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What do we publish in AMC?

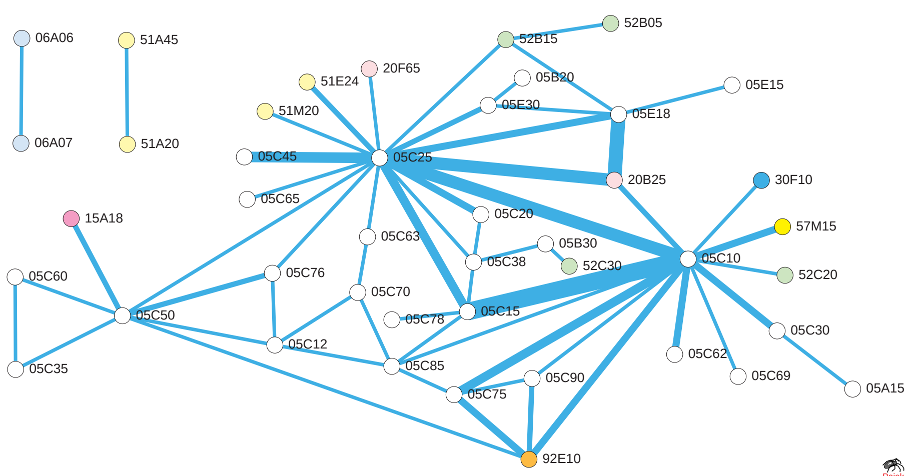
Recently we had to decline a submission to AMC of a paper that we were unable to handle. We did not even send it to a quick assessment, let alone to referees. The reason was not the quality of the paper, but its content and scope, which were far outside the interests of our editors, referees and authors. Our declared scope is clear: we are looking for high quality papers that cover at least two different subject fields, at least one of which is within discrete mathematics. Some easy statistical analysis tells us to what extent this has been realised in practice, and we hope this will be also of help to authors who are considering AMC as a potential venue for their papers.

In the first seven years of AMC we have published 204 papers, totalling 2872 pages. This is a little more than 14 pages per paper. The minimum average page length so far was 12.05, in volume 1, while the maximum average page length was 16.20, in volume 5. Approx. 43% of these papers used at least two different 2-digit MSC codes, while 52% used at least two different 3-letter MSC codes. A little over 85% of the papers used both primary and secondary classifications.

A large majority of the primary 2-digit classification codes came under 05 (Combinatorics), followed by 52 (Convex and discrete geometry), 51 (Geometry) and 20 (Group theory and generalizations). In decreasing order, the other 2-digit primary codes were 57 (Manifolds and cell complexes), 68 (Computer science), 06 (Order, lattices, ordered algebraic structures), 91 (Game theory, economics, social and behavioral sciences), 01 (History and biography), 92 (Biology and other natural sciences) and 47 (Operator theory).

The ten most frequent 3-letter classification codes were: 05C (Graph theory), 05E (Algebraic combinatorics), 20B (Permutation groups), 52B (Polytopes and polyhedra), 05B (Designs and configurations), 51E (Finite geometry and special incidence structures), 52C (Discrete geometry), 57M (Low-dimensional topology), 92E (Chemistry), 20F (Structure and classification of infinite and finite groups), and 06A (Ordered sets).

The most frequent pairs of 5-letter MSC codes declared in a paper are shown as edges in the figure below, which comes from some analysis performed by Vladimir Batagelj using Pajek. The threshold value for inclusion of a pair as an edge was set to 2.



Among the 5-letter classifications by far the most frequent were 05C25 (Graphs and abstract algebra), 05C10 (Planar graphs; geometric and topological aspects of graph theory), followed by 05C15 (Colorings of graphs and hypergraphs), 05E18 (Group actions on combinatorial structures), 05C12 (Distance in graphs), 20B25 (Finite automorphism groups of algebraic, geometric and combinatorial structures), 05C50 (Graphs and linear algebra), 05C76 (Graph operations), 05C75 (Structural characterization of families of graphs), 05C45 (Eulerian and Hamiltonian graphs), 05E30 (Association schemes, strongly regular graphs), and 05C85 (Graph algorithms).

These figures show that we publish mostly papers in algebraic and topological graph theory, with discrete and convex geometry also having significant presence in AMC. We note, however, that papers with less frequent MSC codes still play an important role in world mathematics. According to MathSciNet at the time of writing of this editorial, the most highly cited paper in 05C76 (Graph operations) was published in our journal.

Dragan Marušič and Tomaž Pisanski

Editors In Chief

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Super connectivity of direct product of graphs*

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Abstract

For a graph G, $\kappa(G)$ denotes its connectivity. A graph G is *super connected*, or simply *super-κ*, if every minimum separating set is the neighborhood of a vertex of G, that is, every minimum separating set isolates a vertex. The *direct product* $G_1 \times G_2$ of two graphs G_1 and G_2 is a graph with vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. Let $\Gamma = G \times K_n$, where G is a non-trivial graph and $K_n (n \geq 3)$ is a complete graph on n vertices. In this paper, we show that Γ is not super-κ if and only if either $\kappa(\Gamma) = n\kappa(G)$, or $\Gamma \cong K_{\ell,\ell} \times K_3 (\ell > 0)$.

Keywords: Super connectivity, direct product, vertex-cut.

Math. Subj. Class.: 05C40, 05C76

1 Introduction

Throughout this paper only undirected simple connected graphs without loops and multiple edges are considered. Unless stated otherwise, we follow Bondy and Murty [4] for terminology and definitions.

Let $G = (V(G), E(G))$ be a graph. For two vertices $u, v \in V(G)$, $u \sim v$ means that u is adjacent to v and uv is the edge incident to u and v in G. The set of vertices adjacent to the vertex v is called the *neighborhood* of v and denoted by $N_G(v)$, i.e., $N_G(v) = \{u \mid uv \in E(G)\}$. The *degree* of v is equal to $|N_G(v)|$, denoted by $d_G(v)$. The number $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the *minimum degree* of G. For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. As usual, $K_{m,m}$, (m is a positive integer) denotes the complete bipartite graph; $K_{m,m} - mK_2$ denotes the graph obtained by removing a 1-factor from $K_{m,m}$; K_n denotes the complete graph on n vertices; and $\mathbb{Z}_n$ denotes the ring of integers modulo n.

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A *separating set* of a graph G is a set of vertices whose deletion either disconnects G or reduces G to the trivial graph K_1. The *connectivity* of the graph G is the minimum number of vertices in a separating set of G, and will be denoted by $\kappa(G)$. In particular, $\kappa(K_n) = n-1$, and $\kappa(G) = 0$ if and only if G is disconnected or a K_1. Clearly, $\kappa(G) \leq \delta(G)$. A graph G with minimum degree $\delta(G)$ is *maximally connected* if $\delta(G) = \kappa(G)$.

An interconnection network is often modeled as a graph G, where $V(G)$ is the set of processors and $E(G)$ is the set of communication links in the network. The connectivity $\kappa(G)$ of G is an important measurement for fault-tolerance of the network, and the larger $\kappa(G)$ is, the more reliable the network is. As more refined indices of reliability than connectivity, super connectivity was proposed in [2, 3]. A graph G is *super connected, super-κ,* for short if every minimum separating set isolates a vertex of G.

The *direct product* $G_1 \times G_2$ of two graphs G_1 and G_2 is defined as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. The direct product is also called the Kronecker product, tensor product, cross product, categorical product, or conjunction. As an operation on binary relations, the direct product was introduced by Whitehead and Russell in their Principia Mathematica [21]. It is also equivalent to the direct product of the adjacency matrices of the graphs (see [20]). As one of the four standard graph products [11], the direct product has been studied from several points of view (see, for example, [1, 6, 8, 12, 13, 15]).

The connectivity of the direct product of graphs has also been investigated in several recent publications. For example, Brešar and Špacapan [7] obtained an upper bound and a lower bound on the edge-connectivity of the direct products with some exceptions, and they also obtained several upper bounds on the vertex-connectivity of the direct products of graphs. Mamut and Vumar [14] proved that $\kappa(K_m \times K_n) = (m-1)(n-1)$ where $m \geq n \geq 2$. In [9], it was shown that if $n \geq 3$ and G is a bipartite graph, then $\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\}$, and furthermore, the authors also conjectured that this is true for all nontrivial graph G. Later, this conjecture was confirmed independently by Wang and Wu [17] and Wang and Xue [18].

More recently, several papers dealing with the super-connectivity of direct product of graphs were published. Guo et al. [10] showed that for a bipartite graph G with $\kappa(G) = \delta(G)$, $G \times K_n (n \geq 3)$ is super-κ. In [19], the authors generalized this result by showing that for a nonbipartite graph G with $\kappa(G) = \delta(G)$, $G \times K_n (n \geq 3)$ is super-κ. In [19], the authors also pointed out that Guo et al.'s result is not true when $G = K_{\ell,\ell} (\ell \geq 1)$ and $n = 3$, and they also claimed that except for this case, Guo et al.'s statement is true.

The aim of this article is to determine all graphs G such that $G \times K_n (n \geq 3)$ is not super-κ. The following is the main result.

Theorem 1.1. *Let $\Gamma = G \times K_n$, where $n \geq 3$ and G is a non-trivial graph. Then Γ is not super-κ if and only if one of the following happens.*

(1) *G has a minimum separating set T so that $T \times V(K_n)$ is a minimum separating set of Γ. In particular, $\kappa(\Gamma) = n\kappa(G)$.*

(2) *$\Gamma \cong K_{\ell,\ell} \times K_3 (\ell > 0)$.*

From Theorem 1.1 we can immediately obtain the following corollaries.

Corollary 1.2. [17, 18] *Let $\Gamma = G \times K_n$, where $n \geq 3$ and G is a non-trivial graph. Then $\kappa(\Gamma) = \min\{n\kappa(G), (n-1)\delta(G)\}$.*

Corollary 1.3. [10, 19] *For a maximally connected graph G, $G \times K_n (n \geq 3)$ is not super-κ if and only if $n = 3$ and $G \cong K_{\ell,\ell}(\ell > 0)$.*

2 Proof of Theorem 1.1

We start by introducing some notations.

Notations.

- $\Gamma := G \times K_n$, where $n \geq 3$ and G is a non-trivial graph.
- $V(G) := \{u_i \mid i \in \mathbb{Z}_m\}$.
- $V(K_n) := \mathbb{Z}_n$.
- $V_i := \{u_i\} \times V(K_n), i \in \mathbb{Z}_m$.
- S: a minimum separating set of Γ.
- $\Gamma - S := \bigcup_{i=0}^{s-1} \Gamma_i$, where each Γ_i is a connected component of $\Gamma - S$.
- $W_i := V(\Gamma_i)$.

In the following Lemmas 2.1-2.5, we assume that Γ is not super connected, and S is a minimum separating set of Γ with each component Γ_i of $\Gamma - S$ having at least two vertices.

By the definition, we can obtain the following easy facts.

Lemma 2.1. (1) $\delta(\Gamma) = (n-1)\delta(G)$.

(2) *For any $i \in \mathbb{Z}_m$, V_i is an independent subset of $V(\Gamma)$.*

(3) *If u_{i_0} is adjacent to v_{i_1} in G, then $(u_{i_0}, j) \sim (u_{i_1}, k)$(in Γ) if and only if $j \neq k$. In particular, $\Gamma[V_{i_0} \cup V_{i_1}] \cong K_{n,n} - nK_2$.*

(4) *Let T be a separating set of G. Then $T \times V(K_n)$ is also a separating set of Γ. In particular, $|S| = \kappa(\Gamma) \leq \min\{n\kappa(G), (n-1)\delta(G)\}$.*

(5) *$s \geq 2$ and $|W_i| \geq 2$ for each $i \in \mathbb{Z}_s$.*

Lemma 2.2. *For each $(u_i, j) \in S$, (u_i, j) has at least one neighbor in W_i for each $i \in \mathbb{Z}_s$.*

Proof. Suppose to the contrary that (u_i, j) has no neighbors in W_i for some $i \in \mathbb{Z}_s$. Set $S' = S - \{(u_i, j)\}$. Then W_i must be a component of $\Gamma - S'$. This implies that S' is also a separating set of Γ, contrary to the minimality of S. □

Lemma 2.3. *For two components W_k, W_ℓ, if there exist $(u_i, i') \in W_k$ and $(u_j, j') \in W_\ell$ such that $u_i \sim u_j$(in G), then $i' = j'$ and $W_k \cap V_i = \{(u_i, i')\}$ and $W_\ell \cap V_j = \{(u_j, j')\}$.*

Proof. Since $u_i \sim u_j$(in G), it follows from Lemma 2.1 (3) that $\Gamma[V_i \cup V_j] \cong K_{n,n} - nK_2$. As $\Gamma - S$ is disconnected, there are no edges between W_k and W_ℓ. Consequently, $i' = j'$ and $W_k \cap V_i = \{(u_i, i')\}$ and $W_\ell \cap V_j = \{(u_j, j')\}$. □

Lemma 2.4. *Assume that for each V_i there exists at most one W_j such that $V_i \cap W_j \neq \emptyset$. Then $S = T \times V(K_n)$, where T is a minimum separating set of G. In particular, $\kappa(\Gamma) = n\kappa(G)$.*

Proof. We shall first show the following two claims.

Claim 1 If there exists an $i \in \mathbb{Z}_m$ such that $V_i \cap S \neq \emptyset$ and $V_i \not\subseteq S$, then $|V_i \cap S| = n-1$. Furthermore, for each W_j, there is a V_ℓ such that $|W_j \cap V_\ell| = 1$.

By the assumption, there is a unique $j \in \mathbb{Z}_s$ such that $V_i \cap W_j \neq \emptyset$. By Lemma 2.2, for each vertex, say (u_i, i'), in $V_i \cap S$, there is at least one neighbor, say (u_ℓ, ℓ'), in each W_t with $t \neq j$. By Lemma 2.3, $|V_i \cap W_j| = |V_\ell \cap W_t| = 1$. From our assumption we know that $|V_i \cap S| = |V_\ell \cap S| = n-1$.

Claim 2 For each $i \in \mathbb{Z}_m$, either $V_i \cap S = \emptyset$ or $V_i \subseteq S$.

Suppose on the contrary that there exists an $i \in \mathbb{Z}_m$ such that $V_i \cap S \neq \emptyset$ and $V_i \not\subseteq S$. For each $j \in \mathbb{Z}_s$, let $\Omega_j = \{\ell \in \mathbb{Z}_m \mid |V_\ell \cap W_j| = 1\}$, and set $n_j = |\Omega_j|$. By Claim 1, $n_j > 0$. Without loss of generality, assume that $n_0 \leq n_1 \leq \ldots \leq n_{s-1}$.

Assume that $W_0 \subseteq \bigcup_{\ell \in \Omega_0} V_\ell$. Then for each $(u_i, \ell) \in W_0$, we have $|V_i \cap W_0| = 1$. Combining this with Lemma 2.3, we have for a fixed $(u_i, \ell) \in W_0$, if $u_k \sim u_i$(in G), then $|V_k \cap S| \geq n-1$. As $n \geq 3$, one has

$$|S| \geq |V_i \cap S| + \sum_{u_j \in N_G(u_i)} |V_j \cap S| \geq n - 1 + \delta(G)(n-1) > \kappa(\Gamma).$$

A contradiction occurs.

Now assume that $W_0 \not\subseteq \bigcup_{\ell \in \Omega_0}$. Let $U = \bigcup_{V_i \subseteq S} V_i$, $Z_0 = \bigcup_{\ell \in \Omega_0} V_\ell$, and $Z_1 = \bigcup_{\ell \in \Omega_1} V_\ell$. Set $T = U \cup Z_0$. Clearly, $|Z_0 \cap W_0| = n_0$. Since $n_1 \geq n_0$ and $n \geq 3$, one has $|Z_0 \cap W_0| = n_0 < n_1(n-1) = |Z_1 \cap S|$. Then

$$\begin{aligned} |T| = |U| + |Z_0| &= |U| + |Z_0 \cap S| + |Z_0 \cap W_0| \\ &< |U| + |Z_0 \cap S| + |Z_1 \cap S| \\ &\leq |S|. \end{aligned}$$

Since S is a minimum separating set, $\Gamma - T$ is connected. So there is an edge between $W_0 \setminus T$ and $V(\Gamma) \setminus (T \cup W_0)$. We may assume that $(v_i, j) \in W_0 \setminus T$ is adjacent to $(v_s, k) \in V(\Gamma) \setminus (T \cup W_0)$. Obviously, $(v_s, k) \in S \setminus T$. Since $U = \bigcup_{V_i \subseteq S} V_i$ and $T = U \cup Z_0$, one has $V_s \not\subseteq S$. If $V_s \cap W_0 \neq \emptyset$, then by Claim 1, we must have $|V_s \cap W_0| = 1$ and so $V_s \subseteq Z_0 \subseteq T$. This contradicts the fact that $(v_s, k) \in S \setminus T$. Consequently, $V_s \cap W_0 = \emptyset$. It follows that $V_s \cap W_t \neq \emptyset$ for some $t > 0$. Since $(v_i, j) \sim (v_s, k)$, by Lemma 2.3, $|V_i \cap W_0| = 1$ and so $V_i \subseteq Z_0 \subseteq T$. This contradicts the fact that $(v_i, j) \in W_0 \setminus T$.

Now we are ready to finish the proof. From Claim 2 it follows that $S = T \times V(K_n)$ for some subset T of $V(G)$. Since $n \geq 3$, T is a separating set of G (see [20]). So, $|S| \geq \kappa(G)n$. However, by Lemma 2.1 (4), $|S| \leq n\kappa(G)$. Hence, $|S| = n\kappa(G)$. □

Lemma 2.5. *Assume that there exist a V_i and two different W_{j_0}, W_{j_1} such that $V_i \cap W_k \neq \emptyset$ with $k = j_0, j_1$. Then $n = 3$ and $G \cong K_{\ell,\ell} (\ell > 0)$.*

Proof. Recall that $W_k = V(\Gamma_k)$ with $k = j_0$ or j_1. We shall finish the proof by the following claims.

Claim 1 $V(\Gamma) = W_{j_0} \cup W_{j_1} \cup S$, $|V_i \cap W_{j_0}| = |V_i \cap W_{j_1}| = 1$ and $|V_i \cap S| = n-2$.

By Lemma 2.1 (2), V_i is an independent subset, and by Lemma 2.1 (5), $|W_k| \geq 2$ with $k = j_0$ or j_1. It follows that $V_i \cap W_k \subset W_k$. Since Γ_{j_0} is connected, there exist $(u_i, t_0) \in V_i \cap W_{j_0}$ and $(u_{i_0}, t_0') \in W_{j_0} \setminus (V_i \cap W_{j_0})$ such that $(u_i, t_0) \sim (u_{i_0}, t_0')$. Similarly, there exist $(u_i, t_1) \in V_i \cap W_{j_1}$ and $(u_{i_1}, t_1') \in W_{j_1} \setminus (V_i \cap W_{j_1})$ such that $(u_i, t_1) \sim (u_{i_1}, t_1')$. From Lemma 2.3 we obtain that $t_0' = t_1$, $V_i \cap W_{j_0} = \{(u_i, t_0)\}$ and $V_{i_0} \cap W_{j_0} = \{(u_{i_0}, t_1)\}$, and $t_1' = t_0$, $V_i \cap W_{j_1} = \{(u_i, t_1)\}$ and $V_{i_1} \cap W_{j_1} = \{(u_{i_1}, t_0)\}$ (see Figure 1). In particular, we have $|V_i \cap W_{j_0}| = |V_i \cap W_{j_1}| = 1$.

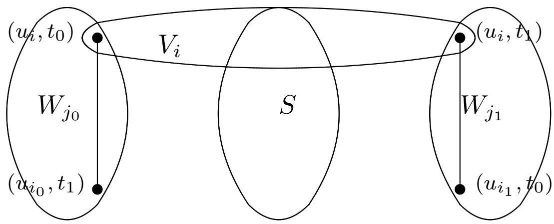

Figure 1: Explanation of the proof of Claim 1

It follows that $|V_i \cap S| \leq n - 2$. If $|V_i \cap S| < n - 2$, then we would have $V_i \cap W_j \neq \emptyset$ for some $j \neq j_0, j_1$. Take $(u_i, t) \in V_i \cap W_j$. Clearly, $t \neq t_0, t_1$. This forces that $(u_i, t) \sim (u_{i_0}, t_1)$, contrary to the fact that Γ_{j_0} and Γ_j are two distinct components. Thus, $|V_i \cap S| = n - 2$.

At last, we shall show that $s = 2$. Suppose to the contrary that $s > 2$. Since $|V_i \cap S| = n - 2 > 0$, we can take $(u_i, j) \in V_i \cap S$. By Lemma 2.2, (u_i, j) has a neighbor, say (u_k, j') in each W_j with $j \neq j_0, j_1$. Since $t_0 \neq t_1$, either $(u_k, j') \sim (u_i, t_0)$ or $(u_k, j') \sim (u_i, t_1)$. This is again contrary to the fact that $\Gamma_{j_0}, \Gamma_{j_1}$ and Γ_j are three distinct components. Thus, $s = 2$ and hence $V(\Gamma) = W_{j_0} \cup W_{j_1} \cup S$.

By Claim 1, we may assume that $V_i \cap W_{j_0} = \{(u_i, t_0)\}$ and $V_i \cap W_{j_1} = \{(u_i, t_1)\}$.

Claim 2 For each $(u_j, \ell) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$, $|V_j \cap S| = n - 1$ or $n - 2$. There is at least one $(u_{i_0}, \ell_0) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$ such that $|V_{i_0} \cap S| = n - 2$.

Take $(u_j, \ell) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$. From Claim 1 we see that $V_i \cap W_{j_1} = \{(u_i, t_1)\}$. By Lemma 2.3, we have $|V_j \cap W_{j_0}| = 1$, implying $|V_j \cap S| \leq n - 1$. If $|V_j \cap S| < n - 1$ then we must have $V_j \cap W_{j_1} \neq \emptyset$. By Claim 1, we have $|V_j \cap S| = n - 2$. Therefore, $|V_j \cap S| = n - 1$ or $n - 2$.

Suppose that for each $(u_j, \ell) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$, we have $|V_j \cap S| = n - 1$. Noting that $n \geq 3$, one has

$$|S| \geq |V_i \cap S| + \sum_{u_j \in N_G(u_i)} |U_j \cap S| \geq n - 2 + \delta(G)(n-1) > \kappa(\Gamma),$$

a contradiction. Thus, there is at least one $(u_{i_0}, \ell_0) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$ such that $|V_{i_0} \cap S| = n - 2$.

Now we know that Claim 2 holds. Since $(u_{i_0}, \ell_0) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$, it follows from Lemma 2.3 that $V_{i_0} \cap W_{j_0} = \{(u_{i_0}, t_1)\}$ and $V_{i_0} \cap W_{j_1} = \{(u_{i_0}, t_0)\}$ (see Figure 2). By the arbitrariness of V_i, Claims 1,2 also hold if we replace V_i by V_{i_0}.

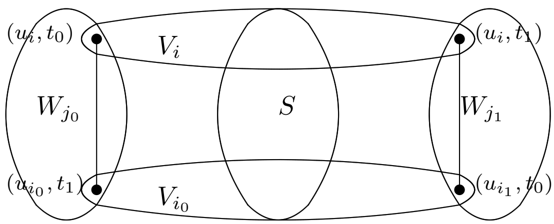

Figure 2: Explanation of Claims 1,2

For the convenience of statement, we shall use the following notations in the remainder of the proof.

Notations

(1) $N_i = N_\Gamma((u_i, t_0)) \cap W_{j_0}$,
(2) $N_{i_0} = N_\Gamma((u_{i_0}, t_1)) \cap W_{j_0}$,
(3) $\Omega_i = \{k \in \mathbb{Z}_m \mid V_k \cap N_i \neq \emptyset\}$,
(4) $\Omega_{i_0} = \{k \in \mathbb{Z}_m \mid V_k \cap N_{i_0} \neq \emptyset\}$,
(5) $\Delta_i = \{k \in \mathbb{Z}_m \mid k \notin \Omega_i, V_k \cap N_\Gamma((u_i, t_0)) \neq \emptyset\}$,
(6) $\Delta_{i_0} = \{k \in \mathbb{Z}_m \mid k \notin \Omega_{i_0}, V_k \cap N_\Gamma((u_{i_0}, t_1)) \neq \emptyset\}$.

It is easy to see that $N_G(u_i) = \{u_k \mid k \in \Omega_i \cup \Delta_i\}$ and $N_G(u_{i_0}) = \{u_k \mid k \in \Omega_{i_0} \cup \Delta_{i_0}\}$. Hence, $|\Omega_i| + |\Delta_i| = d_G(u_i)$ and $|\Omega_{i_0}| + |\Delta_{i_0}| = d_G(u_{i_0})$.

Claim 3 $|N_i| = |\Omega_i|$ and $|N_{i_0}| = |\Omega_{i_0}|$.

By Claim 1, for each $k \in \Omega_i$, we have $|V_k \cap W_{j_0}| = 1$. It follows that $|N_i| = |\Omega_i|$. Similarly, $|N_{i_0}| = |\Omega_{i_0}|$.

Claim 4 Both N_i and N_{i_0} are independent subsets of $V(\Gamma)$.

Take any two vertices, say $(u_{i_1}, t), (u_{i_2}, t')$ in N_i. Since $V_i \cap W_{j_1} = \{(u_i, t_1)\}$, from Lemma 2.3 it follows that $t = t' = t_1$. So, (u_{i_1}, t) is not adjacent to (u_{i_2}, t'). Therefore, N_i is an independent subset of $V(\Gamma)$. Similarly, N_{i_0} is also an independent subset.

Claim 5 $(\bigcup_{k\in\Omega_i\cup\Delta_i} V_k) \cap (\bigcup_{k\in\Omega_{i_0}} V_k) = \emptyset$ and $(\bigcup_{k\in\Omega_{i_0}\cup\Delta_{i_0}} V_k) \cap (\bigcup_{k\in\Omega_i} V_k) = \emptyset$.

Suppose that $(\bigcup_{k\in\Omega_i\cup\Delta_i} V_k) \cap (\bigcup_{k\in\Omega_{i_0}} V_k) \neq \emptyset$. Take $(u_j, t) \in (\bigcup_{k\in\Omega_i\cup\Delta_i} V_k) \cap (\bigcup_{k\in\Omega_{i_0}} V_k)$. Then $V_j \cap N_{i_0} \neq \emptyset$, implying that $V_j \cap W_{j_0} \neq \emptyset$. Assume $(u_j, t') \in V_j \cap W_{j_0}$. Clearly, u_i and u_{i_0} are neighbors of u_j in G. Since $t_0 \neq t_1$, either $(u_j, t') \sim (u_i, t_1)$ or $(u_j, t') \sim (u_{i_0}, t_0)$. This is contrary to the fact that there are no edges between W_{j_0} and W_{j_1}. Thus, $(\bigcup_{k\in\Omega_i\cup\Delta_i} V_k) \cap (\bigcup_{k\in\Omega_{i_0}} V_k) = \emptyset$. Similarly, we have $(\bigcup_{k\in\Omega_{i_0}\cup\Delta_{i_0}} V_k) \cap (\bigcup_{k\in\Omega_i} V_k) = \emptyset$.

Claim 6 Let $k \in \Delta_i \cup \Delta_{i_0}$. Then $V_k \cap W_{j_0} = \emptyset$ and $|V_k \cap S| \geq n - 1$.

Assume $k \in \Delta_i$. Then $u_k \in N_G(u_i)$. If $V_k \cap W_{j_0} \neq \emptyset$, then take $(u_k, t) \in V_k \cap W_{j_0}$. Since $k \notin \Omega_i$, (u_k, t) is not adjacent to (u_i, t_0), and hence $t = t_0$. Consequently, $(u_k, t) \sim (u_i, t_1)$, a contradiction. Thus, $V_k \cap W_{j_0} = \emptyset$. By Lemma 2.3, $|V_k \cap W_{j_1}| \leq 1$, and hence $|V_k \cap S| \geq n - 1$. With a similar argument, we can show that if $k \in \Delta_{i_0}$, then $V_k \cap W_{j_0} = \emptyset$ and $|V_k \cap S| \geq n - 1$.

Claim 7 (1) $n = 3$; (2) $|N_i| = |N_{i_0}| = \delta(G)$; (3) $|\Delta_i| = |\Delta_{i_0}| = 0$; (4) $|S| = 2\delta(G)$; (5) $S = \bigcup_{k\in\Omega_i\cup\Omega_{i_0}}(V_k\cap S)$; (6) for each $k \in \Omega_i\cup\Omega_{i_0}$, $|V_k\cap S| = |V_k\cap W_{j_0}| = |V_k\cap W_{j_1}| = 1$.

By the arbitrariness of V_i and V_{i_0}, we may assume that $|N_i| \le |N_{i_0}|$. By Claim 5,

$$|S| \ge |\bigcup_{k\in\Omega_{i_0}}(V_k\cap S)| + |(\bigcup_{k\in\Omega_i\cup\Delta_i}(V_k\cap S))|. \tag{2.1}$$

By Claim 2, if $k \in \Omega_i \cup \Omega_{i_0}$, then $|V_k \cap S| \ge n-2$, and by Claim 6, if $k \in \Delta_i \cup \Delta_{i_0}$, then $|V_k \cap S| \ge n-1$. It follows that

$$|S| \ge (n-2)|\Omega_{i_0}| + (n-2)|\Omega_i| + (n-1)|\Delta_i|. \tag{2.2}$$

By Claim 3, we have

$$|S| \ge (n-2)|N_{i_0}| + (n-2)|N_i| + (n-1)|\Delta_i|. \tag{2.3}$$

Since $|N_{i_0}| \ge |N_i|$ and $n \ge 3$, we obtain that

$$|S| \ge 2(n-2)|N_i| + (n-1)|\Delta_i| \ge (n-1)d_G(u_i). \tag{2.4}$$

However, by Lemma 2.1 (4), we have $|S| \le (n-1)\delta(G)$. So, in the above four inequalities, "=" must hold. By Eq. (2.4) we obtain that $n = 3$, $|N_i| = |N_{i_0}|$, and $\delta(G) = d_G(u_i)$. Furthermore, for each $k \in \Omega_i\cup\Omega_{i_0}$, $|V_k\cap S| = n-1$, and for each $k \in \Delta_i$, $|V_k\cap S| = n-1$. It follows that

$$|S| = 2|N_i| + 2|\Delta_i| = 2\delta(G). \tag{2.5}$$

To show that $|\Delta_i| = |\Delta_{i_0}| = 0$, we shall first show that

$$(\bigcup_{k\in\Delta_i}V_k)\cap(\bigcup_{k\in\Omega_{i_0}\cup\Delta_{i_0}}V_k) = \emptyset.$$

Suppose on the contrary that for some $k \in \Delta_i$, $V_k\cap(\bigcup_{k\in\Omega_{i_0}\cup\Delta_{i_0}}V_k) \ne \emptyset$. Since $|V_k\cap S| = n-1$, from Claim 6 it follows that $|V_k \cap W_{j_1}| = 1$. Take $(u_k, t) \in V_k \cap W_{j_1}$. Then u_i, u_{i_0} are neighbors of u_k in G. Since $t_0 \ne t_1$, either $(u_k, t) \sim (u_i, t_1)$ or $(u_k, t) \sim (u_{i_0}, t_0)$. This is contrary to the fact that there are no edges between W_0 and W_1. Thus, $(\bigcup_{k\in\Delta_i}V_k)\cap(\bigcup_{k\in\Omega_{i_0}\cup\Delta_{i_0}}V_k) = \emptyset$. It follows that

$$\begin{aligned}|S| &\ge |\textstyle\bigcup_{k\in\Omega_{i_0}\cup\Delta_{i_0}}(V_k\cap S)| + |(\bigcup_{k\in\Omega_i\cup\Delta_i}(V_k\cap S))|\\ &= |\textstyle\bigcup_{k\in\Omega_{i_0}}(V_k\cap S)| + |(\bigcup_{k\in\Omega_i\cup\Delta_i}(V_k\cap S))| + |\bigcup_{k\in\Delta_{i_0}}(V_k\cap S)|\\ &\ge 2|N_i| + 2|\Delta_i| + 2|\Delta_{i_0}|\\ &= 2\delta(G) + 2|\Delta_{i_0}|.\end{aligned}$$

Combining this with Eq. (2.5) we obtain that $|\Delta_{i_0}| = 0$. Since $d_G(u_i) = \delta(G)$, we have $d_G(u_{i_0}) \ge d_G(u_i)$. Recall that $|N_i| + |\Delta_i| = d_G(u_i)$ and $|N_{i_0}| + |\Delta_{i_0}| = d_G(u_{i_0})$. Since $|N_i| = |N_{i_0}|$, one has $|\Delta_{i_0}| \ge |\Delta_i|$, implying $|\Delta_i| = 0$.

At last, from Eq. (2.1) it can be deduced that

$$S = (\bigcup_{k\in\Omega_{i_0}}(V_k\cap S)) \cup (\bigcup_{k\in\Omega_i}(V_k\cap S)). \tag{2.6}$$

Claim 8 $W_{j_0} = N_i \cup N_{i_0}$.

Suppose that $W_{j_0} \neq N_i \cup N_{i_0}$. Since Γ_0 is a component of $\Gamma - S$, we can take a vertex, say (v_{k_1}, t), in $W_{j_1} - (N_i \cup N_{i_0})$ such that (v_{k_1}, t) is adjacent to some vertex, say (v_{k_2}, t'), in $N_i \cup N_{i_0}$. Since $(v_{k_2}, t') \in N_i \cup N_{i_0}$, by Claim 7 $|V_{k_2} \cap W_{j_1}| = 1$. By Lemma 2.3, we have $|V_{k_1} \cap W_{j_0}| = 1$. By Claim 1, we have $V_{k_1} \cap S \neq \emptyset$. From Eq. (2.6) we see that $k_1 \in \Omega_{i_0} \cup \Omega_i$, and hence $(v_{k_1}, t) \in N_i \cup N_{i_0}$, a contradiction. Thus, $W_{j_0} = N_i \cup N_{i_0}$.

