiv \pm 1 \pmod 5$, or $q = p^2 \equiv -1 \pmod 5$ with p an odd prime;*

(7) $\mathsf{PSL}(2,r)$*, where $q = r^m$ with m an odd prime;*

(8) $\mathsf{PGL}(2,r)$*, where $q = r^2$.*

By [2, Theorem 2], one may easily derive the maximal subgroups of $\mathsf{PGL}(2,p)$.

Lemma 2.4. *Let $T = \mathsf{PGL}(2,p)$ with $p \geq 5$ a prime. Then a maximal subgroup of T is isomorphic to one of the following groups:*

(1) $\mathbb{Z}_p : \mathbb{Z}_{p-1}$*;*

(2) $\mathsf{D}_{2(p+1)}$*;*

(3) $\mathsf{D}_{2(p-1)}$*, where $p \geq 7$;*

(4) S_4*, where $p \equiv \pm 3 \pmod 8$;*

(5) $\mathsf{PSL}(2,p)$*.*

A group G is called *perfect* if $G = G'$, the commutator subgroup; and an extension $G = N.H$ is called a *central extension* if $N \subseteq Z(G)$, the center of G. If a group G is perfect and $G/Z(G)$ is isomorphic to a simple group T, then G is called a covering group of T. Schur [25] showed that a simple (and, more generally, perfect) group T possesses a universal covering group G with the property that every covering group of T is a homomorphic image of G, in this case, the center $Z(G)$ is called the *Schur multiplier* of T, denoted by $\mathsf{Mult}(T)$, see [12, P. 43]. The Schur multipliers of nonabelian simple groups are known (see [12, P. 302]), and the following lemma is easy to prove (see [23, Lemma 2.11]).

Lemma 2.5. *Let $G = N.T$, where N is a cyclic group and T is a nonabelian simple group. Then $G = N.T$ is a central extension. Further, $G = NG'$ and $G' = M.T$, where M is contained in $G' \cap N$ and is isomorphic to a subgroup of* $\mathsf{Mult}(T)$.

The following lemma characterizes the vertex stabilizers of 7-valent symmetric graphs, see [13, Theorem 1.1].

Lemma 2.6. *Let Γ be a connected 7-valent (X, s)-arc-transitive graph, where $X \leq \mathsf{Aut}\Gamma$ and $s \geq 1$. Then one of the following holds, where $\alpha \in V\Gamma$.*

(1) If X_α is soluble, then $s \leq 3$ and $|X_\alpha| \mid 252$. Further, the couple (s, X_α) is listed in Table 2.

Table 2: Soluble vertex-stabilizers of 7-valent-symmetric graphs.

s	1	2	3
X_α	$\mathbb{Z}_7, \mathsf{D}_{14}, \mathsf{F}_{21}, \mathsf{D}_{14} \times \mathbb{Z}_2$, $\mathsf{F}_{21} \times \mathbb{Z}_3$	$\mathsf{AGL}(1,7), \mathsf{AGL}(1,7) \times \mathbb{Z}_2$, $\mathsf{AGL}(1,7) \times \mathbb{Z}_3$	$\mathsf{AGL}(1,7) \times \mathbb{Z}_6$

Table 3: Insoluble vertex-stabilizers of 7-valent symmetric graphs.

s	2	3
X_α	$\mathsf{PSL}(3,2), \mathsf{ASL}(3,2)$, $\mathsf{ASL}(3,2){\times}\mathbb{Z}_2, \mathsf{A}_7, \mathsf{S}_7$	$\mathsf{PSL}(3,2){\times}\mathsf{S}_4, \mathsf{A}_7{\times}\mathsf{A}_6$, $\mathsf{S}_7{\times}\mathsf{S}_6, (\mathsf{A}_7{\times}\mathsf{A}_6){:}\mathbb{Z}_2$, $\mathbb{Z}_2^6{:}(\mathsf{SL}(2,2){\times}\mathsf{SL}(3,2))$, $\mathbb{Z}_2^{20}{:}(\mathsf{SL}(2,2){\times}\mathsf{SL}(3,2))$
$\lvert X_\alpha\rvert$	$2^3{\cdot}3{\cdot}7, 2^6{\cdot}3{\cdot}7$, $2^7{\cdot}3{\cdot}7, 2^3{\cdot}3^2{\cdot}5{\cdot}7, 2^4{\cdot}3^2{\cdot}5{\cdot}7$	$2^6{\cdot}3^2{\cdot}7, 2^6{\cdot}3^4{\cdot}5^2{\cdot}7$, $2^8{\cdot}3^4{\cdot}5^2{\cdot}7, 2^7{\cdot}3^4{\cdot}5^2{\cdot}7$, $2^{10}{\cdot}3^2{\cdot}7, 2^{24}{\cdot}3^2{\cdot}7$

(2) If X_α is insoluble, then $|X_\alpha| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$. Further, the couple (s, X_α) lies in Table 3.

Analyzing a graph in terms of its normal quotients is a typical method for studying vertex-transitive graphs. Let Γ be an X-vertex-transitive graph with $X \le \mathsf{Aut}\Gamma$, and suppose that X has a normal subgroup N which is intransitive on $V\Gamma$. Denote by $V\Gamma_N$ the set of all N-orbits on $V\Gamma$. Then the *normal quotient graph* Γ_N of Γ induced by N is defined as the graph with vertex set $V\Gamma_N$, and B is adjacent to C in Γ_N if and only if there exist vertices $\beta \in B$ and $\gamma \in C$ such that β is adjacent to γ in Γ. In particular, if for any adjacent vertices B and C in $V\Gamma_N$, the induced subgraph $[B, C] \cong m\mathsf{K}_2$ is a perfect matching, where $m = |B| = |C|$, then Γ is called a *regular cover* (or *normal cover*) of Γ_N.

The following theorem gives a basic reduction method for studying vertex-transitive locally primitive graphs (see [18, Lemma 2.5]), which slightly improves a nice result of Praeger [24, Theorem 4.1]. Recall that, a graph Γ is called X*-locally primitive* if the vertex stabilizer X_α acts primitively on the neighbour set $\Gamma(\alpha)$ for each $\alpha \in V\Gamma$. Obviously, symmetric graphs with odd prime valency are locally primitive.

Theorem 2.7. *Let Γ be an X-vertex-transitive locally primitive graph, where $X \le \mathsf{Aut}\Gamma$, and let $N \lhd X$ have at least three orbits on $V\Gamma$. Then the following statements hold.*

(i) N is semi-regular on $V\Gamma$, $X/N \le \mathsf{Aut}\Gamma_N$, and Γ is a regular N-cover of Γ_N;

(ii) $X_\alpha \cong (X/N)_\gamma$, where $\alpha \in V\Gamma$ and $\gamma \in V\Gamma_N$;

(iii) Γ is (X, s)-arc-transitive if and only if Γ_N is $(X/N, s)$-arc-transitive, where $1 \le s \le 5$ or $s = 7$.

Symmetric graphs of prime-valency and order twice a prime are known, see [3].

Lemma 2.8. *Let Γ be a connected symmetric graph of odd prime valency p and order $2r$ with r a prime. Then one of the following statements holds.*

(1) $\Gamma \cong \mathbf{O}_2$ *and* $\mathsf{Aut}\Gamma \cong \mathsf{S}_5$*;*

(2) $\Gamma \cong \mathsf{K}_{2r}$ *with* $p = 2r - 1$*, and* $\mathsf{Aut}\Gamma \cong \mathsf{S}_{2r}$*;*

(3) $\Gamma \cong \mathsf{K}_{r,r}$ *with* $p = r$*, and* $\mathsf{Aut}\Gamma \cong \mathsf{S}_r \wr \mathsf{S}_2$*;*

(4) $\Gamma \cong \mathrm{CD}_{2r,p,k}$ *(which, up to isomorphism, is independent of the choice of* k *in this case), where* $p \mid r - 1$*, and one of the following statements holds.*

(i) $(r, p) = (7, 3)$ *and* $\mathsf{Aut}\Gamma \cong \mathsf{PGL}(2, 7)$*;*

(ii) $(r, p) = (11, 5)$ *and* $\mathsf{Aut}\Gamma \cong \mathsf{PGL}(2, 11)$*;*

(iii) $(r, p) \neq (7, 3)$ *and* $(11, 5)$*, and* $\mathsf{Aut}\Gamma \cong \mathsf{D}_{2r} : \mathbb{Z}_p$*.*

Lemma 2.9 ([19, Theorem 1.1])**.** *Let* Γ *be a connected* 7*-valent symmetric graph of order* $2pq$ *with* $p > q$ *odd primes. Then one of the following holds:*

(1) $\mathsf{Aut}\Gamma \cong \mathsf{PSL}(2, p)$ *with* $p \geq 13$*;*

(2) $q = 7$ *or* $7 \mid p - 1$*,* $7 \mid q - 1$*, and* $\Gamma \cong \mathrm{CD}_{2pq,7,k}$ *(as in Example* 1.1*).*

3 A lemma and an example

In this section, we give a technical lemma and introduce an example appearing in Theorem 1.3.

The following is an assertion regarding simple groups, its proof depends on the classification of simple groups, see [12, P. 134-136].

Lemma 3.1. *Let* m *be an odd square-free integer with at least three prime factors, and let* T *be a nonabelian simple group such that* $28m \mid |T|$ *and* $|T| \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7m$*. Then the couple* $(T, |T|)$ *is listed in Table 4, where* p *in part* 4 *is the largest prime factor of* m *and* $p \geq 13$*.*

Proof. If T is a sporadic simple group, by [12, P. 135-136], $T = \mathsf{M}_{22}$, M_{23}, M_{24}, J_1, HS or Ru, as in part 1 of Table 4. If $T = \mathsf{A}_n$ is an alternating group, since 3^6 does not divide $|T|$ and $3^6 \mid |\mathsf{A}_{15}|$, we have $n \leq 14$, it then easily follows that $T = \mathsf{A}_{11}$, A_{12}, A_{13} or A_{14}, as in part 2 of Table 4.

Now, suppose that T is a simple group of Lie type defined on the r^e-elements field $\mathsf{GF}(r^e)$, where r is a prime. If T is of exceptional Lie type, by [12, P. 135], $T \cong \mathsf{Sz}(512)$ or ${}^3\mathsf{D}_4(2)$, as listed in part 3 of Table 4. Consider the case where T is of classical Lie type. Since $r^e \mid |T|$, we have that $e = 1$ if $r > 7$, and $e \leq 2$ if $r = 7$, by [12, P. 135], which give rise to examples $T = \mathsf{PSL}(2, p)$ with $p \geq 13$ a prime (noting that $\mathsf{PSL}(2, p)$ with $p = 5, 7, 11$ does not satisfy the hypothesis of Lemma 3.1) and $T = \mathsf{PSL}(2, 49)$. If $r = 5$, as $5^4 \nmid |T|$, we conclude from [12, P. 135] that $T = \mathsf{PSL}(2, 125)$. For the case where $r \leq 3$, since 3^6, 5^4 and 7^3 do not divide $|T|$, by [12, P. 135] and with the help of Magma [1], we conclude that T is isomorphic to one of the groups listed in part 4 of Table 4. □

Given a permutation group G, a direct computation by Magma program [1] can determine all orbital graphs of G (see [7, P. 66] for the definition of orbital graph), or in other words, can determine all symmetric graphs which admit G as an arc-transitive automorphism group. It is then easy to have the following example.

Example 3.2. There is a unique connected 7-valent symmetric graph of order 330, denoted by $\mathcal{C}_{330}$, which admits M_{22} or $\mathsf{M}_{22}.\mathbb{Z}_2$ as an arc-transitive automorphism group. The graph $\mathcal{C}_{330}$ satisfies the conditions in Row 2 of Table 1.

Table 4: Nonabelian simple groups T with $28m \mid |T|$ and $|T| \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7m$.

Part	T	$\lvert T\rvert$
1	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
	M_{23}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11{\cdot}23$
	M_{24}	$2^{10}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}11{\cdot}23$
	J_1	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}19$
	HS	$2^9{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}11$
	Ru	$2^{14}{\cdot}3^3{\cdot}5^3{\cdot}7{\cdot}13{\cdot}29$
2	A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$
	A_{12}	$2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$
	A_{13}	$2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11{\cdot}13$
	A_{14}	$2^{10}{\cdot}3^5{\cdot}5^2{\cdot}7^2{\cdot}11{\cdot}13$
3	$\mathsf{Sz}(512)$	$2^{18}{\cdot}5{\cdot}7{\cdot}13{\cdot}37{\cdot}73{\cdot}109$
	${}^3\mathsf{D}_4(2)$	$2^{12}{\cdot}3^4{\cdot}7^2{\cdot}13$
4	$\mathsf{PSL}(2,p)$	$p(p^2-1)/2$
	$\mathsf{PSL}(2,49)$	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$
	$\mathsf{PSL}(2,125)$	$2^2{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}31$
	$\mathsf{PSL}(2,2^6)$	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$
	$\mathsf{PSL}(2,2^9)$	$2^9{\cdot}3^3{\cdot}7{\cdot}19{\cdot}73$
	$\mathsf{PSL}(2,2^{12})$	$2^{12}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13{\cdot}17{\cdot}241$
	$\mathsf{PSL}(2,2^{15})$	$2^{15}{\cdot}3^2{\cdot}7{\cdot}11{\cdot}31{\cdot}151{\cdot}331$
	$\mathsf{PSL}(2,2^{18})$	$2^{18}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}13{\cdot}19{\cdot}37{\cdot}73{\cdot}109$
	$\mathsf{PSL}(2,2^{21})$	$2^{21}{\cdot}3^2{\cdot}7^2{\cdot}43{\cdot}127{\cdot}337{\cdot}5419$
	$\mathsf{PSL}(2,2^{24})$	$2^{24}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13{\cdot}17{\cdot}97{\cdot}241{\cdot}257{\cdot}673$
	$\mathsf{PSL}(3,8)$	$2^9{\cdot}3^2{\cdot}7^2{\cdot}73$
	$\mathsf{PSL}(3,16)$	$2^{12}{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}13{\cdot}17$
	$\mathsf{PSL}(3,64)$	$2^{18}{\cdot}3^4{\cdot}5{\cdot}7^2{\cdot}13{\cdot}19{\cdot}73$
	$\mathsf{PSL}(4,4)$	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$
	$\mathsf{PSL}(5,2)$	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$
	$\mathsf{PSL}(5,4)$	$2^{20}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11{\cdot}17{\cdot}31$
	$\mathsf{PSL}(6,2)$	$2^{15}{\cdot}3^4{\cdot}5{\cdot}7^2{\cdot}31$
	$\mathsf{PSL}(7,2)$	$2^{21}{\cdot}3^4{\cdot}5{\cdot}7^2{\cdot}31{\cdot}127$
	$\mathsf{PSp}(6,4)$	$2^{18}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}13{\cdot}17$
	$\mathsf{PSp}(8,2)$	$2^{16}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17$
	$\mathsf{PSp}(4,8)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7^2{\cdot}13$
	$\mathsf{P\Omega}(7,4)$	$2^{18}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}13{\cdot}17$
	$\mathsf{P\Omega}(9,2)$	$2^{16}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17$
	$\mathsf{P\Omega}^+(10,2)$	$2^{20}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17{\cdot}31$
	$\mathsf{P\Omega}^-(8,2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
	$\mathsf{P\Omega}^-(8,4)$	$2^{24}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}13{\cdot}17{\cdot}257$

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 which in particular gives a partial proof of Theorem 1.3.

Let Γ be a connected symmetric graph of odd prime valency p and square-free order n, and let G be a soluble arc-transitive automorphism group of Γ. Since Γ is of odd valency, n is even. Set $n = 2p_1p_2\cdots p_t$, with $p_1, p_2, \ldots, p_t$ distinct odd primes.

If $t = 1$, by Lemma 2.8, $\Gamma \cong \mathrm{CD}_{2p_1,p,k}$ (as in part (2) of Theorem 1.2) or $\Gamma \cong \mathsf{K}_{p,p}$. If $\Gamma \cong \mathsf{K}_{p,p}$, then $G \leq \mathsf{Aut}\Gamma \cong \mathsf{S}_p \wr \mathbb{Z}_2$, as G is arc-transitive on Γ, $\mathbb{Z}_p^2 \lhd G$ and $G \not\subseteq \mathsf{S}_p^2$, it is then routine to show that $G \cong (((\mathbb{Z}_p : \mathbb{Z}_{l_1}) \times (\mathbb{Z}_p : \mathbb{Z}_{l_2})).\mathbb{Z}_r).\mathbb{Z}_2$, where $l_ir \mid p-1$ for $i = 1, 2$, as in part (1) of Theorem 1.2.

Suppose $t \geq 2$ in the following. Let F be the Fitting subgroup of G. By Lemma 2.2, $F \neq 1$. As $|V\Gamma| = 2p_1p_2\cdots p_t$, G has no nontrivial normal Sylow a-subgroup, where $a \notin \{2, p_1, p_2, \ldots, p_t\}$ is a prime, hence

$$F = \mathbf{O}_2(G) \times \mathbf{O}_{p_1}(G) \times \cdots \times \mathbf{O}_{p_t}(G),$$

where $\mathbf{O}_2(G)$ and $\mathbf{O}_{p_i}(G)$ with $i = 1, 2, \ldots, t$ denote the largest normal 2- and p_i-subgroups of G, respectively.

For each prime $q \in \{2, p_1, p_2, \ldots, p_t\}$, since $t \geq 2$, $\mathbf{O}_q(G)$ has at least six orbits on $V\Gamma$, by Theorem 2.7, $\mathbf{O}_q(G)$ is semi-regular on $V\Gamma$, so is F and $\mathbf{O}_q(G) \leq \mathbb{Z}_q$. Hence $F \leq \mathbb{Z}_n$ is cyclic and $C_G(F) = F$ by Lemma 2.2.

If F is transitive on $V\Gamma$, then $F \cong \mathbb{Z}_n$ is regular on $V\Gamma$ and Γ is a Cayley graph of F. Set $\Gamma = \mathsf{Cay}(F, S)$, where $S = S^{-1} \subseteq F \setminus \{1\}$ with size $|S| = p$. Since $F \lhd G$, by [11, Lemma 2.9], $G \leq F{:}\mathsf{Aut}(F, S)$, so $\mathsf{G}_{\mathbf{1}} \leq \mathsf{Aut}(F, S) \leq \mathsf{Aut}(F)$ is transitive on $\Gamma(\mathbf{1}) = S$, where $\mathbf{1}$ denotes the vertex of Γ corresponding to the identity element of F, thus elements in S have the same order, say h. Clearly, $h \neq 2$ as F has a unique involution. If $h > 2$, as $S = S^{-1}$, $|S|$ is even, which is a contradiction.

If F has at least three orbits on $V\Gamma$, then Theorem 2.7 implies that the normal quotient graph Γ_F is G/F-arc-transitive; however, by Lemma 2.2, $G/F = G/C_G(F) \leq \mathsf{Aut}(F)$ is abelian, it forces that G/F is regular on $V\Gamma_F$, and so G/F is not transitive on $A\Gamma$, also yielding a contradiction.

Thus, F has exactly two orbits on $V\Gamma$, and $F \cong \mathbb{Z}_{\frac{n}{2}}$. Because $t \geq 2$, F has a nontrivial normal subgroup $K \cong \mathbb{Z}_{p_2p_3\ldots p_t}$. Since $K \lhd G$ has $2p_1$ orbits on $V\Gamma$, by Theorem 2.7, Γ_K is a G/K-arc-transitive graph of valency p and order $2p_1$, and Γ is a regular K-cover of Γ_K. Such covers have been classified by [21, Theorem 1.1], hence the triple (Γ, K, Γ_K) (as $(\Gamma, \mathbb{Z}_n, \Sigma)$ there) satisfies parts (1)–(5) of [21, Theorem 1.1]. Since $|K| \neq 2$, parts (1)–(3) are impossible. For part (4), since n is square-free, $p_1 \nmid |K|$, by [22, Theorem 1.1], $\Gamma \cong \mathrm{CD}_{n,p,k}$. For part (5), noting that $p_1 \nmid |K|$, part (5)(ii) is not possible, we also have $\Gamma \cong \mathrm{CD}_{n,p,k}$. Finally, the last statement in part (2) of Theorem 1.2 is true by [9, Theorem 3.1]. This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

We will prove Theorem 1.3 in this final section.

Let Γ be a connected 7-valent symmetric graph of square-free order n. Since Γ is of odd valency, n is even, so we may write

$$n := 2m = 2p_1p_2\ldots p_t,$$

where $p_1, p_2, \ldots, p_t$ are distinct odd primes. Let $\mathsf{A} = \mathsf{Aut}\Gamma$.

Lemma 5.1. *If $t \le 2$, then Theorem 1.3 is true.*

Proof. If $t = 1$, by Lemma 2.8, $\Gamma \cong \mathrm{CD}_{2p_1,7,k}$ (as in part (1) of Theorem 1.3), or $\Gamma \cong \mathsf{K}_{7,7}$ (as in Row 1 of Table 1).

If $t = 2$, by Lemma 2.9, $\Gamma \cong \mathrm{CD}_{2p_1p_2,7,k}$ (as in part (1) of Theorem 1.3), or $\mathsf{A} \cong \mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$ with $p \ge 13$ a prime, satisfying part (3) of Theorem 1.3. □

Thus, assume $t \ge 3$ in the following, and assume inductively that Theorem 1.3 is true for the graph which satisfies assumption of Theorem 1.3 and is of order less than n. Let $\alpha \in V\Gamma$. By Lemma 2.6, $|\mathsf{A}_\alpha| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$, hence $|\mathsf{A}| = |\mathsf{A}_\alpha||V\Gamma|$ divides $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7m$. Let R be the soluble radical of A, that is, the largest soluble normal subgroup of A. Obviously, the soluble radical of A/R equals 1.

The next lemma treats the case $R = 1$.

Lemma 5.2. *Suppose $R = 1$ and $t \ge 3$. Then either $\mathsf{Aut}\Gamma \cong \mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$ with $p \ge 13$ a prime such that $p(p^2-1) \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7n$, as in part (2) of Theorem 1.3; or $\Gamma \cong \mathcal{C}_{330}$ and $\mathsf{Aut}\Gamma \cong \mathsf{M}_{22}.2$, as in Row 2 of Table 1.*

Proof. Let N be a minimal normal subgroup of A, and let $C = C_{\mathsf{A}}(N)$. Since $R = 1$, $N = T^d$ and $|N| = |T|^d$ divides $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7m$, where T is a nonabelian simple group and $d \ge 1$.

Claim 1. $C = 1$.

Assume, on the contrary, $C \ne 1$. Then C is insoluble as $R = 1$. If C is semi-regular on $V\Gamma$, then $|C| \mid n$, so C is of square-free order and hence soluble, which is a contradiction. Thus $C_\alpha \ne 1$. Since Γ is connected and $C \lhd \mathsf{A}$, we have $1 \ne C_\alpha^{\Gamma(\alpha)} \lhd \mathsf{A}_\alpha^{\Gamma(\alpha)}$, so $7 \mid |C_\alpha|$. Arguing similarly, one may have $7 \mid |N_\alpha|$. Now, since $N \cap C = 1$, $\langle N, C\rangle = N \times C \lhd \mathsf{A}$, so $N_\alpha \times C_\alpha \lhd \mathsf{A}_\alpha$, hence $7^2 \mid |\mathsf{A}_\alpha|$, which is a contradiction by Lemma 2.6. Therefore, $C = 1$.

Claim 2. A is almost simple and the tuple $(T, |T|)$ is listed in Table 4.

As discussed above, $7 \mid |N_\alpha|$. Then by Theorem 2.7, N has at most two orbits on $V\Gamma$, hence m divides $|N : N_\alpha|$, we further conclude that $7m \mid |N|$, $7 \mid |T|$ and $m \mid |T|$.

Without a loss of generality, let p_t be the largest prime dividing n. As $t \ge 3$, $p_t \ge 7$, and as $m^d = (p_1p_2\cdots p_t)^d$ divides $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7p_1p_2\cdots p_t$, we have $d \le 2$. If $d = 2$, the only possibility is $t = 3$ and $m = 3\cdot5\cdot7$, so $|T|^2 \mid 2^{25}\cdot3^5\cdot5^3\cdot7^2$, hence $|T| \mid 2^{12}\cdot3^2\cdot5\cdot7$; recall that $m \mid |T|$, by [15, Theorem III], $T \cong \mathsf{A}_l$ with $l = 7$ or 8, and $N \cong \mathsf{A}_l^2$. By Claim 1, $C = 1$, then Lemma 2.1 implies $\mathsf{A} = \mathsf{A}/C \lesssim \mathsf{Aut}(N) \cong \mathsf{S}_l \wr \mathbb{Z}_2$, and as $N \cong \mathsf{A}_l^2$ is a minimal normal subgroup of A, we conclude that $\mathsf{A} \cong \mathsf{A}_l \wr \mathbb{Z}_2$, $(\mathsf{A}_l \wr \mathbb{Z}_2).\mathbb{Z}_2$ or $\mathsf{S}_l \wr \mathbb{Z}_2$. Since $|\mathsf{A}_\alpha| = \frac{|\mathsf{A}|}{210}$, a direct computation by Magma [1] shows that no graph Γ exists in this case, a contradiction. Thus, $d = 1$ and $N = T$, and by Lemma 2.1, $\mathsf{A} \le \mathsf{Aut}(T)$ is almost simple. Recall that $|T|$ divides $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7m$ and $7m$ divides $|T|$, and noting that $4 \mid |T|$ as T is nonabelian simple, we have $28m \mid |T|$. By Lemma 3.1, the couple $(T, |T|)$ is listed in Table 4.

Now, we will analyse all the candidates of T in Table 4, thus proving Lemma 5.2. Recall that $n = 2m$ and $|T : T_\alpha| = m$ or $2m$. Denote by $\mathsf{Out}(T)$ the outer automorphism group of T.

Assume $T \cong \mathsf{PSL}(2,p)$ with $p \geq 13$ a prime. Then $p(p^2-1) \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7n$, and as $\mathsf{Out}(T) \cong \mathbb{Z}_2$ (see [12, P. 135]), we have $\mathsf{A} \cong \mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$, the lemma is true.

Assume $T \cong \mathsf{M}_{22}$. Then $m = 3 \cdot 5 \cdot 11 = 165$ and $n = 330$. Since $\mathsf{Out}(\mathsf{M}_{22}) \cong \mathbb{Z}_2$, $\mathsf{A} \cong \mathsf{M}_{22}$ or $\mathsf{M}_{22}.\mathbb{Z}_2$. By Example 3.2, $\Gamma \cong \mathcal{C}_{330}$, satisfying the conditions in Row 2 of Table 1.

Assume $T \cong \mathsf{M}_{23}$. Then $m = 3 \cdot 11 \cdot 23, 5 \cdot 11 \cdot 23$ or $3 \cdot 5 \cdot 11 \cdot 23$. Since $\mathsf{Out}(\mathsf{M}_{23}) = 1$, $\mathsf{A} = T \cong \mathsf{M}_{23}$ and so $|T : T_\alpha| = 2m = 1518, 2530$ or 7590. However, by [4], M_{23} has no subgroup with index $1518, 2530$ or 7590, a contradiction.

Assume $T \cong \mathsf{J}_1$. Then $m = 627, 1045$ or 3135. Since $\mathsf{Out}(\mathsf{J}_1) = 1$, we have $\mathsf{A} = T$ and $|T : T_\alpha| = 2m = 1254, 2090$ or 6270. By [4], J_1 has no subgroup with index $1254, 2090$ or 6270, which is a contradiction.

Suppose $T \cong \mathsf{A}_{12}$. Then $m = 165$ and $|T : T_\alpha| = 165$ or 330. By [4], A_{12} has no subgroup with index 165 or 330, yielding a contradiction.

Suppose $T \cong \mathsf{PSL}(2,49)$. Then $m = 105$ and $|T : T_\alpha| = 105$ or 210, it follows $|T_\alpha| = 560$ or 280 respectively. By Lemma 2.3, $\mathsf{PSL}(2,49)$ has no subgroup with order 560 or 280, a contradiction.

Suppose $T \cong \mathsf{PSL}(2,2^{24})$. Then $\mathsf{Out}(T) \cong \mathbb{Z}_{24}$ by [12, P. 135], it follows $\mathsf{A} \cong \mathsf{PSL}(2,2^{24}).\mathbb{Z}_r$ with $r \mid 24$, and $|\mathsf{A}| = 2^{24}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13{\cdot}17{\cdot}97{\cdot}241{\cdot}257{\cdot}673r$. Hence $m = 3{\cdot}13{\cdot}17{\cdot}97{\cdot}241{\cdot}257{\cdot}673$, $5{\cdot}13{\cdot}17{\cdot}97{\cdot}241{\cdot}257{\cdot}673$ or $3{\cdot}5{\cdot}13{\cdot}17{\cdot}97{\cdot}241{\cdot}257{\cdot}673$. For the first case, $|\mathsf{A}_\alpha| = 2^{23}{\cdot}3{\cdot}5{\cdot}7r$, which is impossible by Lemma 2.6. For the second case, $|\mathsf{A}_\alpha| = 2^{23}{\cdot}3^2{\cdot}7r$, by Lemma 2.6, the only possibility is $r = 2$ and $\mathsf{A}_\alpha \cong \mathbb{Z}_2^{20} : (\mathsf{SL}(2,2) \times \mathsf{SL}(3,2))$; for the last case, we have $|\mathsf{A}_\alpha| = 2^{23}{\cdot}3{\cdot}7r$, by Lemma 2.6, the only possibility is $r = 6$ and $\mathsf{A}_\alpha \cong \mathbb{Z}_2^{20} : (\mathsf{SL}(2,2) \times \mathsf{SL}(3,2))$. However, by Lemma 2.3, both $\mathsf{PSL}(2,2^{24}).\mathbb{Z}_2$ and $\mathsf{PSL}(2,2^{24}).\mathbb{Z}_6$ have no subgroup isomorphic to $\mathbb{Z}_2^{20} : (\mathsf{SL}(2,2) \times \mathsf{SL}(3,2))$, which is a contradiction.

Suppose $T \cong {}^3\mathsf{D}_4(2)$. Since $\mathsf{Out}(T) \cong \mathbb{Z}_3$ (see [4]), $\mathsf{A} \cong {}^3\mathsf{D}_4(2)$ or ${}^3\mathsf{D}_4(2).\mathbb{Z}_3$, and so $|\mathsf{A}| = 2^{12}{\cdot}3^4{\cdot}7^2{\cdot}13$ or $2^{12}{\cdot}3^5{\cdot}7^2{\cdot}13$ respectively, implying $m = 3{\cdot}7{\cdot}13$. Now, $|\mathsf{A}_\alpha| = \frac{|\mathsf{A}|}{2m} = 2^{11}{\cdot}3^3{\cdot}7$ or $2^{11}{\cdot}3^4{\cdot}7$, which is impossible by Lemma 2.6.

Arguing similarly as above, one may prove that no graph Γ exists for all other candidates for T in Table 4 (the results have been checked by Magma [1]). $\square$

We finally consider the case where A is insoluble and $R \neq 1$ by the following lemma.

Lemma 5.3. *Suppose that* A *is insoluble,* $R \neq 1$ *and* $t \geq 3$. *Then no graph* Γ *exists.*

Proof. Let M be a minimal soluble normal subgroup of A. Then $M \cong \mathbb{Z}_r^d$, where r is a prime and $d \geq 1$. Since $t \geq 3$, M has at least $2 \cdot 3 \cdot 5 = 30$ orbits on $V\Gamma$, so, by Theorem 2.7, M is semi-regular on $V\Gamma$ (so $d = 1$ and $r \in \{2, p_1, p_2, \ldots, p_t\}$), Γ_M is a 7-valent A/M-arc-transitive graph of order $\frac{2m}{r}$, and Γ is an arc-transitive regular $\mathbb{Z}_r$-cover of Γ_M.

If $r = 2$, then Γ_M is arc-transitive of odd order m and odd valency 7, which is impossible.

Thus, $r = p_i$ with $i \in \{1, 2, \ldots, t\}$, and Γ_M is a 7-valent A/M-arc-transitive graph of order $\frac{2m}{p_i}$. Recall that we assume by inductive hypothesis that Theorem 1.3 is true for all graphs which satisfy the assumptions of Theorem 1.3 and are of order less than n, so Γ_M satisfies Theorem 1.3. Noting that A is insoluble and M is soluble, A/M is insoluble, so is $\mathsf{Aut}(\Gamma_M)$. Then, checking the graphs in Theorem 1.3, we conclude that the soluble radical of $\mathsf{Aut}(\Gamma_M)$ equals 1, and one of the following holds:

(1) $\Gamma_M \cong \mathcal{C}_{330}$ and $\mathsf{Aut}(\Gamma_M) \cong \mathsf{M}_{22}.\mathbb{Z}_2$;

(2) $\mathsf{Aut}(\Gamma_M) \cong \mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$ with $p \geq 13$ a prime and $t \geq 3$.

Since A/M acts arc-transitively on Γ_M, we have $7|V\Gamma_M|$ divides $|\mathsf{A}/M|$.

For case (1), then $|V\Gamma_M| = 330$, and $7 \cdot 330 = 2310$ divides $|\mathsf{A}/M|$, since $\mathsf{A}/M \leq \mathsf{M}_{22}.\mathbb{Z}_2$, by [4], we have $\mathsf{A}/M \cong \mathsf{M}_{22}$ or $\mathsf{M}_{22}.\mathbb{Z}_2$. Let X be a normal subgroup of A such that $X \cong M.\mathsf{M}_{22} \cong \mathbb{Z}_r.\mathsf{M}_{22}$. Since $(r, |V\Gamma_M|) = (r, 330) = 1$, we have $r \neq 2$ and 3. Then, as $|\mathsf{Mult}(\mathsf{M}_{22})| = 12$ (see [4]), Lemma 2.5 implies $X \cong \mathbb{Z}_r \times \mathsf{M}_{22}$ and $X' \cong \mathsf{M}_{22}$. Since $|X'| = |\mathsf{M}_{22}|$ does not divide $|V\Gamma|$, $X' \lhd \mathsf{A}$ is not semi-regular on $V\Gamma$, and X' has at most two orbits on $V\Gamma$ by Theorem 2.7. Because Γ is connected and $1 \neq X'_\alpha \lhd \mathsf{A}_\alpha$, $1 \neq (X'_\alpha)^{\Gamma(\alpha)} \lhd \mathsf{A}_\alpha^{\Gamma(\alpha)}$, it follows $7 \mid |X'_\alpha|$. Hence r divides $\frac{|X'|}{7} = 2^7 \cdot 3^2 \cdot 5 \cdot 11$, which is a contradiction as $(r, |V\Gamma_M|) = (r, 330) = 1$.

We next consider case (2). Since A/M is insoluble and $7 \mid |\mathsf{A}/M|$, by Lemma 2.4, $\mathsf{A}/M \cong \mathsf{PSL}(2,p)$ or $\mathsf{PGL}(2,p)$. Let $B/M \lhd \mathsf{A}/M$ such that $B/M \cong \mathsf{PSL}(2,p)$. Since $\mathsf{Mult}(\mathsf{PSL}(2,p)) \cong \mathbb{Z}_2$ (see [12, P. 302]) and $r \geq 3$, Lemma 2.5 implies that $B' \cong \mathsf{PSL}(2,p)$ and $B = M \times B'$. Since $B, B' \lhd \mathsf{A}$ are insoluble, both B and B' have at most two orbits on $V\Gamma$. In particular, m divides $|B'|$.

If $r > 7$, since $|\mathsf{A}|$ divides $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7m$, and $|B| = |M \times B'| = r|B'|$ divides $|\mathsf{A}|$, we have $r \nmid |B'|$, which is a contradiction to m dividing $|B'|$.

Assume $r = 7$. Since Γ is connected and $1 \neq B'_\alpha \lhd \mathsf{A}_\alpha$, we have $7 \mid |B'_\alpha|$. Then, as $|B':B'_\alpha| = m$ or $2m$ is divisible by 7, we further conclude that 7^2 divides $|B'| = |B/M|$. However, since $|B/M:(B/M)_\delta| = \frac{m}{7}$ or $\frac{2m}{7}$, which is not divisible by 7, we have $7^2 \mid |(B/M)_\delta|$, so $7^2 \mid |(\mathsf{A}/M)_\delta|$, which is a contradiction by Lemma 2.6.

Assume finally $r = 3$ or 5. Since $B/M \cong B'$ has at most two orbits on $V\Gamma_N$, and B' has at most two orbits on $V\Gamma$, we have $|B/M:(B/M)_\delta| = \frac{m}{r}$ or $\frac{2m}{r}$, and $|B':B'_\alpha| = m$ or $2m$. It follows that $r \mid |(B/M)_\delta|$ and so $r \mid |(\mathsf{A}/M)_\delta|$. Also, as A/M acts arc-transitively on Γ_M, $7 \mid |(\mathsf{A}/M)_\delta|$, hence $7r \mid |(\mathsf{A}/M)_\delta|$. Suppose $r = 3$. If $(\mathsf{A}/M)_\delta$ is soluble, then $(\mathsf{A}/M)_\delta$ is listed in part (1) of Lemma 2.6, and as $21 \mid |(\mathsf{A}/M)_\delta|$, we have $(\mathsf{A}/M)_\delta \geq \mathsf{F}_{21}$; however, since $\mathsf{A}/M \leq \mathsf{PGL}(2,p)$ and $p \neq 7$, by Lemma 2.4, $\mathsf{PGL}(2,p)$ has no soluble subgroup containing a subgroup isomorphic to F_{21}, a contradiction. If $(\mathsf{A}/M)_\delta$ is insoluble, noting that $\mathsf{A}/M \leq \mathsf{PGL}(2,p)$, we have $(\mathsf{A}/M)_\delta \cong \mathsf{PSL}(2,p)$ or A_5. For the first case, $|V\Gamma_M| = |\mathsf{A}/M:(\mathsf{A}/M)_\delta| = 2$, which is impossible. For the latter case, $7 \nmid |(\mathsf{A}/M)_\delta|$, also a contradiction. Suppose now $r = 5$. Then $35 \mid |(\mathsf{A}/M)_\delta|$, and Lemma 2.6 implies that $(\mathsf{A}/M)_\delta$ is insoluble, so $(\mathsf{A}/M)_\delta \cong \mathsf{PSL}(2,p)$ or A_5 as $\mathsf{A}/M \leq \mathsf{PGL}(2,p)$. Now, the same arguments as above draw a contradiction. □

Theorem 1.3 now follows directly from Theorem 1.2 and Lemmas 5.1–5.3.

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Rank 4 toroidal hypertopes

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Abstract

We classify the regular toroidal hypertopes of rank 4. Their automorphism groups are the quotients of infinite irreducible Coxeter groups of euclidean type with 4 generators. We also prove that there are no toroidal chiral hypertopes of rank 4.

Keywords: Regularity, chirality, toroidal, thin geometries, hypermaps, abstract polytopes.

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1 Introduction

A toroidal polytope is an abstract polytope that can be seen as a tessellation on a torus. By abstract polytope we mean a combinatorial structure resembling a classical polytope described by incidence relationships. Highly symmetric types of these polytopes are well known and understood, in particular the regular and chiral toroidal polytopes have been classified for rank 3 by Coxeter in 1948 [5], see also [6], and for any rank by McMullen and Schulte [10]. Regular toroidal polytopes and also regular toroidal hypertopes, which we define below, are strongly related to a special class of Coxeter groups, the infinite irreducible Coxeter groups of euclidean type which are also known as affine Coxeter groups (see, for example [11, page 73]). The symmetry groups of regular tessellations of euclidean space are precisely the affine Coxeter groups with string diagrams (see [11, Theorem 3B5]).

When we talk about a tessellation we mean, informally, a locally finite collection of polytopes which cover $\mathbb{E}^n$ in a face-to-face manner. A toroidal polytope can then be seen as a "quotient" of a tessellation by linearly independent translations. For a precise definition of a toroidal polytope see [8]. The concept of a hypertope has recently been introduced by Fernandes, Leemans and Weiss (see [7]). A hypertope can be seen as a generalization of

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a polytope. Or, from another perspective, as a generalization of a hypermap. For more infomation on hypermaps see [4]. In this paper we will classify the rank 4 regular toroidal hypertopes.

Each affine Coxeter group in rank 4 (which are usually denoted by $\widetilde{C_3}, \widetilde{B_3}$ and $\widetilde{A_3}$), as we shall see, can be associated with the group $\widetilde{C_3} = [4, 3, 4]$, the symmetry group of the cubic tessellation of $\mathbb{E}^3$. The *Coxeter Complex*, denoted by $\mathcal{C}$, of $\widetilde{C_3}$ can be seen as the simplicial complex obtained by the barycentric subdivision of the cubic tessellation $\{4, 3, 4\}$. The Coxeter complex for the other two rank 4 affine Coxeter groups can be obtained by doubling the rank 3 simplicies for $\widetilde{B_3}$ and quadrupling them for $\widetilde{A_3}$. For details on the construction of $\mathcal{C}$ see [9, Section 6.5] or [11, Section 3B]. We note that $\mathcal{C}$ partitions $\mathbb{E}^3$.

A *regular toroidal hypertope* (see Section 2 for a precise definition) can be seen as a quotient $\mathcal{C}/\Lambda_I$ by a normal subgroup of translations, denoted Λ_I where I represents a generating set identifying the normal subgroup. In particular the quotient induced by a normal subgroup of translations in the string affine Coxeter group $\widetilde{C_3}$ yields the three families of regular rank 4 toroidal polytopes, while the other two affine Coxeter groups with non-string diagrams do not yield regular polytopes, but as we shall see below, regular hypertopes.

2 C-groups and hypertopes

Details of the concepts we review here are given in [7] and [11]. A *C-group of rank p* is a pair (G, S) such that G is a group and $S = \{r_0, \dots, r_{p-1}\}$ is a generating set of involutions of G that satisfy the following property:

$$\forall I, J \subseteq \{0, \dots, p-1\}, \langle r_i : i \in I\rangle \cap \langle r_j : j \in J\rangle = \langle r_k : k \in I \cap J\rangle.$$

This is known as the *intersection property* which will be referred to later.

A subgroup of G generated by a subset of S is called a *parabolic subgroup*. A parabolic subgroup generated by a single element of S is called *minimal* and a parabolic subgroup generated by all but one element of S is called *maximal*. For $J \subseteq \{0, \dots, p-1\}$, we define $G_J := \langle r_j\colon j \in J\rangle$ and $G_i := \langle r_j\colon r_j \in S, r_j \neq r_i\rangle$.

A C-group is a *string C-Group* if $(r_i r_j)^2 = 1_G$ for all i, j with $|i - j| > 1$. A *Coxeter diagram* $C(G, S)$ of a C-group (G, S) is a graph whose vertex set is S and two vertices, r_i and r_j are joined by an edge labelled by $o(r_i r_j)$, the order of $r_i r_j$. We use the convention that if an edge is labeled 2 it is dropped and not labeled if the order of the product of the corresponding generators is 3. Thus the Coxeter diagram of a string C-group is a string.

Affine Coxeter groups are C-groups and those with string diagrams are associated with toroidal polytopes. Hypertopes are generalizations of polytopes and we can, however, find toroidal hypertopes whose automorphism groups are quotients of any affine Coxeter group. We start with the definition of an *incidence system*.

Definition 2.1. An *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called *elements* of Γ;
- I is a set whose elements are called *types* of Γ;
- $t\colon X \to I$ is a *type function* that associates to each element $x \in X$ of Γ a type $t(x) \in I$;

- $*$ is a binary relation of X called *incidence*, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

A *flag* is a set of pairwise incident elements of Γ and the *type* of a flag F is $\{t(x) \colon x \in F\}$. A *chamber* is a flag of type I. An element x is said to be *incident* to a flag F when x is incident to all elements of F and we write $x * F$.

Definition 2.2. An *incidence geometry* is an incidence system Γ where every flag is contained in a chamber. The *rank* of Γ is the cardinality of I.

Let $\Gamma := (X, *, t, I)$ be an incidence system and F a flag of Γ. The *residue* of F in Γ is the incidence system $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- $X_F := (x \in F \colon x * F, x \notin F)$;
- $I_F := I \backslash t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F.

If each residue of rank at least 2 of Γ has a connected incidence graph then Γ is said to be *residually connected*. Γ is *thin* when every residue of rank 1 contains exactly 2 elements.

Furthermore, Γ is *chamber-connected* when for each pair of chambers C and C', there exists a sequence of chambers $C =: C_0, C_1, \ldots, C_n := C'$ such that $|C_i \cap C_{i+1}| = |I| - 1$ (here we say that C_i and C_{i+1} are $adjacent$). An incidence system is *strongly chamber-connected* when all of its residues of rank at least 2 are chamber-connected.

Proposition 2.3 ([7, Proposition 2.1])**.** *Let Γ be a thin incidence geometry. Then Γ is residually connected if and only if Γ is strongly chamber-connected.*

A *hypertope* is a strongly chamber-connected thin incidence geometry. To reinforce the relationship between polytopes and hypertopes we will sometimes refer to the *elements* of a hypertope Γ as *hyperfaces* of Γ, and *elements of type I* as *hyperfaces of type I*.

Let $\Gamma := (X, *, t, I)$ be an incidence system. An *automorphism* of Γ is a mapping $\alpha \colon (X, I) \to (X, I) : (x, t(x)) \mapsto (\alpha(x), t(x))$ where

- α is a bijection on X inducing a bijection on I;
- for each $x, y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x, y \in X$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism α is *type-preserving* when, for each $x \in X, t(\alpha(x)) = t(x)$. We denote by $Aut(\Gamma)$ the group of automorphisms of Γ and by $Aut_I(\Gamma)$ is the group of type-preserving automorphisms of Γ.

An incidence system Γ is *flag transitive* if $Aut_I(\Gamma)$ is transitive on all flags of type J for each $J \subseteq I$. It is *chamber-transitive* if $Aut_I(\Gamma)$ is transitive on all chambers of Γ. Furthermore, it is *regular* if the action of $Aut_I(\Gamma)$ is semi-regular and transitive.

Proposition 2.4 ([7, Proposition 2.2])**.** *Let Γ be an incidence geometry. Γ is chamber-transitive if and only if it is flag-transtive.*

A *regular hypertope* is a flag transitive hypertope. We note that every abstract regular polytope is a regular hypertope. The last concept we introduce here before we construct all rank 4 regular toroidal hypertopes is that of coset geometries.

Proposition 2.5 ([14]). *Let p be a positive integer and $I := \{1, \dots, p\}$ a finite set. Let G be a group together with a family of subgroups $(G_i)_{i \in I}$, X the set consisting of all cosets $G_i g, g \in G, i \in I$ and $t\colon X \to I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by :*

$$G_i g_2 * G_j g_2 \text{ if and only if } G_i g_1 \cap G_j g_2 \text{ is non-empty in } G.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having a chamber. Moreover, the group G acts by right multiplication as a group of type-preserving automorphisms of Γ. Finally, the group G is transitive on the flags of rank less than 3.

Whenever Γ is constructed as in the above proposition it is written as $\Gamma(G; (G_i)_{i \in I})$ and if it is an incidence geometry it is called a *coset geometry*. If G acts transitively on all chambers of Γ (thus also flags of any type) we say that G is *flag transitive* on Γ or that Γ is flag transitive.

Now we note that we can construct a coset geometry $\Gamma(G; (G_i)_{i \in I})$ using a C-group (G, S) or rank p by setting $G_i := \langle r_j : r_j \in S, j \in I \backslash \{i\} \rangle$ for all $i \in I := \{0, \dots, p-1\}$.

We introduce the following proposition which lets us know that constructions we use produce regular hypertopes.

Proposition 2.6 ([7, Theorem 4.6]). *Let $(G, \{r_0, \dots, r_{p-1}\})$ be a C-group of rank p and let $\Gamma := \Gamma(G; (G_i)_{i \in I})$ with $G_i := \langle r_j : r_j \in S, j \in I \backslash \{i\} \rangle$ for all $i \in I := \{0, \dots, p-1\}$. If Γ is flag transitive, then Γ is a regular hypertope.*

Henceforth, we restrict our considerations to rank 4. Let $G = \langle r_0, r_1, r_2, r_3 \rangle$ be an affine Coxeter group where each r_i is a reflection through an associated affine hyperplane, H_i. Let $\mathcal{C}$ be the Coxeter complex of G formed from the hyperplanes $H_i's$. Here r_1, r_2 and r_3 will stabilize a point which, without loss of generality, can be assumed to be the origin o in $\mathbb{E}^3$. Then the maximal parabolic subgroup G_0 is a finite crystallographic subgroup, which is a group that leaves a central point fixed. For details, see [3, pages 108–109]. The normal vectors to the reflection planes of the generators of G_0 are called the $fundamental$ $roots$. The images of the fundamental roots under G_0 form a $root\ system$ for G_0.

The lattice, Λ, generated by the root system is called the $root\ lattice$, and the fundamental roots form the integral basis for Λ. The region enclosed by the fundamental roots is called the *fundamental region*. This lattice gives us (and can be identified with) the translation subgroup $T \leq G$ generated by the root lattice of G_0, note that $G = G_0 \rtimes T$ [3]. For convenience we identify the translations with its vectors in addition a lattice also corresponds with its generating translation.

If I is a set of linearly independent translations in T, and let $T_I \leq T$ be the subgroup generated by I. Then the sublattice $\Lambda_I \leq \Lambda$ is the lattice induced by oT_I, the orbit of the origin under T_I. We note that $\mathcal{C}$ is a regular hypertope and each simplex in $\mathcal{C}$ represents a chamber where each vertex of the simplex is an element of a different type. In rank 4, when the quotient $\mathcal{C}/\Lambda_I$ is a regular hypertope, we say it is a $regular\ toroidal\ hypertope$ of rank 4. $\mathcal{C}/\Lambda_I$ is a regular hypertope (and thus a regular toroidal hypertope) when Λ_I is large enough to ensure the corresponding group satisfies the intersection condition and when Λ_I invariant under G_0, i.e. $r_i \Lambda_I r_i = \Lambda_I$ for $i = 1, 2, 3$. It is important to note that, in addition to denoting a lattice, Λ_I is also denotes a set of vectors as well as a translation subgroup of G along those vectors. If I consists of all permutations and changes in sign of the coordinates of some vector $\mathbf{s}$ then we will write $\Lambda_{\mathbf{s}}$.

3 Toroidal polytopes constructed from the group $\widetilde{C}_3 = [4, 3, 4]$

We begin, necessarily, with generating regular toroidal hypertopes (or, in this case, polytopes) whose automorphism groups are quotients of the group $\widetilde{C}_3$, the affine Coxeter Group $[4, 3, 4]$. As generators of $[4, 3, 4]$ we take ρ_1, ρ_2, ρ_3 to be reflections in the hyperplanes with normal vectors $(1, -1, 0), (0, 1, -1), (0, 0, 1)$ respectively, and ρ_0 the reflection in the plane through $(1/2, 0, 0)$ with normal vector $(1, 0, 0)$ (see Figure 1). Then,

$$\begin{aligned}(x, y, z)\rho_0 &= (1 - x, y, z),\\ (x, y, z)\rho_1 &= (y, x, z),\\ (x, y, z)\rho_2 &= (x, z, y),\\ (x, y, z)\rho_3 &= (x, y, -z).\end{aligned} \tag{3.1}$$

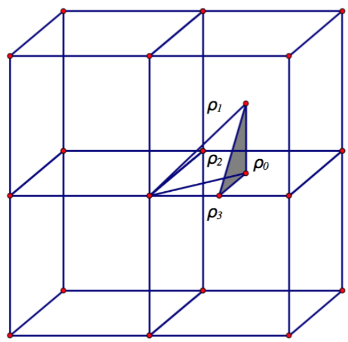

Figure 1: Fundamental simplex of $[4, 3, 4]$.

In this case, the construction described in Section 2 will yield the regular polytopes since $[4, 3, 4]$ is a string group. We denote by τ the corresponding tessellation $\{4, 3, 4\}$ of the Euclidean plane by cubes and by T it's full translation subgroup, where T is generated by the usual basis vectors, $T = \langle(1, 0, 0), (0, 1, 0), (0, 0, 1)\rangle$.

Let H_i be the planes fixed by ρ_i. The simplex bounded by the reflection planes H_i is a fundamental simplex of $[4, 3, 4]$ and is denoted ε, it is a simplex in the Coxeter complex of $\widetilde{C}_3$. Let F_i be the vertex of the fundamental simplex not on H_i then $\{F_0, F_1, F_2, F_3\}$ represents a flag of τ, and the set of all j-faces of $\tau = \{4, 3, 4\}$ is represented by the orbit of F_j under $\widetilde{C}_3$.

The regular polytope which results from factoring the regular tesselation $\tau = \{4, 3, 4\}$ by a subgroup Λ of T which is normal in $[4, 3, 4]$, is denoted by τ/Λ (as above).

We let $\Lambda_{\mathbf{s}}$ be the translation subgroup (or lattice) generated by the vector **s** and its images under the stabilization of the origin in $[4, 3, 4]$ and hence under permutations and changes of sign of its coordinates. The regular polytope $\tau/\Lambda_{\mathbf{s}}$ is denoted by $\{4, 3, 4\}_{\mathbf{s}} := \{4, 3, 4\}/\Lambda_{\mathbf{s}}$ and the corresponding group $[4, 3, 4]/\Lambda_{\mathbf{s}}$ is written as $[4, 3, 4]_{\mathbf{s}}$. The following Lemma lists all possible such subgroups of T.

Lemma 3.1. *Let Λ be a subgroup of T, and if for every $\boldsymbol{a} \in \Lambda$, the image of $\boldsymbol{a}$ under all changes of sign and permutations of coordinates (which is conjugation of $\boldsymbol{a}$ by the* *stabilization of the origin in* $[4,3,4]$*) is also in* Λ*, then* $\Lambda = \langle (x,0,0),(0,x,0),(0,0,x)\rangle$, $\langle (x,x,0),(-x,x,0),(0,-x,x)\rangle$ *or* $\langle (x,x,x),(2x,0,0),(0,2x,0)\rangle$.

Proof. As adapted from page 165 from Abstract Regular Polytopes [11].

Let s be the smallest positive integer from all coordinates of vectors in Λ, then we can assume that $(s,s_2,s_3) \in \Lambda$. Then $(-s,s_2,s_3) \in \Lambda$ and thus $2se_1 \in \Lambda$ and so too are each $2se_i$. By adding and subtracting multiples of these we can find a vector all of whose coordinates are values between $-s$ and $+s$. It then follows that Λ is generated by the all permutations of $(s^k, 0^{3-k})$ with all changes of sign for some $k \in \{1,2,3\}$. (Note that in rank n, k can be only $1,2$ or n. Since otherwise $(s^k,0^{n-k}) - (0,s^k,0^{n-k-1}) \in \Lambda$ and so $(s,s,0^{n-2}) \in \Lambda$ if k is odd or $(s,0^{n-1})$ is if k is even. Though $n=3$ in rank 4.)

If $k=1$ then we have the first basis mentioned in the Lemma, the second if $k=2$ and the third when $k=3$. □

It follows that $\Lambda_{\mathbf{s}} = s\Lambda_{(1^k,0^{n-l}),}$ and thus, as can be seen in [11, Theorem 6D1], we have the following theorem.

Theorem 3.2. *The only regular toroidal polytopes constructed from* $[4,3,4]$ *are* $\{4,3,4\}_{\mathbf{s}}$ *where* $\mathbf{s} = (s,0,0),(s,s,0)$ *or* (s,s,s) *and* $s \geq 2$.

Proof. Since conjugation of vectors in Λ by ρ_1,ρ_2 and ρ_3 are precisely all permutations of coordinates and changes of sign, this theorem follows directly from Lemma 3.1. □

The following theorem also appears in [11] along with its proof. This theorem describes the group of each toroid. To arrive at the following result (and subsequent related results in sections 4 and 5) we note that the mirror of reflection ρ_0 is $x=1/2$ while the mirrors for ρ_1, ρ_2 are $x=y$ and $y=z$ respectively and the mirror for ρ_3 is $z=0$.

Theorem 3.3 ([11, Theorem 6D4])**.** *Let* $\mathbf{s} = (s^k,0^{3-k})$*, with* $s \geq 2$ *and* $k=1,2,3$*. Then the group* $[4,3,4]_{\mathbf{s}}$ *is the Coxeter group* $[4,3,4] = \langle \rho_0,\rho_1,\rho_2,\rho_3\rangle$*, where the generators are specified in (3.1), factored out by the single extra relation which is*

$$\begin{aligned}(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1)^s &= id, \text{ if } k=1,\\ (\rho_0\rho_1\rho_2\rho_3\rho_2)^{2s} &= id, \text{ if } k=2,\\ (\rho_0\rho_1\rho_2\rho_3)^{3s} &= id, \text{ if } k=3.\end{aligned}$$

As explained in [11], a geometric argument can be used to verify the intersection property for these groups when $s \geq 2$. However, note that $[4,3,4]_{\mathbf{s}}$ does not satisfy the intersection condition when $s \leq 1$ and thus is not a C-Group. We show the breakdown of the intersection condition for $s=1$ by way of example for $k=1$ where cases for $k=2,3$ follow similar arguments.

When $s=1$, the identity $\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1 = id$ tells us that $\rho_0 \in \langle \rho_1,\rho_2,\rho_3\rangle$ so G does not satisfy the intersection property.

4 Toroidal hypertopes whose automorphism group is $\widetilde{B}_3$ $(= S_n)$

Let $\{\rho_0,\rho_1,\rho_2,\rho_3\}$ be the set of generators of $[4,3,4]$ as in the previous section and ε the corresponding fundamental simplex. We can double this fundamental simplex by replacing the generator ρ_0 with $\widetilde{\rho_0} = \rho_0\rho_1\rho_0$. Then $\widetilde{\rho_0}$ is a reflection through the hyperplane through the point $(1,0,0)$ with normal vector $(1,1,0)$. The transformation of a general vector by $\widetilde{\rho_0}$ is

$$(x,y,z)\widetilde{\rho_0} = (1-y, 1-x, z). \tag{4.1}$$

Then $\{\widetilde{\rho_0}, \rho_1, \rho_2, \rho_3\}$ generates $\widetilde{B}_3$, a subgroup of $[4,3,4]$ of index 2. The Coxeter diagram for this group is the non-linear diagram in Figure 2. In this section we let $G(= \widetilde{B}_3) := \langle\widetilde{\rho_0}, \rho_1, \rho_2, \rho_3\rangle$ and let $\mathcal{C}(\widetilde{B}_3)$ be the Coxeter complex of $G = \widetilde{B}_3$.

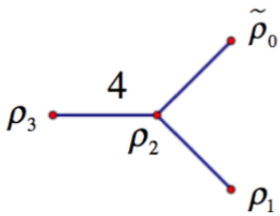

Figure 2: Coxeter diagram for $\widetilde{B_3}$.

The fundamental simplex of $\widetilde{B}_3$ is the simplex in Figure 3 bounded by the planes H_1, H_2, H_3 (fixed by ρ_1, ρ_2, ρ_3 respectively) and H_0 (now fixed by $\widetilde{\rho_0}$). Let, as above, F_i be the vertices of the fundamental simplex opposite to H_i. The orbit of each vertex, F_j of the fundamental simplex of $\widetilde{B}_3$ represents the set of hyperfaces of type j. Since this fundamental simplex shares vertices F_0, F_2 and F_3 with the fundamental simplex of $\langle\rho_0, \rho_1, \rho_2, \rho_3\rangle$ we will use the same names for hyperfaces as the names in Section 3, namely, vertices, faces and facets. Though the orbit or F_1 (which is isomorphic to the orbit of F_0 since the maximal parabolic subgroups generated by excluding ρ_1 or $\widetilde{\rho_0}$ are isomorphic) will be called hyperedges.

Now the translation subgroup of G is different from the one translation subgroup in the previous section since the set of vertices of $\{4,3,4\}$ now represent vertices and hyperedges (hyperfaces of type 0 and 1 respectively). The translation subgroup associated with this fundamental simplex is $T = \langle(1,1,0), (-1,1,0), (0,-1,1)\rangle$.

We then note that the translation by vector $(1,1,0)$ is the transformation (by right multiplication) $w_1 = \widetilde{\rho_0}\rho_2\rho_3\rho_2\rho_1\rho_2\rho_3\rho_2$, $(-1,1,0)$ is $w_2 = \rho_1\rho_2\rho_3\rho_2\widetilde{\rho_0}\rho_2\rho_3\rho_2$ and $(0,-1,1)$ is $w_3 = \rho_2\rho_3\rho_2\rho_1\rho_2\rho_1\widetilde{\rho_0}\rho_2\rho_3\rho_1$.

Now, to form a root lattice Λ we have the freedom to choose the crystollographic subgroup G_0 by fixing either a vertex or a hyperedge (see [3, pages 108–109]). We choose to leave out $\widetilde{\rho_0}$ since this reflection does not fix F_0. Doing so leaves $[3,4]$ as the subgroup we are conjugating with, which is the same as was for $[4,3,4]$. We also note that if we chose ρ_1 rather than $\widetilde{\rho_0}$ then the result is functionally the same since we are still conjugating by $[3,4] = \langle\widetilde{\rho_0}, \rho_2, \rho_3\rangle$ and this corresponds to forming a torus with its corners at hyper-edges.

We now note that although the same conditions are satisfied as in Lemma 3.1, T is now a different set. So instead we have the following lemma.

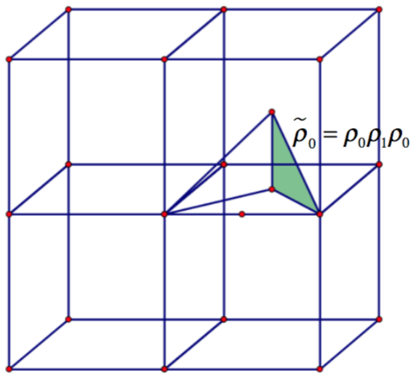

Figure 3: Fundamental simplex of $\widetilde{B}_3$.

Lemma 4.1. *If $T = \langle(1,1,0),(-1,1,0),(0,-1,1)\rangle$, $\Lambda \leq T$ a subgroup and if for every $\boldsymbol{a} \in \Lambda$, the image of $\boldsymbol{a}$ under changes of sign and permutations of coordinates is also in Λ, then $\Lambda = \langle(2x,0,0),(0,2x,0),(0,0,2x)\rangle$, $\langle(x,x,0),(-x,x,0),(0,-x,x)\rangle$ or $\langle(2x,2x,2x),(4x,0,0),(0,4x,0)\rangle$.*

Proof. We will only modify the proof of Lemma 3.1. In that proof we arrive at a generating set $(s^k, 0^{3-k})$ for each $k \in \{1,2,3\}$, given that T is different than the translation subgroup of Section 3.

Similar arguments to the ones used in the proof to Lemma 3.1 can now be used to show that for $k = 1$ or $k = 3$, s is even. For $k = 2$, Λ is generated by permutations and changes of sign of $(s,s,0)$. This needs no further examination since it is clearly in T. $\square$

As in the previous section, we describe the groups that will be used to construct each of the toroids. We denote by $G_{\mathbf{s}}$ the quotient $\widetilde{B}_3/\Lambda_{\mathbf{s}}$. Earlier we noted w_1 as the translation $(1,1,0)$ while $(\widetilde{\rho}_0\rho_2\rho_3\rho_2\rho_1)^2$ is the translation $(2,0,0)$ and $(\widetilde{\rho}_0\rho_2\rho_3\rho_1\rho_2\rho_3)^3$ is the translation $(2,2,2)$. And now that the the mirror for $\widetilde{\rho}_0$ is $y = 1 - x$.

Theorem 4.2. *Let $\mathbf{s} = (2s,0,0), (s,s,0)$ with $s \geq 2$ or $(2s,2s,2s)$ with $s \geq 1$. Then the group $G_{\mathbf{s}} = \widetilde{B}_3/\Lambda_{\mathbf{s}}$ is the Coxeter group $\widetilde{B}_3 = \langle\widetilde{\rho}_0,\rho_1,\rho_2,\rho_3\rangle$ with Coxeter diagram in Figure 2, factored out by the single extra relation which is*

$$(\widetilde{\rho}_0\rho_2\rho_3\rho_2\rho_1)^{2s} = id \text{ if } \mathbf{s} = (2s,0,0),$$
$$(\widetilde{\rho}_0\rho_2\rho_3\rho_2\rho_1\rho_2\rho_3\rho_2)^{s} = id \text{ if } \mathbf{s} = (s,s,0),$$
$$(\widetilde{\rho}_0\rho_2\rho_3\rho_1\rho_2\rho_3)^{3s} = id \text{ if } \mathbf{s} = (2s,2s,2s).$$

Here, as in Section 3, we have that $G_{\mathbf{s}}$ fails the intersection property for small enough s. However, because the fundamental simplex is doubled, this time when $\mathbf{s} = (2s,2s,2s)$, $G_{\mathbf{s}}$ satisfies the intersection condition for $s \geq 1$ while $s \geq 2$ is still necessary for the other two vectors. Verifying that $G_{\mathbf{s}}$ fails the intersection condition for $s = 1$ when $\mathbf{s} = (2s,0,0)$ and $(s,s,0)$ follows similar calculations as those done in Section 3. Namely, when $s = 1$ for the first vector, we arrive at the identity $\widetilde{\rho_0}\rho_2\rho_3\rho_2\widetilde{\rho_0} = \rho_1\rho_2\rho_3\rho_2\rho_1$ and for the second vector we have the identity $\widetilde{\rho_0}\rho_2\rho_3\rho_2 = \rho_2\rho_3\rho_2\rho_1$. Which violates the intersection condition.

MAGMA [1] can be used to verify that $G_{\mathbf{s}}$ satisfies the intersection condition when $\mathbf{s} = (4,0,0) = (2s,0,0), (2,2,0) = (s,s,0)$ or $(2,2,2)$. To see that the it also satisfies the intersection condition for greater values of s can be seen with a geometric argument.

The orbit of a base chamber of each parabolic subgroup of $G_{\mathbf{s}}$ can be seen as a collection of chambers which are duplicated at each of the 8 corners of the boundaries of Λ_s. For instance, the subgroup $\langle \rho_1, \rho_2, \rho_3 \rangle$ consists of chambers forming octahedra centred around corner vertex.

Given the collection of chambers in two such subgroups, there will always be some intersection between the collections occurring at the same corner (someones it's just the base chamber itself). However, If $G_{\mathbf{s}}$ fails the intersection condition, then there will be an intersection with the chambers of one subgroup centred around one corner that intersect with the chambers of the other subgroup on another corner.

So, given a particular s where $G_{\mathbf{s}}$ satisfies the intersection condition, by increasing s, the corners of $\Lambda_{\mathbf{s}}$ get further and further apart. So if there are no such intersections for some s, then for larger s there will not be either.

Adopting a similar notation as in the previous section and using $\Lambda_{\mathbf{s}}$ as defined in Section 2, we now have the following theorem.

Theorem 4.3. *The regular toroidal hypertopes of rank 4 constructed from* $G(= \widetilde{B}_3) = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \rho_3 \rangle$*, where the generators are specified in* (3.1) *and (4.1), are* $\mathcal{C}(\widetilde{B}_3)/\Lambda_{\mathbf{s}}$ *where* $\mathcal{C}(\widetilde{B}_3)$ *is the Coxeter complex of* $\widetilde{B}_3$ *and* $\mathbf{s} = (2s,0,0), (s,s,0)$ *with* $s \geq 2$ *or* $(2s,2s,2s)$ *with* $s \geq 1$.

Proof. To begin we need to find an $\mathbf{s}$ and corresponding $\Lambda_{\mathbf{s}}$ that is invariant under conjugation by a subgroup of G which is the symmetry group of "vertex"-figure (by vertex we mean, the element that the translations begin from). In this case our subgroup ends up being $[3,4]$ as was described before Lemma 4.1.

Now, since we are conjugating by $[3,4] = \langle \rho_1, \rho_2, \rho_3 \rangle$, $\Lambda_{\mathbf{s}}$ must contain all permutations and changes of sign of any vector in $\Lambda_{\mathbf{s}}$ (which we discovered in the proof of Theorem 3.2 which is also on page 165 of [11]). Thus, by Lemma 4.1, $\mathbf{s} = (2s,0,0), (s,s,0)$ or $(2s,2s,2s)$. However, we still do not know if this construction yields a regular hypertope. To do this, we start by noting that the Coxeter complex $\mathcal{C}(\widetilde{B}_3)$ formed from G is precisely the hypertope $\Gamma(G;(G_i)_{i\in I})$ (the construction of which follows from [7]).

So we need to show that $\mathcal{C}(\widetilde{B}_3)$ is flag transitive (or, equivalently, chamber transitive). To do so we will note the rank 3 residue $\Gamma_{\widetilde{0}} := \Gamma(G_{\widetilde{0}};(G_{\{\widetilde{0},i\}})_{i\in\{1,2,3\}})$. This is isomorphic to the cube, a regular polyhedron, which is flag transitive.

So we pick to chambers in $\Gamma(G;(G_i)_{i\in I}) = \mathcal{C}(\widetilde{B}_3)$ which can be written as $C_1 = \{G_{\widetilde{0}}g_0, G_1g_1, G_2g_2, G_3g_3\}$ and $C_2 = \{G_{\widetilde{0}}h_0, G_1h_1, G_2h_2, G_3h_3\}$ for some $g_i, h_i \in G$. Then, since $G = G_{\widetilde{0}} \rtimes T$ and T acts transitively on elements of type $\widetilde{0}$ there is a translation $t \in G$ such that $C_1t = \{G_{\widetilde{0}}h_0, X, Y, Z\}$ which is some chamber that shares the same element of type $\widetilde{0}$ as C_2. Then the chambers C_1t and C_2 are both in some rank 3 residue which is isomorphic to $\Gamma_{\widetilde{0}}$. Since this residue is flag transitive, there is some element, $g \in G$ such that $C_1tg = C_2$. Thus $\mathcal{C}(\widetilde{B}_3)$ is chamber transitive and thus flag transitive. So, by Proposition 4.6 from [7] this is a regular hypertope.

So now we want to know if $\Gamma(G';(G'_i)_{i\in I})$ is a regular hypertope where G' is the group $G/\Lambda_{\mathbf{s}}$ where $s \geq 2$ (since otherwise G' fails the intersection condition and the resulting construction fails to be thin). Just as before, we take two chambers Φ and Ψ from $\Gamma(G'; (G'_i)_{i\in I})$. Then to each of these chambers we can associate a family of chambers Φ' and Ψ' in $\mathcal{C}(\widetilde{B_3})$. Since $\mathcal{C}(\widetilde{B_3})$ is chamber transitive for each $\Phi_j \in \Phi'$ and $\Psi_k \in \Psi'$ there exists $\varphi_{jk} \in G$ where $\Phi_j\varphi_{jk} = \Psi_k$. In particular there exist chambers $\Phi_1 \in \Phi'$ and $\Psi_1 \in \Psi'$ in $\mathcal{C}(\widetilde{B_3})$ where, since $\Lambda_{\mathbf{s}}$ is invariant under G, $\Phi_1\psi = \Psi_1$ and $\psi \in G'$. We can see this by noting that Φ_1 and Ψ_1 are the members of their respective families which lie inside the fundamental region of $\Lambda_{\mathbf{s}}$.

Thus $\Gamma(G'; (G'_i)_{i\in I})$ is chamber transitive and thus face transitive, so is also a regular hypertope by Proposition 2.6.

For the other two possibilities of Λ, we need only change the added relations, but because the relations were chosen specifically, they will also generate regular hypertopes. □

5 Toroidal hypertopes whose automorphism group is $\widetilde{A}_3$ $(= P_n)$

We can show that this group is, yet again a subgroup of $[4, 3, 4]$ by doubling the fundamental simplex a second time (this can be seen geometrically in Figure 5) and now defining $\widetilde{\rho_3} = \rho_3\rho_2\rho_3$ which is a reflection in the plane through $(1, 1, -1)$ with normal vector $(0, 1, 1)$. Transformation of a general vector by $\widetilde{\rho_3}$ is

$$(x, y, z)\widetilde{\rho_3} = (x, -z, -y). \tag{5.1}$$

Now we let $G(= \widetilde{A_3}) := \langle\widetilde{\rho_0}, \rho_1, \rho_2, \widetilde{\rho_3}\rangle$ and $\mathcal{C}(\widetilde{A_3})$ be the Coxeter complex of G. The defining relations for G are implicit in the Coxeter diagram in Figure 4.

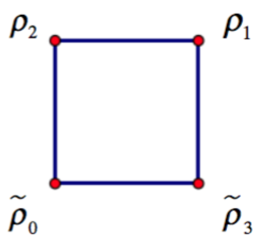

Figure 4: Coxeter diagram for $\widetilde{A_3}$.

Here the fundamental simplex of $\widetilde{A_3}$ is a tetrahedron bounded by the planes H_i (fixed by ρ_i). This fundamental simplex shares the planes fixed by $\widetilde{\rho_0}, \rho_1, \rho_2$ with the fundamental simplex of $\widetilde{B_3}$ as well as the corresponding vertices. The stabilizers of each vertex of the fundamental simplex are also isomorphic since all maximal parabolic subgroups of $\widetilde{A}_3$ are pairwise isomorphic. This implies that the set of hyperfaces of types i and j are isomorphic for each $i, j \in \{0, 1, 2, 3\}$.

This fundamental simplex gives us the same translation subgroup as we had in the previous Section. Though now we must use the new generators to find the translations. We define $w_1 = \widetilde{\rho_0}\rho_2\rho_1\widetilde{\rho_3}\rho_1\rho_2 = (1, 1, 0)$, $w_2 = \rho_1\rho_2\widetilde{\rho_3}\widetilde{\rho_0}\widetilde{\rho_3}\rho_2 = (-1, 1, 0)$ and $w_3 = \rho_2\rho_1\widetilde{\rho_0}\widetilde{\rho_3}\widetilde{\rho_0}\rho_1 = (0, -1, 1)$.

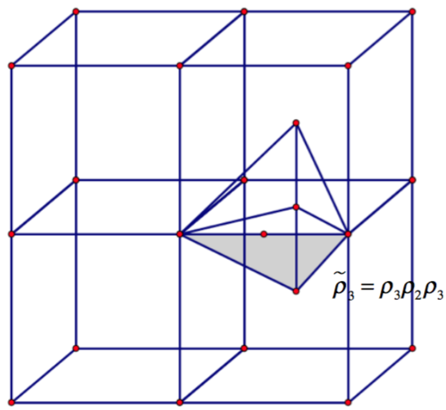

Figure 5: Fundamental simplex of $\widetilde{A_3}$.

Using these translations, for a translation $(a, b, c) \in \Lambda$, we have that $\rho_1(a, b, c)\rho_1 = (b, a, c)$. In a similar way, conjugating by ρ_2 yields (a, c, b) and conjugating by $\widetilde{\rho_3}$ yields $(a, -c, -b)$. So if we conjugate by $\rho_1\rho_2\rho_1$ then we get (c, b, a) and so Λ must have all permutations. Now, from the previous we know Λ must also contain $(a, -b, -c)$ and adding this to (a, b, c) gives $(2a, 0, 0)$, which then subtracted from (a, b, c) is $(-a, b, c)$ and so with all permutations means that Λ must have all permutations and changes of sign.

With this group, we leave out $\widetilde{\rho_0}$ to form the crystollographic subgroup G_0. Though a curiosity of this group is that we use any generator of $\widetilde{B_3}$ to form a crystollographic subgroup and still finish with the same objects. With each choice simply changing where we draw the boundary of the torus. This leaves ρ_1, ρ_2 and $\widetilde{\rho_3}$ with which to conjugate Λ. As in the regular case, ρ_1 and ρ_2 show us that Λ must consist of all permutations of the coordinates of vectors.

If (a, b, c) is a general vector in Λ then $\widetilde{\rho_3}$ tells us that $(-c, b, -a)$ must also be in Λ and then so also must $(-a, b, -c)$. Adding that to our original general vector tells us that $(0, 2b, 0)$ is also included. So, subtracting that from the general vector finally gives us $(a, -b, c)$. Note that this can just as easily be done with either a or c with some simple permutations.

As in the previous section, we describe the groups of each of the toroids. Earlier we noted w_1 as the translation $(1, 1, 0)$ while $(\widetilde{\rho_0}\rho_2\widetilde{\rho_3}\rho_1)^2$ is the translation $(2, 0, 0)$ and $(\widetilde{\rho_0}\rho_2\rho_1\widetilde{\rho_3})^{3s}$ is the translation $(2, 2, 2)$. And now that the the mirror for $\widetilde{\rho_0}$ is $y = 1 - x$ while the mirror for $\widetilde{\rho_3}$ is $y = -z$.

Theorem 5.1. *Let* $\boldsymbol{s} = (2s, 0, 0), (s, s, 0)$ *with* $s \geq 2$ *or* $(2s, 2s, 2s)$ *with* $s \geq 1$. *Then the group* $G_{\boldsymbol{s}} = \widetilde{A_3}/\Lambda_{\boldsymbol{s}}$ *is the Coxeter group* $\widetilde{A}_3 = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \widetilde{\rho_3} \rangle$ *(with Coxeter group specifed in Figure 4), factored out by the single extra relation which is*

$$(\widetilde{\rho_0}\rho_2\widetilde{\rho_3}\rho_1)^{2s} = id \text{ if } \boldsymbol{s} = (2s, 0, 0),$$
$$(\widetilde{\rho_0}\rho_2\rho_1\widetilde{\rho_3}\rho_1\rho_2)^{s} = id \text{ if } \boldsymbol{s} = (s, s, 0),$$
$$(\widetilde{\rho_0}\rho_2\rho_1\widetilde{\rho_3})^{3s} = id \text{ if } \boldsymbol{s} = (2s, 2s, 2s).$$

For the same reasons as in Section 4, the intersection condition is satisfied for $\mathbf{s} = (2s, 2s, 2s)$ when $s \geq 1$.

Theorem 5.2. *The regular toroidal hypertopes of rank 4 induced by* $G(= \widetilde{A_3}) = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \widetilde{\rho_3} \rangle$ *(where the generators are specified in (3.1), (4.1) and (5.1)) are* $\mathcal{C}(\widetilde{A_3})/\Lambda_{\mathbf{s}}$ *where* $\mathcal{C}(\widetilde{A_3})$ *is the Coxeter complex of* $\widetilde{A_3}$ *and* $\mathbf{s} = (2s, 0, 0), (s, s, 0)$ *for* $s \geq 2$ *or* $(2s, 2s, 2s)$ *with* $s \geq 1$.

Proof. We first show that $\mathcal{C}(\widetilde{A_3})$ is a regular hypertope, which requires showing that it is flag transitive. In the same manner as the proof of Theorem 4.3 we need only show each rank 3 residue is flag transitive, since all rank 3 residues are regular tetrahedra $\mathcal{C}(\widetilde{A_3})$ is flag transitive. The translation subgroup is the same as in the previous Section and conjugating Λ by $\rho_1, \rho_2, \widetilde{\rho_3}$ gives all permutations and changes in sign of a general vector in Λ, the same arguments for Lemma 4.1 and Theorem 4.3 will prove this theorem. □

6 Non-existence of rank 4 chiral hypertopes

Here we recall that for an abstract polytope to be chiral its automorphism group must have two orbits when acting on flags and that adjacent flags are in different orbits. Chiral polytopes of any rank are examined in depth in [13]. The existence of these objects in any rank was proved in [12]. There is also a notion of chirality in hypermaps as well, see for example, [2]. Similarly we say for a hypertope to be chiral if its automorphism group action has two chamber orbits and adjacent chambers are in different orbits [7].

As in Section 2, given an affine Coxeter group G and associated Coxeter complex $\mathcal{C}$, we define a subgroup $G_0 \leq G$ as the maximal parabolic subgroup fixing the origin. Then, given a set I of linearly independent translations in G and T_I, the translation subgroup generated by I then we call the lattice Λ_I the lattice induced by the orbit of the origin under T_I. When Λ_I is invariant under the rotation subgroup ${G_0}^+$ but there is no automorphism group of G that interchanges adjacent chambers, then in rank 4 we say that the quotient $\mathcal{C}/\Lambda_I$ is a *chiral toroidal hypertope* of rank 4.

The proof that there are no chiral toroids of rank 4 for the group $[4, 3, 4]$ comes from page 178 from [11] and the same proof can adapted for the other two rank 4 affine Coxeter groups. The basic idea for the proof is that since $\mathcal{C}/\Lambda$ is chiral, Λ is invariant under the rotation group $[3, 4]^+$, so Λ contains vectors that are compositions of an even number of permutations with an even number of sign changes or all compositions of an odd number of permutations with an odd number of sign changes. It then goes to show that if $(a, b, c) \in \Lambda$ then $(b, a, c) \in \Lambda$, which is the image of (a, b, c) under an odd permutation, which is a contradiction. Therefore no such Λ can exist.

We will use the same method to show the same is true for the other two groups.

Theorem 6.1. *There are no rank 4 chiral toroidal hypertopes.*

Proof. In [11] it was shown that there are no rank 4 hypertopes constructed from $[4, 3, 4]$, so we show for constructions from $\widetilde{B}_3$ and $\widetilde{A}_3$. In previous sections we found that if Λ is a subgroup of the translations that is invariant under conjugation by the stabilizer of the origin in $\widetilde{B}_3$ and $\widetilde{A}_3$ with $(a, b, c) \in \Lambda$, then Λ contains all permutations and changes of sign of (a, b, c), just as it did with the stabilizer in $[4, 3, 4]$.

Thus conjugation of Λ by the stabilizer of the rotation subgroup of each of these groups is all compositions of even permutations with an even number of sign changes or all compositions of odd permutations with an odd number of sign changes, just as for $[4, 3, 4]$.

So the same arguments and calculations from page 178 in [11] still hold and show that $(b, a, c) \in \Lambda$ and we develop the same contradiction. $\square$

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On 2-distance-balanced graphs*

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Abstract

Let n denote a positive integer. A graph Γ of diameter at least n is said to be n*-distance-balanced* whenever for any pair of vertices u, v of Γ at distance n, the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. In this article we consider $n = 2$ (e.g. we consider 2*-distance-balanced graphs*). We show that there exist 2-distance-balanced graphs that are not 1-distance-balanced (e.g. distance-balanced). We characterize all connected 2-distance-balanced graphs that are not 2-connected. We also characterize 2-distance-balanced graphs that can be obtained as cartesian product or lexicographic product of two graphs.

Keywords: n-distance-balanced graph, cartesian product, lexicographic product.

Math. Subj. Class.: 05C12, 05C76

1 Introductory remarks

A graph Γ is *distance-balanced* if for each pair u, v of adjacent vertices of Γ the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Although these graphs are interesting from the purely graph-theoretical point of view, they also have applications in other areas of research, such as mathematical chemistry and communication networks. It is for that reason that they have been studied from various different points of view in the literature.

Distance-balanced graphs were first studied by Handa [9] in 1999. The name *distance-balanced*, however, was introduced nine years later by Jerebic, Klavžar and Rall [12]. The

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family of distance-balanced graphs is very rich (for instance, every distance-regular graph as well as every vertex-transitive graph has this property [13]). In the literature these graphs were studied from various purely graph-theoretic aspects such as symmetry [13], connectivity [9, 16] or complexity aspects of algorithms related to such graphs [6], to name just a few. However, it turns out that these graphs have applications in other areas, such as mathematical chemistry (see for instance [3, 11, 12]) and communication networks (see for instance [3]).