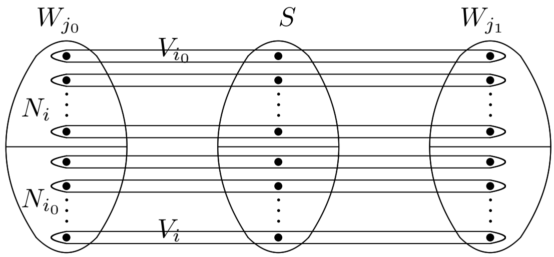

Figure 3: Explanation of Claims 8,9

Claim 9 $m = |G| = 2\delta(G)$.

By Claim 8, $|W_{j_0}| = 2\delta(G)$. So, $m \geq |W_{j_0}| = 2\delta(G)$. Suppose that $m > 2\delta(G)$. By Claim 7, for each $k \in \Omega_i \cup \Omega_{i_0}$, $|V_k \cap S| = |V_k \cap W_{j_0}| = |V_k \cap W_{j_1}| = 1$, and $S = \bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap S)$. This implies that $\bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$ is a proper subset of W_{j_1}. By the connectedness of Γ_1, take an edge e in Γ_1 such that one end, say (u_{k_1}, t), of e is in $\bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$ and the other end, say (u_{k_2}, t'), is in $W_{j_1} \setminus \bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$. By Claim 7, $|V_{k_1} \cap W_{j_0}| = 1$, and by Lemma 2.3, we have $|V_{k_2} \cap W_{j_1}| = 1$. By Claim 1, $|V_{k_2} \cap S| \geq 1$. It follows from Eq. (2.6) that $k_2 \in \Omega_{i_0} \cup \Omega_i$. This forces that $(u_{k_2}, t') \in \bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$, a contradiction.

Claim 10 $G \cong K_{\ell,\ell}$, where $\ell = \delta(G)$.

Clearly, $\{u_k \mid k \in \Omega_i \cup \Omega_{i_0}\} \subseteq V(G)$. By Claim 9, $m = |G| = 2\delta(G)$. It follows that $V(G) = \{u_k \mid k \in \Omega_i \cup \Omega_{i_0}\}$. Set $B_0 = \{u_k \mid k \in \Omega_i\}$ and $B_1 = \{u_k \mid k \in \Omega_{i_0}\}$. Take any two vertices, say u_{k_1} and u_{k_2}, in B_0. Suppose $u_{k_1} \sim u_{k_2}$. By Claim 7, we may assume that $V_{k_i} \cap W_{j_0} = \{(u_{k_i}, d_i)\}$ with $i = 1$ or 2. From Claim 4 we obtain that (u_{k_1}, d_1) is not adjacent to (u_{k_2}, d_2), and hence $d_1 = d_2$. Since $u_{k_1} \sim u_{k_2}$, (u_{k_1}, d_1) is adjacent to all the remaining vertices in V_{k_2}. Again, by Claim 7, we get that $|V_{k_2} \cap W_{j_1}| = 1$. This implies that there is an edge between W_{j_0} and W_{j_1}, a contradiction. Therefore, u_{k_1} and u_{k_2} are nonadjacent. By the arbitrariness of u_{k_1} and u_{k_2}, we get that B_0 is an independent subset of $V(G)$. Similarly, B_1 is also an independent subset of $V(G)$. It follows that G must be a bipartite graph with two partition sets B_0 and B_1. By Claims 3,7, we know that $|B_0| = |B_1| = \delta(G)$. This means that $G \cong K_{\ell,\ell}$, where $\ell = \delta(G)$. □

Lemma 2.6. *Let ℓ be a positive integer. Then $K_{\ell,\ell} \times K_3$ is not super-κ.*

Proof. Let $B_0 = \{v_i \mid i \in \mathbb{Z}_\ell\}$ and $B_1 = \{u_i \mid i \in \mathbb{Z}_\ell\}$ be the two partition sets of $K_{\ell,\ell}$. Set $V(K_3) = \mathbb{Z}_3$. Let $S = V(K_{\ell,\ell}) \times \{1\}$. Clearly, $|S| = 2\ell$. By [9], $\kappa(K_{\ell,\ell} \times K_3) = 2\ell$.

Set $W_0 = (B_0 \times \{0\}) \cup (B_1 \times \{2\})$ and $W_1 = (B_0 \times \{2\}) \cup (B_1 \times \{0\})$. Clearly, $V(K_{\ell,\ell} \times K_3) = S \cup W_0 \cup W_1$. It is also easy to see that $\Gamma[W_i] \cong K_{\ell,\ell}$ for $i = 0, 1$.

Furthermore, in $K_{\ell,\ell} \otimes K_3$ there are no edges between W_0 and W_1. It follows that $K_{\ell,\ell} \times K_3 - S$ is disconnected with no isolated vertices. Therefore, $K_{\ell,\ell} \times K_3$ is not super-κ. $\square$

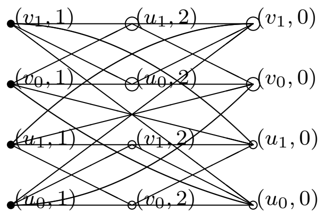

Figure 4: $K_{2,2} \times K_3$

Proof of Theorem 1.1. By Lemmas 2.4 and 2.5, we can get the necessity. For the sufficiency, by Lemma 2.6, $K_{\ell,\ell} \times K_3$ is not super-κ.

Now assume that $\kappa(\Gamma) = n\kappa(G)$. Suppose to the contrary that Γ is super-κ. Then $\kappa(\Gamma) = \delta(\Gamma) = (n-1)\delta(G)$, and hence $(n-1)\delta(G) = n\kappa(G)$. So, $\kappa(G) < \delta(G)$. Let T be a minimum separating set of G. Then $G - T$ has no isolated vertices. By Lemma 2.1 (4), $T \times V(K_n)$ is a separating set of Γ. Clearly, $|T \times V(K_n)| = n\kappa(G)$. So, $T \times V(K_n)$ is also a minimum separating set of G. Since Γ is super-κ, $T \times V(K_n)$ must be the neighborhood of some vertex, say (u_i, j). Let $u_k \in T$. Then $(u_k, j) \in T \times V(K_n)$, and hence $(u_i, j) \sim (u_k, j)$. This is clearly impossible by the definition of the direct product of graphs. Thus, Γ is not super-κ. $\square$

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Mixed fault diameter of Cartesian graph bundles II

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Abstract

The mixed fault diameter $\mathcal{D}_{(p,q)}(G)$ is the maximum diameter among all subgraphs obtained from graph G by deleting p vertices and q edges. A graph is (p,q)+connected if it remains connected after removal of any p vertices and any q edges. Let F be a connected graph with the diameter $\mathcal{D}(F) > 1$, and B be (p,q)+connected graph. Upper bounds for the mixed fault diameter of Cartesian graph bundle G with fibre F over the base graph B are given. We prove that if $q > 0$, then $\mathcal{D}_{(p+1,q)}(G) \leq \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$, and if $q = 0$ and $p > 0$, then $\mathcal{D}_{(p+1,0)}(G) \leq \mathcal{D}(F) + \max\{\mathcal{D}_{(p,0)}(B), \mathcal{D}_{(p-1,1)}(B)\}$.

Keywords: Mixed fault diameter, Cartesian graph bundle, interconnection network, fault tolerance.

Math. Subj. Class.: 05C12, 05C40, 68M10, 68M15, 68R10

1 Introduction

Graph products and bundles belong to a class of frequently studied interconnection network topologies. For example meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some other well-known interconnection network topologies are Cartesian graph bundles, for example twisted hypercubes [9, 12] and multiplicative circulant graphs [23].

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In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and that the delays in communication must not be too long. Furthermore, an interconnection network should be fault tolerant, because practical communication networks are exposed to failures of network components. Both failures of nodes and failures of connections between them happen and it is desirable that a network is robust in the sense that a limited number of failures does not break down the whole system. A lot of work has been done on various aspects of network fault tolerance, see for example the survey [8] and the more recent papers [16, 24, 26]. In particular the fault diameter with faulty vertices, which was first studied in [18], and the edge fault diameter have been determined for many important networks recently [2, 3, 4, 5, 10, 11, 19, 25]. Usually either only edge faults or only vertex faults are considered, while the case when both edges and vertices may be faulty is studied rarely. For example, [16, 24] consider Hamiltonian properties assuming a combination of vertex and edge faults. In recent work on fault diameter of Cartesian graph products and bundles [2, 3, 4, 5], analogous results were found for both fault diameter and edge fault diameter. However, the proofs for vertex and edge faults are independent, and our effort to see how results in one case may imply the others was not successful. A natural question is whether it is possible to design a uniform theory that covers simultaneous faults of vertices and edges. Some basic results on edge, vertex and mixed fault diameters for general graphs appear in [6]. In order to study the fault diameters of graph products and bundles under mixed faults, it is important to understand generalized connectivities. Mixed connectivity which generalizes both vertex and edge connectivity, and some basic observations for any connected graph are given in [13]. We are not aware of any earlier work on mixed connectivity. A closely related notion is the connectivity pairs of a graph [7], but after Mader [20] showed the claimed proof of generalized Menger's theorem is not valid, work on connectivity pairs seems to be very rare.

An upper bound for the mixed fault diameter of Cartesian graph bundles is given in [14] that in some case also improves previously known results on vertex and edge fault diameters on these classes of Cartesian graph bundles [2, 5]. However these results address only the number of faults given by the connectivity of the fibre (plus one vertex), while the connectivity of the graph bundle can be much higher when the connectivity of B is substantial. It seems obvious that the upper bound from [14] can be improved. In this paper we provide an upper bound that takes into account the mixed connectivity of the base graph B, i.e. the number of faults allowed is given by the connectivity of the base graph (plus one vertex), thus complementing the result of [14]. We show by examples that the bounds of the new result are tight. In addition, in some cases Theorem 4.6 also improves previously known results on vertex and edge fault diameters on these classes of Cartesian graph bundles [2, 5].

The rest of the paper is organized as follows. General definitions, in particular of the connectivities, are given in section Preliminaries. The third section introduces graph bundles and recalls relevant previous results. In Section 4, the proof of the main theorem is given, followed by a short discussion.

2 Preliminaries

A *simple graph* $G = (V, E)$ is determined by a *vertex set* $V = V(G)$ and a set $E = E(G)$ of (unordered) pairs of vertices, called *edges*. As usual, we will use the short notation

uv for edge $\{u, v\}$. For an edge $e = uv$ we call u and v its *endpoints*. It is sometimes convenient to consider the union of *elements* of a graph, $S(G) = V(G) \cup E(G)$. Given $X \subseteq S(G)$ then $S(G) \setminus X$ is a subset of elements of G. However, note that in general $S(G) \setminus X$ may not induce a graph. As we need notation for subgraphs with some missing (faulty) elements, we formally define $G \setminus X$, the subgraph of G after deletion of X, as follows:

Definition 2.1. Let $X \subseteq S(G)$, and $X = X_E \cup X_V$, where $X_E \subseteq E(G)$ and $X_V \subseteq V(G)$. Then $G \setminus X$ is the subgraph of $(V(G), E(G) \setminus X_E)$ induced on vertex set $V(G) \setminus X_V$.

A *walk* between vertices x and y is a sequence of vertices and edges v_0, e_1, v_1, e_2, v_2, $\ldots$, v_{k-1}, e_k, v_k where $x = v_0$, $y = v_k$, and $e_i = v_{i-1}v_i$ for each i. A walk with all vertices distinct is called a *path*, and the vertices v_0 and v_k are called the *endpoints* of the path. The *length* of a path P, denoted by $\ell(P)$, is the number of edges in P. The *distance* between vertices x and y, denoted by $d_G(x, y)$, is the length of a shortest path between x and y in G. If there is no path between x and y we write $d_G(x, y) = \infty$. The *diameter* of a connected graph G, $\mathcal{D}(G)$, is the maximum distance between any two vertices in G. A path P in G, defined by a sequence $x = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = y$ can alternatively be seen as a subgraph of G with $V(P) = \{v_0, v_1, v_2, \ldots, v_k\}$ and $E(P) = \{e_1, e_2, \ldots, e_k\}$. Note that the reverse sequence gives rise to the same subgraph. Hence we use P for a path either from x to y or from y to x. A graph is *connected* if there is a path between each pair of vertices, and is *disconnected* otherwise. In particular, K_1 is by definition disconnected. The *connectivity* (or *vertex connectivity*) $\kappa(G)$ of a connected graph G, other than a complete graph, is the smallest number of vertices whose removal disconnects G. For complete graphs is $\kappa(K_n) = n - 1$. We say that G is *k-connected* (or *k-vertex connected*) for any $0 < k \leq \kappa(G)$. The *edge connectivity* $\lambda(G)$ of a connected graph G, is the smallest number of edges whose removal disconnects G. A graph G is said to be *k-edge connected* for any $0 < k \leq \lambda(G)$. It is well known that (see, for example, [1], page 224) $\kappa(G) \leq \lambda(G) \leq \delta_G$, where δ_G is smallest vertex degree of G. Thus if a graph G is k-connected, then it is also k-edge connected. The reverse does not hold in general.

Here we are interested in mixed connectivity that generalizes both vertex and edge connectivity. Note that the definition used here slightly differs from the definition used in a previous work [13].

Definition 2.2. Let G be any connected graph. A graph G is (p, q)+*connected*, if G remains connected after removal of any p vertices and any q edges.

We wish to remark that the mixed connectivity studied here is closely related to connectivity pairs as defined in [7]. Briefly speaking, a connectivity pair of a graph is an ordered pair (k, ℓ) of two integers such that there is some set of k vertices and ℓ edges whose removal disconnects the graph and there is no set of $k - 1$ vertices and ℓ edges or of k vertices and $\ell - 1$ edges with this property. Clearly (k, ℓ) is a connectivity pair of G exactly when: (1) G is $(k - 1, \ell)$+connected, (2) G is $(k, \ell - 1)$+connected, and (3) G is not (k, ℓ)+connected. In fact, as shown in [13], (2) implies (1), so (k, ℓ) is a connectivity pair exactly when (2) and (3) hold.

From the definition we easily observe that any connected graph G is $(0, 0)$+connected, $(p, 0)$+connected for any $p < \kappa(G)$ and $(0, q)$+connected for any $q < \lambda(G)$. In our notation $(i, 0)$+connected is the same as $(i + 1)$-connected, i.e. the graph remains connected after removal of any i vertices. Similarly, $(0, j)$+connected means $(j + 1)$-edge connected, i.e. the graph remains connected after removal of any j edges.

Clearly, if G is a (p, q)+connected graph, then G is (p', q')+connected for any $p' \leq p$ and any $q' \leq q$. Furthermore, for any connected graph G with $k < \kappa(G)$ faulty vertices, at least k edges are not working. Roughly speaking, graph G remains connected if any faulty vertex in G is replaced with a faulty edge. It is known [13] that if a graph G is (p, q)+connected and $p > 0$, then G is $(p - 1, q + 1)$+connected. Hence for $p > 0$ we have a chain of implications: (p, q)+connected $\Longrightarrow$ $(p - 1, q + 1)$+connected $\Longrightarrow \ldots \Longrightarrow$ $(1, p + q - 1)$+connected $\Longrightarrow$ $(0, p + q)$+connected, which generalizes the well-known proposition that any k-connected graph is also k-edge connected. Therefore, a graph G is (p, q)+connected if and only if $p < \kappa(G)$ and $p + q < \lambda(G)$.

Note that by our definition the complete graph K_n, $n \geq 2$, is $(n-2, 0)$+connected, and hence (i, j)+connected for any $i + j \leq n - 2$. Graph K_2 is $(0, 0)$+connected, and mixed connectivity of K_1 is not defined.

If for a graph G $\kappa(G) = \lambda(G) = k$, then G is (i, j)+connected exactly when $i + j < k$. However, if $2 \leq \kappa(G) < \lambda(G)$, the question whether G is (i, j)+connected for $1 \leq i < \kappa(G) \leq i + j < \lambda(G)$ is not trivial. The example below shows that in general the knowledge of $\kappa(G)$ and $\lambda(G)$ is not enough to decide whether G is (i, j)+connected.

Example 2.3. For graphs on Fig. 1 we have $\kappa(G_1) = \kappa(G_2) = 2$ and $\lambda(G_1) = \lambda(G_2) = 3$. Both graphs are $(1, 0)$+connected $\Longrightarrow$ $(0, 1)$+connected, and $(0, 2)$+ connected. Graph G_1 is not $(1, 1)$+connected, while graph G_2 is.

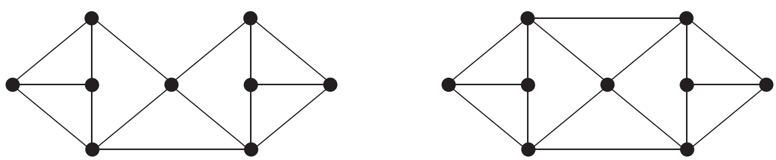

Figure 1: Graphs G_1 and G_2 from Example 2.3.

Definition 2.4. Let G be a k-edge connected graph and $0 \leq a < k$. The *a-edge fault diameter* of G is

$$\mathcal{D}^E_a(G) = \max\{\mathcal{D}(G \setminus X) \mid X \subseteq E(G), |X| = a\}.$$

Definition 2.5. Let G be a k-connected graph and $0 \leq a < k$. The *a-fault diameter* (or *a-vertex fault diameter*) of G is

$$\mathcal{D}^V_a(G) = \max\{\mathcal{D}(G \setminus X) \mid X \subseteq V(G), |X| = a\}.$$

Note that $\mathcal{D}^E_a(G)$ is the largest diameter among the diameters of subgraphs of G with a edges deleted, and $\mathcal{D}^V_a(G)$ is the largest diameter over all subgraphs of G with a vertices deleted. In particular, $\mathcal{D}^E_0(G) = \mathcal{D}^V_0(G) = \mathcal{D}(G)$, the diameter of G. For $p \geq \kappa(G)$ and for $q \geq \lambda(G)$ we set $\mathcal{D}^V_p(G) = \infty$, $\mathcal{D}^E_q(G) = \infty$, as some of the subgraphs are not vertex connected or edge connected, respectively.

It is known [6] that for any connected graph G the inequalities below hold.

1. $\mathcal{D}(G) = \mathcal{D}_0^E(G) \leq \mathcal{D}_1^E(G) \leq \mathcal{D}_2^E(G) \leq \ldots \leq \mathcal{D}_{\lambda(G)-1}^E(G) < \infty$.
2. $\mathcal{D}(G) = \mathcal{D}_0^V(G) \leq \mathcal{D}_1^V(G) \leq \mathcal{D}_2^V(G) \leq \ldots \leq \mathcal{D}_{\kappa(G)-1}^V(G) < \infty$.

Definition 2.6. Let G be a (p,q)+connected graph. The (p,q)-*mixed fault diameter* of G is

$$\mathcal{D}_{(p,q)}(G) = \max\{\mathcal{D}(G \setminus (X \cup Y)) \mid X \subseteq V(G), Y \subseteq E(G), |X| = p, |Y| = q\}.$$

Note that by Definition 2.6 the endpoints of edges of set Y can be in X. In this case we may get the same subgraph of G by deleting p vertices and fewer than q edges. It is however not difficult to see that the diameter of such subgraph is smaller than or equal to the diameter of some subgraph of G where exactly p vertices and exactly q edges are deleted. So the condition that the endpoints of edges of set Y are not in X need not to be included in Definition 2.6. The mixed fault diameter $\mathcal{D}_{(p,q)}(G)$ is the largest diameter among the diameters of all subgraphs obtained from G by deleting p vertices and q edges, hence $\mathcal{D}_{(0,0)}(G) = \mathcal{D}(G)$, $\mathcal{D}_{(0,a)}(G) = \mathcal{D}_a^E(G)$ and $\mathcal{D}_{(a,0)}(G) = \mathcal{D}_a^V(G)$.

Let $\mathcal{H}_a^V = \{G \setminus X \mid X \subseteq V(G), |X| = a\}$ and $\mathcal{H}_b^E = \{G \setminus X \mid X \subseteq E(G), |X| = b\}$. It is easy to see that

1. $\max\{\mathcal{D}_b^E(H) \mid H \in \mathcal{H}_a^V\} = \mathcal{D}_{(a,b)}(G)$,
2. $\max\{\mathcal{D}_a^V(H) \mid H \in \mathcal{H}_b^E\} = \mathcal{D}_{(a,b)}(G)$.

In previous work [6] on vertex, edge and mixed fault diameters of connected graphs the following theorem has been proved.

Theorem 2.7. *Let G be (p,q)+connected graph and $p > 0$.*

- *If $q > 0$, then $\mathcal{D}_{p+q}^E(G) \leq \mathcal{D}_{(1,p+q-1)}(G) \leq \cdots \leq \mathcal{D}_{(p,q)}(G)$.*
- *If $q = 0$, then $\mathcal{D}_p^E(G) \leq \mathcal{D}_{(1,p-1)}(G) \leq \cdots \leq \mathcal{D}_{(p-1,1)}(G) \leq \mathcal{D}_p^V(G) + 1$.*

3 Mixed fault diameter of Cartesian graph bundles

Cartesian graph bundles are a generalization of Cartesian graph products, first studied in [21, 22]. Let G_1 and G_2 be graphs. The *Cartesian product* of graphs G_1 and G_2, $G = G_1 \Box G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$. Vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$. For further reading on graph products we recommend [15].

Definition 3.1. Let B and F be graphs. A graph G is a *Cartesian graph bundle with fibre F over the base graph B* if there is a *graph map* $p : G \to B$ such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F, and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \Box K_2$.

More precisely, the mapping $p : G \to B$ maps graph elements of G to graph elements of B, i.e. $p : V(G) \cup E(G) \to V(B) \cup E(B)$. In particular, here we also assume that the vertices of G are mapped to vertices of B and the edges of G are mapped either to vertices or to edges of B. We say an edge $e \in E(G)$ is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*. The mapping p will also be called the *projection* (of the bundle G to its base B). Note that each edge $e = uv \in E(B)$ naturally induces an isomorphism $\varphi_e : p^{-1}(\{u\}) \to p^{-1}(\{v\})$ between two fibres. It may be interesting to note that while it is well-known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) [15], there may be many different graph bundle representations of the same graph [29]. Here we assume that the bundle representation is given. Note that in some cases finding a representation of G as a graph bundle can be found in polynomial time [17, 27, 28, 29, 30, 31]. For example, one of the easy classes are the Cartesian graph bundles over triangle-free base [17]. Note that a graph bundle over a tree T (as a base graph) with fibre F is isomorphic to the Cartesian product $T \Box F$ (not difficult to see, appears already in [21]), i.e. we can assume that all isomorphisms φ_e are identities. For a later reference note that for any path $P \subseteq B$, $p^{-1}(P)$ is a Cartesian graph bundle over the path P, and one can define coordinates in the product $P \Box F$ in a natural way.

In recent work on fault diameter of Cartesian graph products and bundles [2, 3, 4, 5], analogous results were found for both fault diameter and edge fault diameter.

Theorem 3.2. *[2] Let G be a Cartesian bundle with fibre F over the base graph B, graph F be $(a, 0)$+connected and graph B be $(b, 0)$+connected. Then*

$$\mathcal{D}^V_{a+b+1}(G) \leq \mathcal{D}^V_a(F) + \mathcal{D}^V_b(B) + 1.$$

Theorem 3.3. *[5] Let G be a Cartesian bundle with fibre F over the base graph B, graph F be $(0, a)$+connected and graph B be $(0, b)$+connected. Then*

$$\mathcal{D}^E_{a+b+1}(G) \leq \mathcal{D}^E_a(F) + \mathcal{D}^E_b(B) + 1.$$

Before writing a theorem on bounds for the mixed fault diameter we recall a theorem on mixed connectivity.

Theorem 3.4. *[13] Let G be a Cartesian graph bundle with fibre F over the base graph B, graph F be (p_F, q_F)+connected and graph B be (p_B, q_B)+connected. Then Cartesian graph bundle G is $(p_F + p_B + 1, q_F + q_B)$+connected.*

In recent work [14], an upper bound for the mixed fault diameter of Cartesian graph bundles, $\mathcal{D}_{(p+1,q)}(G)$, in terms of mixed fault diameter of the fibre and diameter of the base graph is given. Theorem 3.5 improves results 3.2 and 3.3 for $b = 0$.

Theorem 3.5. *[14] Let G be a Cartesian graph bundle with fibre F over the base graph B, where graph F is (p, q)+connected, $p + q > 0$, and B is a connected graph with diameter $\mathcal{D}(B) > 1$. Then we have:*

- *If $q > 0$, then $\mathcal{D}_{(p+1,q)}(G) \leq \mathcal{D}_{(p,q)}(F) + \mathcal{D}(B)$.*
- *If $q = 0$, then $\mathcal{D}^V_{p+1}(G) \leq \max\{\mathcal{D}^V_p(F), \mathcal{D}_{(p-1,1)}(F)\} + \mathcal{D}(B)$.*

Here we prove a similar result for an upper bound for the mixed fault diameter of Cartesian graph bundles, $\mathcal{D}_{(p+1,q)}(G)$, in terms of diameter of the fibre and mixed fault diameter of the base graph. We consider mixed fault diameter of Cartesian graph bundle G with connected fibre F. If the graph B is (p, q)+connected then Cartesian graph bundle with connected fibre F over the base graph B is at least $(p + 1, q)$+connected. Theorem 4.6 improves results 3.2 and 3.3 for $a = 0$.

4 Proof of the main theorem

Before stating and proving the main theorem, we prove several lemmas and introduce some notation used in this section.

Let G be a Cartesian graph bundle with fibre F over the base graph B. The *fibre of vertex* $x \in V(G)$ is denoted by F_x, formally, $F_x = p^{-1}(\{p(x)\})$. We will also use notation $F(u)$ for the fibre of the vertex $u \in V(B)$, i.e. $F(u) = p^{-1}(\{u\})$. Note that $F_x = F(p(x))$. We will also use shorter notation $x \in F(u)$ for $x \in V(F(u))$.
Let $u, v \in V(B)$ be distinct vertices, and Q be a path from u to v in B, and $x \in F(u)$. Then the *lift of the path* Q *to the vertex* $x \in V(G)$, $\tilde{Q}_x$, is the path from $x \in F(u)$ to a vertex in $F(v)$, such that $p(\tilde{Q}_x) = Q$ and $\ell(\tilde{Q}_x) = \ell(Q)$. Let $x, x' \in F(u)$. Then $\tilde{Q}_x$ and $\tilde{Q}_{x'}$ have different endpoints in $F(v)$ and are disjoint paths if and only if $x \neq x'$. In fact, two lifts $\tilde{Q}_x$ and $\tilde{Q}_{x'}$ are either disjoint $\tilde{Q}_x \cap \tilde{Q}_{x'} = \emptyset$ or equal, $\tilde{Q}_x = \tilde{Q}_{x'}$. We will also use notation $\tilde{Q}$ for lifts of the path Q to any vertex in $F(u)$.
Let Q be a path from u to v and $e = uw \in E(Q)$. We will use notation $Q \setminus e$ for the subpath from w to v, i.e. $Q \setminus e = Q \setminus \{u, e\} = Q \setminus \{u\}$.
Let G be a graph and $X \subseteq S(G)$ be a set of elements of G. A path P from a vertex x to a vertex y *avoids* X in G, if $S(P) \cap X = \emptyset$, and it *internally avoids* X, if $(S(P) \setminus \{x, y\}) \cap X = \emptyset$.

We will use Lemma 4.1 in following proofs.

Lemma 4.1. *Let F and B be connected graphs, $\mathcal{D}(F) > 1$, and let G be a Cartesian graph bundle with fibre F over the base graph B. Let $x, y \in V(G)$ be two vertices, such that $p(x) \neq p(y)$, and let Q be a path from $p(x)$ to $p(y)$ in B. Then there are (at least) two internally vertex-disjoint paths from x to y in $p^{-1}(Q) = F\Box Q \subseteq G$ of lengths at most $\mathcal{D}(F) + \ell(Q)$.*

Proof. Let G be a Cartesian graph bundle with connected fibre F, $\mathcal{D}(F) > 1$, over the connected base graph B. Let $x, y \in V(G)$, $p(x) \neq p(y)$, and Q be a path from $p(x)$ to $p(y)$ in B. Let $x' \in F_y$ be the endpoint of $\tilde{Q}_x$.

- If $x' = y$, then there are two paths

  $$P_1 : x \xrightarrow{\tilde{Q}} y,\ P_2 : x \to s \xrightarrow{\tilde{Q}} s' \to y,$$

  where $s \in F_x$ and $s' \in F_y$ are neighbors of x and y, respectively. Paths P_1, P_2 are internally vertex-disjoint paths from x to y in $p^{-1}(Q)$ and $\ell(P_1) = \ell(Q)$, $\ell(P_2) = 1 + \ell(Q) + 1 \leq \mathcal{D}(F) + \ell(Q)$.

- If $x' \neq y$, then there are two paths

  $$P_1 : x \xrightarrow{\tilde{Q}} x' \xrightarrow{P} y,\ P_2 : x \xrightarrow{P'} y' \xrightarrow{\tilde{Q}} y,$$

  where P is a path from x' to y inside of the fibre F_y of length $\ell(P) \leq \mathcal{D}(F)$, $y' \in F_x$ is the endpoint of $\tilde{Q}_y$ and P' is a path inside of the fibre F_x of length $\ell(P') \leq \mathcal{D}(F)$. Paths P_1, P_2 are internally vertex-disjoint paths from x to y in $p^{-1}(Q)$ and $\ell(P_i) \leq \ell(Q) + \mathcal{D}(F), i = 1, 2$.

□

Lemma 4.2. *Let G be a Cartesian graph bundle with fibre F over the base graph B, the graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and the graph B be $(p, 0)$+connected, $p > 0$. Then*

$$\mathcal{D}_{(p+1,0)}(G) = \mathcal{D}^V_{p+1}(G) \leq \mathcal{D}(F) + \max\{\mathcal{D}^V_p(B), \mathcal{D}_{(p-1,1)}(B)\}.$$

Proof. Let F be a connected graph, $\mathcal{D}(F) > 1$, the graph B be $(p, 0)$+connected, $p > 0$, and let G be a Cartesian graph bundle with fibre F over the base graph B. By Theorem 3.4, the Cartesian graph bundle G is $(p + 1, 0)$+connected. Let $X \subseteq V(G)$ be a set of faulty vertices, $|X| = p + 1$, and let $x, y \in V(G) \setminus X$ be two distinct nonfaulty vertices in G. We shall consider the distance $d_{G\setminus X}(x, y)$. Note that as graph B is $(p, 0)$+connected and $p > 0$, it is also $(p - 1, 1)$+connected and $\mathcal{D}_{(p-1,1)}(B) \geq 2$.

- Suppose first that x and y are in the same fibre, i.e. $p(x) = p(y)$.
  If $|X \cap V(F_x)| = 0$, then $d_{G\setminus X}(x, y) \leq \mathcal{D}(F)$. If $|X \cap V(F_x)| > 0$, then outside of fibre F_x there are at most p faulty vertices. As a graph B is $(p, 0)$+connected, there are at least $p+1$ neighbors of vertex $p(x)$ in B. Therefore there exist a neighbor v of vertex $p(x)$ in B, such that $|X \cap F(v)| = 0$, and there is a path $x \to x' \stackrel{P}{\to} y' \to y$, which avoids X, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G\setminus X}(x, y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}(F) + \mathcal{D}_{(p-1,1)}(B)$.
- Now assume that x and y are in distinct fibres, i.e. $p(x) \neq p(y)$.
  Denote $X_B = \{v \in V(B) \setminus \{p(x), p(y)\}; |X \cap F(v)| > 0\}$. We distinguish two cases.

  1. If $p \leq |X_B| \leq p + 1$, then let $X'_B \subseteq X_B$ be an arbitrary subset of X_B with $|X'_B| = p$. The subgraph $B \setminus X'_B$ is a connected graph and there exist a path Q from $p(x)$ to $p(y)$ with $\ell(Q) \leq \mathcal{D}^V_p(B)$. In $p^{-1}(Q)$ there is at most one faulty vertex. By Lemma 4.1 there are two internally vertex-disjoint paths from x to y in $p^{-1}(Q)$ and at least one of them avoids the faulty element, thus $d_{G\setminus X}(x, y) \leq \ell(Q) + D(F) \leq \mathcal{D}^V_p(B) + D(F)$.
  2. If $|X_B| < p$, then the subgraph $B \setminus X_B$ is (at least) $(1, 0)$+connected, thus also $(0, 1)$+connected.
     If the vertex $p(y)$ is not a neighbor of $p(x)$, then there is a path Q from $p(x)$ to $p(y)$ in B with $2 \leq \ell(Q) \leq \mathcal{D}^V_{p-1}(B) \leq \mathcal{D}^V_p(B)$ that internally avoids X_B. Let $v \in V(Q)$ be a neighbor of $p(x)$, $e' = p(x)v$. Then there is a path $x \to x' \stackrel{P}{\to} y' \stackrel{\tilde{Q}\setminus e'}{\to} y$, which avoids X, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G\setminus X}(x, y) \leq 1 + \mathcal{D}(F) + \mathcal{D}^V_p(B) - 1 = \mathcal{D}(F) + \mathcal{D}^V_p(B)$.
     If $e = p(x)p(y) \in E(B)$, then $B \setminus (X_B \cup \{e\})$ is a connected graph and there is a path Q' from $p(x)$ to $p(y)$ with $2 \leq \ell(Q') \leq \mathcal{D}_{(p-1,1)}(B)$ that internally avoids X_B. Similar as before $d_{G\setminus X}(x, y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_{(p-1,1)}(B) - 1 = \mathcal{D}(F) + \mathcal{D}_{(p-1,1)}(B)$.

□

Example 4.3. Lemma 4.2 for $p = 1$ reads:

$$\mathcal{D}^V_2(G) \leq \mathcal{D}(F) + \max\{\mathcal{D}^V_1(B), \mathcal{D}^E_1(B)\}.$$

1. Let $B = K_4 \setminus \{e\}$. Then $\mathcal{D}(B) = \mathcal{D}_1^V(B) = \mathcal{D}_1^E(B) = 2$. The 2-vertex fault diameter of Cartesian graph product $P_3 \Box B$ is $\mathcal{D}_2^V(P_3 \Box B) = \mathcal{D}(P_3) + \mathcal{D}_1^V(B) = 2 + 2 = 4$.
2. The 2-vertex fault diameter of Cartesian graph product $P_3 \Box K_3$ is $\mathcal{D}_2^V(P_3 \Box K_3) = \mathcal{D}(P_3) + \mathcal{D}_1^E(K_3) = 2 + 2 = 4$.

In both examples the bound of Lemma 4.2 is tight.

Lemma 4.4. *Let G be a Cartesian graph bundle with fibre F over the base graph B, the graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and the graph B be $(0,q)$+connected, $q > 0$. Then*

$$\mathcal{D}_{(1,q)}(G) \leq \mathcal{D}(F) + \mathcal{D}_q^E(B) = \mathcal{D}(F) + \mathcal{D}_{(0,q)}(B).$$

Proof. Let F be a connected graph, $D(F) > 1$, and B be $(0,q)$+connected graph, $q > 0$. Then $\mathcal{D}_q^E(B) \geq 2$ and by Theorem 3.4, the Cartesian graph bundle G with fibre F over the base graph B is $(1,q)$+connected. Let $a \in V(G)$ be the faulty vertex and $Y \subseteq E(G)$ be the set of faulty edges, $|Y| = q$. Denote the set of degenerate edges in Y by Y_D, and the set of nondegenerate edges by Y_N, $Y = Y_N \cup Y_D$, $p(Y_D) \subseteq V(B)$, $p(Y_N) \subseteq E(B)$. Denote the set of faulty elements by $X = \{a\} \cup Y$. Let $x, y \in V(G) \setminus \{a\}$ be two arbitrary distinct nonfaulty vertices in G. We shall find an upper bound for the distance $d_{G \setminus X}(x,y)$.

- Suppose first that x and y are in the same fibre, i.e. $p(x) = p(y)$.
  If $|F_x \cap X| = 0$, then $d_{G \setminus X}(x,y) \leq \mathcal{D}(F)$. If $|F_x \cap X| > 0$, then outside of fibre F_x there are at most q faulty elements. As the graph B is $(0,q)$+connected, there are at least $q+1$ neighbors of vertex $p(x)$ in B. Therefore there exist a neighbor v of vertex $p(x)$ in B, such that $p(x)v \notin p(Y_N)$ and $|F(v) \cap (\{a\} \cup Y_D)| = 0$, and there is a path $x \to x' \overset{P}{\to} y' \to y$ which avoids X, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$. Thus $d_{G \setminus X}(x,y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}(F) + \mathcal{D}_q^E(B)$.
- Now assume that x and y are in distinct fibres, i.e. $p(x) \neq p(y)$.
  Let $B' = B \setminus p(Y_N)$. As $|p(Y_N)| \leq q - |Y_D|$, the subgraph B' is at least $(0, |Y_D|)$+ connected and $p^{-1}(B')$ does not contain nondegenerate faulty edges, $\left|p^{-1}(B') \cap Y_N\right| = 0$.
  Let $Y' = \{p(x)v \in E(B'); |F(v) \cap (\{a\} \cup Y_D)| > 0\}$. We distinguish two cases.

  1. Let $a \in V(F_x) \cup V(F_y)$, and without of loss of generality assume $a \in V(F_x)$. Then $|Y'| \leq |Y_D|$ and the subgraph $B' \setminus Y' = B \setminus (Y' \cup p(Y_N))$ is a connected graph. Therefore there exists a path Q from $p(x)$ to $p(y)$ in B of length $1 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$, and for the neighbor $v \in V(Q)$ of the vertex $p(x)$, $e = p(x)v$, there is no faulty elements in the fibre $F(v)$. Note that the path $Q \setminus e$ avoids $p(a)$.

     If $v = p(y)$ there is a path $x \to x' \overset{P}{\to} y$, where $x' \in F_y$ and $\ell(P) \leq \mathcal{D}(F)$, that avoids faulty elements, thus $d_{G \setminus X}(x,y) \leq 1 + \mathcal{D}(F) < \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

     If $v \neq p(y)$ there is a path $x \to x' \overset{P}{\to} y' \overset{\tilde{Q} \setminus e}{\to} y$, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$, which avoids faulty elements, thus $d_{G \setminus X}(x,y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_q^E(B) - 1 = \mathcal{D}(F) + \mathcal{D}_q^E(B)$.
  2. If $a \notin V(F_x) \cup V(F_y)$, we distinguish three cases.

(a) Suppose $|(F_x \cup F_y) \cap Y_D| = 0$. There exist a path Q from $p(x)$ to $p(y)$ in $B' \subseteq B$ of length $\ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$. By Lemma 4.1 there are two internally vertex-disjoint paths from x to y in $p^{-1}(Q)$, that avoid Y and at least one of them avoids faulty vertex a, thus $d_{G\setminus X}(x,y) \leq \ell(Q) + D(F) \leq \mathcal{D}_q^E(B) + D(F)$.

(b) Suppose, that exactly one of fibres F_x, F_y contains faulty edges, without of loss of generality let $|F_x \cap Y_D| > 0$ and $|F_y \cap Y_D| = 0$. Then $|Y'| \leq |Y_D|$ and the subgraph $B' \setminus Y' = B \setminus (Y' \cup p(Y_N))$ is a connected graph. There exist a path Q from $p(x)$ to $p(y)$ in B of length $1 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$ and for the neighbor $v \in Q$ of vertex $p(x)$, $e = p(x)v$, there is no faulty elements in the fibre $F(v)$.
If $v = p(y)$ then $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) < \mathcal{D}(F) + \mathcal{D}_q^E(B)$.
If $v \neq p(y)$, let $v' \in F(v)$ be a neighbor of x. As $|(F(v) \cup F_y) \cap Y_D| = 0$, similar as in (a) there is a path from v' to y in $p^{-1}(Q \setminus e)$ of length at most $\mathcal{D}(F) + \mathcal{D}_q^E(B) - 1$, that avoids faulty elements, thus $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_q^E(B) - 1 = \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

(c) At last, suppose $|F_x \cap Y_D| > 0$ and $|F_y \cap Y_D| > 0$.

i. Assume $d_{B'}(p(x), p(y)) = 1$. In this case $p(x)p(y) \in Y'$, and $|Y'| \leq |Y_D|$ as $|F_x \cap Y_D| > 0$. Thus the subgraph $B' \setminus Y'$ is connected, and there exists a path Q from $p(x)$ to $p(y)$ in B of length $2 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$, and for the neighbor $v \in Q$ of vertex $p(x)$, $e = p(x)v$, there is no faulty elements in the fibre $F(v)$.

If $\ell(Q) = 2$, then there is a path $x \to x' \overset{P}{\to} y' \to y$, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$, which avoids faulty elements, thus
$d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

If $3 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, then the path $x \to x' \overset{P'}{\to} y' \to s \to y$ where $x', y' \in F(v)$, $\ell(P') \leq \mathcal{D}(F)$, and $s \in V(F_x)$, avoids faulty elements, thus $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + 1 + 1 \leq \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

ii. Assume, $d_{B'}(p(x), p(y)) = 2$. Then there is at least one common neighbor of vertices $p(x)$ and $p(y)$ in B'. If there exist a common neighbor v of vertices $p(x)$ and $p(y)$ in B' for which there is no faulty elements in the fibre $F(v)$, then as before $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}(F) + \mathcal{D}_q^E(B)$. Otherwise suppose, there is some common neighbor w of vertices $p(x)$ and $p(y)$ in B' for which $a \notin F(w)$ and $|F(w) \cap Y_D| > 0$. As $|Y'| \leq |Y_D| - 1$ the subgraph $B' \setminus Y'$ is (at least) $(0,1)$+connected graph. If vertex $p(a)$ is a neighbor of $p(y)$ in B', $e' = p(y)p(a) \subseteq E(B')$, then also $B' \setminus (Y' \cup \{e'\})$ is a connected graph. Therefore there exist a path Q from $p(x)$ to $p(y)$ in B of length $3 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$, and for the neighbor $u \in V(Q)$ of vertex $p(x)$, there is no faulty elements in the fibre $F(u)$, and for the neighbor $v \in V(Q)$ of vertex $p(y)$, $v \neq p(a)$.

If $\ell(Q) = 3 \leq \mathcal{D}_q^E(B)$, then there is a path $x \to x' \overset{P}{\to} y' \to s \to y$, where $x', y' \in F(u)$, $\ell(P) \leq \mathcal{D}(F)$, $s \in F(v)$, which avoids faulty elements, thus $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + 1 + 1 \leq \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

If $4 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, then the path $x \to x' \overset{P'}{\to} y' \to s \to s' \to y$, where $x', y' \in F(u)$, $\ell(P') \leq \mathcal{D}(F)$, $s \in V(F_x)$, $s' \in F(w)$, avoids

faulty elements, thus $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + 1 + 2 \leq \mathcal{D}(F) + \mathcal{D}_q^E(B)$.
The last case to consider is when $p(a)$ is the only common neighbor of vertices $p(x)$ and $p(y)$ in B'.
Let $Y'' = \{p(y)v \in E(B'); |F(v) \cap (\{a\} \cup Y_D)| > 0\}$. As $p(a)$ is the only common neighbor of vertices $p(x)$ and $p(y)$, $|Y' \cup Y''| \leq |Y_D|$ and the subgraph $B' \setminus (Y' \cup Y'')$ is a connected graph. Therefore there exist a path Q from $p(x)$ to $p(y)$ in B of length $3 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$, and for neighbors $u \in V(Q)$ of vertex $p(x)$ and $v \in V(Q)$ of vertex $p(y)$, there is no faulty elements in fibres $F(u)$ and $F(v)$. Let $x' \in F(u)$ be a neighbor of x and $y' \in F(v)$ be a neighbor of y. As in (a) there is a path from x' to y' in $p^{-1}(Q \setminus \{p(x), p(y)\}$ of length at most $\mathcal{D}(F) + \mathcal{D}_q^E(B) - 2$, that avoids faulty elements, thus $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_q^E(B) - 2 + 1 = \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

iii. Finally, suppose $d_{B'}(p(x), p(y)) \geq 3$. As there is no common neighbor of vertices $p(x)$ and $p(y)$ in B', $|Y' \cup Y''| \leq |Y_D| - 2 + 1 = |Y_D| - 1$ and as before there exist a path Q from $p(x)$ to $p(y)$ in B of length $3 \leq \ell(Q) \leq \mathcal{D}_q^E(B)$, that avoids $p(Y_N)$, and for both neighbors $u \in V(Q)$ of vertex $p(x)$ and $v \in V(Q)$ of vertex $p(y)$, there is no faulty elements in fibres $F(u) \cup F(v)$, thus $d_{G\setminus X}(x,y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_q^E(B) - 2 + 1 = \mathcal{D}(F) + \mathcal{D}_q^E(B)$.

□

Lemma 4.5. *Let G be a Cartesian graph bundle with fibre F over the base graph B, the graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and the graph B be (p,q)+connected, $q > 0$. Then*

$$\mathcal{D}_{(p+1,q)}(G) \leq \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B).$$

Proof. The case when $p = 0$ is already proved by Lemma 4.4. So let us assume $p > 0$. Let the graph F be a connected graph, $\mathcal{D}(F) > 1$, and the graph B be (p,q)+connected, $p, q > 0$. Then $\mathcal{D}_{(p,q)}(B) \geq 2$ and by Theorem 3.4, the Cartesian graph bundle G with fibre F over the base graph B is $(p+1,q)$+connected. Let $X \subseteq V(G)$ be the set of faulty vertices, $|X| = p+1$, and $Y \subseteq E(G)$ be the set of faulty edges, $|Y| = q$. Denote the set of degenerate edges in Y by Y_D, and the set of nondegenerate edges by Y_N, $Y = Y_N \cup Y_D$, $p(Y_D) \subseteq V(B)$, $p(Y_N) \subseteq E(B)$. Let $x, y \in V(G) \setminus X$ be two distinct nonfaulty vertices in G. We shall determine an upper bound for the distance $d_{G\setminus(X\cup Y)}(x,y)$.

- Suppose first that x and y are in the same fibre, i.e. $p(x) = p(y)$.
  If $|F_x \cap (X \cup Y_D)| = 0$, then $d_{G\setminus(X\cup Y)}(x,y) \leq \mathcal{D}(F)$. If $|F_x \cap (X \cup Y_D)| > 0$, then there are at most $p+q$ faulty elements outside of the fibre F_x. As the graph B is (p,q)+connected, there are at least $p+q+1$ neighbors of vertex $p(x)$ in B. Therefore there exists a neighbor v of vertex $p(x)$ in B, such that $p(x)v \notin p(Y_N)$ and $|F(v) \cap (X \cup Y_D)| = 0$, and there is a path $x \to x' \overset{P}{\to} y' \to y$, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$, which avoids $X \cup Y$. Thus $d_{G\setminus(X\cup Y)}(x,y) \leq 1 + \mathcal{D}(F) + 1 \leq \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$.

- Now assume that x and y are in distinct fibres, i.e. $p(x) \neq p(y)$.
  Let $X_B = \{v \in V(B) \setminus \{p(x), p(y)\}; |F(v) \cap X| > 0\}$. We distinguish two cases.

  1. If $p \leq |X_B| \leq p+1$, then let $X'_B \subseteq X_B$, $|X'_B| = p$. The subgraph $B \setminus X'_B$ is $(0,q)$+connected and there is at most one faulty vertex and q faulty edges in $p^{-1}(B \setminus X'_B)$. By Lemma 4.4 there is a path from x to y in $p^{-1}(B \setminus X'_B)$ with length at most $\mathcal{D}(F) + \mathcal{D}^E_q(B \setminus X'_B)$, that avoids faulty elements, thus $d_{G \setminus (X \cup Y)}(x,y) \leq \mathcal{D}(F) + \mathcal{D}^E_q(B \setminus X'_B) = \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$.
  2. Suppose $|X_B| < p$. Let $Y_B = \{p(x)v \in E(B); |F(v) \cap Y_D| > 0\}$ and $B' = B \setminus (X_B \cup Y_B \cup p(Y_N))$. Then the subgraph B' is (at least) $(1,0)$+connected, thus also $(0,1)$+connected.
     If $d_{B'}(p(x), p(y)) \geq 2$, then there is a path Q from $p(x)$ to $p(y)$ in $B' \subseteq B$ with $2 \leq \ell(Q) \leq \mathcal{D}_{(p-1,q)}(B) \leq \mathcal{D}_{(p,q)}(B)$ that internally avoids X_B, it avoids $p(Y_N)$, and for the neighbor $v \in V(Q)$ of vertex $p(x)$, $e' = p(x)v$, there is no faulty elements in the fibre $F(v)$. Therefore there is a path $x \to x' \stackrel{P}{\to} y' \stackrel{\tilde{Q} \setminus e'}{\to} y$, where $x', y' \in F(v)$ and $\ell(P) \leq \mathcal{D}(F)$, which avoids $X \cup Y$, thus $d_{G \setminus X \cup Y}(x,y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B) - 1 = \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$.
     If $d_{B'}(p(x), p(y)) = 1$, $e = p(x)p(y) \in E(B')$, then the subgraph $B' \setminus \{e\}$ is a connected graph and there is a path Q' from $p(x)$ to $p(y)$ with $2 \leq \ell(Q') \leq \mathcal{D}_{(p-1,q+1)}(B) \leq \mathcal{D}_{(p,q)}(B)$ that internally avoids X_B, it avoids $p(Y_N)$, and for the neighbor $v \in V(Q)$ of vertex $p(x)$, there is no faulty elements in the fibre $F(v)$, and as before $d_{G \setminus X \cup Y}(x,y) \leq 1 + \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B) - 1 = \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$.

□

Theorem 4.6. *Let G be a Cartesian graph bundle with fibre F over the base graph B, the graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and the graph B be (p,q)+connected, $p+q > 0$. Then we have:*

- *If $q > 0$, then $\mathcal{D}_{(p+1,q)}(G) \leq \mathcal{D}(F) + \mathcal{D}_{(p,q)}(B)$.*
- *If $q = 0$, then $\mathcal{D}_{(p+1,0)}(G) = \mathcal{D}^V_{p+1}(G) \leq \mathcal{D}(F) + \max\{\mathcal{D}^V_p(B), \mathcal{D}_{(p-1,1)}(B)\}$.*

Proof. The statement of Theorem 4.6 follows from Lemma 4.2 (case $q = 0$), Lemma 4.4 (case $p = 0$), and Lemma 4.5 (for positive p and q). □

Remark 4.7. Let G be a Cartesian graph bundle with fibre F over the base graph B, the graph F be a connected graph with diameter $\mathcal{D}(F) > 1$, and the graph B be $(p,0)$+connected, $p > 0$. By Theorem 4.6 we have an upper bound for the (vertex) fault diameter $\mathcal{D}^V_{p+1}(G) \leq \mathcal{D}(F) + \mathcal{D}^V_p(B) + 1$ for any graph B. Similarly, $\mathcal{D}^V_{p+1}(G) \leq \mathcal{D}(F) + \mathcal{D}^V_p(B)$ if $\mathcal{D}_{(p-1,1)}(B) \leq \mathcal{D}^V_p(B)$ holds.

Next corollary easily follows from Theorems 3.5 and 4.6.

Corollary 4.8. *Let both graphs F and B be (p,q)+connected, $p+q > 0$, $\mathcal{D}(F) > 1$, $\mathcal{D}(B) > 1$, and let G be a Cartesian graph bundle with fibre F over the base graph B. Then we have:*

- *If $q > 0$, then $\mathcal{D}_{(p+1,q)}(G) \leq \max\{\mathcal{D}(F) + \mathcal{D}_{(p,q)}(B), \mathcal{D}_{(p,q)}(F) + \mathcal{D}(B)\}$,*

- *If* $q = 0$*, then* $\mathcal{D}^V_{p+1}(G) \leq \max\{\mathcal{D}(F) + \mathcal{D}^V_p(B), \mathcal{D}^V_p(F) + \mathcal{D}(B)\} + 1$*, and* $\mathcal{D}^V_{p+1}(G) \leq \max\{\mathcal{D}(F)+\mathcal{D}^V_p(B), \mathcal{D}^V_p(F)+\mathcal{D}(B)\}$*, if* $\mathcal{D}_{(p-1,1)}(F) \leq \mathcal{D}^V_p(F)$ *and* $\mathcal{D}_{(p-1,1)}(B) \leq \mathcal{D}^V_p(B)$ *hold.*

We conclude with a conjecture. We know that a Cartesian graph bundle with fibre F over the base graph B, where graph F is (p_F, q_F)+connected, $p_F + q_F > 0$, and where graph B is (p_B, q_B)+connected, $p_B + q_B > 0$, is $(p_B + p_F + 1, q_B + q_F)$+connected [13]. An upper bound for the mixed fault diameter where the number of allowed faulty elements would be the maximal possible may be the following:

Conjecture 4.9. *Let* G *be a Cartesian graph bundle with fibre* F *over the base graph* B*, where the graph* F *is* (p_F, q_F)*+connected,* $p_F + q_F > 0$*, and where the graph* B *is* (p_B, q_B)*+connected,* $p_B + q_B > 0$*. Then*

$$\mathcal{D}_{(p_B+p_F+1,q_B+q_F)}(G) \leq \mathcal{D}_{(p_F,q_F)}(F) + \mathcal{D}_{(p_B,q_B)}(B) + 1.$$

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An atlas of subgroup lattices of finite almost simple groups

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Abstract

We provide algorithms to compute and produce subgroup lattices of finite permutation groups. We discuss the problem of naming groups and we propose an algorithm that automatizes the naming of groups, together with possible ways of refinement. Finally we announce an atlas of subgroup lattices for a large collection of finite almost simple groups made available online.

Keywords: Computational methods in group theory, lattices of subgroups

Math. Subj. Class.: 20B40, 20E15

1 Introduction

The Classification of the Finite Simple Groups (CFSG) emphasizes the importance of the finite simple groups in Group Theory. It is one of the most impressive achievements in the history of Mathematics. We refer to [26] and the references provided there for a broad literature on this wonderful theorem. Among the amazing achievements in this branch of Mathematics, we find the ATLAS of Finite Groups [15] as well as the online version of the ATLAS of Finite Group Representations [1].

Over the years, the finite simple groups have received a lot of attention with respect to the study of geometry. The Theory of Buildings due to Jacques Tits, who was awarded the Abel Prize in 2008, illustrates this perfectly. We refer for instance to [2] and references

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provided there. Much work has been done in this respect with the study of incidence geometries associated to finite almost simple groups (we refer to [7, 9, 20] and references cited there for a large documentation on this aspect).

The computations of subgroup lattices and tables of marks of permutation groups, and in particular sporadic simple groups, have been a subject of interest for many decades, linked among others to the search for a unified geometric interpretation of all finite simple groups. Joachim Neubüser gave in [23] the first algorithm that was implemented later on in the computational software CAYLEY and its successor MAGMA. Francis Buekenhout computed in 1984 the lattices of M_{11} and J_1 [5]. Then, Herbert Pahlings did the lattice of J_2 [24] in 1987. In 1988, Buekenhout and Sarah Rees produced the lattice of M_{12} (see [6] and [16] for a few corrections). In 1991, Pfeiffer computed the table of marks of J_3 and in 1997, those of M_{22}, M_{23} M_{24}, McL [25]. Also in 1997, Merkwitz got the tables of marks of He and Co_3. In 1998, Derek Holt computed all conjugacy classes of subgroups of $\mathrm{O'N}$ (personal communication). In a more general setting again, John Cannon, Bruce Cox and Derek Holt described in [11] a new algorithm to compute the conjugacy classes of subgroups of a given group that was used in MAGMA until 2005. Progresses on the computation of maximal subgroups of a given group by Cannon and Holt [12] led Leemans to a much faster algorithm to compute the subgroup lattice of a given group that is now available in MAGMA. In 2007, Leemans computed the full subgroup lattices of HS, Ru, Suz, $\mathrm{O'N}$, Co_2 and Fi_{22} using permutation degree reduction at each step of the computation. Recently, Naughton and Pfeiffer produced a new algorithm to compute the table of marks of a cyclic extension of a group [22].

The knowledge of the subgroup lattice of a group G is a powerful tool to study the symmetrical objects on which G acts. For instance, in [10], [13] and [21], the authors build flag-transitive coset geometries of ranks 2, 3 and 5 for $\mathrm{O'N}$ by identifying boolean lattices in the subgroup lattice of $\mathrm{O'N}$. In [14], the authors develop an algorithm in order to count the number of regular maps on which a finite group G acts regularly. This algorithm makes an intensive use of the knowledge of the subgroup lattice of G. Then they illustrate the algorithm on the group $\mathrm{O'N}$. In a more general approach, [17] discusses the problem of enumerating regular objects with a given automorphism group. The authors introduce, among other things, the Möbius function for a group G as a tool to enumerate regular objects. The knowledge of the Möbius function of G relies on the knowledge of the full subgroup lattice of G.

In the spirit of contributing to the study of the finite simple groups, we present here an algorithm that determines the subgroup lattice of a given permutation group. This algorithm, designed by Leemans in 2007, proves itself to be lighter in memory and sometimes even faster than the already implemented function `SubgroupLattice` in the software MAGMA [4]. We also present an algorithm to determine structures of groups with a computer. Those two algorithms allow us to produce an atlas of subgroup lattices for a large number of finite almost simple groups. The atlas is made available online at

http://homepages.ulb.ac.be/~tconnor/atlaslat

Our paper is organised as follows. In section 2, we present the algorithm that computes the subgroup lattice of a permutation group G. It features a systematic reduction of the permutation degree of the subgroups of G and a possibility to start the computation of the subgroup lattice of G with partial information of the lattice already available. This permits to compute the subgroup lattice of very large groups, like the O'Nan sporadic group $\mathrm{O'N}$ or even its automorphism group, currently out of reach with the `SubgroupLattice` function of MAGMA, but also Co_2 and Fi_{22}. In section 3 we discuss the problem of describing the structure of a group in an efficient way. We present an algorithm that provides a structure for a group, based on a choice of suitable normal subgroups. In section 4 we present our atlas of subgroup lattices as an application of the algorithms presented and discussed in this paper.

2 The subgroup lattice of a permutation group

We refer to [18] as a reference on subgroup lattices. A lattice is a partially ordered set, or poset, any two of whose elements a, b have a least upper bound $a \cup b$ and a greatest lower bound $a \cap b$. The subgroups of a group G may be taken as the elements of a lattice $L(G)$ under the operations of union and intersection. The poset of conjugacy classes of subgroups forms also a lattice: two conjugacy classes A and B are such that $A \supseteq B$ provided that any subgroup of B is contained in some subgroup of A. We call this lattice the subgroup lattice of G, rather than the lattice of conjugacy classes of subgroups of G for the sake of brevity, and we denote it with $\Lambda(G)$. Our terminology is also the one used in MAGMA. This lattice can be refined with the length of each conjugacy class of subgroups. Moreover, given two conjugacy classes of subgroups $A \supset B$, we define n_{AB} to be the number of subgroups of class B contained in any subgroup of class A; alike we define n_{BA} to be the number of subgroups of class A containing a subgroup of class B. Consider the set N of numbers n_{XY} for every couple of classes $\{X, Y\}$ such that $X \subset Y$ or $X \supset Y$ and there does not exist Z such that $X \subset Z \subset Y$ or $X \supset Z \supset Y$. The subgroup lattice $\Lambda(G)$ together with the length of each conjugacy class and the set N is called the weighted subgroup lattice of G.

We describe in this section a powerful and natural algorithm to compute the weighted subgroup lattice of a given group G. The correctness of this algorithm is obvious.

Start with a set `classes` which is empty and a set `sgr` containing just one element, namely the group G for which we want to compute the subgroup lattice. While `sgr` is nonempty, pick one element H out of `sgr` and put it in `classes`. Obviously, it is G the first time. Reduce the permutation degree of H and let $\phi : H \to \tilde{H}$ be an isomorphism between H and $\tilde{H}$ where $\tilde{H}$ has a reduced permutation degree. Compute the maximal subgroups of $\tilde{H}$ and for each maximal $\tilde{M}$, add $M := \phi^{-1}(\tilde{M})$ to `sgr` provided there is no subgroup in `sgr` conjugate to M in G. During that process, keep track of inclusions of respective subgroups considered. At the end of this process, in `classes` there is one representative of each conjugacy class of subgroups of G. Moreover, we also have the maximal inclusions between classes. So the subgroup lattice is determined. The weighted subgroup lattice can be determined in the process by computing weighted inclusions at each step.

A MAGMA implementation of the algorithm described above to compute the subgroup lattice of a given group is available on the webpage of the atlas. Observe that we use the `DegreeReduction` function in MAGMA to get ϕ and $\tilde{H}$ for every subgroup H above. This improvement can save a lot of time and memory. For instance, consider $L_3(7) : 2$, one of the maximal subgroups of the O'Nan sporadic group O′N, acting on 122760 points (the smallest permutation representation of $O'N$). Then MAGMA v.2.19 needs 13 seconds and more than 200 Mb of memory to compute its maximal subgroups on a computer running at 2.9 GHz. If we reduce the degree of $L_3(7) : 2$ on 5586 points by using the

`DegreeReduction` function, then MAGMA computes them in less than half a second and takes about 20 Mb of memory.

Our implementation has three main advantages.

1. For permutation groups of large degree, say at least 1000, our algorithm will perform faster;
2. Our algorithm will also need less memory for these groups;
3. This algorithm permits to compute the subgroup lattices of a group G unreachable for the `SubgroupLattice` function of MAGMA when MAGMA does not know the maximal subgroups of G, as for instance the O'Nan group, Fi_{22}, Co_2, etc. Indeed, feeding the function with the maximal subgroups, or even part of the subgroup lattice for groups like $\mathrm{Aut}(\mathrm{ON})$, of the group permits to proceed further.

On the other end, of course, if the permutation degree of the group is small, MAGMA will tend to work faster as it is based on our algorithm without the degree reduction and the degree reduction step will slow down the process instead of speeding it up.

3 The structures of a group

3.1 Preliminary remarks

Given any finite group G, it is always desirable to identify G in some sense. This identification can be done for instance in a geometrical way by determining the action of G on some set or by algebraic means. In particular, most finite simple groups can be named after their action on some structured set or after the mathematician that discovered them (like the Suzuki groups or most of the sporadic groups). However some groups carry very different names, depending on the incarnation of the group that the context requires to emphasise. This is the case for instance of $U_4(2)$. Indeed,

$$S_4(3) \cong U_4(2) \cong O_5(3) \cong O_6^-(3) \cong W(E_6).$$

Each of the names of this group emphasizes one of its actions on a structured set of particular interest. Therefore, when speaking about this group, one has to choose carefully the name that should be used depending on the context. This observation means that one has to be aware of possible isomorphisms between different incarnations of a group.

In MAGMA, there exists a database of finite simple groups. Given a simple group G, one can thus ask MAGMA to name G by using the function `NameSimple`. This function returns a triple of integers that permits to identify G as a group of one of the infinite families of finite simple groups, or as one of the sporadic groups. Many non simple groups can also be identified in a canonical way. This is the case of most of the almost simple groups for instance, but also the case of the dihedral groups, or the groups $\mathrm{AGL}(n, q)$. Abelian groups are also identified easily by a name thanks to the classification theorem of abelian groups. However, most of the finite groups are not almost simple, and identifying them in an efficient way by a name can be tricky. For instance, Leemans exhibited two non isomorphic primitive groups in [19] that satisfy the following property: they have isomorphic posets of conjugacy classes of subgroups and for each normal subgroup N of the first, there is a normal subgroup isomorphic to N in the second group such that the quotients by N are isomorphic. In other words, it is not possible to make a difference between those two groups by giving them names based on any quotient by a normal subgroup. This shows that the taxonomy of groups is a difficult and possibly not solvable problem. Hence we should not look for a deterministic algorithm that gives names to groups since it is readily impossible.

The case of p-groups is also particularly difficult to handle. For instance, there are roughly 50 billions pairwise non-isomorphic groups of order 1024 and hence, finding a way to give distinct names to each of them is hopeless, unless we decide to assign a number to each of them, as is done for instance in the SMALLGROUPS database provided by [3].

3.2 Algorithmic approach

Let G be a group and let N be a normal subgroup of G. Denote by Q the quotient group G/N. Then G can be written as $N.Q$ where the dot " . " denotes an extension that can be split (that is, a direct or a semi direct product) or non split. We denote a direct product by " $\times$ ", a semi direct product by " : " and a non split extension by " $\cdot$ ".

We recall that a *composition series* for G is a sequence of subgroups $H_i, i \in \{0, \ldots, n+1\}$ such that

$$1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_n \triangleleft H_{n+1} = G$$

where all inclusions are strict, i.e. H_i is a maximal normal subgroup of H_{i+1}. This is equivalent to require that the *composition factors* $Q_i = H_{i+1}/H_i$ are simple groups, $i = 0, \ldots, n$. Clearly the group G can be written $H_n.Q_n$. Alike, H_n can be written $H_{n-1}.Q_{n-1}$ and thus G can be written $(H_{n-1}.Q_{n-1}).Q_n$. Proceeding inductively, we can finally write

$$G = (\ldots(Q_0.Q_1).Q_2)\ldots).Q_n.$$

However in order to reduce the notations, we always suppose that the products are left associate and we can thus avoid to write parentheses whenever there is no possible confusion. Therefore by $G \cong A.B.C$ we mean $G \cong (A.B).C$.

The Jordan–Hölder theorem states that every finite group has a unique composition series up to the order of the terms [26]. Obviously, two non isomorphic groups can have the same composition series. This is the case for instance of S_5 and $A_5 \times 2$. In this particular example, it is not enough to use the composition series of those two groups to distinguish them. However $S_5 \cong A_5 : 2$ but $S_5 \ncong A_5 \times 2$.

On basis of the previous observations, we detail an algorithm that produces a name for a group G in terms of a product of its composition factors. We detail afterwards an improved algorithm that we actually used in order to produce the lattices of our atlas. First of all, given $N \triangleleft G$ we need to check whether the extension $N.Q$ is split or not, i.e. $G \cong N : Q$ or $G \cong N \cdot Q$, where $Q \cong G/N$ as usual. If N is a maximal normal subgroup of G, then Q is simple. We can use the database of simple groups in MAGMA to identify Q and give it a name. We can now easily extract the following algorithm from the previous observations. If G is simple, we are done. Suppose G is not simple. Compute a composition series of G and the corresponding composition factors. At step $n - i + 1$, identify the simple group Q_i and check whether $H_i.Q_i$ is split or non split. If it is split, check moreover if the extension is a direct product. The procedure returns the group G written as $G \cong Q_0 ._0 Q_1 ._1 \ldots Q_{n-1} ._{n-1} Q_n$. where $._i$ is a symbol in $\{\times, :, \cdot\}$. Applying this procedure to S_5 for instance would produce $A_5 : 2$. Applying it to the dihedral group D_{40} would produce $5 \times 2 \cdot 2 : 2$. Unfortunately, this could also be the result after applying this algorithm to $5 \times D_8$. Finally applying it to an elementary abelian group 2^5 would produce $2 \times 2 \times 2 \times 2 \times 2$.

```
if G is simple then identify G
else if G is in the database then identify G
else if G has a 'desirable property' then identify G
else
 compute the list L of normal subgroups of G
 for each subgroup N in L in decreasing order do
  if N is simple or has a 'desirable property' or is in the database then
   identify N and identify the extension between N and G/N
   proceed inductively on G/N
  else if G/N has a 'desirable property' or is in the database then
   identify G/N and identify the extension between N and G/N
   proceed inductively on N
  else if no N and no G/N is desirable then
   take the largest N and proceed inductively
```

Figure 1: An improved naming algorithm

There is an obvious improvement of this algorithm. The guideline is that some non-simple groups can be identified in a canonical way like the symmetric groups or the dihedral groups for instance. Moreover in the process of building the Atlas that we describe in this article, we observed for example that the group $S_3 \times S_3$ would not be named correctly most of the time, or A_4 would be written $2^2 : 3$. Therefore we produced a database of selected groups that our algorithm checks prior to computing the list of normal subgroups of G. The algorithm also checks possible isomorphisms of G with 'classical' groups (like the symmetric groups or the dihedral groups, for instance). If G is not immediately identifiable, our algorithm computes the list of normal subgroups of G. If a normal subgroup N or a quotient G/N is appealing then our algorithm would select it and proceed inductively.