Another interesting fact is that these graphs can be characterized by properties that do not seem to have much in common with the original definition from [12]. For example, the distance-balanced graphs coincide with *self-median* graphs, that is graphs for which the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex (see [4]). In [1], distance-balanced graphs are called *transmission regular*. Finally, even order distance-balanced graphs possess yet another nice property, making them what are called *equal opportunity graphs* (see [3] for the definition).

In distance-balanced graphs one only considers pairs of adjacent vertices. However, it is very natural to extend the definition to the pairs of nonadjacent vertices. This generalized concept of n-*distance-balanced* graphs (see Section 2 for the definition) was first introduced by Boštjan Frelih in 2014 [8] (we point out that certain other generalizations of this concept, where one still focuses just on pairs of adjacent vertices, have also been considered in the recent years [10, 14, 15]). The n-distance-balanced graphs and their properties were extensively studied in [17]. They are also the main topic in the paper [7], but in this paper some of the stated results do not hold. We comment on one of these problems later (see Remark 5.1).

In this article we consider 2-*distance-balanced graphs*. We now summarize our results. After some preliminaries in Section 2, we show in Section 3 that there exist 2-distance-balanced graphs that are not 1-distance-balanced (e.g. distance-balanced). It was shown in [9] that every distance-balanced graph is 2-connected. It turns out that not all 2-distance-balanced graphs are 2-connected. However, we characterize all connected 2-distance-balanced graphs that are not 2-connected.

In [12] distance-balanced cartesian products and distance-balanced lexicographic products of two graphs were characterized. We characterize 2-distance-balanced cartesian products and 2-distance-balanced lexicographic products of two graphs in Section 4 and 5, respectively.

2 Preliminaries

In this section we review some basic definitions that we will need later. Throughout this paper, all graphs are assumed to be finite, undirected, without loops and multiple edges. Given a graph Γ let $V(\Gamma)$ and $E(\Gamma)$ denote its vertex set and edge set, respectively.

For $v \in V(\Gamma)$ we denote the set of vertices adjacent to v by $N_\Gamma(v)$. If the number $|N_\Gamma(v)|$ is independent of the choice of $v \in V(\Gamma)$, then we call this number the *valency of* Γ and we denote it by k_Γ (or simply by k if the graph Γ is clear from the context). In this case we say that Γ is *regular with valency* k or k-*regular*.

For $u, v \in V(\Gamma)$ we denote the distance between u and v by $\partial_\Gamma(u, v)$ (or simply by $\partial(u, v)$ if the graph Γ is clear from the context). The *diameter* $\max\{\partial_\Gamma(u, v) \mid u, v \in V(\Gamma)\}$ of Γ will be denoted by D_Γ (or simply by D if the graph Γ is clear from the context). For any pair of vertices $u, v \in V(\Gamma)$ we let W_{uv}^Γ be the set of vertices of Γ that are closer to u than to v, that is

$$W_{uv}^{\Gamma} = \{w \in V(\Gamma) \mid \partial_{\Gamma}(u, w) < \partial_{\Gamma}(v, w)\}.$$

Let n denote a positive integer. A graph Γ of diameter at least n is said to be *n-distance-balanced*, if $|W_{uv}^{\Gamma}| = |W_{vu}^{\Gamma}|$ for any $u, v \in V(\Gamma)$ at distance n. The distance-balanced graphs are n-distance-balanced graphs for $n = 1$.

For $W \subseteq V(\Gamma)$ the subgraph of Γ induced by W is denoted by $\langle W\rangle$ (we abbreviate $\Gamma - W = \langle V(\Gamma) \setminus W\rangle$). A *vertex cut* of a connected graph Γ is a set $W \subseteq V(\Gamma)$ such that $\Gamma - W$ is disconnected. A vertex cut of size k is called a *k-cut*. A graph is said to be *k-connected* if it has at least $k + 1$ vertices and the size of the smallest vertex cut is at least k. If a vertex cut consists of a single vertex v, then v is called the *cut vertex*.

We complete this section by defining the cartesian product and the lexicographic product of graphs G and H. In both cases, the vertex set of the product is $V(G) \times V(H)$. Pick $(g_1, h_1), (g_2, h_2) \in V(G) \times V(H)$.

In the *cartesian product* of G and H, denoted by $G \Box H$, (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and h_1, h_2 are adjacent in H, or $h_1 = h_2$ and g_1, g_2 are adjacent in G. Note that the cartesian product is commutative.

In the *lexicographic product* of G and H, denoted by $G[H]$, (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and h_1, h_2 are adjacent in H, or g_1, g_2 are adjacent in G.

3 On the connectivity of 2-distance-balanced graphs

In this section we characterize connected 2-distance-balanced graphs that are not 2-connected (Corollary 3.4). As a consequence, using the well known fact that an arbitrary connected distance-balanced graph is at least 2-connected (see [9]), we construct an infinite family of 2-distance-balanced graphs that are not distance-balanced.

Let G be an arbitrary (not necessary connected) graph, and let c be a vertex that does not belong to the set of vertices of G. We construct a graph, denoted by $\Gamma(G, c)$, with the set of vertices

$$V(\Gamma(G, c)) = V(G) \cup \{c\}$$

and the set of edges

$$E(\Gamma(G, c)) = E(G) \cup \{cv \mid v \in V(G)\}.$$

This graph is obviously connected. Next theorem follows directly from the construction of $\Gamma(G, c)$.

Theorem 3.1. *G is not connected if and only if $\Gamma(G, c)$ is not 2-connected.* □

We show that regularity of G is a sufficient condition for $\Gamma(G, c)$ to be 2-distance-balanced.

Theorem 3.2. *If G is a regular graph that is not a complete graph, then $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.*

Proof. Assume that G is a k-regular graph that is not a complete graph. Let $G_1, G_2, \dots, G_n$ be its connected components for some positive integer n. If G is connected, then $n = 1$, otherwise G has at least two connected components. Since G is not a complete graph, it is clear that the diameter of Γ equals 2, which means that two arbitrary vertices of Γ are either adjacent or they are at distance 2.

There are two different types of vertices at distance 2 in Γ. The first type is when both vertices at distance 2 belong to the same connected component of G. The second type is when vertices at distance 2 belong to different connected components of G.

Let G_i be an arbitrary connected component of G. Let $v_1, v_2 \in V(G_i)$ be arbitrary vertices at distance 2 in Γ. We count vertices that are closer to v_1 than to v_2 in Γ and vertices that are closer to v_2 than to v_1 in Γ. We get

$$W_{v_1v_2}^{\Gamma} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2))).$$

It follows that

$$\left|W_{v_1v_2}^{\Gamma}\right| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|.$$

Changing the roles of vertices v_1 and v_2, we get

$$\left|W_{v_2v_1}^{\Gamma}\right| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|.$$

Since G is regular, the number of vertices that are closer to v_1 than to v_2 in Γ equals the number of vertices that are closer to v_2 than to v_1 in Γ.

Let now G_i and G_j be arbitrary different connected components of a disconnected graph G. Pick arbitrary $v_1 \in V(G_i)$ and $v_2 \in V(G_j)$. Obviously these two vertices are at distance 2 in Γ. Observe that

$$W_{v_1v_2}^{\Gamma} = \{v_1\} \cup N_{G_i}(v_1) \quad \text{and} \quad W_{v_2v_1}^{\Gamma} = \{v_2\} \cup N_{G_j}(v_2).$$

Since every connected component of a k-regular graph is also a k-regular (induced) subgraph, it follows that

$$\left|W_{v_1v_2}^{\Gamma}\right| = 1 + k \quad \text{and} \quad \left|W_{v_2v_1}^{\Gamma}\right| = 1 + k,$$

where k is the valency of G. So the number of vertices that are closer to v_1 than to v_2 in Γ equals the number of vertices that are closer to v_2 than to v_1 in Γ. Since this is true for an arbitrary pair of vertices at distance 2 in Γ, this graph is 2-distance-balanced. □

Next we prove that every connected 2-distance-balanced graph, that is not 2-connected, is isomorphic to $\Gamma(G, c)$ for some regular graph G that is not connected.

Theorem 3.3. *Let Γ be a connected 2-distance-balanced graph that is not 2-connected. Then Γ is isomorphic to $\Gamma(G, c)$ for some disconnected regular graph G.*

Proof. Since Γ is not 2-connected, there exists a cut vertex $c \in V(\Gamma)$. Let $G_1, G_2, \dots, G_n$ be connected components of $G = \Gamma - \{c\}$, $n \geq 2$. We want to prove that G is regular and that the cut vertex c is adjacent to every other vertex in Γ. To do this we will first prove some partial results.

First we claim that the cut vertex c is adjacent to every vertex in a connected component G_ℓ of G for at least one integer ℓ, $1 \leq \ell \leq n$. Suppose that this is not true. Let G_i and G_j be two different connected components of G. Then there exist $v_2 \in V(G_i)$ and $u_2 \in V(G_j)$, both at distance 2 from c in Γ. This means that there exists $v_1 \in V(G_i)$ that is adjacent to c and v_2 in Γ, and there exists $u_1 \in V(G_j)$ that is adjacent to c and u_2 in Γ. If we compare the set of vertices that are closer to c than to v_2 in Γ and the set of vertices that are closer to v_2 than to c in Γ, we get

$$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \quad \text{and} \quad W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\}.$$

It follows that

$$1 + |V(G_j)| \leq \left|W^{\Gamma}_{cv_2}\right| \quad \text{and} \quad \left|W^{\Gamma}_{v_2c}\right| \leq |V(G_i)| - 1.$$

Since Γ is 2-distance-balanced, we get

$$|V(G_j)| \leq |V(G_i)| - 2. \tag{3.1}$$

Similarly as above (changing vertex v_2 with u_2) we get

$$|V(G_i)| + 2 \leq |V(G_j)|. \tag{3.2}$$

However, inequalities (3.1) and (3.2) imply

$$|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2,$$

a contradiction. It follows that the cut vertex $c \in V(\Gamma)$ is adjacent to every vertex in $V(G_\ell)$ for at least one integer ℓ, $1 \leq \ell \leq n$. Without loss of generality we may assume that $\ell = 1$.

Next we claim that the induced subgraph G_1 of Γ is regular. Pick some $u \in V(G) \setminus V(G_1)$ that is adjacent to the cut vertex c in Γ. Since c is adjacent to every vertex in $V(G_1)$, the distance between u and an arbitrary $v \in V(G_1)$ equals 2 in Γ. Pick $v \in V(G_1)$. Notice that

$$W^{\Gamma}_{vu} = \{v\} \cup (N_\Gamma(v) \setminus \{c\}).$$

It follows that

$$\left|W^{\Gamma}_{vu}\right| = 1 + |N_\Gamma(v)| - 1 = |N_{G_1}(v)| + 1.$$

Pick $w \in V(G_1)$. Since Γ is 2-distance-balanced and c is adjacent to every vertex of $V(G_1)$, we get the following sequence of equalities

$$\begin{aligned} |N_{G_1}(v)| + 1 = |N_\Gamma(v)| &= \left|W^{\Gamma}_{vu}\right| = \left|W^{\Gamma}_{uv}\right| = \left|W^{\Gamma}_{uw}\right| \\ &= \left|W^{\Gamma}_{wu}\right| = |N_\Gamma(w)| = |N_{G_1}(w)| + 1. \end{aligned}$$

So

$$|N_{G_1}(v)| = |N_{G_1}(w)|$$

for arbitrary $v, w \in V(G_1)$. From now on we may assume that the induced subgraph G_1 of Γ is k-regular. This also means that every vertex in $V(G_1)$ has valency $k+1$ in Γ.

Our next step is to show that the cut vertex $c \in V(\Gamma)$ is adjacent to every vertex in $V(G) \setminus V(G_1)$. Suppose that this is not true. Then there exists some vertex u_2 in a connected component G_ℓ of G, $2 \leq \ell \leq n$, that is at distance 2 from c in Γ. Without loss of generality we can take $\ell = 2$. Consequently there exists some $u_1 \in V(G_2)$ that is adjacent to both c and u_2 in Γ. Pick an arbitrary $v \in V(G_1)$. We have already proved that the valency of an arbitrary vertex in $V(G_1)$ is $k+1$ in Γ. Now we count vertices that are closer to v than to u_1 in Γ. Since

$$W^{\Gamma}_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\}),$$

we get

$$\left|W^{\Gamma}_{vu_1}\right| = 1 + k + 1 - 1 = k + 1.$$

In addition, for vertices that are closer to c than to u_2 in Γ, we have

$$W^{\Gamma}_{cu_2} \supseteq V(G_1) \cup \{c\}.$$

It follows that

$$\left|W^{\Gamma}_{cu_2}\right| \geq |V(G_1)| + 1 \geq k + 2. \tag{3.3}$$

Consider the distance partition of Γ according to adjacent vertices c and u_1 that is shown in Figure 1. The symbol D^i_j denotes the set of vertices that are at distance i from u_1 and at distance j from c in Γ. Define a set

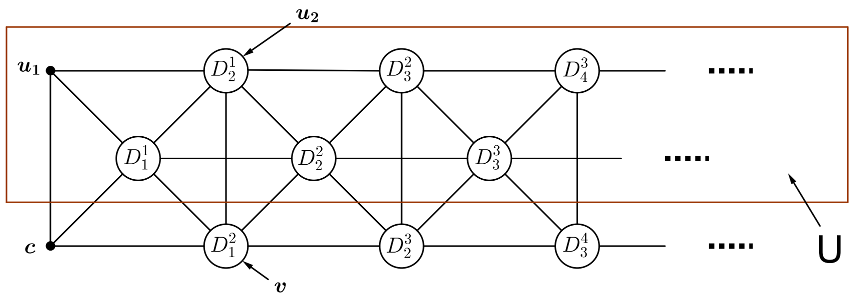

Figure 1: The distance partition of Γ according to adjacent vertices c and u_1.

$$U = \bigcup_{i=1}^{D} = \left(D^{i-1}_i \cup D^i_i\right),$$

where D denotes the diameter of Γ. First we show that $W^{\Gamma}_{u_2c} \subseteq U$. Recall that for $u, v \in V(\Gamma)$, $\partial(u, v)$ denotes the distance between vertices u and v. Pick an arbitrary $w \in W^{\Gamma}_{u_2c}$. Since u_1, u_2 are, by the assumption, adjacent vertices in Γ, the triangle inequality tells us that

$$\partial(u_1, w) \in \{\partial(u_2, w) - 1, \partial(u_2, w), \partial(u_2, w) + 1\}.$$

If we consider all three cases, we get

$$\begin{aligned}
\partial(u_1, w) &= \partial(u_2, w) - 1 < \partial(c, w),\\
\partial(u_1, w) &= \partial(u_2, w) < \partial(c, w),\\
\partial(u_1, w) &= \partial(u_2, w) + 1 \leq \partial(c, w).
\end{aligned}$$

Each considered case gives us that $w \in U$ and so $W^{\Gamma}_{u_2c} \subseteq U$. Note also that $U \subseteq V(G_2)$. Now we show that $U \subseteq W^{\Gamma}_{u_1v}$ (recall that v is an abitrary vertex in $V(G_1)$). Let w be an arbitrary vertex in U, which means that w is also in $V(G_2)$. We get that

$$\partial(u_1, w) \leq \partial(c, w) < \partial(v, w),$$

since vertices v and c are adjacent in Γ, and v is not in $V(G_2)$. It follows that $w \in W^{\Gamma}_{u_1v}$, and so $U \subseteq W^{\Gamma}_{u_1v}$. From relations $W^{\Gamma}_{u_2c} \subseteq U \subseteq W^{\Gamma}_{u_1v}$, we get that $W^{\Gamma}_{u_2c} \subseteq W^{\Gamma}_{u_1v}$ and so

$$\left|W^{\Gamma}_{u_2c}\right| \leq \left|W^{\Gamma}_{u_1v}\right| = \left|W^{\Gamma}_{vu_1}\right| = k + 1. \tag{3.4}$$

By taking into account inequalities (3.3) and (3.4), and since Γ is 2-distance-balanced, we get

$$k+2 \leq |W^{\Gamma}_{cu_2}| = |W^{\Gamma}_{u_2c}| \leq k+1,$$

which is a contradiction. This shows that the cut vertex $c \in V(\Gamma)$ is adjacent to all vertices in $V(G)$.

It remains to prove that the induced subgraph G_ℓ $(2 \leq \ell \leq n)$ of Γ is k-regular. Without loss of generality assume $\ell = 2$. Since we already know that the cut vertex c is adjacent to every vertex in Γ, an arbitrary vertex u in $V(G_2)$ is at distance 2 from an arbitrary vertex v in $V(G_1)$ in Γ. Observe that

$$W^{\Gamma}_{uv} = \{u\} \cup (N_\Gamma(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$$

and

$$W^{\Gamma}_{vu} = \{v\} \cup (N_\Gamma(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v).$$

This means that

$$\left|W^{\Gamma}_{uv}\right| = 1 + |N_{G_2}(u)| \quad \text{and} \quad \left|W^{\Gamma}_{vu}\right| = 1 + k.$$

Since Γ is 2-distance balanced, it follows that $\left|W^{\Gamma}_{uv}\right| = \left|W^{\Gamma}_{vu}\right|$ and so $|N_{G_2}(u)| = k$ for an arbitrary vertex $u \in V(G_2)$. Therefore, G_2 is regular and has the same valency k as the induced subgraph G_1. It follows that G is regular and this completes the proof. $\square$

The characterization of all connected 2-distance-balanced graphs that are not 2-connected follows immediately from Theorems 3.1, 3.2 and 3.3.

Corollary 3.4. *Let Γ be a connected graph. Then Γ is 2-distance-balanced and not 2-connected if and only if it is isomorphic to $\Gamma(G, c)$ for some disconnected regular graph G.* $\square$

4 2-distance-balanced cartesian product

Throughout this section let G and H be graphs and let $\Gamma = G \square H$ be the cartesian product of G and H. We characterize connected 2-distance-balanced cartesian products of graphs G and H (see Theorem 4.4). It follows from the definition that the cartesian product Γ is connected if and only if G and H are both connected. In order to avoid trivialities we assume that $|V(G)| \geq 2$ and $|V(H)| \geq 2$.

Recall that

$$\partial_\Gamma((g_1, h_1), (g_2, h_2)) = \partial_G(g_1, g_2) + \partial_H(h_1, h_2) \tag{4.1}$$

for arbitrary $(g_1, h_1), (g_2, h_2) \in V(\Gamma)$. Since we are dealing with 2-distance-balanced cartesian products of graphs, we are interested in vertices at distance 2. It follows from equality (4.1), that there exist three different types of vertices at distance 2 in Γ. We now state these three types and we will refer to them later. Let (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ be vertices at distance 2 in Γ. We say that these two vertices are of type

- $G2$, if $h_1 = h_2$ and $\partial_G(g_1, g_2) = 2$,
- $H2$, if $g_1 = g_2$ and $\partial_H(h_1, h_2) = 2$,
- $GH2$, if $\partial_G(g_1, g_2) = \partial_H(h_1, h_2) = 1$.

Note that vertices of type $G2$ ($H2$, respectively) do not exist if G (H, respectively) is a complete graph. Denote the set of vertices that are at equal distance from g_1 and g_2 in G by $E^G_{g_1g_2}$, and the set of vertices that are at equal distance from h_1 and h_2 in H by $E^H_{h_1h_2}$.

We first prove three lemmas that we will need later in the proof of the main theorem of this section.

Lemma 4.1. *Let (g_1, h) and (g_2, h) be arbitrary vertices of type $G2$ in $\Gamma = G\square H$. Then*

$$\left|W^{\Gamma}_{(g_1,h)(g_2,h)}\right| = |H|\left|W^G_{g_1g_2}\right| \quad \textit{and} \quad \left|W^{\Gamma}_{(g_2,h)(g_1,h)}\right| = |H|\left|W^G_{g_2g_1}\right|.$$

Proof. Let (a, x) be an arbitrary vertex of Γ. It follows from the equality (4.1) that

$$\partial_\Gamma((g_1, h),(a, x)) = \partial_G(g_1, a) + \partial_H(h, x)$$

and

$$\partial_\Gamma((g_2, h),(a, x)) = \partial_G(g_2, a) + \partial_H(h, x).$$

So (a, x) is closer to (g_1, h) than to (g_2, h) in Γ if and only if a is closer to g_1 than to g_2 in G. Since $(a, x) \in V(\Gamma)$ was an arbitrary vertex, this means that

$$\left|W^{\Gamma}_{(g_1,h)(g_2,h)}\right| = |H|\left|W^G_{g_1g_2}\right|.$$

Similarly we get that

$$\left|W^{\Gamma}_{(g_2,h)(g_1,h)}\right| = |H|\left|W^G_{g_2g_1}\right|. \qquad \square$$

Lemma 4.2. *Let (g, h_1) and (g, h_2) be arbitrary vertices of type $H2$ in $\Gamma = G\square H$. Then*

$$\left|W^{\Gamma}_{(g,h_1)(g,h_2)}\right| = |G|\left|W^H_{h_1h_2}\right| \quad \textit{and} \quad \left|W^{\Gamma}_{(g,h_2)(g,h_1)}\right| = |G|\left|W^H_{h_2h_1}\right|.$$

Proof. Similar to the proof of Lemma 4.1. $\square$

Lemma 4.3. *Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of type $GH2$ in $\Gamma = G\square H$. Then*

$$\left|W^{\Gamma}_{(g_1,h_1)(g_2,h_2)}\right| = \left|E^H_{h_1h_2}\right|\left|W^G_{g_1g_2}\right| + \left|W^H_{h_1h_2}\right|\left|W^G_{g_1g_2} \cup E^G_{g_1g_2}\right|$$

and

$$\left|W^{\Gamma}_{(g_2,h_2)(g_1,h_1)}\right| = \left|E^H_{h_1h_2}\right|\left|W^G_{g_2g_1}\right| + \left|W^H_{h_2h_1}\right|\left|W^G_{g_2g_1} \cup E^G_{g_1g_2}\right|.$$

Proof. Let (a, x) be an arbitrary vertex of Γ. It follows from the equality (4.1) that

$$\partial_\Gamma((g_1, h_1),(a, x)) = \partial_G(g_1, a) + \partial_H(h_1, x) \tag{4.2}$$

and

$$\partial_\Gamma((g_2, h_2),(a, x)) = \partial_G(g_2, a) + \partial_H(h_2, x). \tag{4.3}$$

There are three different cases according to the distance of h_1 and h_2 from x in H.

In the first case let $\partial_H(h_1, x) = \partial_H(h_2, x)$. From equalities (4.2) and (4.3) we get that

$$\partial_\Gamma((g_1, h_1),(a, x)) < \partial_\Gamma((g_2, h_2),(a, x)) \iff \partial_G(g_1, a) < \partial_G(g_2, a).$$

This is true for exactly those $(a, x) \in V(\Gamma)$, for which $a \in W^G_{g_1 g_2}$. Similarly we get that

$$\partial_\Gamma((g_2, h_2), (a, x)) < \partial_\Gamma((g_1, h_1), (a, x)) \iff \partial_G(g_2, a) < \partial_G(g_1, a).$$

And this is true for exactly those $(a, x) \in V(\Gamma)$, for which $a \in W^G_{g_2 g_1}$.

In the second case let $\partial_H(h_1, x) < \partial_H(h_2, x)$. Since h_1 and h_2 are adjacent in H, it is obvious that $\partial_H(h_2, x) = \partial_H(h_1, x) + 1$. From equalities (4.2) and (4.3) we get that

$$\partial_\Gamma((g_1, h_1), (a, x)) < \partial_\Gamma((g_2, h_2), (a, x)) \iff \partial_G(g_1, a) < \partial_G(g_2, a) + 1.$$

This is true for exactly those $(a, x) \in V(\Gamma)$, for which $a \in W^G_{g_1 g_2} \cup E^G_{g_1 g_2}$. Similarly we get that

$$\partial_\Gamma((g_2, h_2), (a, x)) < \partial_\Gamma((g_1, h_1), (a, x)) \iff \partial_G(g_2, a) + 1 < \partial_G(g_1, a).$$

But such vertices do not exist, since $\partial_G(g_1, a) \leq \partial_G(g_2, a) + 1$ by the triangle inequality.

In the third case let $\partial_H(h_2, x) < \partial_H(h_1, x)$. Similarly as above we get that (a, x) is closer to (g_2, h_2) that to (g_1, h_1) if and only if $a \in W^G_{g_2 g_1} \cup E^G_{g_1 g_2}$, and that (a, x) is never closer to (g_1, h_1) that to (g_2, h_2).

It follows from the above comments that

$$W^\Gamma_{(g_1,h_1)(g_2,h_2)} = \left(E^H_{h_1 h_2} \times W^G_{g_1 g_2}\right) \bigcup \left(W^H_{h_1 h_2} \times \left(W^G_{g_1 g_2} \cup E^G_{g_1 g_2}\right)\right)$$

and

$$W^\Gamma_{(g_2,h_2)(g_1,h_1)} = \left(E^H_{h_1 h_2} \times W^G_{g_2 g_1}\right) \bigcup \left(W^H_{h_2 h_1} \times \left(W^G_{g_2 g_1} \cup E^G_{g_1 g_2}\right)\right).$$

The result follows. □

Next theorem gives the characterization of connected 2-distance-balanced cartesian products of graphs G and H.

Theorem 4.4. *The cartesian product $\Gamma = G \square H$ is a connected 2-distance-balanced graph if and only if each of G, H is either a connected 2-distance-balanced and 1-distance-balanced graph, or a complete graph.*

Proof. We first prove that if each of G, H is either a connected 2-distance-balanced and 1-distance-balanced graph or a complete graph, then Γ is a connected 2-distance-balanced graph.

Let us assume that G and H are connected 2-distance-balanced and 1-distance-balanced graphs. The connectivity of Γ follows from the connectivity of G and H. In this case all three types of vertices at distance 2 are present in Γ.

Let (g_1, h) and (g_2, h) be arbitrary vertices of type $G2$ in Γ. Since G is, by the assumption, 2-distance-balanced and since vertices g_1, g_2 are at distance 2 in G, it follows from Lemma 4.1 that

$$\left|W^\Gamma_{(g_1,h)(g_2,h)}\right| = |H| \left|W^G_{g_1 g_2}\right| = |H| \left|W^G_{g_2 g_1}\right| = \left|W^\Gamma_{(g_2,h)(g_1,h)}\right|.$$

So for arbitrary vertices $(g_1, h), (g_2, h) \in V(\Gamma)$ of type $G2$, the number of vertices that are closer to (g_1, h) than to (g_2, h) in Γ equals the number of vertices that are closer to (g_2, h) than to (g_1, h) in Γ.

If (g, h_1) and (g, h_2) are arbitrary vertices of type $H2$ in Γ, then similarly as above (using Lemma 4.2 instead of Lemma 4.1) we find that the number of vertices that are closer to (g, h_1) than to (g, h_2) in Γ equals the number of vertices that are closer to (g, h_2) than to (g, h_1) in Γ.

Let $(g_1, h_1), (g_2, h_2) \in V(\Gamma)$ be arbitrary vertices of type $GH2$ in Γ. Since G and H are both, by the assumption, 1-distance-balanced, and since g_1, g_2 are adjacent in G and h_1, h_2 are adjacent in H, we have

$$\left|W^G_{g_1g_2}\right| = \left|W^G_{g_2g_1}\right| \quad \text{and} \quad \left|W^H_{h_1h_2}\right| = \left|W^H_{h_2h_1}\right|.$$

It follows from Lemma 4.3 that

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right|$$

for arbitrary vertices of type $GH2$ in Γ. So we proved that if G and H are both connected 2-distance-balanced and 1-distance-balanced graphs, then the cartesian product $\Gamma = G \Box H$ is a connected 2-distance-balanced graph. Note that since G and H are 1-distance-balanced graphs, it follows that the cartesian product $\Gamma = G \Box H$ is also 1-distance-balanced (see [12, Proposition 4.1]).

If one (or both) of G, H is a complete graph, then the proof that $\Gamma = G \Box H$ is a connected 2-distance balanced graph is similar to the proof above. The only diference is that we do not have to consider vertices of type $G2$ ($H2$, respectively).

Assume now that $\Gamma = G \Box H$ is a connected 2-distance-balanced graph. The connectivity of G and H follows from the connectivity of Γ. If G and H are complete graphs, then we are done. Therefore we assume that at least one of G or H is not a complete graph. First we show that in this case G and H are 2-distance-balanced graphs provided they are not complete.

Assume that G is not a complete graph. For an arbitrary $h \in V(H)$ and arbitrary $g_1, g_2 \in V(G)$ that are at distance 2 in G, consider $(g_1, h), (g_2, h) \in V(\Gamma)$. Note that $\partial_\Gamma((g_1, h), (g_2, h)) = 2$ by (4.1) and that

$$\left|W^\Gamma_{(g_1,h)(g_2,h)}\right| = |H|\left|W^G_{g_1g_2}\right| \quad \text{and} \quad \left|W^\Gamma_{(g_2,h)(g_1,h)}\right| = |H|\left|W^G_{g_2g_1}\right|$$

by Lemma 4.1. Since Γ is 2-distance-balanced, it follows that $\left|W^G_{g_1g_2}\right| = \left|W^G_{g_2g_1}\right|$, so also G is a 2-distance-balanced graph. Due to commutativity of the cartesian product, if H is not a complete graph, we can similarly show that H is a 2-distance-balanced graph.

Finally we show that G and H are also 1-distance-balanced graphs. Pick arbitrary adjacent vertices g_1, g_2 of G and arbitrary adjacent vertices h_1, h_2 of H, and note that (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ are at distance 2. Since Γ is 2-distance-balanced, it follows that

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right|.$$

From Lemma 4.3 we get that

$$\begin{aligned}\left|E^H_{h_1h_2}\right|\left(\left|W^G_{g_1g_2}\right| - \left|W^G_{g_2g_1}\right|\right) = \left|W^H_{h_2h_1}\right|&\left|W^G_{g_2g_1} \cup E^G_{g_1g_2}\right| \\ &- \left|W^H_{h_1h_2}\right|\left|W^G_{g_1g_2} \cup E^G_{g_1g_2}\right|. \quad (4.4)\end{aligned}$$

Assume that G is not a 1-distance-balanced graph. Then we could choose g_1, g_2 in such a way that $\left|W^G_{g_1 g_2}\right| > \left|W^G_{g_2 g_1}\right|$. As a consequence we also have that

$$\left|W^G_{g_1 g_2} \cup E^G_{g_1 g_2}\right| > \left|W^G_{g_2 g_1} \cup E^G_{g_1 g_2}\right|.$$

It follows from (4.4) that $\left|W^H_{h_2 h_1}\right| > \left|W^H_{h_1 h_2}\right|$. Consider now vertices $(g_1, h_2), (g_2, h_1)$, which are also at distance 2 in Γ. Similar argument as above shows that $\left|W^H_{h_2 h_1}\right| < \left|W^H_{h_1 h_2}\right|$, which is a contradiction. So G is a 1-distance balanced graph. Since the cartesian product is commutative, the proof that H is a 1-distance balanced graph is analogous to the proof for G. □

5 2-distance-balanced lexicographic product

Throughout this section let G and H be graphs and let $\Gamma = G[H]$ be the lexicographic product of G and H. It follows from the definition that the lexicographic product Γ is connected if and only if G is connected. In order to avoid trivialities we assume that $|V(G)| \geq 2$ and $|V(H)| \geq 2$. We characterize connected 2-distance-balanced lexicographic products of G and H (see Theorem 5.4).

Remark 5.1. A more general result about the characterization of connected n-distance-balanced lexicographic products of G and H as in Theorem 5.4 is stated in [7, Theorem 3.4]. But the result is not correct for at least $n = 2$. As a counterexample, let both G and H be paths on 3 vertices, which are connected graphs. Observe that G is 2-distance-balanced, and that H is locally regular (in a sense that any non-adjacent vertices in H have the same number of neighbours). By [7, Theorem 3.4], $G[H]$ is 2-distance-balanced. However, one can easily check that the $G[H]$ is not 2-distance-balanced.

Notice that there exist two different types of vertices at distance 2 in Γ. We now state these two types and we will refer to them in the proof of the Theorem 5.4. Let (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ be vertices at distance 2 in Γ. We say that this two vertices are of type

- $G2$, if $\partial_G(g_1, g_2) = 2$,
- $H2$, if $g_1 = g_2$ and $\partial_H(h_1, h_2) \geq 2$.

It follows from the definition that there exist vertices of type $G2$ in Γ if and only if G is connected non-complete graph. Similarly, there exist vertices of type $H2$ in Γ if and only if H is non-complete graph.

The following two lemmas will be used in the proof of the main theorem of this section.

Lemma 5.2. *Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of type $G2$ in $\Gamma = G[H]$. Then*

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = 1 + |N_H(h_1)| + \left(\left|W^G_{g_1 g_2}\right| - 1\right) |V(H)|$$

and

$$\left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right| = 1 + |N_H(h_2)| + \left(\left|W^G_{g_2 g_1}\right| - 1\right) |V(H)|.$$

Proof. Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of type $G2$ in Γ. Clearly, (g_1, h_1) is closer to itself than to (g_2, h_2). Now consider vertices of Γ of type (g_1, h), where $h \neq h_1$. Note that $\partial_\Gamma((g_1, h), (g_2, h_2)) = 2$, and so $(g_1, h) \in W^\Gamma_{(g_1,h_1)(g_2,h_2)}$ if and only if $h \in N_H(h_1)$. Finally, consider vertices of Γ of type (g, h), where $g \neq g_1$. Then $\partial_\Gamma((g_1, h_1), (g, h)) = \partial_G(g_1, g)$, and so $(g, h) \in W^\Gamma_{(g_1,h_1)(g_2,h_2)}$ if and only if $g \in W^G_{g_1 g_2} \setminus \{g_1\}$. It follows that

$$W^\Gamma_{(g_1,h_1)(g_2,h_2)} = \{(g_1, h_1)\} \cup (\{g_1\} \times N_H(h_1)) \cup ((W^G_{g_1 g_2} \setminus \{g_1\}) \times V(H)).$$

Similarly we get

$$W^\Gamma_{(g_2,h_2)(g_1,h_1)} = \{(g_2, h_2)\} \cup (\{g_2\} \times N_H(h_2)) \cup ((W^G_{g_2 g_1} \setminus \{g_2\}) \times V(H)).$$

The result follows. □

Lemma 5.3. *Let (g, h_1) and (g, h_2) be arbitrary vertices of type $H2$ in $\Gamma = G[H]$. Then*

$$\left|W^\Gamma_{(g,h_1)(g,h_2)}\right| = 1 + |N_H(h_1)| - |N_H(h_1) \cap N_H(h_2)|$$

and

$$\left|W^\Gamma_{(g,h_2)(g,h_1)}\right| = 1 + |N_H(h_2)| - |N_H(h_1) \cap N_H(h_2)|.$$

Proof. Let (g, h_1) and (g, h_2) be arbitrary vertices of type $H2$ in Γ, and let (g', h') be an arbitrary vertex of Γ. Note that if $g \neq g'$ then $\partial_\Gamma((g, h_1), (g', h')) = \partial_\Gamma((g, h_2), (g', h'))$. Assume therefore that $g' = g$. But it is clear that in this case $(g, h') \in W^\Gamma_{(g,h_1)(g,h_2)}$ if and only if $\partial_H(h_1, h') \leq 1 < \partial_H(h_2, h')$. It follows that

$$W^\Gamma_{(g,h_1)(g,h_2)} = \{(g, h_1)\} \cup (\{g\} \times (N_H(h_1) \setminus (N_H(h_1) \cap N_H(h_2)))).$$

Similarly we get

$$W^\Gamma_{(g,h_2)(g,h_1)} = \{(g, h_2)\} \cup (\{g\} \times (N_H(h_2) \setminus (N_H(h_1) \cap N_H(h_2)))).$$

The result follows. □

Next theorem gives the characterization of connected 2-distance-balanced lexicographic products of graphs G and H.

Theorem 5.4. *The lexicographic product $\Gamma = G[H]$ is a connected 2-distance-balanced graph if and only if one of the following (i), (ii) holds:*

(i) G is a connected 2-distance-balanced graph and H is a regular graph.

(ii) G is a complete graph, H is not a complete graph, and each connected component of the complement of H induces a regular subgraph of the complement of H.

Proof. We first prove that if one of (i), (ii) holds, then Γ is a connected 2-distance-balanced graph. The connectivity of Γ follows from the connectivity of G.

Assume that (i) holds. Take arbitrary (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ of type $G2$. Since G is a 2-distance-balanced graph and H is a regular graph, we have that $\left|W^G_{g_1 g_2}\right| = \left|W^G_{g_2 g_1}\right|$ and $|N_H(h_1)| = |N_H(h_2)|$. It follows from Lemma 5.2 that

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right|$$

for arbitrary vertices of type $G2$ in Γ.

Take now arbitrary (g,h_1), $(g,h_2) \in V(\Gamma)$ of type $H2$. Since, by the assumption, H is a regular graph, we have that $|N_H(h_1)| = |N_H(h_2)|$. It follows from Lemma 5.3 that

$$\left|W^{\Gamma}_{(g,h_1)(g,h_2)}\right| = \left|W^{\Gamma}_{(g,h_2)(g,h_1)}\right|$$

for arbitrary vertices of type $H2$ in Γ. So, if (i) holds then Γ is a connected 2-distance-balanced graph.

Assume that (ii) holds. Then G is a complete graph and H is not a complete graph, so we only have vertices of type $H2$ in Γ. Let us denote the complement of H by $\overline{H}$. Let (g,h_1), $(g,h_2) \in V(\Gamma)$ be arbitrary vertices of type $H2$. Note that this implies that h_1, h_2 are not adjacent in H, and so h_1, h_2 are adjacent in $\overline{H}$. As a consequence, h_1, h_2 are contained in the same connected component of $\overline{H}$. It follows that $|N_{\overline{H}}(h_1)| = |N_{\overline{H}}(h_2)|$, and consequently also $|N_H(h_1)| = |N_H(h_2)|$. It follows from Lemma 5.3 that

$$\left|W^{\Gamma}_{(g,h_1)(g,h_2)}\right| = \left|W^{\Gamma}_{(g,h_2)(g,h_1)}\right|.$$

So, if (ii) holds then Γ is a connected 2-distance-balanced graph.

Assume now that the lexicographic product $\Gamma = G[H]$ is a connected 2-distance-balanced graph. The connectivity of G follows from the connectivity of Γ. In what follows we first treat the case where G is not a complete graph, and then the case when G is a complete graph.

Suppose that G is not a complete graph. Take arbitrary $g_1, g_2 \in V(G)$ at distance 2 in G. Then $(g_1,h), (g_2,h) \in V(\Gamma)$ are of type $G2$ in Γ for an arbitrary $h \in V(H)$. Since Γ is, by the assumption, a 2-distance-balanced graph, it follows from Lemma 5.2 that $\left|W^G_{g_1g_2}\right| = \left|W^G_{g_2g_1}\right|$ for arbitrary vertices at distance 2 in G. So, G is a connected 2-distance-balanced graph. For arbitrary $h_1, h_2 \in V(H)$ and arbitrary $g_1, g_2 \in V(G)$ at distance 2 in G, consider $(g_1,h_1), (g_2,h_2) \in V(\Gamma)$. These two vertices are of type $G2$ in Γ. Since Γ is, by the assumption, a 2-distance-balanced graph and we already know that G is also 2-distance-balanced graph, it follows from Lemma 5.2 that $|N_H(h_1)| = |N_H(h_2)|$ for arbitrary two vertices in H. So, H is a regular graph and (i) holds.

From now on let G be a complete graph. Since Γ is not a complete graph, it follows that also H is not a complete graph. This means that all vertices at distance 2 in Γ are of type $H2$. We want to show that in this case each connected component of the complement of H induces a regular subgraph of the complement of H.

Let $h_1, h_2 \in V(H)$ be arbitrary vertices at distance greater or equal than 2 in H (that is, vertices h_1, h_2 are not adjacent in H). Observe that $(g,h_1), (g,h_2) \in V(\Gamma)$ are of type $H2$ for an arbitrary $g \in V(G)$. From Lemma 5.3 we get that $|N_H(h_1)| = |N_H(h_2)|$, and consequently also $|N_{\overline{H}}(h_1)| = |N_{\overline{H}}(h_2)|$. This shows that any adjacent vertices of $\overline{H}$ have the same valency in $\overline{H}$, and therefore each connected component of $\overline{H}$ induces a regular subgraph of $\overline{H}$. $\square$

We finish our paper with a suggestion for further research. A fullerene is a cubic planar graph having all faces 5- or 6-cycles. Examples include the dodecahedron and generalized Petersen graph $\mathrm{GP}(12, 2)$. Dodecahedron is distance-regular, and so it is n-distance-balanced for every $1 \leq n \leq 5$ (recall that the diameter of dodecahedron is 5). On the other hand, the diameter of $\mathrm{GP}(12, 2)$ is also 5, but $\mathrm{GP}(12, 2)$ is n-distance-balanced only for $n = 5$, see [17]. Therefore, it would be interesting to know, which fullerenes are n-distance-balanced at least for some values of n (for example, for $n \in \{1, 2, D\}$, where D is the diameter of a fullerene in question). For more on fullerenes see [2, 5, 18].

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The arc-types of Cayley graphs*

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Abstract

Let X be a finite vertex-transitive graph of valency d, and let A be the full automorphism group of X. Then the *arc-type* of X is defined in terms of the sizes of the orbits of the action of the stabiliser A_v of a given vertex v on the set of arcs incident with v. Specifically, the arc-type is the partition of d as the sum $n_1 + n_2 + \cdots + n_t + (m_1 + m_1) + (m_2 + m_2) + \cdots + (m_s + m_s)$, where $n_1, n_2, \ldots, n_t$ are the sizes of the self-paired orbits, and $m_1, m_1, m_2, m_2, \ldots, m_s, m_s$ are the sizes of the non-self-paired orbits, in descending order.

In a recent paper, it was shown by Conder, Pisanski and Žitnik that with the exception of the partitions $1 + 1$ and $(1 + 1)$ for valency 2, every such partition occurs as the arc-type of some vertex-transitive graph. In this paper, we extend this to show that in fact every partition other than 1, $1 + 1$ and $(1 + 1)$ occurs as the arc-type of infinitely many connected finite Cayley graphs with the given valency d. As a consequence, this also shows that for every $d > 2$, there are infinitely many finite zero-symmetric graphs (or GRRs) of valency d.

Keywords: Symmetry type, vertex-transitive graph, arc-transitive graph, Cayley graph, zero-symmetric graph, Cartesian product, covering graph.

Math. Subj. Class.: 05E18, 20B25, 05C75, 05C76

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1 Introduction

Vertex-transitive graphs hold a significant place in mathematics, and within this, a major role is played by *Cayley graphs*, which represent groups in a very natural way. A Cayley graph can be defined as any graph that admits some group of automorphisms which acts regularly (sharply-transitively) on the vertices of the graph. Equivalently, a Cayley graph can be constructed from the regular permutation representation of a group G, with vertices taken as the elements of G and edges indicating the effect of a subset $S \subseteq G$ (by left multiplication). The set $S \cup S^{-1}$ consists of the elements of G that take the identity vertex to one of its neighbours.

It often happens that the automorphism group of a connected finite Cayley graph itself acts regularly on vertices. Any Cayley graph with this property is called a *zero-symmetric* graph, or a *graphical regular representation* of the group G, or briefly, a *GRR*. But of course the automorphism group of a Cayley graph X may be much larger than the vertex-regular subgroup G, and can sometimes even be the full symmetric group on the vertex-set (when the graph is null or complete). Intermediate cases, with $\mathrm{Aut}(X)$ larger than G but smaller than $\mathrm{Sym}(G)$, as well as other kinds of vertex-transitive graphs, fall into a number of different and interesting classes of graphs, including those that are arc-transitive (or *symmetric*), and those that are half-arc-transitive (which are vertex-transitive and edge-transitive but not arc-transitive).

A means of classifying vertex-transitive graphs was given in a recent paper by Conder, Pisanski and Žitnik [3], using what is known as the *arc-type* of the graph. This can be defined as follows.

Let X be a d-valent vertex-transitive graph, with automorphism group A, let A_v be the stabiliser in A of any vertex v of X, and consider the orbits of A_v on the set of arcs (v, w) with initial vertex v. The A_v-orbit of any arc (v, w) can be 'paired' with the A_v-orbit of the arc (v, w') whenever (v, w') lies in the same orbit of A as the reverse arc (w, v), and if those two A-obits are the same, then we say the A_v-orbit of (v, w) is 'self-paired'. Then the arc-type of X is the partition Π of its valency d as the sum

$$\Pi \;=\; n_1 + n_2 + \cdots + n_t + (m_1 + m_1) + (m_2 + m_2) + \cdots + (m_s + m_s) \qquad (\dagger)$$

where $n_1, n_2, \ldots, n_t$ are the sizes of the self-paired orbits of A_v on arcs with initial vertex v, and $m_1, m_1, m_2, m_2, \ldots, m_s, m_s$ are the sizes of the non-self-paired orbits, in descending order. Similarly, the *edge-type* of X is the partition of d as the sum of the sizes of the orbits of A_v on edges incident with v, and can be found by simply replacing each bracketed term $(m_j + m_j)$ by $2m_j$, for $1 \le j \le s$.

For example, if X is arc-transitive, then its arc-type is simply d, while if X is half-arc-transitive, then its valency d is even and its arc-type is $(\frac{d}{2} + \frac{d}{2})$, and X is a GRR if and only if all the terms n_i and m_j in its arc-type are 1.

The authors of [3] also answered the natural question of which arc-types occur for a given valency d. Every vertex-transitive graph of valency 2 is a union of cycles and is therefore arc-transitive, with arc-type 2. Hence in particular, the partitions $1 + 1$ and $(1 + 1)$ of 2 do not occur as the arc-type of a vertex-transitive graph. It was shown in [3] that these are the only exceptional cases. Using a construction that takes Cartesian products of pairwise 'relatively prime' vertex-transitive graphs, Conder, Pisanski and Žitnik proved that in all other cases, every partition of d as given in ($\dagger$) occurs as the arc-type of some vertex-transitive graph X of valency d.

In this paper, we prove a much stronger theorem, namely that every such partition other than 1, $1+1$ and $(1+1)$ occurs as the arc-type of infinitely many connected finite Cayley graphs. (This answers a question posed by Joy Morris at the 2015 PhD Summer School in Discrete Mathematics, in Rogla, Slovenia.) As corollaries, we find that every standard partition of a positive integer d is realisable as the edge-type of infinitely many connected finite Cayley graphs of valency d, except for 1 and $1+1$ (when $d \leq 2$), and that for every $d > 2$, there are infinitely many finite zero-symmetric graphs of valency d.

To prove our main theorem, we adopt the same approach as taken in [3], but show there are infinitely many Cayley graphs that can be used in the construction as building blocks with the required basic type. In particular, we show that the half-arc-transitive Bouwer graphs $B(m,k,n)$ and the 'thickened covers' used in [3] are Cayley graphs, and we construct some new families of Cayley graphs with various arc-types as well.

We begin by setting notation and giving some further background in Section 2. Then in Section 3 we briefly summarise what has to be done to prove our main theorem, which we proceed to do in Section 4. We complete the paper with the consequence for zero-symmetric graphs in Section 5.

2 Preliminaries and further background

2.1 Notation

All the graphs we consider in this paper are finite, simple, undirected and non-trivial (in the sense of containing at least one edge). Given a graph X, we denote by $V(X)$, $E(X)$ and $A(X)$ the set of vertices, the set of edges, and the set of arcs of X, respectively. We denote an edge with vertices u and v by $\{u,v\}$, and an arc from u to v by (u,v).

The automorphism group of X is denoted by $\mathrm{Aut}(X)$. Note that the action of $\mathrm{Aut}(X)$ on the vertex-set $V(X)$ also induces an action of $\mathrm{Aut}(X)$ on the edge-set $E(X)$ and one on the arc-set $A(X)$. If the action of $\mathrm{Aut}(X)$ is transitive on the vertex-set, edge-set, or arc-set, then we say that X is *vertex-transitive*, *edge-transitive* or *arc-transitive*, and sometimes abbreviate this to 'VT', 'ET' or 'AT', respectively.

Obviously, vertex-transitive graphs are always regular. Moreover, because a disconnected vertex-transitive graph consists of pairwise isomorphic connected components, we may restrict our attention here to connected graphs. An arc-transitive graph is often also called *symmetric*. A graph is called *half-arc-transitive* if it is vertex-transitive and edge-transitive, but not arc-transitive. The valency of every half-arc-transitive graph is necessarily even; see [11, p. 59].

Now let G be a group, and let S be a subset of G that is inverse-closed and does not contain the identity element. Then the *Cayley graph* $\mathrm{Cay}(G,S)$ is the graph with vertex-set G, and with vertices u and v being adjacent if and only if $vu^{-1} \in S$ (or equivalently, $v = xu$ for some $x \in S$). Since we require S to be inverse-closed, this Cayley graph is undirected, and since S does not contain the identity, the graph has no loops. Also $\mathrm{Cay}(G,S)$ is regular, with valency $|S|$, and is connected if and only if S generates G. Furthermore, it is easy to see that G acts as a group of automorphisms of $\mathrm{Cay}(G,S)$ by right multiplication, and this action is transitive on vertices, with trivial stabiliser, and hence sharply-transitive (or regular). In particular, $\mathrm{Cay}(G,S)$ is vertex-transitive.

More generally, a graph X is a Cayley graph for the group G if and only if G acts regularly on $V(X)$ as a group of automorphisms of X. This is very well known — see [10] for example.

2.2 Cartesian products and (relatively) prime graphs

Given a pair of graphs X and Y (which might or might not be distinct), the Cartesian product $X \,\square\, Y$ is a graph with vertex set $V(X) \times V(Y)$, such that two vertices (x, y) and (u, v) are adjacent in $X \,\square\, Y$ if and only if $x = u$ and y is adjacent with v in Y, or $y = v$ and x is adjacent with u in X. This definition can be extended to the Cartesian product $X_1 \,\square \cdots \square\, X_k$ of a larger number of graphs $X_1, \dots, X_k$, which are then called the *factors*.

A graph X is called *prime* (with respect to the Cartesian product) if it is not isomorphic to the Cartesian product of a pair of smaller, non-trivial graphs. Every connected graph can be decomposed as a Cartesian product of prime graphs, in a way that is unique up to reordering and isomorphism of the factors; see [6, Theorem 4.9] for a proof. Then two graphs can be said to be *relatively prime* (with respect to the Cartesian product) if there is no non-trivial graph that is a factor of both. Note that two prime graphs are relatively prime unless they are isomorphic.

For the construction in [3] and here, we need a number of other properties of the Cartesian product, and some ways in which we can tell if a given graph is prime with respect to the Cartesian product. We summarise these as follows:

Proposition 2.1.

(a) *The Cartesian product operation $\square$ is associative and commutative.*

(b) *A Cartesian product graph is connected if and only if all its factors are connected.*

(c) *If $X_1, \dots, X_k$ are regular graphs with valencies $d_1, \dots, d_k$, then their Cartesian product $X_1 \,\square \cdots \square\, X_k$ is also regular, with valency $d_1 + \cdots + d_k$.*

(d) *The Cartesian product of Cayley graphs is a Cayley graph.*

(e) *If $X_1, \dots, X_k$ are connected graphs that are pairwise relatively prime, then $\mathrm{Aut}(X) \cong \mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_k)$.*

(f) *A Cartesian product of connected graphs is vertex-transitive if and only if all its factors are vertex-transitive.*

(g) *If $X_1, \dots, X_k$ are non-trivial connected vertex-transitive graphs with arc-types $\tau_1, \dots, \tau_k$, and $X_1, \dots, X_k$ are pairwise relatively prime, then the arc-type of their Cartesian product $X = X_1 \,\square \cdots \square\, X_k$ is $\tau_1 + \cdots + \tau_k$.*

Proof. Parts (a) to (c) are easy, and part (d) follows by induction from the fact that

$$\mathrm{Cay}(G, S) \,\square\, \mathrm{Cay}(H, T) = \mathrm{Cay}(G \times H,\, (S \times \{1_H\}) \cup (\{1_G\} \times T)).$$

Proofs of parts (e) and (f) can be found in [6], and part (g) was proved in [3]. □

Proposition 2.2. *Let X be a Cartesian product of non-trivial connected graphs. Then:*

(a) *Every edge of X lies in some 4-cycle in X.*

(b) *All the edges in any cycle of length 3 in X belong to the same factor of X.*

(c) *If (x, y, z, w) is any 4-cycle in X, then the edges $\{x, y\}$ and $\{z, w\}$ belong to the same factor of X, as do the edges $\{y, z\}$ and $\{x, w\}$.*

(d) [The square property] *If two edges are incident in X but do not belong to the same factor of X, then there exists a unique 4-cycle in X that contains both of these edges, and this 4-cycle has no diagonals.*

Proof. Part (a) is easy, and others were proved in [7], for example. □

Corollary 2.3. *Let X be a connected graph. If some edge of X is not contained in any 4-cycle (and in particular, if X has no 4-cycles), then X is prime.*

2.3 Thickened covers

Let X be any simple graph, F any subset of the edge-set of X, and m any positive integer. Then the authors of [3] defined the *thickened m-cover* of X over F as the graph $X(F, m)$ that has vertex-set $V(X) \times \mathbb{Z}_m$, and edges of two types:

(a) an edge from (u, i) to (v, i), for every $i \in \mathbb{Z}_m$ and every $\{u, v\} \in E(X) \setminus F$,

(b) an edge from (u, i) to (v, j), for every $(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_m$ and every $\{u, v\} \in F$.

One can think of this graph as being obtained from X by replacing each vertex of X by m vertices, and each edge by the complete bipartite graph $K_{m,m}$ whenever the edge lies in F, or by mK_2 (a set of m 'parallel' edges) whenever the edge does not lie in F.

For example, the thickened 2-cover of the cycle graph C_6 over one of its 1-factors is shown in Figure 1.

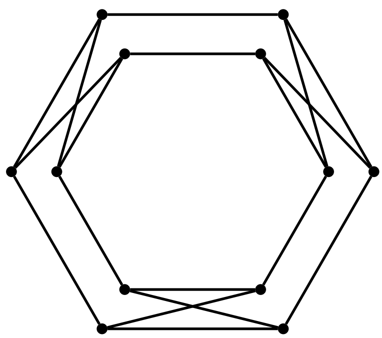

Figure 1: A thickened 2-cover of C_6 (over a 1-factor).

It was shown in [3] that if X is a vertex-transitive graph, and F is a union of orbits of $\mathrm{Aut}(X)$ on edges of X, then $X(F, m)$ is vertex-transitive for every $m \geq 2$. We can take this further, as follows:

Proposition 2.4. *If $X = \mathrm{Cay}(G, S)$ is a Cayley graph, and F is an orbit of G on edges of X, then the thickened cover $Y = X(F, m)$ is a Cayley graph for $G \times \mathbb{Z}_m$.*

Proof. We show that Y is exactly the same as the Cayley graph $\mathrm{Cay}(G \times \mathbb{Z}_m, W)$, where multiplication in the group $G \times \mathbb{Z}_m$ is given by $(g, i)(h, j) = (gh, i + j)$ for all $g, h \in G$ and $i, j \in \mathbb{Z}_m$, and W is the union of the two sets

$$W_1 = \{(s, 0) : s \in S,\ \{1_G, s\} \notin F\} \text{ and } W_2 = \{(t, i) : \{1_G, t\} \in F,\ i \in \mathbb{Z}_m\}.$$

Take any edge of Y of the first kind, say from (u, i) to (v, i) where $\{u, v\} \in E(X) \setminus F$. Then $v = su$ for some $s \in S$, and it follows that $(v, i) = (su, i) = (s, 0)(u, i)$, with $\{1_G, s\} = \{u, su\}u^{-1} = \{u, v\}u^{-1} \notin F$. Conversely, if $s \in S$ and $\{1_G, s\} \notin F$ then $\{u, su\} = \{1_G, s\}u \notin F$, and so $(s, 0)(u, i) = (su, i)$ is adjacent to (u, i), for all u and i.

Similarly, for any edge of the second kind, from (u,i) to (v,j) with $\{u,v\} \in F$, we have $v = tu$ for some $t \in S$ and so $(v,j) = (t, j-i)(u,i)$ with $\{1_G, t\} = \{u,v\}u^{-1} \in F$, and conversely, if $\{1_G, t\} \in F$ (where $t \in S$), then $\{u, tu\} = \{1_G, t\}u \in F$, and therefore $(t, j-i)(u,i) = (tu, j)$ is adjacent to (u,i), for all u, i and j. □

Also we need some other information about thickened covers, taken from [3]. The *fibre* over a vertex u of X is the set $\{(u,i) : i \in \mathbb{Z}_m\}$ of vertices of $X(F,m)$, and any element of this set is said to *project onto* u. Similarly the fibre over an edge $\{u,v\}$ of X is the set $\{\{(u,i),(v,i)\} : i \in \mathbb{Z}_m\}$ of edges of $X(F,m)$ when $\{u,v\} \in E(X) \setminus F$, or the set $\{\{(u,i),(v,j)\} : i,j \in \mathbb{Z}_m\}$ when $\{u,v\} \in F$, and any element of this set is said to project onto $\{u,v\}$. The fibre over an arc is defined similarly.

Proposition 2.5. *Let X be a vertex-transitive graph, and let F be a union of edge-orbits of X, with the property that every edge in F joins vertices from two different components of $X \setminus F$. Then for every two arcs (x,y) and (u,v) from the same arc-orbit of X, any two arcs of $X(F,m)$ that project onto (x,y) and (u,v) respectively lie in the same arc-orbit of $X(F,m)$, for all $m \geq 2$.*

Proof. See [3, Theorem 7.6]. □

2.4 Bouwer graphs

The first known infinite family of half-arc-transitive graphs of arbitrary even valency greater than 2 was constructed by Bouwer [2] in 1970. These graphs were a sub-family of a wider class of graphs, which we now denote by $B(k,m,n)$, defined as follows.

Let m and n be any integers such that $2^m \equiv 1 \bmod n$, with $m \geq 2$ and $n \geq 3$, and also let k be any integer such that $k \geq 2$. Then the vertices of $B(k,m,n)$ may be taken as the k-tuples $(a,\mathbf{b}) = (a, b_2, b_3, \ldots, b_k)$ with $a \in \mathbb{Z}_m$ and $b_j \in \mathbb{Z}_n$ for $2 \leq j \leq k$, with any two such vertices being adjacent if and only if they can be written as $(a,\mathbf{b})$ and $(a+1,\mathbf{c})$ where either $\mathbf{c} = \mathbf{b}$, or $\mathbf{c} = (c_2, c_3, \ldots, c_k)$ differs from $\mathbf{b} = (b_2, b_3, \ldots, b_k)$ in just one position, say position j, where $c_j = b_j + 2^a$.

Bouwer himself proved in [2] that every such graph is connected, edge-transitive and vertex-transitive, with valency $2k$. He also proved that the graphs $B(k,6,9)$ are half-arc-transitive, and his theorem was extended recently by Conder and Žitnik [4], who proved that $B(k,m,n)$ is arc-transitive only when $n = 3$, or $(k,n) = (2,5)$, or $(k,m,n) = (2,3,7)$ or $(2,6,7)$ or $(2,6,21)$. In particular, it follows that $B(k,m,n)$ is half-arc-transitive whenever $m > 6$ and $n > 5$. Moreover, as shown in [4], if $m > 6$ and $n > 7$, then $B(k,m,n)$ has girth 6, and hence in that case, $B(k,m,n)$ is prime.

These prime graphs gave the infinite family of half-arc-transitive graphs with arc-type $(k+k)$, for each $k \geq 2$, used in Lemma 8.2 of [3]. We can take this further, by proving the following (which a referee has also pointed out was proved very recently by Ramos Rivera and Šparl in [9]):

Proposition 2.6. *Every Bouwer graph $B(k,m,n)$ is a Cayley graph.*

Proof. First note that n is odd, since $2^m \equiv 1 \bmod n$. Now let G be the semi-direct product $\mathbb{Z}_m \ltimes \mathbb{Z}_n^{k-1}$, where a generator of the complement $\mathbb{Z}_m$ acts by conjugation from the right on the kernel $\mathbb{Z}_n^{k-1}$ in the same way as component-wise multiplication by 2. Also let R be the set of all elements of G of the form $(1,\mathbf{b})$, where $\mathbf{b}$ is either the zero vector $\mathbf{0}$ in $\mathbb{Z}_n^{k-1}$, or one of the elementary basis vectors $\mathbf{e}_j$ (with all its entries being 0 except for a 1 in position j). The k elements of R are non-involutions, whose inverses are the elements of the form $(-1, \mathbf{d})$ where $\mathbf{d} = \mathbf{0}$ or $-2^{-1}\mathbf{e}_i$ for some i. It follows that the $2k$-valent Cayley graph $\mathrm{Cay}(G, R \cup R^{-1})$ is isomorphic to the Bouwer graph $B(k, m, n)$, for if $(a, \mathbf{b}) = (a, b_2, b_3, \ldots, b_k)$ is any vertex, then $(1, \mathbf{0})(a, \mathbf{b}) = (a{+}1, \mathbf{b})$, and $(1, \mathbf{e}_i)(a, \mathbf{b}) = (a + 1, \mathbf{b} + 2^a\mathbf{e}_i)$ for all i. $\square$

3 Main theorem and overview of the proof

As indicated in the Introduction, our main theorem and its first immediate corollary are as follows:

Theorem 3.1. *For any positive integer d, let Π be any partition of d as given in* $(\dagger)$. *Then Π occurs as the arc-type of infinitely many connected finite Cayley graphs of valency d, except when Π is one of the partitions* 1, $1+1$ *and* $(1+1)$ *in the cases with $d \le 2$.*

Corollary 3.2. *With the exception of* 1 *and* $1+1$ (*in the cases with $d \le 2$*), *every standard partition of a positive integer d is realisable as the edge-type of infinitely many connected finite Cayley graphs of valency d.*

Corollary 3.2 follows easily from Theorem 3.1. To prove Theorem 3.1, we use much of the proof of the theorem in [3] showing that every such partition is the arc-type of at least one vertex-transitive graph of valency d. In that proof, the given partition Π was written as a sum of 'basic' partitions, each having one of a number of forms, and then a VT graph with arc-type Π was constructed as a Cartesian product of pairwise relatively prime graphs with arc-types of the associated forms.

This required a good supply of prime vertex-transitive graphs with particular arc-types as 'building blocks', and the following were sufficient.

(a) Arc-type m: infinitely many prime connected VT graphs, for each integer $m \ge 2$;

(b) Arc-type $(m+m)$: infinitely many prime connected VT graphs, for each $m \ge 2$;

(c) Arc-type $m+1$: infinitely many prime connected VT graphs, for each $m \ge 2$;

(d) Arc-type $1+(1+1)$: at least two prime connected VT graphs;

(e) Arc-type $m+(1+1)$: at least one prime connected VT graph, for each $m \ge 2$;

(f) Arc-type $1+(m+m)$: at least one prime connected VT graph, for each $m \ge 2$;

(g) Arc-type $(1+1)+(1+1)$: infinitely many prime connected VT graphs;

(h) Arc-type $(m{+}m){+}(1{+}1)$: at least one prime connected VT graph, for each $m \ge 2$;

(i) Arc-type $1+1+1$: infinitely many prime connected VT graphs;

(j) Arc-type $1+1+(1+1)$: at least one prime connected VT graph;

(k) Arc-type $1+1+1+1$: at least one prime connected VT graph;

(l) Arc-type $(1+1)+(1+1)+(1+1)$: at least one prime connected VT graph.

Now to extend this to a proof of our theorem, we need *infinitely many connected finite Cayley graphs* of each of the basic forms listed in cases (a) to (l) above.

Such infinite families were provided explicitly for cases (g) and (i) in Lemmas 8.6 and 8.8 of [3]. Also in cases (d), (e) and (g), a single vertex-transitive graph was produced for each m in Lemmas 8.4, 8.5 and 8.7 of [3], as a thickened cover of a particular Cayley graph over an edge-orbit. These are Cayley graphs, by Proposition 2.4, but we have to produce infinitely many of them, for each $m \geq 2$.

Hence it remains for us to find infinitely many connected finite Cayley graphs in the cases (a)–(f), (h), and (j)–(l) above. We do that in the next section. Specifically, we construct new families of Cayley graphs for cases (a), (d) and (j)–(l), we use the Bouwer graphs for case (b), we show that the thickened covers used in [3] for case (c) are Cayley graphs, and we show that thickened covers of the graphs in cases (d) and (g) provide infinitely many Cayley graphs for cases (e), (f) and (h).

4 Proof of main theorem

As noted earlier, all we need to do to prove Theorem 3.1 is show that there exist infinitely many prime connected finite Cayley graphs with each of the arc-types in the cases listed in the previous section, and then the rest follows by the same argument as in [3, Section 9]. We do this case-by-case below. For completeness, we give a brief description of the Cayley graphs in the cases that do not require any further analysis, and we give more detailed arguments for the rest.

Case (a): Arc-type m, for all $m \geq 2$.

For $m = 2$, we can take the family of all cycle graphs C_n with $n \geq 5$. These graphs have arc-type 2, and since they contain no 4-cycles, by part (a) of Proposition 2.2 they are all prime (with respect to the Cartesian product).

For $m \geq 3$, we construct an infinite family of arc-transitive prime connected finite Cayley graphs of valency m using the same groups as for this case in [3, Lemma 8.1].

We know by Macbeath's theorem [8] that for every prime $p > m$, the simple group $G = \mathrm{PSL}(2, p)$ is generated by elements x and y such that $x^2 = y^m = (xy)^{m+4} = 1$. Now take S to be the set $\{x, y^{-1}xy, y^{-2}xy^2, \ldots, y^{-(m-1)}xy^{m-1}\}$ of all conjugates of x by powers of y, and let $X = \mathrm{Cay}(G, S)$.

The elements of S are distinct involutions (since G has trivial centre), and so X has valency $|S| = m$. Moreover, the subgroup generated by S is normal in $\langle x, y\rangle = G$, because $x \in S$ and conjugation by y permutes the elements of S among themselves. Hence S generates G, and therefore X is connected. But also conjugation by y induces an automorphism of X that fixes the identity vertex and cyclically permutes its m neighbours among themselves, and so X is arc-transitive. Hence X has arc-type m.

Finally, X is prime, for if it were the Cartesian product of two relatively prime graphs Y and Z, then its arc-type m would be the sum of the arc-types of Y and Z, and if it were the kth Cartesian power of some prime graph Y, then we would find that $|V(Y)|^k = |V(X)| = |G| = |\,\mathrm{PSL}(2, p)| = p(p^2 - 1)/2$, which can occur only if $k = 1$.

Case (b): Arc-type $(m + m)$, for all $m \geq 2$.

If n and r are any integers such that $2^r \equiv 1 \bmod n$, with $n > 7$ and $r > 6$, then by Lemma 8.2 of [3], the Bouwer graph $B(m, r, n)$ is a prime half-arc-transitive graph with arc-type $(m + m)$, for every $m \geq 2$. Also by Proposition 2.6 above, this graph is a Cayley graph. Hence in particular, the Bouwer graph $B(m, r, n)$ is a prime Cayley graph with arc-type $(m + m)$, whenever $m \geq 2$, $r > 6$ and $n > 7$.

Case (c): Arc-type $m + 1$, for all $m \geq 2$.

By Theorem 7.5 of [3], for every integer $m \geq 2$ and every integer $n \geq 3$, the thickened m-cover of the n-cycle C_{2n} over one of its 1-factors is a prime VT graph with arc-type

$m+1$ (that is, with two self-paired arc orbits of lengths m and 1). This thickened cover is also a Cayley graph, as we show below.

Let a and b be canonical generators for the dihedral group D_{mn} of order $2mn$, satisfying $a^{mn} = b^2 = (ab)^2 = 1$, and define $Y_{mn} = \mathrm{Cay}(D_{mn}, S)$ where S is the set $\{b, ba^n, ba^{2n}, \ldots, ba^{(m-1)n}, ba\}$, consisting of $m+1$ involutions. Now let η be the natural epimorphism from D_{mn} to $D_n \cong D_{mn}/C_m$ with kernel $\langle a^n\rangle \cong C_m$, and let $\overline{S}$ and $\overline{x}$ be the images of S and any $x \in D_{mn}$ under η. Then η induces a graph homomorphism from Y_{mn} to $\mathrm{Cay}(D_n, \overline{S}) = \mathrm{Cay}(D_n, \{\overline{b}, \overline{ba}\})$, which is clearly a cycle of length $|D_n| = 2n$.

Moreover, the pre-image of an edge of the form $\{\overline{x}, \overline{bx}\}$ is a complete bipartite subgraph of order $2m$ with m^2 edges $\{xz, bxw\}$ for $z, w \in \langle a^n\rangle \cong C_m$, while the pre-image of an edge of the form $\{\overline{x}, \overline{bax}\}$ is a subgraph of order $2m$ with m parallel edges $\{xz, baz\}$ for $z \in \langle a^n\rangle \cong C_m$.

Hence Y_{mn} is isomorphic to the m-thickened cover of C_{2n} used in Theorem 7.5 of [3], and so is a prime Cayley graph with arc-type $m+1$, for all $m \geq 2$ and all $n \geq 3$.

Case (d): Arc-type $1 + (1 + 1)$.

Let p be any prime such that $p \equiv 1 \bmod 4$, with $p > 5$, and let k be any integer such that $k^2 \equiv -1 \bmod p$. Now take G to be the semi-direct product $C_p \rtimes_k C_4$, which is generated by elements a and b such that $a^p = b^4 = 1$ and $b^{-1}ab = a^k$. Note that conjugation by b^2 inverts a, while $bab^{-1} = a^{-k}$.

Now let $X = \mathrm{Cay}(G, \{b, b^{-1}, ab^2\})$. This graph is 3-valent, since ab^2 is an involution, and connected, since $\langle b, ab^2\rangle = G$. It is also non-bipartite, because if it were bipartite, then its parts would be preserved by the only subgroup of index 2 in G, namely the subgroup generated by a and b^2, but that cannot happen since there is an edge from 1 to ab^2. We will show that X is prime and has arc-type $1 + (1 + 1)$, for all p and k.

First, note that the arcs $(1, b)$, $(1, b^{-1})$ and $(1, ab^2)$ can each be extended to a path of length 2 in two ways, namely to $(1, b, b^2)$, $(1, b, ab^3)$, $(1, b^{-1}, b^2)$, $(1, b^{-1}, ab)$, $(1, ab^2, a^{-k}b^3)$ and $(1, ab^2, a^kb)$. It follows that the edge $\{1, ab^2\}$ lies in no 4-cycle, and in particular, that X is prime. Moreover, the edges $\{1, b\}$ and $\{1, b^{-1}\}$ lie in a single 4-cycle up to reversal, namely $(1, b, b^2, b^{-1})$, and so $\{1, ab^2\}$ lies in a different edge orbit from $\{1, b\}$ and $\{1, b^{-1}\}$. On the other hand, the edges $\{1, b\}$ and $\{1, b^{-1}\}$ lie in the same orbit of $\mathrm{Aut}(X)$, since right multiplication by b^{-1} takes the former to the latter. Hence the edge-type of X is $2 + 1$, and the arc-type of X must be $1 + (1 + 1)$ or $2 + 1$.

Next, we consider the stabiliser A_1 in $A = \mathrm{Aut}(X)$ of the identity vertex 1. By the above observations, A_1 fixes the vertex ab^2, as well as b^2 (the only other common neighbour of b and b^{-1}) and a^{-1} (the vertex opposite ab^2 in the 4-cycle $(ab^2, a^{-k}b^{-1}, a^{-1}, a^kb)$ containing ab^2). By induction and connectedness, A_1 fixes every vertex in the orbit of the subgroup H of G generated by a and b^2. The latter subgroup has index 2 in G, with coset representatives 1 and b, and if β is any element of A_1 that also fixes the vertex b, then by vertex-transitivity, β fixes every vertex in the orbit of the coset bH, and hence fixes every vertex, so β is trivial. It follows that $|A_1| \leq 2$, and furthermore, since $A = GA_1$ (with $G \cap A_1 = \{1\}$), we find that G has index 1 or 2 in A. Hence in particular, G is normal in A (a fact which also follows from a theorem by Zhou and Feng [12, Theorem 2.3] on 3-valent Cayley graphs of order $4p$, for p prime).

Now suppose that A_1 is non-trivial. Then there exists an automorphism $\theta \in A_1$ such that $A = \langle G, \theta\rangle$, and moreover, θ has order 2 and must swap the neighbours b and b^{-1} of 1, and fix ab^2. Hence conjugation of G by θ fixes ab^2 and swaps b with b^{-1} (as elements of G). It follows that θ fixes b^2 and hence also fixes $(ab^2)b^2 = a$, but then $a^k = (a^k)^\theta =$

$(b^{-1}ab)^\theta = bab^{-1} = a^{-k}$, and so a^k has order 2, contradiction.

Thus A_1 is trivial, and $\mathrm{Aut}(X) = A = G$, so X is a GRR, with arc-type $1 + (1 + 1)$.

Case (e): Arc-type $m + (1 + 1)$, for all $m \geq 2$.

For any prime $p \equiv 1 \bmod 4$ with $p > 5$, and any square root k of $-1 \bmod p$, let X be the Cayley graph for $C_p \rtimes_k C_4$ produced in case (d) above. This graph has two edge-orbits, one of length $4p$ containing the edge $\{1, b\}$, and the other of length $2p$ containing the edge $\{1, ab^2\}$, where a and b are generators for $G = C_p \rtimes_k C_4$ satisfying the relations $a^p = b^4 = 1$ and $b^{-1}ab = a^k$.

Now let $Y_m = X(F, m)$ be the thickened m-cover of X over F, where F is the smaller of the two edge-orbits of X. Then Y_m is regular of valency $m+2$, and is a Cayley graph, by Proposition 2.4, so all we have to do is show that Y_m is prime and has arc-type $m+(1+1)$. We do this in much the same way as was done for the single example (for each m) in [3, Lemma 8.4].

First, we note that $X \setminus F$ is a union of quadrangles (unordered 4-cycles), and every edge of F joins vertices from different quadrangles. Hence by Proposition 2.5, we find that all edges in a fibre over an edge in $E(X) \setminus F$ lie in the same edge-orbit of Y_m, and all edges in a fibre over an edge in F lie in the same edge-orbit. In particular, all edges of the form $\{(1, 0), (ab^2, i)\}$ for $i \in \mathbb{Z}_m$ lie in the same edge-orbit of Y_m. Also multiplication by $(b, 0)$ puts $\{(1, 0), (b^{-1}, 0)\}$ in the same edge-orbit as $\{(1, 0), (b, 0)\}$.

On the other hand, up to reversal the edge $\{(1, 0), (b, 0)\}$ lies in just one 4-cycle, namely $((1, 0), (b, 0), (b^2, 0), (b^{-1}, 0))$, while the edge $\{(1, 0), (ab^2, 0)\}$ lies in $(m - 1)^2$ distinct 4-cycles, namely $((1, 0), (ab^2, 0), (1, j), (ab^2, \ell))$ for $j, \ell \in \mathbb{Z}_m \setminus \{0\}$. Hence if $m > 2$ then $\{(1, 0), (b, 0)\}$ cannot lie in the same orbit as $\{(1, 0), (ab^2, 0)\}$. Similarly, when $m = 2$, up to reversal the edge $\{(1, 0), (b, 0)\}$ lies in precisely four 6-cycles, namely $((1, 0), (b, 0), (ab^{-1}, j), (b, 1), (1, 1), (ab^2, \ell))$ for $j, \ell \in \{0, 1\}$, while the edge $\{(1, 0), (ab^2, 0)\}$ lies in eight 6-cycles, viz. $((1, 0), (ab^2, 0), (1, 1), (b^\epsilon, 1), (ab^{-\epsilon}, j), (b^\epsilon, 0))$ and $((1, 0), (ab^2, 0), (a^{\epsilon k}b^\epsilon, 0), (a^{1-\epsilon k}b^{-\epsilon}, j), (a^{\epsilon k}b^\epsilon, 1), (ab^2, 1))$ for $\epsilon = \pm 1$ and $j \in \{0, 1\}$, and again we find that the edge $\{(1, 0), (b, 0)\}$ cannot lie in the same orbit as the edge $\{(1, 0), (ab^2, 0)\}$.

Hence the edge-type of Y_m is $m + 2$, and its arc-type is $m + 2$ or $m + (1 + 1)$.

Next, consider the stabiliser $A_{(1,0)}$ in $A = \mathrm{Aut}(Y_m)$ of the vertex $(1, 0)$. We know that $A_{(1,0)}$ preserves the set of m neighbours of $(1, 0)$ of the form (ab^2, i) for $i \in \mathbb{Z}_m$, and as a consequence, $A_{(1,0)}$ must preserve the set of all paths of length 2 of the form $((1, 0), (ab^2, i), y)$. For any such i, the third vertex y is either $(a^{-k}b^{-1}, i)$, or $(a^k b, i)$, or $(1, j)$ for some $j \in \mathbb{Z}_m \setminus \{0\}$. Moreover, if $y = (a^{-k}b^{-1}, i)$ or $(a^k b, i)$, then there is just one path of length 2 from $(1, 0)$ to y, while if $y = (1, j)$ for some j, then there are m distinct paths of length 2 from $(1, 0)$ to y. Hence $A_{(1,0)}$ must preserve the set of all vertices $(1, j)$ with $j \in \mathbb{Z}_m \setminus \{0\}$, and so $A_{(1,0)}$ preserves the fibre over $(1, 0)$.

By vertex-transitivity, the same thing holds for every vertex, and so $\mathrm{Aut}(Y_m)$ permutes the fibres over vertices of X. Hence every automorphism of Y_m can be projected to an automorphism of X. In particular, since X has arc-type $1 + (1 + 1)$, no automorphism can take the arc $((1, 0), (b, 0))$ to the arc $((1, 0), (b^{-1}, 0))$, and thus Y_m has arc-type $m+(1+1)$.

Finally, we show that Y_m is prime. To do this, consider any decomposition of Y into Cartesian factors, which are connected and vertex-transitive, by Proposition 2.1. The edge $\{(1, 0), (b, 0)\}$ does not lie in a 4-cycle with any of the m edges of the form $\{(1, 0), (ab^2, i)\}$ for $i \in \mathbb{Z}_m$, and so by part (d) of Proposition 2.2, all of those m edges must lie in the same factor of Y_m as $\{(1, 0), (b, 0)\}$, say Z. The same argument holds for the edge

$\{(1,0),(b^{-1},0)\}$, and so this edge must lie in Z as well. Hence Z contains all $m+2$ edges incident with the vertex $(1,0)$. By vertex-transitivity and connectedness, all edges of Y_m lie in Z, so $Z = Y_m$, and therefore Y_m is prime.

Case (f): Arc-type $1 + (m + m)$, for all $m \geq 2$.

This case is similar to the previous one, except that we let $Y_m = X(F, m)$ be the thickened m-cover of X where this time F is the larger of the two edge-orbits of X. Again, Y_m is a Cayley graph, by Proposition 2.4, but of valency $2m + 1$, and all we have to do is show that Y_m is prime and has arc-type $1 + (m + m)$.

The neighbours of the vertex $(1,0)$ are the $2m$ vertices of the form (b,i) or (b^{-1},i) where $i \in \mathbb{Z}_m$, plus the single vertex $(ab^2,0)$. It is easy to see that every edge of the form $\{(1,0),(b^{\pm 1},i)\}$ lies in many different 4-cycles, while the edge $\{(1,0),(ab^2,0)\}$ lies in no 4-cycles at all. In particular, this shows that Y_m is prime, and that the vertex $(ab^2,0)$ is fixed by the stabiliser $A_{(1,0)}$ of $(1,0)$ in $A = \mathrm{Aut}(Y_m)$. Moreover, $X \setminus F$ is a union of non-incident edges, and so by Proposition 2.5, all arcs of the form $((1,0),(b,i))$ lie in the same arc-orbit of Y_m, and the same holds for all arcs of the form $((1,0),(b^{-1},i))$. Hence the arc-type of Y_m is either $2m + 1$ or $1 + (m + m)$.

To prove that the arc-type is $1 + (m + m)$, again we consider the local effect of the stabiliser $A_{(1,0)}$ on vertices at short distance from the vertex $(1,0)$.

We know that $A_{(1,0)}$ preserves the set of $2m$ neighbours of $(1,0)$ of the form $(b^{\pm 1},i)$ for $i \in \mathbb{Z}_m$, and fixes the neighbour $(ab^2,0)$. In particular, $A_{(1,0)}$ must preserve the set of all paths of length 2 of the form $((1,0),(b^{\pm 1},i),y)$. This time the third vertex y is either (ab^3,i), or (ab,i), or $(1,\ell)$ or (b^2,ℓ) for some $\ell \in \mathbb{Z}_m$, and in the first two cases, there is just one such path of length 2 from $(1,0)$ to y, while if $y = (1,\ell)$ or (b^2,ℓ) for some ℓ, then there are $2m$ such paths. Also each vertex v of the form $(1,\ell)$ or (b^2,ℓ) lies at distance 3 from the vertex $(ab^2,0)$ fixed by $A_{(1,0)}$, via the $2m$ paths $(v,(b^\epsilon,j),(1,0),(ab^2,0))$ with $\epsilon = \pm 1$ and $j \in \mathbb{Z}_m$. Moreover, if v is one of the vertices of the form $(1,\ell)$, then there are $2m$ additional paths, namely $((1,\ell),(ab^2,\ell),(a^{\epsilon k}b^\epsilon,j),(ab^2,0))$ for $\epsilon = \pm 1$ and $j \in \mathbb{Z}_m$, but there are no such additional paths from a vertex of the form (b^2,ℓ).

It follows that no element of $A_{(1,0)}$ can take a vertex of the form $(1,\ell)$ to one of the form (ab^3,i) or (ab,i) or (b^2,ℓ'), and therefore $A_{(1,0)}$ preserves the fibre over $(1,0)$.

By vertex-transitivity, the same thing holds for every vertex, and hence as before, every automorphism of Y_m can be projected to an automorphism of X. In particular, since X has arc-type $1 + (1 + 1)$, no automorphism can take the arc $((1,0),(b,0))$ to the arc $((1,0),(b^{-1},0))$, and thus Y_m has arc-type $1 + (m + m)$.

Case (g): Arc-type $(1 + 1) + (1 + 1)$.

By Lemma 8.6 of [3], if p is any prime number with $p \equiv 1 \bmod 6$, if k is a primitive 6th root of $1 \bmod p$, and G is the semi-direct product $C_p \rtimes_k C_6$, generated by two elements a and b of orders p and 6 such that $b^{-1}ab = a^k$, then the Cayley graph $\mathrm{Cay}(G, \{b, b^{-1}, ab^2, (ab^2)^{-1}\})$ is prime and has arc-type $(1 + 1) + (1 + 1)$. In fact, the edges $\{1, ab^2\}$ and $\{1, (ab^2)^{-1}\}$ lie in 3-cycles, but the edges $\{1, b\}$ and $\{1, b^{-1}\}$ do not.