4 The atlas

For every almost simple group of order at most 1,000,000 appearing in the online version of the Atlas of Finite Groups [1], we computed its subgroup lattice with the MAGMA implementation of our algorithm, available on the homepage mentioned below. Given such a lattice Λ, we ran the algorithm described in Figure 1 on every subgroup in Λ. We also proceeded in this way for some groups of order larger than 1,000,000 like some large sporadic groups. The result is an atlas of more than a hundred subgroup lattices of almost simple groups with a structure provided for every subgroup of each group. The atlas of subgroup lattices is available online at

http://homepages.ulb.ac.be/~tconnor/atlaslat.

Groups are subdivided in several families, namely almost simple groups of sporadic type, alternating type, linear type, symplectic type, orthogonal type, unitary type and exceptional Lie type. For each group G in the atlas, a `pdf` file containing the subgroup lattice of G is available for download.

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Further biembeddings of twofold triple systems

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Abstract

We construct face two-colourable triangulations of the graph $2K_n$ in an orientable surface; equivalently biembeddings of two twofold triple systems of order n, for all $n \equiv 16$ or $28 \pmod{48}$. The biembeddings come from index 1 current graphs lifted under a group $\mathbb{Z}_{n/4} \times \mathbb{K}_4$.

Keywords: Biembedding, orientable surface, twofold triple system.

Math. Subj. Class.: 05B07, 05C10

1 Introduction

A complete graph K_n has a triangulation in an orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$, [5]. In such a triangulation the number of faces around each vertex is $n-1$, and so if $n-1$ is even, i.e. $n \equiv 3$ or $7 \pmod{12}$, it may be possible to colour each face using one of two colours, say black or white, so that no two faces of the same

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colour are adjacent. In such a case we say that the triangulation is *(properly) face two-colourable*. In this case the set of faces of each colour class forms a *Steiner triple system of order* n, STS(n) for short, i.e. a collection of triples which have the property that every pair is contained in precisely one triple. We say that the two STS(n)s are *biembedded* in the (orientable) surface. An obvious question therefore is whether, for each $n \equiv 3$ or $7 \pmod{12}$, there is a biembedding of some pair of STS(n)s in an orientable surface. The answer is in the affirmative; the case $n \equiv 3 \pmod{12}$ is dealt with in [5] and the case $n \equiv 7 \pmod{12}$ in [6].

In [2] an extension of the above was considered. The necessary condition for the graph $2K_n$, i.e. the graph on n vertices with each pair of vertices joined by two edges, to have a triangular embedding in an orientable surface is $n \equiv 0$ or $4 \pmod 6$. Such triangulations may be face two-colourable, in which case each colour class forms a *twofold triple system of order* n, TTS(n) for short, i.e. a collection of triples which have the property that every pair is contained in precisely two triples. Again we say that the two TTS(n)s are *biembedded* in the (orientable) surface. For such biembeddings to admit a cyclic automorphism it is necessary and sufficient that $n \equiv 4 \pmod{12}$ [2] and a complete solution was provided in that paper. However the method is rather complex. In this paper, for the case $n \equiv 28 \pmod{48}$, we give a much simpler construction based on cyclic biembeddings of Steiner triple systems of order $12m + 7$, $m \geq 0$. These were found by Youngs [6] and index 1 current graphs corresponding to these solutions are readily accessible. They can be found in [3]. The biembeddings of the TTS(n)s which we obtain from these biembeddings of Youngs however are not cyclic; they have an automorphism group $\mathbb{Z}_{12m+7} \times \mathbb{K}_4$ where $\mathbb{K}_4$ is the Klein 4-group.

Further we extend our method to find new biembeddings of TTS(n)s for $n \equiv 16 \pmod{48}$; these have an automorphism group $\mathbb{Z}_{12m+4} \times \mathbb{K}_4$. Finally iterating this latter process we obtain biembeddings of twofold triple systems of order $4^s(12m + 4)$ with an automorphism group $\mathbb{Z}_{12m+4} \times (\mathbb{K}_4)^s$, $s \geq 1$, $m \geq 0$.

We will also construct our biembeddings from index 1 current graphs lifted under the appropriate current group G of order g. These will satisfy the following properties, which are sufficient to construct a biembedding of a pair of TTS(n)s in an orientable surface, [5], [4], [3].

(i) Each vertex has degree 3.

(ii) Each edge is assigned a current from the set $G \setminus \{0\}$ so that each current appears exactly once. Note that a current of i in one direction is equivalent to a current of $-i$ in the opposite direction.

(iii) At each vertex, the sum of the directed currents is the identity (*Kirchoff's current law, KCL*).

(iv) The directions (clockwise or anticlockwise) assigned to each vertex are such that a *complete circuit* is formed, that is, one in which every edge of the graph is traversed in each direction exactly once.

(v) The graph is bipartite.

Hence, such a current graph has $2(g - 1)/3$ vertices and $g - 1$ edges. We use a *Möbius ladder graph* with $(g - 1)/3$ "rungs". Set $u := (g - 1)/3$. The formal definition of such a graph is as follows. The vertex set is $\{x_i, y_i \mid 0 \leq i \leq u - 1\}$ and the edge set is $\{\{x_i, y_i\}, \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid 0 \leq i \leq u - 2\} \cup \{\{x_{u-1}, y_{u-1}\}, \{x_0, y_{u-1}\},$

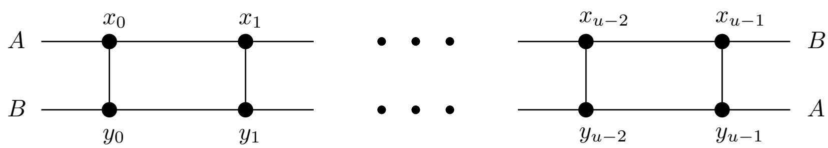


Figure 1: A Möbius ladder graph

$\{x_{u-1}, y_0\}\}$. In order to obtain a complete circuit, vertices x_i, $0 \leq i \leq u-1$, and y_{u-1} are assigned a clockwise direction and the vertices y_i, $0 \leq i \leq u-2$, an anticlockwise direction.

In this paper we represent these graphs as shown in Figure 1, where the directions of rotation are not indicated but implicit as defined above.

We build the Möbius ladder graphs with currents assigned to the edges, so that Conditions (ii) and (iii) are met, from *gadgets*, i.e. edge labelled subgraphs which we link together by concatenation; we will define the linking of two gadgets D_1 followed by D_2 by $D_1 : D_2$ and the sequential linking of k gadgets by $[D_i :]_{1 \leq i \leq k}$.

For ease of notation we will label the elements of $\mathbb{K}_4$ as x, y, z and 0, such that

$$x + x = y + y = z + z = x + y + z = 0. \tag{1.1}$$

Hence, the identity element in the group $\mathbb{Z}_n \times \mathbb{K}_4$ is $(0, 0)$.

2 The case n $\equiv$ 28 (mod 48)

Let $v = n/4 = 12m + 7$, $m \geq 0$. From [6] there exists a Möbius ladder graph that lifts under $\mathbb{Z}_v$ to yield a biembedding of a pair of STS(v)s. Let $\mathcal{L}$ be such a graph.

We begin by labelling Figure 1 as follows, $x_i = v_i$, $y_i = v_{i+2m+1}$, $0 \leq i \leq 2m$. Thus the bipartition of $\mathcal{L}$ consists of the sets of even-indexed and odd-indexed vertices respectively.

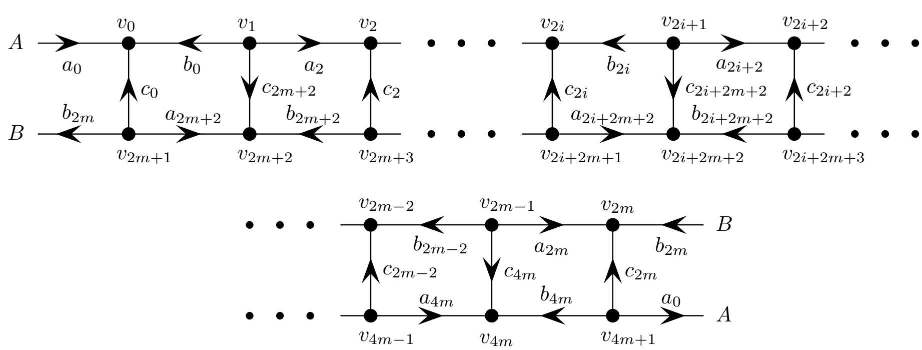


Figure 2: Vertex and edge labels of $\mathcal{L}$.

Noting that by replacing a directed edge label by its inverse in the opposite direction, we can arrange the directed edge labels so that they point in to even-indexed vertices and out of odd-indexed vertices. Thus the directed edge labels will be taken to be as shown in Figure 2.

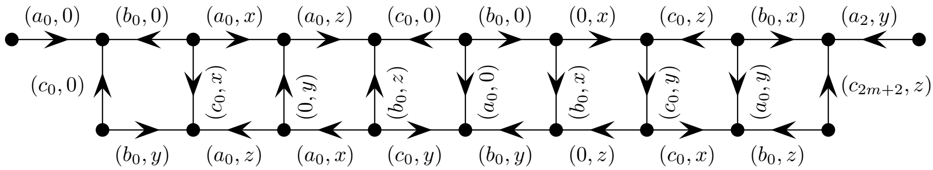
As $\mathcal{L}$ satisfies Kirchoff's current law we have the following equations.

$$a_i + b_i + c_i \equiv 0 \pmod{12m+7}, \tag{2.1}$$

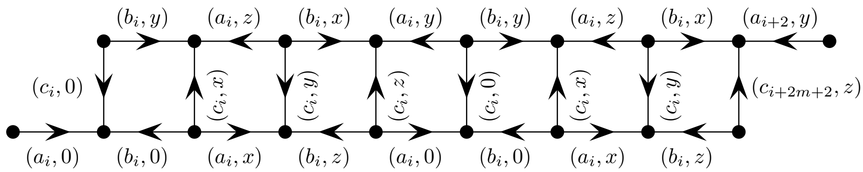
$$a_{i+2} + b_i + c_{i+2m+2} \equiv 0 \pmod{12m+7}. \tag{2.2}$$

We will require the following gadgets where indices are taken modulo $4m+2$:

X



Y_i



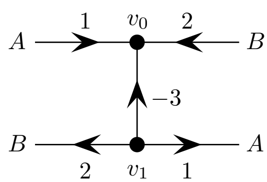
Equations (1.1), (2.1) and (2.2) verify that the gadgets X and, for $1 \le j \le 2m$, Y_{2j} satisfy KCL. Now consider the Möbius ladder graph $X : [Y_{2j} :]_{1 \le j \le 2m}$. Equations (1.1), (2.1) and (2.2) verify that this graph also satisfies KCL.

Hence, we can lift under the group $\mathbb{Z}_{12m+7} \times \mathbb{K}_4$. Thus, we can construct a biembedding of a pair of TTS($48m+28$)s, $m \ge 0$, with $\mathbb{Z}_{12m+7} \times \mathbb{K}_4$ as an automorphism group.

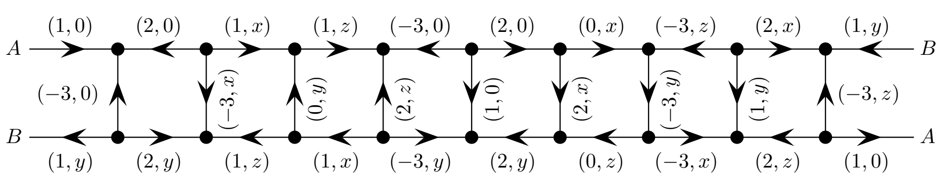
We conclude this section by giving two examples.

Example 2.1. Let $m = 0$, $v = 7$ and $n = 28$.

A Möbius ladder graph $\mathcal{L}$, which yields the well known toroidal biembedding of a pair of STS(7)s, has $a_0 = 1$, $b_0 = 2$ and $c_0 = -3$, i.e.

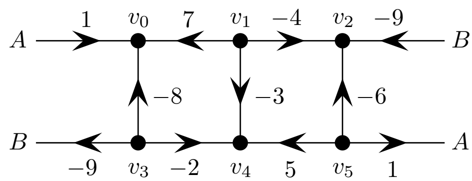


In this case our construction gives just the gadget X, labelled as follows.

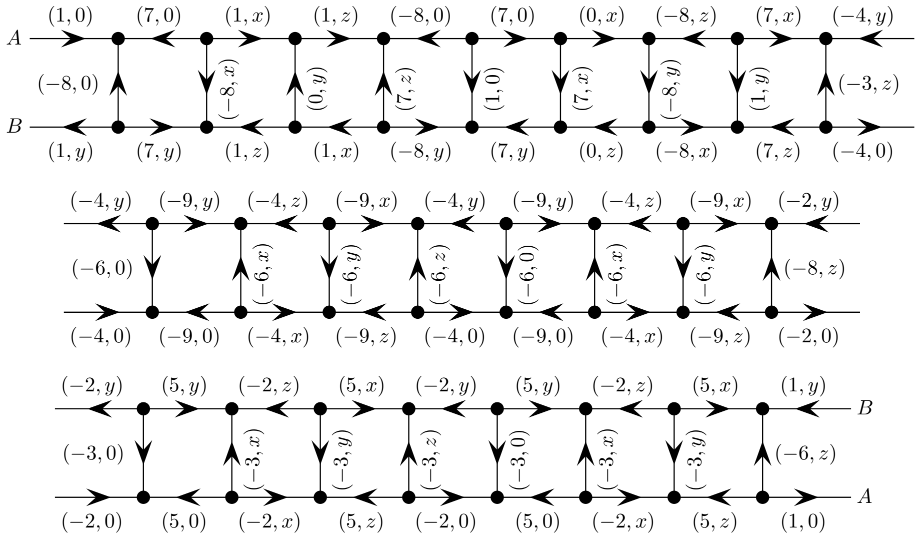


Example 2.2. Let $m = 1$, $v = 19$ and $n = 76$.

A Möbius ladder graph, $\mathcal{L}$, yielding a biembedding of a pair of STS(19), is as follows.

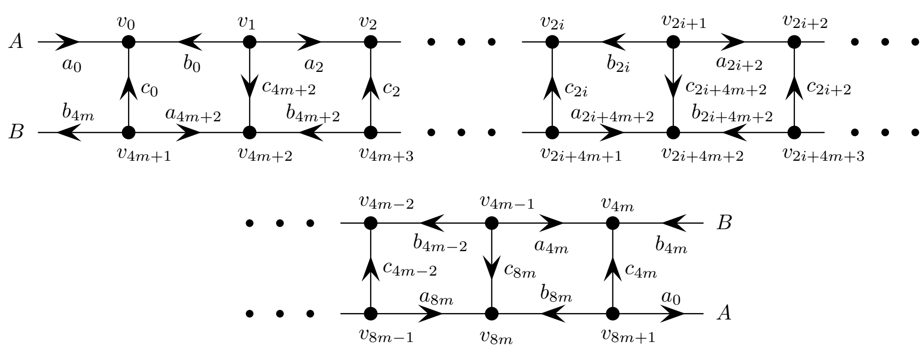


Thus $(a_0, b_0, c_0) = (1, 7, -8)$, $(a_2, b_2, c_2) = (-4, -9, -6)$ and $(a_4, b_4, c_4) = (-2, 5, -3)$. Our construction gives the following Möbius ladder graph.



3 The case $\mathrm{n} \equiv 16 \pmod{48}$

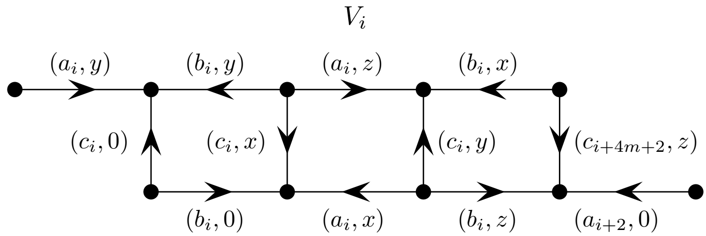
Let $v = n/4 = 12m + 4$, $m \geq 0$. In [2] a Möbius ladder graph $\mathcal{L}$ based on the Colbourn and Colbourn difference triples, [1], on the set $\mathbb{Z}_{12m+4} \setminus \{0\}$ was constructed. In that paper the ladders were lifted under the cyclic group $\mathbb{Z}_{12m+4}$ to yield a biembedding of a pair of TTS(v)s. Similarly to Section 2 label the vertices and edges of $\mathcal{L}$ as follows (taking indices modulo $8m + 2$):



where v_0 corresponds to the difference triple $3m+1, 3m+1, 6m+2$. Note that this means that the vertices with even indices correspond to the Colbourn and Colbourn difference triples.

Without loss of generality, in $\mathcal{L}$, either $a_0 = c_0 = 3m+1$ and $b_0 = 6m+2$ or $a_0 = b_0 = 3m+1$ and $c_0 = 6m+2$. Both of these cases occur in the Möbius ladder graphs constructed in [2], depending on the residue class of v modulo 72 and we consider them separately in Subsections 3.1 and 3.2, respectively.

We will make use of the following gadget where $2 \leq i \leq 8m$:



Note that in this case, because the initial Möbius ladder $\mathcal{L}$ yields a biembedding of a pair of twofold triple systems V_i, is simpler than the gadget used in Section 2. In fact it is "half" that gadget.

As $\mathcal{L}$ satisfies Kirchoff's current law we have the following two equations.

$$a_i + b_i + c_i \equiv 0 \pmod{12m+4}, \quad (3.1)$$
$$a_{i+2} + b_i + c_{i+4m+2} \equiv 0 \pmod{12m+4}. \quad (3.2)$$

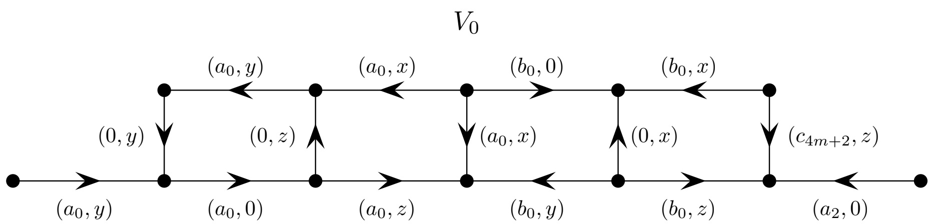
These two equations together with Equation (1.1) verify that V_i also satisfies KCL.

3.1 $\mathbf{a_0 = c_0 = 3m + 1}$ and $\mathbf{b_0 = 6m + 2}$

In this case it follows, from [2], that $\mathcal{L}$ yields the following two equations

$$(6m+2) + c_{4m+2} + a_2 \equiv 0 \pmod{12m+4}, \quad (3.3)$$
$$b_{8m} + c_{4m} + (3m+1) \equiv 0 \pmod{12m+4}. \quad (3.4)$$

In this case we will require the following gadget:



Equations (1.1) and (3.3) verify that V_0 satisfies KCL.

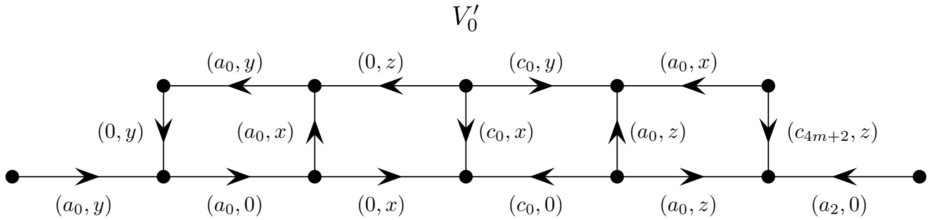
As $a_0 = 3m+1$, Equations (1.1), (3.1), (3.2), (3.3) and (3.4) verify that the Möbius ladder graph $V_0 : [V_i :]_{i=2j,\, 1 \leq j \leq 4m}$ satisfies KCL.

3.2 $\mathbf{a_0 = b_0 = 3m + 1}$ and $\mathbf{c_0 = 6m + 2}$

In this case it follows, from [2], that $\mathcal{L}$ yields the following two equations

$$(3m+1) + c_{4m+2} + a_2 \equiv 0 \,(\text{mod } 12m+4), \tag{3.5}$$
$$b_{8m} + c_{4m} + (3m+1) \equiv 0 \,(\text{mod } 12m+4). \tag{3.6}$$

In this case we require the following gadget:



Equations (1.1) and (3.5) verify that V_0' satisfies KCL.

As $a_0 = 3m + 1$, Equations (1.1), (3.1), (3.2), (3.5) and (3.6) verify that the Möbius ladder graph $V_0' : [V_i :]_{i=2j,\, 1\le j\le 4m}$ satisfies KCL.

Thus, we have constructed $(\mathbb{Z}_{12m+4} \times \mathbb{K}_4)$-biembeddings of a pair of TTS($48m+16$)s, $m \ge 0$.

Finally note that the gadget V_0 contains a vertex with currents $(3m+1, x)$, $(3m+1, x)$ and $(6m+2, 0)$ pointing outwards. Similarly the gadget V_0' contains a vertex with currents $(3m+1, z)$, $(3m+1, z)$ and $(6m+2, 0)$ also pointing outwards. We call this vertex α. Our constructed Möbius ladder graphs $\mathcal{L}'$ contain just one of these two gadgets. By reversing the directions on all of the edges of $\mathcal{L}'$, labelling the vertex α as v_0 and the edge with label $(6m + 2, 0)$ as b_0, and extending in the same manner as above, the construction, using the gadget V_0, can be reapplied. Repeated application of this process yields $(\mathbb{Z}_{12m+4} \times (\mathbb{K}_4)^s)$-biembeddings of a pair of TTS($4^s(12m + 4)$)s, $s \ge 1$, $m \ge 0$.

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Edmonds maps on the Fricke-Macbeath curve

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Abstract

In 1985, L. D. James and G. A. Jones proved that the complete graph K_n defines a clean dessin d'enfant (the bipartite graph is given by taking as the black vertices the vertices of K_n and the white vertices as middle points of edges) if and only if $n = p^e$, where p is a prime. Later, in 2010, G. A. Jones, M. Streit and J. Wolfart computed the minimal field of definition of them. The minimal genus $g > 1$ of these types of clean dessins d'enfant is $g = 7$, obtained for $p = 2$ and $e = 3$. In that case, there are exactly two such clean dessins d'enfant (previously known as Edmonds maps), both of them defining the Fricke-Macbeath curve (the only Hurwitz curve of genus 7) and both forming a chiral pair. The uniqueness of the Fricke-Macbeath curve asserts that it is definable over $\mathbb{Q}$, but both Edmonds maps cannot be defined over $\mathbb{Q}$; in fact they have as minimal field of definition the quadratic field $\mathbb{Q}(\sqrt{-7})$. It seems that no explicit models for the Edmonds maps over $\mathbb{Q}(\sqrt{-7})$ are written in the literature. In this paper we start with an explicit model X for the Fricke-Macbeath curve provided by Macbeath, which is defined over $\mathbb{Q}(e^{2\pi i/7})$, and we construct an explicit birational isomorphism $L : X \to Z$, where Z is defined over $\mathbb{Q}(\sqrt{-7})$, so that both Edmonds maps are also defined over that field.

Keywords: Riemann surface, algebraic curve, dessin d'enfant.

Math. Subj. Class.: 30F20, 30F10, 14Q05, 14H45, 14E05

1 Introduction

A dessin d'enfant D on a closed orientable surface is given by a bipartite map on it (vertices will be colored black and white). The dessin d'enfant is called clean if the white vertices have all valence 2.

A Belyi curve is an irreducible non-singular complex projective algebraic curve (i.e. a closed Riemann surface) S admitting a non-constant meromorphic map $\beta : S \to \widehat{\mathbb{C}}$ with

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at most three branch values; which we assume to be inside the set $\{\infty, 0, 1\}$; we say that (S, β) is a Belyi pair. Two Belyi pairs (S_1, β_1) and (S_2, β_2) are called equivalent, denoted this by the symbol $(S_1, \beta_1) \cong (S_2, \beta_2)$, if there is an isomorphism $f : S_1 \to S_2$ so that $\beta_2 \circ f = \beta_1$.

A subfield $\mathbb{K}$ of $\overline{\mathbb{Q}}$ is called a field of definition of a Belyi pair (S, β) if there an equivalent Belyi pair $(\widehat{S}, \widehat{\beta})$ where $\widehat{S}$ and $\widehat{\beta}$ are both defined over $\mathbb{K}$. As a consequence of Belyi's theorem [1], the field of algebraic numbers $\overline{\mathbb{Q}}$ is a field of definition of every Belyi pair.

Each Belyi pair (S, β) defines a dessin d'enfant on S by taking the edges as the components of $\beta^{-1}((0, 1))$, the black vertices as the points in $\beta^{-1}(0)$ and the white vertices as the points in $\beta^{-1}(1)$. Conversely, as a consequence of the uniformization theorem, every dessin d'enfant on a closed orientable surface induces a (unique up to isomorphisms) Riemann surface structure (being a Belyi curve) and a Belyi map so that the original dessin d'enfant is homotopic to the one associated to the Belyi pair [11, 15].

A field of definition of a dessin d'enfant is a field of definition of the corresponding Belyi pair.

As there is a natural (faithful) action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the collection of Belyi pairs [13], it also provides a (faithful) action on dessins d'enfant. This action may help in the study of the internal structure of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in terms of combinatorial data.

Let us consider a dessin d'enfant D, which is defined by the Belyi pair (S, β). By Belyi's theorem, we may assume that both S and β are defined over $\overline{\mathbb{Q}}$. The field of moduli of D is then defined as the fixed field of the subgroup $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : (S, \beta) \cong (S^\sigma, \beta^\sigma)\}$. The field of moduli of D is always contained in any field of definition of it, but it may be that the field of moduli is not a field of definition of it. Both, the computation of the field of moduli of a dessin d'enfant and to decide if the dessin d'enfant can be defined over it, are in general difficult problems. If the dessin d'enfant is regular, that is, the Belyi map β is a Galois branched cover, then J. Wolfart [19] proved that D can be defined over its field of moduli. Also, if the dessin d'enfant has no non-trivial automorphisms, then it is definable over its field of moduli as a consequence of Weil's descent theorem [16]. So, the problem to decide if the field of moduli is a field of definition appears when it has non-trivial automorphisms but it is non-regular.

In 1985, L. D. James and G. A. Jones [10] proved that the complete graph K_n defines a clean dessin d'enfant (the bipartite graph is given by taking as the black vertices the vertices of K_n and the white vertices as middle points of edges) if and only if $n = p^e$, where p is a prime. Later, in 2010, G. A. Jones, M. Streit and J. Wolfart [12] computed the minimal field of definition of such clean dessins d'enfant. The minimal genus $g > 1$ of these types of clean dessins d'enfants is $g = 7$, obtained for $p = 2$ and $e = 3$. In that case, there are exactly two (non-equivalent) such dessins (previously known as Edmonds maps), both of them defining the Fricke-Macbeath curve (the only Hurwitz curve of genus 7) and both forming a chiral pair. The uniqueness of the Fricke-Macbeath curve asserts that it is definable over $\mathbb{Q}$, but each of the two Edmonds maps cannot be defined over $\mathbb{Q}$; they have as minimal field of definition the quadratic field $\mathbb{Q}(\sqrt{-7})$ [12]. No explicit models for the Edmonds maps over $\mathbb{Q}(\sqrt{-7})$ seems to be written in the literature.

In Section 2 we recall an explicit model X for the Fricke-Macbeath curve provided by Macbeath, which is defined over $\mathbb{Q}(e^{2\pi i/7})$, and describe both Edmonds maps. We also provide (as matter of interest for specialists) two different equations, over $\mathbb{Q}$, for the Fricke-Macbeath curve which were independently obtained by Bradley Brock (personal communication) and by Maxim Hendriks in his Ph.D. Thesis [7]. In Section 3 we provide an explicit birational isomorphism $L : X \to Z$, where Z is defined over $\mathbb{Q}(\sqrt{-7})$. In this model we obtain that the two Belyi maps defining the two Edmonds maps are defined over $\mathbb{Q}$; in particular, this provides explicit models for both Edmonds maps over $\mathbb{Q}(\sqrt{-7})$ as desired. In Section 4 we provide an explicit birational isomorphism $\widehat{L} : X \to W$, where W is defined over $\mathbb{Q}$. Unfortunately, the explicit equations over $\mathbb{Q}$ are not very simple (they are long ones) and they can be found in [9]. In Section 5 we construct a generalized Fermat curve $\widehat{S}$ of type $(2,6)$ [5] that covers the Fricke-Macbeath curve and we provide a description of the three elliptic curves appearing in the equations of X given by Macbeath. Another model of the Fricke-Macbeath curve is also described.

2 Macbeath's equations of the Fricke-Macbeath curve and the two Edmonds maps

In this section we recall equations of the Fricke-Macbeath curve, obtained by Macbeath in [14], and we describe both Edmonds maps discovered in [12]. As a matter of interest to specialists, we also describe two different models over $\mathbb{Q}$, one obtained by Bradley Brock (personal communication) and the other by Maxim Hendriks in his Ph.D. Thesis [7].

2.1 Hurwitz curves

It is well known that $|\mathrm{Aut}(S)| \leq 84(g-1)$ (Hurwitz upper bound) if S is a closed Riemann surface of genus $g \geq 2$. In the case that $|\mathrm{Aut}(S)| = 84(g-1)$, one says that S is a Hurwitz curve. In this last situation, the quotient orbifold $S/\mathrm{Aut}(S)$ has signature $(0; 2, 3, 7)$, that is, $S = \mathbb{H}^2/\Gamma$, where Γ is a torsion free normal subgroup of finite index in the triangular Fuchsian group $\Delta = \langle x, y : x^2 = y^3 = (xy)^7 = 1\rangle$ acting as isometries of the hyperbolic plane $\mathbb{H}^2$.

Wiman [17] noticed that there is no Hurwitz curve in each genera $g \in \{2, 4, 5, 6\}$ and there is exactly one Hurwitz curve (up to isomorphisms) of genus three, this being Klein's quartic $x^3y + y^3z + z^3x = 0$; whose automorphisms group is the simple group $\mathrm{PSL}(2, 7)$ (of order 168).

2.2 Macbeath's equations of the Fricke-Macbeath curve

In [14] Macbeath observed that in genus seven there is exactly one (up to isomorphisms) Hurwitz curve, called the Fricke-Macbeath curve; its automorphisms group is the simple group $\mathrm{PSL}(2, 8)$, consisting of 504 symmetries. In the same paper, Macbeath computed the following explicit equations over $\mathbb{Q}(\rho)$, where $\rho = e^{2\pi i/7}$, for the Fricke-Macbeath curve (involving three particular elliptic curves):

$$X = \left\{ \begin{array}{l} y_1^2 = (x-1)(x-\rho^3)(x-\rho^5)(x-\rho^6) \\ \\ y_2^2 = (x-\rho^2)(x-\rho^4)(x-\rho^5)(x-\rho^6) \\ \\ y_4^2 = (x-\rho)(x-\rho^3)(x-\rho^4)(x-\rho^5) \end{array} \right\} \subset \mathbb{C}^4. \quad (2.1)$$

In Section 5 we provide a rough explanation about the elliptic curves in the above equations (different from the approach given in [14]) in geometric terms of the highest regular branched Abelian cover of the orbifold X/G of signature $(0; 2, 2, 2, 2, 2, 2, 2)$.

Another interesting fact on the Fricke-Macbeath curve is that its jacobian variety is isogenous to E^7 where E is the elliptic curve with j-invariant $j(E) = 1792$ (E does not have complex multiplication); see, for instance, [2]. There are not to many explicit examples of Riemann surfaces whose jacobian variety is isogenous to the product of elliptic curves (see [6]).

2.3 Equations over $\mathbb{Q}$ of the Fricke-Macbeath curve

The uniqueness up to isomorphisms of the Fricke-Macbeath curve asserts that its field of moduli is the field of rational numbers $\mathbb{Q}$. As quasiplatonic curves can be defined over their fields of moduli [19] and Hurwitz curve are quasiplatonic curves, it follows that the Fricke-Macbeath curve can be defined over $\mathbb{Q}$. When the author put a first version of this paper in Arxiv [9] we didn't know of explicit equations of the Fricke-Macbeath curve over $\mathbb{Q}$. Later, Bradley Brock sent me an e-mail in which he told me that, using some suitable change of coordinates on the above equations for X, he was able to compute a plane equation for X over $\mathbb{Q}$, with some simple nodes as singularities, given as

$$1 + 7xy + 21x^2y^2 + 35x^3y^3 + 28x^4y^4 + 2x^7 + 2y^7 = 0.$$

An automorphism of order 7 is given by $b(x,y) = (\rho x, \rho^{-1}y)$ and one of order two is given by $a_1(x,y) = (y,x)$.

The following model over $\mathbb{Q}$, for the Fricke-Macbeath curve, was recently computed by Maxim Hendriks in his Ph.D. Thesis [7]

$$\left\{\begin{array}{c}
-x_1x_2 + x_1x_0 + x_2x_6 + x_3x_4 - x_3x_5 - x_3x_0 - x_4x_6 - x_5x_6 = 0,\\
x_1x_3 + x_1x_6 - x_2^2 + 2x_2x_5 + x_2x_0 - x_3^2 + x_4x_5 - x_4x_0 - x_5^2 = 0,\\
x_1^2 - x_1x_3 + x_2^2 - x_2x_4 - x_2x_5 - x_2x_0 - x_3^2 + x_3x_6 + 2x_5x_0 - x_0^2 = 0,\\
x_1x_4 - 2x_1x_5 + 2x_1x_0 - x_2x_6 - x_3x_4 - x_3x_5 + x_5x_6 + x_6x_0 = 0,\\
x_1^2 - 2x_1x_3 - x_2^2 - x_2x_4 - x_2x_5 + 2x_2x_0 + x_3^2 + x_3x_6 + x_4x_5 + x_5^2 - x_5x_0 - x_6^2 = 0,\\
x_1x_2 - x_1x_5 - 2x_1x_0 + 2x_2x_3 - x_3x_0 - x_5x_6 + 2x_6x_0 = 0,\\
-2x_1x_2 - x_1x_4 - x_1x_5 + 2x_1x_0 + 2x_2x_3 - 2x_3x_0 + 2x_5x_6 - x_6x_0 = 0,\\
2x_1^2 + x_1x_3 - x_1x_6 + 3x_2x_0 + x_4x_5 - x_4x_0 - x_5^2 + x_6^2 - x_0^2 = 0,\\
2x_1^2 - x_1x_3 + x_1x_6 + x_2^2 + x_2x_0 + x_3^2 - 2x_3x_6 + x_4x_5 - x_4x_0 + x_5^2 - 2x_5x_0 + x_6^2 + x_0^2 = 0,\\
x_1^2 + x_1x_3 - x_1x_6 + 2x_2x_5 - 3x_2x_0 + 2x_3x_6 + x_4^2 + x_4x_5 - x_4x_0 + x_6^2 + 3x_0^2 = 0
\end{array}\right\} \subset \mathbb{P}^6.$$

In Section 4 we provide an explicit birational isomorphism $\widehat{L} : X \to W$, where W is defined over $\mathbb{Q}$. The explicit form of $\widehat{L}$ may be used to compute explicit equation for W; this can be done with MAGMA [3].

2.4 A description of the two Edmonds maps

In the above model X of the Fricke-Macbeath curve it is easy to see a group $\mathbb{Z}_2^3 \cong G = \langle A_1, A_2, A_3 \rangle < \mathrm{Aut}(X)$ where

$$A_1(x, y_1, y_2, y_4) = (x, -y_1, y_2, y_4),$$

$$A_2(x, y_1, y_2, y_4) = (x, y_1, -y_2, y_4),$$

$$A_3(x, y_1, y_2, y_4) = (x, y_1, y_2, -y_4).$$

The quotient Riemann orbifold X/G has signature $(0; 2, 2, 2, 2, 2, 2, 2)$, that is, is the Riemann sphere with exactly 7 cone points of order 2.

An automorphism of order 7 of the Fricke-Macbeath curve is given in such a model by

$$B(x, y_1, y_2, y_4) = \left(\rho x, \rho^2 y_2, \rho^2 y_4, \rho^2 \frac{y_1 y_2}{(x-\rho^5)(x-\rho^6)}\right).$$

The automorphism B normalizes G and it induces, on the orbifold $X/G = \widehat{\mathbb{C}}$, the rotation $T(x) = \rho x$. Moreover, $X/\langle G, B\rangle$ has signature $(0; 2, 7, 7)$, that is, the group $\langle G, B\rangle$ defines a regular dessin d'enfant (X, β), where $\beta(x, y_1, y_2, y_4) = x^7$ (this is one of the two Edmonds maps, but is defined over $\mathbb{Q}(\rho)$).

We may also see that X admits the following anticonformal involution

$$J(x, y_1, y_2, y_4) = \left(\frac{1}{\overline{x}}, \frac{\overline{y_1}}{\overline{x}^2}, \frac{\rho^5 \overline{y_2}}{\overline{x}^2}, \frac{\rho^3 \overline{y_4}}{\overline{x}^2}\right).$$

It can be seen that $JBJ = B$ and $JA_jJ = A_j$, for $j = 1, 2, 3$. In this way, one gets another regular dessin d'enfant (X, δ), where $\delta(x, y_1, y_2, y_4) = 1/x^7$ (this is the other Edmonds map, again defined over $\mathbb{Q}(\rho)$).

As $\delta = C \circ \beta \circ J$, where $C(x) = \overline{x}$, we have that the two regular dessins d'enfant described above are chirals.

3 An explicit model of the Edmonds maps over $\mathbb{Q}(\sqrt{-7})$

In this section we will construct an explicit biregular isomorphism $L : X \to Z$, where Z is defined over $\mathbb{Q}(\sqrt{-7})$, so that both Edmonds maps are defined over such a field.

Note that $\mathbb{Q}(\sqrt{-7}) = \mathbb{Q}(\rho + \rho^2 + \rho^4)$ since $\rho + \rho^2 + \rho^4 = \frac{1}{2}(\sqrt{-7} - 1)$. Most of the computations have been carried out with MAGMA [3] or with MATHEMATICA [20].

3.1

Let $N = \mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q}(\sqrt{-7})) = \langle \tau \rangle \cong \mathbb{Z}_3$, where $\tau(\rho) = \rho^2$. If we set

$$\vec{x} = (x_1, x_2, x_3, x_4) = (x, y_1, y_2, y_4),$$

then it is not difficult to check that $\{f_e = I, f_\tau, f_{\tau^2}\}$ is a Weil datum (i.e., they satisfies the Weil co-cycle condition in Weil's descent theorem [16]) with respect to the Galois extension $\mathbb{Q}(\rho)/\mathbb{Q}(\sqrt{-7})$, where I denotes the identity and

$$f_\tau(\vec{x}) = \left(x, y_1, y_4, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}\right),$$

$$f_{\tau^2}(\vec{x}) = \left(x, y_1, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}, y_2\right).$$

3.2

Let us consider the rational map

$$\Phi_1 : X \to \mathbb{C}^{12}$$

$$(x, y_1, y_2, y_4) \mapsto (\vec{x}, \vec{w}, \vec{v}),$$

where

$$\vec{w} = (w_1, w_2, w_3, w_4) = f_\tau(\vec{x}),$$

$$\vec{v} = (v_1, v_2, v_3, v_4) = f_{\tau^2}(\vec{x}).$$

We may see that Φ_1 produces a birational isomorphism between X and $\Phi_1(X)$ (its inverse is just the projection on the $\vec{x}$-coordinate). Equations defining the algebraic curve $\Phi_1(X)$ are the following ones

$$\Phi_1(X) = \left\{ \begin{array}{l} x_2^2 = (x_1 - 1)(x_1 - \rho^3)(x_1 - \rho^5)(x_1 - \rho^6) \\ x_3^2 = (x_1 - \rho^2)(x_1 - \rho^4)(x_1 - \rho^5)(x_1 - \rho^6) \\ x_4^2 = (x_1 - \rho)(x_1 - \rho^3)(x_1 - \rho^4)(x_1 - \rho^5) \\ w_1 = x_1,\ w_2 = x_2,\ w_3 = x_4,\ w_4 = \dfrac{x_3 x_4}{(x_1 - \rho^4)(x_1 - \rho^5)}, \\ v_1 = x_1,\ v_2 = x_2,\ v_3 = \dfrac{x_3 x_4}{(x_1 - \rho^4)(x_1 - \rho^5)},\ v_4 = x_3 \end{array} \right\} \tag{3.1}$$

3.3

We consider the linear permutation action of N on the coordinates of $\mathbb{C}^{12}$ defined by

$$\Theta_1(\tau)(\vec{x}, \vec{w}, \vec{v}) = (\vec{w}, \vec{v}, \vec{x}).$$

Let us notice that the stabilizer of $\Phi_1(X)$, with respect to the above permutation action, is trivial since

$$\{\eta \in N : \Theta_1(\eta)(\Phi_1(X)) = \Phi_1(X)\} = \{\eta \in N : X^\eta = X\} = \{e\}.$$

3.4

Each $\theta \in \mathrm{Gal}(\mathbb{C})$ induces a natural bijection

$$\widehat{\theta} : \mathbb{C}^{12} \to \mathbb{C}^{12} : (y_1, \ldots, y_{12}) \mapsto (\theta(y_1), \ldots, \theta(y_{12})).$$

It is not hard to see that $\widehat{\theta}(X) = X^\theta$.

3.5

If $\theta \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\sqrt{-7}))$, then we denote by θ_N is projection in N. With this notation, we see that the following diagram commutes (see also [8])

$$\begin{array}{lcl} X & \stackrel{\Phi_1}{\to} & \Phi_1(X) \\ \downarrow f_{\theta_N} & & \downarrow \Theta_1(\theta_N) \\ X^{\theta_N} & \stackrel{\Phi_1^{\theta_N}}{\to} & \Theta_1(\theta_N)(\Phi_1(X)) = \Phi_1^{\theta_N}(X^{\theta_N}) = \Phi_1(X)^{\theta_N} \\ \downarrow \widehat{\theta}^{-1} & & \downarrow \widehat{\theta}^{-1} \\ X & \stackrel{\Phi_1}{\to} & \Phi_1(X) \end{array} \tag{3.2}$$

and, for every $\eta, \theta \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\sqrt{-7})))$, that

$$(*) \quad \Theta_1(\eta_N) \circ \widehat{\theta} = \widehat{\theta} \circ \Theta_1(\eta_N).$$

3.6

A generating set of invariant polynomials for the linear action $\Theta_1(N)$ can be obtained with MAGMA as

$$\begin{array}{rclrcl} t_1 & = & x_1+w_1+v_1, & t_2 & = & x_2+w_2+v_2 \\ t_3 & = & x_3+w_3+v_3, & t_4 & = & x_4+w_4+v_4 \\ t_5 & = & x_1^2+w_1^2+v_1^2, & t_6 & = & x_2^2+w_2^2+v_2^2 \\ t_7 & = & x_3^2+w_3^2+v_3^2, & t_8 & = & x_4^2+w_4^2+v_4^2 \\ t_9 & = & x_1^3+w_1^3+v_1^3, & t_{10} & = & x_2^3+w_2^3+v_2^3 \\ t_{11} & = & x_3^3+w_3^3+v_3^3, & t_{12} & = & x_4^3+w_4^3+v_4^3 \end{array}$$

The map

$$\Psi_1 : \mathbb{C}^{12} \to \mathbb{C}^{12}$$

$$(\vec{x}, \vec{w}, \vec{v}) \mapsto (t_1, ..., t_{12})$$

clearly satisfies the following properties:

$$\begin{cases} \Psi_1^{\tau^j} = \Psi_1, \; j = 0, 1, 2; \\ \\ \Psi_1 \circ \Theta_1(\tau^j) = \Psi_1, \; j = 0, 1, 2. \end{cases} \tag{3.3}$$

Also (as we have chosen a set of generators of the invariant polynomials for the action of $\Theta_1(N)$), it holds that Ψ_1 is a branched regular cover with Galois group N. It turns out that, if we set $Z_1 = \Psi_1(\Phi_1(X))$ and $L_1 = \Psi_1 \circ \Phi_1$, then

$$L_1 : X \to Z_1$$

is a birational isomorphism (since the stabilizer of $\Phi_1(X)$ is trivial).

3.7

If $\eta \in N$, then

$$Z_1^{\eta} = L_1(X)^{\eta} = L_1^{\eta}(X^{\eta}) = \Psi_1^{\eta} \circ \Phi_1^{\eta}(X^{\eta}) = \Psi_1 \circ \Theta_1(\eta)(\Phi_1(X)) = \\ \Psi_1 \circ \Phi_1(X) = L_1(X) = Z_1,$$

so Z_1 can be defined by polynomials with coefficient over $\mathbb{Q}(\sqrt{-7})$.

3.8

Next, we proceed to compute explicit equations for Z_1 and the inverse $L_1^{-1} : Z_1 \to X$.

The following equalities hold:

$$x_1 = \frac{t_1}{3}, \quad x_2 = \frac{t_2}{3}, \quad t_4 = t_3$$

$$(*) \; x_4 = \frac{(t_3 - x_3)(\frac{t_1}{3} - \rho^4)(\frac{t_1}{3} - \rho^5)}{x_3 + (\frac{t_1}{3} - \rho^4)(\frac{t_1}{3} - \rho^5)}$$

$$t_5 = \frac{t_1^2}{3}, \quad t_6 = \frac{t_2^2}{3}, \quad t_8 = t_7$$

$$(**)\ x_4^2 = \frac{(t_7 - x_3^2)(\frac{t_1}{3} - \rho^4)^2(\frac{t_1}{3} - \rho^5)^2}{x_3^2 + (\frac{t_1}{3} - \rho^4)^2(\frac{t_1}{3} - \rho^5)^2}$$

$$t_9 = \frac{t_1^3}{9}, \quad t_{10} = \frac{t_2^3}{9}, \quad t_{12} = t_{11}$$

$$(***)\ x_4^3 = \frac{(t_{11} - x_3^3)(\frac{t_1}{3} - \rho^4)^3(\frac{t_1}{3} - \rho^5)^3}{x_3^3 + (\frac{t_1}{3} - \rho^4)^3(\frac{t_1}{3} - \rho^5)^3}$$

Equality $(*)$ permits to obtain x_4 uniquely in terms of t_1 and x_3 and the equation

$$x_2^2 = (x_1 - 1)(x_1 - \rho^3)(x_1 - \rho^5)(x_1 - \rho^6)$$

provides a polynomial equation (relating t_1 and t_2) given by $P_1(t_1, t_2, t_3, t_7, t_{11}) = 0$, where

$$P_1(t_1, t_2, t_3, t_7, t_{11})$$
$$||$$
$$-81+27(1+(\rho+\rho^2+\rho^4))t_1+9t_1^2-3(\rho+\rho^2+\rho^4)t_1^3-t_1^4+9t_2^2 \in \mathbb{Q}(\sqrt{-7})[t_1, t_2, t_3, t_7, t_{11}].$$

Equation

$$x_3^2 = (x_1 - \rho^2)(x_1 - \rho^4)(x_1 - \rho^5)(x_1 - \rho^6)$$

permits to obtain the new equation

$$(1)\ x_3^2 = (t_1 - 3\rho^2)(t_1 - 3\rho^4)(t_1 - 3\rho^5)(t_1 - 3\rho^6)/81,$$

and the equation

$$x_4^2 = (x_1 - \rho)(x_1 - \rho^3)(x_1 - \rho^4)(x_1 - \rho^5)$$

provides the equation

$$(2)\ x_4^2 = (t_1 - 3\rho)(t_1 - 3\rho^3)(t_1 - 3\rho^4)(t_1 - 3\rho^5)/81.$$

In this way, by replacing the above values for x_3^2 and x_4^2 (obtained in (1) and (2)) in the equality $(**)$, we obtain the polynomial equation $P_2(t_1, t_2, t_3, t_7, t_{11}) = 0$, where

$$P_2(t_1, t_2, t_3, t_7, t_{11})$$
$$||$$
$$27 + 27(\rho + \rho^2 + \rho^4) - 18t_1 - 3(1 + (\rho + \rho^2 + \rho^4))t_1^2 - 2t_1^3 - t_1^4 + 27t_7 \in \mathbb{Q}(\sqrt{-7})[t_1, t_2, t_3, t_7, t_{11}].$$

Now, if we replace, in equality $(***)$, x_3^3 by $x_3(x_1-\rho^2)(x_1-\rho^4)(x_1-\rho^5)(x_1-\rho^6)/81$ and x_4^3 by $x_4(t_1 - 3\rho)(t_1 - 3\rho^3)(t_1 - 3\rho^4)(t_1 - 3\rho^5)/81$, where x_4 is given in $(*)$, then we obtain a polynomial which is of degree one in the variable x_3.

$x_3 = (-9\rho^2(-162t_1 - 18t_1^3 + 4t_1^5 - 243(1+t_{11}) + t_1^2(27 - 54t_3) + 6t_1^4 t_3) + 3(729 + 18t_1^4 - 6t_1^5 - 27t_1^3(-6+t_3) - t_1^6(-2+t_3) + 243t_1(3+t_3) + 81t_1^2(2+t_{11}+t_3)) + \rho^3(2187 -$

$t_1^7+27t_1^4(-6+t_3)+9t_1^5(-3+t_3)+486t_1^2t_3+81t_1^3(1+t_3)+729t_1(1+2t_3))+\rho^5(2187+27t_1^4+12t_1^6+t_1^7-729t_1(-1+t_{11}-t_3)+729t_1^2t_3+81t_1^3(5+t_3)+9t_1^5(1+2t_3))+\rho(2916t_1+3t_1^6-t_1^7-81t_1^3(-6+t_3)-2187(-2+t_3)-27t_1^4(-2+t_3)+9t_1^5(2+t_3)+243t_1^2(5+2t_3))+\rho^4(2187+t_1^7-729t_1(-3+t_{11}-2t_3)-81t_1^3(-1+t_3)+27t_1^4(1+t_3)+9t_1^5(-1+2t_3)+243t_1^2(1+3t_3)))/(9(t_1^5-243t_{11}+27t_1^2(-1+t_3)+81t_1t_3+9t_1^3t_3+3t_1^4t_3+\rho(3+t_1)(-81+18t_1^2-9t_1^3+2t_1^4+27t_1t_3)+27\rho^2t_1(3+t_1^2+t_1(3+t_3))+\rho^4t_1(243+3t_1^3+t_1^4+9t_1^2(-1+t_3)+27t_1(3+t_3))+\rho^5(-6t_1^4+t_1^5+243(1+t_3)+81t_1(2+t_3)+9t_1^3(2+t_3)+27t_1^2(3+t_3))+\rho^3t_1(162+36t_1^2+6t_1^3+2t_1^4+27t_1(4+t_3))))$

Then, using $(*)$, we obtain

$x_4=-((3\rho^4-t_1)(3\rho^5-t_1)(-\rho^3(-2187-729t_1+t_1^7+243t_1^2t_3(2+t_3)+9t_1^5(3+t_3)+27t_1^4(6+t_3)+81t_1^3(-1+3t_3))+\rho^4(2187+27t_1^4+t_1^7+9t_1^5(-1+t_3)-729t_1(-3+t_{11}+t_3)-243t_1^2(-1+t_3^2)-81t_1^3(-1+t_3^2))+\rho(4374+486t_1^3+54t_1^4+3t_1^6-t_1^7-9t_1^5(-2+t_3)-243t_1^2(-5+t_3^2)-729t_1(-4-t_3+t_3^2))-3(t_1^6(-2+t_3)+3t_1^5(2+t_3)-729(1+t_{11}t_3)-81t_1^2(2+t_{11}+2t_3-t_3^2)+9t_1^4(-2+t_3^2)+243t_1(-3-t_3+t_3^2)+27t_1^3(-6+t_3+t_3^2))-9\rho^2(4t_1^5-243(1+t_{11})+81t_1(-2+t_3)+6t_1^4t_3+9t_1^3(-2+3t_3)+27t_1^2(1+t_3+t_3^2))+\rho^5(12t_1^6+t_1^7-243t_1^2t_3^2+9t_1^5(1+t_3)+27t_1^4(1+2t_3)-81t_1^3(-5+t_3+t_3^2)-2187(-1+t_3+t_3^2)-729t_1(-1+t_{11}+t_3+t_3^2))))/(9(567t_1^3+6t_1^6+t_1^7+\rho(-3+t_1)(-54t_1^3+t_1^6+9t_1^4(-4+t_3)+729(-2+t_3)+243t_1(-2+t_3)-81t_1^2(-2+t_3))+27t_1^4(-7+t_3)+9t_1^5(-5+t_3)+2187(2+t_3)+243t_1^2(-1+2t_3)+729t_1(1+2t_3)+\rho^5(2187+216t_1^4+3t_1^6+2t_1^7+729t_1t_3+729t_1^2(1+t_3)+18t_1^5(2+t_3)+81t_1^3(16+t_3))+\rho^3t_1(9t_1^5+t_1^6+27t_1^3(-4+t_3)+9t_1^4(3+t_3)+81t_1^2(5+t_3)+729(-5+2t_3)+243t_1(-3+2t_3))+\rho^4(2187+6t_1^6+2t_1^7-81t_1^3(-14+t_3)+18t_1^5(-2+t_3)+1458t_1t_3+27t_1^4(5+t_3)+243t_1^2(1+3t_3))-9\rho^2(-243+243t_1-27t_1^3+t_1^5-54t_1^2(-5+t_3)+t_1^4(-9+6t_3)))).$

Now, using such values for x_3 and x_4, and replacing them in (1) and (2) above, we obtain another two polynomials identities $P_3(t_1,t_3,t_7,t_{11})=0$ and $P_4(t_1,t_3,t_7,t_{11})=0$, where these two new polynomials are defined over $\mathbb{Q}(\rho)$ (see [9] for these long polynomials). In this way, we have obtained the following equations over $\mathbb{Q}(\rho)$ for Z_1:

$$Z_1=\left\{\begin{array}{c} t_4=t_3,\ 3t_5=t_1^2,\ 3t_6=t_2^2,\ t_8=t_9\\ 9t_9=t_1^3,\ 9t_{10}=t_2^3,\ t_{12}=t_{11}\\ P_1(t_1,t_2,t_3,t_7,t_{11})=0\\ P_2(t_1,t_2,t_3,t_7,t_{11})=0\\ P_3(t_1,t_2,t_3,t_7,t_{11})=0\\ P_4(t_1,t_2,t_3,t_7,t_{11})=0 \end{array}\right\}\subset\mathbb{C}^{12}$$

Notice that, by the above computations, we have explicitly the inverse of L_1 given as

$$L_1^{-1}:Z_1\to X$$

$$(t_1,...,t_{12})\mapsto(x_1,x_2,x_3,x_4),$$

where x_1, x_2, x_3 and x_4 are in terms of t_1, t_2, t_3, t_7 and t_{11}.

As the variables t_1,..., t_{12} are uniquely determined only by the variables t_1, t_2, t_3, t_7 and t_{11}, if we consider the projection

$$\pi:\mathbb{C}^{12}\to\mathbb{C}^5$$

$$(t_1,...,t_{12})\mapsto(t_1,t_2,t_3,t_7,t_{11}),$$

then

$$L = \pi \circ L_1 : X \to Z$$

$$L_1^*(x, y_1, y_2, y_4)$$

$$||$$

$$\left(3x, 3y_1, y_2 + y_4 + \tfrac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}, y_2^2 + y_4^2 + \tfrac{y_2^2 y_4^2}{(x-\rho^4)^2(x-\rho^5)^2}, y_2^3 + y_4^3 + \tfrac{y_2^3 y_4^3}{(x-\rho^4)^3(x-\rho^5)^3}\right)$$

is a birational isomorphism, where

$$Z = \left\{ \begin{array}{c} P_1(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_2(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_3(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_4(t_1, t_2, t_3, t_7, t_{11}) = 0 \end{array} \right\} \subset \mathbb{C}^5$$

The inverse $L^{-1} : Z \to X$ is given as

$$L^{-1}(t_1, t_2, t_3, t_7, t_{11}) = (x_1, x_2, x_3, x_4).$$

We have obtained equations for Z over $\mathbb{Q}(\rho)$. But, as $Z_1^\eta = Z_1$, for every $\eta \in N$, and π is defined over $\mathbb{Q}$, we may see that $Z^\eta = Z$, for every $\eta \in N$, that is, Z can be defined by polynomials over $\mathbb{Q}(\sqrt{-7})$. To obtain such equations over $\mathbb{Q}(\sqrt{-7})$, we replace each polynomial P_j ($j = 3, 4$) by the new polynomials (with coefficients in $\mathbb{Q}(\sqrt{-7})$)

$$Q_{j,1} = \mathrm{Tr}(P_j),\ Q_{j,2} = \mathrm{Tr}(\rho P_j),\ Q_{j,3} = \mathrm{Tr}(\rho^2 P_j)$$

that is

$$Z = \left\{ \begin{array}{c} P_1(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_2(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_3(t_1, t_2, t_3, t_7, t_{11}) + P_3(t_1, t_2, t_3, t_7, t_{11})^\tau + P_3(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho P_3(t_1, t_2, t_3, t_7, t_{11}) + \rho^2 P_3(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho^4 P_3(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho^2 P_3(t_1, t_2, t_3, t_7, t_{11}) + \rho^4 P_3(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho P_3(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ P_4(t_1, t_2, t_3, t_7, t_{11}) + P_4(t_1, t_2, t_3, t_7, t_{11})^\tau + P_4(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho P_4(t_1, t_2, t_3, t_7, t_{11}) + \rho^2 P_4(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho^4 P_4(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho^2 P_4(t_1, t_2, t_3, t_7, t_{11}) + \rho^4 P_4(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho P_4(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \end{array} \right\} \subset \mathbb{C}^5$$

We have obtained an explicit model Z for the Fricke-Macbeath curve over $\mathbb{Q}(\sqrt{-7})$ together explicit birational isomorphisms $L : X \to Z$ and $L^{-1} : Z \to X$.

3.9

Finally, notice that the regular dessin d'enfant (X, β), given before, is isomorphic to that provided by the pair (Z, β^*), where $\beta^*(t_1, t_2, t_3, t_7, t_{11}) = \beta \circ L^{-1}(t_1, t_2, t_3, t_7, t_{11}) = (t_1/3)^7$; that is, the dessin d'enfant is defined over $\mathbb{Q}(\sqrt{-7})$.

4 An explicit isomorphism $L : X \to W$ where W is defined over $\mathbb{Q}$

Next we explain how to construct an explicit birational isomorphism $\widehat{L} : X \to W$, where W is known to be defined over $\mathbb{Q}$.

Let us consider the explicit model $Z \subset \mathbb{C}^5$ over $\mathbb{Q}(\sqrt{-7})$ constructed above. Let $M = \mathrm{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q}) = \langle \eta \rangle \cong \mathbb{Z}_2$, where η is the complex conjugation. As already noticed, since X admits the anticonformal involution J (defined previously), the curve Z admits the anticonformal involution $T = L \circ J \circ L^{-1}$. It is not difficult to see that by setting $g_e = I$ and $g_\eta = S \circ T$, where $S(t_1, t_2, t_3, t_7, t_{11}) = (\overline{t_1}, \overline{t_2}, \overline{t_3}, \overline{t_7}, \overline{t_{11}})$, we obtain a Weil datum for the Galois extension $\mathbb{Q}(\sqrt{-7})/\mathbb{Q}$. Now, identically as done above, we consider the rational map

$$\Phi_2 : Z \to \mathbb{C}^{10}$$

$$(t_1, t_2, t_3, t_7, t_{11}) \mapsto (t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11})$$

where $g_\eta(t_1, t_2, t_3, t_7, t_{11}) = (s_1, s_2, s_3, s_7, s_{11})$. We may see that Φ_2 induces a birational isomorphism between Z and $\Phi_2(Z)$. In this case,

$$\Phi_2(Z) = \left\{ \begin{array}{c} Q_{1,1}(t_1, t_2, t_3, t_7, t_{11}) = \cdots = Q_{4,3}(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ g_\eta(t_1, t_2, t_3, t_7, t_{11}) = (s_1, s_2, s_3, s_7, s_{11}) \end{array} \right\} \subset \mathbb{C}^{10}.$$

The Galois group M induces the permutation action $\Theta_2(M)$ defined as

$$\Theta_2(\eta)(t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11}) = (s_1, s_2, s_3, s_7, s_{11}, t_1, t_2, t_3, t_7, t_{11})$$

A set of generators for the invariant polynomials (with respect to the previous permutation action) is given by

$$q_1 = t_1 + s_1,\ q_2 = t_2 + s_2,\ q_3 = t_3 + s_3,$$

$$q_4 = t_7 + s_7,\ q_5 = t_{11} + s_{11},\ q_6 = t_1^2 + s_1^2,$$

$$q_7 = t_2^2 + s_2^2,\ q_8 = t_3^2 + s_3^2,\ q_9 = t_7^2 + s_7^2,$$

$$q_{10} = t_{11}^2 + s_{11}^2$$

Then the rational map

$$\Psi_2 : \mathbb{C}^{10} \to \mathbb{C}^{10}$$

$$(t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11}) \mapsto (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10})$$

satisfies the following properties:

$$\left\{ \begin{array}{l} \Psi_2^\eta = \Psi_2; \\ \Psi_2 \circ \Theta_2(\eta) = \Psi_2. \end{array} \right. \tag{4.1}$$

There are two possibilities:

1. $\Phi_2(Z) = \Theta_2(\eta)(\Phi_2(Z))$; in which case $Z^\eta = Z$ and Z will be already defined over $\mathbb{Q}$ (which seems not to be the case); and
2. the stabilizer of $\Phi_2(Z)$ under $\Theta_2(M)$ is trivial.

Under the assumption (2) above, we have that $\Psi_2 : \Phi_2(Z) \to W = \Psi_2(\Phi_2(Z))$ is a biregular isomorphism and that, as before, W is defined over $\mathbb{Q}$. That is, the map $L_1 = \Psi_2 \circ \Phi_2 : Z \to W$ is an explicit biregular isomorphism and W is defined over $\mathbb{Q}$. In this way, $\widehat{L} = L_1 \circ L : X \to W$ is an explicit birational isomorphism as desired.

As R_2 and Z are explicitly given, one may compute explicit equations for W over $\mathbb{Q}(\sqrt{-7})$, say by the polynomials $A_1, ..., A_m \in \mathbb{Q}(\sqrt{-7})[q_1, ..., q_{10}]$ (this may be done with MAGMA [3] or by hands, but computations are heavy and very long). To obtain equations over $\mathbb{Q}$ we replace each A_j (which is not already defined over $\mathbb{Q}$) by the traces $A_j + A_j^{\eta}$ and $iA_j - iA_j^{\eta}$.

5 A remark on the elliptic curves in the model X

5.1 A connection to homology covers

Let us set $\lambda_1 = 1$, $\lambda_2 = \rho$, $\lambda_3 = \rho^2$, $\lambda_4 = \rho^3$, $\lambda_5 = \rho^4$, $\lambda_6 = \rho^5$ and $\lambda_7 = \rho^6$, where $\rho = e^{2\pi i/7}$. If S is the Fricke-Macbeath curve, then there is a regular branched cover $Q : S \to \widehat{\mathbb{C}}$ having deck group $G \cong \mathbb{Z}_2^3$ and whose branch locus is the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$.

Let us consider a Fuchsian group

$$\Gamma = \langle \alpha_1, ..., \alpha_7 : \alpha_1^2 = \cdots = \alpha_7^2 = \alpha_1\alpha_2\cdots\alpha_7 = 1 \rangle$$

acting on the hyperbolic plane $\mathbb{H}^2$ uniformizing the orbifold S/G.

If Γ' denotes the derived subgroup of Γ, then Γ' acts freely and $\widehat{S} = \mathbb{H}^2/\Gamma'$ is a closed Riemann surface. Let $H = \Gamma/\Gamma' \cong \mathbb{Z}_2^6$; a group of conformal automorphisms of $\widehat{S}$. Then there exists a set of generators of H, say a_1,..., a_6, so that the only elements of H acting with fixed points are these and $a_7 = a_1a_2a_3a_4a_5a_6$. In [4, 5] it was noted that $\widehat{S}$ corresponds to the generalized Fermat curve of type $(2, 6)$ (also called the homology cover of S/H)

$$\widehat{S} = \left\{ \begin{array}{rcl} x_1^2 + x_2^2 + x_3^2 & = & 0 \\ \left(\dfrac{\lambda_3 - 1}{\lambda_4 - 1}\right) x_1^2 + x_2^2 + x_4^2 & = & 0 \\ \left(\dfrac{\lambda_4 - 1}{\lambda_5 - 1}\right) x_1^2 + x_2^2 + x_5^2 & = & 0 \\ \left(\dfrac{\lambda_5 - 1}{\lambda_6 - 1}\right) x_1^2 + x_2^2 + x_6^2 & = & 0 \\ \left(\dfrac{\lambda_6 - 1}{\lambda_7 - 1}\right) x_1^2 + x_2^2 + x_7^2 & = & 0 \end{array} \right\} \subset \mathbb{P}^6_{\mathbb{C}},$$

that a_j is just multiplication by -1 at the coordinate x_j and that the regular branched cover $P : \widehat{S} \to \widehat{\mathbb{C}}$ given by

$$P([x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7]) = \frac{x_2^2 + x_1^2}{x_2^2 + \lambda_7 x_1^2} = z$$

has H has its deck group and branch locus given by the set of the $7th$-roots of unity $\{\lambda_1, ..., \lambda_7\}$.

By classical covering theory, there should be a subgroup $K < H$, $K \cong \mathbb{Z}_2^3$, acting freely on $\widehat{S}$ so that there is an isomorphism $\phi : S \to \widehat{S}/K$ with $\phi G \phi^{-1} = H/K$.

Let us also observe that the rotation $R(z) = \rho z$ lifts under P to an automorphism T of $\widehat{S}$ of order 7 of the form

$$T([x_1 : \cdots : x_7]) = [c_1 x_7 : c_2 x_1 : c_3 x_2 : c_4 x_3 : c_5 x_4 : c_6 x_5 : c_7 x_6]$$

for suitable comples numbers c_j. One has that $T a_j T^{-1} = a_{j+1}$, for $j = 1, ..., 6$ and $T a_7 T^{-1} = a_1$. The subgroup K above must satisfy that $TKT^{-1} = K$ as R also lifts to the Fricke-Macbeath curve (as noticed in the Introduction).

5.2 About the elliptic curves in the Fricke-Macbeath curve

The subgroup

$$K^* = \langle a_1 a_3 a_7, a_2 a_3 a_5, a_1 a_2 a_4 \rangle \cong \mathbb{Z}_2^3$$

acts freely on $\widehat{S}$ and it is normalized by T. In particular, $S^* = \widehat{S}/K^*$ is a closed Riemann surface of genus 7 admitting the group $L = H/K^* = \{e, a_1^*, ..., a_7^*\} \cong \mathbb{Z}_2^3$ (where a_j^* is the involution induced by a_j) as a group of automorphisms and it also has an automorphism T^* of order 7 (induced by T) permuting cyclically the involutions a_j^*. As $S^*/\langle L, T^* \rangle = \widehat{S}/\langle H, T \rangle$ has signature $(0; 2, 7, 7)$, we may see that $S = S^*$ and $K = K^*$.

We may see that $L = \langle a_1^*, a_2^*, a_3^* \rangle$ and $a_4^* = a_1^* a_2^*$, $a_5^* = a_2^* a_3^*$, $a_6^* = a_1^* a_2^* a_3^*$ and $a_7^* = a_1^* a_3^*$. In this way, we may see that every involution of H/K is induced by one of the involutions (and only one) with fixed points; so every involution in L acts with 4 fixed points on S.

Let $a_i^*, a_j^* \in H/K$ be any two different involutions, so $\langle a_i^*, a_j^* \rangle \cong \mathbb{Z}_2^2$. Then, by the Riemann-Hurwitz formula, the quotient surface $S/\langle a_i^*, a_j^* \rangle$ is a closed Riemann surface of genus 1 with six cone points of order 2. These six cone points are projected onto three of the cone points of S/H. These points are λ_i, λ_j and λ_r, where $a_r^* = a_i^* a_j^*$. In this way, the corresponding genus one surface is given by the elliptic curve

$$y^2 = \prod_{k \notin \{i,j,r\}} (x - \lambda_k)$$

So, for instance, if we consider $i = 2$ and $j = 3$, then $r = 5$ and the elliptic curve is

$$y_1^2 = (x - 1)(x - \rho^3)(x - \rho^5)(x - \rho^6).$$

If $i = 1$ and $j = 2$, then $r = 4$ and the elliptic curve is

$$y_2^2 = (x - \rho^2)(x - \rho^4)(x - \rho^5)(x - \rho^6).$$

If $i = 1$ and $j = 3$, then $r = 7$ and the elliptic curve is

$$y_4^2 = (x - \rho)(x - \rho^3)(x - \rho^4)(x - \rho^5).$$

We have obtained the three elliptic curves appearing in the Fricke-Macbeath equation (2.1).