Case (h): Arc-type $(m + m) + (1 + 1)$, for all $m \geq 2$.

For any prime $p \equiv 1 \bmod 6$, and any primitive 6th root k of $1 \bmod p$, let X be the Cayley graph produced in case (g) above. This graph has arc-type $(1 + 1) + (1 + 1)$, and its two edge-orbits both have length $4p$, with representatives $\{1, b\}$ and $\{1, ab^2\}$, where a and b are generators for $G = C_p \rtimes_k C_6$ satisfying $a^p = b^6 = 1$ and $b^{-1}ab = a^k$.

Now let $Y_m = X(m, F)$ be the thickened m-cover of X over F, where F is the edge-orbit containing $\{1, b\}$, or equivalently, the set of edges that lie in no 3-cycle. This graph is regular, with valency $2m + 2$, and by Proposition 2.4 is a Cayley graph, so all we have to do is show it is prime and has arc-type $(m + m) + (1 + 1)$. We do this in the same way as was done for the single example (for each m) in [3, Lemma 8.7].

First, $X \setminus F$ is a union of triangles (unordered 3-cycles), and every edge of F joins vertices from different triangles, and it follows that every automorphism of Y_m induces a permutation of the fibres over the edges in $E(X) \setminus F$, and also a permutation of the fibres over the edges in F. On the other hand, Proposition 2.5 tells us that all edges in a fibre over an edge in $E(X) \setminus F$ lie in the same edge-orbit, and all edges in a fibre over an edge in F lie in the same edge-orbit. Hence the edge-type of Y_m is $2m + 2$.

Next, from vertex (u, i) in Y_m there are precisely $2m$ paths from (u, i) to any other vertex (u, ℓ) in the fibre over (u, i), namely those of the form $((u, i), (bu, j), (u, \ell))$ and $((u, i), (b^{-1}u, j), (u, \ell))$ for each $j \in \mathbb{Z}_m$, while on the other hand, there are only one, two or m paths from (u, i) to any other vertex v at distance 2 from (u, i). Hence the stabiliser in $\mathrm{Aut}(Y_m)$ of the vertex (u, i) preserves the fibre over the vertex u, and it follows that $\mathrm{Aut}(Y_m)$ permutes the fibres over the vertices of X.

Thus every automorphism of Y_m can be projected to an automorphism of X, and the arc-type of Y_m is $(m + m) + (1 + 1)$, as required.

Finally, we show that Y_m is prime. If Y_m were the Cartesian product of two relatively prime graphs, then one of them would have arc-type $(1 + 1)$, which is impossible. On the other hand, if Y_m were a proper Cartesian power of some prime graph Z, say $Y_m = Z^r$ with $r \geq 2$, then by part (b) of Proposition 2.2, all edges in a 3-cycle of Y_m would lie in the same factor of Y_m, so Z would contain a 3-cycle, but in that case a vertex of $Y_m = Z^r$ would lie in at least two triangles, contradiction. Thus Y_m is prime.

Case (i): Arc-type $1 + 1 + 1$.

By Lemma 8.8 of [3], if n is any odd integer greater than 11, and G is the dihedral group D_n, generated by two elements x and y satisfying $x^2 = y^n = 1$ and $xyx = y^{-1}$, then $\mathrm{Cay}(G, \{x, xy, xy^3\})$ is prime and has arc-type $1 + 1 + 1$.

Case (j): Arc-type $1 + 1 + (1 + 1)$.

This is similar to case (d). Let p be any prime such that $p \equiv 1 \bmod 4$, with $p > 5$, let k be any integer such that $k^2 \equiv -1 \bmod p$, and let G be the semi-direct product $C_p \rtimes_k C_4$, generated by two elements a and b such that $a^p = b^4 = 1$ and $b^{-1}ab = a^k$. Now take $S = \{b, b^{-1}, ab^2, a^2b^2\}$, which consists of an inverse pair of elements of order 4 and two involutions (as conjugation by b^2 inverts a), and let $X = \mathrm{Cay}(G, \{b, b^{-1}, ab^2, a^2b^2\})$.

Then X is 4-valent and connected, since $\langle b, ab^2 \rangle = G$, and is also non-bipartite, just as in case (d). We will show that X is prime and has arc-type $1 + 1 + (1 + 1)$.

First, by considering the vertices at distance 2 from the identity we see that up to reversal, the edges $\{1, b\}$ and $\{1, b^{-1}\}$ lie in a single 4-cycle, namely $(1, b, b^2, b^{-1})$, while each of the edges $\{1, ab^2\}$ and $\{1, a^2b^2\}$ lies in no 4-cycle. In particular, it follows from the latter observation that X is prime.

Also as before, the edges $\{1, b\}$ and $\{1, b^{-1}\}$ lie in the same edge-orbit. On the other hand, the edges $\{1, ab^2\}$ and $\{1, a^2b^2\}$ lie in different edge orbits, because up to reversal the edge $\{1, ab^2\}$ lies in four 5-cycles, namely those of the form $(1, ab^2, u, v, w)$ with $(u, v, w) = (a, b^2, b)$, (a, b^2, b^{-1}), (a^kb, a^{-1}, a^2b^2) and $(a^{-k}b^{-1}, a^{-1}, a^2b^2)$, while the edge $\{1, a^2b^2\}$ lies in only two 5-cycles, namely those of the form $(1, a^2b^2, u, v, w)$

with $(u, v, w) = (a^{-1}, a^k b, ab^2)$ and $(a^{-1}, a^{-k}b^{-1}, ab^2)$. Hence the edge-type of X is $2 + 1 + 1$.

This also implies that every automorphism of X preserves the set $T = \{x, a^2b^2x\}$ of all edges corresponding to left multiplication by the element $a^2b^2 \in S$, and hence induces an automorphism of the subgraph obtained by removing those edges, namely $\mathrm{Cay}(G, S \setminus T) = \mathrm{Cay}(G, \{b, b^{-1}, ab^2\})$. By case (d), however, the latter subgraph is a GRR, with automorphism group G, and so every such automorphism is given by right multiplication by some element of G. It follows that $G = \mathrm{Aut}(X)$, and hence X is also a GRR, and has arc-type $1 + 1 + (1 + 1)$.

Case (k): Arc-type $1 + 1 + 1 + 1$.

For any integer $n > 15$, let G be the dihedral group D_n, generated by elements a and b such that $a^n = b^2 = (ab)^2 = 1$, and take $S = \{b, ba, ba^2, ba^5\}$. Then since S consists of four involutions and G is generated by b and ba, the graph $X = \mathrm{Cay}(G, S)$ is 4-valent and connected. We show that X is prime and has arc-type $1 + 1 + 1 + 1$.

First, the paths of length 2 in X starting at the identity vertex 1 are $(1, b, a^j)$ for $j \in \{-1, -2, -5\}$, and $(1, ba, a^j)$ for $j \in \{1, -1, -4\}$, and $(1, ba^2, a^j)$ for $j \in \{2, 1, -3\}$, and $(1, ba^5, a^j)$ for $j \in \{5, 4, 3\}$. By considering the final vertex of each of these, we see that the vertex 1 lies in only two 4-cycles up to reversal, namely $(1, b, a^{-1}, ba)$ and $(1, ba, a, ba^2)$. Hence the edges $\{1, b\}$ and $\{1, ba^2\}$ lie in just one 4-cycle, while $\{1, ba\}$ lies in two 4-cycles, and $\{1, ba^5\}$ lies in no 4-cycles at all. In particular, X is prime, and also X has edge-type $1 + 1 + 1 + 1$ or $2 + 1 + 1$, with each of $\{1, ba\}$ and $\{1, ba^5\}$ lying in different orbits from each other and from $\{1, b\}$ and $\{1, ba^2\}$.

Next, multiplying by b, we find that $ba^5b = a^{-5}$ plays the same role for the vertex b as ba^5 does for the vertex 1, namely that $\{b, a^{-5}\}$ is the only edge incident with b that lies in no 4-cycle. Now consider the cycles of length 6 containing one of the paths $(ba^5, 1, b)$, $(a^{-5}, b, 1)$ and $(ba^5, 1, ba^2)$. An easy calculation shows there are precisely three 6-cycles of the form $(ba^5, 1, b, u, v, w)$, namely with $(u, v, w) = (a^{-1}, ba^4, a^3)$, (a^{-1}, ba^4, a^4) and (a^{-1}, ba^3, a^3), and similarly, there are three 6-cycles of the form $(a^{-5}, b, 1, u, v, w)$, namely with $(u, v, w) = (ba, a^{-4}, ba^{-3})$, (ba, a^{-4}, ba^{-4}) and (ba^2, a^{-3}, ba^{-3}), but there are seven 6-cycles of the form $(ba^5, 1, ba^2, u, v, w)$, namely with $(u, v, w) = (a, ba^3, a^3)$, (a, ba^6, a^4), (a, ba^6, a^5), (a^2, ba^3, a^3), (a^2, ba^4, a^3), (a^2, ba^4, a^4) and (a^2, ba^7, a^5).

In fact, up to reversal the edge $\{1, b\}$ lies in 16 different 6-cycles altogether, while the edge $\{1, ba^2\}$ lies in 20 different 6-cycles, but this takes more work to verify.

Both calculations show that the edge $\{1, ba^2\}$ cannot lie in the same orbit as $\{1, b\}$ under $\mathrm{Aut}(X)$, and it follows that X has edge-type and arc-type $1 + 1 + 1 + 1$.

Case (l): Arc-type $(1 + 1) + (1 + 1) + (1 + 1)$.

This is somewhat similar to case (g). Let p be any prime with $p \equiv 1 \bmod 6$, but this time where $p > 7$, let k be a primitive 6th root of $1 \bmod p$, with $k^3 \equiv -1 \bmod p$, and let G be the semi-direct product $C_p \rtimes_k C_6$, generated by two elements a and b of orders p and 6 such that $b^{-1}ab = a^k$. Now take $S = \{b, ab^2, a^2b^2, b^{-1}, a^{-k^2}b^4, a^{-2k^2}b^4\}$, which consists of the elements b, ab^2 and a^2b^2 and their inverses, and let $X = \mathrm{Cay}(G, S)$. Then clearly X is 6-valent and connected. We will show that X is prime, and has arc-type $(1 + 1) + (1 + 1) + (1 + 1)$, for all p.

First we note that $\{1, s\}$ and $\{1, s\}s^{-1} = \{1, s^{-1}\}$ lie in the same edge orbit of X, for each $s \in S$. Hence X has at most three distinct edge orbits.

Next, up to reversal the edge $\{1, b\}$ lies in just two 4-cycles, namely $(1, b, ab^3, a^{-k^2}b^4)$

and $(1, b, a^2b^3, a^{-2k^2}b^4)$, and multiplying by b^{-1} gives the two 4-cycles containing the edge $\{1, b^{-1}\}$ as $(1, b^{-1}, a^{-k^2}b^3, ab^2)$ and $(1, b^{-1}, a^{-k^2}b^3, a^2b^2)$. Each of the other four edges incident with the vertex 1 is contained in only one 4-cycle (up to reversal), namely one of the four just listed for $\{1, b\}$ and $\{1, b^{-1}\}$. Hence the orbit of the edges $\{1, b^{\pm 1}\}$ under $\mathrm{Aut}(X)$ is different from the orbit(s) of $\{1, s^{\pm 1}\}$ for $s = ab^2$ and $s = a^2b^2$.

Also the edge $\{1, ab^2\}$ lies in five 5-cycles, viz. those of the form $(1, ab^2, u, v, w)$ with $(u, v, w) = (a^kb, a^{k-1}, a^2b^2)$, $(a^kb, a^{-k}b^{-1}, a^{-k^2}b^4)$, $(a^{-k^2}b^3, ab, b^{-1})$, (a^{-k+1}, b^2, b), and $(a^{-k^2}b^4, ab^3, b)$, while the edge $\{1, a^2b^2\}$ lies in only four 5-cycles, namely those of the form $(1, a^2b^2, u, v, w)$ with $(u, v, w) = (a^{k-1}, a^kb, ab^2)$, $(a^{2k}b, a^{-2k}b^{-1}, a^{-2k^2}b^4)$, $(a^{-2k^2}b^3, a^2b, b^{-1})$ and $(a^{-2k^2}b^4, a^2b^3, b)$. Hence the orbit of the edges $\{1, (ab^2)^{\pm 1}\}$ is different from the orbit of $\{1, (a^2b^2)^{\pm 1}\}$, and so the edge-type of X is $2 + 2 + 2$.

But furthermore, if there exists an automorphism of X that fixes the vertex 1 and swaps b with b^{-1}, then that automorphism must swap the two 4-cycles $(1, b, ab^3, a^{-k^2}b^4)$ and $(1, b, a^2b^3, a^{-2k^2}b^4)$ with the two 4-cycles $(1, b^{-1}, a^{-k^2}b^3, ab^2)$ and $(1, b^{-1}, a^{-k^2}b^3, a^2b^2)$, and hence must swap ab^2 with $a^{-k^2}b^4 = (ab^2)^{-1}$ and swap a^2b^2 with $a^{-2k^2}b^4 = (a^2b^2)^{-1}$. Similarly, if if there exists an automorphism that fixes 1 and swaps ab^2 with $a^{-k^2}b^4$, then it must swap the 4-cycle $(1, b^{-1}, a^{-k^2}b^3, ab^2)$ with the 4-cycle $(1, b, ab^3, a^{-k^2}b^4)$, and hence must swap b with b^{-1}, and the same holds for a^2b^2 and $a^{-2k^2}b^4$.

It follows that any automorphism that fixes the vertex 1 must either fix all its six neighbours, or induce the triple transposition $(b, b^{-1})(ab^2, a^{-k^2}b^4)(a^2b^2, a^{-2k^2}b^4)$ on them. By vertex-transitivity, the analogous thing happens at every vertex, and an easy argument then shows that the stabiliser A_1 in $A = \mathrm{Aut}(X)$ of the vertex 1 acts faithfully on its neighbourhood, and therefore $|A_1| = 1$ or 2.

Now suppose that $|A_1| = 2$. Then $|A| = |GA_1| = 2|G|$, and so G is normal in A. Hence if θ is any non-trivial element of A_1, then θ normalises G, and so induces an automorphism of $G = \langle a, b\rangle$. Moreover, as θ fixes the vertex 1 and acts non-trivially on its neighbourhood, we find that θ swaps b with b^{-1}, and ab^2 with $a^{-k^2}b^4 = (ab^2)^{-1}$. In turn, this implies that θ swaps $a = (ab^2)b^{-2}$ with $a^{-k^2}b^4b^2 = a^{-k^2}$, but then we find that $a^k = (a^k)^\theta = (b^{-1}ab)^\theta = ba^{-k^2}b^{-1} = a^{-k}$, and so a^k has order 2, contradiction.

Thus A_1 is trivial, and $\mathrm{Aut}(X) = G$, so X has arc-type $(1 + 1) + (1 + 1) + (1 + 1)$.

Finally, X cannot be the Cartesian product of two smaller graphs that are relatively prime, since those would have to be connected and vertex-transitive, and one of them would have arc-type $(1 + 1)$, which is impossible. Also X cannot be a Cartesian power of some smaller VT graph, since its order $6p$ is not a non-trivial power of any integer. Hence X is prime, as required.

Accordingly, we have infinitely many connected finite Cayley graphs with each of the basic arc-types, and this completes the proof of Theorem 3.1 and Corollary 3.2.

For the benefit of the reader (and for possible future reference), we summarise some of the details of the basic arc-types used here, in Table 1.

5 A consequence for zero-symmetric graphs

Another consequence of Theorem 3.1 is the following:

Corollary 5.1. *For every integer $d > 2$, there exist infinitely many finite zero-symmetric graphs (or GRRs) of valency d.*

Table 1: Summary of some Cayley graphs with the basic arc-types.

Arc-type	Cayley graphs
m	Cycle graphs C_n ($n \geq 5$) for $m = 2$, and Cayley graphs for $\mathrm{PSL}(2, p)$ via m conjugate involutions for $m \geq 3$
$(m{+}m)$	Bouwer graphs $B(m, r, n)$ with $n > 7$ and $r > 6$
$m{+}1$	Thickened m-cover of C_{2n} over a 1-factor
$1{+}(1{+}1)$	3-valent Cayley graph for $C_p \rtimes_k C_4$ (for prime p)
$m{+}(1{+}1)$	Thickened cover of Cayley graph for $C_p \rtimes_k C_4$
$1{+}(m{+}m)$	Thickened cover of Cayley graph for $C_p \rtimes_k C_4$
$(1{+}1){+}(1{+}1)$	4-valent Cayley graph for $C_p \rtimes_k C_6$ (for prime p)
$(m{+}m){+}(1{+}1)$	Thickened cover of Cayley graph for $C_p \rtimes_k C_6$
$1{+}1{+}1$	3-valent Cayley graph for dihedral groups D_n
$1{+}1{+}(1{+}1)$	4-valent Cayley graph for $C_p \rtimes_k C_4$ (for prime p)
$1{+}1{+}1{+}1$	4-valent Cayley graph for dihedral groups D_n
$(1{+}1){+}(1{+}1){+}(1{+}1)$	6-valent Cayley graph for $C_p \rtimes_k C_6$ (for prime p)

This is not at all surprising, but appears to be new, in the sense that we cannot find the statement or something similar in the literature on GRRs or zero-symmetric graphs. It is shown in [5, Theorem 3.10.4] that there exists a GRR of valency d for the symmetric group S_{d+1} whenever the latter group can be generated by an 'asymmetric' set of d transpositions. The latter happens for all $d > 5$, but gives only finitely many GRRs with given valency d. On the other hand, it is clear that larger sets of involutory generators for dihedral or symmetric or other groups will give GRRs, even if this does not appear to have been explicitly proved elsewhere.

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Coloring properties of categorical product of general Kneser hypergraphs*

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Abstract

More than 50 years ago Hedetniemi conjectured that the chromatic number of categorical product of two graphs is equal to the minimum of their chromatic numbers. This conjecture has received a considerable attention in recent years. Hedetniemi's conjecture was generalized to hypergraphs by Zhu in 1992. Hajiabolhassan and Meunier, in 2016, introduced the first nontrivial lower bound for the chromatic number of categorical product of general Kneser hypergraphs and using this lower bound, they verified Zhu's conjecture for some families of hypergraphs. In this paper, we shall present some colorful type results for the coloring of categorical product of general Kneser hypergraphs, which generalize the Hajiabolhassan-Meunier result. Also, we present a new lower bound for the chromatic number of categorical product of general Kneser hypergraphs which can be much better than the Hajiabolhassan-Meunier lower bound. Using this lower bound, we enrich the family of hypergraphs satisfying Zhu's conjecture.

Keywords: Categorical product, chromatic number, Hedetniemi's conjecture, general Kneser hypergraph.

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1 Introduction and main results

For two graphs G and H, their categorical product $G \times H$ is the graph defined on the vertex set $V(G) \times V(H)$ such that two vertices (g, h) and (g', h') are adjacent whenever $gg' \in E(G)$ and $hh' \in E(H)$. The categorical product is the product involved in the famous long-standing conjecture posed by Hedetniemi which states that the chromatic number of $G \times H$ is equal to the minimum of $\chi(G)$ and $\chi(H)$. It was shown that the conjecture is true for several families of graphs, but it is wide open in general (see Tardif [21] and Zhu [23]). In spite of being investigated in several articles, there is no substantial progress in solving this conjecture. This conjecture was generalized to the case of hypergraphs by Zhu [22].

A hypergraph $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a set of vertices, and $E(\mathcal{H})$ is a family of nonempty subsets of $V(\mathcal{H})$. The elements of $E(\mathcal{H})$ are called edges. All hypergraphs considered in the paper have no multiple edges and $E(\mathcal{H})$ is thus a usual set. For a subset $S \subseteq V(\mathcal{H})$, *the subhypergraph induced by* S, denoted by $\mathcal{H}[S]$, is a hypergraph with vertex set S and edge set $\{e \in E(\mathcal{H}) : e \subseteq S\}$. A hypergraph $\mathcal{H}$ is said to be r-uniform if $E(\mathcal{H})$ is a family of r-subsets of $V(\mathcal{H})$. In particular, a 2-uniform hypergraph is called a simple graph. From now on, by a graph we mean a simple graph. An r-uniform hypergraph $\mathcal{H}$ is called *r-partite* if $V(\mathcal{H})$ can be written as a union of r pairwise disjoint subsets (parts) $U_1, \ldots, U_r$ such that each edge of $\mathcal{H}$ intersects each part U_i in one vertex. An r-partite hypergraph is called *complete* if it contains all possible edges. Also, it is said to be *balanced* if $|U_i| - |U_j| \leq 1$ for each $i, j \in [r]$.

Let $\mathcal{H}$ be a hypergraph and r be an integer, where $r \geq 2$. For pairwise disjoint subsets $U_1, \ldots, U_r \subseteq V(\mathcal{H})$, the hypergraph $\mathcal{H}[U_1, \ldots, U_r]$ is defined to be a subhypergraph of $\mathcal{H}$ whose vertex set is $\cup_{i=1}^r U_i$ and whose edge set consists of all edges of $\mathcal{H}$ which are contained in $\cup_{i=1}^r U_i$ and have exactly one element in each U_i. Note that $\mathcal{H}[U_1, \ldots, U_r]$ is an r-uniform r-partite hypergraph.

A proper coloring of a hypergraph $\mathcal{H}$ is an assignment of colors to the vertices of $\mathcal{H}$ such that there is no monochromatic edge. The chromatic number of a hypergraph $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the smallest number k such that there exists a proper coloring of $\mathcal{H}$ with k colors. If there is no such a k, we define the chromatic number to be infinite. Let c be a proper coloring of a complete r-partite hypergraph $\mathcal{H}$ with parts $U_1, \ldots, U_r$. The hypergraph $\mathcal{H}$ is *colorful* (with respect to the coloring c) whenever for each $i \in [r]$, the vertices in U_i receive different colors, that is, $|c(U_i)| = |U_i|$ for each $i \in [r]$.

Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two hypergraphs. For $i = 1, 2$, the projection π_i is defined by $\pi_i \colon (v_1, v_2) \mapsto v_i$. The categorical product of two hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, defined by Dörfler and Waller in 1980 [10], is the hypergraph $\mathcal{H}_1 \times \mathcal{H}_2$ with vertex set $V_1 \times V_2$ and edge set

$$\{e \subseteq V_1 \times V_2 : \pi_1(e) \in E_1, \pi_2(e) \in E_2\}.$$

In 1992, Zhu [22] proposed the following conjecture as a generalization of Hedetniemi's conjecture.

Conjecture 1.1 ([22]). *Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two hypergraphs. Then*

$$\chi(\mathcal{H}_1 \times \mathcal{H}_2) = \min\{\chi(\mathcal{H}_1), \chi(\mathcal{H}_2)\}.$$

One can easily derive a proper coloring of $\mathcal{H}_1 \times \mathcal{H}_2$ from a proper coloring of $\mathcal{H}_1$ or of $\mathcal{H}_2$. Therefore the hard part is to show that $\chi(\mathcal{H}_1 \times \mathcal{H}_2) \geq \min\{\chi(\mathcal{H}_1), \chi(\mathcal{H}_2)\}$. Let

$\mathcal{F}$ be a subhypergraph of $\mathcal{H}_1 \times \mathcal{H}_2$ with the same vertex set and whose edge set consists of minimal edges of $\mathcal{H}_1 \times \mathcal{H}_2$. It is clear that any proper coloring of $\mathcal{F}$ is also a proper coloring of $\mathcal{H}_1 \times \mathcal{H}_2$. This observation shows that Conjecture 1.1 is a generalization of Hedetniemi's conjecture.

For an integer r and a hypergraph $\mathcal{H}$, the *r-colorability defect of $\mathcal{H}$*, denoted by $\mathrm{cd}^r(\mathcal{H})$, is the minimum number of vertices that shall be removed from $\mathcal{H}$ so that the hypergraph induced by the remaining vertices admits a proper coloring with r colors.

Let $Z_r = \{\omega, \omega^2, \ldots, \omega^r\}$ be a multiplicative cyclic group of order r with generator ω. For $X = (x_1, \ldots, x_n) \in (Z_r \cup \{0\})^n$, a sequence $x_{i_1}, \ldots, x_{i_m}$ with $1 \leq i_1 < \cdots < i_m \leq n$ is called an *alternating subsequence of X* if $x_{i_j} \neq 0$ for each $j \in [m]$ and $x_{i_j} \neq x_{i_{j+1}}$ for each $j \in [m-1]$. The *alternation number of X*, denoted by $\mathrm{alt}(X)$, is the length of the longest alternating subsequence of X. We set $\mathbf{0} = (0, \ldots, 0)$ and define $\mathrm{alt}(\mathbf{0}) = 0$. Also, for an $X = (x_1, \ldots, x_n) \in (Z_r \cup \{0\})^n$ and for $\varepsilon \in Z_r$, define $X^\varepsilon = \{i : x_i = \varepsilon\}$. Note that the r-tuple $(X^\varepsilon)_{\varepsilon \in Z_r}$ uniquely determines X and vice versa. Therefore, with abuse of notations, we can write $X = (X^\varepsilon)_{\varepsilon \in Z_r}$. The notation $|X|$ stands for the number of nonzero coordinates of X, i.e., $|X| = \sum_{\varepsilon \in Z_r} |X^\varepsilon|$. For two vectors $X, Y \in (Z_r \cup \{0\})^n$, we write $X \subseteq Y$ whenever $X^\varepsilon \subseteq Y^\varepsilon$ for each $\varepsilon \in Z_r$.

For a hypergraph $\mathcal{H}$ and a bijection $\sigma\colon [n] \to V(\mathcal{H})$, the *$r$-alternation number of $\mathcal{H}$ with respect to the permutation σ* is defined as follows:

$$\mathrm{alt}^r_\sigma(\mathcal{H}) = \max\left\{\mathrm{alt}(X) : E(\mathcal{H}[\sigma(X^\varepsilon)]) = \emptyset \text{ for all } \varepsilon \in Z_r\right\}.$$

The *r-alternation number* of $\mathcal{H}$, denoted by $\mathrm{alt}^r(\mathcal{H})$, is equal to $\min_\sigma \mathrm{alt}^r_\sigma(\mathcal{H})$ where the minimum is taken over all bijections $\sigma\colon [n] \to V(\mathcal{H})$ (for more details see [3]).

For any hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ and positive integer $r \geq 2$, *the general Kneser hypergraph* $\mathrm{KG}^r(\mathcal{H})$ is an r-uniform hypergraph whose vertex set is $E(\mathcal{H})$ and whose edge set is the set of all r-subsets of $E(\mathcal{H})$ containing r pairwise disjoint edges of $\mathcal{H}$. Note that by this notation the well-known Kneser hypergraph $\mathrm{KG}^r(n,k)$ is the Kneser hypergraph $\mathrm{KG}^r\left([n], \binom{[n]}{k}\right)$. For $r = 2$, we will rather use $\mathrm{KG}(\mathcal{H})$ than $\mathrm{KG}^r(\mathcal{H})$.

Lovász in 1978, by using tools from algebraic topology, proved that $\chi(\mathrm{KG}(n,k)) = n - 2k + 2$. His paper showed an inspired and deep application of algebraic topology in combinatorics [15]. As a generalization of this result and to confirm a conjecture of Erdős [11], Alon, Frankl, and Lovász [5] proved that the chromatic number of $\mathrm{KG}^r(n,k)$ is equal to $\left\lceil \frac{n-(k-1)r}{r-1} \right\rceil$. A different kind of generalization of Lovász's theorem has been obtained by Dol'nikov [9]. He proved that

$$\chi(\mathrm{KG}(\mathcal{H})) \geq \mathrm{cd}^2(\mathcal{H}).$$

Then, in 1992, Kříž [13] extended the both latter results by proving that

$$\chi(\mathrm{KG}^r(\mathcal{H})) \geq \left\lceil \frac{\mathrm{cd}^r(\mathcal{H})}{r-1} \right\rceil.$$

Alishahi and Hajiabolhassan [3] introduced the alternation number as an improvement of colorability defect. Using the Z_p-Tucker lemma, they proved that

$$\chi(\mathrm{KG}^r(\mathcal{H})) \geq \left\lceil \frac{|V(\mathcal{H})| - \mathrm{alt}^r(\mathcal{H})}{r-1} \right\rceil.$$

It can be verified that $|V(\mathcal{H})| - \mathrm{alt}^r(\mathcal{H}) \geq \mathrm{cd}^r(\mathcal{H})$ and the inequality is often strict [3]. Therefore, the preceding lower bound for chromatic number surpasses the Dol'nikov-Kříž lower bound. Recently, by an innovative use of the Z_p-Tucker lemma, Hajiabolhassan and Meunier [12] extended the Alishahi-Hajiabolhassan result (as well as the Dol'nikov-Kříž result) to the categorical product of general Kneser hypergraphs as follows.

Theorem A ([12])**.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_t$ be hypergraphs and r be an integer, where $r \geq 2$. Then*

$$\chi(\mathrm{KG}^r(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^r(\mathcal{H}_t)) \geq \left\lceil \frac{1}{r-1} \min_{i \in [t]} (|V(\mathcal{H}_i)| - \mathrm{alt}^r(\mathcal{H}_i)) \right\rceil .$$

Using Theorem A, Hajiabolhassan and Meunier introduced new families of hypergraphs satisfying Zhu's conjecture.

From another point of view, Simonyi and Tardos [20] generalized the Dol'nikov result. Indeed, they proved that for any hypergraph $\mathcal{H}$, if $t = \mathrm{cd}^2(\mathcal{H})$, then any proper coloring of $\mathrm{KG}(\mathcal{H})$ contains a complete bipartite subgraph $K_{\lfloor t/2 \rfloor, \lceil t/2 \rceil}$ such that all vertices of this subgraph receive different colors and these different colors occur alternating on the two parts of the bipartite graph with respect to their natural order. Then, this result as well as the Dol'nikov-Kříž result was extended to Kneser hypergraphs by Meunier [19] as the following theorem.

Theorem B ([19])**.** *Let $\mathcal{H}$ be a hypergraph and p be a prime number. Any proper coloring of $\mathrm{KG}^p(\mathcal{H})$ contains a colorful, balanced, and complete p-partite subhypergraph $\mathcal{F}$ with $\mathrm{cd}^p(\mathcal{H})$ vertices.*

It should be mentioned that, in his paper [19], Meunier also generalized Theorem B and proved that this theorem remains true by replacing $\mathrm{cd}^p(\mathcal{H})$ with $|V(\mathcal{H})| - \mathrm{alt}^p(\mathcal{H})$. In his proof, Meunier used a Z_q-generalization of a theorem by Ky Fan which is stated in terms of chain maps. Later, by introducing an appropriate generalization of the Z_p-Tucker lemma, the present second author [2] gave a simple proof for Meunier's result. Moreover, several extensions of Meunier's result can be found in [2]. Another common generalization of the Simonyi-Tardos result and a result by Chen [7, Theorem 7] can be found in [4].

As an improvement of r-colorability defect, the equitable r-colorability defect was introduced in [1]. For a hypergraph $\mathcal{H}$, the *equitable r-colorability defect of $\mathcal{H}$*, denoted by $\mathrm{ecd}^r(\mathcal{H})$, is the minimum number of vertices that shall be removed so that the subhypergraph induced by the remaining vertices admits a proper equitable r-coloring, i.e., a proper r-coloring in which the sizes of color classes differ by at most one. Clearly, $\mathrm{ecd}^r(\mathcal{H}) \geq \mathrm{cd}^r(\mathcal{H})$. As a generalization of Theorem B, it was proved [1] that any proper coloring of $\mathrm{KG}^p(\mathcal{H})$ contains a colorful, balanced, and complete p-partite subhypergraph $\mathcal{F}$ with $\mathrm{ecd}^p(\mathcal{H})$ vertices. It is not difficult to construct a hypergraph $\mathcal{H}$ for which $\mathrm{ecd}^r(\mathcal{H}) - \mathrm{cd}^r(\mathcal{H})$ is arbitrary large. Surpassing the Dol'nikov-Kříž lower bound, Abyazi Sani and Alishahi [1] proved

$$\chi(\mathrm{KG}^r(\mathcal{H})) \geq \left\lceil \frac{\mathrm{ecd}^r(\mathcal{H})}{r-1} \right\rceil .$$

It is worth mentioning that they indeed proved a more general result which in particular implies the prior lower bound. To be more specific, they gave a new lower bound for the chromatic number of a generalization of Kneser hypergraphs introduced by Ziegler which improves substantially Ziegler's lower bound [24, 25]. Furthermore, they compared their lower bound with the Dol'nikov-Kříž lower bound and the Alishahi-Hajiabolhassan lower bound. In this regard, it was shown that there is a family of hypergraphs $\mathscr{H}$ such that for each hypergraph $\mathcal{H} \in \mathscr{H}$,

$$\chi(\mathrm{KG}^r(\mathcal{H})) = \left\lceil \frac{\mathrm{ecd}^r(\mathcal{H})}{r-1} \right\rceil,$$

while $\chi(\mathrm{KG}^r(\mathcal{H})) - \left\lceil \frac{\mathrm{cd}^r(\mathcal{H})}{r-1} \right\rceil$ and $\chi(\mathrm{KG}^r(\mathcal{H})) - \left\lceil \frac{|V(\mathcal{H})|-\mathrm{alt}^r(\mathcal{H})}{r-1} \right\rceil$ are both unbounded for the hypergraphs $\mathcal{H}$ in $\mathscr{H}$. Although there are hypergraphs $\mathcal{H}$ for which $\mathrm{ecd}^r(\mathcal{H}) - (|V(\mathcal{H})| - \mathrm{alt}^r(\mathcal{H}))$ is arbitrary large, one can construct some hypergraphs $\mathcal{H}$ making $(|V(\mathcal{H})| - \mathrm{alt}^r(\mathcal{H})) - \mathrm{ecd}^r(\mathcal{H})$ arbitrary large, see [1].

As the main results of this paper, motivated by the preceding discussion, we simultaneously extend the results by Abyazi Sani and Alishahi [1] and by Hajiabolhassan and Meunier [12] to the following theorems.

Theorem 1.2. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_t$ be hypergraphs. Let p be a prime number and*

$$\eta = \max\left\{ \min_{i\in[t]} \mathrm{ecd}^p(\mathcal{H}_i),\ \min_{i\in[t]} \left(|V(\mathcal{H}_i)| - \mathrm{alt}^p(\mathcal{H}_i)\right) \right\}.$$

Any proper coloring of $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$ contains a colorful, balanced, and complete p-partite subhypergraph $\mathcal{F}$ with η vertices.

Remark. The question of whether Theorem 1.2 holds for an arbitrary positive integer r instead of a prime number p is an interesting open question.

Let c be the proper coloring with color set $[C]$. Let $\mathcal{F}$ be the colorful, balanced, and complete p-partite subhypergraph whose existence is ensured by Theorem 1.2. Clearly, any color appears in at most $p-1$ vertices of $\mathcal{F}$. Consequently, the previous theorem implies

$$\chi(\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)) \geq \left\lceil \frac{\eta}{p-1} \right\rceil \geq \left\lceil \frac{1}{p-1} \min_{i\in[t]} \mathrm{ecd}^p(\mathcal{H}_i) \right\rceil,$$

which can be extended for an arbitrary $r \geq 2$ as follows.

Theorem 1.3. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_t$ be hypergraphs and r be a positive integer, where $r \geq 2$. Then*

$$\chi(\mathrm{KG}^r(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^r(\mathcal{H}_t)) \geq \left\lceil \frac{1}{r-1} \min_{i\in[t]} \mathrm{ecd}^r(\mathcal{H}_i) \right\rceil.$$

Example. In what follows, by introducing some hypergraphs, we compare the two lower bounds presented in Theorems A and 1.3. Let n, k, r and a be positive integers, where $n \geq rk$, $n > a$ and $r \geq 2$. Define $\mathcal{H}(n,k,a)$ to be a hypergraph with vertex set $[n]$ and edge set

$$\left\{B \subseteq [n] : |B| = k \text{ and } B \not\subseteq [a]\right\}.$$

Let $\mathrm{KG}^r(n,k,a)$ denote the hypergraph $\mathrm{KG}^r(\mathcal{H}(n,k,a))$. It was proved [1, Proposition 7] that if either $a \leq 2k-2$ or $a \geq rk-1$, then $\chi\left(\mathrm{KG}^r(n,k,a)\right) = \left\lceil \frac{n-\max\{a,r(k-1)\}}{r-1} \right\rceil$. Indeed, for $a \geq rk-1$, it was proved that

$$\chi\left(\mathrm{KG}^r(\mathcal{H}(n,k,a))\right) = \left\lceil \frac{\mathrm{ecd}^r(\mathcal{H}(n,k,a))}{r-1} \right\rceil = \left\lceil \frac{n-a}{r-1} \right\rceil.$$

One should notice that the chromatic number of $\mathrm{KG}^r(\mathcal{H}(n,k,a))$ was left open for several values of a with $2k-1 \leq a \leq rk-2$. Note that Theorem 1.3 implies the validity of Zhu's conjecture for the family of hypergraphs $\mathrm{KG}^r(n,k,a)$ provided that $a \geq rk-1$. What is interesting about the hypergraph $\mathrm{KG}^r(\mathcal{H}(n,k,a))$ is the fact that for $r \geq 4$ and $a \geq rk-1$, the value of $\mathrm{ecd}^r(\mathcal{H}(n,k,a)) - (n - \mathrm{alt}^r(\mathcal{H}(n,k,a)))$ is unbounded. Thus, by the lower bound presented in Theorem A, we cannot derive that the family of hypergraphs $\mathrm{KG}^r(n,k,a)$ satisfies Zhu's conjecture. On the other hand, there is a family $\mathscr{H}$ of hypergraphs (see [1]) such that for $\mathcal{H} \in \mathscr{H}$, the value of $\big(n - \mathrm{alt}^r(\mathcal{H}(n,k,a))\big) - \mathrm{ecd}^r(\mathcal{H}(n,k,a)\big)$ is unbounded. Hence, Theorem A and Theorem 1.3 introduce two somehow complementary lower bounds.

2 Proofs

This section is devoted to the proofs of Theorem 1.2 and Theorem 1.3. In the first subsection, we define some necessary tools which will be needed in the rest of the paper. We assume basic knowledge in topological combinatorics. For more details, see [16].

2.1 Notations and tools

A *simplicial complex* is a pair (V,K) where V is a finite nonempty set and K is a family of nonempty subsets of V such that for each $A \in K$, if $\emptyset \neq B \subseteq A$, then $B \in K$. Respectively, the set V and the family K are called *vertex set* and *simplex set* of the simplicial complex (V,K). For simplicity of notation and since we can assume that $V = \cup_{A\in K} A$, with no ambiguity, we can point to a simplicial complex (V,K) just by its simplex set K. *The barycentric subdivision of* K, denoted by $\mathrm{sd}\, K$, is a simplicial complex whose vertices are the simplices of K and whose simplices are the chains of simplices of K ordered by inclusion.

Let V and W be two sets. We write $V \uplus W$ for the set $V \times \{1\} \cup W \times \{2\}$. Let K and L be two simplicial complexes with vertex sets V and W, respectively. We define $K * L$, *the join of* K *and* L, to be a simplicial complex with vertex set $V \uplus W$ and simplex set $\{A \uplus B : A \in K, B \in L\}$. The join operation is obviously associative: if K, L, M are simplicial complexes, then the simplicial complexes $K*(L*M)$ and $(K*L)*M$ are the same up to a natural relabeling of their vertices. This allows us, if we do not care about the names of the vertices, to use $K*L*M$ for both of $K*(L*M)$ and $(K*L)*M$. The n-fold join of K, denoted by K^{*n}, is a simplicial complex obtained by joining n copies of K. By relabeling the vertices of K^{*n}, we assume that K^{*n} has vertex set $V(K) \times [n]$ where for each vertex $(v,i) \in V(K) \times [n]$, the index i indicates that the vertex v is considered as a vertex of the ith copy of K.

For a prime number p, we also consider Z_p as a simplicial complex with vertex set Z_p and simplex set $\big\{\{\omega\},\{\omega^2\},\ldots,\{\omega^p\}\big\}$. Clearly Z_p^{*n} is a simplicial complex whose vertex set is $Z_p \times [n]$ and whose simplices are all nonempty subsets $A \subseteq Z_p \times [n]$ such that for each $i \in [n]$, the number of ε's for which $(\varepsilon,i) \in A$ is at most one. This observation implies that the simplex set of Z_p^{*n} can be identified with the set $(Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, i.e., for each simplex A in Z_p^{*n}, define $A \mapsto (x_1,\ldots,x_n)$ where $x_i = \varepsilon$ if $(\varepsilon,i) \in A$ and $x_i = 0$ otherwise. Also, the simplicial complex σ_{p-2}^{p-1} is a simplicial complex with vertex set Z_p and with simplex set consisting of all nonempty proper subsets of Z_p. Note that $\left(\sigma_{p-2}^{p-1}\right)^{*n}$ is a simplicial complex with vertex set $Z_p \times [n]$ and $\emptyset \neq \tau \subseteq Z_p \times [n]$ is a simplex of

$\left(\sigma_{p-2}^{p-1}\right)^{*n}$ if and only if $|\tau \cap (Z_p \times \{i\})| \leq p-1$ for each $i \in [n]$. It is clear that $\left(\sigma_{p-2}^{p-1}\right)^{*n}$ is a free simplicial complex where for each $\varepsilon \in Z_p$ and $(\varepsilon', i) \in Z_p \times [n]$, the action is defined by $\varepsilon \cdot (\varepsilon', i) = (\varepsilon \cdot \varepsilon', i)$. Let $\tau \in \left(\sigma_{p-2}^{p-1}\right)^{*n}$ be a simplex. For each $\varepsilon \in Z_p$, define $\tau^\varepsilon = \{(\varepsilon, j) : (\varepsilon, j) \in \tau\}$. Also, define

$$\ell(\tau) = p \cdot h(\tau) + |\{\varepsilon \in Z_p : |\tau^\varepsilon| > h(\tau)\}|,$$

where $h(\tau) = \min_{\varepsilon \in Z_p} |\tau^\varepsilon|$. As stated above, each $X \in (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ represents a simplex in $Z_p^{*n} \subseteq \left(\sigma_{p-2}^{p-1}\right)^{*n}$ and vice versa. Therefore, speaking about $h(X)$ and $\ell(X)$ is meaningful. Indeed, we have

$$h(X) = \min_{\varepsilon \in Z_p} |X^\varepsilon| \qquad \text{and} \qquad \ell(X) = p \cdot h(X) + |\{\varepsilon \in Z_p : |X^\varepsilon| > h(X)\}|.$$

Note that Z_p acts freely on $(Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ by the action $\varepsilon \cdot X = (\varepsilon \cdot x_1, \ldots, \varepsilon \cdot x_n)$, where $\varepsilon \cdot 0$ is defined to be 0 for each $\varepsilon \in Z_p$.

Now, we are ready to present the proof of Theorem 1.2. For simplicity, we first assume that $\eta = \min_{i \in [t]} \operatorname{ecd}^p(\mathcal{H}_i)$ and then, in Subsection 2.2.2, we sketch the proof for $\eta = \min_{i \in [t]}(|V(\mathcal{H}_i)| - \operatorname{alt}^p(\mathcal{H}_i))$. The proof will follow by applying Dold's theorem on a Z_p-equivariant simplicial map

$$\begin{array}{rccc} \lambda\colon & \operatorname{sd}(Z_p^{*n}) & \longrightarrow & Z_p^{*m} \\ & X & \longmapsto & (s(X), \nu(X)) \end{array}$$

with $n = \sum_{i=1}^t |V(\mathcal{H}_i)|$ and m as small as possible. Indeed, Dold's theorem implies that if there is such a map λ, then $m \geq n$. It is worth noting that the idea of using Dold's theorem or some of it specializations such as the Z_p-Tucker lemma has been used in several articles initiated by a fascinating paper of Matoušek [17]. For instance, see [1, 4, 6, 7, 12, 18, 19, 24]. Usually, the most challenging task in using Dold's theorem is how to define the map λ, especially the sign part $s(X)$. In what follows, we show that some of the techniques used in these works can be fruitfully mixed and extended to get a common generalization. However, some additional tricks are introduced to make these techniques work together. In particular, in our approach, we use a different way to define the sign map $s(X)$ and also we appropriately modify the value function $\nu(X)$. Being more specific, to define the map λ, we partition $\operatorname{sd}(Z_p^{*n}) = (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ into two subsets Σ_1 and Σ_2, where Σ_2 is the set of vectors $X \in \operatorname{sd}(Z_p^{*n})$ such that for each $j \in [t]$ and $\varepsilon \in Z_p$, the set $\left\{i \in [n_j] \colon x_{i+\sum_{j'=1}^{j-1} n_{j'}} = \varepsilon\right\}$ contains some edge of $\mathcal{H}_j = ([n_j], E_j)$, and hence X^ε somehow contains a vertex of the hypergraph $\operatorname{KG}^p(\mathcal{H}_1) \times \cdots \times \operatorname{KG}^p(\mathcal{H}_t)$. For each $X \in \Sigma_2$, we define $\nu(X) \in \{\alpha+1, \ldots, m\}$, where $\alpha = n-\eta+p-1$, according to a given proper coloring of $\operatorname{KG}^p(\mathcal{H}_1) \times \cdots \times \operatorname{KG}^p(\mathcal{H}_t)$ and we define $s(X) \in Z_p$ with the help of an auxiliary sign map $s_3(-)$. Defining $\lambda(X)$ for the remaining vectors X, i.e., $X \in \Sigma_1$, is even more difficult and technical which will be done by the use of two auxiliary sign maps $s_1(-)$ and $s_2(-)$. A larger value of η will allow us to make α smaller and consequently m smaller, giving a better bound in the end.

2.2 Proof of Theorem 1.2

When $\eta = 0$, there is nothing to prove. If $1 \leq \eta \leq p-1$, then consider pairwise disjoint sets $U_1, \ldots, U_p \subseteq V(\operatorname{KG}^p(\mathcal{H}_1) \times \cdots \times \operatorname{KG}^p(\mathcal{H}_t))$ such that $|U_i| = 1$ for $i \leq \eta$ and

$|U_i| = 0$ otherwise. Note that for at least one i, we have $U_i = \emptyset$. In view of the definitions, the subhypergraph $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)[U_1, \ldots, U_p]$ which has no edge is clearly balanced and p-partite. Furthermore, for any proper coloring of $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$, this subhypergraph is colorful which is desired. Henceforth, we assume that $\eta \geq p$.

For simplicity of notation, assume that $\mathcal{H}_1 = ([n_1], E_1), \ldots, \mathcal{H}_t = ([n_t], E_t)$ and moreover, set $n = \sum_{i=1}^t n_i$. For each $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, let $X(1) \in (Z_p \cup \{0\})^{n_1}$ be the first n_1 coordinates of X, $X(2) \in (Z_p \cup \{0\})^{n_2}$ be the next n_2 coordinates of X, and so on, up to $X(t) \in (Z_p \cup \{0\})^{n_t}$ be the last n_t coordinates of X. Also, for each $j \in [t]$, define $A_j(X)$ to be the set of signs $\varepsilon \in Z_p$ such that $X(j)^\varepsilon$ contains at least one edge of $\mathcal{H}_j$. We remind that $X(j)^\varepsilon$ is the set of all $i \in [n_j]$ such that $x_{i+\sum_{j'=1}^{j-1} n_{j'}} = \varepsilon$. Define

$$\Sigma_1 = \left\{X \in (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\} : A_j(X) \neq Z_p \text{ for at least one } j \in [t]\right\}$$

and

$$\Sigma_2 = \left\{X \in (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\} : A_j(X) = Z_p \text{ for all } j \in [t]\right\}.$$

Note that for an $X \in (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ and for each $j \in [t]$, if we set $S = \cup_{\varepsilon \in Z_p} X(j)^\varepsilon$, then $X(j) = \big(X(j)^\varepsilon\big)_{\varepsilon \in Z_p}$ can be thought of as a partition of vertices of $\mathcal{H}_j[S]$ into p color classes, i.e., the vertices in $X(j)^\varepsilon$ receive the color ε. Intuitively, the value $h(X(j))$ is then the size of the smallest color class, $\ell(X(j))$ is the maximum possible number of vertices colored by an equitable sub-coloring (not necessarily proper), while $A_j(X)$ is the set of colors $\varepsilon \in Z_p$ for which there is an ε-monochromatic edge in $\mathcal{H}_j[S]$.

2.2.1 Proof of Theorem 1.2 when $\eta = \min_{i \in [t]} \mathrm{ecd}^p(\mathcal{H}_i)$

In what follows, we first define two sign maps s_1 and s_2 playing important roles in the proof. These two maps will help us to define $s(X)$ for each $X \in \Sigma_1$.

Definition of $s_1(-)$. Let $X \in \Sigma_1$ be a vector such that $A_j(X) \in \{\emptyset, Z_p\}$ for each $j \in [t]$. Define

$$B_j(X) = \begin{cases} X(j) & \text{if } A_j(X) = Z_p, \\ \{\varepsilon : X(j)^\varepsilon \neq \emptyset\} & \text{if } A_j(X) = \emptyset \text{ and } h(X(j)) = 0, \\ \widetilde{X(j)} & \text{if } A_j(X) = \emptyset \text{ and } h(X(j)) > 0, \end{cases}$$

where $\widetilde{X(j)} \in (Z_p \cup \{0\})^{n_j} \setminus \{\mathbf{0}\}$ and for each $\varepsilon \in Z_p$, we have

$$\widetilde{X(j)}^\varepsilon = \begin{cases} X(j)^\varepsilon & \text{if } |X(j)^\varepsilon| = h(X(j)), \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $B_j(X)$ may be of two different natures: a vector in $(Z_p \cup \{0\})^{n_j} \setminus \{\mathbf{0}\}$ or a proper subset of Z_p. Now, set $\boldsymbol{B}(X) = \big(B_1(X), \ldots, B_t(X)\big)$ and

$$L_1 = \left\{\boldsymbol{B}(X) : X \in \Sigma_1 \text{ and } A_j(X) \in \{\emptyset, Z_p\} \text{ for all } j \in [t]\right\}.$$

Note that L_1 is a subset of

$$\Big(\big(Z_p \cup \{0\}\big)^{n_1} \cup \big(2^{Z_p} \setminus \{Z_p\}\big)\Big) \times \cdots \times \Big(\big(Z_p \cup \{0\}\big)^{n_t} \cup \big(2^{Z_p} \setminus \{Z_p\}\big)\Big) \setminus (\{\mathbf{0}, \emptyset\} \times \cdots \times \{\mathbf{0}, \emptyset\}).$$

For an $\varepsilon \in Z_p$ and a vector $\boldsymbol{B} = (B_1, \ldots, B_t) \in L_1$, we define

$$\varepsilon \cdot \boldsymbol{B} = (\varepsilon \cdot B_1, \ldots, \varepsilon \cdot B_t),$$

where

$$\varepsilon \cdot B_i = \begin{cases} (\varepsilon \cdot x_1, \ldots, \varepsilon \cdot x_{n_i}) & \text{if } B_i = (x_1, \ldots, x_{n_i}) \in (Z_p \cup \{0\})^{n_i} \setminus \{\mathbf{0}\}, \\ \{\varepsilon \cdot z : z \in B_i\} & \text{if } B_i \subsetneq Z_p. \end{cases}$$

With respect to this action, one can simply check that L_1 is closed and free and furthermore, $\boldsymbol{B}(-)$ is a Z_p-equivariant map, i.e., $\boldsymbol{B}(\varepsilon \cdot X) = \varepsilon \cdot \boldsymbol{B}(X)$ for each $\varepsilon \in Z_p$ and for each $X \in \Sigma_1$ such that $A_j(X) \in \{\emptyset, Z_p\}$ for each $j \in [t]$. Now, let $s_1 \colon L_1 \to Z_p$ be an arbitrary Z_p-equivariant map. Note that such a map can be defined by choosing one representative in each orbit and defining the value of the map arbitrarily on this representative.

Definition of $s_2(-)$. Clearly Z_p acts freely on

$$L_2 = 2^{Z_p} \times \cdots \times 2^{Z_p} \setminus (\{\emptyset, Z_p\} \times \cdots \times \{\emptyset, Z_p\})$$

by the action $\varepsilon \cdot (C_1, \ldots, C_t) = (\varepsilon \cdot C_1, \ldots, \varepsilon \cdot C_t)$, where $\varepsilon \cdot C_i = \{\varepsilon \cdot z : z \in C_i\}$. Similar to the definition of $s_1(-)$, let $s_2 \colon L_2 \to Z_p$ be an arbitrary Z_p-equivariant map.

Set $\alpha = n - \min_{i \in [t]} \operatorname{ecd}^p(\mathcal{H}_i) + p - 1$. Note that since $\min_{i \in [t]} \operatorname{ecd}^p(\mathcal{H}_i) \geq p$, we have $\alpha < n$. For every $j \in [t]$, define the function $\nu_j \colon (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\} \to \mathbb{N}$ as follows:

$$\nu_j(X) = \begin{cases} |X(j)| & \text{if } A_j(X) = Z_p, \\ |A_j(X)| + \max \Big\{ \ell(Z) : Z \subseteq X(j) \text{ and} \\ \qquad E(\mathcal{H}_j[Z^\varepsilon]) = \emptyset \text{ for all } \varepsilon \in Z_p \Big\} & \text{if } A_j(X) \neq Z_p. \end{cases}$$

We remind the reader that $|X(j)|$ denotes the number of nonzero coordinates in $X(j)$. Now, let $\nu(X) = \sum_{j=1}^{t} \nu_j(X)$.

Defining the map $\boldsymbol{\lambda_1}$. Set $\alpha = n - \min_{\varepsilon \in Z_p} \operatorname{ecd}^p(\mathcal{H}_i) + p - 1$. Define the map

$$\begin{array}{rccc} \lambda_1 : & \Sigma_1 & \longrightarrow & Z_p \times \{1, \ldots, \alpha\} \\ & X & \longmapsto & (s(X), \nu(X)). \end{array}$$

For defining $s(X)$, we consider the following different cases.

- If for each $j \in [t]$, we have $A_j(X) \in \{\emptyset, Z_p\}$, then $s(X) = s_1\big(\boldsymbol{B}(X)\big)$.
- If for some $j \in [t]$, we have $A_j(X) \notin \{\emptyset, Z_p\}$, then $s(X) = s_2\big(A_1(X), \ldots, A_t(X)\big)$.

Lemma 2.1. *The map λ_1 is a Z_p-equivariant map with no $X, Y \in \Sigma_1$ such that $X \subseteq Y$, $\nu(X) = \nu(Y)$ and $s(X) \neq s(Y)$.*

Proof. Clearly, λ_1 is a Z_p-equivariant map since the two maps $s_1(-)$ and $s_2(-)$ are Z_p-equivariant and $\nu(\varepsilon \cdot X) = \nu(X)$ for all $\varepsilon \in Z_p$. For a contradiction, suppose that X and Y are two vectors in Σ_1 such that $X \subseteq Y$, $\nu(X) = \nu(Y)$ and $s(X) \neq s(Y)$. Since $X \subseteq Y$, we have $X(j) \subseteq Y(j)$ and consequently, $A_j(X) \subseteq A_j(Y)$ for each $j \in [t]$. Additionally, $X(j) \subseteq Y(j)$ implies that

$$\big\{\ell(Z) : Z \subseteq X(j) \text{ and } E(\mathcal{H}_j[Z^\varepsilon]) = \emptyset \quad \forall \varepsilon \in Z_p\big\} \subseteq \\ \big\{\ell(Z) : Z \subseteq Y(j) \text{ and } E(\mathcal{H}_j[Z^\varepsilon]) = \emptyset \quad \forall \varepsilon \in Z_p\big\}.$$

Thus, $\nu_j(X) \leq \nu_j(Y)$ for each $j \in [t]$. Therefore, the equality $\nu(X) = \nu(Y)$ implies $\nu_j(X) = \nu_j(Y)$. This equality together with above discussion results in $A_j(X) = A_j(Y)$ for each $j \in [t]$. This observation leads us to the following cases.

(I) $A_j(X) \in \{\emptyset, Z_p\}$ for each j. Therefore, $s(X) = s_1\big(\boldsymbol{B}(X)\big)$. Since $A_j(X) = A_j(Y)$ for each j, we have $s(Y) = s_1\big(\boldsymbol{B}(Y)\big)$, Consequently, the fact that $s(X) \neq s(Y)$ implies that $\boldsymbol{B}(X) \neq \boldsymbol{B}(Y)$. Now, let j_0 be the smallest integer for which $B_{j_0}(X) \neq B_{j_0}(Y)$. We consider the following different cases.

(1) When $A_{j_0}(X) = A_{j_0}(Y) = Z_p$. In view of the definition of $B_{j_0}(-)$, we have $X(j_0) \subsetneq Y(j_0)$. Therefore, the definition of ν_{j_0} implies that $\nu_{j_0}(X) < \nu_{j_0}(Y)$, which is not possible.

(2) When $A_{j_0}(X) = A_{j_0}(Y) = \emptyset$. Using $\nu_{j_0}(X) = \nu_{j_0}(Y)$, we have $\ell(X(j_0)) = \ell(Y(j_0))$. Therefore,

$$p \cdot h(X(j_0)) + |\{\varepsilon : |X(j_0)^\varepsilon| > h(X(j_0))\}| = \\ p \cdot h(Y(j_0)) + |\{\varepsilon : |Y(j_0)^\varepsilon| > h(Y(j_0))\}|,$$

which clearly implies that $h(X(j_0)) = h(Y(j_0))$ and

$$|\{\varepsilon : |X(j_0)^\varepsilon| > h(X(j_0))\}| = |\{\varepsilon : |Y(j_0)^\varepsilon| > h(Y(j_0))\}|.$$

The fact that $X(j_0) \subseteq Y(j_0)$ results in

$$\{\varepsilon : |X(j_0)^\varepsilon| > h(X(j_0))\} = \{\varepsilon : |Y(j_0)^\varepsilon| > h(Y(j_0))\}.$$

Therefore, in view of the definition of $\boldsymbol{B}(-)$, we have $B_{j_0}(X) = B_{j_0}(Y)$ which is a contradiction.

(II) $A_j(X) \notin \{\emptyset, Z_p\}$ for some $j \in [t]$. Since $s(X) \neq s(Y)$, we have

$$s_2\big(A_1(X), \ldots, A_t(X)\big) \neq s_2\big(A_1(Y), \ldots, A_t(Y)\big).$$

Consequently, we must have $\big(A_1(X), \ldots, A_t(X)\big) \neq \big(A_1(Y), \ldots, A_t(Y)\big)$. Therefore, there is at least one j for which $A_j(X) \neq A_j(Y)$ which is not possible. □

In what follows, we will define some new notations needed in the rest of proof. Let c be a proper coloring of $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$ with color set $[C]$. For each $X \in \Sigma_2$ and each $\varepsilon \in Z_p$, define

$$E^\varepsilon(X) = \Big\{(e_1, \ldots, e_t) \in E_1 \times \cdots \times E_t : e_j \subseteq X(j)^\varepsilon \text{ for each } j \in [t]\Big\}.$$

Note that, in view of the definition of Σ_2, for each $\varepsilon \in Z_p$, we have $E^\varepsilon(X) \neq \emptyset$. Now, set τ_X to be defined as follows:

$$\tau_X = \Big\{(\varepsilon, c(u)) : \varepsilon \in Z_p \text{ and } u = (e_1, \ldots, e_t) \in E^\varepsilon(X)\Big\}.$$

Note that if we choose $u_\varepsilon \in E^\varepsilon(X)$ for each $\varepsilon \in Z_p$, then $\{u_\varepsilon : \varepsilon \in Z_p\}$ is an edge of $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$. Consequently, since c is a proper coloring of $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$, for each $i \in [C]$, there is at least one $\varepsilon \in Z_p$ for which $(\varepsilon, i) \notin \tau_X$. This observation indicates that τ_X is a simplex of $\left(\sigma_{p-2}^{p-1}\right)^{*C}$. Furthermore, since $E^\varepsilon(X) \neq \emptyset$ for each $\varepsilon \in Z_p$, we have $\ell(\tau_X) \geq p$.

Definition of $s_3(-)$. For a positive integer $b \in [C]$, let U_b be the set consisting of all simplices $\tau \in \left(\sigma_{p-2}^{p-1}\right)^{*C}$ such that $|\tau^\varepsilon| \in \{0, b\}$ for each $\varepsilon \in Z_p$. Define $U = \cup_{b=1}^{C} U_b$. Choose an arbitrary Z_p-equivariant map $s_3 \colon U \to Z_p$. Also, for each $\tau \in \left(\sigma_{p-2}^{p-1}\right)^{*C}$ with $h = h(\tau) = \min |\tau^\varepsilon|$, define

$$\widetilde{\tau} = \bigcup_{\varepsilon \,:\, |\tau^\varepsilon| = h} \tau^\varepsilon.$$

Note that $\widetilde{\tau}$ is a sub-simplex of τ which is in U. Therefore, $s_3(\widetilde{\tau})$ is defined.

Defining the map λ_2. Define the map

$$\begin{array}{rccl} \lambda_2\colon & \Sigma_2 & \longrightarrow & Z_p \times \left\{\alpha+1, \ldots, \alpha - p + 1 + \max_{X \in \Sigma_2} \ell(\tau_X)\right\} \\ & X & \longmapsto & (s(X), \nu(X)), \end{array}$$

where $s(X) = s_3(\widetilde{\tau_X})$ and $\nu(X) = \alpha - p + 1 + \ell(\tau_X)$.

Lemma 2.2. *The map λ_2 is a Z_p-equivariant map with no $X, Y \in \Sigma_1$ such that $X \subseteq Y$, $\nu(X) = \nu(Y)$ and $s(X) \neq s(Y)$.*

Proof. Obviously, λ_2 is a Z_p-equivariant map. Suppose for a contradiction that X and Y are two vectors in Σ_2 such that $X \subseteq Y$, $\nu(X) = \nu(Y)$ and $s(X) \neq s(Y)$. In view of the definition of λ_2, we must have $\ell(\tau_X) = \ell(\tau_Y)$. Using the definition of $\ell(-)$, it implies that $h(\tau_X) = h(\tau_Y)$. From the last equality and $\tau_X \subseteq \tau_Y$, we deduce that $\widetilde{\tau_X} = \widetilde{\tau_Y}$ and consequently, $s(X) = s_3(\widetilde{\tau_X}) = s_3(\widetilde{\tau_Y}) = s(Y)$, which is a contradiction. □

In the following lemma, we show that how the existence of an X with large $\ell(X)$ completes the proof.

Lemma 2.3. *If there is an $X \in \Sigma_2$ with $\ell(\tau_X) \geq q$, then $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$ contains a colorful, balanced, and complete p-partite subhypergraph with q vertices.*

Proof. Let $X \in \Sigma_2$ be a vector for which we have $\ell(\tau_X) \geq q$. Let $\tau \subseteq \tau_X$ be a sub-simplex such that $\ell(\tau) = |\tau| = q$. For each $i \in [p]$, set $S_i = \{j \in [C] : (\omega^i, j) \in \tau\}$. First note that $\lfloor \frac{q}{p} \rfloor \leq |S_i| \leq \lceil \frac{q}{p} \rceil$ for each $i \in [p]$. Moreover, it is clear that $\sum_{i=1}^{p} |S_i| = q$. For each $i \in [p]$ and $s \in S_i$, in view of the definitions of $\tau(X)$ and S_i, there is a vertex $u_{i,s} = (e_{i,1}^s, \ldots, e_{i,t}^s)$ of $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)$ such that $c(u_{i,s}) = s$ and $e_{i,j}^s \subseteq X(j)^{\omega^i}$ for each $j \in [t]$. Now, for $i \in [p]$, set $U_i = \{u_{i,s} : s \in S_i\}$. Clearly, $\mathrm{KG}^p(\mathcal{H}_1) \times \cdots \times \mathrm{KG}^p(\mathcal{H}_t)[U_1, \ldots, U_p]$ is the desired subhypergraph. □

Completing the proof of Theorem 1.2 when $\eta = \min_{i \in [t]} \mathrm{ecd}^p(\mathcal{H}_i)$. For completing the proof of Theorem 1.2, we need to use a generalization of the Borsuk-Ulam theorem by Dold, see [8, 16]. Indeed, Dold's theorem implies that if there is a Z_p-equivariant simplicial map from a simplicial Z_p-complex K_1 to a free simplicial Z_p-complex K_2, then the dimension of K_2 should be strictly larger than the connectivity of K_1.

For simplicity of notation, let

$$m = \alpha - p + 1 + \max_{X \in \Sigma_2} \ell(\tau(X)).$$

In view of Lemma 2.3, it suffices to show that

$$\max_{X\in\Sigma_2} \ell(\tau_X) \geq \min_{i\in[t]} \operatorname{ecd}^p(\mathcal{H}_i).$$

To this end, define $\lambda\colon (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\} \to Z_p \times [m]$ such that for each $X \in (Z_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, if $X \in \Sigma_1$, then $\lambda(X) = \lambda_1(X)$, otherwise $\lambda(X) = \lambda_2(X)$. In view of Lemma 2.1 and Lemma 2.2, $\lambda(-)$ is a Z_p-equivariant simplicial map from $\operatorname{sd}(Z_p^{*n})$ to Z_p^{*m}. Consequently, according to Dold's theorem, the dimension of Z_p^{*m} should be strictly larger than the connectivity of $\operatorname{sd}(Z_p^{*n})$, that is $m - 1 > n - 2$ as desired. □

2.2.2 Proof of Theorem 1.2 when $\eta = \min_{i\in[t]}(|V(\mathcal{H}_i)| - \operatorname{alt}^p(\mathcal{H}_i))$

In this subsection, we sketch the proof of Theorem 1.2 for the $\eta = \min_{i\in[t]}(|V(\mathcal{H}_i)| - \operatorname{alt}^p(\mathcal{H}_i))$ case. To this end, we need to slightly modify the proof of Theorem 1.2 in the case of $\eta = \min_{i\in[t]} \operatorname{ecd}^p(\mathcal{H}_i)$ as follows.

- Throughout Subsection 2.2.1, replace $\min_{i\in[t]} \operatorname{ecd}^p(\mathcal{H}_i)$ by $\min_{i\in[t]}(|V(\mathcal{H}_i)| - \operatorname{alt}^p(\mathcal{H}_i))$.
- Use $\operatorname{alt}(-)$ instead of function $\ell(-)$ to define each $\nu_j(X)$.
- For any X such that $A_j(X) \in \{\emptyset, Z_p\}$ for each $j \in [t]$, in the definition of $\lambda_1(X)$, set $s(X)$ to be the first nonzero entry of X.

With the same approach as in Subsection 2.2.1, it is straightforward to check that Lemmas 2.1, 2.2, and 2.3 are still valid with the preceding modifications. Therefore, again applying Dold's theorem leads us to the desired assertion.

2.3 Proof of Theorem 1.3

To prove Theorem 1.3, we reduce this theorem to the prime case of r which is known to be true by the discussion right after Theorem 1.2. One should notice that this reduction is a refinement of the well-known reduction originally due to Kříž [14], which has been used in some other papers as well, for instance see [3, 12, 24, 25]. In what follows, we use a similar approach as in [12].

Lemma 2.4. *Let r' and r'' be two positive integers. If Theorem 1.3 holds for both r' and r'', then it holds also for $r = r'r''$.*

For two positive integers s and C and a hypergraph $\mathcal{H}$, define a new hypergraph $\mathcal{T}_{\mathcal{H},C,s}$ as follows:

$$\begin{array}{rcl} V(\mathcal{T}_{\mathcal{H},C,s}) & = & V(\mathcal{H}) \\ E(\mathcal{T}_{\mathcal{H},C,s}) & = & \Big\{A \subseteq V(\mathcal{H}) : \operatorname{ecd}^s(\mathcal{H}[A]) > (s-1)C\Big\}. \end{array}$$

The following lemma can be proved with a similar approach as in [12, Lemma 3].

Lemma 2.5. *Let r and s be two positive integers. Then*

$$\operatorname{ecd}^{rs}(\mathcal{H}) \leq r(s-1)C + \operatorname{ecd}^r(\mathcal{T}_{\mathcal{H},C,s}).$$

Proof of Lemma 2.4. Using Lemma 2.5 instead of Lemma 3 in the proof of Lemma 1 in [12] leads us to the proof. □

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Coordinatizing n_3 configurations

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Abstract

Given an n_3 configuration, a one-point extension is a technique that constructs $(n+1)_3$ configurations from it. A configuration is *geometric* if it can be realized by a collection of points and straight lines in the plane. Given a geometric n_3 configuration with a planar coordinatization of its points and lines, a method is presented that uses a one-point extension to produce $(n+1)_3$ configurations from it, and then constructs geometric realizations of the $(n+1)_3$ configurations. It is shown that this can be done using only a homogeneous *cubic* polynomial in just *three* variables, *independent* of n. This transforms a computationally intractable problem into a computationally practical one.

Keywords: $(n, 3)$*-configuration, geometric configuration, anti-Pappian, rational coordinatization, elliptic curve.*

Math. Subj. Class.: 51E20, 51E30

1 Projective configurations

A *projective configuration* consists of a set Σ of points and lines, and an incidence relation Π, such that two lines intersect in at most one point. We denote this by (Σ, Π). For example, a triangle with points A, B, C and lines a, b, c can be represented by the pair $(\{A, B, C, a, b, c\}, \{Ab, Ac, Ba, Bc, Ca, Cb\})$. A configuration (Σ, Π) can also be viewed as a bipartite incidence graph of points versus lines. We will always assume that the incidence graph of a configuration is connected. Excellent references on configurations are the recent books by Grünbaum [10], and by Pisanski and Servatius [18].

An n_3-configuration is a projective configuration with n points and n lines such that every line is incident with 3 points, and every point is incident with 3 lines. There is a unique

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7_3-configuration, the Fano configuration, and a unique 8_3-configuration, the Möbius-Kantor configuration.

An n_3 configuration which can be represented by a collection of points and straight lines in the real or rational plane, such that all incidences are respected, and no two points or two lines coincide is termed a *geometric* n_3 configuration. In order to show that an n_3 configuration is geometric, the usual method is to assign suitable homogeneous coordinates to its points and lines. We call this a *coordinatization* of the configuration. A central problem [10] is to characterize which n_3 configurations are geometric configurations, and to find rational coordinatizations [4, 10, 21, 22, 23] of those that are geometric. Grünbaum [9], and [10] (p. 151) has conjectured that an (n_3) configuration that admits a real coordinatization also admits a rational coordinatization. He considers this the single most important outstanding problem in n_3 configurations [11]. Sturmfels and White [22, 23] have shown that all (11_3) and (12_3) configurations have rational coordinatizations. These configurations were originally discovered by Martinetti [17], and Daublebsky von Sterneck [6, 7]. Sturmfels and White and Bokowski [4, 22, 23] found rational coordinatizations by constructing systems of diophantine equations, and then using methods of computer algebra to solve them, in particular, Grassmannian algebras and Gröbner bases.

A coordinatization of an n_3 configuration is usually represented by homogeneous coordinates in the plane, e.g., let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be the homogeneous coordinates of two points, and let $L = (a, b, c)$ be the homogeneous coordinates of a line. Then P_1 and P_2 are incident with L if and only if $P_1 \cdot L = P_2 \cdot L = 0$. Equivalently, L is a multiple of $P_1 \times P_2$. Consequently there is an exterior algebra that the homogeneous coordinates generate. If there are n points and n lines, with $3n$ incidences, there are $6n$ variables, and numerous algebraic constraints that the coordinates must satisfy. Bokowski and Sturmfels [4] used computer-aided algebra to search for rational solutions to these algebraic constraints. Eventually the constraints can be manipulated to produce a homogeneous polynomial with at most $3n$ variables whose zeros characterize the coordinatizations. The polynomial has degree bounded by n. The difficulty of this work led Sturmfels and White [23] to suggest that the problem of finding rational coordinatizations of n_3 configurations may be *recursively undecidable*.

A simpler method of finding a coordinatizing polynomial, without the need of Gröbner bases and the exterior algebra, was presented in Kocay-Szypowski [15]. The degree of the polynomial is still bounded by n. This method was used in Kocay [13] to find a rational coordinatization of the Georges configuration, which is a (25_3) configuration. In Sturmfels and White [22, 23], ad-hoc methods were used to find rational roots of the coordinatizing polynomials for each of the (11_3) and (12_3) configurations. There are 31 (11_3) and 229 (12_3) configurations.

A note on homogeneous polynomials and their zeros: Homogeneous *quadratic* polynomials are well understood, see Conway [5]. It is the theory of quadratic forms. Cubic homogeneous polynomials are much more difficult. When there are three variables, they include the class of *elliptic curves* [20]. The rational points on an elliptic curve form a group. If there are one or more known rational points on the curve, then others can be found by combining them using the group operation. This generates a countable number of points. Mordell's theorem says that these groups are finitely generated, i.e., a finite number of starting points is needed to find the entire group. It does not say what the group is, or whether there are *any* rational points on the curve. And it does not provide a method to determine if there are any rational points on the curve. Because it is relatively easy to do computation in these groups, but simultaneously, there are theoretical difficulties in characterizing them, these groups are used in elliptic curve cryptographic systems [20]. Homogeneous polynomials of degree four or more are much more difficult, apparently not amenable to the same techniques. Thus the degree of the polynomial is important.

The purpose of this paper is to present an algorithm which can be used to construct real or rational coordinatizations of $(n+1)_3$-configurations from coordinatizations of n_3-configurations, by finding the roots (real or rational) of a *cubic homogeneous polynomial in three variables*. The use of a cubic homogeneous polynomial in three variables makes the formerly intractable problem of finding rational coordinatizations computationally practical and efficient. Some of the techniques are similar to methods used in the theory of elliptic curves [3, 20].

An elliptic curve is a cubic polynomial that can be expressed in the form

$$y^2 = ax^3 + bx^2 + cx + d$$

The rational points on an elliptic curve form a group. See [20] for further information on these groups.

Theorem 1.1 (Mordell's theorem)**.** *If a non-singular plane cubic curve has a rational point, then the group of rational points is finitely generated.*

Methods that originated with Diophantus [1] are used to find the rational roots of elliptic curves [20]. We use similar methods to construct coordinatizations of n_3-configurations. As there can be very many rational points on an elliptic curve, there can be also be very many different rational coordinatizations of an n_3 configuration. They are related in a way that is similar to the group operation of an elliptic curve. In general, it seems to be difficult to characterize when a rational coordinatization is possible. However the method presented here is very fast in practice, and can be automated.

We begin with a *1-point extension* [14] in an n_3 configuration, which extends it to an $(n+1)_3$ configuration, and which leads to the coordinatization algorithm. This extension is different from Martinetti's extension [17], which is described in Grünbaum [10] (p. 89). As pointed out in [10], it is in general quite difficult to characterize exactly which configurations are generated by an inductive construction which produces an $(n+1)_3$ configuration from an n_3 configuration. This is true even if the construction can easily be described. In [14] the configurations that can be built using a 1-point extension are characterized.

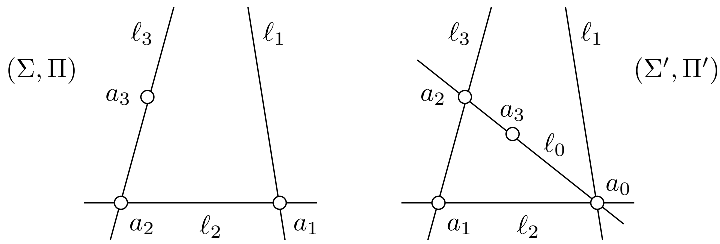
Theorem 1.2 (1-Point Extension)**.** *Let (Σ, Π) be an n_3-configuration. Let a_1, a_2, a_3 be 3 distinct points in Σ, and let ℓ_1, ℓ_2, ℓ_3 be 3 distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$ and $a_3 \in \ell_3$, where $a_3 \notin \ell_1$. We can represent this in tabular form as*

(Σ, Π)	ℓ_1	ℓ_2	ℓ_3	$\cdots$
	a_1	a_1	a_2	$\cdots$
	b_1	a_2	a_3	$\cdots$
	b_2	b_3	b_4	$\cdots$

where the dots indicate other points of the configuration. Here the points in each column are incident with the line at the top of the column. Let ℓ' be the third line containing a_1. Suppose further that if $\ell' \cap \ell_3 \neq \emptyset$, then $\ell' \cap \ell_3 = a_3$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. Define the new incidences as $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3\} \cup \{a_1\ell_3, a_2\ell_0, a_3\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

(Σ', Π')	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\cdots$
	a_2	a_0	a_1	a_1	$\cdots$
	a_3	b_1	a_0	a_2	$\cdots$
	a_0	b_2	b_3	b_4	$\cdots$

Here the dots represent exactly the same points as in the previous table. Then (Σ', Π') is an $(n+1)_3$-configuration. (Refer to Figure 1.)



(a)
initial points and lines

(b)
after 1-point extension

Figure 1: A 1-point extension with 3 points, before (a), after (b).

Example. The Fano configuration can be represented by the following table.

Fano	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	1	2	3	4	5	6	7
	2	3	4	5	6	7	1
	4	5	6	7	1	2	3

Choose ℓ_1, ℓ_2, ℓ_3 as indicated, and choose $a_1 = 2$, $a_2 = 3$, $a_3 = 6$, and let $a_0 = 8$. Notice that the third line containing a_1 is $\ell' = \ell_6$, which intersects ℓ_3 in $a_3 = 6$. Then by Theorem 1.2, the following table represents an 8_3-configuration, which is known to be unique.

8_3-config	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	3	1	2	2	4	5	6	7
	6	4	5	3	5	6	7	1
	8	8	8	4	7	1	2	3

The diagram of Figure 1 illustrates the 1-point extension schematically, showing the incidences altered by the extension. The method uses three points a_1, a_2, a_3 and three lines ℓ_1, ℓ_2, ℓ_3 sequentially incident, with a new point a_0 and line ℓ_0 added. It can be generalized to m points $a_1, a_2, \ldots, a_m$ and m lines $\ell_1, \ell_2, \ldots, \ell_m$ sequentially incident, see Kocay [14] for more details. This is indicated in Figure 2 for $m = 4$. When $m = 4$, the 1-point extension theorem has the following abridged form.

Theorem 1.3 (1-Point Extension with 4 points and 4 lines)**.** *Let (Σ, Π) be an n_3-configuration. Let a_1, a_2, a_3, a_4 be 4 distinct points in Σ, and let $\ell_1, \ell_2, \ell_3, \ell_4$ be 4 distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$, $a_3 = \ell_3 \cap \ell_4$, and $a_4 \in \ell_4$, where $a_3, a_4 \notin \ell_1, \ell_2$, and $a_1 \notin \ell_4$.*

Let ℓ_1' be the third line containing a_1, and ℓ_2' be the third line containing a_2. Suppose further that if $\ell_1' \cap \ell_3 \neq \emptyset$, then $\ell_1' \cap \ell_3 = a_3$; and if $\ell_2' \cap \ell_4 \neq \emptyset$, then $\ell_2' \cap \ell_4 = a_4$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. Define the new incidences as $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3, a_4\ell_4\} \cup \{a_1\ell_3, a_2\ell_4, a_3\ell_0, a_4\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$.

Then (Σ', Π') is an $(n+1)_3$-configuration. (Refer to Figure 2.)

When one point extensions are generated by computer, it is necessary to name them, so that the extensions generated can be identified. We have used the following naming convention. Here a configuration (Σ, Π) is assumed, but is not explicitly indicated in the notation, as this will be clear from the context.

Definition 1.4. A 1-point extension using three lines ℓ_1, ℓ_2, ℓ_3 and three points a_1, a_2, a_3 is denoted $\mathrm{Ext}(\ell_1, \ell_2, \ell_3; a_1, a_2, a_3)$. A 1-point extension using four lines $\ell_1, \ell_2, \ell_3, \ell_4$ and four points a_1, a_2, a_3, a_4 is denoted $\mathrm{Ext}(\ell_1, \ell_2, \ell_3, \ell_4; a_1, a_2, a_3, a_4)$, and so forth.

When the starting n_3 configuration has a real or rational coordinatization, we want to use its coordinatization to find a real or rational coordinatization of the resulting $(n+1)_3$ configuration. Both Theorems 1.2 and 1.3 are needed for the extension algorithm.

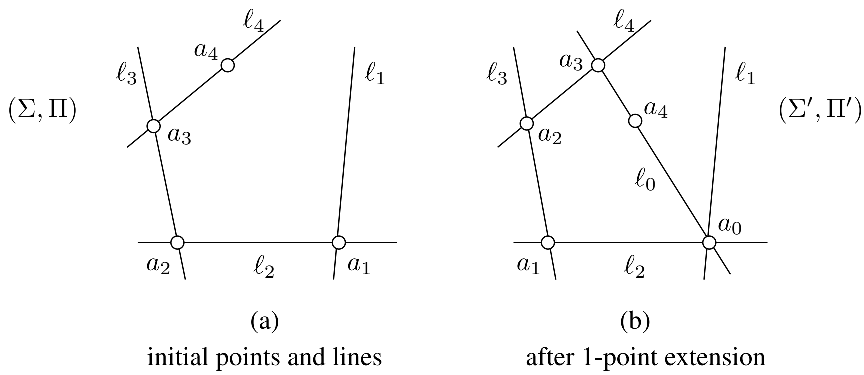


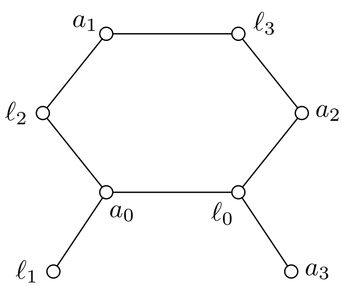
Figure 2: A 1-point extension with 4 points, before (a), after (b).

2 The coordinatization algorithm

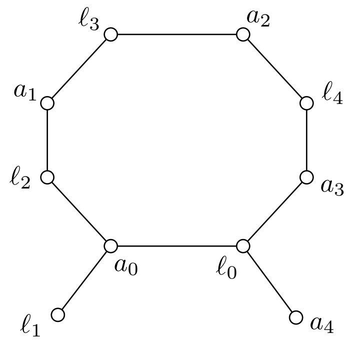
Let the points of a geometric n_3 configuration (Σ, Π) be $\{a_1, a_2, \ldots, a_n\}$ and let the lines be $\{\ell_1, \ell_2, \ldots, \ell_n\}$. Let the homogeneous coordinates of a_i be P_i, and the homogeneous coordinates of ℓ_i be L_i. These can be either real or rational. Then point a_i is incident on line ℓ_j if and only if $P_i \cdot L_j = 0$. Suppose that a 1-point extension is applied to (Σ, Π) to obtain an $(n+1)_3$ configuration (Σ', Π'), using three points and lines of (Σ, Π), as in Figure 1. We can assume that the points and lines are labelled so that the extension uses points a_1, a_2, a_3 and lines ℓ_1, ℓ_2, ℓ_3 as in Figure 1, and adds a_0 and ℓ_0.