5.3 Another model for the Fricke-Macbeath curve

The above description of the Fricke-Macbeath curve in terms of the homology cover $\widehat{S}$ permits to obtain an explicit model. Let us consider now an affine model of $\widehat{S}$, say by taking $x_7 = 1$, which we denote by $\widehat{S}^0$. In this case the involution a_7 is multiplication of all coordinates by -1. A set of generators for the algebra of invariant polynomials in $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ under the natural linear action induced by K is

$$t_1 = x_1^2, t_2 = x_2^2, t_3 = x_3^2, t_4 = x_4^2, t_5 = x_5^2, t_6 = x_6^2, t_7 = x_1x_2x_5, t_8 = x_1x_2, x_3x_6$$

$$t_9 = x_1x_4x_6, t_{10} = x_1x_3x_4x_5, t_{11} = x_2x_4x_5x_6, t_{12} = x_2x_3x_4, t_{13} = x_3x_5x_6.$$

If we set

$$F : \widehat{S}^0 \to \mathbb{C}^{13}$$

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}),$$

then $F(\widehat{S}^0)$ will provide a model for the Fricke-Macbeath (affine) curve S. Equations for such an affine model of S are

$$\left\{\begin{array}{c}
t_1 + t_2 + t_3 = 0\\
\left(\frac{\lambda_3-1}{\lambda_4-1}\right) t_1 + t_2 + t_4 = 0\\
\left(\frac{\lambda_4-1}{\lambda_5-1}\right) t_1 + t_2 + t_5 = 0\\
\left(\frac{\lambda_5-1}{\lambda_6-1}\right) t_1 + t_2 + t_6 = 0\\
\left(\frac{\lambda_6-1}{\lambda_7-1}\right) t_1 + t_2 + 1 = 0\\
t_6t_{10} = t_9t_{13},\ t_6t_7t_{12} = t_8t_{11},\ t_5t_9t_{12} = t_{10}t_{11}\\
t_5t_8 = t_7t_{13},\ t_5t_6t_{12} = t_{11}t_{13},\ t_4t_8 = t_9t_{12}\\
t_4t_7t_{13} = t_{10}t_{11},\ t_4t_6t_7 = t_9t_{11},\ t_3t_{11} = t_{12}t_{13}\\
t_3t_6t_7 = t_8t_{13},\ t_3t_5t_9 = t_{10}t_{13},\ t_3t_5t_6 = t_{13}^2\\
t_3t_4t_7 = t_{10}t_{12},\ t_2t_{10} = t_7t_{12},\ t_2t_9t_{13} = t_8t_{11}\\
t_2t_5t_9 = t_7t_{11},\ t_2t_4t_{13} = t_{11}t_{12},\ t_2t_4t_5t_6 = t_{11}^2\\
t_2t_3t_9 = t_8t_{12},\ t_2t_3t_4 = t_{12}^2,\ t_1t_{12}t_{13} = t_8t_{10}\\
t_1t_{11} = t_7t_9,\ t_1t_6t_{12} = t_8t_9,\ t_1t_5t_{12} = t_7t_{10}\\
t_1t_4t_{13} = t_9t_{10},\ t_1t_4t_6 = t_9^2,\ t_1t_3t_4t_5 = t_{10}^2\\
t_1t_2t_{13} = t_7t_8,\ t_1t_2t_5 = t_7^2,\ t_1t_2t_3t_6 = t_8^2
\end{array}\right\} \subset \mathbb{C}^{13}$$

Of course, one may see that the variables t_2, t_3, t_4, t_5 and t_6 are uniquely determined by the variable t_1. Other variables can also be determined in order to get a lower dimensional model.

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Counterexamples to a conjecture on injective colorings*

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Abstract

An injective coloring of a graph is a vertex coloring where two vertices receive distinct colors if they have a common neighbor. Chen, Hahn, Raspaud, and Wang [3] conjectured that every planar graph with maximum degree $\Delta \geq 3$ admits an injective coloring with at most $\lceil 3\Delta/2 \rceil$ colors. We present an infinite family of planar graphs showing that the conjecture is false for graphs with small or even maximum degree. We conclude this note with an alternative conjecture, which sheds some light on the well-known Wegner's conjecture for the mentioned degrees.

Keywords: Injective coloring, planar graph, square graph.

Math. Subj. Class.: 05C10, 05C15

1 Introduction

An *injective k-coloring* of a graph G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every pair of distinct vertices $u, v \in V(G)$ having a common neighbor. The least k such that G is injectively k-colorable is the *injective chromatic number* of G, denoted by $\chi_i(G)$. Note that this type of coloring is not necessarily proper.

The injective coloring of graphs was first introduced by Hahn, Kratochvíl, Širáň and Sotteau [7] in 2002. The first results on the injective chromatic number of planar graphs were presented by Doyon, Hahn and Raspaud [6] in 2005. As a corollary of the main theorem they obtained that if G is a planar graph of maximum degree Δ and girth $g(G) \geq 7$, then the injective chromatic number is at most $\Delta + 3$. Moreover, if $g(G) \geq 6$ then $\chi_i(G) \leq \Delta+4$, and if $g(G) \geq 5$ then $\chi_i(G) \leq \Delta+8$. Chen, Hahn, Raspaud, and Wang [3] investigated K_4-minor-free graphs and showed that if G is such a graph of maximum degree Δ, then $\chi_i(G) \leq \lceil \frac{3}{2}\Delta \rceil$. Moreover, they conjectured that the same bound holds for all planar graphs:

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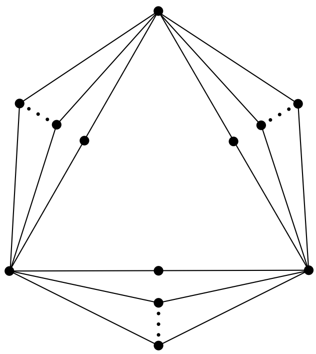

Figure 1: A planar graph with maximum degree Δ and girth 4 that needs $\frac{3}{2}\Delta$ colors for an injective coloring.

Conjecture 1.1 (Chen, Hahn, Raspaud, and Wang). *For every planar graph G with maximum degree $\Delta \geq 1$, it holds that*

$$\chi_i(G) \leq \left\lceil \frac{3}{2}\Delta \right\rceil .$$

The conjecture holds for planar graphs with large girth. There is a number of results (see e.g. [1, 2, 4, 5, 8]) showing that if the girth of a planar graph is at least 5, the injective chromatic number of a graph is at most $\Delta + C$, where C is some small constant. When considering graphs of girth smaller than 5 the injective chromatic number considerably increases. In Fig. 1 a planar graph of girth 4 and injective chromatic number $\frac{3}{2}\Delta$ is depicted.

In this note we present examples of planar graphs with maximum degree $\Delta \geq 4$ and injective chromatic number $\Delta + 5$, for $4 \leq \Delta \leq 7$, and $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, for $\Delta \geq 8$. Thus providing counterexamples to Conjecture 1.1 for planar graphs with maximum degree at most 7 or even.

The central problem regarding the chromatic number of squares of planar graphs is the well-known Wegner's conjecture [9] proposed in 1977. Note that for larger Δ, the bound of this conjecture is one more than that of Conjecture 1.1.

Conjecture 1.2 (Wegner). *Let G be a planar graph with maximum degree Δ. The chromatic number of square graph G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \leq \Delta \leq 7$, and at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, otherwise.*

2 Planar graphs with largest injective chromatic numbers

Using the following result we easily disprove Conjecture 1.1 for every $\Delta \in \{4, 5, 6, 7\}$ and for every even $\Delta \geq 8$. We believe that the bound for graphs with maximum degree 3 is correct, however.

Theorem 2.1. *There exist planar graphs G of maximum degree $\Delta \geq 3$ satisfying the following:*

(*a*) $\chi_i(G) = 5$, *if* $\Delta = 3$;

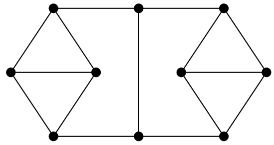

Figure 2: A planar cubic graph with the injective chromatic number 5.

(b) $\chi_i(G) = \Delta + 5$, *if* $4 \leq \Delta \leq 7$;

(c) $\chi_i(G) = \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, *if* $\Delta \geq 8$.

Proof. For $\Delta = 3$, i.e., the case (a) of the theorem, a cubic planar graph with the injective chromatic number equal to 5 was presented in [3] (see Fig. 2). For $\Delta \geq 4$, the following simple characterization will be used:

$(*)$ A graph G has injective chromatic number equal to its order, if and only if

1. G has diameter 2; and
2. every edge of G belongs to a triangle.

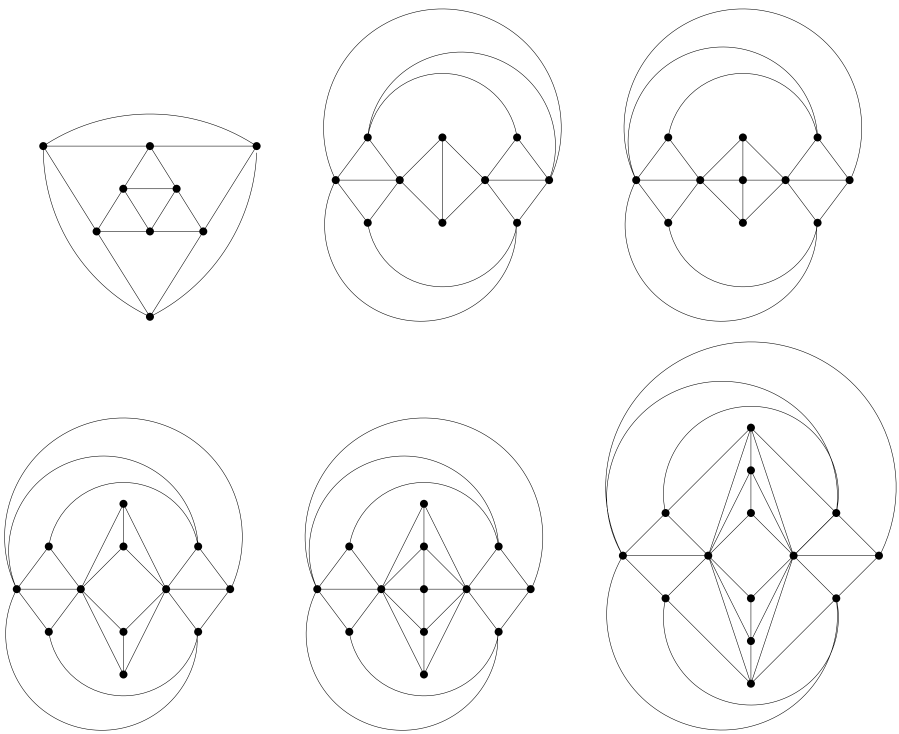

Figure 3: Planar graphs with diameter 2 and maximum degree $\Delta \in \{4, \ldots, 9\}$.

In Fig. 3 planar graphs with maximum degree $\Delta \in \{4, \ldots, 9\}$ are presented. Note that these graphs are of diameter 2 and of orders 9, 10, 11, 12, 13, and 14, respectively.

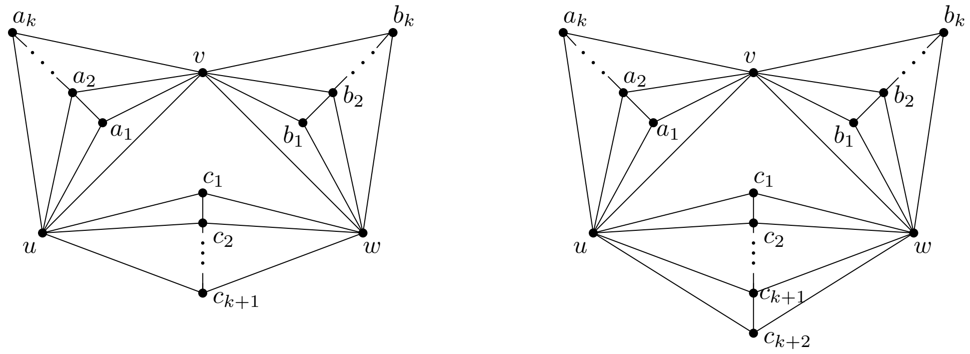

Figure 4: Constructions of diameter 2 planar graphs with maximum degree $\Delta \geq 8$ (even on the left and odd on the right) and the injective chromatic number equal to $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$.

Moreover, they have the property that each of their edges belongs to a 3-cycle. By $(*)$, it follows that each of them has chromatic index equal to its order. As these graphs have maximum degree 4, 5, 6, 7, 8, and 9, respectively, we conclude that each of them satisfies the identity of the case (b) of the theorem. Finally, we consider the case (c) of the theorem. We give constructions for planar graphs with diameter 2, maximum degree $\Delta \geq 8$, and order $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, where each edge belongs to a 3-cycle. Then by $(*)$, the claim (c) of the theorem immediately follows.

We distinguish two cases regarding whether Δ is even or odd (see Fig. 4 for an illustration). In both cases we start with a path uvw. Then we insert the paths $P_a = a_1 a_2 \cdots a_k$, $P_b = b_1 b_2 \cdots b_k$, and $P_c = c_1 c_2 \cdots c_{k+1}$ (if Δ is odd, we introduce also the edge $c_{k+1}c_{k+2}$). These additional edges are providing paths of length two between the vertices u, v, w and the vertices a_i, b_i, and c_i.

The left graph depicted in Fig. 4 has maximum degree $\Delta = 2k + 2$ and the right one has $\Delta = 2k + 3$, for $k \geq 3$. It is easy to see that there is a path of length two between every pair of vertices, thus every vertex in the graph should receive a different color in an injective coloring. □

Let us remark that the presented graphs from the above theorem are not the only ones with such properties. For example in the last construction all the edges of the paths P_a, P_b, and P_c are not really needed to obtain a graph with the desired properties. In fact, it is enough that each vertex of these paths is incident with one edge of the path, so roughly every second is redundant.

We conclude the paper with an attempt to correct Conjecture 1.1 by proposing the following Wegner type conjecture for the injective chromatic number of planar graphs:

Conjecture 2.2. *Let G be a planar graph with maximum degree Δ. Then*

(a) $\chi_i(G) \leq 5$, *if* $\Delta = 3$;

(b) $\chi_i(G) \leq \Delta + 5$, *if* $4 \leq \Delta \leq 7$;

(c) $\chi_i(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, *if* $\Delta \geq 8$.

Since the injective chromatic number is at most equal to the chromatic number of the square of a graph, proving Wegner's conjecture would imply the truth of Conjecture 2.2. If

Wegner's conjecture holds, then the extremal graphs (i.e. the graphs that attain the upper bound) of both conjectures coincide for Δ's from Theorem 2.2. As the injective coloring is a relaxed version of coloring the square one may reason that colorability of the square is mainly conducted by the injective incidence of the vertices.

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Spherical tilings by congruent quadrangles: Forbidden cases and substructures

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Abstract

In this article we show the non-existence of a class of spherical tilings by congruent quadrangles. We also prove several forbidden substructures for spherical tilings by congruent quadrangles. These are results that will help to complete of the classification of spherical tilings by congruent quadrangles.

Keywords: Spherical tiling by congruent quadrangles, monohedral tiling, quadrangulation.

Math. Subj. Class.: 05B45, 05C10, 51M20, 52C20

1 Introduction

In this paper we prove the non-existence of a subclass of spherical tilings by congruent quadrangles which have three equal sides and one side different. We also list several forbidden substructures for this type of spherical tilings.

It follows from Euler's formula that spherical tilings by congruent polygons can only exist for triangles, quadrangles and pentagons.

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In [5], Davies completed the classification of spherical tilings by congruent triangles. He however only gave an outline of the proof and his classification contained several duplicates. Ueno and Agaoka [9] worked out the details of the proof, thus completely solving the classification of spherical tilings by congruent triangles.

Ueno and Agaoka [8] gave several examples of spherical tilings by congruent quadrangles and showed that the classification of these would be considerably harder than the classification of spherical tilings by congruent triangles. Akama and Sakano [7] completed the classification of spherical tilings by congruent kites, darts and rhombi. Since these quadrangles can be subdivided into congruent triangles, they could rely on the classification by Ueno and Agaoka to solve this classification.

The spherical quadrangles can be subdivided into classes based on the cyclic list of edge lengths. Only four of these classes admit a spherical tiling by congruent quadrangles[8]:

1. $aaaa$
2. $aaab$
3. $aabb$
4. $aabc$

The cases handled by Akama and Sakano cover type 1 and type 3.

The remaining two cases are spherical tilings by congruent quadrangles which have three equal sides and one side different (type 2), and spherical tilings by congruent quadrangles which have three different sides and an adjacent pair of sides of the same length (type 4). Akama, Nakumara and Sakano [1, 2, 7] showed that if concave quadrangles are allowed, there exist several tilings which have non-congruent tiles but for which the inner angles and the underlying graph are the same.

In this paper we focus on convex quadrangles of type 2. We show that there exists no spherical tiling by congruent quadrangles of type 2 if the quadrangles are isosceles. Furthermore we show several forbidden substructures for the underlying graph of spherical tilings by congruent quadrangles of type 2.

This paper is organised as follows. We start by giving some general definitions and notations. Then we show the non-existence of spherical tilings by congruent quadrangles of type 2 with isosceles quadrangles. Next we look at the different possible configurations of angles around each vertex and finally we use this to show some forbidden substructures for the underlying graph.

2 Definitions

To simplify the notation we will always express angles in π radians.

A spherical tiling is a subdivision of the unit sphere into spherical polygons. Edges are always assumed to be parts of great circles. All tilings are edge-to-edge tilings.

A spherical quadrangle is of **type 2** if the cyclic list of edge-lengths is $aaab$ (with $a \neq b$). We use the naming convention shown in Figure 1. We only consider convex spherical quadrangles. This means that we always assume

$$0 < \alpha, \beta, \gamma, \delta < 1.$$

Throughout this paper G will always refer to a 2-connected, simple graph on the 2-sphere in which all faces are quadrangles. Let $A(G)$ be the set of ordered pairs (f, v) such that f is a face of G, v is a vertex of G and $v \in f$. A **chart** (G, ϕ), is an ordered pair consisting of a graph G and a function $\phi : A(G) \to \{\alpha, \beta, \gamma, \delta\}$ such that for each face

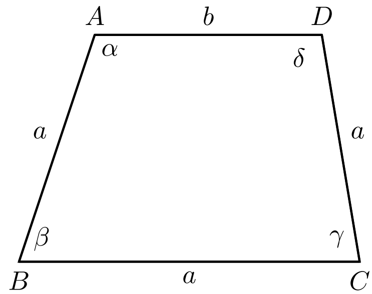

Figure 1: Naming conventions in a spherical quadrangle of type 2.

of the graph, the cyclic list of the inner angles is $(\alpha, \beta, \gamma, \delta)$ or the reverse. These four parameters, $\alpha, \beta, \gamma, \delta$, will take on the role of angles of tiles, so a chart can be seen as a combinatorial spherical tiling by congruent quadrangles. We say a vertex of the tiling has **vertex type** $n_1\alpha + n_2\beta + n_3\gamma + n_4\delta$, if there are n_1 pairs containing v that are mapped to α, n_2 pairs containing v that are mapped to β, n_3 pairs containing v that are mapped to γ, and n_4 pairs containing v that are mapped to δ.

It is clear how a chart can be obtained from a spherical tiling by congruent quadrangles. Vice versa, a chart (G, f) is **solvable**, if there exist values for the four angles such that there is a spherical tiling realising that graph and those values.

There are several conditions that need to be satisfied in order for a spherical tiling by congruent quadrangles to exist. If F is the number of tiles, then the following condition follows from the fact that the area of the tiles need to sum up to the area of the sphere.

$$\alpha + \beta + \gamma + \delta - 2 = \frac{4}{F} \tag{2.1}$$

Lemma 2.1. *In a convex spherical quadrangle of type 2, we have that*

$$\alpha + \delta < 1 + \beta, \tag{2.2}$$

$$\alpha + \beta < 1 + \delta, \tag{2.3}$$

$$\alpha + \delta < 1 + \gamma, \tag{2.4}$$

$$\gamma + \delta < 1 + \alpha. \tag{2.5}$$

Proof. Draw the diagonal as is shown in Figure 2. The area of the triangle ABD is given by

$$\alpha + \beta_1 + \delta_1 - 1.$$

The area of the spherical lune that is formed by the great circles AB and BD is given by $2\beta_1$. Since the area of the triangle is smaller than that of the lune, we have that

$$\alpha + \beta_1 + \delta_1 - 1 < 2\beta_1$$

which can be rewritten as

$$\alpha + \delta_1 < 1 + \beta_1 \tag{2.6}$$

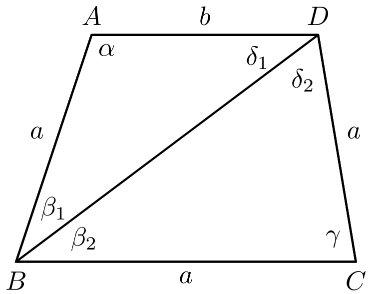

Figure 2: A diagonal in a spherical quadrangle of type 2.

The triangle BDC is an isosceles triangle. This implies that

$$\delta_2 = \beta_2 \tag{2.7}$$

If we combine inequality 2.6 and equation 2.7, we find equation 2.2. Equation 2.3 can be proven using the spherical lune formed by AD and BD. Equation 2.4 and equation 2.5 can be proven by using the other diagonal. □

Lemma 2.2. *In a convex spherical quadrangle of type 2, we have that*

$$\alpha \neq \gamma \tag{2.8}$$

and

$$\delta \neq \beta. \tag{2.9}$$

Proof. Assume that $\delta = \beta$. Draw the diagonal as is shown in Figure 2. The triangle BDC is an isosceles triangle, so we have that $\delta_2 = \beta_2$. This implies that $\delta_1 = \beta_1$, so ABD is an isosceles triangle and $a = b$. This is however a contradiction, so we find that $\delta \neq \beta$. By using the other diagonal, we can prove that $\alpha \neq \gamma$. □

Lemma 2.3. *In a convex spherical quadrangle of type 2, we have that*

$$\alpha = \delta \Leftrightarrow \beta = \gamma. \tag{2.10}$$

Proof. Assume that $\alpha = \delta$. The great circles AB and DC in Figure 1 intersect in two points, N and S. Since $\alpha = \delta$, the triangle ADN is an isosceles triangle, but since the distance from A to B and from D to C is a, then also the triangle BCN is an isosceles triangle and $\beta = \gamma$. The other direction is completely analogous. □

Lemma 2.4. *Let (G, f) be a solvable chart. Let v be a vertex of G with vertex type $n_1\alpha + n_2\beta + n_3\gamma + n_4\delta$ in (G, f), then $n_1 + n_4$ is even.*

Proof. This follows immediately from the fact that each edge of length b that is incident to v contributes exactly two to $n_1 + n_4$ and each angle α and δ at the vertex v corresponds to exactly one edge of length b incident to v. □

The following lemma can easily be proved using Euler's formula.

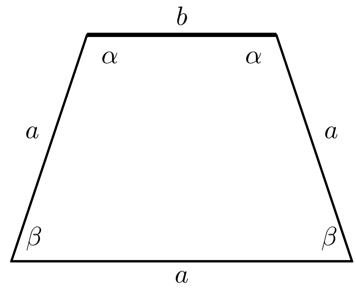

Figure 3: An isosceles quadrangle of type 2

Lemma 2.5. *Let G be a quadrangulation of the sphere. Let V_i (with $3 \leq i \leq \Delta$, where Δ is the largest degree of the G) be the number of vertices in G with degree i, then we have the following equality:*

$$V_3 = 8 + \sum_{i=5}^{\Delta} (i-4)V_i.$$

3 Spherical tilings by congruent isosceles quadrangles of type 2

An isosceles spherical quadrangle of type 2 is a convex spherical quadrangle having the cyclic list of edge-lengths $aaab$ (with $a \neq b$) and in which $\alpha = \delta$ and $\beta = \gamma$. Therefore the cyclic list of the inner angles in a isosceles quadrangle is $(\alpha, \beta, \beta, \alpha)$. An example of such a quadrangle is given in Figure 3.

We can rewrite several of the conditions for general spherical quadrangles of type 2. Equation 2.1 can be rewritten as

$$2\alpha + 2\beta - 2 = \frac{4}{F}. \tag{3.1}$$

The corresponding lemma for Lemma 2.1 is

Lemma 3.1. *In an isosceles spherical quadrangle of type 2, we have that*

$$2\alpha < 1 + \beta. \tag{3.2}$$

The corresponding lemma for Lemma 2.2 is

Lemma 3.2. *In an isosceles spherical quadrangle of type 2, we have that*

$$\alpha \neq \beta \tag{3.3}$$

We now have the tools to prove the main theorem of this section.

Theorem 3.3. *There is no isosceles spherical tiling by congruent quadrangles of type 2.*

Proof. From Lemma 2.5, we know that each quadrangulation contains at least 8 vertices of degree 3. The possible vertex types for a vertex of degree 3 in a spherical tiling by congruent isosceles spherical quadrangles of type 2 are $2\alpha+\beta$ and 3β. There is no isosceles spherical tiling by congruent quadrangles of type 2 with two vertices of degree 3 with a different vertex type, because in that case we would have $\alpha = \beta$, which does not correspond to a quadrangle of type 2 (cf. Lemma 3.2). So all vertices of degree 3 have the same type.

We will examine both possible vertex types.

vertex type $2\alpha + \beta$

We first assume that all vertices of degree 3 have vertex type $2\alpha + \beta$.

As a consequence all vertices of degree $d > 3$ have vertex type $d\beta$ or $d\alpha$. Otherwise there would be a vertex of degree $d > 3$ with vertex type

$$2i\alpha + (d - 2i)\beta$$

with $0 < i < \left\lfloor \frac{d}{2} \right\rfloor$. If we combine this with the vertex type for the vertices of degree 3, then we find that

$$(2i - 2)\alpha + (d - 2i - 1)\beta = 0.$$

Since $\alpha > 0$ and $\beta > 0$, this is equivalent with

$$\begin{cases} 2i - 2 = 0 \\ d - 2i - 1 = 0 \end{cases}$$

But since $d > 3$, this has no solution.

It is also not the case that all vertices are of degree 3, since that would mean that there are more α's than β's.

This means that there are only a limited number of possibilities for different degrees in this situation:

- the quadrangulation has two types of vertices: vertices of degree 3 with vertex type $2\alpha + \beta$ and vertices of degree $d > 3$ with vertex type $d\beta$, or
- the quadrangulation has three types of vertices: vertices of degree 3 with vertex type $2\alpha + \beta$, vertices of degree $d > 3$ with vertex type $d\beta$, and vertices of even degree $d_e > 3$ with vertex type $d_e\alpha$ (d_e is even due to Lemma 2.4).

Assume first that there are only vertices of degree 3 with vertex type $2\alpha + \beta$ and vertices of degree $d > 3$ with vertex type $d\beta$. In this case we get two equations:

$$\begin{cases} 2\alpha + \beta = 2 \\ d\beta = 2 \end{cases}$$

This is equivalent to

$$\begin{cases} \alpha = 1 - \frac{1}{d} \\ \beta = \frac{2}{d} \end{cases}$$

If we substitute these values for α and β in inequality 3.2, we find

$$2 - \frac{2}{d} < 1 + \frac{2}{d}.$$

This is equivalent to

$$d < 4,$$

which contradicts $d > 3$.

Next we assume that there are only vertices of degree 3 with vertex type $2\alpha + \beta$, vertices of degree $d > 3$ with vertex type $d\beta$, and vertices of even degree $d_e > 3$ with vertex type $d_e\alpha$. In this case we get three equations:

$$\begin{cases} 2\alpha + \beta = 2 \\ d\beta = 2 \\ d_e\alpha = 2 \end{cases}$$

This is equivalent to

$$\begin{cases} \frac{4}{d_e} + \frac{2}{d} = 2 \\ \beta = \frac{2}{d} \\ \alpha = \frac{2}{d_e} \end{cases}$$

The first equation has no solution, since $d_e \geq 4$ and $d > 3$.

vertex type 3β

Next we assume that all vertices of degree 3 have vertex type 3β. This means that $\beta = \frac{2}{3}$ and from equation 3.1, we then find that

$$\alpha = \frac{1}{3} + \frac{2}{F} = \frac{F+6}{3F}. \tag{3.4}$$

As 3β is equal to 2, any vertex type that contains a β, has at most 2β. Since there has to be at least one vertex for which the vertex type contains an α, there is a vertex of degree d with one of the following three types:

- $d\alpha$,
- $(d-1)\alpha + \beta$,
- $(d-2)\alpha + 2\beta$.

We examine the three possibilities:

$d\alpha$

Combined with equation 3.4, this gives us

$$d\frac{F+6}{3F} = 2$$

which can be rewritten as

$$6d = (6-d)F.$$

Since d and F are both positive integers, and d is even and larger than 3, we find that this only holds if $d = 4$ and $F = 12$.

$(d-1)\alpha + \beta$

Combined with equation 3.4, this gives us

$$(d-1)\frac{F+6}{3F} = 2 - \frac{2}{3} = \frac{4}{3}$$

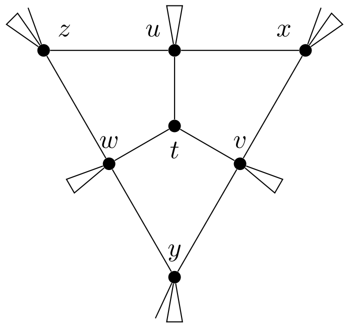

Figure 4: The quadrangles around a vertex t of degree 3.

which can be rewritten as

$$6(d-1) = (5-d)F.$$

Since d and F are both positive integers, and d is odd and larger than 3, we find that this never holds.

$(d-2)\alpha + 2\beta$

Combined with equation 3.4, this gives us

$$(d-2)\frac{F+6}{3F} = 2 - \frac{4}{3} = \frac{2}{3}$$

which can be rewritten as

$$6(d-2) = (4-d)F.$$

Since d and F are both positive integers, and d is even and larger than 3, we find that this never holds.

So the only possibility is a quadrangulation which 12 faces. Such a quadrangulation has 14 vertices, of which at least 8 have degree 3 and vertex type 3β. This already accounts for all of the 24β's, so all remaining 6 vertices have degree 4 and vertex type 4α.

Assume we have a vertex t of degree 3 as is shown in Figure 4. This vertex has vertex type 3β. This means that, in the quadrangle $tuzw$, the angle at vertex t is β and either the angle at the vertex w or the angle at the vertex u is α. Without loss of generality, we can assume that the angle at the vertex u is α. This implies that the vertex type of u is 4α, and we find that this means that the vertex type of both the vertices w and v is 3β. But then the quadrangle $twyv$ has three consecutive angles β. This is a contradiction, so there is no spherical tiling by congruent isosceles quadrangles of type 2 with vertex types 3β and 4α.

This proves that there is no spherical tiling by congruent isosceles spherical quadrangles of type 2. □

	1	2	3	4	5	6	7	8	9	10
1		4.1a				4.1a	4.1a		4.1b	
2	4.1a					4.1a	4.1a	4.1c		4.1b
3					4.1a		4.1c	4.1a	4.1a	
4					4.1a	4.1b		4.1a		4.1a
5			4.1a	4.1a			4.1b		4.1a	
6	4.1a	4.1a		4.1b			4.1a			
7	4.1a	4.1a	4.1c		4.1b	4.1a				
8		4.1c	4.1a	4.1a						4.1a
9	4.1b		4.1a		4.1a					4.1a
10		4.1b		4.1a				4.1a	4.1a	

Table 1: Overview of the combinations of two vertex types for vertices of degree 3. For each impossible combination of vertex type, the corresponding case is given.

4 Vertex types in spherical tilings by arbitrary congruent quadrangles of type 2

Since there are at least 8 vertices of degree 3 (and in most cases even more), it can be interesting to look at the possible vertex types for these vertices, and examine whether certain combination are not possible. Owing to Lemma 2.4, there are ten possible vertex types for vertices of degree 3 in a spherical tiling by congruent quadrangles of type 2:

1) 3β
2) $2\beta+\gamma$
3) $\alpha+\delta+\beta$
4) $2\alpha+\gamma$
5) $2\alpha+\beta$
6) 3γ
7) $2\gamma+\beta$
8) $\alpha+\delta+\gamma$
9) $2\delta+\beta$
10) $2\delta+\gamma$

The last five of these types can be obtained from the first five by interchanging α with δ, and β with γ.

The following lemma shows that several combinations of vertex types for vertices of degree 3 are not possible in a spherical tiling by congruent quadrangles of type 2. Table 1 gives an overview of all combinations.

Lemma 4.1. *There is no spherical tiling by congruent quadrangles of type 2 which has any of the following combinations of vertex types:*

a) 3β *and* $2\beta+\gamma$, 3β *and* 3γ, 3β *and* $\beta+2\gamma$, $2\beta+\gamma$ *and* 3γ, $2\beta+\gamma$ *and* $\beta+2\gamma$, $\alpha+\delta+\beta$ *and* $2\alpha+\beta$, $\alpha+\delta+\beta$ *and* $\alpha+\delta+\gamma$, $\alpha+\delta+\beta$ *and* $2\delta+\beta$, $2\alpha+\gamma$ *and* $2\alpha+\beta$, $2\alpha+\gamma$ *and* $\alpha+\delta+\gamma$, $2\alpha+\gamma$ *and* $2\delta+\gamma$, $2\alpha+\beta$ *and* $2\delta+\beta$*;*

b) 3β *and* $2\delta+\beta$, $2\beta+\gamma$ *and* $2\delta+\gamma$, $2\alpha+\gamma$ *and* 3γ, $2\alpha+\beta$ *and* $2\gamma+\beta$*;*

c) $2\beta+\gamma$ *and* $\alpha+\delta+\gamma$, $2\gamma+\beta$ *and* $\alpha+\delta+\beta$*.*

Proof.

a) Each of these combinations either implies that $\alpha = \delta$, or that $\beta = \gamma$. This means that the quadrangle is a isosceles quadrangle of type 2. Due to Theorem 3.3, there are no spherical tilings by congruent quadrangles with such a tile.

b) The first two combinations imply that $\beta = \delta$, but this contradicts inequality 2.9. The last two combinations imply that $\alpha = \gamma$, but this contradicts inequality 2.8.

c) We will only give the proof for $2\gamma + \beta$ and $\alpha + \delta + \beta$. The other case can be obtained by interchanging α with δ, and β with γ.

   When we combine
$$\alpha + \delta + \beta = 2$$
   with equation 2.1, we get
$$\gamma = \frac{4}{F}.$$
   When we combine this with
$$2\gamma + \beta = 2,$$
   we get
$$\beta = 2 - \frac{8}{F}.$$
   Since $\beta < 1$, this implies that $F < 8$. However, if $F = 6$, we have that $\beta = \gamma$. This is not possible due to Lemma 2.3 and Theorem 3.3. So we find that this combination is not possible.

□

Lemma 4.2. *In each spherical tiling by congruent quadrangles of type 2 we have the restrictions on the number of faces that are given in Table 2.*

Proof. We will examine case by case. First we note that for all quadrangulations, we have that $F \geq 6$, so $F \neq 6$ is equivalent to $F > 6$.

- **Vertex type 1 and vertex type 3**
  In this case we have the following system of equations:
$$\begin{cases} 3\beta = 2 \\ \alpha + \beta + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$
  The last equation in this system corresponds to equation 2.1. If we subtract the second equation from this last equation, we find that
$$\gamma = \frac{4}{F}.$$
  Owing to Lemma 2.3 and Theorem 3.3, we have that $F \neq 6$, because otherwise $\beta = \gamma$. By combining the first two equations in the system, we find that
$$\alpha + \delta = \frac{4}{3}.$$

	1	2	3	4	5	6	7	8	9	10
1			$6 < F < 12$					$F = 6$		$6 < F$
2			$6 < F$							
3	$6 < F < 12$	$6 < F$		$6 < F$		$F = 6$				
4			$6 < F$						$F = 6$	
5						$6 < F$				$F = 6$
6			$F = 6$		$6 < F$			$6 < F < 12$		
7								$6 < F$		
8	$F = 6$					$6 < F < 12$	$6 < F$		$6 < F$	
9				$F = 6$				$6 < F$		
10	$6 < F$				$F = 6$					

Table 2: Combinations of two vertex types for vertices of degree 3. For each combination of vertex type with known restrictions on the number of faces, that restriction is given. The red cells are the impossible combinations which where already given in Table 1.

When we substitute these previous two equalities into inequality 2.4, we find that

$$\frac{4}{3} = \alpha + \delta < 1 + \gamma = 1 + \frac{4}{F},$$

which can be rewritten as

$$F < 12.$$

- **Vertex type 1 and vertex type 8**
In this case we have the following system of equations:

$$\begin{cases} 3\beta = 2 \\ \alpha + \gamma + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

By combining the last two equations, we find that $\beta = \frac{4}{F}$, but together with the first equation of the system, this implies that $F = 6$.

- **Vertex type 1 and vertex type 10**
In this case we have the following system of equations:

$$\begin{cases} 3\beta = 2 \\ \gamma + 2\delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

By substituting the first two equations in the third, we get:

$$\alpha - \delta = \frac{4}{F} - \frac{2}{3}.$$

In combination with Lemma 2.3 and Theorem 3.3, this implies that $F \neq 6$.

- **Vertex type 2 and vertex type 3**
In this case we have the following system of equations:

$$\begin{cases} 2\beta + \gamma = 2 \\ \alpha + \beta + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

The last two equations give us that

$$\gamma = \frac{4}{F},$$

and using the first equation from the system, this then implies that

$$\beta = 1 - \frac{2}{F}.$$

In combination with Lemma 2.3 and Theorem 3.3, these last two equations imply that $F \neq 6$.

- **Vertex type 2 and vertex type 8**
  In this case we have the following system of equations:

$$\begin{cases} 2\beta + \gamma = 2 \\ \alpha + \gamma + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

The last two equations give us that

$$\beta = \frac{4}{F},$$

and using the first equation from the system, this then implies that

$$\gamma = 2 - \frac{8}{F}.$$

In combination with Lemma 2.3 and Theorem 3.3, these last two equations imply that $F \neq 6$.

- **Vertex type 3 and vertex type 4**
  In this case we have the following system of equations:

$$\begin{cases} 2\alpha + \gamma = 2 \\ \alpha + \beta + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

Once again, the last two equations give us that

$$\gamma = \frac{4}{F},$$

and using the first equation from the system, this then implies that

$$\alpha = 1 - \frac{2}{F}.$$

In combination with inequality 2.8, these last two equations imply that $F \neq 6$.

- **Vertex type 4 and vertex type 9**
  In this case we have the following system of equations:

$$\begin{cases} 2\alpha + \gamma = 2 \\ 2\delta + \beta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

This is equivalent to the following system:

$$\begin{cases} \gamma = 2 - 2\alpha \\ \beta = 2 - 2\delta \\ \alpha + \delta = 2 - \frac{4}{F} \end{cases}$$

By combining the last equation in this system with inequalities 2.2 and 2.4, we find that

$$\beta > 1 - \frac{4}{F},$$

and

$$\gamma > 1 - \frac{4}{F}.$$

By combining these inequalities with the first two equations in the system, we find that

$$\alpha < \frac{1}{2} + \frac{2}{F},$$

and

$$\delta < \frac{1}{2} + \frac{2}{F}.$$

If we add up these two inequalities, we get

$$\alpha + \delta < 1 + \frac{4}{F}.$$

If we then combine this last inequality with the last equation of the system, we find the following inequality:

$$2 - \frac{4}{F} < 1 + \frac{4}{F},$$

which is equivalent to

$$F < 8.$$

- **Vertex type 5 and vertex type 10**
  In this case we have the following system of equations:

$$\begin{cases} 2\alpha + \beta = 2 \\ 2\delta + \gamma = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

  This is equivalent to the following system:

$$\begin{cases} \beta = 2 - 2\alpha \\ \gamma = 2 - 2\delta \\ \alpha + \delta = 2 - \frac{4}{F} \end{cases}$$

  Similar to the previous case, we find that

$$\alpha + \delta < 1 + \frac{4}{F}.$$

  Together with the last equation of the system, this implies

$$F < 8.$$

The remaining cases are equivalent to one of these cases by interchanging α with δ and β with γ. □

A question that pops up naturally at this point is which combinations of three vertex types for vertices of degree 3 are possible. There are 12 combinations of three vertex types for vertices of degree 3 which we can not exclude at this point. The remaining combinations can be excluded because they contain one of the combinations of two vertex types for vertices of degree 3 that are not allowed by Table 1. The 12 combinations come in pairs, since interchanging α with δ, and β with γ gives a different combination with the same properties. From these 12 combinations we can also discard combinations (1,5,10) and (5,6,10), since (1,10), resp. (6,5), implies that $6 < F$, and (5,10) implies that $6 = F$. The remaining 10 combinations are

- (1,3,4) and (6,8,9);
- (1,3,10) and (5,6,8);
- (2,3,4) and (7,8,9);
- (1,5,8) and (3,6,10);
- (2,4,9) and (4,7,9).

Lemma 4.3. *There is no spherical tiling by congruent quadrangles of type 2 on more than 8 vertices that contains 3 vertices of degree 3 with pairwise different vertex types.*

Proof. We need to examine the remaining 5 cases stated above.

- (1,3,4): In this case we have the following system of equations

$$\begin{cases} 3\beta = 2 \\ \alpha + \beta + \delta = 2 \\ 2\alpha + \gamma = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = 1 - \frac{4}{F}\beta = \frac{2}{3} \\ \gamma = \frac{8}{F} \\ \delta = \frac{1}{3} + \frac{4}{F} \end{cases}$$

If we combine this with inequality 2.4, we get

$$\frac{4}{3} < \frac{F+8}{F}$$

which is equivalent to

$$F < 8.$$

This is a contradiction because the combination (1,3) implies that $6 < F$.

- (1,3,10): In this case we have the following system of equations

$$\begin{cases} 3\beta = 2 \\ \alpha + \beta + \delta = 2 \\ 2\delta + \gamma = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = \frac{1}{3} + \frac{4}{F}\beta = \frac{2}{3} \\ \gamma = \frac{8}{F} \\ \delta = 1 - \frac{4}{F} \end{cases}$$

If we combine this with inequality 2.4, we get

$$\frac{4}{3} < \frac{F+8}{F}$$

which is equivalent to

$$F < 8.$$

This is a contradiction because the combination (1,3) implies that $6 < F$.

- (2,3,4): In this case we have the following system of equations

$$\begin{cases} 2\beta + \gamma = 2 \\ \alpha + \beta + \delta = 2 \\ 2\alpha + \gamma = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = \beta = 1 - \frac{2}{F} \\ \gamma = \delta = 1 + \frac{2}{F} \end{cases}$$

This is a contradiction with inequality 2.5.

- (1,5,8): In this case we have the following system of equations

$$\begin{cases} 3\beta = 2 \\ 2\alpha + \beta = 2 \\ \alpha + \gamma + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = \beta = \frac{2}{3} \\ \delta = \frac{4}{3} - \gamma \\ F = 6 \end{cases}$$

So we find that a quadrangulation which has this combination, has 8 vertices.

- (2,4,9): In this case we have the following system of equations

$$\begin{cases} 2\beta + \gamma = 2 \\ 2\alpha + \gamma = 2 \\ 2\delta + \beta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = 1 - \frac{2}{F} \\ \beta = \alpha \\ \gamma = 2 - 2\alpha \\ \delta = 2 - 2\alpha \end{cases}$$

If we combine this system with inequality 2.3, we get

$$2 - \frac{4}{F} < 1 + \frac{4}{F},$$

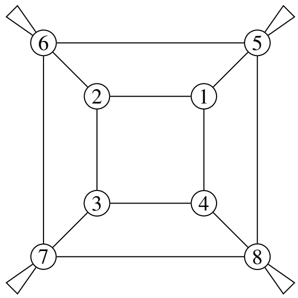

Figure 5: An example of a cubic quadrangle

which is equivalent to

$$F < 8.$$

So we find that a quadrangulation which has this combination, has $F = 6$, which implies that it has 8 vertices.

□

Theorem 4.4. *In a spherical tiling by congruent quadrangles of type 2 there are at most two different vertex types for cubic vertices.*

Proof. An enumeration of all possible angle assignments for the cube shows that, up to equivalence, only one angle assignment admits a spherical tiling by congruent quadrangles of type 2, and this angle assignment has two vertex types: 3β and $\alpha+\gamma+\delta$. Together with Lemma 4.3 this proves the theorem. □

5 Forbidden substructures in spherical tilings by arbitrary congruent quadrangles of type 2

5.1 Cubic quadrangles

A **cubic quadrangle** in a quadrangulation is a quadrangle such that all four vertices have degree 3. Figure 5 shows an example of a cubic quadrangle. In Table 3 an overview of the number of quadrangulations which contain a cubic quadrangle is given. Note that the percentage of quadrangulations which contain a cubic quadrangle increases as the size of the quadrangulations increases.

We prove the following theorem.

Theorem 5.1. *A quadrangulation on more than 8 vertices that contains a cubic quadrangle does not admit a realisation as a spherical tiling by congruent quadrangles of type 2.*

Proof. There are two ways of assigning the edges of length a and b to the edges of the cubic quadrangle and its neighbouring faces. These two ways are shown in Figure 6. If we take the complete quadrangulation into account, then these two ways will of course be realised in different ways, but this is not important for this proof.

In both edge assignments there is at least one cubic vertex that is incident to an edge of length b, and one cubic vertex that is not. Owing to Theorem 4.4, all cubic vertices

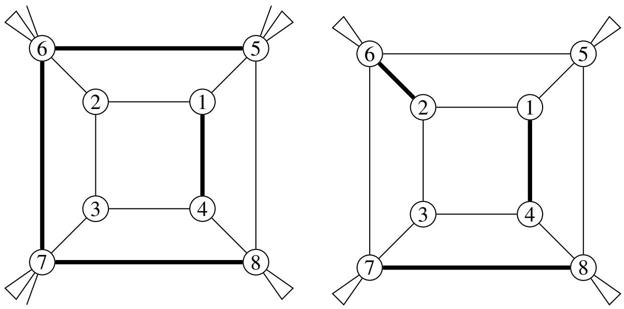

Figure 6: Possible edge assignments for a cubic quadrangle and its neighbouring faces. The bold edge corresponds to an edge of length b.

incident to an edge of length b have the same type, and all cubic vertices not incident to an edge of length b have the same type. This means that we can already fix some angle assignments for both cases in Figure 6. This partial angle assignment is shown in Figure 7. The angles in the cubic quadrangle can be fixed, since interchanging α with δ and β with γ gives the same results. The angles in the face that shares an edge of length b with the cubic quadrangle can be fixed, since the other possible assignment implies that there are two different vertex types for cubic vertices incident to an edge of length b: one containing 2α and one containing 2δ.

We first consider the edge assignment on the left side in Figure 6. The angle assignment for the quadrangle 1562 fixes all remaining angle assignments for the faces neighbouring the cubic quadrangle: either the vertex type of 1 and 4 is $\alpha + \delta + \beta$ and the vertex type of 2 and 3 is $2\gamma + \beta$, or the vertex type of 1 and 4 is $\alpha + \delta + \gamma$ and the vertex type of 2 and 3 is $2\beta + \gamma$. Owing to Lemma 4.1, these combinations are not possible, so this edge assignment is not possible.

Next we consider the edge assignment on the right side in Figure 6. The angle assignment for the quadrangle 1562 fixes all remaining angle assignments for the faces neighbouring the cubic quadrangle: either the vertex type of 1, 2 and 4 is $\alpha + \delta + \beta$ and the vertex type of 3 is 3γ, or the vertex type of 1, 2 and 4 is $\alpha + \delta + \gamma$ and the vertex type of 3 is 3β. Owing to Lemma 4.2, this implies that the quadrangulation has 6 faces, and thus 8 vertices. □

5.2 Cubic tristars

A **cubic tristar** in a quadrangulation is a cubic vertex v such that all three neighbouring vertices have degree 3. The vertex v is called the **central vertex** of the cubic tristar. Figure 8 shows an example of a cubic tristar in a quadrangulation.

Theorem 5.2. *In a spherical tiling by congruent quadrangles of type 2, there is no cubic tristar for which the central vertex is incident to an edge of length b.*

Proof. We use the vertex labels as given in Figure 8. Assume that the edge 12 has length b. This implies that either edge 36 or edge 46 has length b. Both cases are completely analogous, so we will assume without loss of generality that edge 36 has length b.

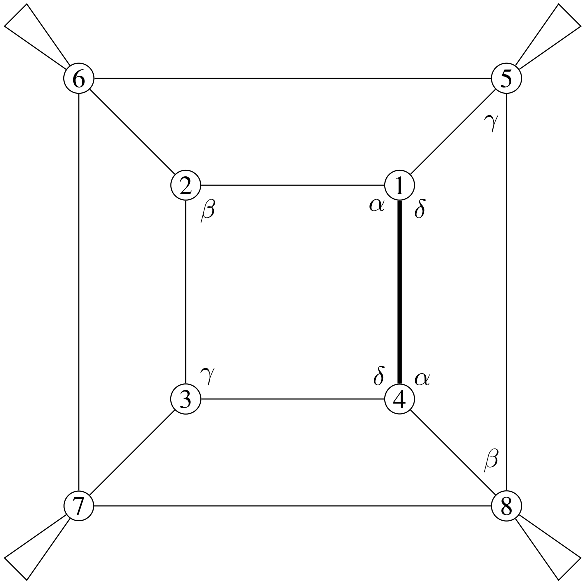

Figure 7: Partial angle assignment for a cubic quadrangle.

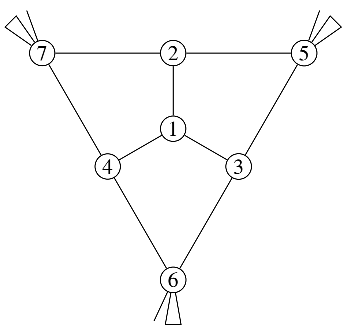

Figure 8: An example of a cubic tristar in a quadrangulation.

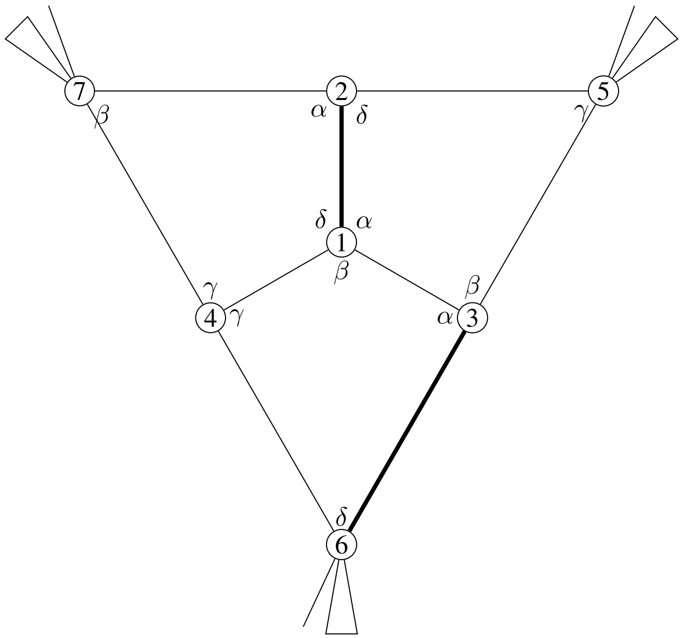

Figure 9: Partial angle assignment for a cubic tristar

We can fix the angle assignment in the quadrangle 1274, since interchanging α with δ and β with γ gives the same results. This also fixes the angle assignment in the quadrangle 1253, since the vertex 1 and the vertex 2 have the same type owing to Theorem 4.4. Since the vertex 3 is incident to an angle β, also the vertex 1 has to be incident to an angle β, and so the angle assignment in the quadrangle 1364 is also fixed. This gives the partial angle assignment shown in Figure 9.

The third angle at vertex 4 is either β or γ. Owing to Lemma 4.1, β is not possible. Owing to Lemma 4.2, γ implies that the quadrangulation has 6 faces, and thus 8 vertices. This proves the theorem. □

6 Conclusion

For the classification of spherical tilings by congruent quadranglesthere remain two open cases: spherical tilings by congruent quadrangles of type 2 and those of type 4. We show that the most symmetric of type 2 quadrangles, i.e., the isosceles quadrangles of type 2, cannot be used to tile the sphere. This might seem surprising, since spherical tilings by congruent quadrangles of type 2 do exist, but it can be explained because being isosceles and tiling the sphere forces the quadrangle to be of type 1.

Next we gave an overview of which vertex types of degree 3 can be used and showed that at most two different types can be used. We also showed that there is no spherical tiling by congruent quadrangles of type 2 for which the underlying graph contains a cubic quadrangle or a cubic tristar containing an edge of length b. As can be seen from Table 3 and Table 4, this excludes already a reasonable percentage of the quadrangulations that can

n	Quadrangulations	contain cubic quadrangle	Percentage
8	1	1	100.00%
10	1	0	0.00%
12	3	1	33.33%
14	12	3	25.00%
16	64	24	37.50%
18	510	210	41.18%
20	5 146	2 208	42.91%
22	58 782	25 792	43.88%
24	716 607	319 553	44.59%
26	9 062 402	4 110 016	45.35%
28	117 498 072	54 277 671	46.19%
30	1 553 048 548	731 637 255	47.11%

Table 3: Overview of quadrangulations on n vertices that contain a cubic quadrangle.

	10	12	14	16	18	20	22	24	26
0	1	2	7	31	217	2 065	22 869	272 106	3 355 499
1				6	68	747	8 804	108 738	1 383 419
2			2	3	15	119	1 249	15 363	201 586
3						5	66	832	11 619
4						2	2	15	259
5									4

Table 4: Overview of the number of cubic tristars in quadrangulations that do not contain a cubic quadrangle. The top row gives the number of vertices, the first column gives the number of cubic tristars and the remaining numbers give how many quadrangulations have that many vertices and that many cubic tristars.

appear as the underlying graph of a spherical tiling by congruent quadrangles of type 2, and also limits the possible charts that can correspond to a spherical tiling by congruent quadrangles of type 2. This is why these results can contribute to the completion of the classification of spherical tilings by congruent quadrangles. Table 3 and Table 4 were constructed using `plantri`[6, 4].

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Embedded graphs whose links have the largest possible number of components

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Abstract

We derive the basic properties of graphs embedded on surfaces of positive genus whose corresponding link diagrams have the largest possible number of components.

Keywords: Embedded graphs, medial, components of links.

Math. Subj. Class.: 05C10, 57M25

1 Introduction

A graph G embedded in a surface determines a link diagram $D(G)$, which has a certain number μ of components.

The relationship between the graph and the link diagram is through the crossing which replaces each edge of the graph, shown in figure 1. In this article, we are only interested in μ. So at each crossing, the choice of the over-crossing strand does not matter, and we are therefore actually considering a link universe rather than a diagram of a particular link. However, we will for simplicity refer to a link diagram.

The relationship between a graph and the number of components in the corresponding link diagram has been studied by several people. It is shown in [6] that

$$T(G;-1,-1) = (-1)^{q(G)}(-2)^{\mu(D(G))-1}, \tag{1.1}$$

where $T(G;x,y)$ is the Tutte polynomial and $q(G)$ is the number of edges in G. In [4], equation (1.1) is generalised to the projective plane and the torus, while in [7] $T(G;-1,-1)$

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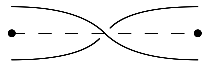

Figure 1: A crossing replacing an edge. The curved lines are strands of the link diagram, and the dashed straight line joining the two vertices is the edge of the graph replaced by the crossing.

is calculated for fans, wheels, and 2-sums of graphs. The number μ is the same as the number of "straight-ahead" walks in medial graphs, as described in [8]. The focus in [3] is to characterize the plane graphs G whose $\mu(D(G))$ is as large as possible, which is the cycle rank plus one; these are the "extremal" graphs. Maximising μ is also our principal interest here, although we will study graphs embedded on various orientable surfaces.

In section 2 we show how μ depends on the blocks of the graph, we note that μ does not change when the graph undergoes a "graph Reidemeister move" or an "embedded" $\mathrm{Y} \leftrightarrow \Delta$ move, and we show that μ cannot change very much when an edge is added to the graph. In section 3 we study plane graphs. Many of our results replicate those in [3], although our emphasis is different because we are preparing to work on other surfaces.

Let g be the genus of a surface. Then section 4 shows how to extend many of the results of section 3 to graphs embedded on surfaces with $g > 0$. In particular, in Theorem 4.3 we show that $\mu \leq f + 2g$, where f is the number of faces in the embedding, so that the extremal graphs now have $\mu = f + 2g$. In Theorem 4.5 we give a list of some graph operations which preserve the extremal property, and in Corollary 4.6 and Theorem 4.7 we give some local consequences of this property.

We finish in section 5 with some observations on other possible values of μ. For plane graphs, the case when μ is equal to the cycle rank is considered in [5], where it is found that this class of graphs is quite severely constrained. We comment on two other cases: the case when $\mu = 1$, and the Petersen and Heawood families.

It is a pleasure to thank Iain Moffatt for many interesting discussions.

2 Basic results

Our first theorem comes from the connected sum operation on links.

Theorem 2.1. *Let G be a graph with blocks $B_1, B_2, \ldots, B_k$. Then*

$$\mu(D(G)) = \sum_{i=1}^{k} \mu(D(B_i)) - (k-1).$$

Proof. For any two adjacent blocks B_i and B_{i+1} of G with common vertex v, the two strands at v must be part of the same component. So splitting G at v into two graphs increases the number of components by one. See figure 2.

Therefore, splitting G into its k blocks increases the number of components by $k-1$, and hence the result. □

Theorem 2.2. *Let G be a graph with a bridge e. Then*

$$\mu(D(G)) = \mu(D(G/e)).$$

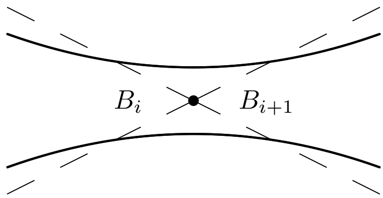

Figure 2: The curved lines are two strands of a single component at a cut vertex of the graph, which separates blocks B_i and B_{i+1}. The dashed straight lines are edges of the graph.

Proof. Let G_1 and G_2 be the two components of $G \setminus \{e\}$, and let B be the block in G containing e. Then by Theorem 2.1

$$\begin{aligned}\mu(D(G)) &= \mu(D(G_1)) + \mu(D(G_2)) + \mu(D(B)) - 2 \\ &= \mu(D(G_1)) + \mu(D(G_2)) - 1,\end{aligned}$$

because $\mu(D(B)) = 1$. However, $G/\{e\}$ consists of blocks G_1 and G_2, so by Theorem 2.1 again

$$\mu(D(G/\{e\})) = \mu(D(G_1)) + \mu(D(G_2)) - 1,$$

and hence the result. □

Theorem 2.3. *Let G be a graph with parallel edges e_1 and e_2 bounding a disc. Then*

$$\mu(D(G)) = \mu(D(G \setminus \{e_1, e_2\})).$$

If, instead, e_1 and e_2 are not parallel edges, but are incident with a common vertex of degree 2, then

$$\mu(D(G)) = \mu(D(G/\{e_1, e_2\})).$$

Proof. This follows immediately from the Reidemeister 2 move (see figure 3) on the link diagrams, which evidently preserves μ. □

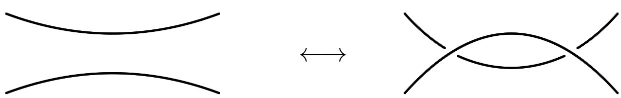

Figure 3: The Reidemeister 2 move on a link diagram.

We next consider the Y $\leftrightarrow \Delta$ moves. These replace a "Y" by a triangle, or vice versa, as in figure 4. For our purposes, we need the graph to be embedded in a surface and the triangle to bound a disc on that surface. Then we refer to the Y $\leftrightarrow \Delta$ move as **embedded**. Otherwise it is an **abstract** Y $\leftrightarrow \Delta$ move.

Theorem 2.4. *If G_1 and G_2 are related by embedded Y $\leftrightarrow \Delta$ moves, then*

$$\mu(D(G_1)) = \mu(D(G_2)).$$

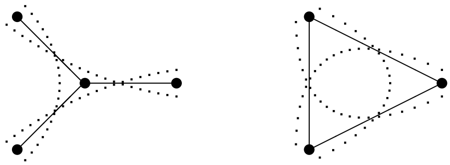

Figure 4: The effect of a Y $\leftrightarrow$ Δ move. In this figure the curved dotted lines are the strands of the link diagram, while the straight lines are the edges of the graph.

Proof. This is evident from figure 4. □

Finally in this section we note that adding an edge cannot change the number of components very much.

Theorem 2.5. *Let e be a new edge connecting two vertices in the same face of a graph G, this face being a disc. Then*

$$\mu(D(G)) - 1 \leq \mu(D(G+e)) \leq \mu(D(G)) + 1.$$

Proof. If e is a loop bounding a disc then

$$\mu(D(G)) = \mu(D(G+e)),$$

so the result holds. If e is a loop not bounding a disc, or e is not a loop, then there are two cases: we refer to figure 5, where the face is labelled F.

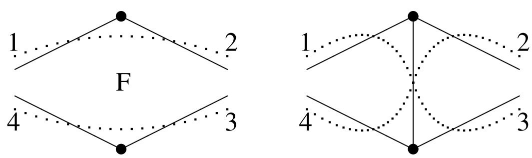

Figure 5: The effect of adding an edge in a face F. The arcs α (joining 1 and 2) and β (joining 3 and 4) may or may not be in different components. (Again, the curved dotted lines are the strands of the link diagram, while the straight lines are the edges of the graph.)

(1) If the arcs α (joining 1 and 2) and β (joining 3 and 4) are contained in different components of $D(G)$, then

$$\mu(D(G+e)) = \mu(D(G)) - 1.$$

(2) If the arcs α and β are contained in the same component of $D(G)$, then there are two further cases.

(a) Along this one component, if the order of the four endpoints of the two arcs α and β is $1, 2, 3, 4$ then

$$\mu(D(G+e)) = \mu(D(G)).$$

(b) If the order of the four endpoints of the two arcs α and β is $1, 2, 4, 3$ then

$$\mu(D(G+e)) = \mu(D(G)) + 1.$$

□

3 Extremal plane graphs

Theorem 3.1. *Let G be a connected plane graph. Then*

$$1 \leq \mu(D(G)) \leq f(G).$$

Proof. Let T be a spanning tree of the graph G. Then $f(T) = 1$ and $\mu(D(T)) = 1$, so the theorem is true for T.

Now add, one by one, edges to T in order to obtain G. The intermediate graphs are

$$G_1, G_2, \ldots, G_{s-1}.$$

We obtain a sequence of graphs

$$T = G_0, G_1, \ldots, G_{s-1}, G_s = G.$$

The insertion of an edge increases the number of faces by exactly one, so for $i = 0, \ldots, s-1$ we have

$$f(G_{i+1}) = f(G_i) + 1 = f(G_0) + i + 1.$$

By Theorem 2.5

$$\begin{aligned} \mu(D(G_{i+1})) &\leq \mu(D(G_i)) + 1 \quad (3.1) \\ &\leq \mu(D(G_0)) + i + 1. \quad (3.2) \end{aligned}$$

Since $\mu(D(G_0)) = f(G_0)$, we must have $\mu(D(G_{i+1})) \leq f(G_{i+1})$ for each i, which means that $\mu(D(G)) \leq f(G)$. □

If G is a connected plane graph then G is called **extremal** if

$$\mu(D(G)) = f(G).$$

A face of a plane graph is called **even** if it has an even number of edges.

Theorem 3.2. *If G is extremal then each face of G is even.*

Proof. Let T be a spanning tree of G, and let

$$T = G_0, G_1, G_2, \ldots, G_s = G$$

be the sequence of graphs in the proof of Theorem 3.1. Since T and G are extremal then each G_i in the sequence is extremal. Otherwise $\mu(D(G_i)) < f(G_i)$, and then from Theorem 2.5 we would have $\mu(D(G_{i+1})) < f(G_{i+1})$ and eventually $\mu(D(G)) < f(G)$.

T has one even face. Suppose that there is a graph in this sequence with an odd face, and let G_{i+1} be the first such graph. $G_{i+1} = G_i \cup e$ where e has been inserted into a necessarily even face in G_i, creating two odd faces f_1 and f_2 in G_{i+1}.

Because all the graphs in the sequence are extremal, we must be in case **2b** of Theorem 2.5. Choose the component of $D(G_{i+1})$ which includes the arc 13. This component contains exactly one of the faces f_1 or f_2, suppose it is f_1, and the component defines an even circuit in the edges of G_{i+1}. But the faces inside this circuit are all even except f_1, because all except f_1 come from G_i, which is a contradiction. □

Corollary 3.3. *If G is extremal then G^* is eulerian.* □

The converse of this corollary is not true. For example, let G be the dual graph of $K_{2,2}$. Then G^* is eulerian but G is not extremal.

Corollary 3.4. *If G is extremal then G is bipartite.* □

Define $\delta(G)$ to be the minimum degree of G.

Theorem 3.5. *If G is extremal, then $\delta(G) < 3$.*

Proof. Let

$$T = G_0, \ldots, G_i, G_{i+1}, \ldots, G_s = G$$

be the sequence of extremal graphs in Theorem 3.2, in which

$$G_0, \ldots, G_i$$

all have $\delta(G) < 3$, while

$$G_{i+1}, \ldots, G_s$$

all have $\delta(G) \geq 3$. Then $\delta(G_i)$ must be 2.

When we add an edge to G_i to get G_{i+1} we contradict Theorem 3.1, because in G_{i+1} the number of faces has increased but the number of components has not: see figure 6. So $i = s$. □

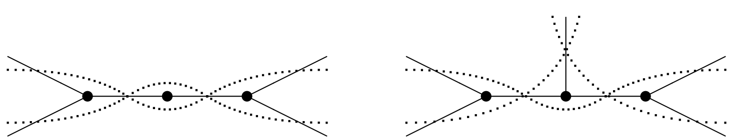

Figure 6: Adding an edge to G_i to obtain G_{i+1}.

Lemma 3.6. *If G is extremal then it has no loops, and any parallel edges must be between cut-vertices.*

Proof. G cannot have a pair e_1, e_2 of parallel edges between vertices which are not cut-vertices, because if it did then

$$\begin{aligned} f(G) &= \mu(D(G)) \\ &= \mu(D(G \setminus \{e_1, e_2\})) \qquad \text{by Theorem 3} \\ &\leq f(G \setminus \{e_1, e_2\}) \qquad \text{by Theorem 3.1} \\ &\leq f(G) - 2. \end{aligned}$$

(This argument fails if the parallel edges are between cut-vertices because Theorem 3.1 needs a connected graph.) Similarly, G cannot have a loop e, because if it did then

$$f(G) = \mu(D(G)) = \mu(D(G \setminus \{e\})) \leq f(G \setminus \{e\}) = f(G) - 1.$$

□

Lemma 3.7. *Let G be connected, simple, and non-trivial. Then G^* is not extremal.*

Proof. If G is connected, simple, and non-trivial then G^* has no vertices of degree 1 or 2. So from Theorem 3.5 G^* is not extremal. □

Lemma 3.8. *Let G be connected, simple, and non-trivial. Then*

$$\mu(D(G)) < p.$$

Proof. From Lemma 3.8 G^* is not extremal. Therefore

$$\mu(D(G^*)) < f(G^*),$$

and hence

$$\mu(D(G)) < p(G)$$

as required. □

Theorem 3.9. *Let G be a simple connected plane graph with a non-trivial dual. Then G is extremal if and only if there is an even number of edges between each pair of vertices of G^*.*

Proof. Suppose there were a pair of vertices of G^* with an odd number of edges joining them. Delete all loops and all pairs of parallel edges in G^*, to obtain a simple graph H (not necessarily connected) with at least one edge, and such that $\mu(D(G^*)) = \mu(D(H)$. Each component of H either has $\mu = 1$ or μ less than the number of vertices in that component, by Lemma 3.8. Therefore

$$\mu(D(G)) = \mu(D(G^*)) = \mu(D(H) < p(H) = p(G^*) = f(G),$$

contradicting that G is extremal.