Let G denote the incidence graph, also known as the *Levi* graph, of (Σ, Π). The subgraph induced by $\{a_1, a_2, a_3, \ell_1, \ell_2, \ell_3\}$ is a path of length five, since $a_3 \notin \ell_1$, and because the girth of the incidence graph must be at least six. After the extension, a_0 and ℓ_0 are added. Let G' be the new incidence graph. The subgraph now induced is illustrated in

Figure 3(a), since the girth of the incidence graph must be at least six. The significant feature of this subgraph is the hexagon induced by $\{a_0, a_1, a_2, \ell_0, \ell_2, \ell_3\}$. We now look for a *shortest* path Q in the incidence graph, not using any edges of the hexagon, from any one of $\{a_0, a_1, a_2\}$ to any one of $\{\ell_0, \ell_2, \ell_3\}$. This is easy to do using a breadth-first search of the incidence graph. Note that the shortest path may possibly contain a_3 and/or ℓ_1. Q must contain at least two internal vertices, i.e., one point and one line. Let the endpoints of Q be a_i and ℓ_j. If u is an internal vertex of Q, then u is not incident with the other vertices ℓ_k on the hexagon (where $k \neq j$), or there would either be a shorter path than Q, or else the girth requirement would not be satisfied. Similarly, u is not incident with the other vertices a_m on the hexagon (where $m \neq i$).



induced subgraph for Figure 1 (b)



induced subgraph for Figure 2 (b)

Figure 3: An induced subgraph of the incidence graph of (Σ', Π') of Figures 1 and 2.

We now have a *theta subgraph* in the incidence graph, that is, two vertices (a_i and ℓ_j), connected by three internally disjoint paths. When $m = 4$, the situation is similar. The vertices $a_1, a_2, a_3, a_4, \ell_1, \ell_2, \ell_3, \ell_4$ of Figure 2(b) determine a path of length 7 in the incidence graph G. After the extension, the subgraph of G' determined by Figure 2(b) is illustrated in Figure 3(b). It is necessary that this be an *induced* subgraph for the coordinatization algorithm. We now look for a *shortest* path Q in the incidence graph, not using any edges of the octagon, from any one of $\{a_0, a_1, a_2, a_3\}$ to any one of $\{\ell_0, \ell_2, \ell_3, \ell_4\}$. Let the endpoints of Q be a_i and ℓ_j. Once again we find that Q must contain at least two internal vertices, and again we have a theta-subgraph, Θ. The algorithm requires that this be an *induced* theta subgraph. The incidence graph is 3-regular, so that vertices a_i and ℓ_j are adjacent only to vertices of Θ. All other vertices of Θ are adjacent to *exactly one* vertex not in Θ. We now look for a coordinatization of (Σ', Π') such that all points and lines have the *same* coordinates as in (Σ, Π), *except* for the points and lines of Θ.

Let the homogeneous coordinates of a_i be (x, y, z), where x, y, z are real or rational indeterminates, according to whether the coordinatization of (Σ, Π) is real or rational. Then Θ contains three internally disjoint paths Q_1, Q_2, Q_3 from a_i to ℓ_j. We follow each path, and execute the following statements, assigning coordinates to its vertices in terms of x, y, z. For each vertex not in Θ, its homogeneous coordinates are those of (Σ, Π). These are known constants. The algorithm below constructs coordinates for the vertices of Θ in terms of x, y, z, by starting at a_i, and successively following each path Q_m of Θ to ℓ_j. Note that if L and L' are homogeneous coordinates of lines, then the cross product $L \times L'$ gives the homogeneous coordinates of the unique point which is the intersection of the two lines. Similarly $P \times P'$ gives the homogeneous coordinates of the unique line containing points with coordinates P and P'.

procedure FOLLOWPATH(a_i, ℓ_j, Q_m)
comment: follow a path Q_m of Θ from a_i to ℓ_j, assigning coordinates

$u \leftarrow a_i$
$v \leftarrow$ first vertex on path Q_m after a_i
while $v \neq \ell_j$
do
- **if** v is a point
  - **then**
    - let ℓ be the unique adjacent line not in Θ
    - let L be the known coordinates of ℓ
    - let L' be the assigned coordinates of u
    - $P \leftarrow L \times L'$
    - assign P as the coordinates of v
  - **else**
    - let a be the unique adjacent point not in Θ
    - let P be the known coordinates of a
    - let P' be the assigned coordinates of u
    - $L \leftarrow P \times P'$
    - assign L as the coordinates of v
- $u \leftarrow v$
- $v \leftarrow$ next vertex on path Q_m after u

comment: every vertex of Q_m except for ℓ_j now has coordinates assigned

Observation. *Once the algorithm* FOLLOWPATH() *has been executed for each path of Θ, all vertices of Θ except for ℓ_j have homogeneous coordinates assigned such that each coordinate is a* linear homogeneous function *of x, y, z.*

There are three vertices of Θ adjacent to ℓ_j. Let their coordinates be P, P' and P''. Define the polynomial $p(x, y, z) = P \cdot P' \times P''$.

Observation. $p(x, y, z)$ *is a* cubic homogeneous polynomial *in x, y, z.*

Note that by projective duality we could equally well follow the paths in the other direction, from ℓ_j to a_i, starting with (x, y, z) as the coordinates of ℓ_j.

Theorem 2.1. *If there is a coordinatization of (Σ', Π') such that all points and lines not in Θ have the same coordinates as in (Σ, Π), then the values of x, y, z must satisfy $p(x, y, z) = 0$.*

Proof. The three points incident on ℓ_j all belong to Θ, with coordinates P, P', P''. Therefore $P \cdot P' \times P'' = p(x, y, z) = 0$. Note that the coordinates of ℓ_j can be taken as any one of $P \times P'$, $P \times P''$ or $P' \times P''$. $\square$

Thus, if there is a coordinatization of (Σ', Π') of the type we are looking for, we can find it by solving $p(x, y, z) = 0$ for x, y, z. In general, there will be many values (x, y, z) with $p(x, y, z) = 0$. They do not all give valid coordinatizations. According to the current coordinatization of (Σ, Π), we want the values to be either real or rational. We will use a method that originated with Diophantus (see [1]), as frequently used in the theory of elliptic curves [3, 20]. Now the groups defined by elliptic curves are used for cryptography, because it is relatively easy to calculate with them, but a characterization of the groups appears to be algorithmically intractable. A similar situation exists in the search for coordinatizations of n_3 configurations. But if we can find suitable values of x, y, z such that $p(x, y, z) = 0$, then a real or rational coordinatization of (Σ', Π') can be relatively easy to find. The method described below works very effectively.

Lemma 2.2. *Let ℓ be any one of the three lines adjacent to a_i in Θ, and let its coordinates be L. Let a be the unique point not in Θ adjacent to ℓ, and let its coordinates be P. If (x, y, z) is set equal to P, then $p(x, y, z) = 0$.*

Proof. If $(x, y, z) = P$, then $L = P \times (x, y, z) = (0, 0, 0)$. Each subsequent vertex on this path in Θ will have coordinates $(0, 0, 0)$, so that ℓ_j will also have coordinates $(0, 0, 0)$. Therefore $p(x, y, z) = 0$. □

As there are three lines in Θ adjacent to a_i, this gives three different points (x, y, z) with $p(x, y, z) = 0$. None of these give coordinatizations of (Σ', Π'), because $(0, 0, 0)$ is not a valid homogeneous coordinate. However, we can now proceed as follows.

Suppose that $p(x, y, z) = 0$, for some value $(x, y, z) = (u, v, w)$. The equation $p(x, y, z) = 0$ defines a *cubic* curve in the projective plane. The tangent line at point (u, v, w) has the equation $x\partial p/\partial x + y\partial p/\partial y + z\partial p/\partial z = 0$, where the partial derivatives are evaluated at (u, v, w). This is a *linear* equation in (x, y, z). As long as at least one partial derivative is non-zero, say $\partial p/\partial z$, we can solve for the associated variable, and obtain $z = -[x\partial p/\partial x + y\partial p/\partial y]/[\partial p/\partial z]$ along the tangent line. This is substituted into the cubic homogeneous polynomial $p(x, y, z) = 0$ to obtain $q(x, y) = 0$, where $q(x, y)$ is a cubic homogeneous polynomial in x, y. At this point, we can divide by y^3 to obtain the cubic polynomial $q(x/y, 1) = 0$ in one variable x/y. Now $q(x/y, 1) = 0$ has three roots, of which one, $x/y = u/v$, is already known (note: if $v = 0$, use $y/x = v/u$ instead). The tangent line has *double contact* (see [3]) with the curve $p(x, y, z) = 0$ at $(x, y, z) = (u, v, w)$. Therefore we can divide $q(x, y)$ by $vx - uy$ *twice* to obtain a *linear* homogenous equation $h(x, y) = 0$. The single root of $h(x, y)$ is then easy to find, even over the rational numbers. Combining this with the expression for z, we obtain another root $(x, y, z) = (u', v', w')$ of $p(x, y, z) = 0$.

This new value for (x, y, z) is now substituted into the coordinates for the vertices of Θ, and the coordinates (which are linear homogeneous functions of x, y, z) of all vertices of Θ are evaluated. It is then quickly determined whether this produces a valid coordinatization of (Σ', Π'). The conditions that must be satisfied are:

1. All points must have inequivalent homogeneous coordinates;
2. All lines must have inequivalent homogeneous coordinates;
3. $P \cdot L = 0$ only if point P is incident with line L.

If some points or lines coincide, or if unwanted incidences are produced, then the method can be repeated, starting from $(x, y, z) = (u', v', w')$. Either a new point (u'', v'', w'') will be found, or else a value previously found will recur, and so forth.

This can be done for each of the three lines in Θ adjacent to a_i, which frequently produces a number of valid coordinatizations of (Σ', Π').

There is still another possibility. The coordinates of any two of the three lines in Θ adjacent to a_i determine a line in the projective plane, intersecting the curve $p(x, y, z) = 0$

in two known points. The third point of intersection is then easy to find. This calculation allows a sequence of points satisfying $p(x, y, z) = 0$ to be found. We can then continue with the tangents from these points, or take any two known roots on the curve to find another. The number of points on the curve that can be generated from the starting values can be either finite or countably infinite, as this is the situation that holds for rational points on elliptic curves (see [20]).

We summarize this method as two theorems.

Theorem 2.3. *Let $(x, y, z) = (u, v, w)$ be a rational solution of the cubic homogeneous polynomial $p(x, y, z) = 0$. If at least one of $\partial p/\partial x, \partial p/\partial y, \partial p/\partial z$ evaluated at $(x, y, z) = (u, v, w)$ is non-zero, then the tangent line $x\partial p/\partial x + y\partial p/\partial y + z\partial p/\partial z = 0$ intersects the curve in another rational point.*

Theorem 2.4. *Let $(x, y, z) = (u_1, v_1, w_1)$ and $(x, y, z) = (u_2, v_2, w_2)$ be two rational solutions of the cubic homogeneous polynomial $p(x, y, z) = 0$. Then the line containing (u_1, v_1, w_1) and (u_2, v_2, w_2) intersects the curve in another rational point.*

In practice, we want at least two of the partial derivatives to be non-zero at $(x, y, z) = (u, v, w)$. For if two of them are zero, then solving for the third variable forces one of x, y, z to be zero. This invariably leads to a solution which does not give a valid coordinatization. (However, it can then be used to find another rational solution.)

Once a valid coordinatization of (Σ', Π') has been found for a suitable value $(x, y, z) = (u, v, w)$, this process can be repeated, and more coordinatizations can be found. In general, numerous coordinatizations for a given configuration can be found in this way. They are inter-related through tangents to the cubic polynomial $p(x, y, z)$, and through lines containing pairs of rational solutions, similar to the relation between points of the group of rational points on an elliptic curve.

Example. We begin with a rational coordinatization of a (9_3) configuration, shown in Figure 4. This is the (9_3) configuration listed as $(9_3)_2$ in Figure 2.2.1 of [10], and as 9.2 in [2]. It is cyclic and self-dual, with an automorphism group of order 9. The two "parallel" lines ℓ_4 and ℓ_8 meet in point a_9 at infinity. Similarly ℓ_5 and ℓ_7 meet in a_8 at infinity, and lines ℓ_1 and ℓ_3 meet in a_2 at infinity. These three points at infinity are all contained in the line ℓ_6, which is the "line at infinity". The drawing is based on the rational coordinatization of the configuration given by the coordinates shown in Table 1.

Table 1: Rational coordinates of the 9_3 configuration of Figure 4.

$P_1 = (2, 4, -3)$	$L_1 = (1, 1, 2)$
$P_2 = (-1, 1, 0)$	$L_2 = (2, -1, 0)$
$P_3 = (1, 2, -3)$	$L_3 = (1, 1, 1)$
$P_4 = (1, 1, -1)$	$L_4 = (0, 3, 2)$
$P_5 = (0, 0, 1)$	$L_5 = (3, 0, 2)$
$P_6 = (1, 0, -1)$	$L_6 = (0, 0, 1)$
$P_7 = (2, 2, 3)$	$L_7 = (1, 0, 1)$
$P_8 = (0, 1, 0)$	$L_8 = (0, 1, 0)$
$P_9 = (1, 0, 0)$	$L_9 = (1, -1, 0)$

A 1-point extension using four points $\mathrm{Ext}(\ell_1, \ell_9, \ell_4, \ell_6; a_4, a_7, a_9, a_8)$, as in Figure 2, is then done. (This example using 4 points was chosen instead of one using 3 points,

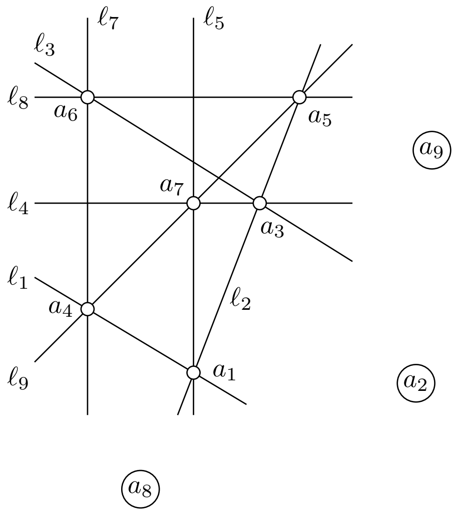


Figure 4: A 9_3 configuration.

because the resulting (10_3) configuration has a "nice" drawing.) Observe that $a_4 = \ell_1 \cap \ell_9$, $a_7 = \ell_9 \cap \ell_4$, $a_9 = \ell_4 \cap \ell_6$, and that $a_8 \in \ell_6$. The third line through a_4 is ℓ_7. It intersects ℓ_6 in a_8, as required for the 1-point extension. The result of the extension is the 10_3 configuration shown in Figure 5. It is $(10_3)_6$ in Grünbaum [10]. Lines ℓ_1, ℓ_3 and ℓ_6 in Figure 5 meet in point a_2 at infinity. Points and lines whose coordinates did not change from (9_3) are drawn in heavier lines. (But note that the scaling of the two diagrams may be slightly different.)

In order to find a rational coordinatization of it, we first find a theta subgraph by searching for a shortest path from one of a_4, a_7, a_9, a_{10} to one of $\ell_9, \ell_4, \ell_6, \ell_{10}$, where a_{10} and ℓ_{10} are the new point and line that were added. The theta subgraph is shown in Figure 6. It consists of the octagon of Figure 3(b) and the shortest path just found. This is partly indicated in Figure 5. The "corners" of the theta subgraph, a_4 and ℓ_{10}, are shaded light grey. With the aid of Figure 6, the paths can be traced out in Figure 5.

We now assign homogeneous coordinates (x, y, z) to ℓ_{10}, as it is one of the "corner" vertices of the theta subgraph, and using the coordinates of Table 1 for the points and lines not in the theta subgraph, we calculate coordinates for those of the theta subgraph in terms of (x, y, z). Each point or line of the theta subgraph (except for the "corner" vertices) is adjacent to exactly one line or point not in the theta subgraph. The adjacent vertices can be determined from Figure 5. The calculated homogeneous coordinates are linear homogeneous forms, shown in Table 2. Note that homogeneous coordinates can be multiplied by a constant without changing the configuration. Therefore sometimes the coordinates in Table 2. were multiplied by -1, or a common factor was removed from the individual coordinates in order to simplify them. We then find that $p(x, y, z) = L_9 \cdot L_7 \times L_4$, which is expanded to

$$p(x, y, z) = -4x^3 + 4x^2y + 4x^2z + 7xz^2 - 16xyz + 11yz^2 - 6z^3$$

where a common factor of six has been removed from each term. The partial derivatives

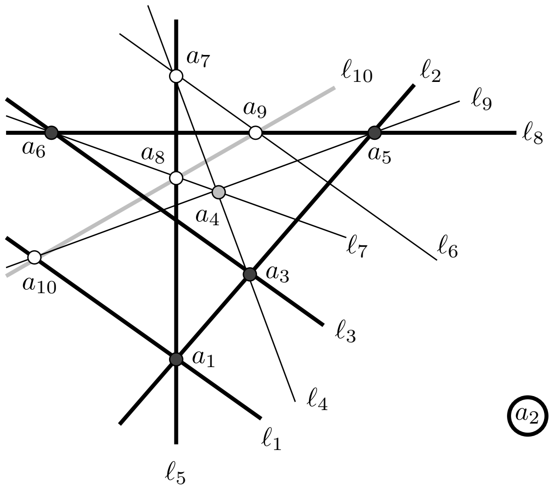


Figure 5: The extended 10_3 configuration.

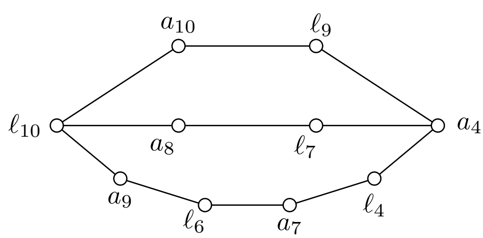


Figure 6: A theta subgraph in the 10_3 configuration.

are

$$\partial p/\partial x = -12x^2 + 8xy + 8xz - 16yz + 7z^2$$
$$\partial p/\partial y = 4x^2 - 16xz + 11z^2$$
$$\partial p/\partial z = 4x^2 - 16xy + 14xz + 22yz - 18z^2$$

We have three known solutions to $p(x, y, z) = 0$, namely

$$(x, y, z) = L_1 = (1, 1, 2) \text{ (which makes } P_{10} = (0, 0, 0)\text{)},$$
$$(x, y, z) = L_5 = (3, 0, 2) \text{ (which makes } P_8 = (0, 0, 0)\text{)},$$
$$(x, y, z) = L_6 = (0, 0, 1) \text{ (which makes } P_9 = (0, 0, 0)\text{)}.$$

The tangent line at $(x, y, z) = L_1 = (1, 1, 2)$ has equation $2x + 4y - 3z = 0$. Solving for $2x = -4y + 3z$, substituting this into $p(x, y, z) = 0$, and removing common factors gives

$$q(y, z) = 4y^3 - 4y^2z + yz^2$$

The point L_1 on the tangent line has $(y, z) = (1, 2)$ so that $q(y, z)$ is divisible twice by $2y - z$. We find that

$$q(y, z) = 6y(2y - z)^2$$

Therefore the third point of intersection of the tangent with $p(x, y, z) = 0$ occurs when $y = 0$. Then since $2x + 4y - 3z = 0$, we can take $z = 2$, and obtain $2x = -4y + 3z = 6$, giving $(x, y, z) = (3, 0, 2)$. This does not give a valid solution, as it makes $P_8 = (0, 0, 0)$.

Table 2: Homogeneous coordinates for the theta subgraph.

L_{10}	$=$	(x, y, z)		
P_{10}	$=$	$L_{10} \times L_1$	$=$	$(2y - z, z - 2x, x - y)$
P_8	$=$	$L_{10} \times L_5$	$=$	$(2y, 3z - 2x, -3y)$
P_9	$=$	$L_{10} \times L_8$	$=$	$(-z, 0, x)$
L_9	$=$	$P_{10} \times P_5$	$=$	$(z - 2x, z - 2y, 0)$
L_7	$=$	$P_8 \times P_6$	$=$	$(2x - 3z, -y, 2x - 3z)$
L_6	$=$	$P_9 \times P_2$	$=$	(x, x, z)
P_7	$=$	$L_6 \times L_5$	$=$	$(-2x, 2x - 3z, 3x)$
L_4	$=$	$P_7 \times P_3$	$=$	$(4x - 3z, x, 2x - z)$

We then try the tangent line at $(x, y, z) = L_5 = (3, 0, 2)$, which has equation $2x + y - 3z = 0$. Solve for $y = 3z - 2x$ and substitute this into $p(x, y, z)$ to obtain

$$q(x, z) = 4x^3 - 16x^2z + 21xz^2 - 9z^3$$

The known solution is $(x, z) = (3, 2)$, so that this is divisible twice by $2x - 3z$, giving

$$q(x, z) = (x - z)(2x - 3z)^2$$

We find that the third intersection point with the curve $p(x, y, z) = 0$ occurs when $x = z$. Without loss of generality, we take $(x, y, z) = (1, 1, 1)$. If we then calculate the coordinates, we find that L_{10} and L_6 both have coordinates $(1, 1, 1)$, which is not acceptable.

However, this gives another rational point on the curve, so we find the tangent line at $(x, y, z) = (1, 1, 1)$. It is $-5x - y + 6z = 0$. We substitute $y = 6z - 5x$ into $p(x, y, z)$ to obtain

$$q(x, z) = 2x^3 - 9x^2z + 12xz^2 - 5z^3$$

The known solution is $(x, z) = (1, 1)$, so that this is divisible twice by $x - z$, giving $q(x, z) = (2x - 5z)(x - z)^2$. The third point of intersection is therefore $(x, y, z) = (5, -13, 2)$. This value of (x, y, z) is then found to give a valid coordinatization of the 10_3 configuration found. The coordinates that result are shown in Table 3.

At this point, the algorithm could continue, and find the tangent line at $(x, y, z) = (5, -13, 2)$ to look for more rational coordinatizations. Or the known rational points on the curve could be taken two at a time, as the line containing two points intersects the curve in a third rational point, and so forth. In practice, very many rational coordinatizations can be found in this way from a single theta subgraph of a single one-point extension of a geometric configuration.

Table 3: Rational coordinates of the 10_3 configuration of Figure 5.

$P_1 = (2, 4, -3)$	$L_1 = (1, 1, 2)$
$P_2 = (-1, 1, 0)$	$L_2 = (2, -1, 0)$
$P_3 = (1, 2, -3)$	$L_3 = (1, 1, 1)$
$P_4 = (-14, -4, 27)$	$L_4 = (14, 5, 8)$
$P_5 = (0, 0, 1)$	$L_5 = (3, 0, 2)$
$P_6 = (1, 0, -1)$	$L_6 = (5, 5, 2)$
$P_7 = (-10, 4, 15)$	$L_7 = (4, 13, 4)$
$P_8 = (-26, -4, 39)$	$L_8 = (0, 1, 0)$
$P_9 = (-2, 0, 5)$	$L_9 = (-2, 7, 0)$
$P_{10} = (-14, -4, 9)$	$L_{10} = (-5, 13, -2)$

We now start from the 10_3 configuration of Figure 5, with the rational coordinatization given in Table 3. There is a one-pont extension $\mathrm{Ext}(\ell_{10}, \ell_6, \ell_3; a_9, a_2, a_6)$ that can be done, resulting in an 11_3 configuration. Its incidence table is given in Table 4. This configuration is isomorphic to configuration $(11_3)X$ in Martinetti [17]. The new point and line added are a_{11} and ℓ_{11}. We use a theta subgraph to find a rational coordinatization of it. The theta subgraph consists of the three paths $[a_9, \ell_3, a_2, \ell_{11}]$, $[a_9, \ell_6, a_{11}, \ell_{11}]$, $[a_9, \ell_8, a_6, \ell_{11}]$.

There are many rational coordinatizations that result. One of them is shown in Table 5. We see that the integer coordinates are getting bigger. This is the single greatest obstacle that the algorithm has to deal with. One of the questions that needs to be addressed is how to limit the number of digits in the integers that arise. It is very easy for integer overflow to occur after several successive extensions have been done.

The Desargues configuration cannot be obtained by a 1-point extension (see [14]). The "anti-Pappian" (see [8, 16]) is the only non-geometric 10_3 configuration. Rational coordinatizations of *all* the other (10_3) configurations, can be easily found using one-point extensions of the (9_3) configurations in this way. Then rational coordinatizations of all the (11_3) configurations can be found from the (10_3) configurations, which then extend to coordinatizations of all the (12_3) configurations. The author has written a computer program to generate coordinatizations from a theta subgraph in a one point extension. It produces *thousands* of them very quickly. Currently the program has to be run individually for each starting configuration, and the resulting output files must be individually collated and then tested for isomorphisms.

Table 4: The 11_3 configuration extended from Figure 5.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}
1	1	2	3	1	7	4	5	4	8	2
2	3	3	4	7	9	6	6	5	10	6
10	5	9	7	8	11	8	9	10	11	11

Table 5: Rational coordinates of the 11_3 configuration extended from Table 3.

$P_1 = (2, 4, -3)$	$L_1 = (1, 1, 2)$
$P_2 = (16, -34, 9)$	$L_2 = (2, -1, 0)$
$P_3 = (1, 2, -3)$	$L_3 = (28, 19, 22)$
$P_4 = (-14, -4, 27)$	$L_4 = (14, 5, 8)$
$P_5 = (0, 0, 1)$	$L_5 = (3, 0, 2)$
$P_6 = (-136, 4, 123)$	$L_6 = (27, -15, 22)$
$P_7 = (-10, 4, 15)$	$L_7 = (4, 13, 4)$
$P_8 = (-26, -4, 39)$	$L_8 = (1, 34, 0)$
$P_9 = (-748, 22, 933)$	$L_9 = (-2, 7, 0)$
$P_{10} = (-14, -4, 9)$	$L_{10} = (-5, 13, -2)$
$P_{11} = (-64, -14, 69)$	$L_{11} = (37, 28, 40)$

3 In practice

Given an n_3 configuration (Σ, Π), it is relatively easy to write a computer algorithm that searches for all possible one-point extensions $\mathrm{Ext}(\ell_1, \ell_2, \ell_3; a_1, a_2, a_3)$ or $\mathrm{Ext}(\ell_1, \ell_2, \ell_3, \ell_4; a_1, a_2, a_3, a_4)$, and extends (Σ, Π) to an $(n + 1)_3$ configuration (Σ', Π'), in all possible ways. We also want a coordinatization of (Σ', Π') when (Σ, Π) is geometric. For *each* extension (Σ', Π') found, the coordinatization algorithm of the previous section can be used to look for a coordinatization of (Σ', Π'). There are various situations that one has to be aware of when programming this.

1. The polynomial $p(x, y, z)$ is a cubic homogeneous polynomial in three variables. Sometimes a cubic polynomial will factor into the product of three linear homogeneous polynomials, or a linear and quadratic polynomial. In these cases the algorithm will not succeed. This happens occasionally in practice. It will usually be detected when the tangent is found. Not every extension (Σ', Π') has a coordinatization extended using a given theta subgraph. However, another theta subgraph can be chosen in this case.
2. The tangent at $(x, y, z) = (u, v, w)$ is a linear homogeneous polynomial. It may be identically 0. In this case the extension does not succeed.
3. The tangent at $(x, y, z) = (u, v, w)$ may be be a monomial, e.g., $x = 0$. This does not tend to produce valid coordinatizations.

4. Suppose that the tangent at $(x, y, z) = (u, v, w)$ is $ax + by + cz = 0$. Solving for one variable, e.g., $cz = -ax - by$ and substituting this into $p(x, y, z)$ gives the reduced polynomial $q(x, y) = 0$, which is divisible twice by $vx - uy$. It can happen that $q(x, y)/(vx - uy)^2$ is a monomial, e.g., $q(x, y) = x(vx - uy)^2$. This gives $x = 0$, from which we find the solution $(x, y, z) = (0, -c, b)$. This frequently occurs as a special case.
5. The general case is when $q(x, y)$ factors into $(vx - uy)^2(rx + sy)$. In this case the solution is $cx = cs, cy = -cr$ and $cz = -as + br$, or equivalently, $(cs, -cr, -as + br)$ is taken as the solution. The majority of solutions fall into this case.
6. The algorithm stores an array of solutions $(x, y, z) = (u, v, w)$ to $p(x, y, z) = 0$. Initially there are three such points (u, v, w) known, and they are known not to give valid coordinatizations of (Σ', Π'). They are placed on the array of solutions. For *each* (u, v, w) on the array, the tangent is used to find another possible solution, which is *appended* to the array. The solutions on the array are then taken in pairs (u_1, v_1, w_1) and (u_2, v_2, w_2), to find more solutions, which are also appended to the array. The algorithm proceeds to build an array of all solutions (u, v, w) that can be obtained by these methods. This is similar to generating the elements of a group. Typically a potentially infinite number of solutions will be found, so that a limit must be placed on the maximum number allowed. The algorithm can stop with the first valid coordinatization found, or it can look for some maximum number of valid coordinatizations. It can easily find *thousands* of valid integer coordinatizations. However, the values of the integers u, v, w rapidly become enormous if a large number of coordinatizations is required, causing integer overflow even when 64-bit integers are used. The author has programmed it to find a maximum of three valid coordinatizations for each extension (Σ', Π') found, using 64-bit integers, and using only one theta subgraph. More theta subgraphs could be chosen.

If (Σ, Π) is an n_3 configuration, there will be various $(n+1)_3$ configurations that can be produced from it by one-point extensions. If (Σ', Π') is such an $(n+1)_3$ configuration, then there are usually very many different extensions of (Σ, Π) that give rise to an isomorphic (Σ', Π'). Each extension will have up to three coordinatizations found. And this same (Σ', Π') may also arise by a one-point extension from another n_3 configuration, which will also produce numerous coordinatizations of (Σ', Π'). The result is thousands of integer coordinatizations for (Σ', Π') when $n = 10, 11$ or 12. Graph isomorphism software is used to distinguish and recognize the various configurations that are produced. The author has used the software of [12], although others could also be used. The configuration is represented by its Levi graph, with an initial partition of vertices into points and lines.

This method of finding coordinatizations is *much* simpler than that of [22, 23] because it only requires finding the roots of cubic homogeneous polynomials with three variables, whereas [23] states that solving their general diophantine equations for (n_3) configurations is likely to be recursively undecidable.

So far, the author has used this method to produce integer coordinatizations of all the geometric (10_3), (11_3) and (12_3) configurations. As n increases, the integer coordinates rapidly tend to have more and more digits, so that it is necessary to filter them somewhat to limit the number of digits in the coordinates. If fixed size integers are used (e.g. 64 bits), overflow can soon occur, which limits the number of coordinatizations found. It is advantageous to choose a coordinatization of (Σ, Π) to extend from, whose coordinates are “small” integers. Very many coordinatizations of (Σ, Π) are then obtained. This is the case with $n = 10, 11, 12, 13$, where thousands of coordinatizations are easily found. If multi-precision integer arithmetic is used, it is likely that coordinatizations can be found for nearly any fixed n.

The number of distinct (13_3) configurations is 2036 (see [10], p. 69). One of these is a Fano-type configuration, as described in [14], and therefore does not arise as a one point extension. Using ad-hoc methods, the author has shown that it is geometric, and in fact has a rational coordinatization. The other 2035 (13_3) configurations can all be constructed as one point extensions of (12_3) configurations. All of them are geometric, and all have rational coordinatizations. The coordinatization algorithm finds many integer coordinatizations of them. One of them was much more difficult than all the others, requiring integer coordinates with up to 22 digits in the intermediate calculations, and 13 in the final coordinates. For this one configuration, the algorithm was carried out by hand using *Maple* [24] as a calculator with unlimited precision. Maple was also used for constructing a coordinatization of the Fano-type configuration. The description of the coordinatizations is too long to include here. An article containing the details is currently in preparation.

4 Additional coordinatizations

Suppose that (Σ, Π) is an n_3 configuration for which an integer coordinatization is known. We would like to find more integer coordinatizations. One method is this.

1. Find an induced theta subgraph Θ in the incidence graph of (Σ, Π). This is most easily done by finding an induced cycle of reasonable length, and then finding a suitable path across the cycle. The path must have odd length.
2. The vertices not in Θ are to keep their current coordinates. One of the vertices of degree three in Θ is chosen to have coordinates (x, y, z), with values to be determined.
3. The algorithm FOLLOWPATH() is used to assign coordinates that are homogeneous linear forms to the vertices of Θ of degree two. A polynomial $p(x, y, z)$ is constructed using the second vertex of degree three of Θ. Solutions of $p(x, y, z) = 0$ are found as in the previous section.

This allows us to find “related” coordinatizations of (Σ, Π). The author has used this method to produce many rational coordinatizations of the (9_3) configurations, which can then be used as starting points for the generation and coordinatization of the (10_3) configurations and beyond. A given Θ may not produce any additional coordinatizations. In general, different choices of Θ will produce different results. This method is less reliable that the extension method of the previous section. The reason seems to be that the polynomial $p(x, y, z)$ frequently has large integer coefficients, resulting in solutions which lead to integer overflow. For some configurations (Σ, Π), no additional coordinatizations are found like this. For others, it gives dozens of new coordinatizations.

5 Real coordinatizations – the anti-Pappian

The previous sections are concerned with using one-point extensions to find rational coordinatizations of n_3 configurations. Theorems 2.3 and 2.4 also apply to real coordinatizations. The *anti-Pappian* [8, 16, 19] is the only (10_3) configuration that is not geometric. It cannot be coordinatized over *any* field, as shown in [8, 16]. However, it can be coordinatized over the quaternions [16].

The anti-Pappian can be obtained by a one-point extension from a geometric (9_3) configuration (it is $(9_3)_3$ in [10] and 9.1 in [2], a self-dual configuration with an automorphism group of order 12). When the extension algorithm is applied to find a coordinatization, it is necessary to divide polynomials. It is easy to divide polynomials with integer coefficients, as the division is always exact. However, when a computer works with real numbers, they are represented as floating point numbers, and round-off error is always present. Consequently division will always leave a non-zero remainder, which is usually very small, even when the division is theoretically exact. A suitably small number is then replaced by zero, e.g. 10^{-9}. When $P_i \cdot L_j$ is evaluated to test for incidence of a point and line, the result will usually not be exactly zero, due to round-off error, even if they are incident. So if $P_i \cdot L_j$ is sufficiently close to zero, it must be considered to be zero. Thus, it is possible to have a point and line not *exactly* incident, but *very, very* close to incident, for example, $|P_i \cdot L_j| \leq 10^{-9}$. Thus, a *near-coordinatization* can be found. *Every* real coordinatization found using floating point numbers is in fact a near-coordinatization.

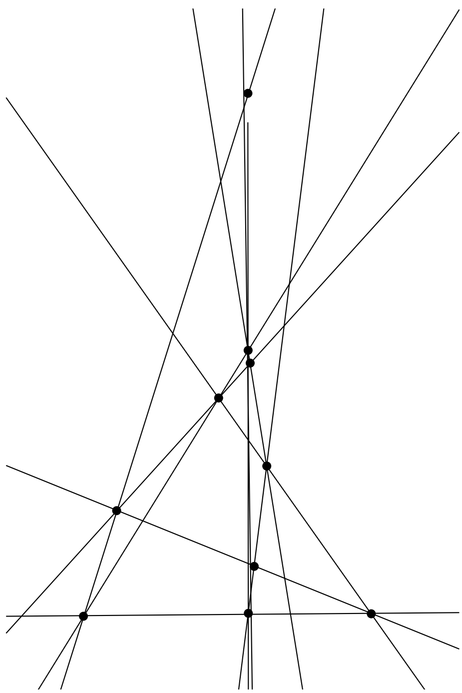


Figure 7: A near-coordinatization of the anti-Pappian configuration.

When the coordinatization algorithm is applied to the extension that produces the anti-Pappian, several near-coordinatizations are found, even though the anti-Pappian cannot be coordinatized over the reals. One of them is shown in Figure 7.

Question. Let ε be a small positive real value, and let Δ be a fixed positive real value, e.g., $\Delta = 1$. How small can ε be chosen so that there is a near-coordinatization of the anti-Pappian configuration such that $|P_i \cdot L_j| \leq \varepsilon$ for all points P_i and lines L_j which are incident, and $|P_i \cdot L_j| \geq \Delta$ if P_i and L_j are non-incident?

Grünbaum [10] (p. 151) also asks whether there are any n_3 configurations with $n > 10$ which are non-geometric? One place to look for them is the *Fano-type* configurations of [14], as they cannot be constructed using a one point extension, and so are not accessible to the cubic-polynomial-based coordinatization algorithm. The smallest Fano-type configuration is the unique (7_3). The next one is a (13_3) configuration (which *is* geometric). Then (14_3).

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Maximum cuts of graphs with forbidden cycles*

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Abstract

For a graph G, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G. For an integer $m \geq 1$ and for a set $\mathcal{H}$ of graphs, let $f(m, \mathcal{H})$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no member of $\mathcal{H}$ as a subgraph. In particular, for a given graph H, we simply write $f(m, H)$ for $f(m, \mathcal{H})$ when $\mathcal{H} = \{H\}$. Let $r > 4$ be a fixed even integer. Alon et al. (2003) conjectured that there exists a positive constant $c(r)$ such that $f(m, C_{r-1}) \geq m/2 + c(r)m^{r/(r+1)}$ for all m. In the present article, we show that $f(m, C_{r-1}) \geq m/2 + c(r)(m^r \log^4 m)^{1/(r+2)}$ for some positive constant $c(r)$ and all m. For any fixed integer $s \geq 2$, we also study the function $f(m, \mathcal{H})$ for $\mathcal{H} = \{K_{2,s}, C_5\}$ and $\mathcal{H} = \{C_4, C_5, \ldots, C_{r-1}\}$, both of which improve the results of Alon et al.

Keywords: $\mathcal{H}$-free graph, partition, maximum cut.

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1 Introduction

All graphs considered here are finite, undirected and have no loops and no parallel edges, unless otherwise specified. All logarithms in this paper are with the natural base e. For a graph G, let $f(G)$ denote the maximum number of edges in a cut of G, that is, the

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maximum number of edges in a bipartite subgraph of G. For an integer m, let $f(m)$ denote the minimum value of $f(G)$, as G ranges over all graphs with m edges. Thus, $f(m)$ is the largest integer f such that any graph with m edges contains a bipartite subgraph with at least f edges.

It is easy to show that $f(m) \geq m/2$ by considering a random bipartition of a graph with m edges. Edwards [10, 11] proved that for every m

$$f(m) \geq \frac{m}{2} + \frac{1}{4}\Big(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\Big), \tag{1.1}$$

and noticed that this is tight when $m = \binom{k}{2}$ for odd integers k. For more information on $f(m)$ and some related topics, we refer the reader to [1, 3, 5, 6, 8, 14, 15, 16, 21, 26, 27, 28]. For survey articles, see [7, 23].

Suppose that $\mathcal{H}$ is a set of graphs. Let $f(m, \mathcal{H})$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no member of $\mathcal{H}$. In particular, for a given graph H, we simply write $f(m, H)$ for $f(m, \mathcal{H})$ when $\mathcal{H} = \{H\}$. It is noted (see, e.g., [2]) that for every fixed graph H there exist positive constants $\epsilon = \epsilon(H)$ and $c = c(H)$ such that $f(m, H) \geq m/2 + cm^{1/2+\epsilon}$ for all m. However, the problem of estimating the error term more precisely is not easy, even for relatively simple graphs H. For example, let $r \geq 4$ be an integer and let H be the cycle C_{r-1}. The case $r = 4$ has been studied extensively. After a series of papers by various researchers [12, 22, 24], Alon [1] proved that $f(m, C_3) = m/2 + \Theta(m^{4/5})$ for all m. For general $r \geq 4$, Alon et al. [4] proposed the following conjecture.

Conjecture 1.1. *For every integer $r \geq 4$, there is a positive constant $c(r)$ such that*

$$f(m, C_{r-1}) \geq \frac{m}{2} + c(r)m^{\frac{r}{r+1}} \tag{1.2}$$

for all m. This is tight, up to the value of $c(r)$, for all $r \geq 4$.

The authors confirmed (1.2) for all odd $r > 4$. In this paper, we consider the conjecture for every even integer $r > 4$ and establish the following theorem.

Theorem 1.2. *For every even integer $r > 4$, there is a positive constant $c(r)$ such that*

$$f(m, C_{r-1}) \geq \frac{m}{2} + c(r)\big(m^r \log^4 m\big)^{\frac{1}{r+2}}$$

for all m.

Alon et al. [4] also studied the function $f(m, H)$ when H is the complete bipartite graph $K_{2,s}$. It is proved that, for every $s \geq 2$, there is a positive constant $c(s)$ such that

$$f(m, K_{2,s}) \geq \frac{m}{2} + c(s)m^{5/6}$$

for all m, and this is tight up to the value of $c(s)$. Now, we consider the function $f(m, \mathcal{H})$ for $\mathcal{H} = \{K_{2,s}, C_5\}$, which improves the above lower bound as follows.

Theorem 1.3. *For each $s \geq 2$, let G be a $\{K_{2,s}, C_5\}$-free graph with m edges. Then there exists a positive constant $c(s)$ such that*

$$f(G) \geq \frac{m}{2} + c(s)m^{6/7}$$

for all m.

Moreover, Alon et al. [2] considered the function $f(m, \mathcal{H})$ for $\mathcal{H} = \{C_3, \ldots, C_{r-1}\}$, and proved that

$$f(m, \mathcal{H}) \geq \frac{m}{2} + c(r)m^{\frac{r}{r+1}}$$

for all m. In the following, we allow the occurence of triangles and get a stronger result.

Theorem 1.4. *Let $r > 4$ be a fixed even integer and $\mathcal{H}_r = \{C_4, \ldots, C_{r-1}\}$. Then there exists a positive constant $c(r)$ such that*

$$f(m, \mathcal{H}_r) \geq \frac{m}{2} + c(r)m^{\frac{r}{r+1}}$$

for all m.

2 Maximum cuts of C_{2k+1}-free graphs

In this section, we give a proof of Theorem 1.2. The goal is to prove that the chromatic number of a C_{2k+1}-free graph is relatively small, since graphs with small chromatic number must have large bipartite subgraphs.

For a graph G, let $\chi(G)$ and $\alpha(G)$ denote the chromatic number and independence number of G, respectively. We need the following lemma, whose easy proof can be found in [1] (see also [2, 12, 21]).

Lemma 2.1. *Let G be a graph with m edges and chromatic number at most χ. Then*

$$f(G) \geq \frac{\chi + 1}{2\chi} m.$$

To find an upper bound on the chromatic number of a C_{2k+1}-free graph, we require a lemma of Jensen and Toft [17] (see also [18]), which is a general lemma on monotone properties. Note that a graph property is called *monotone* if it holds for all subgraphs of a graph which has this property, i.e., is preserved under deletion of edges and vertices.

Lemma 2.2 (Jensen and Toft [17, §7.3])**.** *For $s \geq 1$, let $\psi\colon [s, \infty) \to (0, \infty)$ be a positive continuous non-decreasing function. Suppose that $\mathcal{P}$ is a family of graphs with monotone properties such that $\alpha(G) \geq \psi(|V(G)|)$ for every $G \in \mathcal{P}$ with $|V(G)| \geq s$. Then for every such G with $|V(G)| \geq s$,*

$$\chi(G) \leq s + \int_s^{|V(G)|} \frac{1}{\psi(x)}\, dx.$$

In order to bound $\chi(G)$ by Lemma 2.2, we need bound $\alpha(G)$ of a C_{2k+1}-free graph G in terms of $|V(G)|$. The following well-known Turán's lower bound (see, e.g., [25]) and another two lemmas from [19] and [20] will be used to bound $\alpha(G)$.

Lemma 2.3 (Turán's Lower Bound)**.** *Let G be a graph on n vertices with average degree at most d. Then*

$$\alpha(G) \geq \frac{n}{1 + d}.$$

Lemma 2.4 (Li et al. [19])**.** *Let G be a graph on n vertices with average degree at most d. If the average degree of the subgraph induced by the neighborhood of any vertex is at most a, then*

$$\alpha(G) \geq nF_{a+1}(d),$$

where

$$F_a(x) = \int_0^1 \frac{(1-t)^{1/a}}{a+(x-a)t}\,dt > \frac{\log(x/a)-1}{x},\ (x>0).$$

Lemma 2.5 (Li and Zang [20])**.** *For a fixed integer $k \geq 2$, let G be a C_{2k+1}-free graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\alpha(G) \geq \frac{1}{4k}\Big(\sum_{i=1}^{n} d_i^{\frac{1}{k-1}}\Big)^{\frac{k-1}{k}}.$$

Next, we shall also use the following upper bound, proved by Erdős and Gallai [13], on the maximum number of edges in P_t-free graphs, where P_t stands for a simple path with t vertices.

Lemma 2.6 (Erdős and Gallai [13])**.** *Let G be a P_{t+1}-free graph with n vertices. Then G contains at most $(t-1)n/2$ edges.*

Finally, we give a simple inequality, which is used frequently in our proofs of the following several theorems. We omit the proof details.

Lemma 2.7. *For any real number $x > 0$, we have*

$$x \geq \max\Big\{\log(x+3) - \frac{3}{2}, e\log x\Big\} \tag{2.1}$$

and that the function $g(x) = \log x/x$ is monotonically increasing over the interval $(0, e]$ and decreasing over the interval (e, ∞).

Having finished all the necessary preparations, we are ready to give lower bounds of the independence number of a C_{2k+1}-free graph.

Theorem 2.8. *For any fixed integer $k \geq 2$, let $G = (V, E)$ be a C_{2k+1}-free graph on n vertices with average degree at most d. Then*

$$\alpha(G) \geq \max\Big\{\frac{n\log d}{2kd}, \frac{1}{5k^2}(n^k \log n)^{\frac{1}{k+1}}\Big\}.$$

Proof. First, we prove that

$$\alpha(G) \geq \frac{n\log d}{2kd}.$$

Case 1. $d \leq e^2(2k-1)$. By inequality (2.1), we have

$$2k \geq \log(2k-1) + \frac{5}{2} > \log d + \frac{1}{e} \geq \log d + \frac{\log d}{d} = \frac{(1+d)\log d}{d}.$$

This together with Lemma 2.3 implies that

$$\alpha(G) \geq \frac{n}{1+d} \geq \frac{n\log d}{2kd}.$$

Case 2. $d > e^2(2k-1)$. It follows from inequality (2.1) that $2k-1 \geq 1 + \log(2k-1)$. This together with $d > e^2(2k-1)$ yields that

$$\log d \geq 2 + \log(2k-1) \geq \frac{2k}{2k-1}\big(1 + \log(2k-1)\big),$$

which gives that

$$\log d - \big(1 + \log(2k-1)\big) \geq \frac{\log d}{2k}. \tag{2.2}$$

Since G is C_{2k+1}-free, the subgraph induced by the neighborhood of any vertex of G is P_{2k}-free. By Lemma 2.6, the average degree of any P_{2k}-free graph is at most $2(k-1)$. It follows from Lemma 2.4 and inequality (2.2) that

$$\alpha(G) \geq nF_{2k-1}(d) > \frac{n \log \frac{d}{e(2k-1)}}{d} \geq \frac{n \log d}{2kd},$$

as desired.

Now, we show that

$$\alpha(G) \geq \frac{1}{5k^2}(n^k \log n)^{\frac{1}{k+1}}.$$

Let $v_1, \ldots, v_n$ be the vertices of G such that $d(v_i) = d_i$ for $1 \leq i \leq n$. Set

$$S = \Big\{ v_i \in V : d_i > (n \log^k n)^{\frac{1}{k+1}} \Big\}.$$

If $|S| \geq 2n/5$, then, by Lemma 2.5, we have

$$\alpha(G) \geq \frac{1}{4k}\Big(\sum_{i=1}^{n} d_i^{\frac{1}{k-1}}\Big)^{\frac{k-1}{k}} \geq \frac{1}{4k}\Big(\frac{2n}{5} \cdot (n \log^k n)^{\frac{1}{k^2-1}}\Big)^{\frac{k-1}{k}} \geq \frac{1}{5k^2}(n^k \log n)^{\frac{1}{k+1}}.$$

Suppose that $|S| < 2n/5$. Consider the graph H induced by $V \backslash S$. Clearly, the number of vertices contained in H is at least $3n/5$, and the average degree $d(H)$ of H is at most $(n \log^k n)^{1/(k+1)}$. If $d(H) \leq e$, then the desired result follows immediately from Lemma 2.3. Otherwise, by the preceding result, we obtain

$$\alpha(G) \geq \alpha(H) \geq \frac{3n}{5} \cdot \frac{\log d(H)}{2kd(H)} \geq \frac{1}{5k^2}(n^k \log n)^{\frac{1}{k+1}},$$

where the last inequality holds because the function $g(x) = \log x/x$ is monotonically decreasing over the interval $[e, (n \log^k n)^{1/(k+1)}]$ by Lemma 2.7. This completes the proof of Theorem 2.8. $\square$

With the help of Lemma 2.2 and Theorem 2.8, we establish the following theorem, which plays a key role in our proof of Theorem 1.2. The approach we take is an extension of that by Poljak and Tuza [22].

Theorem 2.9. *For any fixed integer $k \geq 2$, let G be a C_{2k+1}-free graph with $m > 1$ edges. Then*

$$\chi(G) \leq 32(k+1)^3 \Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}}.$$

Proof. Let G be a C_{2k+1}-free graph on n vertices with $m > 1$ edges. If G is bipartite, then $\chi(G) = 2$ and the claim follows. Suppose that $\chi(G) \geq 3$. Without loss of generality, we may assume that G is vertex-critical. Note that each vertex-critical graph has minimal degree at least $\chi(G) - 1$. It follows that the minimal degree of G is at least 2. Thus, we have $m \geq n$. Now, we end the proof by showing the following series of claims.

Claim 1.

$$\chi(G) \le 15k^3\Big(\frac{n}{\log n}\Big)^{\frac{1}{k+1}}.$$

This is trivial for $n < e^2$ as $\chi(G) \le n < e^2$, hence we may assume that $n \ge e^2$. For $x \ge e^2$, define the functions

$$\gamma(x) = 1 - \log^{-1} x \quad \text{and} \quad \psi(x) = \frac{1}{5k^2}(x^k \log x)^{\frac{1}{k+1}}.$$

Clearly, $\gamma(x) \ge 1/2$ for $x \ge e^2$, and $\gamma(x)$, $\psi(x)$ are positive continuous and non-decreasing. By Theorem 2.8, we have $\alpha(G) \ge \psi(n)$. Thus, Lemma 2.2 gives that

$$\begin{aligned}\chi(G) &\le e^2 + \int_{e^2}^{n} \frac{1}{\psi(x)}\, dx \le e^2 + \frac{5k^2}{\gamma(e^2)} \int_{e^2}^{n} \frac{\gamma(x)}{(x^k \log x)^{\frac{1}{k+1}}}\, dx \\ &= e^2 + 10k^2(k+1)\Big(\frac{x}{\log x}\Big)^{\frac{1}{k+1}}\Big|_{e^2}^{n} \le 15k^3\Big(\frac{n}{\log n}\Big)^{\frac{1}{k+1}}.\end{aligned}$$

This completes the proof of Claim 1.

For convenience, we define

$$n^* = \Big(\frac{m^{k+1}}{\log^k m}\Big)^{\frac{1}{k+2}}.$$

Claim 2. $n > n^*$.

Otherwise, assume that $n \le n^*$. By Lemma 2.7, we know the function $g(x) = x/\log x$ is monotonically increasing over the interval (e, ∞) and $\log x \ge e \log\log x$ for each $x > 1$. Note that $m > 1$ (which implies $n \ge 3 > e$). It follows from Claim 1 that

$$\chi(G) \le 15k^3\Big(\frac{n}{\log n}\Big)^{\frac{1}{k+1}} \le 15k^3\Big(\frac{n^*}{\log n^*}\Big)^{\frac{1}{k+1}} \le 32k^3\Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}}.$$

Thus, we get the desired result and complete the proof of Claim 2.

Now, we construct a graph sequence $\mathcal{G} = \{G_i\}_{i\ge 0}$ according to the following procedure, which we will call the $\mathcal{G}$ algorithm. Set $i = 0$, $G_0 = G$ and $n_0 = |V(G_0)|$. Repeat the following steps until $n_i \le n^*$.

- Choose S_i to be a maximum independent set of G_i.
- Set $G_{i+1} = G_i \backslash S_i$ and $n_i = |V(G_i)|$. Increment i.

Let $\ell+1$ be the length of the resulting sequence $\mathcal{G}$. By the $\mathcal{G}$ algorithm, we immediately have $n_\ell \le n^*$ and that G can be colored by at most $\chi(G_\ell) + \ell$ colors. Clearly, we may assume that G_ℓ is vertex-critical. Thus, by Claim 1, for $n_\ell \ge 3$, we have

$$\chi(G_\ell) \le 15k^3\Big(\frac{n_\ell}{\log n_\ell}\Big)^{\frac{1}{k+1}} \le 15k^3\Big(\frac{n^*}{\log n^*}\Big)^{\frac{1}{k+1}} \le 32k^3\Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}}. \tag{2.3}$$

Note that $\chi(G_\ell)$ clearly satisfies the above inequality for $n_\ell \le 2$. In the following, we aim to bound the value of ℓ.

Firstly, we give a lower bound of $|S_i|$. Let $t = \lceil \frac{n}{n^*} \rceil$. It follows from Claim 2 that $t \geq 2$. Let $I = \{0, 1, \ldots, \ell - 1\}$ and $J = \{2, 3, \ldots, t\}$. Note that $n_i > n^* \geq n/t$ for each $i \in I$ by the $\mathcal{G}$ algorithm and the definition of t. Let $v_1, \ldots, v_{n_0}$ be a labelling of the vertices of G_0 such that $S_i = \{v_p : n_{i+1} < p \leq n_i\}$ for each $i \in I$. Denote S the union of S_i for all $i \in I$. Thus, for each $j \in J$, we can define

$$V_j = \Big\{ v_p \in S : \frac{n}{j} < p \leq \frac{n}{j-1} \Big\} \quad \text{and} \quad I_j = \Big\{ i \in I : n_i > \frac{n}{j} \Big\}.$$

Note that $S \backslash S_{\ell-1} \subseteq \cup_{j \in J} V_j \subseteq S$ and $I_2 \subseteq I_3 \subseteq \ldots \subseteq I_t$. Therefore, for each $x \in V_j$, there exists an $i \in I_j$ such that $x \in S_i$. In addition, we have

$$|V_j| \leq \Big\lceil \frac{n}{j-1} - \frac{n}{j} \Big\rceil. \tag{2.4}$$

Claim 3. *For each $i \in I_j \neq \emptyset$,*

$$|S_i| \geq \frac{n^2 \log \frac{2jm}{n}}{4kj^2 m}.$$

Let d_i denote the average degree of G_i for each $i \in I$. Clearly, for each $i \in I_j$, we have $d_i \leq \frac{2m}{n_i} \leq \frac{2jm}{n}$. Suppose that $d_i > e$. Recall that the function $g(x) = \log x / x$ is decreasing over the interval (e, ∞). By Theorem 2.8, we have

$$|S_i| \geq \frac{n_i \log d_i}{2kd_i} \geq \frac{n^2 \log \frac{2jm}{n}}{4kj^2 m}.$$

Otherwise, $d_i \leq e$. It follows from Lemma 2.3 that $|S_i| \geq \frac{n_i}{2k} \geq \frac{n}{2kj}$, which together with the fact that $x \geq \log x$ implies the required result as well. This completes the proof of Claim 3.

Then, for each $x \in S_i$ and $i \in I$, define $w(x) = |S_i|^{-1}$. Therefore, for each $x \in S_i$ and $i \in I_j$, it follows from Claim 3 that

$$w(x) = |S_i|^{-1} \leq \frac{4kj^2 m}{n^2 \log \frac{2jm}{n}} \leq \frac{4kj^2 mn^{-2}}{\log j + \log \frac{m}{n}}.$$

By the definition of $w(x)$ and the above inequality, we immediately have

$$\begin{aligned} \ell - 1 &= \sum_{i \in I \backslash \{\ell-1\}} \sum_{x \in S_i} w(x) \\ &\leq \sum_{j \in J} \sum_{x \in V_j} w(x) \leq \sum_{j=2}^{t} \frac{4kj^2 |V_j| mn^{-2}}{\log j + \log \frac{m}{n}} \leq \sum_{j=2}^{t} \frac{16kmn^{-1}}{\log j + \log \frac{m}{n}}. \end{aligned} \tag{2.5}$$

The last inequality follows from (2.4) and the fact $j \geq 2$.

Finally, we give the following upper bound of ℓ.

Claim 4.

$$\ell - 1 \leq 64(k+1)^2 \Big(\frac{m}{\log^2 m} \Big)^{\frac{1}{k+2}}.$$

By the definition of n^*, we have

$$\frac{n}{n^*} \cdot \frac{m}{n} = \frac{m}{n^*} = (m \log^k m)^{\frac{1}{k+2}}. \tag{2.6}$$

It follows that $\max\{m/n, n/n^*\} > m^{\frac{1}{2(k+2)}}$, and then

$$\max\left\{ \log \frac{m}{n}, \log \frac{n}{n^*} \right\} > \frac{1}{2(k+2)} \log m. \tag{2.7}$$

Suppose that $n/n^* < m/n$. Note that $t - 1 < n/n^*$ by the definition of t. Then, we delete the first term of the denominator of (2.5) and obtain

$$\ell - 1 \leq \sum_{j=2}^{t} \frac{16kmn^{-1}}{\log \frac{m}{n}} \leq \frac{16k(t-1)m}{n \log \frac{m}{n}} < \frac{16km}{n^* \log \frac{m}{n}} \leq 64k^2 \Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}},$$

where the last inequality follows from (2.6) and (2.7); as desired. Otherwise, $n/n^* \geq m/n$. Recall that $t - 1 < n/n^* \leq t$. It follows that

$$\int_2^t \frac{1}{\log x}\, dx \leq \frac{2(t-1)}{\log t} \leq \frac{2n}{n^* \log \frac{n}{n^*}}.$$

Deleting the second term of the denominator in (2.5), we have

$$\ell - 1 \leq \frac{16km}{n} \sum_{j=2}^{t} \frac{1}{\log j} \leq \frac{16km}{n} \int_2^t \frac{1}{\log x}\, dx \leq \frac{32km}{n^* \log \frac{n}{n^*}} \leq 64(k+1)^2 \Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}}.$$

Again, the last inequality follows from (2.6) and (2.7). This completes the proof of Claim 4.

Now, it follows from (2.3) and Claim 4 that

$$\begin{aligned} \chi(G) \leq \chi(G_\ell) + \ell &\leq \big(32k^3 + 64(k+1)^2\big) \Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}} + 1 \\ &\leq 32(k+1)^3 \Big(\frac{m}{\log^2 m}\Big)^{\frac{1}{k+2}}. \end{aligned}$$

Thus, we get the desired result and complete the proof of Theorem 2.9. □

We are now in a position to establish Theorem 1.2.

Proof of Theorem 1.2. Let $r > 4$ be a fixed integer and let G be a C_{r-1}-free graph with m edges. The desired result follows immediately for $m = 1$. Suppose that $m > 1$. Set $r - 1 = 2k + 1$ and $c(r) = 1/(8r^3)$. By Theorem 2.9, we have

$$2\chi(G) \leq 8r^3 \Big(\frac{m}{\log^2 m}\Big)^{\frac{2}{r+2}}.$$

This together with Lemma 2.1 yields that

$$f(G) \geq \frac{m}{2} + c(r)\big(m^r \log^4 m\big)^{\frac{1}{r+2}}.$$

Thus, we complete the proof of Theorem 1.2. □

3 Maximum cuts of $\mathcal{H}$-free graphs

In this section, we obtain lower bounds on the size of the maximum cuts of $\mathcal{H}$-free graphs. Let $G = (V, E)$ be a graph. For a subset $U \subset V$, denote $E(U)$ the set of edges of G spanned by U. We need the following simple lemma from [1, 4, 8].

Lemma 3.1. *Let $G = (V, E)$ be a graph with m edges. Suppose that $U \subset V$ and let G' be the induced subgraph of G on U. If G' has m' edges, then*

$$f(G) \geq f(G') + \frac{m - m'}{2}.$$

Next, we need another result from [4], which provides a very useful lower bound on the size of a maximum cut in an H-free graph for a certain class of graphs H.

Lemma 3.2 (Alon et al. [4]). *There exists an absolute positive constant ϵ such that for every positive constant C there is a $\delta = \delta(C) > 0$ with the following property. Let G be a graph with n vertices (with positive degrees), m edges, and degree sequence $d_1, d_2, \ldots, d_n$. Suppose, further, that the induced subgraph on any set of $d \geq C$ vertices, all of which have a common neighbour, contains at most $\epsilon d^{3/2}$ edges. Then*

$$f(G) \geq \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i}.$$

A graph is *r-degenerate* if every one of its subgraphs contains a vertex of degree at most r. We need the following easy and well-known fact. See, e.g., [1, 2, 4] for a proof.

Lemma 3.3. *Let H be an r-degenerate graph on h vertices. Then there is an ordering $v_1, \ldots, v_h$ of the vertices of H such that for every $1 \leq i \leq h$ the vertex v_i has at most r neighbours v_j with $j < i$.*

Finally, we shall also use the following lower bound in extremal set theory, proved by Corrádi [9], on the size of a set Q from which we can draw N subsets of size at least q such that any two of them share at most λ elements.

Lemma 3.4 (Corrádi [9]). *Let $Q_1, \ldots, Q_N$ be N sets with $|Q_i| \geq q$ for each $i = 1, \ldots, N$, and let Q be their union. If $|Q_i \cap Q_j| \leq \lambda$ for all $i \neq j$, then*

$$|Q| \geq \frac{q^2 N}{q + (N-1)\lambda}.$$

Having finished all the necessary preparations, we are ready to give proofs of Theorems 1.3 and 1.4. Our proofs combine combinatorial and probabilistic techniques, including extensions of ideas that appear in [1, 2, 4].

Proof of Theorem 1.3. For each $s \geq 2$, let $G = (V, E)$ be a $\{K_{2,s}, C_5\}$-free graph on n vertices with m edges. Define $\ell = \lfloor 4sm^{2/7} \rfloor$. The proof proceeds by considering two possible cases depending on the existence of dense subgraphs in G.

Case 1. G is $(\ell - 1)$-degenerate, that is, it contains no subgraph with minimum degree at least ℓ. In this case, we use Lemma 3.2 to bound $f(G)$. By Lemma 3.3, we can get a labelling $v_1, v_2, \dots, v_n$ of the vertices of G such that $d_i^+ < \ell$ for every i, where d_i^+ denotes the number of neighbors v_j of v_i with $j < i$. Note that $\sum_{i=1}^n d_i^+ = m$. Let d_i be the degree of v_i for each $1 \le i \le n$. Then

$$\sum_{i=1}^{n} \sqrt{d_i} \ge \sum_{i=1}^{n} \sqrt{d_i^+} > \frac{\sum_{i=1}^n d_i^+}{\sqrt{\ell}} \ge \frac{1}{2\sqrt{s}} m^{6/7}.$$

Now, we check the condition of Lemma 3.2. For each $v \in V$, let $N(v)$ be the neighborhood of v in G and $N_d(v)$ be any subset of cardinality d of $N(v)$. Denote G_d^v the subgraph induced by $N_d(v)$. Since G is C_5-free, we know that G_d^v contains no path of length 3. It follows from Lemma 2.6 that G_d^v contains at most d edges, which is smaller than $\epsilon d^{3/2}$ for all $d > \epsilon^{-2}$. Thus, by Lemma 3.2, we have

$$f(G) \ge \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i} \ge \frac{m}{2} + \frac{\delta}{2\sqrt{s}} m^{6/7},$$

where $\delta = \delta(\epsilon)$ is a constant, as required.

Case 2. There exists a subset Q of q vertices of G such that the induced subgraph $G[Q]$ has minimum degree at least ℓ. Now, we prove that Q contains a subset Q' such that the induced subgraph $G[Q']$ spans at least $q\ell/4$ edges and is $3t$-colorable for $t = \lceil 4sq/\ell^2 \rceil$.

For fixed $x \in Q$, denote by $S(x)$ the set of vertices in Q which are at distance exactly 2 from x and denote by s_x the size of $S(x)$. We bound s_x by Lemma 3.4.

Claim 5. *$s_x \ge \ell^2/(2s)$ for each $x \in Q$.*

For each $x \in Q$, let $N_Q(x)$ be the neighborhood of x in $G[Q]$. For each $v \in N_Q(x)$, let $Q_v = N_Q(v) \cap S(x)$. Since G is $K_{2,s}$-free, we conclude that $|Q_u \cap Q_v| \le s - 1$ for each pair of vertices $u, v \in N_Q(x)$ and that v is adjacent to at most $s - 1$ vertices in $N_Q(x)$. It follows that $|Q_v| \ge \ell - (s-1) - 1 = \ell - s$. Note that

$$S(x) = \bigcup_{v \in N_Q(x)} Q_v \quad \text{and} \quad |N_Q(x)| \ge \ell \ge 4s.$$

By Lemma 3.4, we obtain

$$s_x = \Big| \bigcup_{v \in N_Q(x)} Q_v \Big| \ge \frac{(\ell - s)^2 |N_Q(x)|}{(\ell - s) + (|N_Q(x)| - 1)(s-1)} \ge \frac{\ell^2}{2s}.$$

This completes the proof of Claim 5.

Let T be a random subset of Q obtained by picking uniformly at random, with repetitions, t vertices of Q. Let Q' be the set of all vertices x in Q such that $S(x) \cap T \ne \emptyset$ and let $G[Q']$ be the induced subgraph of G on Q'.

Claim 6. *There exists a set T such that $G[Q']$ spans at least $q\ell/4$ edges.*

By the definition of Q', for each $x \in Q$, we have

$$\mathbb{P}(x \notin Q') = \Big(1 - \frac{s_x}{q}\Big)^t \le \Big(1 - \frac{\ell^2}{2sq}\Big)^t \le \exp\Big\{-\frac{\ell^2 t}{2sq}\Big\} < \frac{1}{4},$$

where the second inequality follows from Claim 5, and the last inequality holds by noting that $t \geq 4sq/\ell^2$. Thus, for each edge $xy \in E(Q)$, we obtain

$$\mathbb{P}(xy \in E(Q')) = \mathbb{P}(x \in Q') \cdot \mathbb{P}(y \in Q') > \Big(1 - \frac{1}{4}\Big) \cdot \Big(1 - \frac{1}{4}\Big) > \frac{1}{2}.$$

By linearity of expectation, and noting that $|E(Q)| \geq q\ell/2$, we have

$$\mathbb{E}(|E(Q')|) = \sum_{xy \in E(Q)} \mathbb{P}(xy \in E(Q')) \geq \frac{1}{2}|E(Q)| \geq \frac{1}{4}q\ell.$$

Hence, there exists a set T of at most t vertices so that the corresponding graph $G[Q']$ has at least $q\ell/4$ edges. Thus, we complete the proof of Claim 6.

Fix such sets T and Q', let $G' = G[Q']$ and $T = \{u_1, \ldots, u_{t'}\}$, where $1 \leq t' \leq t$. Now we show G' is $3t$-colorable. Define a coloring c of G' in t' colors by coloring each vertex $x \in Q'$ with the smallest index of a vertex from T which belongs to $S(x)$. For each $1 \leq i \leq t'$, let H_i be the subgraph of G' induced by the vertices of Q' with color i.

Claim 7. *For each $1 \leq i \leq t'$, H_i is 3-colorable.*

For each $u_i \in T$ and $v \in N(u_i)$, let H_i^v be the subgraph induced by the neighbors of v with color i in G'. By the above definition and the fact that G is C_5-free, we have the following properties:

- For each $v \in N(u_i)$, H_i^v is P_4-free;
- For each $v_1, v_2 \in N(u_i)$ and $u \in V(H_i^{v_1}) \cap V(H_i^{v_2})$, u is an isolated vertex in both $H_i^{v_1}$ and $H_i^{v_2}$;
- For each $x \in V(H_i^{v_1})$ and $y \in V(H_i^{v_2})$, x and y are nonadjacent in H_i.

Note that H_i is induced by the union of $V(H_i^v)$ over all $v \in N(u_i)$. This together with the above three properties implies that H_i is P_4-free, i.e., 3-colorable. Thus, we complete the proof of Claim 7.

By the definition of c and Claim 7, we conclude that G' is $3t$-colorable. According to Lemma 2.1, it follows that

$$\begin{aligned} f(G') &\geq \frac{|E(Q')|}{2} + \frac{|E(Q')|}{6t} \geq \frac{|E(Q')|}{2} + \frac{q\ell}{24}\left\lceil \frac{4sq}{\ell^2} \right\rceil^{-1} \\ &\geq \frac{|E(Q')|}{2} + \frac{\ell^3}{144s} \geq \frac{|E(Q')|}{2} + \frac{4s^2}{9}m^{6/7}. \end{aligned}$$

The second inequality follows from Claim 6, and the third inequality holds because $q \geq s_x \geq \ell^2/(2s)$ by Claim 5. The above inequality together with Lemma 3.1 gives that

$$f(G) \geq \frac{m - |E(Q')|}{2} + \frac{|E(Q')|}{2} + \frac{4s^2}{9}m^{6/7} = \frac{m}{2} + \frac{4s^2}{9}m^{6/7}.$$

Therefore, the desired result follows immediately from Cases 1 and 2 by setting $c(s) = \min\{\frac{\delta}{2\sqrt{s}}, \frac{4s^2}{9}\}$, completing the proof of Theorem 1.3. $\square$

The proof of Theorem 1.4 is similar to that of Theorem 1.3.

Proof of Theorem 1.4. Let G be an $\mathcal{H}_r$-free graph on n vertices with m edges. Define $\ell = \lfloor 2m^{2/(r+1)} \rfloor$ and proceed as before, by considering two possible cases.

Case 1. G contains no subgraph with minimum degree at least 2ℓ. In this case, we proceed as in the previous proof. Similarly, the induced subgraph of G on any set of common neighbors of a vertex can span only a linear number of edges, as it contains no copy of C_4. Thus, we can apply, again, Lemma 3.2 and conclude, as in the proof of Theorem 1.3, that

$$f(G) \geq \frac{m}{2} + \delta \frac{m}{\sqrt{2\ell}} \geq \frac{m}{2} + \frac{\delta}{2} m^{\frac{r}{r+1}},$$

where $\delta = \delta(\epsilon)$ is also a constant, as needed.

Case 2. There exists a subset Q of q vertices of G such that the induced subgraph $G[Q]$ has minimum degree at least 2ℓ. Here, too, we prove that there exists $Q' \subset Q$ such that the induced subgraph $G[Q']$ spans at least $q\ell/2$ edges and is $2t$-colorable for $t = \lceil q/\ell^k \rceil$.

Let $r = 2k + 2$. Denote by $S_k(x)$ the set of vertices in Q which are at distance exactly k from x and denote by s_x the size of $S_k(x)$. Since the minimal degree of $G[Q]$ is at least 2ℓ and $G[Q]$ contains no cycle of length from 4 to $2k + 1$, it can easily be seen that $s_x \geq 2\ell(2\ell - 2)^{k-1} \geq 2\ell^k$ for each $x \in Q$.

Let T be a random subset of Q obtained by picking, with repetitions, t vertices of Q, each chosen randomly with uniform probability. This together with the fact $s_x \geq 2\ell^k$ yields that the probability that $S_k(x) \cap T$ is empty is at most

$$\Big(1 - \frac{s_x}{q}\Big)^t \leq \Big(1 - \frac{2\ell^k}{q}\Big)^t \leq \exp\Big\{ - \frac{2\ell^k t}{q} \Big\} < \frac{1}{4}.$$

An argument similar to the one used in the proof of Claim 6, the details of which we omit, shows that there exists a set T of at most t vertices so that the corresponding graph $G[Q']$ has at least $q\ell/2$ edges.

Fix such sets T and Q'. Now, we define a coloring c of G' and the induced subgraphs H_i of G' for $1 \leq i \leq |T|$ as in the proof of Theorem 1.3.

Claim 8. *For each* $1 \leq i \leq |T|$, H_i *is the disjoint union of edges modulo isolated vertices.*

For fixed $u_i \in T$ and for each $v \in S_{k-1}(u_i)$, let H_i^v be the subgraph induced by the neighbors of v with color i in G'. By the above definition, and recalling that G contains no cycle of length from 4 to $2k + 1$, we have the following properties: (i) for each $v \in S_{k-1}(u_i)$, H_i^v is P_3-free; (ii) for each $v_1, v_2 \in S_{k-1}(u_i)$, $V(H_i^{v_1}) \cap V(H_i^{v_2}) = \emptyset$; (iii) for each $x \in V(H_i^{v_1})$ and $y \in V(H_i^{v_2})$, x and y are nonadjacent in H_i. It follows from (ii) and (iii) that H_i is the disjoint union of H_i^v over all $v \in S_{k-1}(u_i)$. Thus, by (i), H_i is the disjoint union of edges modulo isolated vertices. This completes the proof of Claim 8.

By the definition of c and Claim 8, we know that G' is $2t$-colorable. Using Lemma 2.1, we conclude that

$$\begin{aligned} f(G') &\geq \frac{|E(Q')|}{2} + \frac{|E(Q')|}{4t} \geq \frac{|E(Q')|}{2} + \frac{q\ell}{8} \Big\lceil \frac{q}{\ell^k} \Big\rceil^{-1} \\ &\geq \frac{|E(Q')|}{2} + \frac{\ell^{k+1}}{12} = \frac{|E(Q')|}{2} + \frac{(\sqrt{2})^r}{12} m^{\frac{r}{r+1}}. \end{aligned}$$

The second inequality follows from the fact $|E(Q')| \geq q\ell/2$, and the third inequality holds because $q \geq s_x \geq 2\ell^k$. Taking Lemma 3.1 into consideration, we obtain

$$f(G) \geq \frac{m - |E(Q')|}{2} + \frac{|E(Q')|}{2} + \frac{(\sqrt{2})^r}{12} m^{\frac{r}{r+1}} = \frac{m}{2} + \frac{(\sqrt{2})^r}{12} m^{\frac{r}{r+1}}.$$

Again, we get the required result from Cases 1 and 2 by choosing $c(s) = \min\{\frac{\delta}{2}, \frac{(\sqrt{2})^r}{12}\}$, which completes the proof of Theorem 1.4. $\square$

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Circular chromatic number of induced subgraphs of Kneser graphs*

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Abstract

Investigating the equality of the chromatic number and the circular chromatic number of graphs has been an active stream of research for last decades. In this regard, Hajiabolhassan and Zhu in 2003 proved that if n is sufficiently large with respect to k, then the Schrijver graph $\mathrm{SG}(n,k)$ has the same chromatic and circular chromatic number. Later, Meunier in 2005 and independently, Simonyi and Tardos in 2006 proved that $\chi(\mathrm{SG}(n,k)) = \chi_c(\mathrm{SG}(n,k))$ if n is even. In this paper, we study the circular chromatic number of induced subgraphs of Kneser graphs. In this regard, we shall first generalize the preceding result to s-stable Kneser graphs for large even n and even s. Furthermore, as a generalization of the Hajiabolhassan-Zhu result, we prove that if n is large enough with respect to k, then any sufficiently large induced subgraph of the Kneser graph $\mathrm{KG}(n,k)$ has the same chromatic number and circular chromatic number.

Keywords: Chromatic number, circular chromatic number, Kneser graph, stable Kneser graph.

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1 Introduction

Throughout the paper, the symbol $[n]$ stands for the set $\{1,\ldots,n\}$. Let n and d be two positive integers. The *circular complete graph* $K_{\frac{n}{d}}$ is a graph with vertex set $[n]$ and two vertices i and j are adjacent whenever $d \le |i-j| \le n-d$. For a graph G, the circular chromatic number of G, denoted by $\chi_c(G)$, is defined as follows:

$$\chi_c(G) \stackrel{\text{def}}{=} \inf\left\{\frac{n}{d} \ : \text{ there is a homomorphism from } G \text{ to } K_{\frac{n}{d}}\right\}.$$

It is known that the infimum can be replaced by minimum. Moreover, one can see that $\chi(G)-1 < \chi_c(G) \le \chi(G)$, see [36]. Therefore, the circular chromatic number is a refinement of the chromatic number. It is a natural question to ask for which graphs G, we have $\chi_c(G) = \chi(G)$. However, it turns out to be a difficult question. Hell [19] proved that the problem of determining whether a graph has the circular chromatic number at most $\frac{n}{d}$ is NP-Hard. Hatami and Tusserkani [18] showed that the problem of determining whether or not $\chi_c(G) = \chi(G)$ is NP-Hard even if the chromatic number is known. Therefore, finding necessary conditions for graphs to have the same chromatic and circular chromatic number turns out to be an interesting problem. This problem has received significant attention, for instance see [1, 17, 36, 37].

For two positive integers n and k, where $n \ge 2k$, the *Kneser graph* $\mathrm{KG}(n,k)$ is a graph with vertex set $\binom{[n]}{k}$, that is, the family of all k-subsets of $[n]$, and two vertices are adjacent if their corresponding k-subsets are disjoint. Kneser in 1955 [23] conjectured that the chromatic number of $\mathrm{KG}(n,k)$ is $n-2k+2$. In 1978, Lovász [26] gave an affirmative answer to Kneser's conjecture. He used algebraic topological tools, giving birth to the field of topological combinatorics. For a positive integer s, a nonempty subset $\mathcal{S}$ of $[n]$ is said to be s-stable if for any two different elements i and j in S, we have $s \le |i-j| \le n-s$. Throughout the paper, the family of all s-stable k-subsets of $[n]$ is denoted by $\binom{[n]}{k}_s$. The subgraph of $\mathrm{KG}(n,k)$ induced by all s-stable k-subsets of $[n]$ is called the *s-stable Kneser graph* and is denoted by $\mathrm{KG}_s(n,k)$. The 2-stable Kneser graph $\mathrm{KG}_2(n,k)$ is known as the Schrijver graph $\mathrm{SG}(n,k)$. Schrijver [31] proved that Schrijver graphs are vertex critical subgraphs of Kneser graphs with the same chromatic number. Meunier [30] showed that for any two positive integers n and k, where $n \ge sk$, the s-stable Kneser graph $\mathrm{KG}_s(n,k)$ can be colored by $n-s(k-1)$ colors and conjectured that the chromatic number is $n-s(k-1)$. Jonsson [22] proved that this conjecture is true provided that $s \ge 4$ and n is sufficiently large with respect to k and s. Also, Chen [12] confirmed Meunier's conjecture for even values of s.

Lovász's theorem [26] has been generalized in several aspects. For a hypergraph $\mathcal{H}$, the *general Kneser graph* $\mathrm{KG}(\mathcal{H})$ is a graph with vertex set $E(\mathcal{H})$ and two vertices are adjacent if their corresponding edges are vertex disjoint. Dol'nikov [13] generalized Lovász's result and proved that the chromatic number of $\mathrm{KG}(\mathcal{H})$ is at least the *colorability defect of* $\mathcal{H}$, denoted by $\mathrm{cd}(\mathcal{H})$, where the colorability defect of $\mathcal{H}$ is the minimum number of vertices which should be excluded from $\mathcal{H}$ so that the induced subhypergraph on the remaining vertices is 2-colorable.

For a vector $X = (x_1,\ldots,x_n) \in \{-,0,+\}^n$, a sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$ ($i_1 < \cdots < i_t$) is called an *alternating subsequence of X with length t* if $x_{i_j} \ne 0$ for each $j \in \{1,\ldots,t\}$ and $x_{i_j} \ne x_{i_{j+1}}$ for each $j \in \{1,\ldots,t-1\}$. The maximum length of an alternating subsequence of X is called the *alternation number of* X, denoted by $\mathrm{alt}(X)$. For $\mathbf{0} \stackrel{\text{def}}{=} (0,\ldots,0)$, we define $\mathrm{alt}(\mathbf{0}) \stackrel{\text{def}}{=} 0$. Also, we define X^+ and X^- to be respectively the indices of positive and negative coordinates of X, i.e.,

$$X^+ \stackrel{\text{def}}{=} \{i \,:\, x_i = +\} \quad \text{and} \quad X^- \stackrel{\text{def}}{=} \{i \,:\, x_i = -\}.$$

Note that both X^+ and X^- are subsets of $[n]$ and by abuse of notation, we can write $X = (X^+, X^-)$. For two vectors $X, Y \in \{-, 0, +\}^n$, by $X \subseteq Y$, we mean $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

Let $\mathcal{H} = (V, E)$ be a hypergraph and $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection. The *alternation number of $\mathcal{H}$ with respect to σ*, denoted by $\mathrm{alt}_\sigma(\mathcal{H})$, is the maximum possible value of an $\mathrm{alt}(X)$ such that $E(\mathcal{H}[\sigma(X^+)]) = E(\mathcal{H}[\sigma(X^-)]) = \varnothing$. Also, the *strong alternation number of $\mathcal{H}$ with respect to σ*, denoted by $\mathrm{salt}_\sigma(\mathcal{H})$, is the maximum possible value of an $\mathrm{alt}(X)$ such that $E(\mathcal{H}[\sigma(X^+)]) = \varnothing$ or $E(\mathcal{H}[\sigma(X^-)]) = \varnothing$. The *alternation number of $\mathcal{H}$*, denoted by $\mathrm{alt}(\mathcal{H})$, and the *strong alternation number of $\mathcal{H}$*, denoted by $\mathrm{salt}(\mathcal{H})$, are respectively the minimum values of $\mathrm{alt}_\sigma(\mathcal{H})$ and $\mathrm{salt}_\sigma(\mathcal{H})$, where the minimum is taken over all bijections $\sigma : [n] \longrightarrow V(\mathcal{H})$. The present first author and Hajiabolhassan [4] proved the following theorem.

Theorem A. For any hypergraph $\mathcal{H}$, we have

$$\chi(\mathrm{KG}(\mathcal{H})) \geq \max\left\{|V(\mathcal{H})| - \mathrm{alt}(\mathcal{H}), |V(\mathcal{H})| - \mathrm{salt}(\mathcal{H}) + 1\right\}.$$

One can simply see that this result improves the aforementioned Dol'nikov's result [13]. Using this lower bound, the chromatic number of several families of graphs is computed, for instance see [2, 3, 5, 6, 8].

In 1997, Johnson, Holroyd, and Stahl [21] proved that $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$ provided that $2k+1 \leq n \leq 2k+2$ or $k = 2$. They also conjectured that the circular chromatic number of Kneser graphs is always equal to their chromatic number. This conjecture has been studied in several articles. Hajiabolhassan and Zhu [17] using a combinatorial method proved that if n is large enough with respect to k, then $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. Later, using algebraic topology, Meunier [29] and Simonyi and Tardos [33] independently confirmed this conjecture for the case of even n. It should be mentioned that the results by Hajiabolhassan and Zhu [17], Meunier [29], and Simonyi and Tardos [33] are indeed proved for the Schrijver graph $\mathrm{SG}(n,k)$. Eventually in 2011, Chen [11] confirmed the Johnson-Holroyd-Stahl conjecture. Chen's proof was simplified by Chang, Liu and Zhu in [10] and by Liu and Zhu in [25]. The present first author, Hajiabolhassan, and Meunier [8] generalized Chen's result to a larger family of graphs. They introduced a sufficient condition for a hypergraph $\mathcal{H}$ having $\chi(\mathrm{KG}(\mathcal{H})) = \chi_c(\mathrm{KG}(\mathcal{H}))$.

Plan. The paper contains two main sections. In Section 2, we shall investigate the coloring properties of stable Kneser graphs. In this regard, we prove the equality of the chromatic number and the circular chromatic number of s-stable Kneser graph $\mathrm{KG}_s(n,k)$ provided that $n \geq (s+2)k - 2$ and both n and s are even. In the last section, we study the circular chromatic number of large induced subgraphs of Kneser graphs. Indeed, it is proved that, for large enough n, any sufficiently large induced subgraph of the Kneser graph $\mathrm{KG}(n,k)$ has the same chromatic number and circular chromatic number. In particular, giving a partial answer to a question posed by Lih and Liu [24], we present a threshold $n(k,s)$ such that for any $n \geq n(k,s)$, the chromatic number and circular chromatic number of $\mathrm{KG}_s(n,k)$ are equal.

2 Chromatic number of stable Kneser graphs

As it is mentioned in the previous section, the chromatic number of s-stable Kneser graph $\mathrm{KG}_s(n,k)$ is determined provided that k and $s \geq 4$ are fixed and n is large enough [22] or s is even [12]. In this section, we first present a generalization of Theorem A. Using this generalization, for any even s, we prove that any proper coloring of s-stable Kneser graph $\mathrm{KG}_s(n,k)$ contains a large colorful complete bipartite subgraph, which immediately gives solutions to the chromatic number of s-stable Kneser graphs $\mathrm{KG}_s(n,k)$. Also, this result concludes that the circular chromatic number of s-stable Kneser graph $\mathrm{KG}_s(n,k)$ equals to its chromatic number provided that $n \geq (s+2)k-2$ and both n and s are even.

Tucker's lemma is a combinatorial counterpart of the Borsuk-Ulam theorem with several useful applications, for instance, see [27, 28].

Lemma A (Tucker's lemma [35]). Let $\lambda\colon \{-,0,+\}^n \setminus \{\mathbf{0}\} \longrightarrow \{\pm 1,\ldots,\pm m\}$ be a map satisfying the following properties:

- it is antipodal: $\lambda(-X) = -\lambda(X)$ for each $X \in \{-,0,+\}^n \setminus \{\mathbf{0}\}$, and
- it has no complementary edges: there are no X and Y in $\{-,0,+\}^n \setminus \{\mathbf{0}\}$ such that $X \subseteq Y$ and $\lambda(X) = -\lambda(Y)$.

Then $m \geq n$.

There are several results concerning the existence of a large complete bipartite multicolored subgraph in any proper coloring of a graph G, see [4, 11, 32, 33, 34]. In what follows, we present a similar type of result with a combinatorial proof. Note that since there is a purely combinatorial proof [28] for Tucker's lemma, any proof using Tucker's lemma with combinatorial approach can be considered as a purely combinatorial proof.

Theorem 2.1. *Let $\mathcal{H}$ be a hypergraph and set $t = \max\left\{|V(\mathcal{H})| - \mathrm{alt}(\mathcal{H}),\, |V(\mathcal{H})| - \mathrm{salt}(\mathcal{H}) + 1\right\}$. For any proper coloring $c\colon V(\mathrm{KG}(\mathcal{H})) \longrightarrow [C]$, there exists a complete bipartite subgraph $K_{\lfloor t/2\rfloor,\lceil t/2\rceil}$ of $\mathrm{KG}(\mathcal{H})$ whose vertices receive different colors and moreover, these different colors occur alternating on the two parts of the bipartite graph with respect to their natural order.*

Proof. Let $\sigma_1, \sigma_2\colon [n] \longrightarrow V(\mathcal{H})$ be two bijections for which we have $\mathrm{alt}(\mathcal{H}) = \mathrm{alt}_{\sigma_1}(\mathcal{H})$ and $\mathrm{salt}(\mathcal{H}) = \mathrm{salt}_{\sigma_2}(\mathcal{H})$. Now, we shall proceed the proof with two different cases, $t = n - \mathrm{alt}(\mathcal{H})$ and $t = n - \mathrm{salt}(\mathcal{H}) + 1$. Assume that $t = n - \mathrm{alt}(\mathcal{H})$ (resp. $t = n - \mathrm{salt}(\mathcal{H}) + 1$). For simplicity of notation, by identifying the set $V(\mathcal{H})$ and $[n]$ via the bijection σ_1 (resp. σ_2), we may assume that $V(\mathcal{H}) = [n]$. For each $X = (X^+, X^-) \in \{-,0,+\}^n \setminus \{\mathbf{0}\}$, define $c(X) \stackrel{\text{def}}{=} (c(X^+), c(X^-)) \in \{-,0,+\}^C$ to be a signed vector, where

$$c(X^+) \stackrel{\text{def}}{=} \left\{c(e) \ :\ e \in E(\mathcal{H}) \ \text{ and } \ e \subseteq X^+\right\}$$

and

$$c(X^-) \stackrel{\text{def}}{=} \left\{c(e) \ :\ e \in E(\mathcal{H}) \ \text{ and } \ e \subseteq X^-\right\}.$$

For each $X \in \{-,0,+\}^n \setminus \{\mathbf{0}\}$, define $\lambda(X)$ as follows.

- If $\mathrm{alt}(X) \leq \mathrm{alt}_{\sigma_1}(\mathcal{H})$ (resp. $\mathrm{alt}(X) \leq \mathrm{salt}_{\sigma_2}(\mathcal{H})$), then define $\lambda(X) = \pm\,\mathrm{alt}(X)$, where the sign is positive if the first nonzero term of X is positive and is negative otherwise.

- If $\mathrm{alt}(X) \geq \mathrm{alt}_{\sigma_1}(\mathcal{H}) + 1$ (resp. $\mathrm{alt}(X) \geq \mathrm{salt}_{\sigma_2}(\mathcal{H}) + 1$), then define $\lambda(X) = \pm(\mathrm{alt}_{\sigma_1}(\mathcal{H}) + \mathrm{alt}(c(X)))$ (resp. $\lambda(X) = \pm(\mathrm{salt}_{\sigma_2}(\mathcal{H}) + \mathrm{alt}(c(X)) - 1)$), where the sign is positive if the first nonzero term of $c(X)$ is positive and is negative otherwise.

One can simply check that λ satisfies the conditions of Lemma A. Consequently, there should be an $X \in \{-, 0, +\}^n \setminus \{\mathbf{0}\}$ such that $|\lambda(X)| = \lambda(X) \geq n$. Clearly, we should have $\mathrm{alt}(X) \geq \mathrm{alt}_{\sigma_1}(\mathcal{H}) + 1$ (resp. $\mathrm{alt}(X) \geq \mathrm{salt}_{\sigma_2}(\mathcal{H}) + 1$). Therefore, the definition of $\lambda(X)$ implies that $\mathrm{alt}(c(X)) \geq n - \mathrm{alt}_{\sigma_1}(\mathcal{H})$ (resp. $\mathrm{alt}(c(X)) \geq n - \mathrm{salt}_{\sigma_2}(\mathcal{H}) + 1$). Let $Z = (Z^+, Z^-) \subseteq c(X)$ be a signed vector such that $\mathrm{alt}(Z) = |Z| = t$, as $\mathrm{alt}(c(X)) \geq t$. Note that if $Z^+ \cup Z^- = \{i_1, i_2, \ldots, i_t\}$, where $1 \leq i_1 < \cdots < i_t \leq C$, then we should have $Z^+ = \{i_j \ : \ j \in [t] \text{ is odd}\}$ and $Z^- = \{i_j \ : \ j \in [t] \text{ is even}\}$. For an $j \in [t]$, if j is odd (resp. even), then according to the definition of $c(X)$, there is an edge $e \in E(\mathcal{H})$ such that $e \subseteq X^+$ (resp. $e \subseteq X^-$) with $c(e) = i_j$. Note that the induced subgraph of $\mathrm{KG}(\mathcal{H})$ on the vertices $\{e_1, \ldots, e_t\}$ contains the desired complete bipartite graph. □

Note that the complete bipartite graph whose existence is guaranteed by Theorem 2.1 is not necessarily an induced subgraph. Also, it is worth mentioning that we here used Tucker's lemma though, in case $t = |V(\mathcal{H})| - \mathrm{alt}(\mathcal{H})$, the previous theorem was proved in [4] by use of Ky Fan's lemma [14].

Let n, k, and s be positive integers, where $n \geq sk$ and s is even. It is not difficult to see that if n is large enough (with respect to s and k), then any 2-stable $(\frac{s}{2}(k-1)+1)$-subset of $[n]$ contains an s-stable k-subset of $[n]$. In the following two lemmas, we shall prove that $n \geq (s+2)k - 2$ would be sufficient for this observation.

Lemma 2.2. *Let s be an even positive integer and let $n = 2s + 2$. If $\mathcal{S}$ is a 2-stable subset of $[n]$ of cardinality $\frac{s}{2} + 1$, then there are $a, a' \in \mathcal{S}$ such that $a - a' \in \{s, s+1, s+2\}$.*

Proof. Without loss of generality, we may assume that $1 \in \mathcal{S}$ and $2s+2 \notin \mathcal{S}$. If $s+1 \in \mathcal{S}$, then there is nothing to prove. Therefore, let $s+1 \notin \mathcal{S}$. For $1 \leq i \leq \frac{s}{2}$, define $B_i = \{2i-1, 2i, 2i+s, 2i+s+1\}$. Therefore, for some i, $1 \leq i \leq \frac{s}{2}$, $|B_i \cap \mathcal{S}| = 2$. Let $a, a' \in B_i \cap \mathcal{S}$, since $\mathcal{S}$ is 2-stable, we have $a - a' \in \{s, s+1, s+2\}$. □

Lemma 2.3. *Let k and n be two positive integers and let s be an even positive integer, where $n \geq (s+2)k - 2$. If $\mathcal{S}$ is a 2-stable subset of $[n]$ of cardinality $\frac{s}{2}(k-1)+1$, then there is an s-stable k-subset of $\mathcal{S}$. In particular,* $\mathrm{salt}\left([n], \binom{n}{k}_s\right) = s(k-1)+1$.

Proof. First note that for given k and s, if the statement is true for some $n \geq k(s+2) - 2$, then it is true for all integers $n' \geq n$. Therefore it is enough to prove the lemma for $n = k(s+2) - 2$.

We use induction on k to prove the lemma. The validity of the lemma when $k = 1$ is trivial and when $k = 2$ it was shown in Lemma 2.2. Thus, we may assume that $k \geq 3$.

If for each $i \in \mathcal{S}$, we have $\{i+s, i+s+1, i+s+2\} \cap \mathcal{S} \neq \varnothing$ (where addition is modulo n), then we can greedily find an s-stable k-subset, and there is nothing to prove. Otherwise, without loss of generality, assume that $n - s - 1 \in \mathcal{S}$ and $n-1, n, 1 \notin \mathcal{S}$.

Set $A_{n-s-1} = \{n-s-1, n-s, \ldots, n\}$. Note that since $n-1, n \notin S$, we have $|A_{n-s-1} \cap \mathcal{S}| = \frac{s}{2} - \beta$, for some $0 \leq \beta \leq \frac{s}{2}$. Now, consider $[n] \setminus A_{n-s-1}$ and $\mathcal{S} \setminus A_{n-s-1}$. Set $\dot{n} = n - (s+2)$ and $\dot{\mathcal{S}} = \mathcal{S} \setminus A_{n-s-1}$. Note that $[\dot{n}]$ and $[n] \setminus A_{n-s-1}$ are equal and since $1 \notin \mathcal{S}$, $\dot{\mathcal{S}}$ is a 2-stable subset of $[\dot{n}]$ of cardinality $\frac{s}{2}(k-2) + \beta + 1$.

Define the s-subset B of $[\dot{n}]$ by

$$B \stackrel{\text{def}}{=} \{n-2s-1, n-2s, \ldots, n-s-2\}.$$

By induction, we may consider the following two cases:

(i) There is an s-stable $(k-1)$-subset of $\dot{\mathcal{S}}$, say $\dot{D}$, which has no element of B. In this case, it is readily verified that $D = \dot{D} \cup \{n-s-1\}$ is an s-stable k-subset of $[n]$, completing the proof in this case.

(ii) There are at least $\beta+1$ s-stable $(k-1)$-subsets of $\dot{\mathcal{S}}$, say $\dot{D}_1, \dot{D}_2, \ldots, \dot{D}_{\beta+1}$, such that each $\dot{D}_i$ has exactly one distinct element of B, say b_i.

Now, consider the 2-stable subset $\{b_1, b_2, \ldots, b_{\beta+1}\} \cup (\mathcal{S} \cap A_{n-s-1})$, by Lemma 2.2, there exist two elements a, b such that $a - b \in \{s, s+1, s+2\}$. Since $n-1, n \notin \mathcal{S}$, both a, b are not in A_{n-s-1}. Hence, we may assume that $a \in A_{n-s-1}$ and $b = b_i$ for some i, $1 \leq i \leq \beta+1$. Let d be the smallest element of $\dot{D}_i$. Since $\dot{D}_i$ is an s-stable $(k-1)$-subset of $[\dot{n}]$, therefore we have $s \leq b - d \leq \dot{n} - s = n - (2s+2)$. On the other hand, $s \leq a - b \leq s+2$. Therefore, $2s \leq a - d \leq n - s$. Therefore, $\dot{D}_i \cup \{a\}$ is an s-stable k-subset of $[n]$ as desired.

Note that for an $X \in \{-,0,+\}^n \setminus \{\mathbf{0}\}$ with $\operatorname{alt}(X) \geq s(k-1)+2$, both X^+ and X^- contain 2-stable subsets of size at least $\frac{s}{2}(k-1)+1$, which implies that both X^+ and X^- contain s-stable subsets of size at least k. This concludes that $\operatorname{salt}\left([n], \binom{n}{k}_s\right) = s(k-1)+1$. □

We remind the reader that Meunier [30] showed that $\mathrm{KG}_s(n,k)$ has a proper coloring with $n - s(k-1)$ colors. Note that if we set $\mathcal{H} = ([n], \binom{[n]}{k}_s)$, then $\mathrm{KG}(\mathcal{H}) = \mathrm{KG}_s(n,k)$. Clearly, using these observations, Lemma 2.3, and Theorem 2.1, we have the next theorem.

Theorem 2.4. *Let k and n be two positive integers and let s be an even positive integer, where $n \geq (s+2)k-2$. Any properly colored $\mathrm{KG}_s(n,k)$ contains a complete bipartite subgraph $K_{\lfloor t/2 \rfloor, \lceil t/2 \rceil}$, where $t = n - s(k-1)$ such that all vertices of this subgraph receive different colors and these different colors occur alternating on the two parts of the bipartite graph with respect to their natural order. In particular, we have $\chi(\mathrm{KG}_s(n,k)) = n - s(k-1)$.*

Let r be a positive integer. For an r-coloring c of a given graph G, a cycle $C = v_1, v_2, \ldots, v_m, v_1$ is called *tight* if for each $i \in [m]$, we have $c(v_{i+1}) = c(v_i) + 1 \pmod{r}$. It is known [36] that $\chi_c(G) = r$ if and only if the graph G is r-colorable and every r-coloring of G contains a tight cycle. In view of this result, to prove the next theorem, it suffices to show that any proper $(n - s(k-1))$-coloring of $\mathrm{KG}_s(n,k)$ contains a tight cycle.

Theorem 2.5. *Let n, k, and s be positive integers, where n and s are even and $n \geq (s+2)k-2$. Then, we have*

$$\chi_c(\mathrm{KG}_s(n,k)) = n - s(k-1).$$

Proof. For simplicity of notation, we set $t = n - s(k-1)$. In view of the former discussion, to prove the assertion, let c be a proper t-coloring of $\mathrm{KG}_s(n,k)$. Consider the complete bipartite subgraph $K_{t/2,t/2}$ of $\mathrm{KG}_s(n,k)$, whose existence is ensured by Theorem 2.4. Clearly, this subgraph contains a tight cycle, which completes the proof. □

The original proof of Lovász of Kneser's conjecture is rather long and complicated [26]. Bárány [9], using Gale's lemma [15], presented a short proof of this result. For $n > 2k$, Gale [15] proved that the set $[n]$ can be identified with a subset of S^{n-2k} in such a way that any open hemisphere contains at least one k-subset of $[n]$ (a vertex of $\mathrm{KG}(n,k)$). Schrijver [31] generalized Gale's lemma to 2-stable k-subsets of $[n]$. He also used this generalization to prove that $\chi(\mathrm{SG}(n,k)) = n - 2k + 2$. For an interesting proof of Gale's lemma, see [16]. Moreover, the present first author and Hajiabolhassan [7] generalized Gale's lemma. For any hypergraph $\mathcal{H} = (V, E)$, they introduce a lower bound for the maximum possible value of m for which there is a subset X of S^m and a suitable identification of V with X such that any open hemisphere of S^m contains an edge of $\mathcal{H}$. The next lemma can be obtained directly from this result. However, for the sake of completeness, we prove it here with a different approach.

Lemma 2.6. *Let k and n be two positive integers and let s be an even positive integer, where $n \geq (s+2)k - 2$. There exists an n-subset X of $S^{n-s(k-1)-2}$ and a suitable identification between X and $[n]$ such that every open hemisphere of $S^{n-s(k-1)-2}$ contains an s-stable k-subset of $[n]$.*

Proof. Set $p = \frac{s}{2}(k-1) + 1$. In view of the generalization of Gale's lemma by Schrijver [31], there exists an n-subset X of S^{n-2p} and an identification of X with $[n]$ such that any open hemisphere of S^{n-2p} contains a 2-stable p-subset of $[n]$. Now, by Lemma 2.3, any 2-stable p-subset contains an s-stable k-subset. This implies that any open hemisphere of $S^{n-s(k-1)-2}$ contains an s-stable k-subset of $[n]$ as desired. $\square$

Simonyi and Tardos [34], using the Tucker-Bacon lemma (Lemma B), proved that if the chromatic number of a graph G equals to a topological lower bound for chromatic number, then for any optimal coloring of G with colors $[C]$ and for any partition $L \uplus M$ of $[C]$, there is a multi-colored complete bipartite subgraph $K_{|L|,|M|}$ of G such that all colors in L are assigned to the vertices of one side of $K_{|L|,|M|}$ and all colors in M are assigned to the vertices of the other side. These kinds of results are known as $K_{l,m}$ type theorems, see [32, 34].

Lemma B (Tucker-Bacon lemma)**.** Let $U_1, U_2, \ldots, U_{d+2}$ be open subsets of the d-sphere S^d such that for any $1 \leq i \leq d+2$, $U_i \cap -U_i = \varnothing$ and also, $U_1 \cup \cdots \cup U_{d+2} = S^d$. Then for any partition $A \cup B = \{1, 2, \ldots, d+2\}$ for which $A \neq \varnothing$ and $B \neq \varnothing$, there is an $x \in S^d$ such that $x \in \cap_{i \in A} U_i$ and $-x \in \cap_{j \in B} U_j$.

In what follows, similar to the Simonyi-Tardos result, using the Tucker-Bacon lemma, we prove a $K_{l,m}$ type theorem for s-stable Kneser graphs provided that n is large and s is even.

Theorem 2.7. *Let n, k, and s be positive integers, where s is even and $n \geq (s+2)k - 2$. Also, let c be a proper coloring of $\mathrm{KG}_s(n,k)$ with colors $\{1, 2, \ldots, n - s(k-1)\}$ and assume that A and B form a partition of $\{1, 2, \ldots, n - s(k-1)\}$. Then there exists a complete bipartite subgraph $K_{l,m}$ of $\mathrm{KG}_s(n,k)$ with parts L and M such that $|L| = l = |A|$, $|M| = m = |B|$ and the vertices in L and M receive different colors from A and B, respectively.*

Proof. By Lemma 2.6, we can identify $[n]$ with a subset of $S^{n-s(k-1)-2}$ such that every open hemisphere of $S^{n-s(k-1)-2}$ contains an s-stable k-subset of $[n]$. For $1 \leq i \leq$

$n - s(k-1)$, define

$$U_i \stackrel{\text{def}}{=} \left\{x \in S^{n-s(k-1)-2} : H(x) \text{ contains a vertex with color } i\right\}.$$

One can see that each U_i is an open set, $U_1, U_2, \ldots, U_{n-s(k-1)}$ covers $S^{n-s(k-1)-2}$ and also none of them contains a pair of antipodal points. Thus, the Tucker-Bacon lemma implies that there is an $x \in S^{n-s(k-1)-2}$ such that $x \in \cap_{i \in A} U_i$ and $-x \in \cap_{j \in B} U_j$. Therefore, in view of the definition of U_i's, for each $i \in A$ (resp. $j \in B$), there is an s-stable k-subset L_i (resp. M_j) of $[n]$ such that $c(L_i) = i$ and $L_i \subseteq H(x)$ (resp. $c(M_j) = j$ and $M_j \subseteq H(-x)$). Note that since $H(x) \cap H(-x) = \varnothing$, for each $i \in A$ and $j \in B$, L_i is adjacent to M_j in $\mathrm{KG}_s(n,k)$, which completes the proof. $\square$

We would like to mention that the idea of our proof is close to the Bárány's proof of Kneser conjecture [9].

3 Circular coloring of induced subgraphs of Kneser graphs

The concept of free coloring of graphs was introduced in [1] by the present first author and Hajiabolhassan as a tool for studying the circular chromatic number of graphs. Indeed, they proved that if the free chromatic number of a graph G is at least twice of its chromatic number, then $\chi(G) = \chi_c(G)$.

An independent set in a graph G is called a *free independent set* if it can be extended to at least two distinct maximal independent sets in G. Clearly, one can see that an independent set F in G is a free independent set if and only if there exists an edge $uv \in E(G)$ such that $(N(u) \cup N(v)) \cap F = \varnothing$. The maximum possible size of a free independent set in G is denoted by $\bar{\alpha}(G)$. Furthermore, a vertex of a graph G is contained in a free independent set if and only if the graph obtained by deleting the closed neighborhood of this vertex has at least one edge (for more details, see [1]). As a natural extension of the chromatic number, we can define the free chromatic number of graphs as follows.

Definition 3.1. The *free chromatic number* of a graph G, denote $\phi(G)$, is the minimum size of a partition of $V(G)$ into free independent sets. If G does not have such a partition, then we set $\phi(G) = \infty$.

The next lemma plays a key role in the rest of the paper.

Lemma C ([1, Lemma 2])**.** Let G be a graph such that $\chi_c(G) = \frac{n}{d}$ with $\gcd(n,d) = 1$. If $d \geq 2$, or equivalently, if $\chi_c(G) \neq \chi(G)$, then $\phi(G) \leq 2\chi(G) - 1$.

Let G be a graph with at least one free independent set. By definition, we have $\phi(G) \geq |V(G)|/\bar{\alpha}(G)$. It was proved by Hilton and Milner [20] that if T is an independent set of $\mathrm{KG}(n,k)$ of size at least

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2,$$

then

$$\bigcap_{A \in T} A = \{i\},$$

for some $i \in [n]$. By using this result of Hilton and Milner, it was proved by Hajiabolhassan and Zhu in [17] that if $n \geq 2k^2(k-1)$, then $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. This result was improved in [1] by proving that we have $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$ for $n \geq 2k^2(k-1) - 2k + 3$. It was also showed in [17] that there is a threshold $n(k)$ such that for $n \geq n(k)$, we have $\chi_c(\mathrm{SG}(n,k)) = \chi(\mathrm{SG}(n,k))$. This gave a positive answer to a question of Lih and Liu [24]. Lih and Liu [24] also posed the question of what is the smallest value of $n(k)$. They proved that $n(k) \geq 2k+2$. One should note that in [17] only the existence of the threshold $n(k)$ is ensured and the authors did not present any upper bound for it.

Using the Hilton-Milner theorem, one can simply see that, for $n > 2k$, the size of any free independent set in the Kneser graph $\mathrm{KG}(n,k)$ is at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} \leq k\binom{n-2}{k-2}$, see [1]. In view of this observation, we generalize the result by Hajiabolhassan and Zhu [17] to the following theorem.

Theorem 3.2. *Let n and k be two positive integers, where $n \geq 2k^2(k-1)$. Let H be an induced subgraph of $\mathrm{KG}(n,k)$ with at least $\frac{2k^2(k-1)}{n}\binom{n}{k}$ vertices. Then H has the same chromatic number and circular chromatic number.*

Proof. Obviously, the assertion holds for $k = 1$. So, let $k \geq 2$. Assume that H is an induced subgraph of $\mathrm{KG}(n,k)$ with at least $\frac{2k^2(k-1)}{n}\binom{n}{k}$ vertices. According to Lemma C, it is enough to show that $\phi(H) \geq 2\chi(H)$. To this end, note that

$$\begin{aligned}
\phi(H) &\geq \frac{|V(H)|}{\bar{\alpha}(H)} \\
&\geq \frac{|V(H)|}{\bar{\alpha}(\mathrm{KG}(n,k))} \\
&\geq \frac{\frac{2k^2(k-1)}{n}\binom{n}{k}}{k\binom{n-2}{k-2}} \\
&\geq \frac{2k^2(k-1)n(n-1)}{nk^2(k-1)},
\end{aligned}$$

therefore $\phi(H) \geq 2n-2 > 2\chi(\mathrm{KG}(n,k)) \geq 2\chi(H)$ as desired. □

In the rest of this section, we will return to the study of s-stable Kneser graphs from Section 2, $\mathrm{KG}_s(n,k)$, but this time we consider $\mathrm{KG}_s(n,k)$ as an induced subgraph of $\mathrm{KG}(n,k)$. We focus on the chromatic number and the circular chromatic number of the s-stable Kneser graph $\mathrm{KG}_s(n,k)$. As a special case of the previous theorem, we introduce a threshold $n(k,s)$ such that for any $n \geq n(k,s)$, we have $\chi(\mathrm{KG}_s(n,k)) = \chi_c(\mathrm{KG}_s(n,k))$. In this regard, we first need to count the number of vertices of $\mathrm{KG}_s(n,k)$.

Let N_i be the number of vertices of $\mathrm{KG}_s(n,k)$ containing i. It is obvious that $N_i = N_j$ for all $i,j \in [n]$. Also, let $A = \{x_1,\ldots,x_k\}$ be a vertex of $\mathrm{KG}_s(n,k)$, where $1 = x_1 < x_2 < \cdots < x_k \leq n$. Define $y_i = x_{i+1} - x_i$ for all $1 \leq i \leq k-1$ and $y_k = n - x_k + 1$. Since $A \in V(\mathrm{KG}_s(n,k))$ and $1 \in A$, we have $y_i \geq s$ for all $i \in [k]$. Also, since $y_1 + y_2 + \cdots + y_k = n$, any vertex A of $\mathrm{KG}_s(n,k)$ with $1 \in A$ leads us to a solution of the following system:

$$\begin{array}{l}
Z_1 + Z_2 + \cdots + Z_k = n \\
Z_i \geq s \quad \text{for each } i \in [n]
\end{array}$$

and vise versa. Note that the number of solutions of the preceding system is $\binom{n-k(s-1)-1}{k-1}$. Consequently, for each $i \in [n]$, we have $N_i = N_1 = \binom{n-k(s-1)-1}{k-1}$ for all $i \in [n]$. By an easy double counting, one can see that

$$|V(\mathrm{KG}_s(n,k))| = \frac{1}{k}\sum_{i=1}^{n} N_i = \frac{n}{k}\binom{n-k(s-1)-1}{k-1}.$$

Theorem 3.3. *If* $n \geq 2k^2(k-1)+(s-1)k(k-1)+1$, *then* $\chi_c(\mathrm{KG}_s(n,k)) = \chi(\mathrm{KG}_s(n,k))$.

Proof. Let X be the number of $(k-1)$-subsets B of the set $[n-1]$ such that $B\cap[(s-1)k] \neq \varnothing$, i.e.,

$$X = \#\left\{B \ :\ B \subseteq [n-1] \text{ and } B \cap [(s-1)k] \neq \varnothing\right\}.$$

Obviously, we have $\binom{n-1}{k-1} = \binom{n-(s-1)k-1}{k-1} + X$. On the other hand, one can check that $X \leq (s-1)k\binom{n-2}{k-2}$, which implies the following inequalities:

$$\begin{aligned}
|V(\mathrm{KG}_s(n,k))| &= \frac{n}{k}\binom{n-k(s-1)-1}{k-1} \\
&\geq \frac{n}{k}\left(\frac{n-1}{k-1} - (s-1)k\right)\binom{n-2}{k-2} \\
&\geq \frac{n}{k(k-1)}\left(n-1-(s-1)k(k-1)\right)\binom{n-2}{k-2}.
\end{aligned}$$

Clearly, the previous inequalities lead us to the following:

$$\begin{aligned}
\phi(\mathrm{KG}_s(n,k)) &\geq \frac{|V(\mathrm{KG}_s(n,k))|}{\bar{\alpha}(\mathrm{KG}_s(n,k))} \\
&\geq \frac{\frac{n}{k(k-1)}\left(n-1-(s-1)k(k-1)\right)\binom{n-2}{k-2}}{k\binom{n-2}{k-2}} \\
&\geq \frac{n}{k^2(k-1)}\left(n-1-(s-1)k(k-1)\right).
\end{aligned}$$

Consequently, we have $\phi(\mathrm{KG}_s(n,k)) \geq 2n \geq 2(n-s(k-1))$ provided that $n \geq 2k^2(k-1)+(s-1)k(k-1)+1$. Considering Lemma C, the proof is completed. □

Note that for $s = 2$, the previous theorem gives an upper bound for the smallest value of the threshold $n(k)$, giving a partial answer to the question posed by Lih and Liu [24].

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Groups of symmetric crosscap number less than or equal to 17

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Abstract

Every finite group G acts on some non-orientable unbordered surfaces. The minimal topological genus of those surfaces is called the symmetric crosscap number of G. It is known that 3 is not the symmetric crosscap number of any group but it remains unknown whether there are other such values, called gaps.

In this paper we obtain the groups with symmetric crosscap number less than or equal to 17. Also, we obtain six infinite families with symmetric crosscap number of the form $12k + 3$.

Keywords: Symmetric crosscap number, Klein surfaces.

Math. Subj. Class.: 57M60, 20F05, 20H10, 30F50

1 Introduction

A Klein surface X is a compact surface endowed with a dianalytic structure [1]. Klein surfaces may be seen as a generalization of Riemann surfaces including bordered and non-orientable surfaces. An orientable unbordered Klein surface is a Riemann surface. Given a Klein surface X of topological genus g with k boundary components the number $p = \eta g + k - 1$ is called the algebraic genus of X, where $\eta = 2$ if X is an orientable surface and $\eta = 1$ otherwise.

In the study of Klein surfaces and their automorphism groups the non-euclidean crystallographic (NEC) groups play an essential role. An NEC group Γ is a discrete subgroup of $\mathcal{G}$ (the full group of isometries of the hyperbolic plane $\mathcal{H}$) with compact quotient $\mathcal{H}/\Gamma$. For a Klein surface X with $p \geq 2$ there exists an NEC group Γ, such that $X = \mathcal{H}/\Gamma$, [27].

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A finite group G of order N is a subgroup of the automorphism group of a Klein surface $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. Every finite group G acts as a subgroup of the automorphism group of some non-orientable surface without boundary, see [7]. The minimum topological genus of these surfaces is called the symmetric crosscap number of G and it is denoted by $\tilde{\sigma}(G)$. Such a surface of topological genus $g \geq 3$ has at most $84(g-2)$ automorphisms. Hence, for each g there is a finite number of groups acting on surfaces of genus g. The systematic study of the symmetric crosscap number was begun by May in [23], although previous results from other authors are also to be noted, see for instance [7, 14, 19].

Four types of inter-related problems arise naturally when dealing with the symmetric crosscap number $\tilde{\sigma}(G)$.

First of all, to obtain $\tilde{\sigma}(G)$ for any given group G, and for the groups belonging to a given infinite family.

Second, to obtain $\tilde{\sigma}(G)$ for all groups G with $o(G) < n$ for a given (small) value of n.

Third, for a given value of g, to obtain all groups G such that $\tilde{\sigma}(G) = g$. Evidently this question is feasible only for low values of g.

Finally, to determine which values of g are in fact $\tilde{\sigma}(G) = g$ for a group G. The set of such values is called the symmetric crosscap spectrum and there exists a conjecture according to which $g = 3$ is the unique positive integer not belonging to the spectrum.

In this paper we deal with the third question. We will study which groups have symmetric crosscap number less than or equal to 17. First, we will indicate all the results we know and then we will make a study of each group with symmetric crosscap number $g \leq 17$ that has not been studied in detail. Also, results on the spectrum are given. The contents of this paper form part of the doctoral thesis of the author, [3].

2 Preliminaries

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation-reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Each NEC group Γ has associated a signature [22]:

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i,1}, \ldots, n_{i,s_i}), i = 1, \ldots, k\}), \tag{2.1}$$

where $g, k, r, m_i, n_{i,j}$ are integers satisfying $g, k, r \geq 0, m_i \geq 2, n_{i,j} \geq 2$. We will denote by $[-]$, $(-)$ and $\{-\}$ the cases when $r = 0$, $s_i = 0$ and $k = 0$, respectively.

The signature determines a presentation of Γ, see [30], by generators x_i $(i = 1, \ldots, r)$; e_i $(i = 1, \ldots, k)$; $c_{i,j}$ $(i = 1, \ldots, k;\ j = 0, \ldots, s_i)$; a_i, b_i $(i = 1, \ldots, g)$ if σ has sign '$+$'; and d_i $(i = 1, \ldots, g)$ if σ has sign '$-$'. These generators satisfy the following relations:

$$x_i^{m_i} = 1; \quad c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1; \quad e_i^{-1}c_{i,0}e_i c_{i,s_i} = 1$$

and

$$\textstyle\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g (a_i b_i a_i^{-1} b_i^{-1}) = 1 \quad \text{if } \sigma \text{ has sign '}+\text{'}$$
$$\textstyle\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1 \quad \text{if } \sigma \text{ has sign '}-\text{'}$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide-reflections.

Every NEC group Γ with signature (2.1) has associated a fundamental region whose area $\mu(\Gamma)$, called area of the group, is:

$$\mu(\Gamma) = 2\pi\left(\eta g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{i,j}}\right)\right), \tag{2.2}$$

with $\eta = 2$ or 1 depending on the sign '$+$' or '$-$' in the signature. An NEC group with signature (2.1) actually exists if and only if the right-hand side of (2.2) is greater than 0. We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the reduced area of Γ.

If Γ is a subgroup of an NEC group Λ of finite index N, then also Γ is an NEC group and the Riemann-Hurwitz formula holds, $|\Gamma| = N|\Lambda|$.

Let X be a non-orientable Klein surface of topological genus $g \geq 3$ without boundary. Then by [28] there exists an NEC group Γ with signature:

$$\sigma(\Gamma) = (g, -, [-], \{-\}), \tag{2.3}$$

such that $X = \mathcal{H}/\Gamma$.

A group Γ with this signature is called a surface NEC group. If G acts as an automorphism group of $X = \mathcal{H}/\Gamma$, then there exists another NEC group Λ such that $G = \Lambda/\Gamma$. From the Riemann-Hurwitz relation we have $g - 2 = o(G)|\Lambda|$, where $o(G)$ denotes the order of G. Then

$$\tilde{\sigma}(G) \leq g = 2 + o(G)|\Lambda|,$$

and so to obtain the symmetric crosscap number of G is equivalent to find a group Λ and an epimorphism $\theta\colon \Lambda \to G$, such that $\Gamma = \ker\theta$ is a surface NEC group (and so, without elements with finite order) and $G = \theta(\Lambda^+)$, where Λ^+ is the subgroup consisting of the orientation-preserving elements of Λ, see [28], and minimal $|\Lambda|$.

The groups having symmetric crosscap numbers 1 and 2 have been classified by T. W. Tucker, [29]. The groups of symmetric crosscap number 1 are C_n, D_n, A_4, S_4 and A_5. We have two families of groups of symmetric crosscap number 2, $C_2 \times C_n$, $n > 2$ even, and $C_2 \times D_n$, n even. It is known that there exists no group of symmetric crosscap number 3, [23]. The groups with symmetric crosscap number 4 and 5 were obtained in [8].

M. D. E. Conder at a conference in Castro-Urdiales in 2010 announced that using computing software, he had obtained the groups of symmetric crosscap number up to 65, in terms of their "SmallGroupLibrary" description. The result of this research is available in his webpage, [9]. The list contains the GAP reference of each group, its symmetric crosscap number and the corresponding NEC group Λ. However, this list gives information neither on the algebraic structure of the groups nor on the epimorphism θ which determines the action of the NEC group Λ over the group G. Throughout the paper, we use extensively this fundamental work by Conder, in order to study which are the concerned groups.

For each group G we have described its algebraic structure, its presentation and the corresponding epimorphism, but here we will only show the algebraic structure and its presentation. In the most complicated cases, we will show also the epimorphism. In the presentations we skip the abelian relations. The full details are to be found in [2] and [3]. For groups of order 32 and 64 we use the notation given by Hall and Senior in [20]. The algebraic identification allows us to know the subgroups structure of the involved groups, and this is essential to determine all the groups that act on a surface of a given genus. Along the article C_n, D_n, DC_n and QA_n denote, respectively, the cyclic, dihedral, dicyclic and quasiabelian groups, for more details see [12, 13].

3 Groups of symmetric crosscap number 6 to 9

In symmetric crosscap number 6 some groups stand out:

1. The group $[80, 46]$: Coxeter described this group of order 80 in [12], where he named it as $(2, 5, 5; 2)$ with presentation and algebraic structure as shown in the table.
2. The group $[160, 234]$: This group contains the previous one of order 80. In [12] it is denoted as $(4, 5 \mid 2, 4)$.

Table 1: Groups of symmetric crosscap number 6.

GAP	G	Relations [+ Generators]	Reference
$[8, 4]$	$DC_2 \simeq Q$	$a^4, a^2b^2, b^{-1}aba$	[23]
$[16, 3]$	$(4, 4 \mid 2, 2)$	$a^4, b^4, (ab)^2, (a^{-1}b)^2$	[15]
$[16, 6]$	QA_4	$a^8, b^2, baba^3$	[15]
$[16, 8]$	L_4	$a^8, b^2, baba^5$	[15]
$[16, 13]$	$\langle 2, 2, 2\rangle_2$	$a^2, b^2, c^2, abcacb, abcbac, bcabac$	[15]
$[16, 14]$	$C_2 \times C_2 \times C_2 \times C_2$	a^2, b^2, c^2, d^2	[19]
$[32, 27]$	$\Gamma_4 a_1$	$a^2, b^2, c^2, d^2, e^2, cecae, dedbe$	
$[32, 43]$	$\Gamma_6 a_1$	$a^8, b^2, c^2, (ab)^2, aca^3c$	
$[80, 46]$	$(2, 5, 5; 2)$	$a^2, b^5, (ab)^5, (a^{-1}b^{-1}ab)^2$	
$[120, 35]$	$C_2 \times A_5$	a^2, [+ (1 2 3 4 5), (1 2 3)]	
$[160, 234]$	$(4, 5 \mid 2, 4)$	$a^4, b^5, (ab)^2, (a^{-1}b)^4$	

Attending to symmetric crosscap number 7, we must analyze the group [72,15], which contains the group of order 36 that appears in the table below (see [26]) and so that the algebraic structure is $((C_2 \times C_2) \rtimes C_9) \rtimes C_2$. In this case, we are going to give the epimorphism. This group has a presentation given by generators a, b, c and relations $a^4 = b^9 = c^2 = (ac)^2 = (cb)^2 = (ab)^2 = cb^{-1}ab^{-1}a^{-2} = 1$. An associated NEC group is Λ with signature $(0; +; [-]; \{(2, 4, 9)\})$ and reduced area $\frac{5}{72}$ and an epimorphism $\theta\colon \Lambda \to G$ is

$$\theta(c_{1,0}) = cb, \quad \theta(c_{1,1}) = ac, \quad \theta(c_{1,2}) = c, \quad \theta(c_{1,3}) = cb.$$

The image of $c_{1,1}c_{1,2}$ is the generator a, the image of $c_{1,2}c_{1,3}$ is the generator b, and finally, c is the image of the element $(c_{1,1}c_{1,2})^2c_{1,2}c_{1,3}(c_{1,1}c_{1,2})^3c_{1,2}c_{1,3}$. So we have the generators as images of orientation-preserving elements, and so that the group acts on a non-orientable surface.

For symmetric crosscap number 8 we just have to emphasize the group of order 504, that is $\mathrm{PSL}(2, 8)$, whose symmetric crosscap number was firstly studied in detail by Wendy Hall in [21].

To end this section, we comment some groups with symmetric crosscap number 9, where we find:

1. The group $[42, 1]$, which we call $\langle 7, 6, 5\rangle$, according to the Coxeter-Moser notation in [13]. It contains G_{21}, which is also a group of this symmetric crosscap number, and so that its algebraic structure is $G_{21} \rtimes C_2$. Its presentation can be expressed in terms of permutations taking $a = (1\ 2\ 3\ 4\ 5\ 6\ 7)$ and $b = (1\ 5\ 4\ 6\ 2\ 3)$.

Table 2: Groups of symmetric crosscap number 7.

GAP	G	Relations [+ Generators]	Reference
$[12, 1]$	DC_3	$a^6, a^3b^2, b^{-1}aba$	[23]
$[24, 8]$	$(4, 6 \mid 2, 2)$	$a^4, b^6, (ab)^2, (a^{-1}b)^2$	[15]
$[36, 3]$	$(C_2 \times C_2) \rtimes C_9$	$a^2, b^2, c^9, [a, b], c^{-1}acb, c^{-1}bcba$	
$[72, 15]$	$((C_2 \times C_2) \rtimes C_9) \rtimes C_2$	$a^4, b^9, c^2, (ac)^2, (cb)^2, (ab)^2, cb^{-1}ab^{-1}a^{-2}$	

Table 3: Groups of symmetric crosscap number 8.

GAP	G	Relations [+ Generators]	Reference
$[24, 5]$	$C_4 \times D_3$	$a^4, b^2, c^2, (bc)^3$	[16]
$[24, 10]$	$C_3 \times D_4$	$a^3, b^2, c^2, (bc)^4$	[16]
$[48, 38]$	$D_3 \times D_4$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^4$	[17]
$[56, 11]$	$(C_2 \times C_2 \times C_2) \rtimes C_7$	$a^7, b^2, c^2, d^2, badca^{-1}, caba^{-1}, daca^{-1}$	
$[504, 156]$	$\mathrm{PSL}(2, 8)$	$a^2, b^3, (ab)^7, ([a, b]^4b)^2$	[21]

2. The group $[168, 42]$ is $\mathrm{PSL}(2, 7)$. In this case, the presentation given in the table can be expressed by permutations $b = (2\,3\,4)(5\,7\,6)$ and $a = (1\,2\,3)(4\,5\,6)$ and relations $a^3 = b^3 = (ab)^4 = (a^{-1}b)^4 = 1$, see [12]. Two more presentations for this group are useful:

   (a) $R^4, S^4, (RS)^2, (R^{-1}S)^3$
   (b) $R^2, S^3, (RS)^7, (R^{-1}S^{-1}RS)^4$

   Studying this group, there are actions given by NEC groups with two different signatures:

   (i) For an NEC group Λ with signature $(0; +; [-]; \{(3, 3, 4)\})$ and reduced area $\frac{1}{24}$, we take the presentation given by permutations. So an associated epimorphism $\theta\colon \Lambda \to G$ is:

   $$\theta(c_{1,0}) = (baba^2)^2,\ \ \theta(c_{1,1}) = (a^2b)^2,\ \ \theta(c_{1,2}) = (ba^2)^2,\ \ \theta(c_{1,3}) = (baba^2)^2$$

   Consider the image of $c_{1,0}c_{1,1}$ and the image of $c_{1,1}c_{1,2}$. Then the image of the element $(c_{1,0}c_{1,1})^2c_{1,1}c_{1,2}c_{1,0}c_{1,1}(c_{1,1}c_{1,2})^2$ is $(1\ 5\ 4\ 3\ 6\ 2\ 7)$, a permutation of order 7. This element, together with the elements of order 3 and order 4, $\theta(c_{1,0}c_{1,1})$ and $\theta(c_{1,2}c_{1,3})$, generate a group of order 84 at least, but $\mathrm{PSL}(2, 7)$ is simple, so it is the full group. So the group G is generated by images of orientation-preserving elements and the group acts on a non-orientable surface.

   (ii) For an NEC group Λ with signature $(0; +; [3]; \{(4)\})$ and reduced area $\frac{1}{24}$, we use the presentation (b). An associated epimorphism $\theta\colon \Lambda \to G$ is:

   $$\theta(x_1) = S,\ \ \theta(e_1) = S^2,\ \ \theta(c_{1,0}) = R,\ \ \theta(c_{1,1}) = SRS^{-1}$$

It is clear that θ is an epimorphism. The element $c_{1,0}x_1$ is orientation-reversing, its seventh power is also orientation-reversing and the image of $(c_{1,0}x_1)^7$ is the identity element, so the group acts on a non-orientable surface.

3. The group $[336, 208]$ has order $336 = 168 \cdot 2$. Then we can guess its algebraic structure is $\mathrm{PSL}(2,7) \rtimes C_2$. We can find a presentation of this group in [10], and an epimorphism θ does exist. Hence this is the group we are looking for.

Table 4: Groups of symmetric crosscap number 9.

GAP	G	Relations [+ Generators]	Reference
$[21,1]$	G_{21}	a^7, b^3, aba^3b^{-1}	[15]
$[30,1]$	$C_5 \times D_3$	$a^5, b^2, c^2, (bc)^3$	[16]
$[30,2]$	$C_3 \times D_5$	$a^3, b^2, c^2, (bc)^5$	[16]
$[42,1]$	$\langle 7,6,5 \rangle$	$a^7, b^6, b^{-1}aba^2$	
$[60,8]$	$D_3 \times D_5$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^5$	[17]
$[168,42]$	$\mathrm{PSL}(2,7)$	$a^3, b^3, (ab)^4, (a^{-1}b)^4$	
$[336,208]$	$\mathrm{PSL}(2,7) \rtimes C_2$	$a^3, b^8, c^2, (ac)^2, (cb)^2, (ab)^2,$ $cb^{-1}(ab^{-2})^3a^{-1}$	

4 Groups of symmetric crosscap number 10 to 17

Firstly, we analyze the groups with symmetric crosscap number 10, where we can find 30 different groups, most of them of order 32, 48 and 64. We just emphasize:

1. For the group $[48, 29]$ we use two presentations, the one given in the table (generators a, b, c and relations $a^2, b^3, c^3, (bc)^4, (ab)^2, (ac)^2, [b, c](bc)^2$) and another one given by generators R, S and relations $R^8, S^3, (RS)^2$, $R^4SR^4S^{-1}$. For this case, three signatures of NEC groups are given:

   (i) For an NEC group Λ with signature $(0; +; [-]; \{(2, 2, 3, 3)\})$ and reduced area $\frac{1}{6}$ we take the presentation given in the table and an epimorphism $\theta\colon \Lambda \to G$ given by

   $$\theta(c_{1,0}) = ac,\ \ \theta(c_{1,1}) = (bc)^2,\ \ \theta(c_{1,2}) = ba,\ \ \theta(c_{1,3}) = a,\ \ \theta(c_{1,4}) = ac$$

   The group acts on a non-orientable surface, because the image of the element $c_{1,2}c_{1,3}$ is the generator b, the image of the element $c_{1,3}c_{1,4}$ is the generator c and the image of the element $c_{1,3}c_{1,1}(c_{1,2}c_{1,4})^2$ is the generator a, so these three images generate the group, and they are images of orientation-preserving elements.

   (ii) For an NEC group Λ with signature $(0; +; [3]; \{(2, 2)\})$ and reduced area $\frac{1}{6}$ we take the second presentation and so an associated epimorphism is $\theta\colon \Lambda \to G$ given by

   $$\theta(x_1) = S,\ \ \theta(e_1) = S^{-1},\ \ \theta(c_{1,0}) = RS,\ \ \theta(c_{1,1}) = R^4,\ \ \theta(c_{1,2}) = SR$$

The images of the elements $(c_{1,1}c_{1,0}e_1)^5$ and x_1 are the generators R and S respectively and both are orientation-preserving elements, so it is a group acting on a non-orientable surface.

$$\theta(x_1) = RS,\ \ \theta(x_2) = S,\ \ \theta(e_1) = SR^{-1},\ \ \theta(c_{1,0}) = R^4$$

The quotient gives a non-orientable surface because the images of the elements $x_1x_2^2$ and x_2 are the generators R and S respectively and both elements are orientation-preserving.

2. The group $[96, 70]$ can be expressed in terms of permutations, by means of the generators $a = (1\ 2)(3\ 4)(5\ 8)(6\ 7)$ and $b = (1\ 5)(2\ 8\ 3\ 6\ 4\ 7)$.
3. We can find the group $[96, 193]$ in [24], called G^*_{48}, but in the presentation given there, one relation is missing. We have added it, as can be seen in Table 5.

In symmetric crosscap number 11 we have to stand out two things: One is that the presentation of group $[108, 15]$ can be expressed in terms of permutations of S_{18} as $a = (4\ 7)(5\ 8)(6\ 9)(13\ 16)(14\ 17)(15\ 18)$ and $b = (1\ 17\ 5\ 14\ 2\ 18\ 6\ 15\ 3\ 16\ 4\ 13)$ $(7\ 12\ 9\ 11\ 8\ 10)$; and the other is that the group $[108, 17]$ is $G^{3,6,6}$ in the notation of [12].

For symmetric crosscap number 12 and 13, we have nothing to remark.

In symmetric crosscap number 14 we find several groups of order 48, and the following groups stand out:

1. The presentation of the group $[72, 43]$ has been deduced from its algebraic structure $(C_3 \times A_4) \rtimes C_2$. We have taken d as the generator of C_2 and we have determined how d acts on the other generators.
2. The same argument has been applied to the group $[96, 89]$, where its algebraic structure $(D_2 \times D_6) \rtimes C_2$ determine its presentation. In this case, e is the generator to add. The presentation is given by generators a, b, c, d, e and relations a^2, b^2, c^2, d^2, e^2, $(ab)^2$, $(cd)^6$, $eabea$, $ecdec$. Let Λ be an associated NEC group with signature $(0; +; [-]; \{(2, 2, 2, 4)\})$ and reduced area $\frac{1}{8}$, so an epimorphism is $\theta\colon \Lambda \to G$ given by
$$\theta(c_{1,0}) = e,\ \ \theta(c_{1,1}) = b,\ \ \theta(c_{1,2}) = a,\ \ \theta(c_{1,3}) = c,\ \ \theta(c_{1,4}) = e$$
The elements $c_{1,0}$, $c_{1,2}$, $c_{1,3}$, $c_{1,1}$ and $(c_{1,4}c_{1,3})^2$ have as images the generators e, a, c, b, d respectively and generate the group. On the other hand the element $(c_{1,0}c_{1,2})^2c_{1,1}$ has as image the identity element and it is orientation-reversing. Thus, the group acts on a non-orientable surface.
3. The same happens for $[96, 115]$ and its algebraic structure is $(C_2 \times D_{12}) \rtimes C_2$, where d is the generator of C_2 and so that we have to determine its relations with the other generators.

In symmetric crosscap number 15, we just note that the group $[1092, 25]$ was obtained in [21] by Wendy Hall, who proved that $\mathrm{PSL}(2, 13)$ is a group of $84(g-2)$ automorphisms of a surface of genus g, and so $g = 15$.

Nothing stands out in symmetric crosscap number 16. But in symmetric crosscap number 17 we have again the same situation that in symmetric crosscap number 14. For the group $[72, 23]$ we have deduced the presentation from its algebraic structure $(C_6 \times D_3) \rtimes C_2$, taking d as the generator of C_2 and obtaining its action on the other generators.

Table 5: Groups of symmetric crosscap number 10.

GAP	G	Relations [+ Generators]	Ref.
$[16,2]$	$C_4 \times C_4$	a^4, b^4	[19]
$[16,4]$	$C_4 \rtimes C_4$	$a^4, b^4, b^{-1}aba$	[15]
$[16,9]$	DC_4	$a^8, a^4b^2, b^{-1}aba$	[23]
$[16,10]$	$C_4 \times C_2 \times C_2$	a^4, b^2, c^2	[19]
$[24,3]$	$\langle 2,3,3\rangle$	$a^3, abab^{-1}a^{-1}b^{-1}$	[15]
$[32,5]$	$\Gamma_2 j_1$	$a^2, b^8, c^2, bcb^{-1}ac$	
$[32,6]$	$\Gamma_7 a_1$	$a^2, b^2, c^2, d^4, bdbad^{-1}, cdcbad^{-1}$	
$[32,7]$	$\Gamma_7 a_2$	$a^8, b^2, c^2, aba^3b, aca^{-1}bc$	
$[32,9]$	$\Gamma_3 a_1$	$a^2, b^8, c^2, bcbac$	
$[32,11]$	$\Gamma_3 e$	$a^4, b^4, c^2, bcba^{-1}c$	
$[32,17]$	$\Gamma_2 k$	a^{16}, b^2, aba^7b	
$[32,19]$	$\Gamma_8 a_2$	a^{16}, b^2, aba^9b	
$[32,28]$	$\Gamma_4 b_1$	$a^2, b^2, c^4, d^2, bdbad, (cd)^2$	
$[32,34]$	$\Gamma_4 a_2$	$a^4, b^4, c^2, (ac)^2, (bc)^2$	
$[32,42]$	$\Gamma_3 b$	$a^8, b^2, c^2, (ac)^2, bcba^4c$	
$[32,46]$	$C_2 \times C_2 \times D_4 \simeq \Gamma_2 a_1$	$a^2, b^2, c^2, d^2, (ab)^2, (cd)^4$	[17]
$[32,49]$	$\Gamma_5 a_1$	$a^4, b^2a^2, c^2a^2, d^2a^2, abab^{-1}, cdcd^{-1}$	
$[48,29]$	$\mathrm{GL}(2,3)$	$a^2, b^3, c^3, (bc)^4, (ab)^2, (ac)^2, [b,c](bc)^2$	
$[48,31]$	$C_4 \times A_4$	a^4, [+ (1 4)(3 2), (1 2 3)]	[18]
$[48,33]$	$\mathrm{SL}(2,3) \rtimes C_2$	$a^2, b^3, c^3, (bc)^4, abac, [b,c]^2(bc)^2$	
$[48,50]$	$(C_2{\times}C_2{\times}C_2{\times}C_2){\rtimes}C_3$	$a^2, b^3, c^3, (cb)^2, (ab^{-1})^3,$ $c^{-1}b^{-1}abca, cbab^{-1}c^{-1}a$	
$[64,128]$	$\Gamma_{15} a_1$	$a^2, b^2, c^2, e^2, f^2, d^2f, [a,b]fd^{-1},$ $[a,c]e, [a,d]f, [b,d]f$	
$[64,134]$	$\Gamma_{26} a_1$	$a^2, b^2, c^2, e^2, f^2, d^2f, [a,b]fd^{-1},$ $[a,c]e, [a,d]f, [b,d]f, [b,e]f, [c,d]f$	
$[64,138]$	$\Gamma_{25} a_1$	$a^2, b^2, c^2, d^2, e^2, f^2, [a,b]d, [a,c]e,$ $[b,e]f, [c,d]f$	
$[64,190]$	$\Gamma_{19} a_1$	$a^2, b^2, c^2, f^2, d^2fe^{-1}, e^2f,$ $[a,b]e^{-1}d^{-1}, [a,c]f, [a,d]fe^{-1},$ $[a,e]f, [b,d]fe^{-1}, [b,e]f$	
$[96,70]$	$((C_2 \times C_2 \times C_2 \times C_2)$ $\rtimes C_3) \rtimes C_2$	$a^2, b^6, (bab^{-1}a)^2, (b^{-2}a)^3$	
$[96,187]$	$(C_2 \times S_4) \rtimes C_2$	$a^4, b^{12}, c^2, (ab)^2, (cb)^2, (ac)^2,$ $cb^{-1}ab^{-1}a^{-2}$	
$[96,193]$	$\mathrm{GL}(2,3) \rtimes C_2$	$a^2, b^8, c^3, (bc)^2, (ac)^2, (ab)^2, b^4cb^4c^{-1}$	
$[96,227]$	$((C_2 \times C_2 \times C_2 \times C_2)$ $\rtimes C_3) \rtimes C_2$	$a^2, b^3, c^2, d^2, e^2, f^2, (ba)^2, cada,$ $cbdb^{-1}, daca, dbdcb^{-1}, eafa,$ $ebfeb^{-1}, faea, fbeb^{-1}$	
$[192,955]$	$(((C_2 \times C_2 \times C_2 \times C_2)$ $\rtimes C_3) \rtimes C_2) \rtimes C_2$	$a^4, b^6, c^2, (ab)^2, (cb)^2, (ac)^2, (ab^{-1})^4,$ $cb^{-1}ab^2a^{-1}b^3a^{-1}$	

Table 6: Groups of symmetric crosscap number 11.

GAP	G	Relations [+ Generators]	Ref.
$[18, 5]$	$C_6 \times C_3$	a^6, b^3	[19]
$[27, 3]$	$(3, 3 \mid 3, 3)$	$a^3, b^3, (ba)^3, (b^{-1}a)^3$	[15]
$[36, 13]$	$C_2 \times ((C_3 \times C_3) \rtimes C_2)$	$a^2, b^3, c^3, d^2, (ba)^2, (ca)^2$	
$[54, 5]$	$(2, 3, 6; 3)$	$a^3, b^6, (ab)^2, (ba^{-1}b)^3$	
$[54, 8]$	$((C_3 \times C_3) \rtimes C_3) \rtimes C_2$	$a^2, b^3, c^2, (b^{-1}a)^2, (ca)^3,$ $(b^{-1}c)^2(bc)^2, (ab^{-1}c)^2bac$	
$[108, 15]$	$((C_3 \times C_3) \rtimes C_3) \rtimes C_4$	$a^2, (b^{-2}a)^3, b^{-1}ab^4ab^{-3},$ $b^{-1}abab^{-2}abab^{-1}a$	
$[108, 17]$	$G^{3,6,6}$	$a^2, b^2, c^2, (ab)^2, (ac)^3, (bc)^6, (abc)^6$	
$[216, 87]$	$(((C_3 \times C_3) \rtimes C_3)$ $\rtimes C_4) \rtimes C_2$	$a^4, b^6, c^2, (ab)^2, (cb)^2, (ac)^2,$ $c(b^{-1}a)^3a$	

Table 7: Groups of symmetric crosscap number 12.

GAP	G	Relations [+ Generators]	Reference
$[20, 1]$	DC_5	$a^{10}, a^5b^2, b^{-1}aba$	[23]
$[40, 5]$	$C_4 \times D_5$	$a^4, b^2, c^2, (bc)^5$	[16]
$[40, 8]$	$(C_{10} \times C_2) \rtimes C_2$	$a^{10}, b^2, (aba)^2, (a^{-1}b)^2(ab)^2$	
$[40, 10]$	$C_5 \times D_4$	$a^5, b^2, c^2, (bc)^4$	[16]
$[40, 12]$	$C_2 \times \langle 5, 4, 2 \rangle$	$a^5, b^4, bab^{-1}a^3$	
$[80, 39]$	$D_5 \times D_4$	$a^2, b^2, c^2, d^2, (ab)^5, (cd)^4$	[17]
$[240, 189]$	$C_2 \times S_5$	a^2, [+ (1 2 3 4 5), (1 2)]	

Table 8: Groups of symmetric crosscap number 13.

GAP	G	Relations [+ Generators]	Reference
$[42, 3]$	$C_7 \times S_3$	a^7, [+ (1 2 3), (1 2)]	[16]
$[42, 4]$	$C_3 \times D_7$	$a^3, b^2, c^2, (bc)^7$	[16]
$[52, 3]$	$C_{13} \rtimes C_4$	$a^4, b^{13}, baba^{-1}$	
$[60, 9]$	$C_5 \times A_4$	a^5, [+ (1 2 3), (1 4)(2 3)]	[18]
$[84, 8]$	$D_3 \times D_7$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^7$	[17]
$[120, 38]$	$(C_5 \times A_4) \rtimes C_2$	$a^4, b^{15}, c^2, (ab)^2, (cb)^2, (ac)^2,$ $cb^{-1}ab^{-1}a^2$	

Table 9: Groups of symmetric crosscap number 14.

GAP	G	Relations [+ Generators]	Ref.
$[16, 12]$	$C_2 \times Q$	$a^4, a^2b^2, c^2, b^{-1}aba$	[23]
$[24, 4]$	$C_2 \times DC_3$	$a^3, b^4, c^2, bab^{-1}a$	[18]
$[24, 7]$	DC_6	$a^{12}, a^6b^2, b^{-1}aba$	[23]
$[24, 15]$	$C_6 \times C_2 \times C_2$	a^6, b^2, c^2	[19]
$[32, 48]$	$\Gamma_2 b$	$a^4, b^2, c^2, d^2, bcba^2c$	
$[36, 11]$	$C_3 \times A_4$	a^3, [+ (1 2)(3 4), (1 2 3)]	[18]
$[36, 12]$	$C_6 \times D_3$	$a^6, b^2, c^2, (bc)^3$	[16]
$[48, 6]$	$C_{24} \rtimes C_2$	$a^{24}, b^2, baba^{13}$	
$[48, 14]$	$(C_{12} \times C_2) \rtimes C_2$	$a^3, b^4, c^4, (bc)^2, (b^{-1}c)^2, c^{-1}aca$	
$[48, 21]$	$C_3 \times (4, 4 \mid 2, 2)$	$a^3, b^4, c^4, (bc)^2, (b^{-1}c)^2$	
$[48, 24]$	$C_3 \times QA_4$	$a^8, b^2, c^3, baba^3$	
$[48, 37]$	$(C_{12} \times C_2) \rtimes C_2$	$a^3, b^2, c^2, d^2, dcbcdb, dcbdbc, bdcdbc, (ba)^2, (da)^2$	
$[48, 43]$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	$a^4, b^6, c^2, (ab)^2, (a^{-1}b)^2$	
$[48, 49]$	$C_2 \times C_2 \times A_4$	a^2, b^2, [+ (1 2)(3 4), (1 2 3)]	
$[48, 51]$	$D_2 \times D_6$	$a^2, b^2, c^2, d^2, (ab)^2, (cd)^6$	[17]
$[72, 42]$	$C_3 \times S_4$	a^3, [+ (1 2), (1 2 3 4)]	
$[72, 43]$	$(C_3 \times A_4) \rtimes C_2$	$a^3, b^2, c^3, d^2, (da)^2, (dc)^2$, [+ $b = (1\ 2)(3\ 4)$, $c = (1\ 2\ 3)$]	
$[72, 44]$	$A_4 \times S_3$	[+ (1 2)(3 4), (1 2 3), (5 6 7), (5 6)]	
$[72, 46]$	$D_3 \times D_6$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^6$	[17]
$[96, 89]$	$(D_2 \times D_6) \rtimes C_2$	$a^2, b^2, c^2, d^2, e^2, (ab)^2, (cd)^6, eabea, ecdec$	
$[96, 115]$	$(C_2 \times D_{12}) \rtimes C_2$	$a^2, b^2, c^2, d^2, (bc)^{12}, dcbdc$	
$[96, 226]$	$C_2 \times C_2 \times S_4$	a^2, b^2, [+ (1 2 3 4), (1 2)]	
$[144, 183]$	$S_3 \times S_4$	[+ (1 2 3), (1 2), (4 5 6 7), (4 5)]	
$[180, 19]$	$A_5 \times C_3$	a^3, [+ (1 3 2 4 5), (2 4 3), (2 4)(1 3)]	
$[360, 121]$	$A_5 \times D_3$	$a^3, b^{10}, c^2, (ab)^2, (cb)^2, (ac)^2, b^{-2}ab^3a^{-1}b^{-4}a^{-1}c$	

Table 10: Groups of symmetric crosscap number 15.

GAP	G	Relations [+ Generators]	Ref.
$[24, 1]$	$\langle -2, 2, 3 \rangle$	$a^8, b^8, (a^3b)^3, a^2b^6, a^2(b^{-1}a^{-1})^3, b^2(b^{-1}a^{-1})^3$	[15]
$[39, 1]$	$C_{13} \rtimes C_3$	$a^3, b^{13}, bab^{10}a^{-1}$	
$[48, 15]$	$(C_3 \times D_4) \rtimes C_2$	$a^2, b^8, c^3, (ab)^2, (ac)^2, b^{-1}cbc$	
$[78, 1]$	$(C_{13} \rtimes C_3) \rtimes C_2$	$a^2, b^3, c^{13}, (ca)^2, cbc^{10}b^{-1}$	
$[1092, 25]$	$\mathrm{PSL}(2, 13)$	$a^3, b^7, c^2, (ab)^2, (cb)^2, (ac)^2, b^{-1}(ab^{-2})^6a^{-1}c$	[21]

Table 11: Groups of symmetric crosscap number 16.

GAP	G	Relations [+ Generators]	Reference
$[28, 1]$	DC_7	$a^{14}, a^7b^2, b^{-1}aba$	[23]
$[56, 4]$	$C_4 \times D_7$	$a^4, b^2, c^2, (bc)^7$	[16]
$[56, 7]$	$(C_{14} \times C_2) \rtimes C_2$	$a^2, b^{14}, (bab)^2, (b^{-1}a)^2(ba)^2$	
$[56, 9]$	$C_7 \times D_4$	$a^7, b^2, c^2, (bc)^4$	[16]
$[72, 16]$	$C_2 \times ((C_2 \times C_2) \rtimes C_9)$	$a^9, b^2, c^2, d^2, bacba^{-1}, caba^{-1}$	
$[112, 31]$	$D_7 \times D_4$	$a^2, b^2, c^2, d^2, (ab)^7, (cd)^4$	[17]
$[144, 109]$	$(C_2 \times ((C_2 \times C_2) \rtimes C_9)) \rtimes C_2$	$a^4, b^{18}, c^2, (ab)^2, (cb)^2, (ac)^2, cb^{-1}ab^{-1}a^{-2}$	

Table 12: Groups of symmetric crosscap number 17.

GAP	G	Relations [+ Generators]	Ref.
$[25, 2]$	$C_5 \times C_5$	a^5, b^5	[14]
$[27, 2]$	$C_9 \times C_3$	a^9, b^3	[14]
$[27, 4]$	$C_9 \rtimes C_3$	a^3, b^9, bab^5a^{-1}	[15]
$[36, 6]$	$C_3 \times DC_3$	$a^{12}, b^3, baba^{-1}$	[18]
$[50, 3]$	$C_5 \times D_5$	$a^5, b^2, c^2, (bc)^5$	[16]
$[50, 4]$	$(C_5 \times C_5) \rtimes C_2$	$a^2, b^5, c^5, (ba)^2, (ca)^2$	
$[54, 3]$	$C_3 \times D_9$	$a^3, b^2, c^2, (bc)^9$	[16]
$[54, 4]$	$C_9 \times D_3$	$a^9, b^2, c^2, (bc)^3$	[16]
$[54, 6]$	$(C_9 \rtimes C_3) \rtimes C_2$	$a^2, b^9, c^3, (ba)^2, cb^7c^{-1}b^{-1}$	
$[54, 7]$	$(C_9 \times C_3) \rtimes C_2$	$a^2, b^3, c^9, (ba)^2, (ca)^2$	
$[68, 3]$	$C_{17} \rtimes C_4$	a^4, b^{17}, bab^4a^{-1}	
$[72, 23]$	$(C_6 \times D_3) \rtimes C_2$	$a^6, b^2, c^2, d^2, (bc)^3, bdcbd, dada^3$	
$[72, 39]$	$(C_3 \times C_3) \rtimes C_8$	$a^8, b^3, c^3, baca^{-1}, cab^{-1}a^{-1}$	
$[100, 12]$	$(C_5 \times C_5) \rtimes C_4$	$a^4, b^5, c^5, bab^3a^{-1}, cac^3a^{-1}$	
$[100, 13]$	$D_5 \times D_5$	$a^2, b^2, c^2, d^2, (ab)^5, (cd)^5$	[17]
$[108, 16]$	$D_3 \times D_9$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^9$	[17]
$[200, 43]$	$(D_5 \times D_5) \rtimes C_2$	$a^4, b^{10}, c^2, (ab)^2, (cb)^2, (ac)^2, [ab, ba], cb^{-1}(ab^{-3})^2a^{-2}$	
$[360, 118]$	A_6	[+ (1 4 2 3 5), (3 5 4), (1 2 4 3)(5 6)]	
$[720, 764]$	$A_6 \rtimes C_2$	$a^3, b^8, c^2, (ab)^2, (cb)^2, (ac)^2, cb^{-1}ab^3ab^{-2}a^{-1}ba^{-1}b^{-3}a^{-1}$	

5 Groups with symmetric crosscap number $12k + 3$

Firstly, the strong symmetric genus is the minimum genus of any Riemann surface on which G acts, preserving orientation. For this parameter, there is a group of every strong symmetric genus, [25]. The symmetric genus is the smallest non-negative integer g such that the group G acts faithfully on a closed orientable surface of genus g (not necessarily preserving orientation). For this parameter, the spectrum includes every non-negative integer $g \not\equiv 8$ or $14 \pmod{18}$, and moreover, if a gap occurs at some $g \equiv 8$ or $14 \pmod{18}$, then the prime-power factorization of $g - 1$ includes some factor $p^e \equiv 5 \pmod 6$, [11].

In the study of the spectrum of the symmetric crosscap number, the groups with symmetric crosscap number of the form $12k + 3$ are very interesting. It is known that for all $n \neq 12k + 3$, there is a finite group with symmetric crosscap number n, see [6]. Conversely, for some values $n = 12k + 3$, it is not known whether there exists a group with symmetric crosscap number n. So that, we can enunciate some theorems whereby we find infinite families of groups whose symmetric crosscap number is of the type $12k + 3$.

The symmetric crosscap numbers obtained in Theorems 5.1 to 5.5, although of $12k+3$ form, were already obtained for other groups, as we can see in the proofs. In the case of Theorem 5.6, also these numbers n were already covered, since the group $C_{7(12k+7)} \rtimes C_3$, in the terms of the statement, has symmetric crosscap number $84k + 51$, see [6]. But they are important because they give more examples of groups of this type of n, helping us to see how these groups act.

Theorem 5.1. *Let* $n = 12k + 3$ *be such that* $n - 2$ *has all its prime factors congruent to* $1 \pmod 3$. *Then*

$$C_{12k+1} \rtimes C_3 \quad \textit{and} \quad (C_{12k+1} \rtimes C_3) \rtimes C_2$$

have symmetric crosscap number n.

Proof. Firstly we have that $C_{12k+1} \rtimes C_3$ has a presentation given by generators a, b such that $a^3 = b^{12k+1} = (ab)^3 = 1$. Now let Λ be an NEC group with signature $(1; -; [3, 3]; \{-\})$, whose reduced area is $\frac{1}{3}$. We can define an epimorphism $\theta\colon \Lambda \to G$ given by

$$\theta(x_1) = a^{-1},\ \ \theta(x_2) = ab,\ \ \theta(d_1) = b^{6k}$$

We have that the images of x_1 and x_1x_2 are the generators a^{-1} and b respectively, and both are preserving-orientation elements, then we have that it is a group that acts on a non-orientable surface. Besides, the NEC group area is minimal ([6]), and so the symmetric crosscap number of $C_{12k+1} \rtimes C_3$ is n.

Now we have $(C_{12k+1} \rtimes C_3) \rtimes C_2$ that has a presentation given by generators a, b, c and relations $a^3 = b^{12k+1} = c^2 = (ab)^3 = 1, ca = ac$ and $bc = cb^{-1}$. Now let Λ be an NEC group with signature $(0; +; [2, 3]; \{(-)\})$, whose reduced area is $\frac{1}{6}$. Therefore, if we define an epimorphism from this NEC group, $(C_{12k+1} \rtimes C_3) \rtimes C_2$ will have symmetric crosscap number less or equal to n. We can define an epimorphism $\theta\colon \Lambda \to G$ given by

$$\theta(x_1) = cb,\ \ \theta(x_2) = b^{-1}a^{-1},\ \ \theta(e_1) = ac,\ \ \theta(c_{1,0}) = c$$

We have that the element $c_{1,0}$, the element $e_1c_{1,0}$ and the element $c_{1,0}x_1$ have as images the generators c, a and b respectively. Besides the element $(e_1c_{1,0})^3$ has as image the identity element and it is orientation-reversing, so we have just proved that the group acts on a non-orientable surface. Because of this epimorphism we can say that $(C_{12k+1} \rtimes C_3) \rtimes C_2$

has symmetric crosscap number at most n. But since it contains $C_{12k+1} \rtimes C_3$, that has symmetric crosscap number n, $\tilde{\sigma}((C_{12k+1} \rtimes C_3) \rtimes C_2) = n$. $\square$

Theorem 5.2. *Let $n = 12k + 3$ be such that $n - 2 = m^2$ is a square. Then:*

(i) $(3, 3 \mid 3, m)$ *has symmetric crosscap number n.*

(ii) There are two groups with algebraic structure $(3, 3 \mid 3, m) \rtimes C_2$*, namely* $(2, 3, 2m; 3)$ *and* $(2, 3, 6; m)$*, that have symmetric crosscap number n.*

Proof. Firstly we have that the group $(3, 3 \mid 3, m)$ of order $3m^2$ has a presentation given by generators a, b and relations $a^3 = b^3 = (ab)^3 = (a^{-1}b)^m = 1$. From [15], we know that this group has symmetric crosscap number $m^2 + 2$.

Now we have two groups with algebraic structure $(3, 3 \mid 3, m) \rtimes C_2$:

(i) The first one, that is the group $(2, 3, 2m; 3)$ in the notation of [12], of order $6m^2$, has a presentation given by generators a, b, c and relations $a^3 = b^3 = c^2 = (ab)^3 = (a^{-1}b)^m = 1, ca = a^2c$ and $cb = b^2c$. Take an NEC group Λ with signature $(0; +; [2]; \{(3, 3)\})$, that has reduced area $\frac{1}{6}$. We define an epimorphism $\theta \colon \Lambda \to G$ given by

$$\theta(x_1) = c, \ \ \theta(e_1) = c, \ \ \theta(c_{1,0}) = ac, \ \ \theta(c_{1,1}) = cb, \ \ \theta(c_{1,2}) = a^{-1}c$$

We have that $\theta(x_1) = c, \theta(c_{1,0}x_1) = a$ and $\theta(x_1c_{1,1}) = b$, and the element $(e_1c_{1,0})^3$ has as image the identity element and it is orientation-reversing. Thereby we have proved that the group acts on a non-orientable surface. Thereupon we have that this group has symmetric crosscap number at most m^2+2, but as it contains $(3, 3 \mid 3, m)$ that has that symmetric crosscap number n, then we have proved that $\tilde{\sigma}((2, 3, 2m; 3)) = n$.

(ii) The second one, that is the group $(2, 3, 6; m)$ in the notation of [12], also with order $6m^2$, has a presentation given by generators a, b, c and relations $a^3 = b^3 = c^2 = (ab)^3 = (a^{-1}b)^m = 1, ac = ca$ and $bc = cb^{-1}$. For an NEC group Λ with signature $(0; +; [2, 3]; \{(-)\})$ and reduced area $\frac{1}{6}$, we define an epimorphism $\theta \colon \Lambda \to G$ given by

$$\theta(x_1) = cb, \ \ \theta(x_2) = b^{-1}a^{-1}, \ \ \theta(e_1) = ac, \ \ \theta(c_{1,0}) = c$$

We have that the element $c_{1,0}$, the element $e_1c_{1,0}$ and the element $c_{1,0}x_1$ have as images the generators c, a and b respectively. Besides, the element $(e_1c_{1,0})^3$ has as image the identity element and it is orientation-reversing, so that we have proved that the group acts on a non-orientable surface. So this group has symmetric crosscap number at most $m^2 + 2$, but as it contains $(3, 3 \mid 3, m)$, that has that symmetric crosscap number, we have proved that $\tilde{\sigma}((2, 3, 6; m)) = n$. $\square$

Theorem 5.3. *Let $n = 12k + 3$ be such that $n - 2 = m^2$ is a square. The symmetric crosscap number of the group $G^{3,6,2m} \approx ((3, 3 \mid 3, m) \rtimes C_2) \rtimes C_2$ is n.*

Proof. The group $G^{3,6,2m}$ of order $12m^2$ has a presentation given by generators a, b, X, c and relations $a^3 = b^3 = X^2 = c^2 = (ab)^3 = (a^{-1}b)^m = 1$, $aX = Xa^{-1}, bX = Xb^{-1}, ac = ca^{-1}$ and $cb = bc$. For an NEC group Λ with signature $(0; +; [-]; \{(2, 2, 2, 3)\})$ we define an epimorphism $\theta \colon \Lambda \to G$, given by

$$\theta(c_{1,0}) = aX, \ \ \theta(c_{1.1}) = Xc, \ \ \theta(c_{1,2}) = c, \ \ \theta(c_{1,3}) = Xb, \ \ \theta(c_{1,4}) = aX$$

We have that the element $c_{1,2}$ has as image the generator c, the element $c_{1,1}c_{1,2}$ has as image the generator X, the element $c_{1,1}c_{1,2}c_{1,3}$ has as image the generator b, and the element $c_{1,4}c_{1,1}c_{1,2}$ has as image the generator a. Moreover, the element $(c_{1,0}c_{1,1}c_{1,2})^3$ has as image the identity element and it is orientation-reversing, so that we have proved that the group acts on a non-orientable surface. The reduced area of the associated NEC group is $\frac{1}{12}$, then we have proved that this group has symmetric crosscap number at most m^2+2, but as it contains $(2,3,2m;3)$ (see [12]), that has the same symmetric crosscap number, then our group has symmetric crosscap number n. □

Theorem 5.4. *Let n be such that $n = 48k+39$. The symmetric crosscap number of $DC_3 \times C_{6k+5}$ and $(DC_3 \times C_{6k+5}) \rtimes C_2$ is n.*

Proof. We have a presentation of the group $(DC_3 \times C_{6k+5}) \rtimes C_2$, given by generators a, b, X, Y and relations $a^4 = b^3 = X^{6k+5} = Y^2 = 1$, $ba = ab^2, aY = Ya^3, XY = YX^{-1}, Yb = b^2Y$ and the rest commute. Let Λ be an NEC group with signature $(0; +; [-]; \{(2,2,3,4(6k+5))\})$, which has reduced area $\frac{8(6k+5)-3}{24(6k+5)}$. So if we can define the adequate epimorphism, we will have that this group has symmetric crosscap number at most $48k+39$, but as it contains $DC_3 \times C_{6k+5}$ that has the same symmetric crosscap number (see [18]), we will be done. Then we take an epimorphism $\theta\colon \Lambda \to G$ given by

$$\theta(c_{1,0}) = YXa,\ \ \theta(c_{1,1}) = a^2,\ \ \theta(c_{1,2}) = a^2Y,\ \ \theta(c_{1,3}) = Yba^2,\ \ \theta(c_{1,4}) = YXa$$

We have that the element $c_{1,1}c_{1,2}$ has as image the generator Y, the element $c_{1,1}c_{1,2}c_{1,3}c_{1,1}$ has as image the generator b. We differentiate between two cases according to the value of k:

(a) If k is even, then we have that the element $(c_{1,3}c_{1,4})^{3(6k+5)+1}$ has as image the generator X and the element $c_{1,1}c_{1,2}c_{1,3}c_{1,1}c_{1,3}c_{1,4}(c_{1,3}c_{1,4})^{(6k+5)-1}$ has as image the generator a.

(b) If k is odd, then we have that the element $(c_{1,3}c_{1,4})^{(6k+5)+1}$ has as image the generator X and the element $c_{1,1}c_{1,2}c_{1,3}c_{1,1}c_{1,3}c_{1,4}(c_{1,3}c_{1,4})^{3(6k+5)-1}$ has as image the generator a.

So in both cases, we have generated the group with images of elements that preserve the orientation, and thus we have proved that it acts on a non-orientable surface. □

Theorem 5.5. *Let $n = 24k+15$. The symmetric crosscap number of $C_3 \rtimes C_{12k+8}$ and $(C_3 \rtimes C_{12k+8}) \rtimes C_2$ is n.*

Proof. We have a presentation of the group $(C_3 \rtimes C_{12k+8}) \rtimes C_2$, given by generators a, b, c and relations $a^3 = b^{8+12k} = c^2 = 1$, $ab = ba^{-1}, ca = a^{-1}c, cb = b^{-1}c$. Let Λ be an NEC group with signature $(0; +; [-]; \{(2,2,3,12k+8)\})$, which has reduced area $\frac{13+24k}{6(12k+8)}$. So if we have an epimorphism, we will have that this group has symmetric crosscap number at most $24k+15$, but as it contains $C_3 \rtimes C_{12k+8}$ that has the same symmetric crosscap number (see [6]), we will be done. Then we take an epimorphism $\theta\colon \Lambda \to G$ given by

$$\theta(c_{1,0}) = cb,\ \ \theta(c_{1,1}) = b^{4+6k},\ \ \theta(c_{1,2}) = ac,\ \ \theta(c_{1,3}) = c,\ \ \theta(c_{1,4}) = cb$$

We have that the element $c_{1,3}c_{1,4}$ has as image the generator b, the element $c_{1,2}c_{1,3}$ has as image the generator a and the element $c_{1,0}c_{1,1}(c_{1,3}c_{1,4})^{3+6k}$ has as image the generator c. So we have generated the group with images of elements that preserve the orientation, and thus we have proved that it acts on a non-orientable surface. □

Theorem 5.6. *Let $n = 84k + 51$ be such that $12k + 7$ has all its prime factors congruent to* $1 \pmod 3$. *Then the symmetric crosscap number of* $C_4 \times (C_{12k+7} \rtimes C_3)$ *is* n.

Proof. Firstly we have to indicate, that within the conditions in the statement, there exist groups with order $36k + 21$ with algebraic structure $C_{12k+7} \rtimes C_3$ and with a presentation given by generators a, b and relations $a^3 = b^{12k+7} = (ab)^3 = 1$. We call c a generator of C_4.

Let Λ be an NEC group with signature $(0; +; [3, 12]; \{(-)\})$ and reduced area $\frac{7}{12}$, and define an associated epimorphism $\theta\colon \Lambda \to G$ given by:

$$\theta(x_1) = ba,\ \ \theta(x_2) = a^{-1}c,\ \ \theta(e_1) = c^{-1}b^{-1},\ \ \theta(c_{1,0}) = c^2$$

The element x_2^9 has as image the generator c, the element x_2^8 has as image the generator a, and the element $x_1x_2^4$ has as image the generator b. So we have generated the group with images of orientation-preserving elements and so that it acts on a non-orientable surface. Therefore the symmetric crosscap number of the group will be at most n.

On the other hand, the group $C_{12k+7} \rtimes C_3$ can be generated by two elements of order 3 and this condition cannot be lowered. Similarly, an element of order 4 is needed to generate the group C_4. Hence the area of Λ is minimal, because one element of order a multiple of 4 and two elements of order a multiple of an odd number are necessary. Thus, the symmetric crosscap number of our group is n. $\square$

So that, we need to study some low k to try to find some clues in order to get new numbers in the spectrum. In the previous section we have studied symmetric crosscap number 15, and in this section we study $12k + 3$ for $k = 2, 3, 4, 5$. For each symmetric crosscap number we give the complete list of all groups with that symmetric crosscap number. For that, we have used the Conder's list and the previous theorems to know the algebraic structure and the presentation of some of the groups. It is important to note that all groups G with $\tilde{\sigma}(G) = 15, 27, 39, 51$ are provided by the results in the current section.

Table 13: Groups of symmetric crosscap number 27.

GAP	G	Relations [+ Generators]	Reference
$[40, 3]$	$C_5 \rtimes C_8$	a^8, b^5, bab^3a^{-1}	[3]
$[75, 2]$	$(3, 3 \mid 3, 5)$	$a^3, b^3, (ab)^3, (a^{-1}b)^5$	[15]
$[150, 5]$	$(2, 3, 10; 3)$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^5, (ca)^2, (cb)^2$	Theorem 5.2
$[150, 6]$	$(2, 3, 6; 5)$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^5, (cb)^2$	Theorem 5.2
$[300, 25]$	$G^{3,6,10}$	$a^3, b^3, c^2, d^2, (ab)^3, (a^{-1}b)^5, (ca)^2, (cb)^2, (ad)^2$	Theorem 5.3

From the group $[40, 3]$, with symmetric crosscap number 27, and from the group $[96, 1]$, with symmetric crosscap number 63, other families have been obtained that cover all numbers of the form $24k + 15$ and $60k + 27$, see [6]. So that, it is totally necessary to know the algebraic structure of the groups we have been studying. Another feature of this study is to obtain the groups which are the full automorphism group of a surface of a given genus. This was already done for $g \leq 5$ in [8], for $g = 6$ in [4] and for $g = 7$ in [5].

Table 14: Groups of symmetric crosscap number 39.

GAP	G	Relations [+ Generators]	Reference
$[60, 1]$	$DC_3 \times C_5$	$a^4, b^3, c^5, baba^{-1}$	[18]
$[111, 1]$	$C_{37} \rtimes C_3$	$a^3, b^{37}, bab^{27}a^{-1}$	Theorem 5.1
$[120, 12]$	$(DC_3 \times C_5) \rtimes C_2$	$a^4, b^3, c^5, d^2, baba^{-1}, (ad)^2, (cd)^2, (db)^2$	Theorem 5.4
$[222, 1]$	$(C_{37} \rtimes C_3) \rtimes C_2$	$a^3, b^{37}, c^2, bab^{27}a^{-1}, (bc)^2$	Theorem 5.1

Table 15: Groups of symmetric crosscap number 51.

GAP	G	Relations [+ Generators]	Reference
$[84, 2]$	$C_4 \times (C_7 \rtimes C_3)$	$a^3, b^7, c^4, bab^5a^{-1}$	Theorem 5.6
$[147, 1]$	$C_{49} \rtimes C_3$	$a^3, b^{49}, bab^{31}a^{-1}$	Theorem 5.1
$[147, 5]$	$(3, 3 \mid 3, 7)$	$a^3, b^3, (ab)^3, (a^{-1}b)^7$	[15]
$[294, 1]$	$(C_{49} \rtimes C_3) \rtimes C_2$	$a^3, b^{49}, c^2, bab^{31}a^{-1}, (ac)^2$	Theorem 5.1
$[294, 7]$	$(2, 3, 14; 3)$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^7, (ca)^2, (cb)^2$	Theorem 5.2
$[294, 14]$	$(2, 3, 6; 7)$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^7, (cb)^2$	Theorem 5.2
$[588, 35]$	$G^{3,6,14}$	$a^3, b^3, c^2, d^2, (ab)^3, (a^{-1}b)^7, (ca)^2,$ $(cb)^2, (ad)^2$	Theorem 5.3

Table 16: Groups of symmetric crosscap number 63.

GAP	G	Relations [+ Generators]	Reference
$[96, 1]$	$C_3 \rtimes C_{32}$	$a^3, b^{32}, abab^{-1}$	Theorem 5.5
$[183, 1]$	$C_{61} \rtimes C_3$	$a^3, b^{61}, bab^{48}a^{-1}$	Theorem 5.1
$[192, 78]$	$(C_3 \rtimes C_{32}) \rtimes C_2$	$a^3, b^{32}, c^2, abab^{-1}, (ca)^2, (cb)^2$	Theorem 5.5
$[366, 1]$	$(C_{61} \rtimes C_3) \rtimes C_2$	$a^{61}, b^3, c^2, aba^{48}b^{-1}, bcb^{-1}c, (ac)^2$	Theorem 5.1

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Saturation number of lattice animals

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Abstract

A matching M in a graph G is maximal if no other matching of G has M as a proper subset. The saturation number of G is the cardinality of any smallest maximal matching in G. In this paper we investigate saturation number for several classes of square and hexagonal lattice animals.

Keywords: Maximal matching, saturation number, lattice animal, polyomino graph, benzenoid graph, coronene.

Math. Subj. Class.: 92E10, 05C70, 05C35, 05C90

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1 Introduction

A lattice animal is any bounded subset of a regular lattice in the plane whose boundary is made of simple closed curve following lattice edges. In this paper we study the saturation number of hexagonal and square lattice animals.

The saturation number $s(G)$ of a graph G is the cardinality of a smallest maximal matching in G. Maximal matchings serve as models of adsorption of dimers (those that occupy two adjacent atoms) to a molecule. It can occur that the bonds in a molecule are not efficiently saturated by dimers, and therefore, their number is below the theoretical maximum. Hence, the saturation number provides an information on the worst possible case of adsorption. Besides in chemistry the saturation number has a number of interesting applications in engineering and networks. The problem of determining the saturation number is equivalent to the problem of finding the edge domination number in a graph. Moreover, if a graph G has an efficient edge dominating set D, it holds $s(G) = |D|$ (see [7]). Previous work on the saturation number includes research on random graphs [8, 9], on benzenoid systems [5], fullerenes [1, 2, 4], and nanotubes [7]. Recent results on related concepts can be found in [3, 6].

2 Preliminaries

A *matching* M in a graph G is a set of edges of G such that no two edges from M share a vertex. A matching M is a *maximum matching* if there is no matching in G with greater cardinality. The cardinality of any maximum matching in G is denoted by $\nu(G)$ and called the *matching number* of G. If every vertex of G is incident with an edge of M, the matching M is called a *perfect matching* (in chemistry perfect matchings are known as *Kekulé structures*).

A matching M in a graph G is *maximal* if it cannot be extended to a larger matching in G. Obviously, every maximum matching is also maximal, but the opposite is generally not true. A matching M is a *smallest maximal matching* if there is no maximal matching in G with smaller cardinality. The cardinality of any smallest maximal matching in G is the *saturation number* of G, denoted by $s(G)$.

The following lemma is very useful for proving lower bounds for the saturation number. The proof can be found in [7]. See also [8, 9].

Lemma 2.1. *Let G be a graph and let A and B be maximal matchings in G. Then $|A| \geq \frac{|B|}{2}$ and $|B| \geq \frac{|A|}{2}$.*

This result implies the lower bound $s(G) \geq \frac{\nu(G)}{2}$. In particular, in graphs with perfect matchings the saturation number cannot be smaller than one quarter of the number of vertices, $s(G) \geq \frac{n}{4}$.

A *polyomino system* consists of a cycle C in the infinite square lattice together with all squares inside C. A *polyomino graph* is the underlying graph of a polyomino system.

A *benzenoid system* consists of a cycle C in the regular infinite hexagonal lattice together with all hexagons inside C. A *benzenoid graph* is the underlying graph of a benzenoid system.

Let G be a benzenoid graph or a polyomino graph. The vertices lying on the outer face of G are called *external*; other vertices, if any, are called *internal*. Graph G without internal vertices is called *catacondensed*. If no inner face in a catacondensed graph G is adjacent to more than two other inner faces, we say that graph G is *unbranched* or that it is a *chain*.

In each chain G there are exactly two inner faces adjacent to one other inner face; those two inner faces are called *terminal*, while any other inner faces are called *interior*. The number of inner faces in chain G is called its *length*. An interior inner face is called *straight* if the two edges it shares with other inner faces are parallel, i.e. opposite to each other. If the shared edges are not parallel, the inner face is called *kinky*. If all interior inner faces of a chain G are straight, the chain is called *linear*.

There is also another terminology, calling straight inner faces *linear*, and kinky inner faces *angular*. By introducing abbreviations L and A, respectively, for linear and angular inner faces, each chain can be represented as a word over the alphabet $\{L, A\}$, with the restriction that the first and the last letter are always L. Such a word is called the LA-sequence of the chain.

A *fullerene* F is a 3-connected 3-regular plane graph such that every face is bounded by either a pentagon or a hexagon. By Euler's formula, it follows that the number of pentagonal faces of a fullerene is exactly 12.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \,\square\, H)$ whenever $ab \in E(G)$ and $x = y$, or, if $a = b$ and $xy \in E(H)$.

3 Polyomino chains and grid graphs

In this section we prove some results regarding the saturation number of polyomino chains and rectangular grids. We start with the linear chain L_n, where n denotes the number of squares. Such chain can be obtained as Cartesian product of the path P_n of length n and K_2. Here P_n is the path on n edges so that $L_n = P_n \,\square\, K_2$. Alternatively, $L_n = P_n \,\square\, P_1$. We draw L_n so that the edges of both copies of P_n are horizontal, see Figure 1.

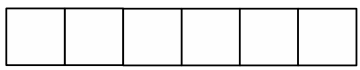

Figure 1: Linear polyomino chain $L_6 = P_6 \,\square\, K_2$.

We start by quoting two facts about the saturation number and the structure of smallest matchings in paths.

Proposition 3.1 ([6]). *Let P_n be a path of length n. Then $s(P_n) = \lceil \frac{n}{3} \rceil$. More precisely,*

$$s(P_n) = \begin{cases} \frac{n}{3}, & 3 \mid n \\ \frac{n+2}{3}, & 3 \mid (n-1) \\ \frac{n+1}{3}, & 3 \mid (n-2). \end{cases}$$

Proposition 3.2. *Let P_n be a path of length n. Then P_n has a smallest maximal matching that leaves at least one of the end-vertices unsaturated.*

Proof. Let n be divisible by 3. We form groups of three consecutive edges and construct a matching M by taking the middle edge of each group. M is obviously a smallest maximal matching and leaves unsaturated both end-vertices of P_n. If $n = 3k + 1$, again consider groups of 3 consecutive edges, take the middle edge in each group and add the sole edge that does not belong to any group. Again the constructed matching is a smallest maximal matching. Finally, when $n = 3k + 2$, construct a matching in the same way by taking the middle edge from each of k groups of three consecutive edges and adding the edge saturating the rightmost vertex. □

Next we show that we can construct a smallest maximal matching in L_n without using vertical edges. This result will enable us to reduce the problem of finding the saturation number of L_n to known results about $s(P_n)$.

Proposition 3.3. *Let M be a maximal matching in L_n containing $k > 0$ vertical edges. Then there is another maximal matching M' in L_n containing $k' < k$ vertical edges such that $|M'| \leq |M|$.*

Proof. We label the vertices in the upper copy of P_n with $u_0, u_1, \ldots, u_n$, from left to right, and vertices in the lower copy with $v_0, v_1, \ldots, v_n$, in the same direction. There are $n + 1$ vertical edges, each of the form u_iv_i for some $0 \leq i \leq n$. See Figure 2.

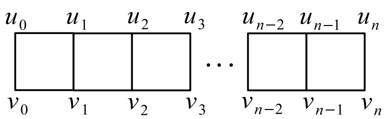

Figure 2: Linear polyomino chain L_n.

Let M be a maximal matching in L_n with $k > 0$ vertical edges and let the leftmost vertical edge in M be the edge u_mv_m. Obviously, m cannot be equal to 1.

We consider first the case $m = 0$. If u_1v_1 is also in M, we construct a matching M' as $M' = M - \{u_0v_0, u_1v_1\} \cup \{u_0u_1, v_0v_1\}$. Obviously, M' is a maximal matching of the same cardinality as M containing $k - 2$ vertical edges.

Let now both neighbors u_1 and v_1 of end-vertices of u_0v_0 be saturated by horizontal edges. Hence, both u_1u_2 and v_1v_2 are in M. Then at least one of u_3 and v_3 must be saturated by an edge of M. Let u_3 be saturated. Then we can construct a matching M' as $M' = M - \{u_0v_0, u_1u_2\} \cup \{u_0u_1\}$. Again, M' is a maximal matching, $|M'| = |M| - 1$ and $k' = k - 1$. The case of saturated v_3 follows by symmetry.

The last case to consider for $m = 0$ is the one in which only one of u_1, v_1 is saturated by a, necessarily horizontal, edge of M. Let it be u_1. Hence, $u_1u_2 \in M$ and $v_1v_2 \notin M$. Then v_3 must be saturated and $M' = M - \{u_0v_0\} \cup \{v_0v_1\}$ is a maximal matching of the same cardinality as M but with one vertical edge less. Hence, the claim holds if the leftmost vertical edge in M is u_0v_0. This case is depicted in Figure 3.

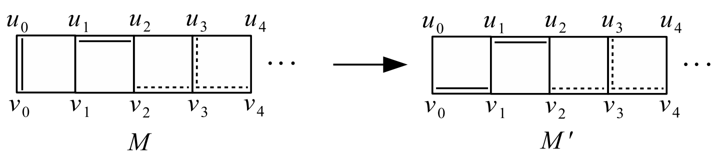

Figure 3: The case when $m = 0$ and only one of u_1,v_1 is saturated by M.

Let now the leftmost vertical edge in M be u_2v_2. Then both u_0u_1 and v_0v_1 must be in M, and at least one of vertices u_3 and v_3 must be saturated by an edge of M. Let it be

u_3. Then the matching M' constructed as $M' = M - \{u_2v_2, v_0v_1\} \cup \{v_1v_2\}$ will be a maximal matching with smaller cardinality and with one vertical edge less than M.

Similar constructions apply when the leftmost vertical edge of M is near the right end of the chain. The simplest is the case $m = n - 1$, when also the rightmost edge u_nv_n must be in M. Then by switching the edges on the rightmost square one readily obtains a maximal matching of the same size as M but without vertical edges. The case $m = n - 2$ forces both horizontal edges $u_{n-1}u_n$ and $v_{n-1}v_n$ to be in M. Then at least one of u_{n-3} and v_{n-3} must be saturated. Let it be v_{n-3}. Then the matching $M' = M - \{u_{n-2}v_{n-2}, u_{n-1}u_n\} \cup \{u_{n-2}u_{n-1}\}$ is a maximal matching of smaller size than M without vertical edges. Remains the case when u_nv_n is the leftmost (and hence the only) vertical edge in M. If only one of u_{n-1}, v_{n-1} is saturated, let us say u_{n-1}, it suffices to switch u_nv_n and $v_{n-1}v_n$ to obtain a maximal matching M' of the same size without vertical edges. If both u_{n-1}, v_{n-1} are saturated, they must be saturated by horizontal edges $u_{n-2}u_{n-1}$ and $v_{n-2}v_{n-1}$, respectively. Also, at least one of u_{n-3} and v_{n-3} must be saturated. Let it be v_{n-3}. Then $M' = M - \{u_nv_n, v_{n-2}v_{n-1}\} \cup \{v_{n-1}v_n\}$ is a maximal matching of smaller size than M but without vertical edges.

Now we can look at the remaining cases in a unified manner. So, let u_mv_m, $3 \leq m \leq n - 3$, be the leftmost vertical edge in a maximal matching M. If $u_{m+1}v_{m+1}$ is also in M, we construct M' by switching the edges on the square $u_m, u_{m+1}, v_{m+1}, v_m$, obtaining a maximal matching of the same size but with two vertical edges less. Hence, we can suppose that $u_{m+1}v_{m+1} \notin M$.

If both u_{m-1} and u_{m+1} are unsaturated, then both v_{m-1} and v_{m+1} must be saturated, necessarily by horizontal edges $v_{m-2}v_{m-1}$ and $v_{m+1}v_{m+2}$, respectively. Further, both u_{m-2} and u_{m+2} must be saturated, again by horizontal edges. The situation is shown in Figure 4.

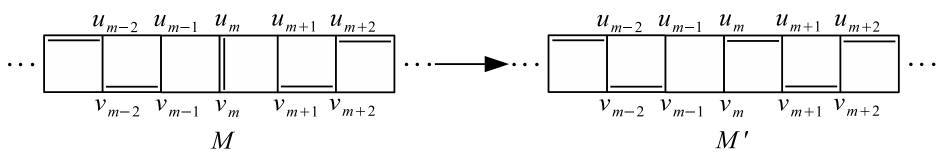

Figure 4: The case when u_{m-1} and u_{m+1} are both unsaturated.

We construct M' as $M' = M - \{u_mv_m\} \cup \{u_mu_{m+1}\}$. Obviously, M' is a maximal matching of the same size as M and with one vertical edge less. The situation in which both v_{m-1} and v_{m+1} are unsaturated follows by symmetry.

It remains to consider the case when at least one of u_{m-1}, u_{m+1} and at least one of v_{m-1}, v_{m+1} are saturated. We construct a new matching M'' by keeping the part of M to the left of u_mv_m, shifting all edges of M that were right of u_mv_m one place to the left (hence, u_lu_{l+1} goes to $u_{l-1}u_l$, v_lv_{l+1} to $v_{l-1}v_l$ and u_lv_l to $u_{l-1}v_{l-1}$ for $m < l \leq n$) and moving u_mv_m to u_nv_n. Obviously, M'' is a maximal matching of the same size as M and with the same number of vertical edges, but with the leftmost vertical edge at some place $l > m$. Let us look at the situation on the right-hand side of L_n.

If $u_{n-1}v_{n-1}$ is in M'', then M' with the desired properties can be obtained by switching edges on the rightmost square of L_n. If $u_{n-1}v_{n-1}$ is not in M'', then also $u_{n-2}v_{n-2}$ cannot be in M, and M' can be constructed in exactly the same manner as when u_nv_n is the only vertical edge in M.

Hence, no matter where in M the leftmost vertical edge appears, we can always construct a maximal matching of the same or smaller size with strictly smaller number of vertical edges. □

Corollary 3.4. *There is a maximal matching in L_n of cardinality $s(L_n)$ without vertical edges.*

Corollary 3.5. $2s(P_n) \leq s(L_n) \leq 2s(P_n) + 1$.

Proof. We know that there is a smallest maximal matching M in L_n (i.e., of the size equal to $s(L_n)$) without vertical edges. Hence all edges of M are horizontal, and each edge belongs to one of two copies of P_n in L_n. If the cardinality of M is smaller than $2s(P_n)$, then at least one of two copies of P_n will contain two adjacent unsaturated vertices. This proves the left inequality.

To prove the right inequality, let us take a smallest maximal matching M_u in the upper copy of P_n. If M_u saturates exactly one end-vertex of P_n, let us take it so that it saturates u_0. Let M_v be a smallest maximal matching in the lower copy of P_n obtained by taking the edges corresponding to the edges of M_u and shifting them one place to the right. Then M_v saturates the vertices in the lower copy of P_n adjacent to the vertices of the upper copy of P_n left unsaturated by M_u. Hence $M = M_u \cup M_v$ is a maximal matching in L_n of size $2s(P_n)$.

It remains to consider the case when all smallest maximal matchings in P_n leave both end-vertices unsaturated. In that case, take two smallest maximal matchings M_u and M_v in upper and lower copy of P_n, respectively, and shift M_v one place to the right so that it saturates the neighbors of the vertices left unsaturated by M_u. That leaves unsaturated both end-vertices of v_0v_1. By adding that edge to the maximal matching constructed from M_u and shifted M_v we obtain a maximal matching of size $2s(P_n) + 1$. □

From this we can get the exact expression for the saturation number of the linear polyomino chain.

Theorem 3.6. *Let L_n be the linear polyomino chain. Then*

$$s(L_n) = \begin{cases} \frac{2n}{3} + 1, & 3 \mid n \\ \frac{2(n+2)}{3}, & 3 \mid (n-1) \\ \frac{2(n+1)}{3}, & 3 \mid (n-2). \end{cases}$$

Proof. If $3 \mid (n-1)$ or $3 \mid (n-2)$ there is a smallest maximal matching M for P_n such that M saturates exactly one end-vertex of P_n. Therefore, it follows from the proof of Corollary 3.5 that $s(L_n) = 2s(P_n)$ and we are done. If $3 \mid n$, the smallest maximal matching of P_n is uniquely defined and it leaves both end-vertices unsaturated. Hence, in this case we obtain $s(L_n) > 2s(P_n)$ and therefore, $s(L_n) = 2s(P_n) + 1$.

Examples of smallest maximal matchings in L_n for all classes of divisibility of the chain length by 3 are given in the Figure 5. □

The above approach can be successfully applied also to obtain non-trivial upper bounds on the saturation number of grid graphs that arise as Cartesian products of two (or more) paths. By taking smallest maximal matchings in all horizontal (or in all vertical) copies of paths in $P_m \,\square\, P_n$, shifting them and adjusting by adding an edge where necessary, and using symmetry, we can obtain following upper bound on $s(P_m \,\square\, P_n)$.

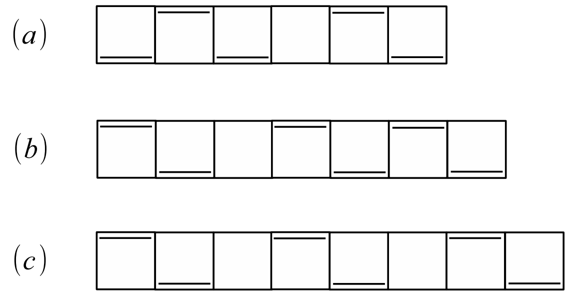

Figure 5: Linear polyomino chains with maximal matchings.

Proposition 3.7. $s(P_m \,\square\, P_n) \leq \min\{(m+1)[s(P_n)+1], (n+1)[s(P_m)+1]\}$.

This upper bound can be improved a bit by exploiting particular relationships between parities and remainders modulo 3 of m and n. See Figure 6 for an example. We believe, however, that our upper bounds capture the asymptotic behavior of the saturation number of rectangular grids.

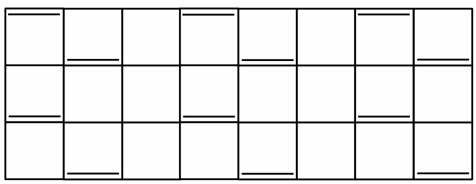

Figure 6: Graph $P_8 \,\square\, P_3$ with a maximal matching.

Now we go back to polyomino chains. In the following theorem we give the exact closed formulas for the saturation number of polyomino chains where all internal squares are kinky.

Theorem 3.8. *Let S_k be a polyomino chain with k squares such that all internal squares are kinky. Then*

$$s(S_k) = \left\lceil \frac{k}{2} \right\rceil + 1.$$

Proof. We consider two cases.

1. Let k be even. Since the number of vertices in S_k is $2k+2$, a perfect matching (which always exists) has $k+1$ edges. Using Lemma 2.1 we obtain that $s(S_k) \geq \frac{k+1}{2}$. Since k is even, we obtain $s(S_k) \geq \left\lceil \frac{k}{2} \right\rceil + 1$. To show the upper bound, we construct a maximal matching M from Figure 7.

   Obviously, $|M| = \frac{k}{2} + 1 = \left\lceil \frac{k}{2} \right\rceil + 1$. Hence, $s(S_k) = \left\lceil \frac{k}{2} \right\rceil + 1$.
2. If k is odd, let M' be a maximal matching from Figure 8.

   Obviously, $|M'| = \frac{k+1}{2} + 1$ and therefore, $s(S_k) \leq \frac{k+1}{2} + 1 = \left\lceil \frac{k}{2} \right\rceil + 1$. Now suppose that there is a maximal matching N for S_k such that $|N| \leq \frac{k+1}{2}$. It is easy to see that at least one of edges e_1, e_2, and e_3 must be in N. Consider the following cases.

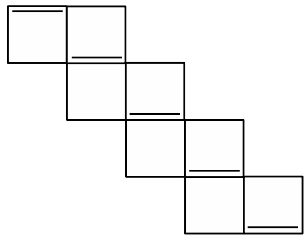

Figure 7: Polyomino chain S_k (k even) with maximal matching M.

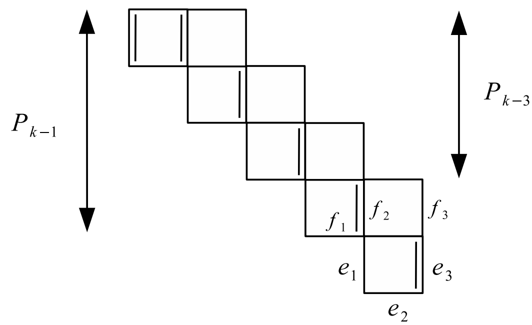

Figure 8: Polyomino chain S_k (k odd) with maximal matching M.

(a) If $e_1 \in N$, then also $e_3 \in N$ or $f_3 \in N$. Therefore, for the graph S_{k-3} (see Figure 8) it must hold $s(S_{k-3}) \leq \frac{k-3}{2}$, which is a contradiction with Case 1.

(b) If $e_2 \in N$, then for the graph S_{k-1} (see Figure 8) it must hold $s(S_{k-1}) \leq \frac{k-1}{2}$, which is a contradiction with Case 1.

(c) If $e_3 \in N$, then also one of the edges e_1, f_1, f_2 must be in N. If $e_1 \in N$ or $f_1 \in N$, then for the graph S_{k-3} (see Figure 8) it must hold $s(S_{k-3}) \leq \frac{k-3}{2}$, which is a contradiction with Case 1. Therefore, suppose that $f_2 \in N$. But in this case we can use similar reasoning and either obtain a contradiction with the Case 1 or eventually obtain a matching M', which is a contradiction since $|M'| > |N|$.

Since we obtain a contradiction in every case, it follows that every maximal matching of S_k has at least $\frac{k+1}{2}+1$ edges. Since $\frac{k+1}{2}+1 = \lceil \frac{k}{2} \rceil + 1$ it follows $s(S_k) \geq \lceil \frac{k}{2} \rceil + 1$ and we are done. □

4 Hexagonal animals

In this section we prove some results regarding the saturation number of benzenoid chains and coronenes.

4.1 Benzenoid chains

A benzenoid chain of length h will be denoted by B_h. If all interior hexagons of a benzenoid chain are straight, the chain is called a *polyacene* and denoted by A_h.

Saturation number of benzenoid chains has been already studied in a recent paper coauthored by one of the present authors [5]. We quote without proof some basic results established there.

Proposition 4.1 ([5]). *Let B_h be a benzenoid chain with h hexagons. Then $s(B_h) \geq h+1$.*

Proposition 4.2 ([5]). *For any h it holds*

$$s(B_h) + 1 \leq s(B_{h+1}) \leq s(B_h) + 2.$$

Proposition 4.3 ([5]). $s(B_h) = h + 1$ *if and only if* $B_h = A_h$.

Let $B_{h,1}$ denote a chain of length $h = k+m$ in which hexagon h_k is kinky and all other hexagons are straight. An example is shown in Figure 9. Furthermore, let $B_{h,k}$ denote a benzenoid chain of length h with exactly k kinky hexagons.

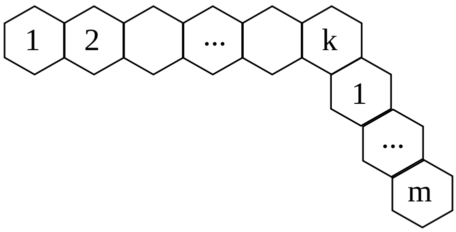

Figure 9: A chain with one kinky hexagon.

Proposition 4.4 ([5]). *For any h it holds*

$$s(B_{h,1}) = h + 2.$$

Hence one kinky hexagon means one more edge in the smallest maximal matching.

The following claim was stated in [5] as Proposition 5.

Proposition 4.5 ([5]). *Let $B_{h,k}$ be a benzenoid chain of length h with k kinky hexagons such that no two kinky hexagons are adjacent. Then $s(B_{h,k}) = h + k + 1$.*

However, we show in Proposition 4.7, Proposition 4.8, and Proposition 4.9 that the above proposition provides only an upper bound for the saturation number, which is evident from the following proposition.

Proposition 4.6. *Let $B_{h,k}$ be a benzenoid chain of length h with k kinky hexagons. Then $s(B_{h,k}) \leq h + k + 1$.*

Proof. Let M be a matching of $B_{h,k}$ obtained by taking all edges shared by two hexagons, one additional edge in each terminal hexagon and all edges connecting vertices of degree two in kinky hexagons. See Figure 10 for an example.

It is easy to see that M is a maximal matching and $|M| = h + k + 1$. Therefore, we are done. □

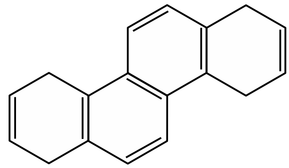

Figure 10: Maximal matching M.

However, in the same graph shown in Figure 10 we can construct a smaller maximal matching by simply taking all vertical edges. Hence $h + k + 1$ is only an upper bound on $s(B_{h,k})$ and it can be improved in particular cases.

Let B_h be a chain of length h. A *straight segment* in B_h is any sequence of consecutive straight hexagons. Equivalently, it is any sub-word made of consecutive L's in the LA sequence of B_h. The number of consecutive straight hexagons is the *length* of the straight segment.

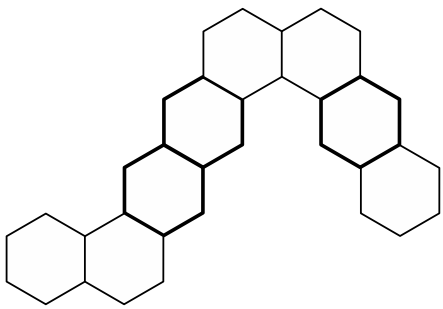

Figure 11: A benzenoid chain B_8 with two straight segments (labelled with bold edges), one of length 2 and one of length 1.

In the following we consider the saturation number of benzenoid chains where all straight segments are of length one and no two kinky hexagons are adjacent. It turns out we have to distinguish between three cases. In all of them the upper bound from Proposition 4.6 is improved.

Proposition 4.7. *Let $B_{2k+1,k}$ be a benzenoid chain such that all straight segments are of length one and no two kinky hexagons are adjacent. Then*

$$s(B_{2k+1,k}) \le \frac{1}{4}\big(10k + 9 - (-1)^k\big).$$

Proof. We build $B_{2k+1,k}$ from left to right by adding blocks of 4 consecutive hexagons at a time. Each block has the form $LALL$ and it is added on the rightmost hexagon of the already constructed chain so that it becomes a kinky hexagon in the new chain. To show an upper bound for the saturation number, we construct a maximal matching M of $B_{2k+1,k}$. For the first four hexagons we need six edges in a maximal matching such that the edge connecting the first and the second block of hexagons is in the matching. See Figure 12.

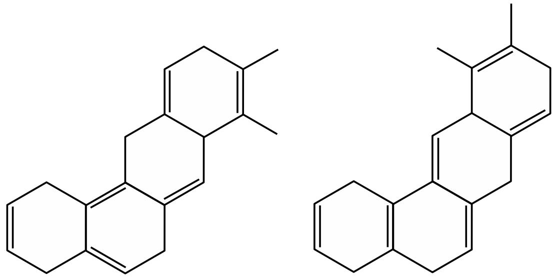

Figure 12: Two possibilities for a maximal matching M of the first block.

Afterwards, we can always add four hexagons at a price of five new edges in a maximal matching. Let l be the number of blocks in $B_{2k+1,k}$. We consider two cases.

- If k is odd, then $4 \mid (2k-2)$ and $l = \frac{2k-2}{4} = \frac{k-1}{2}$ and we have three additional hexagons in a benzenoid chain. For these three hexagons, we need 4 additional edges in a maximal matching. We obtain $|M| = 6 + 5(l-1) + 4 = 10 + 5 \cdot \frac{k-3}{2} = \frac{5k+5}{2}$.
- If k is even, then $l = \frac{2k}{4} = \frac{k}{2}$ and we have one additional hexagon in a benzenoid chain. For this hexagon, we need 1 additional edge in a maximal matching. Therefore, we get $|M| = 6 + 5(l-1) + 1 = 7 + 5 \cdot \frac{k-2}{2} = \frac{5k+4}{2}$.

Combining both cases, we obtain $|M| = \frac{1}{4}\big(10k + 9 - (-1)^k\big)$. □

Proposition 4.8. *Let $B_{2k+2,k}$, $k \in \mathbb{N}$, be a benzenoid chain such that all straight segments are of length one and no two kinky hexagons are adjacent. Then*

$$s(B_{2k+2,k}) \leq \frac{1}{4}\big(10k + 13 - (-1)^k\big).$$

Proof. We build $B_{2k+2,k}$ from left to right by adding blocks of 4 consecutive hexagons at a time. Each block has the form $LALL$ or each block has the form $LLAL$ (before adding) and it is added on the rightmost hexagon of the already constructed chain. Because of the symmetry, we can assume that each block has the form $LALL$ (otherwise we can start from right to left). The new block is added on the last hexagon such that it becomes a kinky hexagon. To show an upper bound for the saturation number, we construct a maximal matching M of $B_{2k+2,k}$. For the first four hexagons we need six edges in a maximal matching such that the edge connecting the first and the second block of hexagons is in the matching. Afterwards, we can always add four hexagons at a price of five new edges in a maximal matching. Let l be the number of blocks in $B_{2k+2,k}$. We consider two cases.

- If k is odd, then $4 \mid (2k+2)$ and $l = \frac{2k+2}{4} = \frac{k+1}{2}$. We obtain $|M| = 6 + 5(l-1) = 6 + 5 \cdot \frac{k-1}{2} = \frac{5k+7}{2}$.
- If k is even, then $l = \frac{(2k+2)-2}{4} = \frac{k}{2}$ and we have two additional hexagons in a benzenoid chain. For these two hexagons, we need 2 additional edges in a maximal matching. Therefore, we get $|M| = 6 + 5(l-1) + 2 = 8 + 5 \cdot \frac{k-2}{2} = \frac{5k+6}{2}$.

Combining both cases, we obtain $|M| = \frac{1}{4}\big(10k + 13 - (-1)^k\big)$. □

Proposition 4.9. *Let $B_{2k+3,k}$, $k \in \mathbb{N}$, be a benzenoid chain such that all straight segments are of length one and no two kinky hexagons are adjacent. Then*

$$s(B_{2k+3,k}) \leq \frac{1}{4}\big(10k + 19 + (-1)^k\big).$$

Proof. We build $B_{2k+3,k}$ from left to right by adding blocks of 4 consecutive hexagons at a time. Each block has the form $LLAL$ (before adding) and it is added on the right-most hexagon of the already constructed chain. The new block is added such that the first hexagon of this block becomes a kinky hexagon. To show an upper bound for the saturation number, we construct a maximal matching M of $B_{2k+3,k}$. For the first four hexagons we need six edges in a maximal matching such that the edge connecting the first and the second block of hexagons is in the matching. Afterwards, we can always add four hexagons at a price of five new edges in a maximal matching (such that the edge connecting that block with the next block is in the matching). Let l be the number of blocks in $B_{2k+3,k}$. We consider two cases.

- If k is odd, then $4 \mid ((2k+3)-1)$ and we have to add one additional hexagon (for this hexagon we need one additional edge). Hence, $l = \frac{2k+2}{4} = \frac{k+1}{2}$. We obtain $|M| = 6 + 5(l-1) + 1 = 7 + 5 \cdot \frac{k-1}{2} = \frac{5k+9}{2}$.
- If k is even, then $4 \mid ((2k+3)-3)$ and we have to add three additional hexagons (for this three hexagons we need four additional edges in a maximal matching). Hence, $l = \frac{2k}{4} = \frac{k}{2}$. Therefore, we get $|M| = 6 + 5(l-1) + 4 = 10 + 5 \cdot \frac{k-2}{2} = \frac{5k+10}{2}$.

Combining both cases, we obtain $|M| = \frac{1}{4}\big(10k + 19 + (-1)^k\big)$. □

4.2 Coronenes

In this section we prove bounds for the saturation number of coronenes. These highly symmetric benzenoid systems have long been attracting the attention of both theoretical and experimental chemists. They are suggested as markers for vehicle emissions, since they are produced by incomplete combustion of organic matter. Coronene H_1 is just a single hexagon, and H_k is obtained from H_{k-1} by adding a ring of hexagons around it. See Figure 13 for an example of coronene H_4.

Proposition 4.10. *Let H_k be a coronene. Then*

$$\frac{3}{2}k^2 \leq s(H_k) \leq \begin{cases} 2k^2, & 3 \mid (k-1) \\ 2k^2 + \frac{4k}{3}, & 3 \mid k \\ 2k^2 + \frac{2k+2}{3}, & 3 \mid (k-2). \end{cases}$$

Proof. Obviously, every coronene has a perfect matching. Since the number of vertices in H_k is $6k^2$, it follows by Lemma 2.1 that $s(H_k) \geq \frac{3}{2}k^2$.

For the upper bound, we will consider just the case when $3 \mid (k-1)$, since the proofs for other two cases are almost the same. To prove this case, we construct a maximal matching M for H_k. In the matching we put all the vertical edges lying in the center layer of the coronene H_k. Since there are $2k-1$ hexagons in the center layer, we obtain $2k$ edges in the matching M. See Figure 13. Next, we continue at the top half of the coronene with alternating non-vertical and vertical edges such that two layers of edges are needed

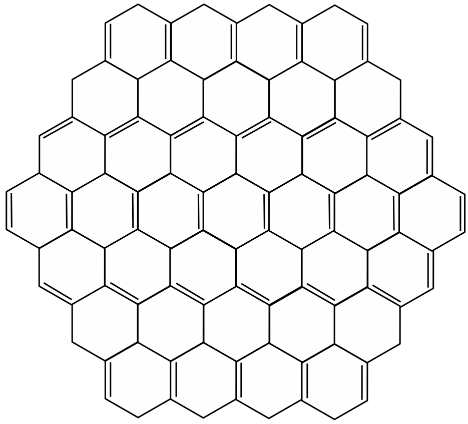

Figure 13: A coronene H_4.

for every three layers of hexagons. Furthermore, for every non-vertical layer of edges we need one additional vertical edge. Let x be the number of edges in M in the top half of the coronene. Then

$$\begin{aligned} x &= (2k-1)+(2k-3)+(2k-4)+(2k-6)+\cdots+(k+3)+(k+1) = \\ &= \frac{2(k-1)}{3}\cdot(2k)-\big(1+3+4+6+\cdots+(k-3)+(k-1)\big) = \\ &= \frac{4k^2-4k}{3}-\left(\frac{k^2-k}{2}-\big(2+5+\cdots+(k-2)\big)\right) = \\ &= \frac{4k^2-4k}{3}-\left(\frac{k^2-k}{2}-\frac{k^2-k}{6}\right) = \\ &= k^2-k. \end{aligned}$$

Finally, we obtain $|M| = 2k+2x = 2k+2(k^2-k) = 2k^2$. □

In the next proposition we improve the lower bound for any $k \geq 7$.

Proposition 4.11. *Let H_k be a coronene where $k > 1$. Then*

$$s(H_k) \geq 2k^2-3k-1.$$

Proof. For any H_k, $k \geq 2$, one can construct a disk-shaped fullerene by taking another copy of H_k and connecting the borders in the following way. We insert $6k$ edges between vertices of degree 2 such that end-vertices lie in different copies of H_k. Obviously, this can be done in such a way that the resulting graph F is planar with only pentagonal and hexagonal faces. Since F is also 3-regular, it is a fullerene with $12k^2$ vertices.

Let M' be a maximal matching in each copy of H_k. Then this matching can be extended to a maximal matching M of a graph F by adding at most $6k$ edges between two copies of H_k. Therefore, $|M| \leq 2|M'|+6k$. From Theorem 4.1 in [2] it follows $|M| \geq \frac{|V(F)|}{3}-2 = 4k^2-2$. Therefore, we obtain $2|M'|+6k \geq 4k^2-2$. Finally, $|M'| \geq 2k^2-3k-1$. □

Concluding remarks

In the paper we have established some bounds and also exact values for the saturation number of certain families of lattice animals. However, there are still many open problems regarding the exact values for the saturation number of different families of graphs.

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Enumerating regular graph coverings whose covering transformation groups are $\mathbb{Z}_2$-extensions of a cyclic group*

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Abstract

Several types of the isomorphism classes of graph coverings have been enumerated by many authors. In 1988, Hofmeister enumerated the double covers of a graph, and this work was extended to n-fold coverings of a graph by the second and third authors. For *regular* coverings of a graph, their isomorphism classes were enumerated when the covering transformation group is a finite abelian or dihedral group. In this paper, we enumerate the isomorphism classes of graph coverings when the covering transformation group is a $\mathbb{Z}_2$-extension of a cyclic group, including generalized quaternion and semi-dihedral groups.

Keywords: Graphs, regular coverings, voltage assignments, enumeration, Möbius functions (on a lattice), group extensions.

Math. Subj. Class.: 05C30, 20F28, 20K27

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1 Introduction

Throughout this paper, all graphs and groups are assumed to be finite. Let G be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v. We use $|X|$ for the cardinality of a set X. The number $\beta = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to as the *Betti number* of G.

Two graphs G and H are *isomorphic* if there exists a one-to-one correspondence between their vertex sets which preserves adjacency, and such a correspondence is called an *isomorphism* between G and H. An *automorphism* of a graph G is an isomorphism of G onto itself. Thus, an automorphism of G is a permutation of the vertex set $V(G)$ which preserves adjacency. Obviously, the automorphisms of G form a permutation group, $\mathrm{Aut}(G)$, under composition, which acts on the vertex set $V(G)$.

A graph $\widetilde{G}$ is called a *covering* of G with projection $p\colon \widetilde{G} \to G$ if there is a surjection $p\colon V(\widetilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})}\colon N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. Also, we sometimes say that the projection $p\colon \widetilde{G} \to G$ is a covering, and an *n-fold covering* if p is n-to-one. A covering $p\colon \widetilde{G} \to G$ is said to be *regular* (simply, $\mathcal{A}$-*covering*) if there is a subgroup $\mathcal{A}$ of the automorphism group $\mathrm{Aut}(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ so that the graph G is isomorphic to the quotient graph $\widetilde{G}/\mathcal{A}$, say by h, and the quotient map $\widetilde{G} \to \widetilde{G}/\mathcal{A}$ is the composition $h \circ p$ of p and h. The *fiber* of an edge or a vertex is its preimage under p.

Two coverings $p_i\colon \widetilde{G}_i \to G$, $i = 1, 2$, are *isomorphic* if there exists a graph isomorphism $\Phi\colon \widetilde{G}_1 \to \widetilde{G}_2$ such that $p_2 \circ \Phi = p_1$, that is, the diagram

$$\begin{tikzcd} \widetilde{G}_1 \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & \widetilde{G}_2 \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

commutes. Such a Φ is called a *covering isomorphism*. A *covering transformation* is just a covering automorphism.

Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. Let $\mathcal{A}$ be a finite group. An *ordinary voltage assignment* (or, $\mathcal{A}$-*voltage assignment*) of G is a function $\phi\colon D(G) \to \mathcal{A}$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of ϕ are called *voltages*, and $\mathcal{A}$ is called the *voltage group*. The *ordinary derived graph* $G \times_\phi \mathcal{A}$ derived from an ordinary voltage assignment $\phi\colon D(G) \to \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$, and as its edge set $E(G) \times \mathcal{A}$, so that an edge (e, g) of $G \times_\phi \mathcal{A}$ joins a vertex (u, g) to $(v, \phi(e)g)$ for $e = uv \in D(G)$ and $g \in \mathcal{A}$. In the (ordinary) derived graph $G \times_\phi \mathcal{A}$, a vertex (u, g) is denoted by u_g and an edge (e, g) is denoted by e_g. The first coordinate projection $p_\phi\colon G \times_\phi \mathcal{A} \to G$ commutes with the left multiplication action of the $\phi(e)$ and the right multiplication action of $\mathcal{A}$ on the fibers, which is free and transitive, so that p_ϕ is a regular $|\mathcal{A}|$-fold covering, called simply an $\mathcal{A}$-*covering*. Moreover, if the covering graph $G \times_\phi \mathcal{A}$ is connected, then the group $\mathcal{A}$ becomes the covering transformation group of the $\mathcal{A}$-covering.

For a group $\mathcal{A}$, let $C^1(G; \mathcal{A})$ denote the set of $\mathcal{A}$-voltage assignments ϕ of G. Choose a spanning tree T of G, and let

$$C^1_T(G; \mathcal{A}) = \{\phi \in C^1(G; \mathcal{A}) : \phi(uv) \text{ is the identity for each } uv \in D(T)\}.$$

Gross and Tucker [4] showed that every $\mathcal{A}$-covering $\widetilde{G}$ of a graph G can be derived from an $\mathcal{A}$-voltage assignment ϕ in $C_T^1(G;\mathcal{A})$, say it *T-reduced.* From now on, let T denote a fixed spanning tree of a graph G, and we consider only an $\mathcal{A}$-voltage assignment ϕ in $C_T^1(G;\mathcal{A})$.

The enumeration problem of coverings became subject of investigation by many authors starting from the classical paper by Hurwitz published more then 100 years ago. In particular, enumeration of graph coverings became possible after the paper by Hall ([6]) published in 1949. In 1988, Hofmeister [8] counted double covers of graphs. Liskovets enumerated connected non-isomorphic coverings of the graph with a given Betti number, see [19, 20]. The number of connected and disconnected coverings were determined by Kwak and Lee in [15]. Later, Kwak, Lee and A. D. Mednykh counted cyclic and dihedral coverings over surfaces and graphs with prescribed topological characteristics, see [16, 17].

Following notations in [14], let $\mathrm{Iso}^R(G;n)$ denote the number of the isomorphism classes of regular (connected or disconnected) n-fold coverings of G, and use $\mathrm{Isoc}^R(G;n)$ for their connected ones. Similarly, let $\mathrm{Iso}(G;\mathcal{A})$ denote the number of the isomorphism classes of (connected or disconnected) $\mathcal{A}$-coverings of G, and use $\mathrm{Isoc}(G;\mathcal{A})$ for their connected ones. By the properties of regularity of coverings, one can see that the number of the isomorphism classes of (connected or disconnected) n-fold regular coverings of a graph G is the sum of numbers of the isomorphism classes of connected d-fold regular coverings of G, where d runs over all divisors of n:

$$\mathrm{Iso}^R(G;n) = \sum_{d|n} \mathrm{Isoc}^R(G;d).$$

Moreover, the number of the isomorphism classes of connected n-fold regular coverings of G is the sum of the numbers of the isomorphism classes of connected $\mathcal{A}$-coverings of G, where $\mathcal{A}$ runs over all non-isomorphic groups of order n:

$$\mathrm{Isoc}^R(G;n) = \sum_{\mathcal{A}} \mathrm{Isoc}(G;\mathcal{A}).$$

Consequently, it just needs to determine the numbers $\mathrm{Isoc}(G;\mathcal{A})$ for every finite group $\mathcal{A}$. Hong, Kwak and Lee [9] obtained an algebraic characterization of two isomorphic graph regular coverings given as follows.

Lemma 1.1. *Let $\phi \in C_T^1(G;\mathcal{A})$ and $\psi \in C_T^1(G;\mathcal{B})$ be any two ordinary voltage assignments in G. If their derived (regular) coverings $p_\phi\colon G\times_\phi \mathcal{A}\to G$ and $p_\psi\colon G\times_\psi \mathcal{B}\to G$ are connected, then they are isomorphic if and only if there exists a group isomorphism $\sigma\colon \mathcal{A}\to\mathcal{B}$ such that $\psi(uv)=\sigma(\phi(uv))$ for all $uv\in D(G)-D(T)$.*

In particular, if two voltages ϕ and ψ in $C_T^1(G;\mathcal{A})$ derive connected coverings, then the derived coverings are isomorphic if and only if there exists a group automorphism $\sigma\in\mathrm{Aut}(\mathcal{A})$ such that

$$\psi(uv)=\sigma(\phi(uv))$$

for all $uv\in D(G)-D(T)$.

With a linear ordering of the cotree edges of G, the set $C_T^1(G;\mathcal{A})$ of T-reduced $\mathcal{A}$-voltage assignments of G can be identified as

$$C_T^1(G;\mathcal{A}) = \mathcal{A}\times\cdots\times\mathcal{A}\ \ (\beta \text{ times}),$$

that is, an $\mathcal{A}$-voltage assignment ϕ of G can be identified as a β-tuple $(g_1, \ldots, g_\beta)$ of group elements $g_i \in \mathcal{A}$. Moreover, such a β-tuple of g's derives a connected covering if and only if it is transitive. It means by definition that the subgroup $\langle g_1, \ldots, g_\beta \rangle$ generated by them acts transitively on the group $\mathcal{A}$ (under the left translation on $\mathcal{A}$), or equivalently $\{g_1, g_2, \ldots, g_\beta\}$ generates the whole group $\mathcal{A}$.

Note that the automorphism group $\mathrm{Aut}(\mathcal{A})$ of $\mathcal{A}$ can act on the set of transitive β-tuples of group elements $g_i \in \mathcal{A}$ coordinatewisely, and any two transitive β-tuples of elements in $\mathcal{A}$ belong to the same orbit under the action if and only if they derive (connected) isomorphic $\mathcal{A}$-coverings, by Lemma 1.1.

Clearly, the $\mathrm{Aut}(\mathcal{A})$-action on the set of transitive β-tuples of group elements $g_i \in \mathcal{A}$ is free (having no fixed element), and hence Burnside's counting Lemma gives a counting formula for $\mathrm{Isoc}(G; \mathcal{A})$ as follows.

Theorem 1.2 ([14]). *For any finite group $\mathcal{A}$,*

$$\mathrm{Isoc}(G; \mathcal{A}) = \frac{|\Omega(\mathcal{A}; \beta)|}{|\mathrm{Aut}(\mathcal{A})|},$$

where $\Omega(\mathcal{A}; \beta) = \{(g_1, g_2, \ldots, g_\beta) \in \mathcal{A}^\beta \mid \{g_1, g_2, \ldots, g_\beta\}$ generates $\mathcal{A}\}$.

Note that the set $\Omega(\mathcal{A}; \beta)$ can be identified as the set of epimorphisms from the free group generated by β elements onto the group $\mathcal{A}$.

To determine the number $\mathrm{Isoc}(G; \mathcal{A})$, we need to estimate $|\mathrm{Aut}(\mathcal{A})|$ and $|\Omega(\mathcal{A}; \beta)|$. The number $|\mathrm{Aut}(\mathcal{A})|$ can certainly be determined for a few groups $\mathcal{A}$. For example, one can refer to [14] for $|\mathrm{Aut}(\mathcal{A})|$ when $\mathcal{A}$ is abelian or dihedral groups. Also, one can see recent two papers [1], [7] for abelian case.

The other number $|\Omega(\mathcal{A}; \beta)|$ can be determined by a direct counting and it can also be determined in terms of the Möbius function defined on the subgroups lattice of $\mathcal{A}$, as shown in [17]. The Möbius function assigns an integer $\mu(\mathcal{K})$ to each subgroup $\mathcal{K}$ of $\mathcal{A}$ by the recursive formula

$$\sum_{\mathcal{H} \geq \mathcal{K}} \mu(\mathcal{H}) = \begin{cases} 1 & \text{if } \mathcal{K} = \mathcal{A}, \\ 0 & \text{if } \mathcal{K} < \mathcal{A}. \end{cases}$$

Jones ([12, 13]) used the Möbius function to count the normal subgroups of a surface group or a crystallographic group, and applied it to count certain covering surfaces. We see that

$$|\mathcal{A}|^\beta = \sum_{\mathcal{K} \leq \mathcal{A}} |\Omega(\mathcal{K}; \beta)|.$$

It follows from the Möbius inversion that

$$|\Omega(\mathcal{A}; \beta)| = \sum_{\mathcal{K} \leq \mathcal{A}} \mu(\mathcal{K}) |\mathcal{K}|^\beta.$$

The next theorem is deduced from Theorem 1.2.

Theorem 1.3. *For any finite group $\mathcal{A}$,*

$$\mathrm{Isoc}(G; \mathcal{A}) = \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{\mathcal{K} \leq \mathcal{A}} \mu(\mathcal{K}) |\mathcal{K}|^\beta.$$

Now, we have two ways of computing $|\Omega(\mathcal{A};\beta)|$, by a direct counting and by using the Möbius function on the subgroups lattice of $\mathcal{A}$. For example, when $\mathcal{A}$ is cyclic or dihedral, $\mathrm{Isoc}(G;\mathcal{A})$ was determined by Kwak, Lee and Mednykh in [17] in terms of the Möbius function. However, it is not easy to determine the Möbius function on the subgroups lattice of any abelian group $\mathcal{A}$. For an abelian group $\mathcal{A}$, $\mathrm{Isoc}(G;\mathcal{A})$ was determined in [14] by a direct counting method.

This paper is organized as follows. In a coming section, we review an extension of a group, giving a classification of $\mathbb{Z}_2$-extensions of a cyclic p-group and a discussion on $\mathbb{Z}_2$-extensions of a cyclic group. In Sections 3 and 4, we determine the number $\mathrm{Isoc}(G;\mathcal{A})$ when $\mathcal{A}$ is a $\mathbb{Z}_2$-extension of a cyclic p-group, or $\mathbb{Z}_2$-extensions of any cyclic group, as main results in this paper. In Section 5, we try to extend our discussion to a $\mathbb{Z}_2$-extension of an abelian group, by considering two special cases of them.

2 Review on extensions of groups

We review briefly an extension of a group with some recent results to use it in this paper.

Let $\mathcal{N}$ and $\mathcal{Q}$ be two groups. A group $\mathcal{A}$ is an *extension* of $\mathcal{N}$ by $\mathcal{Q}$ (or a $\mathcal{Q}$*-extension* of $\mathcal{N}$) if $\mathcal{N}$ is a normal subgroup of $\mathcal{A}$ and the quotient group $\mathcal{A}/\mathcal{N} \cong \mathcal{Q}$. Or equivalently, a sequence

$$1 \to \mathcal{N} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{Q} \to 1$$

is exact. The extension is *split* if $\mathcal{N}$ has a complement in $\mathcal{A}$. By a *complement* of $\mathcal{N}$ in $\mathcal{A}$, we mean a subgroup $\mathcal{H}$ satisfying $\mathcal{A} = \mathcal{N}\mathcal{H}$ and $\mathcal{N} \cap \mathcal{H} = 1$. Otherwise, the extension is *nonsplit*.

Let us assume that the given extension is split. For a complement $\mathcal{H}$ of $\mathcal{N}$ in $\mathcal{A}$, one has $\mathcal{H} \cong \mathcal{A}/\mathcal{N}$. So we can view $\mathcal{Q}$ as a subgroup of $\mathcal{A}$. A trivial case is an (internal) direct product of two groups $\mathcal{N}$ and $\mathcal{Q}$: $\mathcal{A} = \mathcal{N}\mathcal{Q}$ with $\mathcal{N} \cap \mathcal{Q} = 1$ and a trivial commutator $[\mathcal{N},\mathcal{Q}] = 1$. For all nontrivial cases, it holds $\mathcal{A} = \mathcal{N}\mathcal{Q}$ with $\mathcal{N} \cap \mathcal{Q} = 1$, but the commutator $[\mathcal{N},\mathcal{Q}]$ is not trivial and the multiplication in $\mathcal{N}$ is twisted by an action of the elements of $\mathcal{Q}$, that is, for $n_i \in \mathcal{N}$ and $q_j \in \mathcal{Q}$ with $i, j \in \{1, 2\}$,

$$(n_1q_1)(n_2q_2) = n_1(q_1n_2q_1^{-1})q_1q_2 = n_1n_2^{\alpha(q_1)^{-1}}q_1q_2,$$

where $\alpha\colon \mathcal{Q} \to \mathrm{Aut}(\mathcal{N})$ is a homomorphism defined by $\alpha(q)^{-1} = \mathrm{Inn}(q^{-1})$. The *semidirect product* $\mathcal{A} = \mathcal{N} \rtimes_\alpha \mathcal{Q}$ of $\mathcal{N}$ by $\mathcal{Q}$ with respect to α is defined on the set

$$\mathcal{A} = \{(n, q) \mid n \in \mathcal{N}, q \in \mathcal{Q}\}$$

with a multiplication

$$(n_1, q_1)(n_2, q_2) = (n_1n_2^{\alpha(q_1)^{-1}}, q_1q_2).$$

The semidirect product $\mathcal{A} = \mathcal{N} \rtimes_\alpha \mathcal{Q}$ is in fact a group with $(n, q)^{-1} = (n^{-\alpha(q)}, q^{-1})$. If we identify $\mathcal{Q}$ and $\mathcal{N}$ with $\{(1, q) \mid q \in \mathcal{Q}\}$ and $\{(n, 1) \mid n \in \mathcal{N}\}$, respectively, then $\mathcal{A} = \mathcal{N}\mathcal{Q} = \mathcal{Q}\mathcal{N}$ and $\mathcal{N} \cap \mathcal{Q} = 1$. So a semidirect product $\mathcal{A} = \mathcal{N} \rtimes_\alpha \mathcal{Q}$ is a split extension of $\mathcal{N}$ by $\mathcal{Q}$. Consequently an extension of $\mathcal{N}$ by $\mathcal{Q}$ is split if and only if $\mathcal{A}$ is a semidirect product of $\mathcal{N}$ by $\mathcal{Q}$.

A (split or nonsplit) extension of a cyclic group by another cyclic group is called a *metacyclic group*. The next two lemmas are famous in finite group theory, see [11] and [10], respectively.

Lemma 2.1 (Hölder)**.** *Let $\mathcal{A}$ be a metacyclic group which is an extension of a cyclic group of order n by a cyclic group of order m. Then $\mathcal{A}$ has the following presentation*

$$\mathcal{A} = \langle a, b \mid a^n = 1, b^m = a^t, b^{-1}ab = a^r \rangle, \tag{2.1}$$

where n, m, t and r satisfy

$$r^m \equiv 1 \pmod{n}, \quad t(r-1) \equiv 0 \pmod{n}. \tag{2.2}$$

Conversely, for any parameters n, m, t, r satisfying Equation (2.2)*, the relations in Equation* (2.1) *define a metacyclic group which is an extension of a cyclic group of order n by a cyclic group of order m.*

A subgroup $\mathcal{N}$ of $\mathcal{A}$ is a *Hall subgroup* if $|\mathcal{N}|$ is coprime to $|\mathcal{A} : \mathcal{N}|$.

Lemma 2.2 (Schur-Zassenhaus)**.** *Let $\mathcal{N}$ be a normal Hall subgroup of $\mathcal{A}$. Then*

(1) $\mathcal{N}$ has a complement in $\mathcal{A}$.

(2) If $\mathcal{H}$ and $\mathcal{K}$ are two complements of $\mathcal{N}$ in $\mathcal{A}$, then there is an element $n \in \mathcal{N}$ such that $n^{-1}\mathcal{H}n = \mathcal{K}$.

By Lemmas 2.1 and 2.2, one can show that a $\mathbb{Z}_2$-extension of a cyclic p-group with odd prime p is a cyclic or a dihedral group. Now, let $p = 2$.

The following theorems in this section come from an unpublished manuscript [18] Chapter 3 by Kwak and Xu. Since the authors cannot find these theorems in any other sources, we add their proofs in this paper.

Theorem 2.3. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic 2-group $\mathbb{Z}_{2^{n-1}}$ with $n \geq 4$. Then $\mathcal{A}$ is isomorphic to one of following six groups.*

(1) (*the cyclic group*)
$\mathbb{Z}_{2^n} = \langle b \mid a^{2^{n-1}} = 1, b^2 = a \rangle$,

(2) (*the non-cyclic abelian group*)
$\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2 = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a \rangle$,

(3) (*the dihedral group*)
$\mathbb{D}_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$,

(4) (*the generalized quaternion group*)
$\mathbb{Q}_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle$,

(5) (*the ordinary metacyclic group*)
$\mathbb{M}_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+2^{n-2}} \rangle$,

(6) (*the semidihedral group*)
$\mathbb{SD}_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle$.

All the six groups are not isomorphic one another.

Proof. Since (1) and (2) are trivial cases, we assume that $\mathcal{A}$ is not abelian. By Lemma 2.1, $\mathcal{A}$ has the following presentation:

$$\mathcal{A} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^t, b^{-1}ab = a^r \rangle,$$

where t and r satisfy

$$r^2 \equiv 1 \pmod{2^{n-1}}, \quad t(r-1) \equiv 0 \pmod{2^{n-1}}.$$

By $\mathcal{A}$ non-abelian, one has $r \equiv -1$ or $\pm 1 + 2^{n-2} \pmod{2^{n-1}}$, the latter two cases can happen only when $n \geq 4$. If $r \equiv -1$ or $-1+2^{n-2} \pmod{2^{n-1}}$, then $2^{n-1} \mid 2t$ and hence $2^{n-2} \mid t$, it follows that $t \equiv 0$ or $2^{n-2} \pmod{2^{n-1}}$. Now we consider the three cases separately.

(i) $r \equiv -1 \pmod{2^{n-1}}$. In this case we get the dihedral group (3) and the generalized quaternion group (4) depending on $t \equiv 0$ or $2^{n-2} \pmod{2^{n-1}}$, respectively. These two groups are not isomorphic. Note that the following cases (ii) and (iii) happen only when $n \geq 4$. So, when $n = 3$ we have only the above two groups.

(ii) $r \equiv -1 + 2^{n-2} \pmod{2^{n-1}}$. In this case $t \equiv 0 \pmod{2^{n-2}}$. Thus $b^2 = 1$ or $a^{2^{n-2}}$. If $b^2 = a^{2^{n-2}}$, letting $b_1 = ba$, then

$$b_1^2 = (ba)^2 = b^2(b^{-1}ab)a = b^2 a^{-1+2^{n-2}} a = a^{2^{n-2}} a^{2^{n-2}} = 1.$$

Thus we get the group (6).

(iii) $r \equiv 1 + 2^{n-2} \pmod{2^{n-1}}$. In this case, one has $t \cdot 2^{n-2} \equiv 0 \pmod{2^{n-1}}$ which implies that t is even. Let $t = 2s$. Since $n \geq 4$, there is a j satisfying $j(1+2^{n-3}) + s \equiv 0 \pmod{2^{n-2}}$. Let $b_1 = ba^j$. Then

$$b_1^2 = b^2(b^{-1}a^j b)a^j = b^2 a^{j(2+2^{n-2})} = a^{2(j(1+2^{n-3})+s)} = 1.$$

Now the generators a, b_1 satisfy the relations in the group (5), with b instead of b_1.

Finally, we shall show that the mentioned four non-abelian groups are not isomorphic, and we assume that $n \geq 4$. It is easy to see that in these four cases the derived group $\mathcal{A}' = \langle [a,b] \rangle$. We calculate the commutator $[a,b]$ and get

$$[a,b] = a^{-1}b^{-1}ab = \begin{cases} a^{-2} & \text{for the groups (3) and (4),} \\ a^{2^{n-2}} & \text{for the group (5),} \\ a^{-2+2^{n-2}} & \text{for the group (6).} \end{cases}$$

So, one has $|\mathcal{A}'| = 2$ for (5), and $|\mathcal{A}'| = 2^{n-2}$ for the others. It follows that the group (5) is not isomorphic to any one of the rest. To prove the rest three groups are not isomorphic, we calculate the square of the elements of the form ba^i outside $\langle a \rangle$. We have

$$(ba^i)^2 = b^2(b^{-1}a^i b)a^i = \begin{cases} 1 & \text{for the group (3),} \\ a^{2^{n-2}} & \text{for the group (4),} \\ a^{i2^{n-2}} & \text{for the group (6).} \end{cases}$$

This shows that the subgroup of order 2^{n-1} in $\mathcal{A}$ is unique, and outside this subgroup $\langle a \rangle$, all elements are of order 2 in the group (3), order 4 in the group (4), and some are of order 2 and the others are of order 4 in the group (6). Therefore, all the four groups are not isomorphic to one another. $\square$

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n$, where $n = p_0^{\alpha_0} p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ is the prime decomposition with $p_0 = 2$. First, we consider the case that n is odd, that is, $\alpha_0 = 0$.

Theorem 2.4. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ with n odd. Then, $\mathcal{A}$ has a presentation*

$$\mathcal{A} = \langle a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = b^2 = 1, [a_i, a_j] = 1, b^{-1} a_i b = a_i^{r_i} \textit{ for all } i, j \rangle,$$

where $r_i^2 \equiv 1 \pmod{p_i^{\alpha_i}}$ for all i. There are 2^s non-isomorphic such extended groups.

Proof. By Lemma 2.2, $\mathcal{A}$ is split. Since $\mathcal{A}$ is a metacyclic group, by Lemma 2.1, $\mathcal{A}$ has the presentation

$$\mathcal{A} = \langle a, b \mid a^n = b^2 = 1, b^{-1} a b = a^r \rangle,$$

with $r^2 \equiv 1 \pmod{n}$. The action of b on each element of $\mathbb{Z}_n$ by conjugacy is an automorphism of $\mathbb{Z}_n$ of order at most 2. Since $\mathrm{Aut}(\mathbb{Z}_n) \cong \mathrm{Aut}(\mathbb{Z}_{p_1^{\alpha_1}}) \times \cdots \times \mathrm{Aut}(\mathbb{Z}_{p_s^{\alpha_s}})$, the b-conjugation on $\mathbb{Z}_n$ corresponds to an s-tuple $(r_1, \ldots, r_s)$ with $r_i \equiv \pm 1 \pmod{p_i^{\alpha_i}}$ for $i \in \{1, \ldots, s\}$. Thus the s-tuple $(r_1, \ldots, r_s)$ has 2^s choices and $\mathcal{A}$ is presented by

$$\mathcal{A} = \langle a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = b^2 = 1, [a_i, a_j] = 1, b^{-1} a_i b = a_i^{r_i} \text{ for all } i, j \rangle.$$

To finish the proof, it suffices to show that different s-tuples $(r_1, \ldots, r_s)$ give non-isomorphic groups. It is easy to see that $\mathbb{Z}_{p_i^{\alpha_i}}$ is a subgroup of the center of $\mathcal{A}$ if and only if $r_i = 1$. Hence the groups with different s-tuples $(r_1, \ldots, r_s)$ have different center of $\mathcal{A}$. Therefore, there are 2^s non-isomorphic $\mathbb{Z}_2$-extensions of $\mathbb{Z}_n$. □

Next we consider the case of even n. Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ with $p_0 = 2$. We deal with three cases $\alpha_0 = 1, 2$ or $\alpha_0 \geq 3$ in the next theorem. First we determine the Sylow 2-subgroup $\mathcal{S}_0$ of $\mathcal{A}$ which is a $\mathbb{Z}_2$-extension of $\mathbb{Z}_{2^{\alpha_0}} = \langle a_0 \rangle$. This has been done by Theorem 2.3. Namely,

$$\mathcal{S}_0 = \langle a_0, b_0 \mid a_0^{2^{\alpha_0}} = 1, b_0^2 = a_0^{t_0}, b_0^{-1} a_0 b_0 = a_0^{r_0} \rangle,$$

where $t_0 = 0, 1$ or $2^{\alpha-1}$, $r_0 = \pm 1$ or $\pm 1 + 2^{\alpha_0 - 1}$ depending on the types of $\mathcal{S}_0$ in Theorem 2.3. Next, take $b = b_0$. Thus each Sylow 2-subgroup and each element of order at most 2 in $\mathrm{Aut}(\mathbb{Z}_{p_1^{\alpha_1}}) \times \cdots \times \mathrm{Aut}(\mathbb{Z}_{p_s^{\alpha_s}})$ gives a unique $\mathbb{Z}_2$-extension of $\mathbb{Z}_n$.

Theorem 2.5. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ with $p_0 = 2$.*

(1) If $\alpha_0 = 1$, then $\mathcal{A}$ has the following presentations

- *(i) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = b^2 = 1, [a_i, a_j] = 1, b^{-1} a_i b = a_i^{r_i}$ for all $i, j \rangle$, ($\mathcal{S}_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$).*
- *(ii) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = 1, b^2 = a_0, [a_i, a_j] = 1, b^{-1} a_i b = a_i^{r_i}$ for all $i, j \rangle$, ($\mathcal{S}_0 = \mathbb{Z}_4$).*

There are 2^{s+1} non-isomorphic groups.

(2) If $\alpha_0 = 2$, then $\mathcal{A}$ has the following presentations

(i) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = b^2 = 1, [a_i, a_j] = 1, b^{-1}a_ib = a_i^{r_i}$ *for all* $i, j\rangle$, *(*$\mathcal{S}_0 = \mathbb{Z}_4 \times \mathbb{Z}_2$ *or* $\mathbb{D}_8$*).*

(ii) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = 1, b^2 = a_0, [a_i, a_j] = 1, b^{-1}a_ib = a_i^{r_i}$ *for all* $i, j\rangle$, *(*$\mathcal{S}_0 = \mathbb{Z}_8$*).*

(iii) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = 1, b^2 = a_0^2, [a_i, a_j] = 1, b^{-1}a_ib = a_i^{r_i}$ *for all* $i, j\rangle$, *(*$\mathcal{S}_0 = \mathbb{Q}_8$*).*

There are 2^{s+1} *non-isomorphic groups.*

(3) If $\alpha_0 \geq 3$*, then* $\mathcal{A}$ *has the following presentations*

(i) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = b^2 = 1, [a_i, a_j] = 1, b^{-1}a_ib = a_i^{r_i}$ *for all* $i, j\rangle$, *(*$\mathcal{S}_0 = \mathbb{Z}_{2^{\alpha_0}} \times \mathbb{Z}_2,\ \mathbb{D}_{2^{\alpha_0+1}},\ \mathbb{SD}_{2^{\alpha_0+1}},$ *or* $\mathbb{M}_{2^{\alpha_0+1}}$*).*

(ii) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = 1, b^2 = a_0, [a_i, a_j] = 1, b^{-1}a_ib = a_i^{r_i}$ *for all* $i, j\rangle$, *(*$\mathcal{S}_0 = \mathbb{Z}_{2^{\alpha_0+1}}$*).*

(iii) $\mathcal{A} = \langle a_0, a_1, \ldots, a_s, b \mid a_i^{p_i^{\alpha_i}} = 1, b^2 = a_0^{2^{\alpha_0-1}}, [a_i, a_j] = 1, b^{-1}a_ib = a_i^{r_i}$ *for all* $i, j\rangle$, *(*$\mathcal{S}_0 = \mathbb{Q}_{2^{\alpha_0+1}}$*).*

There are $6 \cdot 2^s$ *non-isomorphic groups.*

For each extension group $\mathcal{A}$ appeared so far, the number $\mathrm{Isoc}(G; \mathcal{A})$ shall be determined in the next section.

3 In cases of $\mathbb{Z}_2$-extensions of a cyclic p-group

For each group $\mathcal{A}$ in the classification of $\mathbb{Z}_2$-extensions of a cyclic p-group listed in the previous section, we aim to determine the number $\mathrm{Isoc}(G; \mathcal{A})$ in this section. However, for an abelian or a dihedral group $\mathcal{A}$, it has already been done in [14]. Hence, we need to do it only for each group $\mathcal{A}$ listed in the last three cases of Theorem 2.3. For a $\mathbb{Z}_2$-extension $\mathcal{A}$ of a finite group $\mathcal{H}$, we call an element x *normal type* if $x \in \mathcal{H}$ and *quotient type* otherwise. Note that $\mathcal{H}$ is normal in $\mathcal{A}$, and a product of any two normal type elements is normal type. For any two quotient type elements $ab, a'b$, their product is $aba'b = ab^2b^{-1}a'b$, and hence a product of any two quotient type elements is normal type. A *word* in $\{x_1, \ldots, x_s\}$ is any expression of the form $y_1^{i_1} \cdots y_k^{i_k}$ where $y_1, \ldots, y_k \in \{x_1, \ldots, x_s\}$ and $i_1, \ldots, i_k \in \{1, -1\}$, denoted by $w(x_1, \ldots, x_s)$. The number k is known as the *length* of the word. When writing words, it is common to use exponential notation as an abbreviation.

Lemma 3.1. *Let* $\mathcal{A}$ *be a* $\mathbb{Z}_2$*-extension of a finite group* $\mathcal{H}$*. For a subset* $\mathcal{I}$ *of* $S = \{1, \ldots, \beta\}$*, let*

$$\Omega_{\mathcal{I}}(\mathcal{A}; \beta) = \{(x_1, \ldots, x_\beta) \in \Omega(\mathcal{A}; \beta) : x_i \text{ is quotient type for exactly indices } i \in \mathcal{I}\}.$$

Then $|\Omega(\mathcal{A}; \beta)| = (2^\beta - 1)|\Omega_{\{1\}}(\mathcal{A}; \beta)|$.

Proof. Recall that

$$\Omega(\mathcal{A}; \beta) = \{(x_1, \ldots, x_\beta) \in \mathcal{A}^\beta : \langle x_1, \ldots, x_\beta\rangle = \mathcal{A}\}.$$

For each tuple $(x_1, \dots, x_\beta) \in \Omega(\mathcal{A}; \beta)$, at least one of entries x_i should be quotient type to generate the whole group $\mathcal{A}$. Then,

$$\Omega(\mathcal{A}; \beta) = \bigcup_{\emptyset \neq \mathcal{I} \subseteq S} \Omega_{\mathcal{I}}(\mathcal{A}; \beta), \quad \text{disjoint union,}$$

and

$$|\Omega(\mathcal{A}; \beta)| = \sum_{\emptyset \neq \mathcal{I} \subseteq S} |\Omega_{\mathcal{I}}(\mathcal{A}; \beta)|.$$

For any non-empty subset $\mathcal{I}$ of S, choose an index $j_0 \in \mathcal{I}$ and define a map $\phi\colon \Omega_{\mathcal{I}}(\mathcal{A}; \beta) \to \Omega_{\{j_0\}}(\mathcal{A}; \beta)$ by replacing all quotient type entries x_i for $i \in \mathcal{I}$ by $x_{j_0} x_i$ except x_{j_0}. Then one can see that ϕ is well-defined and bijective. It follows $|\Omega_{\mathcal{I}}(\mathcal{A}; \beta)| = (2^\beta - 1)|\Omega_{\{j_0\}}(\mathcal{A}; \beta)|$. One can assume that $j_0 = 1$ for convenience. □

Lemma 3.2. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a finite group $\mathcal{H}$. If each x_i is normal type except x_1, then $\langle x_1, \dots, x_\beta \rangle = \mathcal{A}$ if and only if $\langle x_1^2, x_2, \dots, x_\beta, x_1^{-1} x_2 x_1, \dots, x_1^{-1} x_\beta x_1 \rangle = \mathcal{H}$.*

Proof. Assume $\langle x_1^2, x_2, \dots, x_\beta, x_1^{-1} x_2 x_1, \dots x_1^{-1} x_\beta x_1 \rangle = \mathcal{H}$ and each x_i is normal type except x_1. Then $\langle x_1, \dots, x_\beta \rangle = \langle x_1, x_1^2, x_2, \dots, x_\beta, x_1^{-1} x_2 x_1, \dots x_1^{-1} x_\beta x_1 \rangle = \langle x_1, \mathcal{H} \rangle = \mathcal{A}$. Now assume that $\langle x_1, \dots, x_\beta \rangle = \mathcal{A}$. For any $g \in \mathcal{A}$, g can be expressed by a word $w(x_1, \dots, x_\beta)$. For odd k, $x_i x_1^k = x_1 \cdot (x_1^{-1} x_i x_1) \cdot (x_1^2)^{(k-1)/2}$ and for even k, $x_i x_1^k = x_i \cdot (x_1^{k/2})$. Rewrite g, one has

$$g = w(x_1, \dots, x_\beta) = x_1^\ell w(x_1^2, x_2, \dots, x_\beta, x_1^{-1} x_2 x_1, \dots, x_1^{-1} x_\beta x_1), \quad \ell = 0, 1.$$

It follows that g is normal type if and only if $\ell = 0$. Therefore, $\langle x_1^2, x_2, \dots, x_\beta, x_1^{-1} x_2 x_1, \dots, x_1^{-1} x_\beta x_1 \rangle = \mathcal{H}$. □

Corollary 3.3. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n$. If each x_i is a normal type element except x_1, then $\langle x_1, \dots, x_\beta \rangle = \mathcal{A}$ if and only if $\langle x_1^2, x_2, \dots, x_\beta \rangle = \mathbb{Z}_n$.*

We determine $|\Omega(\mathcal{A}; \beta)|$ and $|\mathrm{Aut}(\mathcal{A})|$ for each group $\mathcal{A}$ listed in the last three cases of Theorem 2.3 in the following.

Lemma 3.4. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_{2^{n-1}}$ and let $\mathcal{A}$ be non-abelian. Then $|\Omega(\mathcal{A}; \beta)| = 2^{(n-2)\beta+1}(2^\beta - 1)(2^{\beta-1} - 1)$.*

Proof. By Lemma 3.1, it just needs to determine $\Omega_{\{1\}}(\mathcal{A}; \beta)$. By Corollary 3.3, $\langle x_1^2, x_2, \dots, x_s \rangle = \mathbb{Z}_{2^{n-1}}$ if and only if $(x_1, \dots, x_s) \in \Omega_{\{1\}}(\mathcal{A}; \beta)$. By the last three cases of Theorem 2.5, one can assume $b^2 = a^t$ with $t = 0$ or 2^{n-2} for a generator a of $\mathbb{Z}_{2^{n-1}}$. Note that x_1 is quotient type, say $x_1 = ba^i$. Then $x_1^2 = b^2 \cdot b^{-1} a^i b \cdot a^i = b^2 a^{i(1+r)} = a^{t+i(1+r)}$ with $r \in \{-1, \pm 1 + 2^{n-2}\}$. Suppose x_1^2 generates $\mathbb{Z}_{2^{n-1}}$. Then $t + i(1+r) \equiv 1 \pmod 2$. But it is impossible by checking case by case. So $\langle x_2, \dots, x_s \rangle = \mathbb{Z}_{2^{n-1}}$. By $|\Omega(\mathbb{Z}_{2^{n-1}}; \beta - 1)| = 2^{(n-2)(\beta-1)}(2^{\beta-1} - 1)$, which was shown by Kwak et al. in [14], it follows $|\Omega(\mathcal{A}; \beta)| = (2^\beta - 1)2^{n-1}|\Omega(\mathbb{Z}_{2^{n-1}}; \beta - 1)| = 2^{(n-2)\beta+1}(2^\beta - 1)(2^{\beta-1} - 1)$. □

Lemma 3.5. *For $n \geq 4$,*

(1) $|\mathrm{Aut}(\mathbb{Z}_{2^n})| = 2^{n-1}$,

(2) $|\mathrm{Aut}(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2)| = 2^n$,

(3) $|\mathrm{Aut}(\mathbb{D}_{2^n})| = 2^{2n-3}$,

(4) $|\mathrm{Aut}(\mathbb{Q}_{2^n})| = 2^{2n-3}$,

(5) $|\mathrm{Aut}(\mathbb{M}_{2^n})| = 2^n$,

(6) $|\mathrm{Aut}(\mathbb{SD}_{2^n})| = 2^{2n-4}$.

Proof. Since the first three cases have been shown in [14], we only need to show the last three cases. To do this separately, let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_{2^{n-1}}$ and let an automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$ be of the form $a \mapsto a^i b^k, b \mapsto a^j b^\ell$ with $0 \le i, j \le 2^{n-1} - 1$ and $0 \le k, \ell \le 1$.

(4) Since the identity $(a^i b)^2 = bb^{-1}a^i b a^i b = b^2$ gives the orders $o(a^i b) = 4$ and $o(a) = 2^{n-1} \neq 4$ for $n \ge 4$, the image $\sigma(a_i)$ should be of the form a^i with $(i, 2^{n-1}) = 1$. The surjectivity of σ implies that the choices of $\sigma(b)$ are $a^j b$ with $j = 0, \ldots, 2^{n-1} - 1$. Moreover, all of such possible choices $\sigma(a)$ and $\sigma(b)$ satisfy the defining relations of $\mathbb{Q}_{2^n}$. Hence $|\mathrm{Aut}(\mathbb{Q}_{2^n})| = 2^{2n-3}$ by counting the choices of $\sigma(a)$ and $\sigma(b)$, that is, the choices of i, j, k, ℓ.

(5) If $k = 0$, then $\sigma(a) = a^i$ for some i with $(i, 2^{n-1}) = 1$. If $k = 1$, then $\sigma(a) = a^i b$ for some i with $(i, 2^{n-1}) = 1$, because the order preserving condition says $o(a^i b) = o(a) = 2^{n-1}$, and $(a^i b)^m = b^m a^{i(1+\cdots+r^{m-1})}$ for all $m \ge 1$, where $r = 1 + 2^{n-2}$. Next, we determine the possible values of $\sigma(b)$. If $\ell = 0$, then j should be 2^{n-2}. In this case, all possible values $\sigma(a)$ and $\sigma(b)$ do not satisfy the defining relations of $\mathbb{M}_{2^n}$. Thus it should be $\ell = 1$. Now the order condition $o(a^k b) = o(b) = 2$ implies $j = 2^{n-2}$ or 0. Consequently, σ has four different forms.

(i) $a \mapsto a^i, b \mapsto b$,
(ii) $a \mapsto a^i, b \mapsto a^{2^{n-2}} b$,
(iii) $a \mapsto a^i b, b \mapsto b$,
(iv) $a \mapsto a^i b, b \mapsto a^{2^{n-2}} b$.

In these four cases, $\sigma(a)$ and $\sigma(b)$ satisfy the defining relations of $\mathbb{M}_{2^n}$. Therefore, the four different cases give $|\mathrm{Aut}(\mathbb{M}_{2^n})| = 2^n$.

(6) Since $(a^i b)^2 = a^{i \cdot 2^{n-2}}$, one gets $o(a^i b) = 2$ for even i and $o(a^i b) = 4$ for odd i. Hence σ should be of the form $a \mapsto a^i, b \mapsto a^j b$ with $(i, 2^{n-1}) = 1$ and j even. Moreover, all such possible values $\sigma(a)$ and $\sigma(b)$ satisfy the defining relations of $\mathbb{SD}_{2^n}$. So $|\mathrm{Aut}(\mathbb{SD}_{2^n})| = 2^{2n-4}$. □

As a special case, $|\mathrm{Aut}(\mathbb{Q}_8)| = 24$ which is not included in the above lemma. From Theorem 1.2 and Lemmas 3.4 and 3.5, one can get the following theorem.

Theorem 3.6. *For a $\mathbb{Z}_2$-extension of $\mathcal{A}$ a cyclic group $\mathbb{Z}_{2^{n-1}}$ for $n \ge 2$,*

$$\mathrm{Isoc}(G;\mathcal{A}) = \begin{cases} 2^{(\beta-1)(n-1)}(2^\beta - 1) & \text{if } \mathcal{A} \text{ is } \mathbb{Z}_{2^n}, \\ 2^{(\beta-2)(n-2)+(n-3)}(2^\beta - 1)(2^{\beta-1} - 1) & \text{if } \mathcal{A} \text{ is } \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2, \\ 2^{(\beta-2)(n-2)}(2^\beta - 1)(2^{\beta-1} - 1) & \text{if } \mathcal{A} \text{ is } \mathbb{D}_{2^n} \text{ for } n \ge 3, \\ 2^{(\beta-2)}(2^\beta - 1)(2^{\beta-1} - 1)/3 & \text{if } \mathcal{A} \text{ is } \mathbb{Q}_8, \\ 2^{(\beta-2)(n-2)}(2^\beta - 1)(2^{\beta-1} - 1) & \text{if } \mathcal{A} \text{ is } \mathbb{Q}_{2^n} \text{ for } n \ge 4, \\ 2^{(\beta-1)(n-2)-1}(2^\beta - 1)(2^{\beta-1} - 1) & \text{if } \mathcal{A} \text{ is } \mathbb{M}_{2^n} \text{ for } n \ge 4, \\ 2^{(\beta-2)(n-2)+1}(2^\beta - 1)(2^{\beta-1} - 1) & \text{if } \mathcal{A} \text{ is } \mathbb{SD}_{2^n} \text{ for } n \ge 4, \end{cases}$$

where the first three cases were shown in [14].

By using the Möbius function, $\mathrm{Isoc}(G; \mathcal{A})$ can also be determined. For example, for a generalized quaternion group $\mathbb{Q}_{2^n}$, a proper subgroup $\mathcal{S}$ of $\mathbb{Q}_{2^n}$ is isomorphic to $\mathbb{Z}_{2^m}$ or $\mathbb{Q}_{2^m}^{(i)}$, where $\mathbb{Z}_{2^m} = \langle a^{2^{n-m-1}} \rangle$ and $\mathbb{Q}_{2^m}^{(i)} = \langle a^{2^{n-m}}, a^i b \rangle$ for $m \in \{1, \dots, n-1\}$ and $i \in \{0, \dots, 2^{n-m} - 1\}$. From the subgroups lattice of $\mathbb{Q}_{2^n}$, see Figure 1, one has

$$\mu(S) = \begin{cases} 1 & \text{if } \mathcal{S} = \mathbb{Q}_{2^n}, \\ -1 & \text{if } \mathcal{S} = \mathbb{Z}_{2^{n-1}}, \mathbb{Q}_{2^{n-1}}^{(0)} \text{ or } \mathbb{Q}_{2^{n-1}}^{(1)}, \\ 2 & \text{if } \mathcal{S} = \mathbb{Z}_{2^{n-2}}, \\ 0 & \text{otherwise.} \end{cases}$$

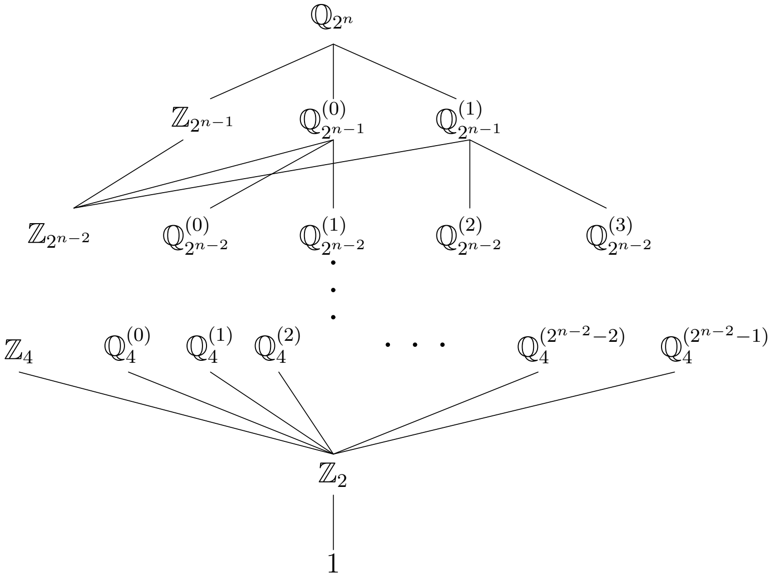

Figure 1: The subgroup lattice of $\mathbb{Q}_{2^n}$.

It follows from Theorem 1.3

$$\mathrm{Isoc}(G; \mathbb{Q}_{2^n}) = \begin{cases} \frac{1}{3}(2^{3\beta-3} - 3 \cdot 2^{2\beta-3} + 2^{\beta-2}) & \text{if } n = 3, \\ \frac{1}{2^{2n-3}}(2^{\beta n} - 3 \cdot 2^{\beta(n-1)} + 2^{\beta(n-2)+1}) & \text{if } n > 3, \end{cases}$$

which coincides with the formula given in Theorem 3.6.

If $\mathcal{A} \cong \mathbb{M}_{2^n}$, then every proper subgroup $\mathcal{S}$ of $\mathbb{M}_{2^n}$ is isomorphic to $\mathbb{Z}_m$ or $\mathbb{M}_{2^m}^{(i)}$ for $m \in \{2, \dots, n-1\}$ and $i \in \{0, 1\}$, where $\mathbb{Z}_{2^m} = \langle a^{2^{n-m-1}} \rangle$, $\mathbb{M}_{2^m}^{(0)} = \langle a^{2^{n-m}}, b \rangle$ and $\mathbb{M}_{2^m}^{(1)} = \langle a^{2^{n-m}}, a^{2^{n-m-1}} b \rangle$. If $m = 1$, then $\mathcal{S}$ is isomorphic to $\mathbb{Z}_2^{(0)} = \langle a^{n-2} \rangle$ or $\mathbb{Z}_2^{(1)} = \langle a^{n-2} b \rangle$. Now from the subgroups lattice of $\mathbb{M}_{2^n}$ illustrated in Figure 2 and $|\mathrm{Aut}(\mathbb{M}_{2^n})| = 2^n$, one can have

$$\mathrm{Isoc}(G; \mathbb{M}_{2^n}) = \frac{1}{2^n}(2^{n\beta} - 3 \cdot 2^{(n-1)\beta} + 2^{(n-2)\beta+1}),$$

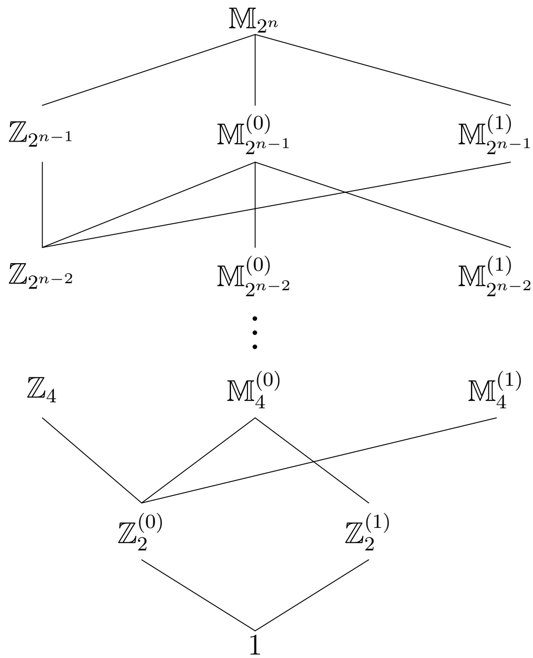

Figure 2: The subgroup lattice of $\mathbb{M}_{2^n}$.

which coincides exactly with the result in Theorem 3.6.

Also, by using the Möbius function, one can show that

$$\mathrm{Isoc}(G;\mathbb{SD}_{2^n}) = \frac{1}{2^{2n-4}}(2^{n\beta} - 3\cdot 2^{(n-1)\beta} + 2^{(n-2)\beta+1}).$$

For some small β and n, the numbers $\mathrm{Isoc}(G;\mathcal{A})$ are tabulated in Table 1.

Table 1: The number Isoc for small β and n.

	Isoc						
(β,n)	$\mathbb{Z}_{2^n}$	$\mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_2$	$\mathbb{D}_{2^n}$	$\mathbb{Q}_{2^n}$	$\mathbb{M}_{2^n}$	$\mathbb{SD}_{2^n}$	$\mathcal{A}$
$(2,3)$	12	3	3	1	0	0	22
$(2,4)$	24	6	3	6	24	6	69
$(2,4)$	48	12	3	6	48	6	123
$(3,3)$	112	42	42	56	0	0	252
$(3,4)$	448	168	84	168	672	168	1708
$(3,5)$	1792	672	168	336	2688	336	5992
$(4,3)$	960	420	420	560	0	0	2360
$(4,4)$	7680	3360	1680	3360	13440	3360	32880
$(4,5)$	61440	26880	6720	13440	107520	13440	229440

4 In cases of $\mathbb{Z}_2$-extensions of any cyclic groups

In this section we determine $\mathrm{Isoc}(G;\mathcal{A})$ for a $\mathbb{Z}_2$-extension $\mathcal{A}$ of a cyclic group $\mathbb{Z}_n$ (of any order n, not necessarily to be a p-group). Again, let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ and let $n = p_0^{\alpha_0} p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime decomposition with $p_0 = 2$. Let the b-conjugation on $\mathbb{Z}_n$ correspond to an $(s+1)$-tuple $(r_0, r_1, \ldots, r_s)$, where $r_0 \in \{\pm 1, \pm 1 + 2^{\alpha_0 - 1}\}$ and $r_i = \pm 1$ for $i \in \{1, \ldots, s\}$ with -1 in exactly t entries $\ell_1, \ldots, \ell_t$. Let $n = 2^{\alpha_0} n_1 n_2$ with $n_1 = \prod_{j=1}^{t} p_{\ell_j}^{\alpha_{\ell_j}}$. Then $\mathcal{A}$ is isomorphic to $\mathcal{B} \times \mathbb{Z}_{n_2}$ where $\mathcal{B}$ is a $\mathbb{Z}_2$-extension of $\mathbb{Z}_{n_1}$ since any element of $\mathbb{Z}_{n_2}$ commutes with each element of $\mathcal{A}$. Since $(|\mathcal{B}|, |\mathbb{Z}_{n_1}|) = 1$, one has $\mathrm{Isoc}(G; \mathcal{B} \times \mathbb{Z}_{n_1}) = \mathrm{Isoc}(G;\mathcal{B}) \cdot \mathrm{Isoc}(G; \mathbb{Z}_{n_1})$, as shown in [14]. Because $\mathrm{Isoc}(G;\mathbb{Z}_{n_1})$ has already been determined, we just need to determine $\mathrm{Isoc}(G;\mathcal{B})$.

Lemma 4.1. *Let $\mathcal{B}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ with $p_0 = 2$ and $s \geq 1$, and let $n = 2^{\alpha_0} m$. Let the b-conjugation on $\mathbb{Z}_n$ correspond to an $(s+1)$-tuple $(r_0, r_1, \ldots, r_s)$, where $r_0 \in \{\pm 1, \pm 1 + 2^{\alpha_0 - 1}\}$ and all other r_i's are -1. Then*

$$|\Omega(\mathcal{B};\beta)| = \begin{cases} (2^\beta - 1) m 2^{\alpha_0 \beta} |\Omega(\mathbb{Z}_m; \beta - 1)| & \text{if } 2^{\alpha_0 + 1} \mid o(b), \\ (2^\beta - 1) m 2^{\alpha_0} |\Omega(\mathbb{Z}_{2^{\alpha_0} m}; \beta - 1)| & \text{otherwise}. \end{cases}$$

Proof. By Theorem 2.5, one can assume $b^2 = a_0^t$ with $t \in \{0, 1, 2^{\alpha_0 - 1}\}$. By Lemma 3.1, it just needs to determine $|\Omega_{\{1\}}(\mathcal{B};\beta)|$. Take $(x_1, \ldots, x_\beta) \in \Omega_{\{1\}}(\mathcal{A};\beta)$. Note that x_1 is a quotient type element and other x_i's are all normal type. Since

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times Z_{p_s^{\alpha_s}},$$

any element of $\mathbb{Z}_n$ can be presented $g_i h_i$ with $g_i \in \mathbb{Z}_{p_0^{\alpha_0}}$ and $h_i \in \prod_{i=1}^{\beta} \mathbb{Z}_{p_i^{\alpha_i}}$. So x_1 can be presented by $g_1 h_1 b$ and other x_i's can be presented by $g_i h_i$. By Corollary 3.3, $\langle x_1, \ldots, x_\beta \rangle = \mathcal{B}$ if and only if $\langle x_1^2, x_2, \ldots, x_\beta \rangle = \mathbb{Z}_n$. By $x_1^2 = (g_1 h_1 b)^2 = b^2 g_1^{1+r_0}$, one has $\langle b^2 g_1^{1+r_0}, g_2 h_2, \ldots, g_\beta h_\beta \rangle = \mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$. Recall that $b^2 = a_0^t \in \mathbb{Z}_{p_0^{\alpha_0}}$ with $p_0 = 2$, then $\langle b^2 g_1^{1+r_0}, g_2, \ldots, g_\beta \rangle = \mathbb{Z}_{p_0^{\alpha_0}}$ and $\langle h_2, \ldots, h_\beta \rangle = \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$. So $(b^2 g_1^{1+r_0}, g_2, \ldots, g_\beta) \in \Omega(\mathbb{Z}_{2^{\alpha_0}};\beta)$, $(h_2, \ldots, h_\beta) \in \Omega(\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}; \beta - 1)$. To count the choice of $(x_1, \ldots, x_\beta)$, equivalently to count the number of $(g_1, \ldots, g_\beta)$ and $(h_1, \ldots, h_\beta)$. When computing $x_1^2 = b^2 g_1^{1+r_0}$, h_1 can be any element of $\prod_{i=1}^{\beta} \mathbb{Z}_{p_i^{\alpha_i}}$, and it follows h_1 has m choices by $m = \prod_{i=1}^{\beta} p_i^{\alpha_i}$. The number of choices of $(h_2, \ldots, h_\beta)$ is equal to $|\Omega(\mathbb{Z}_m; \beta - 1)|$. Hence number of choices of $(h_1, \ldots, h_\beta)$ is $m|\Omega(\mathbb{Z}_m;\beta-1)|$. Now we determine the number of choices of $(g_1, \ldots, g_\beta)$ in the following.

Assume that $2^{\alpha_0+1} \mid o(b)$ and it follows $t = 1$. Then the Sylow 2-subgroup of $\mathcal{B}$ is $\mathbb{Z}_{2^{\alpha_0+1}}$. By Theorem 2.5, $\langle b^2 \rangle = \langle a_0 \rangle = \mathbb{Z}_{2^{\alpha_0}}$ and $r_0 = 1$. Since $g_1 \in \mathbb{Z}_{2^{\alpha_0}}$, one has $\langle b^2 g_1^{1+r_0} \rangle = \langle b^2 g_1^2 \rangle = \langle b^2 \rangle = \mathbb{Z}_{2^{\alpha_0}}$. By $\langle b^2 g_1^{1+r_0}, g_2, \ldots, g_\beta \rangle = \langle b^2 \rangle = \mathbb{Z}_{2^{\alpha_0}}$, $(g_1, \ldots, g_\beta)$ has $2^{\alpha_0 \beta}$ choices. So $|\Omega_{\{1\}}(\mathcal{B};\beta)| = 2^{\alpha_0\beta} m |\Omega(\mathbb{Z}_m;\beta-1)|$, and it follows $|\Omega(\mathcal{B};\beta)| = (2^\beta - 1) m 2^{\alpha_0\beta} |\Omega(\mathbb{Z}_m;\beta-1)|$.

If 2^{α_0+1} does not divide $o(b)$, then, by Theorem 2.5, $b^2 = a_0^t$ with $t \in \{0, 2^{\alpha_0-1}\}$. If $t = 0$, then $b^2 = 1$ and $r_0 \in \{\pm 1, \pm 1 + 2^{\alpha_0 - 1}\}$. It follows $b^2 g_1^{1+r_0} = 1, g_1^2, g_1^{2^{\alpha_0-1}}$ or $g_1^{2+2^{\alpha_0-1}}$. So g^{1+r_0} can not be the generator of $\mathbb{Z}_{2^{\alpha_0}}$. If $t = 2^{\alpha_0-1}$, then $r_0 = -1$. Then $b^2 g_1^{1+r_0} = a_0^{2^{\alpha_0-1}}$, again, $b^2 g_1^{1+r_0}$ can not generate $\mathbb{Z}_{2^{\alpha_0}}$. So $\langle b^2 g_1^{1+r_0}, g_2, \ldots, g_\beta \rangle = \mathbb{Z}_{2^{\alpha_0}}$ if and only if $\langle g_2, \ldots, g_\beta \rangle = \mathbb{Z}_{2^{\alpha_0}}$. It follows $(g_2, \ldots, g_\beta) \in \Omega(\mathbb{Z}_{2^{\alpha_0}};\beta-1)$ and g_1

is any element in $\mathbb{Z}_{2^{\alpha_0}}$. Then $(g_1, \dots, g_\beta)$ has $2^{\alpha_0}|\Omega(\mathbb{Z}_{2^{\alpha_0}}; \beta-1)|$ choices. So $|\Omega(\mathcal{B}; \beta)| = (2^\beta - 1)m2^{\alpha_0}|\Omega(\mathbb{Z}_{2^{\alpha_0}m}; \beta - 1)|$. □

Lemma 4.2. *Let $\mathcal{B}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ with $p_0 = 2$ and $s \geq 1$, and let $n = 2^{\alpha_0}m$. Let the b-conjugation on $\mathbb{Z}_n$ correspond to an $(s+1)$-tuple $(r_0, r_1, \dots, r_s)$, where $r_0 \in \{\pm 1, \pm 1 + 2^{\alpha_0 - 1}\}$ and all other r_i's are -1.*

(1) If $r_0 = 1$, then

$$|\mathrm{Aut}(\mathcal{B})| = \begin{cases} 2^{\alpha_0} m\varphi(n) & \textit{if } 2^{\alpha_0+1} \mid o(b), \\ 2m\varphi(n) & \textit{otherwise}. \end{cases}$$

(2) If $r_0 = -1$, then $|\mathrm{Aut}(\mathcal{B})| = 2^{\alpha_0} m\varphi(n)$.

(3) If $r_0 = 1 + 2^{\alpha_0 - 1}$, then $|\mathrm{Aut}(\mathcal{B})| = 2m\varphi(n)$.

(4) If $r_0 = -1 + 2^{\alpha_0 - 1}$, then $|\mathrm{Aut}(\mathcal{B})| = 2^{\alpha_0 - 1} m\varphi(n)$.

Proof. Again, one can assume that $b^2 = a_0^t$ with $t \in \{0, 1, 2^{\alpha_0 - 1}\}$. For an automorphism σ of $\mathcal{B}$, $\sigma(a_k)$ should be of the form $a_k^{i_k}$ with $(i_k, p_k^{\alpha_k}) = 1$ for $k \in \{1, \dots, s\}$ since σ is order-preserving. Suppose that $\sigma(a_0)$ is quotient type, then $b^{-1}a_k b = a_k$ since $\sigma(a_0)$ commutes with $\sigma(a_k)$ for each k. So $r_i = 1$ for each $i \in \{1, \dots, s\}$, which is a contradiction. Then $\sigma(a_0)$ is normal type, say $\sigma(a_0) = a_0^{i_0}$ with $(i_0, 2^{\alpha_0}) = 1$. Assume $\sigma(b) = a_0^{u_0} a_1^{u_1} \cdots a_s^{u_s} b$. We need to count the number of choices of $u_0, \dots, u_\beta$. By computing, $(a_0^{u_0} a_1^{u_1} \cdots a_s^{u_s} b)^2 = b^2 a_0^{u_0(1+r_0)} \cdots a_\beta^{u_\beta(1+r_\beta)} = b^2 a_0^{u_0(1+r_0)}$. Note that $o(\sigma(b)) = o(b)$ and $o(b)$ is even. By hypothesis, $r_1 = \cdots = r_\beta = -1$, and it follows $(\sigma(b))^2 = b^2 a_0^{u_0(1+r_0)}$. Then u_i can be any element of $\mathbb{Z}_{p_i^{\alpha_i}}$ for $i \in \{1, \dots, s\}$. Now it needs to determine the number of choices of u_0.

(1) If $r_0 = 1$ and $o(b) = 2^{\alpha_0+1}$, then $(\sigma(b))^2 = b^2 a_0^{2u_0}$. By Theorem 2.5, $b^2 = a_0$ in this case. Then $o(b^2 a_0^{2u_0}) = o(b^2) = 2^{\alpha_0}$, and it follows u_0 has 2^{α_0} choices. Hence $|\mathrm{Aut}(\mathcal{B})| = 2^{\alpha_0} m\varphi(n)$. If $r_0 = 1$ and $o(b) = 2$, then u_0 can be 0 or $2^{\alpha_0 - 1}$. So $|\mathrm{Aut}(\mathcal{B})| = 2m\varphi(n)$.

(2) If $r_0 = -1$, then $(\sigma(b))^2 = b^2$ and $o(b)$ is 2 or 4, by Theorem 2.5. So u_0 can be any element of $\mathbb{Z}_{2^{\alpha_0}}$ and has 2^{α_0} choices. It follows $|\mathrm{Aut}(\mathcal{B})| = 2^{\alpha_0} m\varphi(n)$.

(3) If $r_0 = 1 + 2^{\alpha_0 - 1}$, then $o(b) = 2$. So $(\sigma(b))^2 = b^2 a_0^{u_0(2+2^{\alpha_0 - 1})} = a_0^{u_0(2+2^{\alpha_0 - 1})} = 1$. If follows $u_0(2 + 2^{\alpha_0 - 1}) \equiv 0 \pmod{2^{\alpha_0}}$. Then u_0 has 2 choices: 0 or $2^{\alpha_0 - 1}$. Hence $|\mathrm{Aut}(\mathcal{B})| = 2m\varphi(n)$.

(4) If $r_0 = -1 + 2^{\alpha_0 - 1}$, then $o(b) = 2$. So $(\sigma(b))^2 = a_0^{u_0 2^{\alpha_0 - 1}} = 1$. Then $u_0 2^{\alpha_0 - 1} \equiv 0 \pmod{2^{\alpha_0}}$, and it follows u_0 has $2^{\alpha_R 0 - 1}$ choices. Hence $|\mathrm{Aut}(\mathcal{B})| = 2^{\alpha_0 - 1} m\varphi(n)$. □

The next lemma follows from Theorem 1.2 and Lemmas 4.1 and 4.2.

Lemma 4.3. *Let $\mathcal{B}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ and let $n = p_0^{\alpha_0} p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime decomposition with $p_0 = 2$. Let the b-conjugation on $\mathbb{Z}_n$ correspond to an $(s+1)$-tuple $(r_0, r_1, \dots, r_s)$, where $r_0 \in \{\pm 1, \pm 1 + 2^{\alpha_0 - 1}\}$ and all other r_i's are -1.*

(1) *If* $r_0 = 1$,

$$\operatorname{Isoc}(G;\mathcal{B}) = \begin{cases} \frac{1}{\varphi(n)}(2^\beta-1)2^{\alpha_0\beta-\alpha_0}\prod\limits_{i=1}^{s} p_i^{(\alpha_i-1)(\beta-1)}(p_i^{\beta-1}-1) & \text{if } 2^{\alpha_0+1} \mid o(b), \\ \frac{1}{\varphi(n)}(2^\beta-1)2^{\alpha_0-1}\prod\limits_{i=0}^{s} p_i^{(\alpha_i-1)(\beta-1)}(p_i^{\beta-1}-1) & \text{otherwise}. \end{cases}$$

(2) *If* $r_0 = -1$, *then*

$$\operatorname{Isoc}(G;\mathcal{B}) = \frac{1}{\varphi(n)}(2^\beta-1)\prod_{i=0}^{s} p_i^{(\alpha_i-1)(\beta-1)}(p_i^{\beta-1}-1).$$

(3) *If* $r_0 = 1 + 2^{\alpha_0-1}$, *then*

$$\operatorname{Isoc}(G;\mathcal{B}) = \frac{1}{\varphi(n)}2^{\alpha_0-1}(2^\beta-1)\prod_{i=0}^{s} p_i^{(\alpha_i-1)(\beta-1)}(p_i^{\beta-1}-1).$$

(4) *If* $r_0 = -1 + 2^{\alpha_0-1}$, *then*

$$\operatorname{Isoc}(G;\mathcal{B}) = \frac{1}{\varphi(n)}2(2^\beta-1)\prod_{i=0}^{s} p_i^{(\alpha_i-1)(\beta-1)}(p_i^{\beta-1}-1).$$

Now one can get main theorem of this section.

Theorem 4.4. *Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_n \cong \mathbb{Z}_{p_0^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ and let $n = p_0^{\alpha_0}p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ be the prime decomposition with $p_0 = 2$. Let the b-conjugation on $\mathbb{Z}_n$ correspond to an $(s+1)$-tuple $(r_0, r_1, \ldots, r_s)$, where $r_0 \in \{\pm1, \pm1 + 2^{\alpha_0-1}\}$ and $r_i = \pm1$ for $i \in \{1, \ldots, s\}$ with -1 in exactly t entries $\ell_1, \ldots, \ell_t$. Let $\mathcal{J} = \{\ell_1, \ldots, \ell_t\}$, $\mathcal{K} = \{1, \ldots, s\} - \mathcal{J}$ and*

$$\mathfrak{N} = \frac{1}{\varphi(n)}(2^\beta-1)\prod_{i\in\mathcal{J}} p_i^{(\alpha_i-1)(\beta-1)}(p_i^{\beta-1}-1)\prod_{i\in\mathcal{K}} p_i^{(\alpha_i-1)\beta}(p_i^{\beta}-1).$$

Then $\operatorname{Isoc}(G;\mathcal{A}) = \mathfrak{T}\mathfrak{N}$, *where*

$$\mathfrak{T} = \begin{cases} 2^{\alpha_0\beta-\alpha_0} & \text{if } r_0 = 1 \text{ and } 2^{\alpha_0+1} \mid o(b), \\ 2^{(\alpha_0-1)\beta}(2^{\beta-1}-1) & \text{if } r_0 = 1 \text{ and } 2^{\alpha_0+1} \nmid o(b), \\ 2^{(\alpha_0-1)(\beta-1)}(2^{\beta-1}-1) & \text{if } r_0 = -1, \\ 2^{(\alpha_0-1)\beta}(2^{\beta-1}-1) & \text{if } r_0 = 1 + 2^{\alpha_0-1}, \\ 2^{(\alpha_0-1)(\beta-1)-1}(2^{\beta-1}-1) & \text{if } r_0 = 1 + 2^{\alpha_0-1}. \end{cases}$$

Example 4.5. Let $\mathcal{A}$ be a $\mathbb{Z}_2$-extension of a cyclic group $\mathbb{Z}_{1260} \cong \mathbb{Z}_4 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5 \times \mathbb{Z}_7 = \langle a_0\rangle \times \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle$. By Theorem 2.5, the b-conjugation on $\mathbb{Z}_n$ corresponds to a 4-tuple (r_0, r_1, r_2, r_3), where $r_i = \pm1$ for $i \in \{0, 1, 2, 3\}$. Take $(r_0, r_1, r_2, r_3) = (1, -1, -1, 1)$ and $\beta = 3$ as an example. One has $\operatorname{Isoc}(G;\mathcal{A}) = \operatorname{Isoc}(G;\mathcal{B})\operatorname{Isoc}(G;\mathbb{Z}_7)$, where $\mathcal{B}$ is a $\mathbb{Z}_2$-extension of $\mathbb{Z}_{180}$. By Lemmas 4.1 and 4.2, $|\Omega(\mathcal{B};3)| = (2^3-1)|\Omega_1(\mathcal{B};3)| = 34836480$ and $|\operatorname{Aut}(\mathcal{B})| = 4320$. It follows that $\operatorname{Isoc}(G;\mathcal{B}) = 8064$. By $\operatorname{Isoc}(G;\mathbb{Z}_7) = 57$, one gets $\operatorname{Isoc}(G;\mathcal{A}) = 459648$.

5 In cases of $\mathbb{Z}_2$-extensions of an abelian group

Naturally, we are interested in extending the counting problem of the previous two sections to the case of a $\mathbb{Z}_2$-extension of an abelian group. To do this, we need to classify $\mathbb{Z}_2$-extensions of an arbitrary abelian group, but we can not give a complete answer so far, see Section 6. So we just count two special cases, generalized dihedral groups or generalized dicyclic groups.

5.1 With generalized dihedral groups

Let $\mathcal{H}$ be an abelian group. A *generalized dihedral group* $\mathrm{Dih}(\mathcal{H})$, as a $\mathbb{Z}_2$-extension of $\mathcal{H}$, is defined with relations

$$b^2 = 1, b^{-1}ab = a^{-1}, \quad \text{for all } a \in \mathcal{H}.$$

It is a semidirect product of $\mathcal{H}$ and $\mathbb{Z}_2$, with $\mathbb{Z}_2$ acting on $\mathcal{H}$ by inverting elements. When $\mathcal{H}$ is cyclic, $\mathrm{Dih}(\mathcal{H})$ is just a dihedral group.

Lemma 5.1. *$\mathcal{H}$ is a characteristic subgroup of* $\mathrm{Dih}(\mathcal{H})$.

Proof. Take an automorphism $\sigma \in \mathrm{Aut}(\mathrm{Dih}(\mathcal{H}))$. Note that the order of a quotient type element is 2. For any element a of odd order in $\mathrm{Dih}(\mathcal{H})$, $\sigma(a)$ should be normal type since σ is order-preserving. For an element a_0 of even order, suppose that $\sigma(a_0)$ is a quotient type element. Since a_0 commutes with a as an element of odd order, $\sigma(a_0)$ commutes $\sigma(a)$. Then b commutes a, which is a contradiction. Then $\sigma(a) \in \mathcal{H}$ for any $a \in \mathcal{H}$. Hence $\mathcal{H}$ is a characteristic subgroup of $\mathrm{Dih}(\mathcal{H})$. □

Now, $|\mathrm{Aut}(\mathrm{Dih}(\mathcal{H}))| = |\mathcal{H}| \cdot |\mathrm{Aut}(\mathcal{H})|$. By Lemmas 3.1 and 3.2, one can show that $|\Omega(\mathrm{Dih}(\mathcal{H}); \beta)| = (2^\beta - 1)|\Omega_{\{1\}}(\mathcal{H}; \beta - 1)| = (2^\beta - 1)|\mathcal{H}||\Omega(\mathcal{H}; \beta - 1)|$. Each abelian group can be decomposed into direct product of abelian p-group, namely, $\mathcal{H} \cong \mathcal{H}_{p_1} \times \cdots \times \mathcal{H}_{p_s}$ with p_i prime. Then $|\Omega(\mathrm{Dih}(\mathcal{H}); \beta)| = (2^\beta - 1)|\mathcal{H}||\Omega(\mathcal{H}_{p_1}; \beta - 1)| \cdots |\Omega(\mathcal{H}_{p_s}; \beta - 1)|$. It just needs to determine $\mathrm{Isoc}(G; \mathrm{Dih}(\mathcal{H}_p))$ for a prime integer p. Since $|\Omega(\mathcal{H}_p; \beta - 1)|$ is determined in [14], one gets

Theorem 5.2. *For a generalized dihedral group $\mathrm{Dih}(\mathcal{H}_p)$ and $\mathcal{H}_p = m_1\mathbb{Z}_{p^{s_1}} \times \cdots \times m_\ell\mathbb{Z}_{p^{s_\ell}}$ with $m_1, \ldots, m_\ell$ and $s_1, \ldots, s_\ell$ are positive integers satisfying $s_\ell < \cdots < s_1$, one can obtain*

$$\mathrm{Isoc}(G; \mathrm{Dih}(\mathcal{H}_p)) = (2^\beta - 1)p^{f(\beta-1, m_i, s_i)} \frac{\prod_{i=1}^{m} p^{\beta - i} - 1}{\prod_{j=1}^{\ell} \prod_{h=1}^{m_j} p^{m_j - h + 1} - 1},$$

where $m = m_1 + \cdots + m_\ell$ and

$$f(\beta - 1, m_i, s_i) = (\beta - 1 - m)\left(\sum_{i=1}^{\ell} m_i(s_i - 1)\right) + \sum_{i=1}^{\ell-1} m_i \left(\sum_{j=i+1}^{\ell} m_j(s_i - s_j - 1)\right).$$

5.2 With generalized dicyclic groups

A *generalized dicyclic group* $\mathrm{Dic}(\mathcal{H})$, as another $\mathbb{Z}_2$-extension of an abelian group $\mathcal{H}$, is defined with relations

$$b^2 = c, b^{-1}ab = a^{-1},$$

where c is an involution of $\mathcal{H}$ and a is an arbitrary element of $\mathcal{H}$. Similarly, one can have the coming lemma.

Lemma 5.3. *$\mathcal{H}$ is a characteristic group of* $\mathrm{Dic}(\mathcal{H})$. *Hence*

$$|\mathrm{Aut}(\mathrm{Dic}(\mathcal{H}))| = |\mathcal{H}| \cdot |\mathrm{Aut}(\mathcal{H})|.$$

Theorem 5.4. *For a generalized dicyclic group* $\mathrm{Dic}(\mathcal{H}_p)$ *and* $\mathcal{H}_p = m_1\mathbb{Z}_{p^{s_1}} \times \cdots \times m_\ell \mathbb{Z}_{p^{s_\ell}}$ *with $m_1, \ldots, m_\ell$ and $s_1, \ldots, s_\ell$ are positive integers satisfying $s_\ell < \cdots < s_1$, one can obtain*

$$\mathrm{Isoc}(G; \mathrm{Dih}(\mathcal{H}_p)) = 2(2^\beta - 1)p^{f(\beta-1, m_i, s_i)} \frac{\prod_{i=1}^{m} p^{\beta-i} - 1}{\prod_{j=1}^{\ell} \prod_{h=1}^{m_j} p^{m_j-h+1} - 1},$$

where $m = m_1 + \cdots + m_\ell$ and

$$f(\beta-1, m_i, s_i) = (\beta - 1 - m)\left(\sum_{i=1}^{\ell} m_i(s_i - 1)\right) + \sum_{i=1}^{\ell-1} m_i \left(\sum_{j=i+1}^{\ell} m_j(s_i - s_j - 1)\right).$$

6 Further remarks

In this paper, we enumerate the regular coverings of a graph whose covering transformation groups are $\mathbb{Z}_2$-extensions of a cyclic group. However, we could not give a complete answer of this problem if $\mathcal{A}$ is a $\mathbb{Z}_2$-extension of any abelian group $\mathcal{H}$.

However, we cannot answer the same enumeration problem when the cyclic group is replaced by an abelian group, even by an elementary abelian p-group. In fact the difficulty for authors is how to determine all involutions of $\mathrm{Aut}(\mathcal{H})$. The counting problem has studied by many researchers, for example, in [21], it gave a generating function for the number of involutions of $\mathrm{GL}(n, p)$ which is isomorphic to automorphism group of $\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. For more results, see [2], [5], [3] and so on. But it is still hard for us to determine the specific form of each involution of $\mathrm{GL}(n, p)$.

For further possible problems unsolved in this paper, we list in the following.

(1) $\mathrm{Isoc}(G; \mathcal{A})$ if $\mathcal{A}$ is a $\mathbb{Z}_2$-extension of any abelian group.

(2) $\mathrm{Isoc}(G; \mathcal{A})$ if $\mathcal{A}$ is a $\mathbb{Z}_p$-extension of any cyclic group.

(3) $\mathrm{Isoc}(G; \mathcal{A})$ if $\mathcal{A}$ is any metacyclic group.

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Enumeration of hypermaps of a given genus*

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Abstract

This paper addresses the enumeration of rooted and unrooted hypermaps of a given genus. For rooted hypermaps the enumeration method consists of considering the more general family of multirooted hypermaps, in which darts other than the root dart are distinguished. We give functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the vertices containing the distinguished darts. We solve these equations to get parametric expressions of the generating functions of rooted hypermaps of low genus. We also count unrooted hypermaps of given genus by number of darts, vertices, hyperedges and faces.

Keywords: Enumeration, surface, genus, rooted hypermap, unrooted hypermap.

Math. Subj. Class.: 05C30, 05A15

1 Introduction

A *(combinatorial) hypermap* is a triple (D, R, L) where D is a finite set of *darts* and R and L are permutations on D such that the group $\langle R, L\rangle$ generated by R and L acts transitively on D. A *(combinatorial ordinary) map* is a hypermap (D, R, L) whose permutation L is a fixed-point-free involution on D. For a hypermap (resp. map) the orbits of R, L and

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RL (L followed by R) are respectively called *vertices*, *hyperedges* (resp. *edges*) and *faces*. The *degree* of a vertex, edge, hyperedge or face is the number of darts it contains. The equivalence of combinatorial maps and topological maps having been established in [14], we use the word "map" to mean "combinatorial map" throughout this paper. The *genus* g of a map is given by the Euler-Poincaré formula [7]

$$v - e + f = 2(1 - g), \tag{1.1}$$

where v is the number of vertices, e is the number of edges and f is the number of faces. The genus of a hypermap with t darts, v vertices, e hyperedges and f faces was defined in [13] by the formula

$$v + e + f = t + 2(1 - g). \tag{1.2}$$

An *isomorphism* between two maps or hypermaps (D, R, L) and (D', R', L') is a bijection from D onto D' that takes R into R' and L into L'; it corresponds to an orientation-preserving homeomorphism between two topological maps. A *sensed* hypermap (resp. map) is an isomorphism class of hypermaps (resp. maps). We admit the existence of a unique hypermap (resp. map) with an empty set of darts D, called the *empty* hypermap (resp. map). For both of these objects $v = f = 1$ and $g = e = 0$. A *rooted* hypermap (resp. map) is either the empty hypermap (resp. map) or a tuple (D, x, R, L) where (D, R, L) is a non-empty combinatorial hypermap (resp. map) and $x \in D$ is a distinguished dart, called the *root*.

The enumeration of maps and hypermaps has several non-trivial applications. One such application is based on the correspondence between hypermaps and algebraic curves established by the Belyi theorem [16]. For instance, the formula for the number of plane trees was used by A. Zvonkin in the computer generation of Shabat polynomials of bounded degree [16]. Another area where the map enumeration plays an important role is theoretical physics, in particular in 2-dimensional gravitation models. Roughly speaking, map enumeration is used to compute matrix integrals determining the properties of gravitational fields (see for instance the works of B. Eynard [9]). Some hypermaps have been shown to be related to contextuality in quantum physics [21]. Also, A. Mednykh and R. Nedela have applied the enumeration of rooted (resp. unrooted) hypermaps to the enumeration of subgroups (resp. conjugacy classes of subgroups) of the triangle group with three generators x, y, z and the relation $xyz = 1$ [20].

We enumerate rooted hypermaps of a given genus by number of darts, vertices, hyperedges and faces. To do so we consider more general families of rooted hypermaps and bipartite maps, in which other vertices or darts than the root dart are distinguished. We also use the genus-preserving bijection between hypermaps and 2-vertex-coloured bipartite maps presented in [23]. But since bipartite maps have all their **faces** of even degree and we're using the degrees of the **vertices** as parameters, we must instead study the face-vertex dual of a 2-coloured bipartite map, that is, a map whose faces are coloured in two colours (white and black) so that no two faces that share an edge have the same colour. All these maps are *Eulerian* – that is, all their vertices are of even degree – but not all Eulerian maps are 2-face-colourable. For example, the map on the torus with one vertex, one face and two edges is Eulerian because its only vertex is of degree 4, but its face cannot be coloured because it shares both edges with itself. Therefore we call the maps we are studying *face-bipartite*.

A *sequenced (rooted) map* is a rooted map with some vertices other than the *root vertex* (the vertex that contains the root) distinguished from each other and from all the other vertices. The labels that distinguish these vertices can be taken to be $1, 2, \ldots, k$, where k is the number of distinguished vertices. A *sequenced (rooted) hypermap* is defined similarly. We state (in Section 4) a bijective decomposition for the set $\mathcal{H}(g, t, f, e, n, D)$ of sequenced orientable hypermaps of genus g with t darts, f faces and e hyperedges, with the root vertex of degree n and with the sequence of degrees of the distinguished vertices equal to $D = (d_1, d_2, \ldots, d_{|D|})$, where d_i is the degree of the distinguished vertex with label i. We obtain a bijective decomposition of the set $\mathcal{F}(g, e, w, b, n, D)$ of sequenced orientable face-bipartite maps of genus g with e edges, w white faces, b black faces, with the root face of degree $2n$ and with the sequence of half-degrees of the distinguished vertices equal to D. Then we apply face-vertex duality to obtain a bijective decomposition of the corresponding set of 2-coloured bipartite maps with distinguished faces. Next we use the bijection in [23] to obtain a bijective decomposition for hypermaps with distinguished faces, and finally we again apply face-vertex duality to obtain a bijective decomposition of $\mathcal{H}(g, t, f, e, n, D)$.

A *mutirooted hypermap* is a hypermap in which a non-empty sequence of darts with pairwise distinct initial vertices is distinguished. We relate multirooted hypermaps to sequenced hypermaps and thus obtain a recurrence for the number of multirooted hypermaps and functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the initial vertices of the distinguished darts.

The paper is organized as follows. Section 2 fixes some notations, recalls a known decomposition for sequenced rooted maps and describes the bijection between hypermaps and bipartite maps presented in [23]. Sections 3 and 4 respectively enumerate sequenced face-bipartite maps and sequenced rooted hypermaps of a given genus. In Section 5 we consider multirooted hypermaps and we give equations for the generating functions that count these objects. In Section 6 we give functional equations relating the generating functions for rooted hypermaps with that for multirooted hypermaps. Then we show how to solve these equations. In Section 7 we obtain parametric expressions for the generating functions that count rooted hypermaps with a given small positive genus. Section 8 presents enumeration algorithms for sensed unrooted hypermaps counted by number of darts, vertices and hyperedges. Appendix A (resp. B) contains a table for numbers of rooted (resp. unrooted) hypermaps of genus g with d darts, v vertices and e hyperedges for $d \leq 14$.

2 Background

2.1 Notations

We first introduce the notations and conventions we use throughout the paper. Let D and D' be two lists of integers. The inclusion $D' \subseteq D$ means that D' is a sublist of D. In this case $D - D'$ is the complementary sublist of D' in D. For instance, the sublists of $D = [1, 1, 2]$ are the empty list $[\,]$, $[1]$ (twice), $[2]$, $[1, 1]$, $[1, 2]$ (twice) and D itself. Their complementary sublists in the same order are D, $[1, 2]$ (twice), $[1, 1]$, $[2]$, $[1]$ (twice) and $[\,]$. We denote by $D.D'$ the concatenation of the lists D and D'. If i is an integer and D is a list of integers, then $i.D$ is a shortcut for $[i].D$. For $1 \leq j \leq |D|$ we denote by d_j the j-th element of the list D of length $|D|$ and by $D - \{d_j\}$ the list obtained from D by removing its j-th element d_j. Let ρ be a positive integer. The abbreviation $D_{1..\rho}$ denotes the list $[d_1, \ldots, d_\rho]$. The abbreviation $v_{1..\rho}^{D_{1..\rho}}$ denotes $v_1^{d_1} \ldots v_\rho^{d_\rho}$.

The sign $+$ (resp. $\sum$) denotes (resp. generalized) disjoint set union in the following decompositions and (resp. generalized) arithmetic sum in the following equations. By convention, a disjoint set union (resp. sum) over an empty domain is equal to the empty set (resp. zero). For any logical formula φ the notation Δ_φ means the singleton set containing only the empty hypermap or map (depending on the context) and the empty set if φ is false. The notation δ_φ means 1 if φ is true and 0 if φ is false.

2.2 Bijective decomposition of the set of sequenced maps

In 1962 W. T. Tutte [22] presented a bijective decomposition of a planar map with all the vertices distinguished and a root in every vertex. In 1972 T. R. Walsh and A. B. Lehman [27] generalized this decomposition to maps of higher genus and used it to count rooted maps of a given genus by number of vertices and faces. In 1987 D. Arquès [3] used this latter decomposition to find a closed-form formula for the number of rooted maps of genus 1 by number of vertices and faces. In 1991 E. A. Bender and E. A. Canfield [4] presented a more efficient decomposition that roots only a single vertex and distinguishes only as many other vertices as necessary and used it to obtain explicit formulas for counting rooted maps of genus 2 and 3. In 1998 the first author [11] modified this decomposition and used it to obtain a bijective decomposition satisfied by the set $\mathcal{M}(g, e, f, n, D)$ of sequenced orientable maps of genus g with e edges and f faces, with the root vertex of degree n and with D the list of degrees of the distinguished vertices was obtained in [11]. Since this bijective decomposition contains an error, we present the correct bijective decomposition here, and we derive it to make the derivation more accessible than the contents of a Ph. D. thesis.

Theorem 2.1. *The set $\mathcal{M}(g, e, f, n, D)$ of sequenced orientable maps of genus g with e edges and f faces, with the root vertex of degree n and with the list D of degrees of the distinguished vertices is defined by the bijective decomposition*

$$\begin{aligned}
\mathcal{M}(g, e, f, n, D) = \\
&\sum_{\substack{g_1+g_2=g\\ e_1+e_2=e-1\\ f_1+f_2=f\\ n_1+n_2=n-2\\ D_1\subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1) \\
&+ \sum_{p=1}^{n-3} \mathcal{M}(g-1, e-1, f, n-2-p, p.D) \times \{1, \dots, p\} \qquad (2.1) \\
&+ \sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e-1, f, p, D) \\
&+ \sum_{j=1}^{|D|} \mathcal{M}(g, e-1, f, d_j + n - 2, D - \{d_j\}) + \Delta_{(g,e,f,n,D)=(0,0,1,0,[\,])}.
\end{aligned}$$

Proof. If a map m has at least one edge, we reduce by 1 the number of edges by the face-vertex dual of deleting the root edge. There are two cases of this operation, depending upon whether the root edge is a loop or a link, and each of these cases breaks down into two sub-cases.

Case 1: The root edge is a loop. We delete the root edge and split the root vertex into two parts, s_1 and s_2. If r is the root, then s_1 consists of the darts $R(r)$, $R^2(r)$, …, $R^{-1}(L(r))$

and s_2 consists of the darts $R(L(r))$, $R^2(L(r))$, …, $R^{-1}(r)$. This case breaks down into two cases, depending upon whether or not this operation disconnects the map.

Case 1a: This operation disconnects the map into two maps, m_1 containing s_1 and m_2 containing s_2. If m_1 has at least 1 edge, its root is $r_1 = R(r)$, and if m_2 has at least 1 edge, its root is $r_2 = R(L(r))$. Let g_1, e_1, f_1, n_1, D_1 and g_2, e_2, f_2, n_2, D_2 be the parameters of the maps m_1 and m_2, respectively, corresponding to g, e, f, n, D. This operation reduces by 1 the total number of edges; so $e_1 + e_2 = e - 1$. It leaves unchanged the total number of faces because r and $L(r)$ simply get deleted from the cycle(s) of RL (L followed by R) containing them; so $f_1 + f_2 = f$. It increases by 1 the total number of vertices; so from Formula (1.1), which relates the genus of a map to the number of its vertices, faces and edges, it can easily be deduced that $g_1 + g_2 = g$. It decreases by 2 the total number of darts in s_1 and s_2 since r and $L(r)$, which belonged to the root vertex, get eliminated; so $n_1 + n_2 = n - 2$. Finally, D_1 can be any sublist of D and D_2 is just the complementary sublist, denoted by $D - D_1$. This operation is uniquely reversible; so the set of ordered pairs of sequenced maps obtained in this case is

$$\sum_{\substack{g_1+g_2=g\\ e_1+e_2=e-1\\ f_1+f_2=f\\ n_1+n_2=n-2\\ D_1\subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1), \tag{2.2}$$

where Σ means the union of disjoint sets.

Case 1b: This operation does not disconnect the map, but instead turns it into a new map m' with $e - 1$ edges and f faces and, since the number of vertices increases by 1, the genus of m' is $g - 1$, so that this case only occurs when $g \geq 1$. Neither s_1 nor s_2 can be of degree 0 (otherwise the map would be disconnected); so we can choose for m' the root $r_1 = R(r)$ belonging to s_1. Let p be the degree of s_2. Since the sum of the degrees of s_1 and s_2 is $n - 2$, the degree of s_1, the root vertex, is $n - 2 - p$. We distinguish the vertex s_2 so that this operation can be reversed, and we put its degree p at the beginning of the list D, turning it into $p.D$. Now this operation is reversible in p distinct ways, since any of the p darts of s_2 can be chosen to be $R(L(r))$ when we merge the vertices s_1 and s_2 and replace the deleted root edge. Now p can be any integer from 1 up to $n - 3$ (so that $n - 2 - p \geq 1$). For both p and $n - 2 - p$ to be at least 1, n must be at least 4. The set of sequenced maps obtained in this case is

$$\sum_{p=1}^{n-3} \mathcal{M}(g - 1, e - 1, f, n - 2 - p, p.D) \times \{1, \ldots, p\}. \tag{2.3}$$

Case 2: The root edge is a link. We contract the root edge, merging its two incident vertices s_1 containing the root r and s_2 containing $L(r)$ into a single vertex s with root $R(r)$. This operation decreases by 1 the number of edges and doesn't change the number of faces, since r and $L(r)$ simply get deleted from the cycle(s) containing them. Since the number of vertices is decreased by 1, the genus remains the same. This case breaks down into two sub-cases, depending upon whether or not s_2 is one of the distinguished vertices.

Case 2a: The vertex s_2 is not one of the distinguished vertices. Let p be the degree of the new vertex s. Then $p = n - 2 +$ the degree of s_2, and since the degree of s_2 must be at least 1, we have $p \geq n-1$. Also, the new map has $2e-2$ darts; so $p \leq 2e-2$. This operation is uniquely reversible for each value of p; so the set of maps so obtained is

$$\sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e-1, f, p, D). \tag{2.4}$$

Case 2b: The vertex s_2 is one of the distinguished vertices. It can be any one of the $|D|$ distinguished vertices. If it is the jth distinguished vertex, then its degree is d_j. Then since it gets merged with s_1 into the new root vertex, d_j gets dropped from D. Finally, the degree of s is $d_j + n - 2$. This operation too is uniquely reversible; so the set of maps so obtained is

$$\sum_{j=1}^{|D|} \mathcal{M}(g, e-1, f, d_j + n - 2, D - \{d_j\}). \tag{2.5}$$

Finally, suppose that m has no edges. It is of genus 0, has 1 face, its one vertex is of degree 0 and its list D is empty because it has no distinguished vertices; so it constitutes the singleton

$$\Delta_{(g,e,f,n,D)=(0,0,1,0,[\,])}. \tag{2.6}$$

Then $\mathcal{M}(g, e, f, n, D)$ is the disjoint union of the sets given by (2.2) – (2.6). $\square$

2.3 Bipartite maps and hypermaps

To motivate the transformation of (2.2) – (2.6) into the corresponding equations for sequenced hypermaps we briefly describe the bijection in [23] that takes a hypermap h into a 2-coloured bipartite map $m = I(h)$, its *incidence map*. The bijection I takes the darts, vertices and hyperedges of h into the edges, white vertices and black vertices of m. A root (distinguished dart) of h corresponds to a distinguished **edge** of m; to make it correspond to a root of m we impose the condition that a root of m belongs to a white vertex. The permutation R in h corresponds to R in m acting on a dart in a white vertex and the permutation L in h corresponds to R in m acting on a dart in a black vertex. The permutation L in m doesn't correspond to any permutation in h; rather, since it takes a dart belonging to a vertex of one colour into a dart belonging to a vertex of the opposite colour, it toggles R in m between R and L in h. A face (cycle of RL) in h corresponds to a face in m with twice the degree. To see this, we follow one application of RL in h starting with a dart d, which corresponds to an edge in m but we make it correspond to the dart d' in that edge that also belongs to a white vertex. Then the L in h takes d' first into $L(d')$, which belongs to a black vertex, and then into $RL(d')$ and the following R in h takes $RL(d')$ first into $LRL(d')$, which belongs to a white vertex, and then into $RLRL(d')$. Since the genus of a hypermap with t darts, v vertices, e hyperedges and f faces is defined by (1.2), m has the same genus as h.

Since the root of an incidence map of a rooted hypermap must belong to a white vertex, we impose the condition on a rooted 2-face-coloured face-bipartite map that the root belong to a white face and we transform (2.2) – (2.6) into the corresponding bijective decomposition for these maps.

3 Sequenced face-bipartite maps

Let $\mathcal{F}(g,e,w,b,n,D)$ be the set of sequenced orientable face-bipartite maps of genus g with e edges, w white faces, b black faces, with the root face of degree $2n$ and with the list of half-degrees of the distinguished vertices equal to D. For any dart d we denote by $f(d)$ the face containing d and we note that the face $f(R(d)) = f(L(d))$ must have the opposite colour from $f(d)$ because those two faces share the edge $\{d, L(d)\}$.

Theorem 3.1. *The set $\mathcal{F}(g,e,w,b,n,D)$ satisfies the bijective decomposition*

$$\begin{aligned}
&\mathcal{F}(g,e,w,b,n,D) = \\
&\quad \sum_{\substack{g_1+g_2=g\\ e_1+e_2=e-1\\ w_1+b_2=b\\ w_2+b_1=w\\ n_1+n_2=n-1\\ D_1\subseteq D}} \mathcal{F}(g_1,e_1,w_1,b_1,n_1,D_1)\times\mathcal{F}(g_2,e_2,w_2,b_2,n_2,D-D_1)\\
&\quad +\sum_{p=1}^{n-2}\mathcal{F}(g-1,e-1,b,w,n-1-p,p.D)\times\{1,\ldots,p\} \qquad (3.1)\\
&\quad +\sum_{p=n}^{p=e-1}\mathcal{F}(g,e-1,b,w,p,D)\\
&\quad +\sum_{j=1}^{|D|}\mathcal{F}(g,e-1,b,w,d_j+n-1,D-\{d_j\})+\Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[\,])}.
\end{aligned}$$

Proof. **Case 1: The root edge is a loop.** By definition, $f(r)$, where r is the root of the map m, is white, so that since $r_1 = R(r)$, $f(r_1)$ must be black. But when the loop is removed and the vertex s containing r is split, r_1 becomes a root; so $f(r_1)$ must change colour and so must all the faces of the new map m' (in case 1b) or the map m_1 containing r_1 (in case 1a). In case 1a, the other map m_2 has $r_2 = RL(r)$ as a root and $f(r_2)$ is white; so its faces stay the same colour. This implies that in case 1a $w_1 + b_2 = b$ and $w_2 + b_1 = w$, whereas in case 1b w and b switch in going from m to m'.

In case 1a, we have, as for general maps, $g_1 + g_2 = g$, $e_1 + e_2 = e - 1$ and D_1 is any subset of D, but instead of $n_1 + n_2 = n-2$ we have $n_1 + n_2 = n-1$ because the degrees satisfy the equation $2n_1 + 2n_2 = 2n - 2$. The analogue of formula (2.2) is thus

$$\sum_{\substack{g_1+g_2=g\\ e_1+e_2=e-1\\ w_1+b_2=b\\ w_2+b_1=w\\ n_1+n_2=n-1\\ D_1\subseteq D}} \mathcal{F}(g_1,e_1,w_1,b_1,n_1,D_1)\times\mathcal{F}(g_2,e_2,w_2,b_2,n_2,D-D_1). \qquad (3.2)$$

In case 1b, the reduced map m' is still of genus $g-1$ and has $e-1$ edges, but the degree of s_2 is now $2p$ instead of p and the degree of the new root vertex s_1 is $2(n-1-p)$; so the parameter $n-2-p$ in (2.3) changes to $n-1-p$. Also, $1 \le 2p \le 2n-3$, but since $2p$ is even, we have $1 \le p \le n-2$ instead of $1 \le p \le n-3$, and the condition that $n \ge 4$

changes to $n \geq 3$. The analogue of formula (2.3) is thus

$$\sum_{p=1}^{n-2} \mathcal{F}(g-1, e-1, b, w, n-1-p, p.D) \times \{1, \ldots, p\}. \tag{3.3}$$

Case 2: The root edge is a link. Since the new root $R(r)$ belongs to a black face, all the faces change colour; so b and w switch.

In case 2a, we have $2n-1 \leq 2p \leq 2e-2$, but since $2p$ is even, we now have $n \leq p \leq e-1$; so the analogue of (2.4) is

$$\sum_{p=n}^{p=e-1} \mathcal{F}(g, e-1, b, w, p, D). \tag{3.4}$$

In case 2b, the degree of the new root vertex is $2d_j + 2n - 2$; so the analogue of (2.5) is

$$\sum_{j=1}^{|D|} \mathcal{F}(g, e-1, b, w, d_j + n - 1, D - \{d_j\}). \tag{3.5}$$

Finally, the map with no edges has one white face and no black ones; so the analogue of (2.6) is

$$\Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[\,])}. \tag{3.6}$$

□

After deriving this bijective decomposition, we became aware of the article [8], which presents a similar bijective decomposition but for multi-rooted face-bipartite maps, which are like sequenced face-bipartite maps except that every distinguished vertex has a root. However, we present our derivation here for several reasons: it makes our article self-contained, we obtained it independently of [8] and our main purpose is to count hypermaps rather than face-bipartite maps. Now [8] does present a construction that converts a hypermap into a face-bipartite map. However, that construction is not proved and it is far more complicated than the one in [23], which is not cited in [8]. We also recently became aware of the article [6], which generalizes the results of [15] by computing the generating functions for edge-labelled bipartite maps on an orientable surface of genus g with an unbounded number of faces and including the degrees of these faces as parameters.

4 Sequenced rooted hypermaps

Theorem 3.1 holds for rooted 2-coloured bipartite maps with distinguished faces, where e is the number of edges, w is the number of white vertices, b is the number of black vertices, n is half the degree of the root face and D is the list of half-degrees of the distinguished faces. By the bijection described in Section 2.3, it also holds for rooted hypermaps with distinguished faces, where e is the number of darts, w is the number of vertices, b is the number of hyperedges, n is the degree of the root face and D is the list of degrees of the distinguished faces. By duality, the theorem also holds for sequenced hypermaps, where e is the number of darts, w is the number of faces, b is the number of hyperedges, n is the degree of the root vertex and D is the list of degrees of the distinguished vertices. To make the letters correspond to the objects they represent, we change $\mathcal{F}$ to $\mathcal{H}$, e to t, w to f and b to e. We thus obtain the following results.

Theorem 4.1 (Bijective decomposition for sequenced hypermaps). *Let $\mathcal{H}(g,t,f,e,n,D)$ be the set of sequenced orientable hypermaps of genus g with t darts, f faces and e hyperedges, with the root vertex of degree n and with the list of degrees of the distinguished vertices equal to $D=(d_1,d_2,\ldots,d_{|D|})$, where d_i is the degree of the distinguished vertex with label i. The set $\mathcal{H}(g,t,f,e,n,D)$ satisfies the bijective decomposition*

$$\begin{aligned}
&\mathcal{H}(g,t,f,e,n,D) = \\
&\quad \sum_{\substack{g_1+g_2=g\\ t_1+t_2=t-1\\ f_1+e_2=e\\ f_2+e_1=f\\ n_1+n_2=n-1\\ D_1\subseteq D}} \mathcal{H}(g_1,t_1,f_1,e_1,n_1,D_1)\times\mathcal{H}(g_2,t_2,f_2,e_2,n_2,D-D_1)\\
&\quad + \sum_{p=1}^{n-2}\mathcal{H}(g-1,t-1,e,f,n-1-p,p.D)\times\{1,\ldots,p\} \qquad (4.1)\\
&\quad + \sum_{p=n}^{p=t-1}\mathcal{H}(g,t-1,e,f,p,D)\\
&\quad + \sum_{j=1}^{|D|}\mathcal{H}(g,t-1,e,f,d_j+n-1,D-\{d_j\}) + \Delta_{(g,t,f,e,n,D)=(0,0,1,0,0,[\,])}.
\end{aligned}$$

Corollary 4.2 (Recurrence between numbers of sequenced hypermaps). *Let $H(g,t,f,e,n,D)$ be the number of rooted sequenced hypermaps of genus g with t darts, f faces and e hyperedges such that the root vertex is of degree n and D is the list of degrees of the distinguished vertices. Then $H(0,0,1,0,0,[\,])=1$ and if $t\geq 1$, then*

$$\begin{aligned}
&H(g,t,f,e,n,D) = \\
&\quad \sum_{\substack{g_1+g_2=g\\ t_1+t_2=t-1\\ f_1+e_2=e\\ f_2+e_1=f\\ n_1+n_2=n-1\\ D_1\subseteq D}} H(g_1,t_1,f_1,e_1,n_1,D_1)\,H(g_2,t_2,f_2,e_2,n_2,D-D_1)\\
&\quad + \delta_{n\geq 3}\delta_{g\geq 1}\sum_{p=1}^{n-2} p\,H(g-1,t-1,e,f,n-1-p,p.D) \qquad (4.2)\\
&\quad + \sum_{p=n}^{p=t-1} H(g,t-1,e,f,p,D)\\
&\quad + \sum_{j=1}^{|D|} H(g,t-1,e,f,d_j+n-1,D-\{d_j\}).
\end{aligned}$$

5 Multirooted hypermaps

For $\rho\geq 1$ a ρ-rooted hypermap is a hypermap in which a sequence of ρ darts with pairwise distinct initial vertices is distinguished. A multirooted hypermap is a ρ-rooted hypermap for some $\rho \geq 1$. This section addresses the enumeration of multirooted hypermaps.

Theorem 5.1 (Recurrence between numbers of multirooted hypermaps)**.** *Let $H_m(g,t,f,e,D)$ be the number of multirooted hypermaps of genus g with t darts, f faces and e hyperedges such that D is the list of degrees of the distinguished vertices. Then $H_m(0,0,1,0,[\,]) = 1$ and if $t \geq 1$, then*

$$
\begin{aligned}
&H_m(g,t,f,e,n.D) = \\
&\qquad \sum_{\substack{g_1+g_2=g \\ t_1+t_2=t-1 \\ f_1+e_2=e \\ f_2+e_1=f \\ n_1+n_2=n-1 \\ D_1 \subseteq D}} H_m(g_1,t_1,f_1,e_1,n_1.D_1)\, H_m(g_2,t_2,f_2,e_2,n_2.(D-D_1)) \\
&\qquad + \delta_{n\geq 3}\delta_{g\geq 1}\sum_{p=1}^{n-2} H_m(g-1,t-1,e,f,(n-1-p).p.D) \qquad\qquad (5.1)\\
&\qquad + \sum_{p=n}^{p=t-1} H_m(g,t-1,e,f,p.D) \\
&\qquad + \sum_{j=1}^{|D|} d_j\, H_m(g,t-1,e,f,(d_j+n-1).(D-\{d_j\})).
\end{aligned}
$$

Proof. A multirooted hypermap is similar to a sequenced rooted hypermap except that for each distinguished non-root vertex a dart starting from it is distinguished. If the degree of the jth distinguished vertex is d_j, then there are d_j ways of distinguishing a dart of this vertex. It follows that for each sequenced rooted hypermap, there are $\Pi_{j=1}^{|D|} d_j$ multirooted hypermaps. Let $H_m(g,t,f,e,D)$ be the number of multirooted hypermaps of genus g with t darts, f faces and e hyperedges such that such that D is the list of degrees of the initial vertex of the distinguished darts. Then

$$H_m(g,t,f,e,n.D) = H(g,t,f,e,n,D)\, \Pi_{j=1}^{|D|} d_j. \qquad (5.2)$$

Solving (5.2) for $H(g,t,f,e,n,D)$ and substituting into (4.2) proves the theorem. □

For $\rho \geq 1$ let

$$H_g(v_1,\dots,v_\rho,x,y,u,z) = \sum_{\substack{t\geq 0,\, f\geq 1,\, e\geq 0 \\ d_1\geq 1,\dots,d_\rho\geq 1 \\ v=t+2(1-g)-e-f}} H_m(g,t,f,e,D_{1..\rho}) v_{1..\rho}^{D_{1..\rho}} x^f y^e u^v z^t \qquad (5.3)$$

be the generating function that counts multirooted hypermaps of genus g with ρ distinguished darts if $g \geq 0$, and 0 otherwise. For $1 \leq i \leq \rho$, the exponent d_i of the variable v_i in this series is the degree of the initial vertex of the i-th distinguished dart. The exponent f of the variable x is the number of faces, the exponent e of the variable y is the number of hyperedges, the exponent t of the variable z is the number of darts and the exponent v of the variable u is the number of vertices (v is computable from the other parameters by Formula (1.2)).

Corollary 5.2 (Functional equations for multirooted hypermaps)**.** *For $g \geq 0$ and $\rho \geq 1$ the generating functions H_g of multirooted hypermaps of genus g are defined by the following functional equations:*

$$\begin{aligned}
H_g(v_1, W, x, y, u, z) = &\\
\frac{yv_1z}{xu} &\sum_{j=0}^{g} \sum_{X \subseteq W} H_j(v_1, X, y, x, u, z) H_{g-j}(v_1, W - X, x, y, u, z) \\
&+ \frac{v_1 z}{u} H_{g-1}(v_1, v_1, W, y, x, u, z) \\
&+ \frac{v_1 u z}{v_1 - 1} \left(H_g(v_1, W, y, x, u, z) - H_g(1, W, y, x, u, z)\right) \\
&+ v_1 u z \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(v_j \frac{H_g(v_j, W - \{v_j\}, y, x, u, z) - H_g(v_1, W - \{v_j\}, y, x, u, z)}{v_j - v_1} \right) \\
&+ xu\delta_{g=0}\delta_{\rho=1},
\end{aligned} \tag{5.4}$$

where $W = v_2, \ldots, v_\rho$.

Proof. By summation according to (5.3) of the recurrence between numbers of multirooted hypermaps from Theorem 5.1. □

By vertex-hyperedge duality, we have

$$H_g(v_1, W, y, x, u, z) = H_g(v_1, W, x, y, u, z) + \delta_{g=0}\delta_{\rho=1}(yu - xu) \tag{5.5}$$

and thus another functional equation without x, y swaps is:

$$\begin{aligned}
H_g(v_1, W, x, y, u, z) = &\\
\frac{yv_1z}{xu} &\sum_{j=0}^{g} \sum_{X \subseteq W} \Big(\big(H_j(v_1, X, x, y, u, z) + \delta_{j=0}\delta_{|X|=0}(yu - xu)\big) \\
&\qquad\qquad H_{g-j}(v_1, W - X, x, y, u, z) \Big) \\
&+ \frac{v_1 z}{u} H_{g-1}(v_1, v_1, W, x, y, u, z) \\
&+ \frac{v_1 u z}{v_1 - 1} \left(H_g(v_1, W, x, y, u, z) - H_g(1, W, x, y, u, z)\right) \\
&+ v_1 u z \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(v_j \frac{H_g(v_j, W - \{v_j\}, x, y, u, z) - H_g(v_1, W - \{v_j\}, x, y, u, z)}{v_j - v_1} \right) \\
&+ xu\delta_{g=0}\delta_{\rho=1}.
\end{aligned} \tag{5.6}$$

The former equation is given here for maximal generality. However, a consequence of the genus formula (1.2) is that three variables among the four variables x, y, u and z are sufficient. In the remainder of the paper we consider the generating functions

$$H_g(v_1, W, x, y, u) = H_g(v_1, W, x, y, u, 1)$$

with one fewer variable. They are defined by the following functional equations:

$$\begin{aligned}
&H_g(v_1, W, x, y, u) = \\
&\quad \frac{yv_1}{xu} \sum_{j=0}^{g} \sum_{X \subseteq W} \left(H_j(v_1, X, x, y, u) + \delta_{j,0}\delta_{|X|,0}(yu - xu)\right) H_{g-j}(v_1, W - X, x, y, u) \\
&\quad + \frac{v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) \qquad (5.7)\\
&\quad + \frac{v_1 u}{v_1 - 1} \left(H_g(v_1, W, x, y, u) - H_g(1, W, x, y, u)\right) \\
&\quad + v_1 u \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(v_j \frac{H_g(v_j, W - \{v_j\}, x, y, u) - H_g(v_1, W - \{v_j\}, x, y, u)}{v_j - v_1} \right) \\
&\quad + xu\delta_{g=0}\delta_{\rho=1}.
\end{aligned}$$

For $g, \rho \neq 0, 1$, after grouping in the left-hand side the terms containing $H_g(v_1, W, x, y, u)$ in (5.7), one gets

$$\begin{aligned}
&\frac{A(v_1, x, y, u)}{v_1} H_g(v_1, W, x, y, u) = \\
&\quad x(1 - v_1) \sum_{j=0}^{g} \sum_{\substack{X \subseteq W \\ (j, X) \neq (0, []) \\ (j, X) \neq (g, W)}} H_j(v_1, X, x, y, u) H_{g-j}(v_1, W - X, x, y, u) \\
&\quad + \frac{1 - v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) + uH_g(1, W, x, y, u) \\
&\quad + uT_g(v_1, W, x, y, u) \qquad (5.8)
\end{aligned}$$

with

$$A(v, x, y, u) = vu + (1 - v)(1 - yv + xv - 2vH_0(v, x, y, u)/u) \qquad (5.9)$$

and

$$\begin{aligned}
&T_g(v_1, W, x, y, u) = \\
&\quad (1 - v_1) \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \Bigg(\frac{v_j}{v_j - v_1} \Big(H_g(v_j, W - \{v_j\}, x, y, u) \\
&\qquad\qquad - H_g(v_1, W - \{v_j\}, x, y, u) \Big) \Bigg). \qquad (5.10)
\end{aligned}$$

6 Rooted hypermap generating functions

Let $h_g(v, e, f)$ be the number of rooted genus-g hypermaps with v vertices, e hyperedges and f faces. Let

$$H_g(x, y, u) = \sum_{v,e,f \geq 1} h_g(v, e, f) x^v y^e u^f \qquad (6.1)$$

be the ordinary generating function for counting rooted hypermaps on the orientable surface of genus $g \geq 0$, where the exponent of variable x is the number of vertices, the exponent of variable y is the number of hyperedges, and the exponent of variable u is the number of faces.

Rooted hypermaps being 1-rooted hypermaps,

$$H_g(x, y, u) = H_g(1, x, y, u), \tag{6.2}$$

where $H_g(v_1, \ldots, v_\rho, x, y, u)$ is the generating function counting ρ-rooted genus-g hypermaps defined in Section 5 for $\rho \geq 1$.

We first recall in Section 6.1 a known parametric expression of the generating function that counts rooted planar hypermaps. Then we explain in Section 6.2 how to solve the functional equation of the generating functions $H_g(x, y, u)$ that count rooted hypermaps with a given positive genus g.

6.1 Rooted planar hypermaps

The following proposition is a reformulation of [1, Theorem 3], with the correspondence $s = x$, $f = u$ and $a = y$ for variables, $\lambda = p$, $\mu = q$ and $\nu = r$ for parameters, and $H_0 = sf(1 + J)$ for generating functions.

Proposition 6.1 ([1]). *The ordinary generating function $H_0(x, y, u)$ that counts rooted planar hypermaps by number of vertices (exponent of x), hyperedges (exponent of y) and faces (exponent of u) is the unique solution of the following parametric system:*

$$H_0(x, y, u) = 1 + pqr(1 - p - q - r) \tag{6.3}$$

with

$$\begin{cases} x = p(1 - q - r) \\ u = q(1 - p - r) \\ y = r(1 - p - q). \end{cases} \tag{6.4}$$

Proof. The generating function $H_0(v, x, y, u)$ that counts rooted planar hypermaps (genus 0) by number of vertices (exponent of x), hyperedges (exponent of y), faces (exponent of u) and degree of the root vertex (exponent of v) satisfies the functional equation

$$\begin{aligned} H_0(v, x, y, u) \quad &= \quad \frac{yv}{xu}\left(H_0(v, x, y, u) + yu - xu\right) H_0(v, x, y, u) \\ &\quad + \frac{vu}{v-1}\left(H_0(v, x, y, u) - H_0(1, x, y, u)\right) + xu \end{aligned} \tag{6.5}$$

obtained by instantiation of (5.7) with $g = 0$, $\rho = 1$ and $v_1 = v$.

This equation can be solved by the *quadratic method* [10, page 515]. The idea is to define auxiliary functions $A(v, x, y, u)$ and $B(v, x, y, u)$ by (5.9) and

$$B(v, x, y, u) \quad = \quad A(v, x, y, u)^2 \tag{6.6}$$

and look for a function $V(x, y, u)$ such that

$$A(V(x, y, u), x, y, u) = 0, \tag{6.7}$$

implying that $B(V(x, y, u), x, y, u) = 0$ and $\partial_v B(v, x, y, u)_{|v=V(x,y,u)} = 0$.

We get from (6.5), (5.9) and (6.6) that

$$\begin{aligned} &B(v, x, y, u) = \\ &\quad 1 - 2yv - 2xv - 2v^3y - 2v^3x - 2v^2u + v^4y^2 - 2v^3y^2 + y^2v^2 + v^4x^2 \\ &\quad - 2v^3x^2 + x^2v^2 + v^2u^2 + 4v^3yx - 2yv^2x - 2yv^2u + 2v^3yu - 2v^4yx \\ &\quad - 2v^3xu + 2xv^2u + 4v^2x + 4v^2y + 2vu + 4v^3H_0(1, x, y, u) \\ &\quad - 4v^2H_0(1, x, y, u) - 2v + v^2. \end{aligned} \tag{6.8}$$

The constraints $B(V(x,y,u),x,y,u)=0$ and $\partial_v B(v,x,y,u)_{|v=V(x,y,u)}=0$ respectively are

$$\begin{aligned}1 &- 2yV - 2xV - 2V^3y - 2V^3x - 2V^2u + V^4y^2 - 2V^3y^2 + y^2V^2\\ &+ V^4x^2 - 2V^3x^2 + x^2V^2 + V^2u^2 + 4V^3yx - 2yV^2x - 2yV^2u\\ &+ 2V^3yux - 2V^4y - 2V^3xu + 2xV^2u + 4V^2x + 4V^2y + 2Vu\\ &+ 4V^3H_0(1,x,y,u) - 4V^2H_0(1,x,y,u) - 2V + V^2 = 0 \end{aligned} \tag{6.9}$$

and

$$\begin{aligned}&-2 + 8yV + 8xV + 4V^3y^2 - 6y^2V^2 + 4V^3x^2 - 6x^2V^2 - 6V^2x\\ &- 6V^2y - 4Vu + 2y^2V + 2x^2V + 2Vu^2 - 4yVu + 4xVu - 4yVx\\ &+ 12yV^2x + 6yV^2u - 8V^3yx - 6xV^2u + 2V - 2x - 2y + 2u = 0. \end{aligned} \tag{6.10}$$

It can be checked that both equations are satisfied by

$$V = 1/(1-q) \tag{6.11}$$

with x, u, y and $H_0(1,x,y,u)$ related to p, q and r by (6.4) and (6.3). □

6.2 Rooted hypermaps with positive genus

The following additional notations are used in this section. Let ρ be a positive integer. Let $H_j[n_1,\ldots,n_\rho]$ denote the partial derivative of the function $H_j(v_1,\ldots,v_\rho,x,y,u)$ with respect to the variables $v_1,\ldots,v_\rho$ to the respective orders $n_1,\ldots,n_\rho$, computed at $v_1=\ldots=v_\rho=V$. The abbreviation $[\rho]$ denotes the list $[2,\ldots,\rho]$ if $\rho\geq 2$ and the empty list $[\,]$ if $\rho=1$. The abbreviation $N_{[\rho]}$ denotes the list $[n_2,\ldots,n_\rho]$. For any sublist $X\subseteq[\rho]$ of $[\rho]$, $[\rho]-X$ denotes the sublist of the elements of $[\rho]$ that are not in X, N_X denotes the list of those n_i in $N_{[\rho]}$ such that i is in X and N_j denotes the list $[n_2,\ldots,n_{j-1},n_{j+1},\ldots,n_\rho]$.

6.2.1 Equation for rooted hypermaps and recurrence relations

The special case of Formula (5.8) for $g\geq 1$, $\rho=1$ and $v_1=V$ is the following formula:

$$uH_g(1,x,y,u) = (V-1)\left(x\sum_{j=1}^{g-1}H_j(V,x,y,u)H_{g-j}(V,x,y,u) + H_{g-1}(V,V,x,y,u)/u\right)$$

i.e.

$$uH_g(1,x,y,u) = (V-1)\left(x\sum_{j=1}^{g-1}H_j[0]H_{g-j}[0] + H_{g-1}[0,0]/u\right). \tag{6.12}$$

In order to derive from (6.12) a value for $H_g(1,x,y,u)$, we are looking for a value for $H_j[0]$, $H_{g-j}[0]$ and $H_{g-1}[0,0]$. More generally, we will derive from the following proposition a closed form for the expressions $H_g[n_1,\ldots,n_\rho]$.

Proposition 6.2. *For $g \geq 0$, $\rho \geq 1$ and $n_1, \ldots, n_\rho \geq 0$ the function $H_g[n_1, \ldots, n_\rho]$ is defined by*

$$\begin{aligned}
&\frac{(n_1+1)A[1]}{V} H_g[n_1, N_{[\rho]}] = \\
&\sum_{\substack{i+j+k=n_1+1 \\ i>0,\, k<n_1}} \binom{n_1+1}{i,j} \frac{(-1)^{j+1}\, j!}{V^{j+1}} A[i] H_g[k, N_{[\rho]}] \\
&+ x \sum_{\substack{k+l+m=n_1+1 \\ 0\leq j\leq g \\ X\subseteq[\rho] \\ (j,X)\neq(0,[\,]) \\ (j,X)\neq(g,[\rho])}} \binom{n_1+1}{k,l} M[m] H_j[k, N_X] H_{g-j}[l, N_{[\rho]-X}] \\
&+ \frac{1}{u} \sum_{i+j+k=n_1+1} \binom{n_1+1}{i,j} M[k] H_{g-1}[i, j, N_{[\rho]}] \qquad (6.13)\\
&+ u \sum_{j=2}^{\rho} \frac{(n_1+1)!n_j!}{(n_1+n_j+2)!} \Big(n_j F_g[n_1+n_j+2, N_j] \\
&\qquad\qquad + \tfrac{V(n_j+1)}{n_1+n_j+3} F_g[n_1+n_j+3, N_j] \Big),
\end{aligned}$$

where

$$F_g(v_1, \ldots, v_h, x, y, u) = L(v_1) H_g(v_1, \ldots, v_h, x, y, u) \qquad (6.14)$$

for $h \geq 1$, $M(v) = 1 - v$ and $L(v) = v(1-v)$.

Proof. Equation (6.13) is obtained from Equation (5.8) as follows:

1. Partial derivation of (5.8) with respect to the variables $v_1, v_2, \ldots, v_\rho$ to the respective orders $n_1+1, n_2, \ldots, n_\rho$.
2. Evaluation of this differential equation at $v_1 = \cdots = v_\rho = V$. The function $H_g[n_1+1, \ldots, n_\rho]$ is multiplied by $A[0]$ in the resulting equation, and $A[0]$ is known to be zero (6.7). The functions $T_g[\ldots]$ are replaced by expressions with the functions $F_g[\ldots]$ thanks to Lemma 6.3 below.
3. In the left-hand side of the resulting equation, isolation of the single term involving the function $H_g[n_1, \ldots, n_\rho]$.

By inspection one can check that the right-hand side of (6.13) depends only on some functions $H_g[k, n_2, \ldots, n_\rho]$ with $k < n_1$, some functions $H_g[n'_1, \ldots, n'_{\rho'}]$ with $\rho' < \rho$ and some functions $H_j[\ldots]$ for $j < g$. Thus, (6.13) in a recursive definition of the family of functions $H_g[n_1, \ldots, n_\rho]$ for $g \geq 0$, $\rho \geq 1$ and $n_1, \ldots, n_\rho \geq 0$. ☐

The following lemma relates the partial derivatives of T_g at $v = V$ with the ones of F_g.

Lemma 6.3. *For $\rho \geq 2$ and $g, n_1, \ldots, n_\rho \geq 0$,*

$$\begin{aligned}
&T_g[n_1+1, N_{[\rho]}] = \\
&\sum_{j=2}^{j=\rho} \frac{(n_1+1)!n_j!}{(n_1+n_j+2)!} \Big(n_j F_g[n_1+n_j+2, N_j] \\
&\qquad\qquad + \tfrac{V(n_j+1)}{n_1+n_j+3} F_g[n_1+n_j+3, N_j] \Big). \qquad (6.15)
\end{aligned}$$

Proof. We can easily prove that

$$\frac{\partial}{\partial v_j}\left[\frac{(v_j - v_1)H_g(v_1,[\rho]-\{v_j\},x,y,u)}{v_j - v_1}\right] = 0. \tag{6.16}$$

Then, $T_g(v_1,\dots,v_\rho,x,y,u)$ equals

$$\begin{aligned}\sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j}\Bigg((v_j - v_1)^{-1}\Big(&v_j(1-v_1)H_g(v_j,[\rho]-\{v_j\},x,y,u)\\ &- v_1(1-v_1)H_g(v_1,[\rho]-\{v_j\},x,y,u)\Big)\Bigg).\end{aligned} \tag{6.17}$$

It also holds that

$$\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}}\left[\frac{v_j(v_j - v_1)H_g(v_j,[\rho]-\{v_j\},x,y,u)}{v_j - v_1}\right] \quad = \quad 0, \tag{6.18}$$

so that $\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}}T_g(v_1,\dots,v_\rho,x,y,u)$ equals

$$\begin{aligned}\sum_{j=2}^{j=r} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1}\partial v_j}\Bigg((v_j - v_1)^{-1}\Big(&v_j(1-v_j)H_g(v_j,[\rho]-\{v_j\},x,y,u)\\ &- v_1(1-v_1)H_g(v_1,[\rho]-\{v_j\},x,y,u)\Big)\Bigg)\end{aligned} \tag{6.19}$$

i.e.

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1}\partial v_j}\left(\frac{F_g(v_j,[\rho]-\{v_j\},x,y,u) - F_g(v_1,[\rho]-\{v_j\},x,y,u)}{v_j - v_1}\right). \tag{6.20}$$

Formula (6.15) is a consequence of

$$\frac{\partial^{n_1+n_2}}{\partial x_1^{n_1}\partial x_2^{n_2}}\left(\frac{\psi(x_1)-\psi(x_2)}{x_1 - x_2}\right)_{x_1=x_2=a} = \frac{n_1!n_2!}{(n_1+n_2+1)!}\psi^{(n_1+n_2+1)}(a). \tag{6.21}$$

□

The formula

$$F_g[n,N] = \sum_{k+l=n}\binom{n}{k}L[k]H_g[l,N] \tag{6.22}$$

is an easy consequence of (6.14). Thus the right-hand side of (6.13) only depends on some functions $H_g[k,\dots,n_\rho]$ with $k < n_1$, some functions $H_g[n'_1,\dots,n'_{\rho'}]$ with $\rho' < \rho$, some functions $H_j[\dots]$ for $j < g$ and some functions $A[i]$. A relation between $A[i]$ and some functions $H_0[j]$ is established in Section 6.2.2.

6.2.2 Case $g = 0$ and $\rho = 1$

The function $A[i]$ can be related to some functions $H_0[j]$ as follows: With $M(v) = 1 - v$ and $L(v) = v(1-v)$, Equation (5.9) is

$$A(v,x,y,u) = vu + M(v) + L(v)(-y + x - 2xH_0(v,x,y,u)). \tag{6.23}$$

Its instantiation at $v = V$ gives

$$H_0[0] = \frac{1-q}{1-q-r}. \tag{6.24}$$

For $k \geq 1$, the k-th partial derivative of (6.23) in v is

$$\begin{aligned} \frac{\partial^k}{\partial v^k} A(v,x,y,u) &= \frac{\partial^k}{\partial v^k}(vu) + \frac{\partial^k}{\partial v^k} M(v) \\ &\quad + \frac{\partial^k}{\partial v^k}\left[L(v)(-y+x-2xH_0(v,x,y,u))\right] \end{aligned} \tag{6.25}$$

and its instantiation in $v = V$ is

$$\begin{aligned} A[k] &= \frac{\partial^k}{\partial v^k}(vu)_{|v=V} + M[k] \\ &\quad + \sum_{i+j=k} \binom{k}{i} L[i] \left(\frac{\partial^j}{\partial v^j}(-y+x-2xH_0(v,x,y,u))_{|v=V} \right). \end{aligned} \tag{6.26}$$

Solving (6.26) for $k = 1$ gives

$$H_0[1] = \frac{(1-q)^2(A[1]+1-p-q-r)}{2pq(1-q-r)}. \tag{6.27}$$

For $k \geq 2$, one gets

$$A[k] = -2x \sum_{i+j=k} \binom{k}{i} L[i] H_0[j]$$

since $M[k] = 0$, i.e.

$$A[k] = -2x \left(L[0]H_0[k] + kL[1]H_0[k-1] + \frac{k(k-1)}{2} L[2]H_0[k-2] \right) \tag{6.28}$$

since $L[k] = 0$ if $k \geq 3$.

7 Explicit formulas for small genera

This section proposes explicit parametric expressions for the generating functions that count rooted hypermaps of small positive genus. In Section 7.1 we count by number of vertices, hyperedges and faces; the number of darts can be obtained from these parameters by Formula (1.2). In Section 7.2 we count by number of darts alone.

7.1 Rooted hypermap series enumerated with three parameters

For $g = 1, \ldots, 5$ we have computed an explicit expression of $H_g(x, y, u)$ parameterized by p, q and r, with $x = p(1-q-r)$, $u = q(1-p-r)$ and $y = r(1-p-q)$, by application of formulas in Section 6. For $g \geq 3$, the expressions are too large to be included in the present text, but a Maple file with all the generating functions up to genus 5 is available from the first author on request.

A parametric expression of $H_1(x,y,u)$ is

$$H_1(x,y,u) = \frac{p\,q\,r\,(1-p)\,(1-q)\,(1-r)}{\left[(1-p-q-r)^2-4pqr\right]^2}. \tag{7.1}$$

This expression can be derived from [2, Theorem 3], with the correspondence $s=x$, $f=u$, and $a=y$ between variables and the correspondence $H_1(x,y,u) = xuK_1(1,x,y,u)$ between generating functions.

A parametric expression of $H_2(x,y,u)$ is

$$H_2(x,y,u) = \frac{p\,q\,r\,(1-p)\,(1-q)\,(1-r)\,P_2(p,q,r)}{\left[(1-p-q-r)^2-4pqr\right]^7} \tag{7.2}$$

where

$$\begin{aligned}
P_2(p,q,r) = {} & 76p^6q^2r^2 - 8p^4q^4r^2 - 8p^4q^2r^4 + 76p^2q^6r^2 - 8p^2q^4r^4 + 76p^2q^2r^6 \\
& + 40p^7qr - 76p^6q^2r - 76p^6qr^2 - 112p^5q^3r - 228p^5q^2r^2 - 112p^5qr^3 \\
& + 8p^4q^4r + 16p^4q^3r^2 + 16p^4q^2r^3 + 8p^4qr^4 - 112p^3q^5r + 16p^3q^4r^2 \\
& + 40p^3q^3r^3 + 16p^3q^2r^4 - 112p^3qr^5 - 76p^2q^6r - 228p^2q^5r^2 \\
& + 16p^2q^4r^3 + 16p^2q^3r^4 - 228p^2q^2r^5 - 76p^2qr^6 + 40pq^7r - 76pq^6r^2 \\
& - 112pq^5r^3 + 8pq^4r^4 - 112pq^3r^5 - 76pq^2r^6 + 40pqr^7 + p^8 - 20p^7q \\
& - 20p^7r - 35p^6q^2 - 64p^6qr - 35p^6r^2 + 56p^5q^3 + 396p^5q^2r + 396p^5qr^2 \\
& + 56p^5r^3 + 140p^4q^4 + 264p^4q^3r + 393p^4q^2r^2 + 264p^4qr^3 + 140p^4r^4 \\
& + 56p^3q^5 + 264p^3q^4r - 92p^3q^3r^2 - 92p^3q^2r^3 + 264p^3qr^4 + 56p^3r^5 \\
& - 35p^2q^6 + 396p^2q^5r + 393p^2q^4r^2 - 92p^2q^3r^3 + 393p^2q^2r^4 + 396p^2qr^5 \\
& - 35p^2r^6 - 20pq^7 - 64pq^6r + 396pq^5r^2 + 264pq^4r^3 + 264pq^3r^4 \\
& + 396pq^2r^5 - 64pqr^6 - 20pr^7 + q^8 - 20q^7r - 35q^6r^2 + 56q^5r^3 \\
& + 140q^4r^4 + 56q^3r^5 - 35q^2r^6 - 20qr^7 + r^8 + 6p^7 + 105p^6q + 105p^6r \\
& + 21p^5q^2 - 116p^5qr + 21p^5r^2 - 420p^4q^3 - 821p^4q^2r - 821p^4qr^2 \\
& - 420p^4r^3 - 420p^3q^4 - 648p^3q^3r - 316p^3q^2r^2 - 648p^3qr^3 - 420p^3r^4 \\
& + 21p^2q^5 - 821p^2q^4r - 316p^2q^3r^2 - 316p^2q^2r^3 - 821p^2qr^4 + 21p^2r^5 \\
& + 105pq^6 - 116pq^5r - 821pq^4r^2 - 648pq^3r^3 - 821pq^2r^4 - 116pqr^5 \\
& + 105pr^6 + 6q^7 + 105q^6r + 21q^5r^2 - 420q^4r^3 - 420q^3r^4 + 21q^2r^5 \\
& + 105qr^6 + 6r^7 - 49p^6 - 189p^5q - 189p^5r + 315p^4q^2 + 479p^4qr \\
& + 315p^4r^2 + 910p^3q^3 + 1162p^3q^2r + 1162p^3qr^2 + 910p^3r^3 + 315p^2q^4 \\
& + 1162p^2q^3r + 720p^2q^2r^2 + 1162p^2qr^3 + 315p^2r^4 - 189pq^5 + 479pq^4r \\
& + 1162pq^3r^2 + 1162pq^2r^3 + 479pqr^4 - 189pr^5 - 49q^6 - 189q^5r \\
& + 315q^4r^2 + 910q^3r^3 + 315q^2r^4 - 189qr^5 - 49r^6 + 112p^5 + 70p^4q \\
& + 70p^4r - 770p^3q^2 - 876p^3qr - 770p^3r^2 - 770p^2q^3 - 1380p^2q^2r \\
& - 1380p^2qr^2 - 770p^2r^3 + 70pq^4 - 876pq^3r - 1380pq^2r^2 - 876pqr^3 \\
& + 70pr^4 + 112q^5 + 70q^4r - 770q^3r^2 - 770q^2r^3 + 70qr^4 + 112r^5
\end{aligned}$$

$$
\begin{aligned}
&- 105p^4 + 210p^3q + 210p^3r + 735p^2q^2 + 1034p^2qr + 735p^2r^2 + 210pq^3 \\
&+ 1034pq^2r + 1034pqr^2 + 210pr^3 - 105q^4 + 210q^3r + 735q^2r^2 + 210qr^3 \\
&- 105r^4 + 14p^3 - 315p^2q - 315p^2r - 315pq^2 - 672pqr - 315pr^2 + 14q^3 \\
&- 315q^2r - 315qr^2 + 14r^3 + 49p^2 + 175pq + 175pr + 49q^2 + 175qr \\
&+ 49r^2 - 36p - 36q - 36r + 8.
\end{aligned}
$$

Remark: For $g = 0$, the formula

$$H_0(x, y, u) = pqr(1 - p - q - r) \tag{7.3}$$

can be derived from [1], with the correspondence $s = x$, $f = u$, and $a = y$ between variables and the correspondence $H_0(x, y, u) = xuK_0(1, x, y, u)$ between generating functions.

7.2 Rooted hypermap series enumerated by number of darts

Let $H_g(z)$ be the ordinary generating function of rooted hypermaps on the orientable surface of genus $g \geq 0$, where the exponent of variable z is the number d of darts.

7.2.1 Generating functions

For g from 0 to 6, a parametric expression of $H_g(z)$, where $z = \tau(1-2\tau)$ and $\tau = 0$ when $z = 0$, is

$$
\begin{aligned}
H_0(z) &= \frac{\tau^3\,(1-3\,\tau)}{z^2}, && (7.4)\\
H_1(z) &= \frac{\tau^3}{(1-\tau)\,(1-4\,\tau)^2}, && (7.5)\\
H_2(z) &= \frac{4\,z^2\,\tau^3\,(51\,\tau^4 - 77\,\tau^3 + 48\,\tau^2 - 15\,\tau + 2)}{(1-\tau)^5\,(1-4\,\tau)^7}, && (7.6)\\
H_3(z) &= \frac{4\,z^4\,\tau^3\,P_3(z)}{(1-\tau)^9\,(1-4\,\tau)^{12}}, && (7.7)\\
H_4(z) &= \frac{4\,z^6\,\tau^3\,P_4(z)}{(1-\tau)^{13}\,(1-4\,\tau)^{17}}, && (7.8)\\
H_5(z) &= \frac{4\,z^8\,\tau^3\,P_5(z)}{(1-\tau)^{17}\,(1-4\,\tau)^{22}}, && (7.9)\\
H_6(z) &= \frac{4\,z^{10}\,\tau^3 P_6(z)}{(1-\tau)^{21}\,(1-4\,\tau)^{27}}, && (7.10)
\end{aligned}
$$

with

$$
\begin{aligned}
P_3(z) = {}& 28496\,\tau^9 - 36888\,\tau^8 - 13164\,\tau^7 + 61676\,\tau^6 - 61872\,\tau^5 + 35172\,\tau^4 \\
& - 13168\,\tau^3 + 3360\,\tau^2 - 552\,\tau + 45,
\end{aligned}
$$

$$\begin{aligned}
P_4(z) = {} & 32375616\,\tau^{14} + 15509760\,\tau^{13} - 243313744\,\tau^{12} + 442844592\,\tau^{11} \\
& - 389268768\,\tau^{10} + 170357328\,\tau^{9} + 1281984\,\tau^{8} - 53553072\,\tau^{7} \\
& + 39814032\,\tau^{6} - 17597520\,\tau^{5} + 5541192\,\tau^{4} - 1320920\,\tau^{3} + 239697\,\tau^{2} \\
& - 30456\,\tau + 2016, \\
P_5(z) = {} & 61742404608\,\tau^{19} + 239043447552\,\tau^{18} - 1163002515456\,\tau^{17} \\
& + 1403096348736\,\tau^{16} + 338393916800\,\tau^{15} - 2962590413376\,\tau^{14} \\
& + 4243997599488\,\tau^{13} - 3552865706240\,\tau^{12} + 2000782619136\,\tau^{11} \\
& - 761565230016\,\tau^{10} + 165542511744\,\tau^{9} + 7568059872\,\tau^{8} \\
& - 23295865824\,\tau^{7} + 11016156244\,\tau^{6} - 3336459144\,\tau^{5} + 761835465\,\tau^{4} \\
& - 141393220\,\tau^{3} + 21738240\,\tau^{2} - 2490480\,\tau + 151200
\end{aligned}$$

and

$$\begin{aligned}
P_6(z) = {} & 178054771302400\,\tau^{24} + 1584534210564096\,\tau^{23} - 4933663711730688\,\tau^{22} \\
& - 2073822560019456\,\tau^{21} + 28025505345377280\,\tau^{20} \\
& - 55010184951564288\,\tau^{19} + 54283457920223232\,\tau^{18} \\
& - 22997164994372352\,\tau^{17} - 13439214645718272\,\tau^{16} \\
& + 31734000656779264\,\tau^{15} - 29719458122609664\,\tau^{14} \\
& + 18704646148809216\,\tau^{13} - 8736443315384448\,\tau^{12} \\
& + 3098312828500416\,\tau^{11} - 813298324826016\,\tau^{10} + 138473163256176\,\tau^{9} \\
& - 4043551301232\,\tau^{8} - 6580517850696\,\tau^{7} + 2630924485729\,\tau^{6} \\
& - 626336383104\,\tau^{5} + 112079088144\,\tau^{4} - 17314508592\,\tau^{3} + 2485496880\,\tau^{2} \\
& - 284717376\,\tau + 17107200.
\end{aligned}$$

We have also computed the generating functions for $7 \leq g \leq 11$. Their expressions are too large to be included in the present text, but a Maple file is available from the first author on request.

A. Mednykh and R. Nedela used our formulas (7.4) to (7.7) to find explicit formulas for the number of rooted hypermaps for genus $g = 0, 1, 2$ and 3 [19].

7.3 Other parameterization

In a private communication to the second author, P. Zograf suggests the parameterization

$$z = \frac{t}{(1+2t)^2}. \tag{7.11}$$

After adding the condition that $t = 0$ when $z = 0$, it corresponds to

$$t = \frac{1 - 4z - \sqrt{1-8z}}{8z}. \tag{7.12}$$

These two parameterizations are equivalent. The one can be transformed into the other by means of the following substitutions:

$$\tau = \frac{t}{1+2t} \tag{7.13}$$

and

$$t = \frac{\tau}{1 - 2\tau}. \tag{7.14}$$

By means of these substitutions, the following parametric expressions in t can be obtained from the parametric expressions (7.4) – (7.10) for $H_g(t)$ in τ:

$$\begin{aligned}
H_0(z) &= t\,(1-t),\\
H_1(z) &= \frac{t^3}{(1+t)(1-2t)^2},\\
H_2(z) &= \frac{4\,t^5\,(1+2t)\,(t^4 - t^3 + 6\,t^2 + t + 2)}{(1+t)^5(1-2t)^7},\\
H_3(z) &= 4\,t^7\,(1+2t)\,(1+t)^{-9}(1-2t)^{-12}\,(80\,t^9 - 120\,t^8 + 1500\,t^7 + 1036\,t^6\\
&\quad + 3768\,t^5 + 2820\,t^4 + 2288\,t^3 + 1008\,t^2 + 258\,t + 45),\\
H_4(z) &= 4\,t^9\,(1+2t)\,(1+t)^{-13}(1-2t)^{-17}\,(16768\,t^{14} - 33536\,t^{13}\\
&\quad + 653776\,t^{12} + 786480\,t^{11} + 4358016\,t^{10} + 6151056\,t^9 + 10059552\,t^8\\
&\quad + 10217040\,t^7 + 8418240\,t^6 + 5227024\,t^5 + 2365888\,t^4 + 800128\,t^3\\
&\quad + 181665\,t^2 + 25992\,t + 2016),\\
H_5(z) &= 4\,t^{11}\,(1+2t)\,(1+t)^{-17}(1-2t)^{-22}\,(6732800\,t^{19} - 16832000\,t^{18}\\
&\quad + 450011520\,t^{17} + 773106240\,t^{16} + 5764983552\,t^{15} + 11910647232\,t^{14}\\
&\quad + 29130502912\,t^{13} + 46090300928\,t^{12} + 63452543616\,t^{11}\\
&\quad + 68713116608\,t^{10} + 60654218080\,t^9 + 43591208976\,t^8\\
&\quad + 25142796864\,t^7 + 11637842232\,t^6 + 4232899206\,t^5 + 1181820745\,t^4\\
&\quad + 245635580\,t^3 + 35501760\,t^2 + 3255120\,t + 151200),\\
H_6(z) &= 4\,t^{13}\,(1+2t)\,(1+t)^{-21}(1-2t)^{-27}\,(4424052736\,t^{24} - 13272158208\,t^{23}\\
&\quad + 452750478336\,t^{22} + 1012254206976\,t^{21} + 9488911137792\,t^{20}\\
&\quad + 25803592571904\,t^{19} + 83891900050944\,t^{18} + 180120643165440\,t^{17}\\
&\quad + 346626234587904\,t^{16} + 535272874975232\,t^{15} + 701152993531392\,t^{14}\\
&\quad + 771688966862592\,t^{13} + 716686355273472\,t^{12} + 563018634260736\,t^{11}\\
&\quad + 372549313187520\,t^{10} + 207088794784752\,t^9 + 96021082581732\,t^8\\
&\quad + 36765061031004\,t^7 + 11475757049569\,t^6 + 2863185376896\,t^5\\
&\quad + 556090776432\,t^4 + 80913152016\,t^3 + 8274846384\,t^2 + 536428224\,t\\
&\quad + 17107200).
\end{aligned}$$

For $0 \le g \le 3$, these expressions correspond to $F_g(t)$ in Zograf's communication. Moreover, they reveal an extra factorization by $4(1+2t)$ for $g \ge 2$.

8 Efficient enumeration of rooted and sensed unrooted hypermaps by number of darts, vertices and hyperedges

We recall that a sensed map or hypermap is an equivalence class of (unrooted) maps or hypermaps under orientation-preserving isomorphism.

Before enumerating sensed hypermaps we first need to enumerate rooted hypermaps. We use an efficient method of counting rooted hypermaps by number of darts, faces, vertices and hyperedges or, equivalently [23], 2-coloured bipartite maps rooted at a white vertex by number of edges, faces, white vertices and black vertices, presented by Kazarian and Zograf [15], and then count sensed 2-coloured bipartite maps and hypermaps with the same parameters using the same method we used [26, 12] to count sensed maps by number of edges, faces and vertices. The recurrence (formula (11) in [15]), with f changed to H, is as follows. Define $H_{g,d}$ to be a homogeneous polynomial in the three variables t, u, and v. The coefficient of $t^f u^b v^w$ in $H_{g,d}$ is the number of 2-coloured bipartite maps of genus g with d edges, f faces, b black vertices and w white vertices rooted at a white vertex or, equivalently, the number of rooted hypermaps of genus g with d darts, f faces, b hyperedges and w vertices. Then $H_{0,1} = tuv$ and

$$\begin{aligned}(d+1)H_{g,d} = \\ &(2d-1)(t+u+v)H_{g,d-1} \\ &+ (d-2)\left(2(tu+tv+uv) - (t^2+u^2+v^2)\right) H_{g,d-2} \\ &+ (d-1)^2(d-2)H_{g-1,d-2} + \sum_{i=0}^{g}\sum_{j=1}^{d-3}(4+6j)(d-2-j)H_{i,j}H_{g-i,d-2-j}.\end{aligned} \tag{8.1}$$

In [26] we collaborated with Mednykh to enumerate rooted and sensed maps. Mednykh enumerated maps of genus up to 11 by number of edges alone, while we enumerated maps of genus up to 10 by number of edges and vertices. The method we used to enumerate rooted maps is presented in [25]. The method we used to enumerate sensed maps is based on Liskovets' refinement [17] of the method Mednykh and Nedela used to enumerate sensed map of genus up to 3 by number of edges [18]. Later we used a more efficient method of enumerating rooted maps, presented in [5], to enumerate rooted and sensed maps of genus up to 50 [12].

To describe here the modifications we made to pass from maps to 2-coloured bipartite maps we need to briefly discuss a few of the concepts described in more detail in [26]. All the automorphisms of a map on an orientable surface are periodic. If the period is $L > 1$, then the automorphism divides the map into L isomorphic copies of a smaller map, called the *quotient map*. Most of the *cells* (vertices, edges and faces) are in orbits of length L under the automorphism; those that aren't are called *branch points*. For example, if a map is drawn on the surface of a sphere which undergoes a rotation through $360/L$ degrees, the two cells through which the axis of rotation pass are fixed; so they are each in an orbit of length 1 for any L. For maps of higher genus, not all the branch points are on orbits of length 1. For example, if a torus is represented as a square with opposite edges identified in pairs, and is rotated by 90 degrees (period 4), then the centre of the square is a branch point of orbit length 1 and so is the point represented by all four corners of the square, but the middle of the sides of the square are two branch points of orbit length 2: the point represented by the middle of both vertical sides of the square is taken by the rotation onto the point represented by the middle of both horizontal sides, and vice versa; so it takes two rotations to take either of these points back onto itself. Also, if the middle of an edge is a branch point, then the quotient map contains half of that edge – a *dangling semi-edge*.

An automorphism of a map M of genus G is characterized by the following parameters: the period L, the genus g of its quotient map and the number of branch points of each orbit length. If each orbit length is replaced by its *branch index* (L divided by the orbit length), we obtain what is called an *orbifold signature* in [18]. In [18] a method is presented for determining which orbifold signatures could characterize an automorphism of a map of genus G (a *G-admissible orbifold*) and how many such automorphisms could be characterized by that orbifold signature; a variant of that method is presented in [17], and this is the one we use except that we deal with orbit lengths instead of branch indices. The method used in [18] to enumerate sensed maps of genus G with E edges by number of edges can be roughly described as follows. For each G-admissible orbifold O, let g be the genus of the quotient map, L be the period and q_i be the number of branch points with branch index i. Then the number $\nu_O(d)$ of rooted maps with d darts that could serve as a quotient map for an automorphism with that signature once the branch points are pasted onto the map in all possible ways is given by

$$\nu_O(d) = \sum_{s=0}^{q_2} \binom{d}{s} \binom{(d-s)/2+2-2g}{q_2-s, q_3, \ldots, q_L} N_g((d-s)/2), \tag{8.2}$$

where $N_g(n)$ is the number of rooted maps of genus g with n edges (0 if n is not an integer). Here s is the number of dangling semi-edges in the quotient map m, all of which must be in orbits of length $L/2$ so that they represent normal edges in the original map M. The binomial coefficient is the number of ways of inserting dangling semi-edges into the rooted map multiplied by $d/(d-s)$ because there are d ways to root the map once the dangling edges have been inserted and only $d-s$ ways to root it without the dangling edges. The multinomial coefficient is the number of ways to distribute the branch points with the various branch indices among the non-edges of the quotient map; the number at the top of the multinomial coefficient is the number of non-edges and is given by the Euler-Poincaré formula (1.1). Then the number of sensed maps of genus G with E edges is

$$\frac{1}{2E} \sum_{L|E} \sum_{O} \mathrm{Epi}_0(\pi_1(O), Z_L)\, \nu_O(2E/L), \tag{8.3}$$

where O runs over all the G-admissible orbifolds with period L and $\mathrm{Epi}_0(\pi_1(O), Z_L)$ is the number of automorphisms that have the orbifold signature of O.

In [26] we distributed the branch points that aren't on dangling semi-edges among the vertices and faces separately. The quotient map of a bipartite map can't contain any dangling semi-edges; otherwise the lifted map would have an edge joining two vertices of the same colour. Here we distribute the branch points among the white vertices, black vertices and faces, and, like in [26], we don't use a formula like (8.3); instead we compute the contribution of each orbifold signature to the number of sensed 2-coloured bipartite maps whose number of white vertices, black vertices, faces and edges are allowed to vary within a user-defined upper bound on the number of edges.

Suppose that the quotient map is of genus g and has w white vertices, b black vertices and f faces. Then the number e of edges can be calculated from the formula

$$f - e + w + b = 2(1-g) \tag{8.4}$$

and the number d of darts is $2e$. Suppose also that among the branch points of orbit length i, w_i are on a white vertex, b_i are on a black vertex and f_i are in a face. We denote by w_L, b_L and f_L the number of white vertices, black vertices and faces, respectively, that do not contain a branch point. The original map will have W white vertices, B black vertices and F faces, where

$$W = \sum_{i=1}^{L} i w_i, \quad B = \sum_{i=1}^{L} i b_i \quad \text{and} \quad F = \sum_{i=1}^{L} i f_i, \tag{8.5}$$

and the total number E of edges is equal to $L\,e = F + W + B - 2(1-g)$.

The binomial coefficient in (8.2) disappears because the quotient map can't contain any dangling semi-edges. The multinomial coefficient must be replaced by the number of ways to distribute the branch points among the white vertices, black vertices and faces. Then (8.2) becomes

$$\nu_O(d,w,b,f) = \binom{w}{w_1, w_2, \ldots, w_L}\binom{b}{b_1, b_2, \ldots, b_L}\binom{f}{f_1, f_2, \ldots, f_L} N_g(d,w,b,f), \quad (8.6)$$

where d is the number of edges in the quotient maps on both sides of the formula (or the number of darts in the corresponding hypermaps) and $N_g(d,w,b,f)$ is the number of 2-coloured bipartite maps with d edges with w white vertices, b black vertices and f faces, rooted at a white vertex. For this number to be positive, the sum of all the w_i cannot exceed w with a similar bound on the sum of all the b_i and the sum of all the f_i; so w, b and f each starts at its respective sum and increases by 1 until the number E of edges in the original map exceeds a user-defined maximum. With each increase of w, b or f, one of the multinomial coefficients in (8.6) gets updated using a single multiplication and division. The product of these three multinomial coefficients must be computed for all sets of non-negative integers such that for each i, $w_i + b_i + f_i$ is equal to the total number of branch points of orbit length i.

Once (8.6) is multiplied by the number of automorphisms with the current orbifold signature, we get the contribution of that signature and the numbers w_i, b_i and f_i to E times the number of sensed 2-coloured bipartite maps of genus G with E edges, F faces, B black vertices and W white vertices. This contribution is added to the appropriate element of an array, initially 0, and when all the contributions have been tallied, for each E, F, W and B the corresponding array element is divided by E (not $2E$ because the root must be incident to a white vertex) to give the number of sensed 2-coloured bipartite maps of genus G with E edges, F faces, B black vertices and W white vertices or, equivalently, the number of sensed hypermaps of genus G with E darts, F faces, B hyperedges and W vertices.

This enumeration was done with a program written in C++ using CLN to treat big integers. It enumerated rooted and sensed hypermaps of genus up to 24 with up to 50 darts as fast as it could display the numbers on the screen. The numbers coincide with those obtained by generating the hypermaps [24]. The source code is available from the second author on request.

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A First numbers of rooted hypermaps

The following sections show the numbers h of rooted hypermaps of genus g with d darts, v vertices, e edges and $d - v - e + 2(1 - g)$ faces, for $g \leq 6$ and $d \leq 14$.

A.1 Genus 0

d	v	e	f	h
1	1	1	1	1
1			sum	1
2	1	1	2	1
2	1	2	1	1
2	2	1	1	1
2			sum	3
3	1	1	3	1
3	1	2	2	3
3	2	1	2	3
3	1	3	1	1
3	2	2	1	3
3	3	1	1	1
3			sum	12
4	1	1	4	1
4	1	2	3	6
4	2	1	3	6
4	1	3	2	6
4	2	2	2	17
4	3	1	2	6
4	1	4	1	1
4	2	3	1	6
4	3	2	1	6
4	4	1	1	1
4			sum	56
5	1	1	5	1
5	1	2	4	10
5	2	1	4	10
5	1	3	3	20
5	2	2	3	55
5	3	1	3	20
5	1	4	2	10
5	2	3	2	55
5	3	2	2	55
5	4	1	2	10
5	1	5	1	1
5	2	4	1	10
5	3	3	1	20
5	4	2	1	10
5	5	1	1	1
5			sum	288
6	1	1	6	1
6	1	2	5	15
6	2	1	5	15
6	1	3	4	50
6	2	2	4	135
6	3	1	4	50
6	1	4	3	50
6	2	3	3	262
6	3	2	3	262
6	4	1	3	50
6	1	5	2	15
6	2	4	2	135
6	3	3	2	262
6	4	2	2	135
6	5	1	2	15
6	1	6	1	1
6	2	5	1	15
6	3	4	1	50
6	4	3	1	50
6	5	2	1	15
6	6	1	1	1
6			sum	1584
7	1	1	7	1
7	1	2	6	21
7	2	1	6	21
7	1	3	5	105
7	2	2	5	280
7	3	1	5	105
7	1	4	4	175
7	2	3	4	889
7	3	2	4	889
7	4	1	4	175
7	1	5	3	105
7	2	4	3	889
7	3	3	3	1694
7	4	2	3	889
7	5	1	3	105
7	1	6	2	21
7	2	5	2	280
7	3	4	2	889
7	4	3	2	889
7	5	2	2	280
7	6	1	2	21
7	1	7	1	1
7	2	6	1	21
7	3	5	1	105
7	4	4	1	175
7	5	3	1	105
7	6	2	1	21
7	7	1	1	1
7			sum	9152
8	1	1	8	1
8	1	2	7	28
8	2	1	7	28
8	1	3	6	196
8	2	2	6	518
8	3	1	6	196
8	1	4	5	490
8	2	3	5	2436
8	3	2	5	2436
8	4	1	5	490
8	1	5	4	490
8	2	4	4	3985
8	3	3	4	7500
8	4	2	4	3985
8	5	1	4	490
8	1	6	3	196
8	2	5	3	2436
8	3	4	3	7500
8	4	3	3	7500
8	5	2	3	2436
8	6	1	3	196
8	1	7	2	28
8	2	6	2	518
8	3	5	2	2436
8	4	4	2	3985
8	5	3	2	2436
8	6	2	2	518
8	7	1	2	28
8	1	8	1	1
8	2	7	1	28
8	3	6	1	196
8	4	5	1	490
8	5	4	1	490
8	6	3	1	196
8	7	2	1	28
8	8	1	1	1
8			sum	54912
9	1	1	9	1
9	1	2	8	36
9	2	1	8	36
9	1	3	7	336
9	2	2	7	882
9	3	1	7	336
9	1	4	6	1176
9	2	3	6	5754
9	3	2	6	5754
9	4	1	6	1176
9	1	5	5	1764
9	2	4	5	13941
9	3	3	5	26004
9	4	2	5	13941
9	5	1	5	1764
9	1	6	4	1176
9	2	5	4	13941
9	3	4	4	42015
9	4	3	4	42015
9	5	2	4	13941
9	6	1	4	1176
9	1	7	3	336
9	2	6	3	5754
9	3	5	3	26004
9	4	4	3	42015
9	5	3	3	26004
9	6	2	3	5754
9	7	1	3	336
9	1	8	2	36
9	2	7	2	882
9	3	6	2	5754
9	4	5	2	13941
9	5	4	2	13941
9	6	3	2	5754
9	7	2	2	882
9	8	1	2	36
9	1	9	1	1
9	2	8	1	36

9	3	7	1	336
9	4	6	1	1176
9	5	5	1	1764
9	6	4	1	1176
9	7	3	1	336
9	8	2	1	36
9	9	1	1	1
9			sum	339456
10	1	1	10	1
10	1	2	9	45
10	2	1	9	45
10	1	3	8	540
10	2	2	8	1410
10	3	1	8	540
10	1	4	7	2520
10	2	3	7	12180
10	3	2	7	12180
10	4	1	7	2520
10	1	5	6	5292
10	2	4	6	40935
10	3	3	6	75840
10	4	2	6	40935
10	5	1	6	5292
10	1	6	5	5292
10	2	5	5	60626
10	3	4	5	179860
10	4	3	5	179860
10	5	2	5	60626
10	6	1	5	5292
10	1	7	4	2520
10	2	6	4	40935
10	3	5	4	179860
10	4	4	4	288025
10	5	3	4	179860
10	6	2	4	40935
10	7	1	4	2520
10	1	8	3	540
10	2	7	3	12180
10	3	6	3	75840
10	4	5	3	179860
10	5	4	3	179860
10	6	3	3	75840
10	7	2	3	12180
10	8	1	3	540
10	1	9	2	45
10	2	8	2	1410
10	3	7	2	12180
10	4	6	2	40935
10	5	5	2	60626
10	6	4	2	40935
10	7	3	2	12180
10	8	2	2	1410
10	9	1	2	45
10	1	10	1	1
10	2	9	1	45
10	3	8	1	540
10	4	7	1	2520
10	5	6	1	5292
10	6	5	1	5292
10	7	4	1	2520
10	8	3	1	540
10	9	2	1	45
10	10	1	1	1
10			sum	2149888
11	1	1	11	1
11	1	2	10	55
11	2	1	10	55
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A.2 Genus 1

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14	3	4	7	1035514340
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14	5	2	7	375707570
14	6	1	7	38648610
14	1	7	6	38648610
14	2	6	6	512104880
14	3	5	6	2020140430
14	4	4	6	3126887407
14	5	3	6	2020140430
14	6	2	6	512104880
14	7	1	6	38648610
14	1	8	5	21471450
14	2	7	5	375707570
14	3	6	5	2020140430
14	4	5	5	4475516612
14	5	4	5	4475516612
14	6	3	5	2020140430
14	7	2	5	375707570
14	8	1	5	21471450
14	1	9	4	6441435
14	2	8	4	145864355
14	3	7	4	1035514340
14	4	6	4	3126887407
14	5	5	4	4475516612
14	6	4	4	3126887407
14	7	3	4	1035514340
14	8	2	4	145864355
14	9	1	4	6441435
14	1	10	3	975975
14	2	9	3	28283255
14	3	8	3	259750218
14	4	7	3	1035514340
14	5	6	3	2020140430
14	6	5	3	2020140430
14	7	4	3	1035514340
14	8	3	3	259750218
14	9	2	3	28283255
14	10	1	3	975975
14	1	11	2	65065
14	2	10	2	2407405
14	3	9	2	28283255
14	4	8	2	145864355
14	5	7	2	375707570
14	6	6	2	512104880
14	7	5	2	375707570
14	8	4	2	145864355
14	9	3	2	28283255
14	10	2	2	2407405
14	11	1	2	65065
14	1	12	1	1365
14	2	11	1	65065
14	3	10	1	975975
14	4	9	1	6441435
14	5	8	1	21471450
14	6	7	1	38648610
14	7	6	1	38648610
14	8	5	1	21471450
14	9	4	1	6441435
14	10	3	1	975975
14	11	2	1	65065
14	12	1	1	1365
14			sum	47168678571

A.3 Genus 2

d	v	e	f	h
5	1	1	1	8
5			sum	8
6	1	1	2	84
6	1	2	1	84
6	2	1	1	84
6			sum	252
7	1	1	3	469
7	1	2	2	1183
7	2	1	2	1183
7	1	3	1	469
7	2	2	1	1183
7	3	1	1	469
7			sum	4956
8	1	1	4	1869
8	1	2	3	8526
8	2	1	3	8526
8	1	3	2	8526
8	2	2	2	21229
8	3	1	2	8526
8	1	4	1	1869
8	2	3	1	8526
8	3	2	1	8526
8	4	1	1	1869
8			sum	77992
9	1	1	5	5985
9	1	2	4	42588
9	2	1	4	42588
9	1	3	3	77028
9	2	2	3	189999
9	3	1	3	77028
9	1	4	2	42588
9	2	3	2	189999
9	3	2	2	189999
9	4	1	2	42588
9	1	5	1	5985
9	2	4	1	42588
9	3	3	1	77028
9	4	2	1	42588
9	5	1	1	5985
9			sum	1074564
10	1	1	6	16401
10	1	2	5	167013
10	2	1	5	167013
10	1	3	4	471240
10	2	2	4	1154095
10	3	1	4	471240
10	1	4	3	471240
10	2	3	3	2068070
10	3	2	3	2068070
10	4	1	3	471240
10	1	5	2	167013
10	2	4	2	1154095
10	3	3	2	2068070
10	4	2	2	1154095
10	5	1	2	167013
10	1	6	1	16401
10	2	5	1	167013
10	3	4	1	471240
10	4	3	1	471240
10	5	2	1	167013
10	6	1	1	16401
10			sum	13545216
11	1	1	7	39963
11	1	2	6	550011
11	2	1	6	550011
11	1	3	5	2221065
11	2	2	5	5409019
11	3	1	5	2221065
11	1	4	4	3465000
11	2	3	4	15014846
11	3	2	4	15014846
11	4	1	4	3465000
11	1	5	3	2221065
11	2	4	3	15014846
11	3	3	3	26717482
11	4	2	3	15014846
11	5	1	3	2221065
11	1	6	2	550011
11	2	5	2	5409019
11	3	4	2	15014846
11	4	3	2	15014846
11	5	2	2	5409019
11	6	1	2	550011
11	1	7	1	39963
11	2	6	1	550011
11	3	5	1	2221065
11	4	4	1	3465000
11	5	3	1	2221065
11	6	2	1	550011
11	7	1	1	39963
11			sum	160174960
12	1	1	8	88803
12	1	2	7	1585584
12	2	1	7	1585584
12	1	3	6	8654646
12	2	2	6	20981337
12	3	1	6	8654646
12	1	4	5	19324305
12	2	3	5	82897296
12	3	2	5	82897296
12	4	1	5	19324305
12	1	5	4	19324305
12	2	4	4	128420004
12	3	3	4	227256510
12	4	2	4	128420004
12	5	1	4	19324305
12	1	6	3	8654646
12	2	5	3	82897296
12	3	4	3	227256510
12	4	3	3	227256510
12	5	2	3	82897296
12	6	1	3	8654646
12	1	7	2	1585584
12	2	6	2	20981337
12	3	5	2	82897296
12	4	4	2	128420004
12	5	3	2	82897296
12	6	2	2	20981337
12	7	1	2	1585584
12	1	8	1	88803
12	2	7	1	1585584
12	3	6	1	8654646
12	4	5	1	19324305
12	5	4	1	19324305
12	6	3	1	8654646
12	7	2	1	1585584
12	8	1	1	88803
12			sum	1805010948
13	1	1	9	183183
13	1	2	8	4114110
13	2	1	8	4114110
13	1	3	7	29135106
13	2	2	7	70367479
13	3	1	7	29135106
13	1	4	6	87933846
13	2	3	6	374127663
13	3	2	6	374127663
13	4	1	6	87933846
13	1	5	5	125855730
13	2	4	5	824962502
13	3	3	5	1453414846
13	4	2	5	824962502
13	5	1	5	125855730
13	1	6	4	87933846
13	2	5	4	824962502
13	3	4	4	2239280420
13	4	3	4	2239280420
13	5	2	4	824962502
13	6	1	4	87933846
13	1	7	3	29135106
13	2	6	3	374127663
13	3	5	3	1453414846
13	4	4	3	2239280420
13	5	3	3	1453414846
13	6	2	3	374127663
13	7	1	3	29135106
13	1	8	2	4114110
13	2	7	2	70367479
13	3	6	2	374127663
13	4	5	2	824962502
13	5	4	2	824962502
13	6	3	2	374127663
13	7	2	2	70367479
13	8	1	2	4114110
13	1	9	1	183183
13	2	8	1	4114110
13	3	7	1	29135106
13	4	6	1	87933846
13	5	5	1	125855730
13	6	4	1	87933846
13	7	3	1	29135106
13	8	2	1	4114110
13	9	1	1	183183
13			sum	19588944336
14	1	1	10	355355
14	1	2	9	9798789
14	2	1	9	9798789
14	1	3	8	87291204
14	2	2	8	210164227
14	3	1	8	87291204
14	1	4	7	341825484
14	2	3	7	1444432612

14	3	2	7	1444432612
14	4	1	7	341825484
14	1	5	6	661320660
14	2	4	6	4286172247
14	3	3	6	7523770016
14	4	2	6	4286172247
14	5	1	6	661320660
14	1	6	5	661320660
14	2	5	5	6100939726
14	3	4	5	16427471172
14	4	3	5	16427471172
14	5	2	5	6100939726
14	6	1	5	661320660
14	1	7	4	341825484
14	2	6	4	4286172247
14	3	5	4	16427471172
14	4	4	4	25199010256
14	5	3	4	16427471172
14	6	2	4	4286172247
14	7	1	4	341825484
14	1	8	3	87291204
14	2	7	3	1444432612
14	3	6	3	7523770016
14	4	5	3	16427471172
14	5	4	3	16427471172
14	6	3	3	7523770016
14	7	2	3	1444432612
14	8	1	3	87291204
14	1	9	2	9798789
14	2	8	2	210164227
14	3	7	2	1444432612
14	4	6	2	4286172247
14	5	5	2	6100939726
14	6	4	2	4286172247
14	7	3	2	1444432612
14	8	2	2	210164227
14	9	1	2	9798789
14	1	10	1	355355
14	2	9	1	9798789
14	3	8	1	87291204
14	4	7	1	341825484
14	5	6	1	661320660
14	6	5	1	661320660
14	7	4	1	341825484
14	8	3	1	87291204
14	9	2	1	9798789
14	10	1	1	355355
14			sum	206254571236

A.4 Genus 3

d	v	e	f	h
7	1	1	1	180
7			sum	180
8	1	1	2	3044
8	1	2	1	3044
8	2	1	1	3044
8			sum	9132
9	1	1	3	26060
9	1	2	2	63600
9	2	1	2	63600
9	1	3	1	26060
9	2	2	1	63600
9	3	1	1	26060
9			sum	268980
10	1	1	4	152900
10	1	2	3	659340
10	2	1	3	659340
10	1	3	2	659340
10	2	2	2	1595480
10	3	1	2	659340
10	1	4	1	152900
10	2	3	1	659340
10	3	2	1	659340
10	4	1	1	152900
10			sum	6010220
11	1	1	5	696905
11	1	2	4	4606910
11	2	1	4	4606910
11	1	3	3	8141100
11	2	2	3	19571123
11	3	1	3	8141100
11	1	4	2	4606910
11	2	3	2	19571123
11	3	2	2	19571123
11	4	1	2	4606910
11	1	5	1	696905
11	2	4	1	4606910
11	3	3	1	8141100
11	4	2	1	4606910
11	5	1	1	696905
11			sum	112868844
12	1	1	6	2641925
12	1	2	5	24656775
12	2	1	5	24656775
12	1	3	4	66805310
12	2	2	4	159762815
12	3	1	4	66805310
12	1	4	3	66805310
12	2	3	3	280514670
12	3	2	3	280514670
12	4	1	3	66805310
12	1	5	2	24656775
12	2	4	2	159762815
12	3	3	2	280514670
12	4	2	2	159762815
12	5	1	2	24656775
12	1	6	1	2641925
12	2	5	1	24656775
12	3	4	1	66805310
12	4	3	1	66805310
12	5	2	1	24656775
12	6	1	1	2641925
12			sum	1877530740
13	1	1	7	8691683
13	1	2	6	108452916
13	2	1	6	108452916
13	1	3	5	414918075
13	2	2	5	988043771
13	3	1	5	414918075
13	1	4	4	636184120
13	2	3	4	2646424729
13	3	2	4	2646424729
13	4	1	4	636184120
13	1	5	3	414918075
13	2	4	3	2646424729
13	3	3	3	4623070842
13	4	2	3	2646424729
13	5	1	3	414918075
13	1	6	2	108452916
13	2	5	2	988043771
13	3	4	2	2646424729
13	4	3	2	2646424729
13	5	2	2	988043771
13	6	1	2	108452916
13	1	7	1	8691683
13	2	6	1	108452916
13	3	5	1	414918075
13	4	4	1	636184120
13	5	3	1	414918075
13	6	2	1	108452916
13	7	1	1	8691683
13			sum	28540603884
14	1	1	8	25537655
14	1	2	7	409732895
14	2	1	7	409732895
14	1	3	6	2096068975
14	2	2	6	4973691275
14	3	1	6	2096068975
14	1	4	5	4538348815
14	2	3	5	18733893115
14	3	2	5	18733893115
14	4	1	5	4538348815
14	1	5	4	4538348815
14	2	4	4	28579309570
14	3	3	4	49719495672
14	4	2	4	28579309570
14	5	1	4	4538348815
14	1	6	3	2096068975
14	2	5	3	18733893115
14	3	4	3	49719495672
14	4	3	3	49719495672
14	5	2	3	18733893115
14	6	1	3	2096068975
14	1	7	2	409732895
14	2	6	2	4973691275
14	3	5	2	18733893115
14	4	4	2	28579309570
14	5	3	2	18733893115
14	6	2	2	4973691275
14	7	1	2	409732895
14	1	8	1	25537655
14	2	7	1	409732895
14	3	6	1	2096068975
14	4	5	1	4538348815
14	5	4	1	4538348815
14	6	3	1	2096068975
14	7	2	1	409732895
14	8	1	1	25537655
14			sum	404562365316

A.5 Genus 4

d	v	e	f	h
9	1	1	1	8064
9			sum	8064
10	1	1	2	193248
10	1	2	1	193248
10	2	1	1	193248
10			sum	579744
11	1	1	3	2286636
11	1	2	2	5458464
11	2	1	2	5458464
11	1	3	1	2286636
11	2	2	1	5458464
11	3	1	1	2286636
11			sum	23235300
12	1	1	4	18128396
12	1	2	3	75220860
12	2	1	3	75220860
12	1	3	2	75220860
12	2	2	2	178462816
12	3	1	2	75220860
12	1	4	1	18128396
12	2	3	1	75220860
12	3	2	1	75220860
12	4	1	1	18128396
12			sum	684173164
13	1	1	5	109425316
13	1	2	4	687238552
13	2	1	4	687238552
13	1	3	3	1194737544
13	2	2	3	2820651496
13	3	1	3	1194737544
13	1	4	2	687238552
13	2	3	2	2820651496
13	3	2	2	2820651496
13	4	1	2	687238552
13	1	5	1	109425316
13	2	4	1	687238552
13	3	3	1	1194737544
13	4	2	1	687238552
13	5	1	1	109425316
13			sum	16497874380
14	1	1	6	539651112
14	1	2	5	4736419688
14	2	1	5	4736419688
14	1	3	4	12465308856
14	2	2	4	29310854804
14	3	1	4	12465308856
14	1	4	3	12465308856
14	2	3	3	50713072144
14	3	2	3	50713072144
14	4	1	3	12465308856
14	1	5	2	4736419688
14	2	4	2	29310854804
14	3	3	2	50713072144
14	4	2	2	29310854804
14	5	1	2	4736419688
14	1	6	1	539651112
14	2	5	1	4736419688
14	3	4	1	12465308856
14	4	3	1	12465308856
14	5	2	1	4736419688
14	6	1	1	539651112
14			sum	344901105444

A.6 Genus 5

d	v	e	f	h
11	1	1	1	604800
11			sum	604800
12	1	1	2	19056960
12	1	2	1	19056960
12	2	1	1	19056960
12			sum	57170880
13	1	1	3	292271616
13	1	2	2	686597184
13	2	1	2	686597184
13	1	3	1	292271616
13	2	2	1	686597184
13	3	1	1	292271616
13			sum	2936606400
14	1	1	4	2961802480
14	1	2	3	11947069680
14	2	1	3	11947069680
14	1	3	2	11947069680
14	2	2	2	27934773440
14	3	1	2	11947069680
14	1	4	1	2961802480
14	2	3	1	11947069680
14	3	2	1	11947069680
14	4	1	1	2961802480
14			sum	108502598960

A.7 Genus 6

d	v	e	f	h
13	1	1	1	68428800
13			sum	68428800
14	1	1	2	2699672832
14	1	2	1	2699672832
14	2	1	1	2699672832
14			sum	8099018496

These tables extend to 14 darts the part of Appendix B of [24] about rooted hypermaps.

B First numbers of unrooted hypermaps

The following sections show the numbers H of unrooted hypermaps of genus g with d darts, v vertices, e edges and $d - v - e + 2(1 - g)$ faces, for $g \leq 6$ and $d \leq 14$.

B.1 Genus 0

d	v	e	f	H
1	1	1	1	1
1		sum		1
2	1	1	2	1
2	1	2	1	1
2	2	1	1	1
2		sum		3
3	1	1	3	1
3	1	2	2	1
3	2	1	2	1
3	1	3	1	1
3	2	2	1	1
3	3	1	1	1
3		sum		6
4	1	1	4	1
4	1	2	3	2
4	2	1	3	2
4	1	3	2	2
4	2	2	2	5
4	3	1	2	2
4	1	4	1	1
4	2	3	1	2
4	3	2	1	2
4	4	1	1	1
4		sum		20
5	1	1	5	1
5	1	2	4	2
5	2	1	4	2
5	1	3	3	4
5	2	2	3	11
5	3	1	3	4
5	1	4	2	2
5	2	3	2	11
5	3	2	2	11
5	4	1	2	2
5	1	5	1	1
5	2	4	1	2
5	3	3	1	4
5	4	2	1	2
5	5	1	1	1
5		sum		60
6	1	1	6	1
6	1	2	5	3
6	2	1	5	3
6	1	3	4	10
6	2	2	4	24
6	3	1	4	10
6	1	4	3	10
6	2	3	3	46
6	3	2	3	46
6	4	1	3	10
6	1	5	2	3
6	2	4	2	24
6	3	3	2	46
6	4	2	2	24
6	5	1	2	3
6	1	6	1	1
6	2	5	1	3
6	3	4	1	10
6	4	3	1	10
6	5	2	1	3
6	6	1	1	1
6		sum		291
7	1	1	7	1
7	1	2	6	3
7	2	1	6	3
7	1	3	5	15
7	2	2	5	40
7	3	1	5	15
7	1	4	4	25
7	2	3	4	127
7	3	2	4	127
7	4	1	4	25
7	1	5	3	15
7	2	4	3	127
7	3	3	3	242
7	4	2	3	127
7	5	1	3	15
7	1	6	2	3
7	2	5	2	40
7	3	4	2	127
7	4	3	2	127
7	5	2	2	40
7	6	1	2	3
7	1	7	1	1
7	2	6	1	3
7	3	5	1	15
7	4	4	1	25
7	5	3	1	15
7	6	2	1	3
7	7	1	1	1
7		sum		1310
8	1	1	8	1
8	1	2	7	4
8	2	1	7	4
8	1	3	6	26
8	2	2	6	67
8	3	1	6	26
8	1	4	5	64
8	2	3	5	309
8	3	2	5	309
8	4	1	5	64
8	1	5	4	64
8	2	4	4	505
8	3	3	4	946
8	4	2	4	505
8	5	1	4	64
8	1	6	3	26
8	2	5	3	309
8	3	4	3	946
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B.2 Genus 1

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12	6	5	1	165262
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12	8	3	1	19694
12	9	2	1	1976
12	10	1	1	62
12			sum	57167260
13	1	1	11	77
13	1	2	10	3080
13	2	1	10	3080
13	1	3	9	38115
13	2	2	9	94281
13	3	1	9	38115
13	1	4	8	203280
13	2	3	8	898051
13	3	2	8	898051
13	4	1	8	203280
13	1	5	7	533610
13	2	4	7	3661896
13	3	3	7	6542368
13	4	2	7	3661896
13	5	1	7	533610
13	1	6	6	731808
13	2	5	6	7221592
13	3	4	6	20047636
13	4	3	6	20047636
13	5	2	6	7221592
13	6	1	6	731808
13	1	7	5	533610
13	2	6	5	7221592
13	3	5	5	28831218
13	4	4	5	44800675
13	5	3	5	28831218
13	6	2	5	7221592
13	7	1	5	533610
13	1	8	4	203280
13	2	7	4	3661896
13	3	6	4	20047636
13	4	5	4	44800675
13	5	4	4	44800675
13	6	3	4	20047636
13	7	2	4	3661896
13	8	1	4	203280
13	1	9	3	38115
13	2	8	3	898051
13	3	7	3	6542368
13	4	6	3	20047636
13	5	5	3	28831218
13	6	4	3	20047636
13	7	3	3	6542368
13	8	2	3	898051
13	9	1	3	38115
13	1	10	2	3080
13	2	9	2	94281
13	3	8	2	898051
13	4	7	2	3661896
13	5	6	2	7221592
13	6	5	2	7221592
13	7	4	2	3661896
13	8	3	2	898051
13	9	2	2	94281
13	10	1	2	3080
13	1	11	1	77
13	2	10	1	3080
13	3	9	1	38115
13	4	8	1	203280
13	5	7	1	533610
13	6	6	1	731808
13	7	5	1	533610
13	8	4	1	203280
13	9	3	1	38115
13	10	2	1	3080
13	11	1	1	77
13			sum	438644841
14	1	1	12	99
14	1	2	11	4659
14	2	1	11	4659
14	1	3	10	69765
14	2	2	10	172040
14	3	1	10	69765
14	1	4	9	460245
14	2	3	9	2020530
14	3	2	9	2020530
14	4	1	9	460245
14	1	5	8	1533950
14	2	4	8	10419653
14	3	3	8	18554641
14	4	2	8	10419653
14	5	1	8	1533950
14	1	6	7	2760990
14	2	5	7	26837442
14	3	4	7	73967488
14	4	3	7	73967488
14	5	2	7	26837442
14	6	1	7	2760990
14	1	7	6	2760990
14	2	6	6	36580432
14	3	5	6	144298902
14	4	4	6	223353280
14	5	3	6	144298902
14	6	2	6	36580432
14	7	1	6	2760990
14	1	8	5	1533950
14	2	7	5	26837442
14	3	6	5	144298902
14	4	5	5	319684549
14	5	4	5	319684549
14	6	3	5	144298902
14	7	2	5	26837442
14	8	1	5	1533950
14	1	9	4	460245
14	2	8	4	10419653
14	3	7	4	73967488
14	4	6	4	223353280
14	5	5	4	319684549
14	6	4	4	223353280
14	7	3	4	73967488
14	8	2	4	10419653
14	9	1	4	460245
14	1	10	3	69765
14	2	9	3	2020530
14	3	8	3	18554641
14	4	7	3	73967488
14	5	6	3	144298902
14	6	5	3	144298902
14	7	4	3	73967488
14	8	3	3	18554641
14	9	2	3	2020530
14	10	1	3	69765
14	1	11	2	4659
14	2	10	2	172040
14	3	9	2	2020530
14	4	8	2	10419653
14	5	7	2	26837442
14	6	6	2	36580432
14	7	5	2	26837442
14	8	4	2	10419653
14	9	3	2	2020530
14	10	2	2	172040
14	11	1	2	4659
14	1	12	1	99
14	2	11	1	4659
14	3	10	1	69765
14	4	9	1	460245
14	5	8	1	1533950
14	6	7	1	2760990
14	7	6	1	2760990
14	8	5	1	1533950
14	9	4	1	460245
14	10	3	1	69765
14	11	2	1	4659
14	12	1	1	99
14			sum	3369276867

B.3 Genus 2

d	v	e	f	H
5	1	1	1	4
5			sum	4
6	1	1	2	16
6	1	2	1	16
6	2	1	1	16
6			sum	48
7	1	1	3	67
7	1	2	2	169
7	2	1	2	169
7	1	3	1	67
7	2	2	1	169
7	3	1	1	67
7			sum	708
8	1	1	4	237
8	1	2	3	1072
8	2	1	3	1072
8	1	3	2	1072
8	2	2	2	2664
8	3	1	2	1072
8	1	4	1	237
8	2	3	1	1072
8	3	2	1	1072
8	4	1	1	237
8			sum	9807
9	1	1	5	667
9	1	2	4	4736
9	2	1	4	4736
9	1	3	3	8560
9	2	2	3	21113
9	3	1	3	8560
9	1	4	2	4736
9	2	3	2	21113
9	3	2	2	21113
9	4	1	2	4736
9	1	5	1	667
9	2	4	1	4736
9	3	3	1	8560
9	4	2	1	4736
9	5	1	1	667
9			sum	119436
10	1	1	6	1649
10	1	2	5	16725
10	2	1	5	16725
10	1	3	4	47164
10	2	2	4	115478
10	3	1	4	47164
10	1	4	3	47164
10	2	3	3	206895
10	3	2	3	206895
10	4	1	3	47164
10	1	5	2	16725
10	2	4	2	115478
10	3	3	2	206895
10	4	2	2	115478
10	5	1	2	16725
10	1	6	1	1649
10	2	5	1	16725
10	3	4	1	47164
10	4	3	1	47164
10	5	2	1	16725
10	6	1	1	1649
10			sum	1355400
11	1	1	7	3633
11	1	2	6	50001
11	2	1	6	50001
11	1	3	5	201915
11	2	2	5	491729
11	3	1	5	201915
11	1	4	4	315000
11	2	3	4	1364986
11	3	2	4	1364986
11	4	1	4	315000
11	1	5	3	201915
11	2	4	3	1364986
11	3	3	3	2428862
11	4	2	3	1364986
11	5	1	3	201915
11	1	6	2	50001
11	2	5	2	491729
11	3	4	2	1364986
11	4	3	2	1364986
11	5	2	2	491729
11	6	1	2	50001
11	1	7	1	3633
11	2	6	1	50001
11	3	5	1	201915
11	4	4	1	315000
11	5	3	1	201915
11	6	2	1	50001
11	7	1	1	3633
11			sum	14561360
12	1	1	8	7417
12	1	2	7	132202
12	2	1	7	132202
12	1	3	6	721382
12	2	2	6	1748723
12	3	1	6	721382
12	1	4	5	1610617
12	2	3	5	6908644
12	3	2	5	6908644
12	4	1	5	1610617
12	1	5	4	1610617
12	2	4	4	10702449
12	3	3	4	18938994
12	4	2	4	10702449
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12	3	4	3	18938994
12	4	3	3	18938994
12	5	2	3	6908644
12	6	1	3	721382
12	1	7	2	132202
12	2	6	2	1748723
12	3	5	2	6908644
12	4	4	2	10702449
12	5	3	2	6908644
12	6	2	2	1748723
12	7	1	2	132202
12	1	8	1	7417
12	2	7	1	132202
12	3	6	1	721382
12	4	5	1	1610617
12	5	4	1	1610617
12	6	3	1	721382
12	7	2	1	132202
12	8	1	1	7417
12			sum	150429819
13	1	1	9	14091
13	1	2	8	316470
13	2	1	8	316470
13	1	3	7	2241162
13	2	2	7	5412883
13	3	1	7	2241162
13	1	4	6	6764142
13	2	3	6	28779051
13	3	2	6	28779051
13	4	1	6	6764142
13	1	5	5	9681210
13	2	4	5	63458654
13	3	3	5	111801142
13	4	2	5	63458654
13	5	1	5	9681210
13	1	6	4	6764142
13	2	5	4	63458654
13	3	4	4	172252340
13	4	3	4	172252340
13	5	2	4	63458654
13	6	1	4	6764142
13	1	7	3	2241162
13	2	6	3	28779051
13	3	5	3	111801142
13	4	4	3	172252340
13	5	3	3	111801142
13	6	2	3	28779051
13	7	1	3	2241162
13	1	8	2	316470
13	2	7	2	5412883
13	3	6	2	28779051
13	4	5	2	63458654
13	5	4	2	63458654
13	6	3	2	28779051
13	7	2	2	5412883
13	8	1	2	316470
13	1	9	1	14091
13	2	8	1	316470
13	3	7	1	2241162
13	4	6	1	6764142
13	5	5	1	9681210
13	6	4	1	6764142
13	7	3	1	2241162
13	8	2	1	316470
13	9	1	1	14091
13			sum	1506841872
14	1	1	10	25405
14	1	2	9	700045
14	2	1	9	700045
14	1	3	8	6235526
14	2	2	8	15012496
14	3	1	8	6235526
14	1	4	7	24417030
14	2	3	7	103175785

14	3	2	7	103175785
14	4	1	7	24417030
14	1	5	6	47238510
14	2	4	6	306159286
14	3	3	6	537417269
14	4	2	6	306159286
14	5	1	6	47238510
14	1	6	5	47238510
14	2	5	5	435785878
14	3	4	5	1173398706
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14	5	2	5	435785878
14	6	1	5	47238510
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14	2	6	4	306159286
14	3	5	4	1173398706
14	4	4	4	1799940644
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14	6	2	4	306159286
14	7	1	4	24417030
14	1	8	3	6235526
14	2	7	3	103175785
14	3	6	3	537417269
14	4	5	3	1173398706
14	5	4	3	1173398706
14	6	3	3	537417269
14	7	2	3	103175785
14	8	1	3	6235526
14	1	9	2	700045
14	2	8	2	15012496
14	3	7	2	103175785
14	4	6	2	306159286
14	5	5	2	435785878
14	6	4	2	306159286
14	7	3	2	103175785
14	8	2	2	15012496
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14	1	10	1	25405
14	2	9	1	700045
14	3	8	1	6235526
14	4	7	1	24417030
14	5	6	1	47238510
14	6	5	1	47238510
14	7	4	1	24417030
14	8	3	1	6235526
14	9	2	1	700045
14	10	1	1	25405
14			sum	14732613116

B.4 Genus 3

d	v	e	f	H
7	1	1	1	30
7			sum	30
8	1	1	2	385
8	1	2	1	385
8	2	1	1	385
8			sum	1155
9	1	1	3	2900
9	1	2	2	7070
9	2	1	2	7070
9	1	3	1	2900
9	2	2	1	7070
9	3	1	1	2900
9			sum	29910
10	1	1	4	15308
10	1	2	3	65972
10	2	1	3	65972
10	1	3	2	65972
10	2	2	2	159608
10	3	1	2	65972
10	1	4	1	15308
10	2	3	1	65972
10	3	2	1	65972
10	4	1	1	15308
10			sum	601364
11	1	1	5	63355
11	1	2	4	418810
11	2	1	4	418810
11	1	3	3	740100
11	2	2	3	1779193
11	3	1	3	740100
11	1	4	2	418810
11	2	3	2	1779193
11	3	2	2	1779193
11	4	1	2	418810
11	1	5	1	63355
11	2	4	1	418810
11	3	3	1	740100
11	4	2	1	418810
11	5	1	1	63355
11			sum	10260804
12	1	1	6	220244
12	1	2	5	2054974
12	2	1	5	2054974
12	1	3	4	5567550
12	2	2	4	13314231
12	3	1	4	5567550
12	1	4	3	5567550
12	2	3	3	23377106
12	3	2	3	23377106
12	4	1	3	5567550
12	1	5	2	2054974
12	2	4	2	13314231
12	3	3	2	23377106
12	4	2	2	13314231
12	5	1	2	2054974
12	1	6	1	220244
12	2	5	1	2054974
12	3	4	1	5567550
12	4	3	1	5567550
12	5	2	1	2054974
12	6	1	1	220244
12			sum	156469887
13	1	1	7	668591
13	1	2	6	8342532
13	2	1	6	8342532
13	1	3	5	31916775
13	2	2	5	76003367
13	3	1	5	31916775
13	1	4	4	48937240
13	2	3	4	203571133
13	3	2	4	203571133
13	4	1	4	48937240
13	1	5	3	31916775
13	2	4	3	203571133
13	3	3	3	355620834
13	4	2	3	203571133
13	5	1	3	31916775
13	1	6	2	8342532
13	2	5	2	76003367
13	3	4	2	203571133
13	4	3	2	203571133
13	5	2	2	76003367
13	6	1	2	8342532
13	1	7	1	668591
13	2	6	1	8342532
13	3	5	1	31916775
13	4	4	1	48937240
13	5	3	1	31916775
13	6	2	1	8342532
13	7	1	1	668591
13			sum	2195431068
14	1	1	8	1824323
14	1	2	7	29267487
14	2	1	7	29267487
14	1	3	6	149721473
14	2	2	6	355267058
14	3	1	6	149721473
14	1	4	5	324171185
14	2	3	5	1338142324
14	3	2	5	1338142324
14	4	1	5	324171185
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14	3	3	4	3551405485
14	4	2	4	2041388556
14	5	1	4	324171185
14	1	6	3	149721473
14	2	5	3	1338142324
14	3	4	3	3551405485
14	4	3	3	3551405485
14	5	2	3	1338142324
14	6	1	3	149721473
14	1	7	2	29267487
14	2	6	2	355267058
14	3	5	2	1338142324
14	4	4	2	2041388556
14	5	3	2	1338142324
14	6	2	2	355267058
14	7	1	2	29267487
14	1	8	1	1824323
14	2	7	1	29267487
14	3	6	1	149721473
14	4	5	1	324171185
14	5	4	1	324171185
14	6	3	1	149721473
14	7	2	1	29267487
14	8	1	1	1824323
14			sum	28897471080

B.5 Genus 4

d	v	e	f	H
9	1	1	1	900
9			sum	900
10	1	1	2	19344
10	1	2	1	19344
10	2	1	1	19344
10			sum	58032
11	1	1	3	207876
11	1	2	2	496224
11	2	1	2	496224
11	1	3	1	207876
11	2	2	1	496224
11	3	1	1	207876
11			sum	2112300
12	1	1	4	1510846
12	1	2	3	6268712
12	2	1	3	6268712
12	1	3	2	6268712
12	2	2	2	14872428
12	3	1	2	6268712
12	1	4	1	1510846
12	2	3	1	6268712
12	3	2	1	6268712
12	4	1	1	1510846
12			sum	57017238
13	1	1	5	8417332
13	1	2	4	52864504
13	2	1	4	52864504
13	1	3	3	91902888
13	2	2	3	216973192
13	3	1	3	91902888
13	1	4	2	52864504
13	2	3	2	216973192
13	3	2	2	216973192
13	4	1	2	52864504
13	1	5	1	8417332
13	2	4	1	52864504
13	3	3	1	91902888
13	4	2	1	52864504
13	5	1	1	8417332
13			sum	1269067260
14	1	1	6	38547144
14	1	2	5	338317960
14	2	1	5	338317960
14	1	3	4	890383128
14	2	2	4	2093639428
14	3	1	4	890383128
14	1	4	3	890383128
14	2	3	3	3622371084
14	3	2	3	3622371084
14	4	1	3	890383128
14	1	5	2	338317960
14	2	4	2	2093639428
14	3	3	2	3622371084
14	4	2	2	2093639428
14	5	1	2	338317960
14	1	6	1	38547144
14	2	5	1	338317960
14	3	4	1	890383128
14	4	3	1	890383128
14	5	2	1	338317960
14	6	1	1	38547144
14			sum	24635879496

B.6 Genus 5

d	v	e	f	H
11	1	1	1	54990
11			sum	54990
12	1	1	2	1588218
12	1	2	1	1588218
12	2	1	1	1588218
12			sum	4764654
13	1	1	3	22482432
13	1	2	2	52815168
13	2	1	2	52815168
13	1	3	1	22482432
13	2	2	1	52815168
13	3	1	1	22482432
13			sum	225892800
14	1	1	4	211558928
14	1	2	3	853365360
14	2	1	3	853365360
14	1	3	2	853365360
14	2	2	2	1995345826
14	3	1	2	853365360
14	1	4	1	211558928
14	2	3	1	853365360
14	3	2	1	853365360
14	4	1	1	211558928
14			sum	7750214770

B.7 Genus 6

d	v	e	f	H
13	1	1	1	5263764
13			sum	5263764
14	1	1	2	192834612
14	1	2	1	192834612
14	2	1	1	192834612
14			sum	578503836

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Groups and Graphs, Designs and Dynamics

Yichang, China, August 12 – 25, 2019

http://math.sjtu.edu.cn/conference/G2D2

China Three Gorges University is organizing the International Conference and PhD-Master's Summer School on "Groups and Graphs, Designs and Dynamics" (G2D2). All scientific activities will take place in the Three Gorges Mathematical Research Center at China Three Gorges University, Yichang, China during August 12 – 25, 2019. The summer school part of G2D2 consists of 4 short courses and 4 colloquium talks; its conference part consists of about 20 invited talks and some contributed talks.

G2D2 is concerned with all aspects of mathematics, especially those relating to simple structures and simple processes. We will bring together experts and students to exchange ideas and to enrich their mathematical horizons. Four short courses and four colloquium talks will let participants see order and simplicity from possibly new perspectives and share insights with experts. We will also schedule invited talks (45 minutes) and contributed talks (25 minutes) with topics ranging from coding theory, design theory, ergodic theory, graph theory, group theory, matrix theory, optimization theory, and quantum information theory to symbolic dynamics.

Selected papers based on talks in G2D2 will be published in a special issue of *The Art of Discrete and Applied Mathematics*.

The guest editors of the special issue:

- **Alexander Ivanov**, Imperial College London, UK
- **Elena Konstantinova**, Sobolev Institute of Mathematics, Novosibirsk State University, Russia
- **Jack Koolen**, University of Science and Technology of China, China
- **Yaokun Wu**, Shanghai Jiao Tong University, China

Confirmed short courses:

- **Rosemary A. Bailey & Peter Cameron**, University of St Andrews, UK, *Laplacian Eigenvalues and Optimality*
- **Mike Boyle & Scott Schmieding**, University of Maryland and Northwestern University, USA, *Symbolic Dynamics and the Stable Algebra of Matrices*
- **Tullio Ceccherini-Silberstein**, Università del Sannio, Italy, *Topics in Representation Theory*
- **Nobuaki Obata**, Tohoku University, Japan, *Spectral Analysis of Growing Graphs – A Quantum Probability Point of View*

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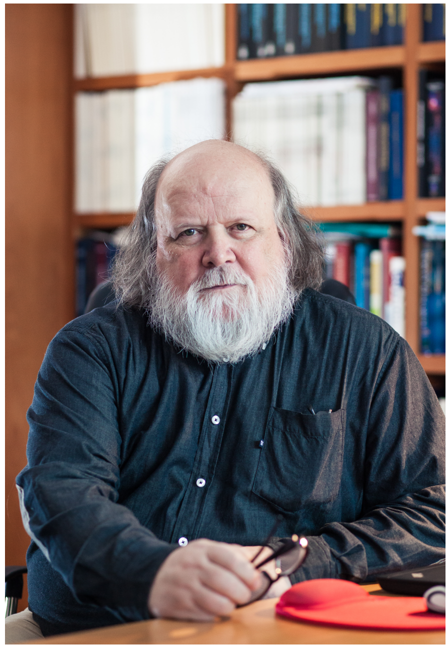
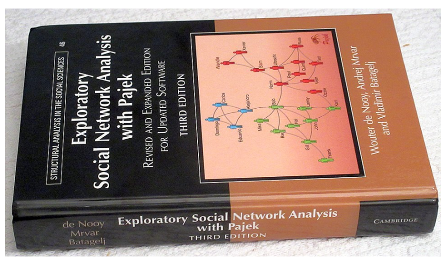
Vladimir Batagelj is 70



Vladimir Batagelj

Vladimir Batagelj, known as Vlado to his friends, is one of the most prolific Slovenian scientists. Although he has a PhD in mathematics and has worked most of his active life at the Department of Mathematics of the University of Ljubljana where he is now Professor Emeritus, he has very broad research interests and a passion for teaching. Over ten years before obtaining his PhD, Vlado published a solo paper on quadratic hash method in the distinguished journal Communications of the ACM. When I was a graduate student of computer science at Penn State, one of our textbooks cited his paper. I was very proud to tell my fellow graduate students that Vlado and I attended the same courses as undergraduates. Actually, during compulsory military service we shared a room in barracks in Zagreb for a year.

Vlado's scientific work has been cited over 11000 times in Google Scholar; over 5400 times since 2013. His most cited work, with over 3300 citations, is his book *Exploratory Social Network Analysis with Pajek*, written together with W. de Nooy and A. Mrvar. The book was also translated into Chinese and Japanese. The revised and expanded 3rd edition of this successful textbook was published by Cambridge University Press this year. *Pajek* is a highly successful, freely available software package for large networks analysis, authored by Vlado and his former PhD student Andrej Mrvar and used widely in social sciences.



Exploratory Social Network Analysis with Pajek, 3rd Edition

Vlado is one of the pioneers of discrete mathematics and theoretical computer science in Slovenia, who chartered his academic course on his own and works on problems that he finds interesting. Nevertheless, he understands the vital need for a nation of 2M to receive fresh knowledge in its own language. Vlado is the author of over 20 textbooks in Slovenian, covering a wide range of topics, from TEX to Combinatorics and Discrete Mathematics.

Vlado, I wish you a very happy birthday and many more productive years!

Tomo (Tomaž Pisanski)

ARS MATHEMATICA
CONTEMPORANEA

Branko Grünbaum, Geometer

Branko Grünbaum passed away on September 14, 2018, just a few weeks short of his 89th birthday. Dr. Grünbaum was an early contributor to this journal, and was a major influence on a lot of the people who have been involved with it over the years. He contributed an article [6] that appeared in its second issue that remains one of the most cited papers to have appeared here, helping raise the profile of AMC, and his most recent contribution is currently available online and will appear in 2019 [1].

Branko and Zdenka Grünbaum at the author's wedding in 2002.

Branko was a prodigious author. Over the course of his career he published over 250 articles and several books. Probably the most influential was his book *Convex Polytopes* [2], which first appeared in 1967. This was an indispensable reference for mathematicians working in the theory of convex polytopes, linear programming, and related combinatorial problems in geometry for at least the next two decades. Its value came not only from the thoroughness of his treatment, but the care and skill he applied in presenting some of the latest ideas and techniques in the study of convex polytopes, and the wealth of material he had collected from sometimes obscure references and then presented in an approachable and clear style. It introduced the world to Micha Perles' theory of Gale diagrams and included Branko's easy to follow proof of Steinitz's Theorem on convex polyhedra. The text also included numerous open problems and spurred much subsequent activity. The text was so esteemed as a reference that a second addition was assembled and prepared by Voker Kaibel, Victor Klee and Günter Ziegler, and released in 2003 [5]. In the new edition the original text was presented in its entirety and supplemented with commentaries at the end of each section, these provide insight into more recent developments and discuss the status of open problems discussed in the original text. There was a long period prior to the publication of the second edition when copies were incredibly hard to obtain, and I was once told it was the most stolen book in mathematics as a consequence.

His volume *Tilings and Patterns*, with Geoffrey C. Shephard [9], was also very influential from the moment of its publication in 1987. It is filled with beautiful diagrams and interesting mathematical results and it inspired many researchers. It also functioned very well as a coffee table book!

To give you a sense of the scope of the reach of his work, these two texts alone have over 1000 citations on MathSciNet, involving 1209 distinct authors!

In addition to his work on convex polytopes and tilings, he also inspired many mathematicians to take up the study of configurations and arrangements of points and lines. His 1972 monograph *Arrangements and Spreads* was based on a series of keynote lectures he gave at the *Conference on Convexity and Combinatorial Geometry* at the University of Oklahoma [10]. To help set the stage he began his lectures by reading part of *McElligot's Pool* by Dr. Seuss [11], a book he had read to his sons when they were small, enjoining his audience to join him in the unexpected adventures that awaited in this subject area if only they would use their imaginations. His fascination with arrangements and configurations continued into his retirement, resulting in the publication of the graduate text *Configurations of Points and Lines* in 2009 [7]. This text is an essential reference on the subject, covering the key developments in the study of configurations since their introduction in 1876 and presenting many open problems that have inspired a new generation of mathematicians to take up their investigation. At least seventeen different articles in this journal alone have listed it as a reference.

Branko was also well known for having an encyclopedic knowledge about the state of the field for a wide variety of topics in discrete geometry. His office had rows of cases filled with note cards with bibliographic information and notes on the many articles he had read over the years. People were always writing him to ask what might be known about questions they were interested in, and he often had excellent references to point them to (this was especially true before MathSciNet became popular). He was equally well known for disseminating open problems in geometry, a testimony to which are the 58 articles he wrote for Geombinatorics, a journal devoted to the discussion of open problems in combinatorial and discrete geometry.

Branko also had a talent for spotting mistakes. Many of his articles contain corrections to the literature, and he often used finding such mistakes as a springboard for reopening and exploring old questions from a new perspective. Probably the most famous mistake he ever caught was in the logo of the Mathematical Association of America. From the period from 1971-1985, the official logo of the MAA — a drawing of a supposedly regular icosahedron — was drawn in such a way that it could not have been the product of *any* geometric projection onto the plane, a point Branko made in the pages of Mathematics Magazine [4]. The MAA, much to its credit, immediately revised its logo and started using one with greater respect for geometry. Unfortunately, Branko caught them using the bad one again a few years later, but a follow up letter from him on the question seems to have permanently resolved the issue.

A recurring theme in Branko's writing was the importance of teaching *geometry*, and not as some highly refined and abstract activity, but through teaching the study of geometry as an area of applied mathematics. Tied closely to this was his concern that as mathematicians we have a responsibility to communicate our ideas and our proofs in a manner that not only achieves the desired result — such as proving a theorem — but doing so in a way that preserves the inherent beauty of the objects under investigation and provides genuine insight into what motivates their study [3]. He was deeply concerned by the approach of the Bourbaki to geometric subjects, and Dieudonné's famous slogan "Euclid must go!" epitomized a movement to treat geometry as a purely formal and abstract subject (so much so that the only diagram in any of the texts on geometry published by the Bourbaki is of a

Coxeter-Dynkin diagram).

Now that you know something of the mathematician, I'd like to say something about the history of the man. Branko was born on October 2, 1929 in the small city of Osijek, in what was then the Kingdom of Yugoslavia and is now Croatia; Zdenka Bienenstock was born there a year later. Zdenka's family, and all of the family on Branko's father's side were Jewish. When World War II came to Yugoslavia in 1941 it uprooted their lives. Zdenka survived the war hidden in a Catholic convent, but the rest of her entire extended family was killed, many in Auschwitz. Branko's mother was Catholic, and his family survived the war by moving to live with his maternal grandmother, benefitting from protections given to families in mixed marriages in Croatia. Branko and Zdenka met after the war while high school students, and soon fell in love. Branko was admitted to the university in Zagreb, but quickly realized that he might not be able to demonstrate sufficient ardor for Marxism-Leninism and could be potentially denied a degree or future employment. This, combined with his father's experience of having been forced to "donate" his share in a successful business to the local government soured him on the idea of staying in Yugoslavia. In 1948, the Communist regime arranged for Jews wishing to emigrate from Yugoslavia to register for transport to Israel. When it turned out that a ship really did arrive and it was announced that there would be a second opportunity to emigrate the following year, Branko convinced his family and Zdenka to seize the opportunity, arriving in Haifa, Israel in July 1949.

As was the case for many immigrants to Israel at the time, conditions were very difficult, but both Branko and Zdenka were determined to resume their studies. In the fall of 1950, Branko quit a job in Tel Aviv to go to Jerusalem to study mathematics. In 1954 he received his M.Sc., and he and Zdenka married on June 30, 1954. In the fall of 1955 Branko was called to active duty in the Israeli Air Force, where he worked in the Operations Research unit; meanwhile Zdenka earned her M.Sc. in Chemistry. Their first son, Rami, was born in 1956. Branko completed his Ph.D. in 1957 and in 1958 he was discharged from the military. Soon afterward he was awarded a scholarship to the Institute for Advanced Study in Princeton, NJ, where he and his family spent two years. In the fall of 1960 he obtained a visiting appointment at the University of Washington in Seattle, where their second son Daniel was born in November. While they were planning their return to Israel where Branko had accepted a position as a lecturer at Hebrew University, they learned his marriage to Zdenka was annulled because he was not legally Jewish according to Orthodox interpretation (his mother having not been a Jew), so he and Zdenka remarried at the City Hall in Seattle before moving to Jerusalem. Within three years Branko had been promoted to Associate Professor. He spent the summer of 1963 in Seattle as a visitor at the University of Washington, and he spent a sabbatical in 1965-66 at Michigan State University as a visiting professor.

The story in the news that another Israeli immigrant from a mixed marriage had her passport and citizenship revoked in 1966 for reasons similar to those used to annul Branko and Zdenka's marriage resulted in them deciding not to return to Israel, even though this meant Zdenka would be unable to complete her Ph.D. in Chemistry. Branko joined the faculty at the University of Washington as a full professor in 1966. He retired in 2001, but continued to teach and work with graduate students as an emeritus professor (Leah Berman and I were his last two doctoral students at the University of Washington, completing our degrees in 2002).

I would like to close by saying a bit about what it was like being his student. I moved to

Seattle in 1997 at the suggestion of Marjorie Senechal. I was immediately welcomed into a vibrant and generous community of discrete geometers. At the core of this community were Vic Klee and Branko Grünbaum, both of whom provided me with much valuable advice and guidance during my time there, and both of whom were unfailingly kind to me.

As a mathematics graduate student, a visit to Branko's office was like a visit to the candy store. His office was filled with models he had built over the years to help him think through geometric problems. They covered shelves and hung from the ceiling tiles on bits of string (which I'm sure caused the fire marshal fits of apoplexy), and they were colorful and intriguing. It seemed like every time I went into his office I noticed something new, and he was always happy to explain the math behind the model and pull a copy of a preprint from his filing cabinet of the paper that had provided the need for the model in the first place. Questions I brought to Branko were often answered by him pulling a model from a shelf to illustrate a point, and would lead us into a discussion of other questions the model helped to illuminate. One of my most prized possessions is a model he built, and I was immensely proud when he asked me to contribute a copy of a model I had built for my own research into his collection. The garage at his house was equally a treasure trove of mathematical models, and interesting examples often made their way from his home to lecture halls at the university. In this way I learned the importance of visualization and model building as tools to gain deeper insight into geometric questions, and as an important step in verifying my understanding of mathematical ideas (if I couldn't build it, I clearly didn't understand it).

When I began preparing my first paper for publication, I got invaluable advice from Branko about what I should be trying to achieve in my writing. He believed strongly that an article should be written as an invitation to engage in a conversation with the author(s). This means making sure it has the necessary background, trying to make the writing as clear and engaging as possible, and asking thought provoking questions. Because a paper is a conversation between the author and the reader, even solo authored papers should use "we". He encouraged the inclusion of conjectures, because they excited the reader to a challenge. He enjoyed and fostered collaboration at every turn. The slight exception to this was co-authoring papers with his students (especially *while* they were his students), because he wanted to make sure readers gave us the credit instead of assuming the significant contributions were his. This was a little frustrating to me because I wanted to lower my Erdős number *and* get my Grünbaum number down to 1, so I've always been a bit envious of my wife Leah Berman, who was a student of his at the same time, who co-authored a paper with him a few years after we graduated.

After completing my Ph.D., Branko continued to be a significant presence and influence on my life and career. His signature graces our marriage contract, and he and Zdenka welcomed us into their home when we would visit Seattle. He and Zdenka always made us feel welcome and cherished, and they doted on our children. I was always a little in awe of how much in love they seemed, even after 61 years of marriage. Zdenka sadly passed away in her sleep on December 28, 2015.

Having been his student has been a constant source of open doors for me. Anywhere I go in the world of discrete geometry, I am always greeted with warmth and delight when someone learns I was his student. One of my favorite interactions along these lines was when I first met János Pach while he was at MSRI; when he learned I was Branko's student he said that Branko "always had great taste in problems." That always struck me as very high praise indeed. It was also clear that many mathematicians I met not only respected him for his achievements and contributions to the field, but also treasured his company, hospitality and generosity — that they held in high esteem not just the mathematician but the man.

There is an old Jewish tradition that no one is truly gone as long as their memory and name survive. I will treasure his memory and the influence he has had on the course of my life for the remainder of my days. May his name be a blessing to you as well.

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†I would like to thank Rami Grünbaum for providing a copy of a transcript of a speech Branko gave in 2017 about his life with Zdenka that was used in the preparation of this memoriam, and for confirming certain dates and details. The author takes full responsibility for any errors that remain. I also would like to thank Dr. Marilyn Breen who provided important background on the *Conference on Convexity and Combinatorial Geometry* at the University of Oklahoma.

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