If G^* has an even number of edges between each pair of vertices, then we delete all pairs of parallel edges in G^*, to obtain a graph with no edges. By Theorem 3 this does not change the number of components, and so

$$\mu(D(G)) = \mu(D(G^*)) = v(G^*) = f(G).$$

□

Theorem 3.10. *Let G be a connected plane graph. Then the following statements are true.*

(a) *Let e be a bridge of G. Then G/e is extremal if and only if G is extremal.*

(b) *Let v be a vertex of degree 2 with exactly one adjacent vertex. Then $G \setminus v$ is extremal if and only if G is extremal.*

(c) *Let v be a vertex of degree 2 with two different adjacent vertices x and y. Then $G/\{v,x\}/\{v,y\}$ is extremal if and only if G is extremal.*

(d) *G is extremal if and only if each block of G is extremal.*

(e) *Let G be extremal and e not a bridge in G. Then $G \setminus e$ is extremal.*

Proof. **(a)** This follows from Theorem 2.2.

(b) This follows from Theorem 3.

(c) This follows from Theorem 3.

(d) Let $B_1, B_2, \ldots, B_k$ be the blocks of G, and suppose that G is extremal. Then from Theorem 2.1 we have

$$\begin{aligned}\sum_{i=1}^{k} \mu(D(B_i)) - k + 1 &= \mu(D(G)) \\ &= f(G) \\ &= \sum_{i=1}^{k} f(B_i) - k + 1.\end{aligned}$$

Therefore

$$\sum_{i=1}^{k} \mu(D(B_i)) = \sum_{i=1}^{k} f(B_i),$$

and so each B_i is extremal. The converse is proved similarly.

(e) Since e is not a bridge, $G \setminus e$ is a connected plane graph and

$$f(G) = f(G \setminus e) + 1.$$

By Theorem 2.5

$$\mu(D(G)) \leq \mu(D(G \setminus e)) + 1,$$

but

$$\mu(D(G)) = f(G) = f(G \setminus e) + 1,$$

and so

$$\mu(D(G \setminus e)) \geq f(G \setminus e).$$

Hence by Theorem 3.1 $G \setminus e$ is extremal. □

Lemma 3.11. *Let G be extremal. Then each component of $D(G)$ only ever crosses itself on a bridge.*

Proof. Suppose a component of $D(G)$ crosses itself on the edge e in G, not a bridge. By Theorem 3.10, $G \setminus e$ is extremal.

When we delete e the number of components stays the same but the number of faces drops. This is impossible because $G \setminus e$ is extremal. □

Note that in the following theorem the graph $G_i/\{x_i, y_i\}$ is obtained from G_i by identifying the vertices x_i and y_i.

Theorem 3.12. *Let G be a plane graph. G is extremal if and only if it satisfies one of the following conditions.*

(1) $G = K_1$

(2) *G has a bridge e such that $G \setminus e$ consists of two extremal graphs.*

(3) *G has edges $e_i = x_i y_i$ for $i = 1, 2$ such that $G \setminus \{e_1, e_2\}$ consists of two disjoint graphs G_1 and G_2 with $x_i, y_i \in V(G_i)$ and $G_i/\{x_i, y_i\}$ extremal.*

Proof. Denote by f, f_1, and f_2 the numbers of faces of G, G_1, and G_2 respectively. Similarly, denote by μ, μ_1, and μ_2 the numbers of components in their link diagrams. Let G be an extremal graph, so that $\mu = f$, and suppose that G has at least one edge.

If G has a bridge e, with $G \setminus e = G_1 \cup G_2$, then

$$f = f_1 + f_2 - 1.$$

because G_1 and G_2 share a common face. Now by Theorem 2.1

$$\mu = \mu_1 + \mu_2 - 1.$$

Since G is extremal

$$\mu_1 + \mu_2 - 1 = \mu = f = f_1 + f_2 - 1.$$

Therefore

$$\mu_1 + \mu_2 - 1 = f_1 + f_2 - 1,$$

which means that

$$\mu_1 + \mu_2 = f_1 + f_2.$$

Since $\mu_i \leq f_i$ for each i, we must now have

$$\mu_1 = f_1$$

and

$$\mu_2 = f_2$$

as required.

Next, let G be an extremal graph without a bridge. By Theorem 3.5 it must have a vertex v with degree less than 3. However, if $d(v) = 0$ then $G = K_1$, and if $d(v) = 1$ we have a bridge. So $d(v) = 2$. Now there are two cases.

(a) If v is adjacent to two distinct vertices x_2 and y_2, as in figure 7, then $G_1 = K_1$ (the vertex v) and $x_2 \neq y_2$ in G_2. Clearly $\mu_1 = f_1$. Suppose μ_2 is the number of components of the link diagram of $G_2/\{x_2, y_2\}$, and f_2 is the number of faces of $G_2/\{x_2, y_2\}$. Then $\mu_2 = \mu$ and $f_2 = f$ because the identification of x and y does not affect the number of components of the link diagram of G_2 or the number of faces of G_2, which means that $G_2/\{x_2, y_2\}$ is extremal.

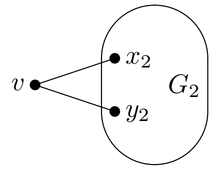

Figure 7: The case $G_1 = K_1$, adjacent to two distinct vertices in G_2.

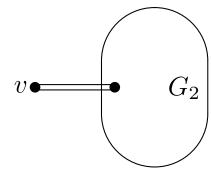

Figure 8: The case $G_1 = K_1$, adjacent twice to a vertex in G_2.

(b) If v is a vertex adjacent twice to another vertex, as in figure 8, then $G_1 = K_1$ as before, and since $x_2 = y_2$ then $\mu = \mu_2 + 1$ and $f = f_2 + 1$. Since G is extremal then $G_2/\{x_2, y_2\}$ is extremal.

Conversely, suppose that one of the three conditions holds. Then we will show that G is extremal.

(1) If $G = K_1$ then G is extremal because $\mu(G) = f(G) = 1$.

(2) If G consists of the two extremal graphs G_1, G_2 and the bridge e between them, then $\mu_1 = f_1$ and $\mu_2 = f_2$ and since e is a bridge then $\mu = \mu_1 + \mu_2 - 1$. There is a common face between G_1 and G_2, so $f = f_1 + f_2 - 1$, which gives $f = f_1 + f_2 - 1 = \mu_1 + \mu_2 - 1 = \mu$. Therefore G is extremal.

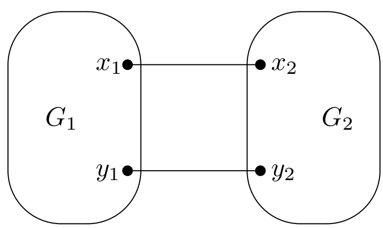

Figure 9: The construction of G from G_1 and G_2.

(3) Suppose that the plane graph G is constructed from two connected plane graphs G_1 and G_2 by adding two new edges e_1 and e_2, where $e_1 = (x_1, x_2)$, $e_2 = (y_1, y_2)$ and x_i, $y_i \in G_i$, as in figure 9. Let μ_i be the number of components of the link diagram of $G_i/\{x_i, y_i\}$ and f_i the number of faces of $G_i/\{x_i, y_i\}$. Then

$$f = f_1 + f_2 - 2 \tag{3.3}$$

because we will get two new faces, one in f_1 and another one in f_2, when we identify x_i and y_i. In order to count the components in the various link diagrams, start with $G_i/\{x_i, y_i\}$ and then "split" the vertices into x_i and y_i, obtaining the arrangement shown in figure 10.

Hence

$$\mu = \mu_1 + \mu_2 - 2. \tag{3.4}$$

From equations (3.3) and (3.4), $\mu = f$. □

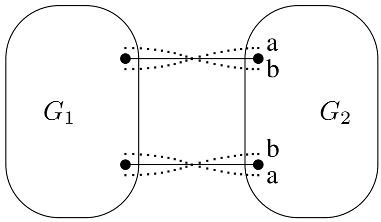

Figure 10: The two components, a and b, crossing from G_1 to G_2.

We finish this section by describing ways of constructing new extremal graphs using the operations of 2-sum and tensor product. (These are natural operations on graphs, but are perhaps most easily defined on matroids: see [2].)

Let G and H be any graphs, with distinguished edges e and f. The **2-sum** $G \oplus_2 H$ along e and f is obtained by identifying the edges e and f to form a new edge, which is then deleted.

The **tensor product** $G \otimes H$ is obtained by taking the 2-sum of G with H along *each* edge of G and the edge f in H.

For example, when H is the triangle graph, constructing $G \otimes H$ amounts to putting a new vertex of degree two in each edge of G. In this case, the embedding of $G \otimes H$ is induced from that of G, but this only happens because H is so symmetrical. In general, there may be more than one embedding of $G \otimes H$ for any given embedding of G.

Theorem 3.13. *Let G be any connected plane graph and H be an odd cycle. Then $G \otimes H$ is extremal.* □

Proof. This follows from the Reidemeister 2 move on the link diagrams. □

Theorem 3.14. *Let G be a tree and H be extremal. Then the tensor product $G \otimes H$, in which the distinguished edge in H is not a bridge, is extremal.*

Proof. This follows from Theorem 3.10, part (d). □

4 Extremal graphs on surfaces of genus g

Here we will often restrict to **cellular embeddings**, in which the interior of each face of the embedded graph is homeomorphic to an open disc. (For plane graphs this implies connectedness, of course.)

Given an embedded graph G, a spanning subgraph ψ which is connected, has just one face, and is cellularly embedded, is called a **pseudo-tree** of G. A pseudo-tree can be obtained from any cellularly embedded graph by iteratively deleting edges that lie on two faces, until no such edge can be found.

Firstly, let G be cellularly embedded on the torus. Then each block of G is a connected plane graph except for one, which must be cellularly embedded on the torus.

Theorem 4.1. *If ψ is a pseudo-tree cellularly embedded on the torus, then $\mu(D(\psi)) \leq 3$.*

Proof. We reduce ψ as follows.

- Contract all bridges in ψ. This leaves $\mu(D(\psi))$ unchanged, by Theorem 2.2.
- For each vertex of degree two in ψ whose edges go to distinct vertices, contract both these edges. By Theorem 3 this also leaves $\mu(D(\psi))$ unchanged.

ψ has one meridian M and one longitude L. $M \cap L$ must be non-empty, and if it had more than one connected component then ψ would have more than one face. Now, up to extra meridians or longitudes, there are only four possibilities for the reduced ψ, shown in figure 11, and by inspection the link diagrams for A, B, C, and D have $\mu = 2, 1, 1$, and 3 respectively. E is the same as A, with an extra meridian. □

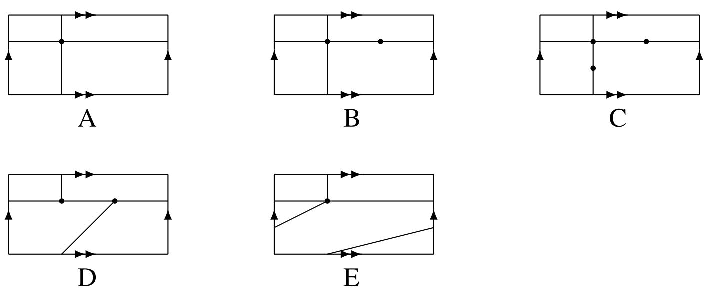

Figure 11: A, B, C, and D are the four possibilities for the reduced ψ. E is the same as A, with an extra meridian.

Theorem 4.2. *Let ψ be a pseudo-tree embedded on a surface of genus g. Then*

$$\mu(D(\psi)) \leq 1 + 2g.$$

Proof. The result is clearly true when $g = 0$.

Now suppose that for any pseudo-tree ψ_g embedded on a surface S_g of genus g, we have

$$\mu(D(\psi_g)) \leq 1 + 2g.$$

Let ψ_{g+1} be a pseudo-tree embedded on S_{g+1}, a surface of genus $g + 1$. We will show that

$$\mu(D(\psi_{g+1})) \leq 1 + 2(g + 1) = 3 + 2g.$$

In other words we will show that ψ_{g+1} has at most two more components than ψ_g.

Consider one of the handles of S_{g+1}, and let L and M be the longitude and meridian cycles in ψ_{g+1} for this handle.

Choose an edge e_L in L, but not in M, and then delete this edge. By Theorem 2.5

$$\mu(D(\psi_{g+1})) \leq \mu(D(\psi_{g+1} \setminus e_L)) + 1.$$

Repeat this process for M by choosing an edge e_M in M, but not in L, to get

$$\mu(D(\psi_{g+1} \setminus e_L)) \leq \mu(D(\psi_{g+1} \setminus \{e_M, e_L\})) + 1.$$

These two deletions yield a graph denoted ψ_g which is no longer a pseudo-tree on S_{g+1}. It is, however, a pseudo-tree on the surface of genus g obtained from S_{g+1} by removing the handle under consideration. We now have

$$\mu(D(\psi_{g+1})) \leq \mu(D(\psi_g)) + 2,$$

as required. □

Theorem 4.3. *Let G be a graph cellularly embedded on a surface of genus g. Then*

$$1 \leq \mu(D(G)) \leq f(G) + 2g.$$

Proof. Let ψ be a pseudo-tree of the graph G. Then $f(\psi) = 1$, and by Theorem 4.2 we have $\mu(D(\psi)) \leq 1 + 2g$, which means the theorem is true for ψ.

Now add edges to ψ, one by one, in order to obtain G. We obtain a sequence of graphs

$$\psi = G_0, G_1, \ldots, G_{s-1}, G_s = G.$$

The insertion of an edge increases the number of faces by exactly one, so for $i = 0, \ldots, s-1$ we have

$$f(G_{i+1}) = f(G_i) + 1 = f(G_0) + i + 1.$$

By Theorem 2.5

$$\begin{aligned} \mu(D(G_{i+1})) &\leq \mu(D(G_i)) + 1 && (4.1) \\ &\leq \mu(D(G_0)) + i + 1. && (4.2) \end{aligned}$$

Since $\mu(D(G_0)) \leq f(G_0) + 2g$, we must have

$$\begin{aligned} \mu(D(G_{i+1})) &\leq \mu(D(G_0)) + i + 1 && (4.3) \\ &\leq f(G_0) + 2g + i + 1 && (4.4) \\ &\leq f(G_{i+1}) + 2g. && (4.5) \end{aligned}$$

So $\mu(D(G_{i+1})) \leq f(G_{i+1}) + 2g$ for each i, and hence the result. □

If G is a graph cellularly embedded on a surface of genus g then G is called **extremal** if

$$\mu(D(G)) = f(G) + 2g.$$

Theorem 4.4. *If ψ is a spanning pseudo-tree of the extremal graph G, then ψ is extremal.*

Proof. Adding edges to ψ one by one we obtain a sequence of graphs

$$\psi = G_0, G_1, \ldots, G_s = G.$$

In particular,

$$G_{i-1} = G_i \setminus e,$$

where e is not a bridge. Suppose that G_i is extremal. Then

$$\mu(D(G_i)) = f(G_i) + 2g. \tag{4.6}$$

Also,

$$f(G_{i-1}) = f(G_i) - 1. \tag{4.7}$$

By Theorem 2.5

$$\begin{aligned} \mu(D(G_{i-1})) &\geq \mu(D(G_i)) - 1 \\ &= f(G_i) + 2g - 1 \\ &= f(G_{i-1}) + 2g. \end{aligned}$$

Now by Theorem 4.3

$$\mu(D(G_{i-1})) = f(G_{i-1}) + 2g,$$

and so G_{i-1} is extremal. Hence the result, by induction on i. □

With two small modifications, Theorem 3.10 is also true for graphs cellularly embedded on surfaces of genus g:

Theorem 4.5. *Let G be a graph cellularly embedded on a surface of genus g. Then the following statements are true.*

(a) *Let e be a bridge of G. Then G/e is extremal if and only if G is extremal.*

(b) *Let v be a vertex of degree 2 with exactly one adjacent vertex, the two edges joining these vertices bounding a disc. Then $G \setminus v$ is extremal if and only if G is extremal.*

(c) *Let v be a vertex of degree 2 with two different adjacent vertices x and y. Then $G/\{v, x\}/\{v, y\}$ is extremal if and only if G is extremal.*

(d) *G is extremal if and only if each block of G is extremal.*

(e) *Let G be extremal and e such that $G \setminus e$ is cellularly embedded. Then $G \setminus e$ is extremal.*

Proof. This is exactly as in Theorem 3.10. In part (e) we note that each block of G is a connected plane graph except for one, which must be cellularly embedded on the surface. □

It follows that Lemma 3.11 is true for extremal graphs cellularly embedded on surfaces of genus g:

Corollary 4.6. *Let G be an extremal graph cellularly embedded on a surface of genus g. Then each component of $D(G)$ only ever crosses itself on a bridge.*

For any vertex $v \in V(G)$ we can define its degree $d(v)$ and we can also count the number of components of the link diagram of G which pass close to v, denoting this by $\mu(v)$.

Theorem 4.7. *Let G be an extremal graph cellularly embedded on a surface of genus g, and let $v \in V(G)$, not a cut vertex. Then $d(v) = \mu(v)$.*

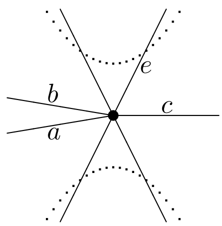

Figure 12: Two arcs, from the same component of the link diagram, passing close to v. There may be many other edges such as a, b, and c incident with v, not shown here.

Proof. Suppose that $d(v) > \mu(v)$. Then there would be two arcs (from the same component of the link diagram) passing close to v, as in figure 12. None of the edges incident with v can be a bridge, or v would be a cut vertex, so by part **e** of Theorem 4.5 $G \setminus e$ is also torus extremal.

This process can be repeated until our two arcs, passing close to v, cross the same edge incident with v, as in figure 13. But this contradicts Corollary 4.6. Hence the result. □

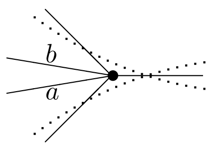

Figure 13: Two arcs, from the same component of the link diagram, passing close to v and crossing the same edge incident with v. There may be many other edges such as a and b incident with v, not shown here.

5 Concluding remarks

It may also be possible to establish results like Theorem 3.2 and Corollary 3.4 for graphs cellularly embedded on surfaces of positive genus. They certainly appear to be true on the torus:

Conjecture 5.1. *If G is an extremal graph cellularly embedded on a torus then each face of G is even.*

Conjecture 5.2. *If G is an extremal graph cellularly embedded on a torus then G is bipartite.*

Next, let us make a few observations about plane graphs for which μ takes its *smallest* possible value. We leave the proofs of the results to the reader. Evidently, if G is a tree then $\mu(D(G)) = 1$. Similarly, if G is an odd cycle or its dual then $\mu(D(G)) = 1$.

Theorem 5.3. *If G is any cycle and $H = K_3^*$, then $\mu(D(G \oplus_2 H)) = 1$.* □

In the next two theorems, the two-sum can be taken at any edge of G, and in fact it can be replaced by the tensor product.

Theorem 5.4. *Let G be a plane graph with $\mu(D(G)) = 1$ and H be an even cycle or its dual. Then $\mu(D(G \oplus_2 H)) = 1$.* □

Theorem 5.5. *Let G be a tree and H be any cycle. Then $\mu(D(G \oplus_2 H)) = 1$.* □

We finish by asking whether there are interesting families of graphs having specific values of μ greater than 1 but less than the maximum. Recall that the Petersen family $\mathcal{P}$ of graphs are those obtainable from K_6 by abstract $\mathrm{Y} \leftrightarrow \Delta$ moves. $\mathcal{P}$ has 7 members, including the Petersen graph itself, and it is of interest partly because of the following intriguing result [9]. (An intrinsically linked graph is one in which all spatial embeddings contain a non-splittable 2-component link.)

Theorem 5.6 (Robertson, Seymour, Thomas)**.** *$\mathcal{P}$ is the minor-minimal family for intrinsically linked graphs.*

The graphs in $\mathcal{P}$ can all be cellularly embedded on the torus, but these embeddings are not unique. Suppose we focus on K_6, and suppose we restrict to *embedded* $\mathrm{Y} \leftrightarrow \Delta$ moves. Then, for any particular embedding of K_6 we will obtain a subfamily of $\mathcal{P}$ whose graphs all have the same value of μ. Our preliminary results are indicated in the table below, where we have used the graph names given in [1]. (P_{10} is the Petersen graph.)

the choice of embedding	μ	the family obtained using embedded $\mathrm{Y} \leftrightarrow \Delta$ moves
(a)	3	$\mathcal{P}$
(b)	3	$\mathcal{P} \setminus \{P_{10}, Q_8\}$
(c)	3	$\mathcal{P} \setminus \{Q_8\}$
(d)	5	$\mathcal{P} \setminus \{Q_8\}$
(e)	5	$\mathcal{P} \setminus \{P_{10}, Q_8\}$
(f)	5	$\mathcal{P} \setminus \{P_{10}\}$
(g)	7	$\mathcal{P} \setminus \{P_{10}\}$

Similarly, the Heawood family $\mathcal{H}$ of graphs are those obtainable from K_7 by abstract $\mathrm{Y} \leftrightarrow \Delta$ moves. It has 20 members, which can all be cellularly embedded on the torus. It is shown in [1] that 14 of the graphs in $\mathcal{H}$ are intrinsically knotted. Again, choosing specific embeddings and restricting to embedded $\mathrm{Y} \leftrightarrow \Delta$ moves will yield subfamilies of graphs with constant μ values.

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Alternating plane graphs

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Abstract

A plane graph is called alternating if all adjacent vertices have different degrees, and all neighboring faces as well. Alternating plane graphs were introduced in 2008. This paper presents the previous research on alternating plane graphs.

There are two smallest alternating plane graphs, having 17 vertices and 17 faces each. There is no alternating plane graph with 18 vertices, but alternating plane graphs exist for all cardinalities from 19 on. From a small set of initial building blocks, alternating plane graphs can be constructed for all large cardinalities. Many of the small alternating plane graphs have been found with extensive computer help.

Theoretical results on alternating plane graphs are included where all degrees have to be from the set $\{3, 4, 5\}$. In addition, several classes of "weak alternating plane graphs" (with vertices of degree 2) are presented.

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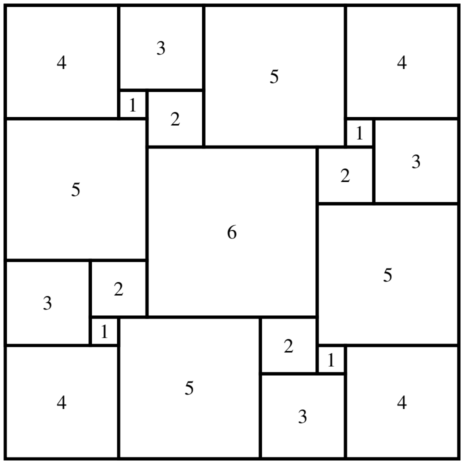

Figure 1: Karl Scherer: Squaring the square with 21 alternating squares.

Keywords: Plane graph, alternating degrees, exhaustive search, heuristic search.

Math. Subj. Class.: 05C10, 05C75

1 Introduction

The concept of alternating plane graphs was introduced by I. Althöfer in January 2008. Years before, he had seen K. Scherer's squarings of a square, in particular the nice symmetric one where 21 small squares exactly fill a square of side length 16 in such a way that no two squares with the same side length join an edge or a vertex (see Figure 1). Scherer called such arrangements "alternating tilings". The 21-solution is the second smallest such object. This concept of alternating tilings formed the inspiration for the definition of alternating plane graphs. A large portion of the history of the development of this concept can be found at [8].

The paper is organized as follows. In Section 2 we give the necessary definitions. In Section 3 several theorems about different types of alternating plane graphs are proven. In Section 4 and Section 5 we describe exhaustive and heuristic searches for alternating plane graphs. Section 6 gives an overview of the alternating plane graphs constructed by hand and by these searches. Section 7 and Section 8 deal with several techniques to construct large alternating plane graphs. In Section 9 we describe a relaxation of the definition of alternating plane graphs.

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2 Definition

Note that a planar graph is a graph that can be embedded in the plane without crossing edges. A plane graph is a particular embedding of a planar graph.

Definition 2.1. A plane graph is called an *alternating plane graph*, when the following conditions are fulfilled:

- There are no adjacent vertices with the same degree.
- There are no adjacent faces with the same size.
- Each vertex has degree at least 3.
- Each face has size at least 3.

Note that the exterior face is also considered to be a face and also needs to satisfy the conditions above.

The following lemma follows immediately from the definition.

Lemma 2.2. *If G is a 3-edge-connected alternating plane graph, then the dual of G is also an alternating plane graph.*

For a 2-edge-connected alternating plane graph that is not 3-edge-connected, the dual is not a simple graph, and therefore the dual is not an alternating plane graph. Note that an alternating plane graph is always at least 2-edge-connected, since plane graph with edge connectivity 1 contains a face that is adjacent to itself.

3 Theoretical results

Definition 3.1. An alternating plane graph is called an $(x_1, \ldots, x_n)$*-alternating plane graph* if all vertices have degree $x_1, \ldots, x_{n-1}$ or x_n and all faces have $x_1, \ldots, x_{n-1}$ or x_n sides.

3.1 Results for $(3, 4, 5)$-alternating plane graphs

Let v_i denote the number of vertices of degree i, and let f_j denote the number of faces with j sides.

Theorem 3.2. *If G is a $(3, 4, 5)$-alternating plane graph, then $v_3 = f_3$, $v_4 = f_4$ and $v_5 = f_5$.*

Proof. Suppose G is a $(3, 4, 5)$-alternating plane graph.

Summing the edges over the vertices shows that the total number of edges equals $\frac{3v_3+4v_4+5v_5}{2}$, while summing over the faces shows that it is $\frac{3f_3+4f_4+5f_5}{2}$. So we have

$$\frac{3v_3 + 4v_4 + 5v_5}{2} = \frac{3f_3 + 4f_4 + 5f_5}{2} \tag{3.1}$$

Euler's formula gives

$$2(v_3 + v_4 + v_5) + 2(f_3 + f_4 + f_5) = \frac{3v_3 + 4v_4 + 5v_5}{2} + \frac{3f_3 + 4f_4 + 5f_5}{2} + 4$$

which simplifies to

$$v_3 + f_3 = v_5 + f_5 + 8 \tag{3.2}$$

The rest of the proof is based on counting (i, j)-combinations, that is, the number of instances of a vertex of degree i incident with a face with j sides. For example, each vertex of degree 3 must be incident with a triangle, a quadrilateral and a pentagon, and each triangle must be incident with one vertex of each degree (3, 4 and 5). So we see that the number of (3, 3)-combinations must be equal to v_3, but it must also be equal to f_3, so we can deduce that

$$v_3 = f_3 \tag{3.3}$$

Note that (3.1) and (3.3) together implies that $v_5 - f_5$ must be divisible by 4.

Due to parity, each pentagon is incident with at least one vertex of each degree too, while a quadrilateral might have just two values represented.

Counting (3, 5)-combinations shows that

$$f_5 \leq v_3 \tag{3.4}$$

while a dual argument (counting (5, 3)-combinations) shows that

$$v_5 \leq f_3 \tag{3.5}$$

Combining (3.1), (3.2), (3.3), (3.4) and (3.5) gives us five possibilities:

$$\begin{aligned} v_5 = v_3 - 8,\ f_5 = v_3 \\ v_5 = v_3 - 6,\ f_5 = v_3 - 2 \\ v_5 = v_3 - 4,\ f_5 = v_3 - 4 \\ v_5 = v_3 - 2,\ f_5 = v_3 - 6 \\ v_5 = v_3,\ f_5 = v_3 - 8 \end{aligned}$$

Let a_i denote the number of vertices of degree 5 incident with exactly one face with i sides (and two of each of the other two types). Adding these up gives the total number of vertices of degree 5; i.e., $a_3 + a_4 + a_5 = v_5$. Since a vertex of degree 5 is incident with either 1 or 2 triangles, the number of (5,3)-combinations is $a_3 + 2a_4 + 2a_5$, and since each triangle is incident with exactly one vertex of degree 5, we have $a_3 + 2a_4 + 2a_5 = f_3 = v_3$. Thus, $a_3 = 2v_5 - v_3$ and $a_4 + a_5 = v_3 - v_5$. It follows that

$$0 \leq a_5 \leq v_3 - v_5 \tag{3.6}$$

Similarly, with b_j denoting the number of pentagons incident with exactly one vertex of degree j, a dual argument shows that $b_3 = 2f_5 - v_3$ and $b_4 + b_5 = v_3 - f_5$, and it follows that

$$0 \leq b_5 \leq v_3 - f_5 \tag{3.7}$$

The number of (5,5)-combinations is $2a_3 + 2a_4 + a_5 = a_5 + 2(v_5 - a_5) = 2v_5 - a_5$, and a dual argument shows that it is also equal to $2f_5 - b_5$. So we have

$$a_5 - b_5 = 2(v_5 - f_5) \tag{3.8}$$

Now let us inspect the possible values of v_5 and f_5 found earlier in view of (3.6), (3.7) and (3.8):

v_5	f_5		$2(v_5 - f_5)$
$v_3 - 8$	v_3	$b_5 = 0$	-16
$v_3 - 6$	$v_3 - 2$	$b_5 \leq 2$	-8
$v_3 - 4$	$v_3 - 4$		0
$v_3 - 2$	$v_3 - 6$	$a_5 \leq 2$	8
v_3	$v_3 - 8$	$a_5 = 0$	16

Since a_5 and b_5 are both non-negative integers, it is clear that (3.8) is only possible if $v_5 = f_5 = v_3 - 4$.

Of course, (3.1) now also gives $v_4 = f_4$ and the proof is complete. □

We can go a little bit further and show that if $p(i,j)$ denotes the number of (i,j)-combinations, then $p(i,j) = p(j,i)$ for all i,j in $\{3,4,5\}$. We have $p(3,j) = v_3$ and $p(i,3) = f_3$ for each i,j, and we know that $v_3 = f_3$. So it remains to prove that $p(4,5) = p(5,4)$. But $p(4,5) = 2b_3 + b_4 + 2b_5$ and $p(5,4) = 2a_3 + a_4 + 2a_5$, so it suffices to verify that $a_i = b_i$ for every i, and this is immediate from the proof of the theorem.

Corollary 3.3. *There is no* $(3,4,5)$*-alternating graph with fewer than 17 vertices.*

Proof. Let $r = v_3 = f_3$. So there are r vertices of degree 3, $r - 8$ vertices incident with a triangle, two quadrilaterals and two pentagons, and 4 other vertices of degree 5, and correspondingly for the dual objects. An immediate consequence is that r is at least 8. The number of edges incident with a triangle is $3r$. In addition, each of the r vertices of degree 3 is incident with an edge that is not incident with any triangles. The r edges thus obtained are all distinct, since each one is only incident with one vertex of degree 3. Hence, there are at least $4r \geq 32$ edges, and the result then follows from Euler's formula.

□

For larger r, we can get a better estimate for the number of edges than $4r$ by considering pentagons instead. There are $r - 4$ pentagons, contributing $5r - 20$ edges, and as above, r distinct edges incident with a triangle, a quadrilateral, and a vertex of degree 3. This gives a lower bound of $6r - 20$ edges. Any edges not counted so far must be incident with a triangle, a quadrilateral and two vertices of degrees 4 and 5. The number of such edges is bounded by the number of triangles, which gives an upper bound of $7r - 20$ edges in total.

Furthermore, since we have that the number of edges is

$$\frac{3f_3 + 4f_4 + 5f_5}{2},$$

it follows that the number of quadrilaterals is in the interval $[r - 5, \frac{3}{2}r - 5]$.

3.2 X, Y-alternating plane graphs

Definition 3.4. An alternating plane graph is called an *X, Y-alternating plane graph* if there are exactly X different vertex degrees and Y different face sizes.

Let $d_1, \ldots, d_X$ be the different degrees of vertices sorted in ascending order:

$$3 \leq d_1 < d_2 < \ldots < d_X,$$

and let $s_1, \ldots, s_Y$ be the different sizes of faces sorted in ascending order:

$$3 \le s_1 < s_2 < \ldots < s_Y.$$

Again let v_{d_i}, resp. f_{s_i}, be the number of vertices with degree d_i, resp. faces with size s_i. We denote the total number of vertices, resp. edges and faces, by V, resp. E and F.

This means we have

$$\sum_{i=1}^{X} v_{d_i} = V \qquad \sum_{i=1}^{X} d_i v_{d_i} = 2E$$
$$\sum_{i=1}^{Y} f_{s_i} = F \qquad \sum_{i=1}^{Y} s_i f_{s_i} = 2E$$

Substituting this in Euler's formula gives

$$4\sum_{i=1}^{X} v_{d_i} + 4\sum_{i=1}^{Y} f_{s_i} = \sum_{i=1}^{X} d_i v_{d_i} + \sum_{i=1}^{Y} s_i f_{s_i} + 8$$

which simplifies to

$$\sum_{i=1}^{X}(4-d_i)v_{d_i} + \sum_{i=1}^{Y}(4-s_i)f_{s_i} = 8. \tag{3.9}$$

Since these are all positive numbers and d_i and s_i are at least 3, this formula gives us that at least one of d_1 and s_1 is equal to 3.

It is immediately clear that X and Y are at least 2. Assume that G is a $2,Y$-alternating plane graph. This means that there are only two different vertex degrees and they form a 2-colouring of the vertices. So G is bipartite and thus contains no odd cycles. This also implies that all s_i are even and thus $s_1 \neq 3$. This gives us that $d_1 = 3$. Substituting this information in (3.9) gives

$$v_{d_1} + (4-d_2)v_{d_2} + \underbrace{\sum_{i=1}^{Y}(4-s_i)f_{s_i}}_{<0} = 8. \tag{3.10}$$

Note that $d_1 = 3$ also implies that Y is at least 3.

If $X = 2$, we also have that $d_1 v_{d_1} = d_2 v_{d_2} = E$, since each vertex is only adjacent to vertices with a different degree. So we have that

$$3v_{d_1} = d_2 v_{d_2}. \tag{3.11}$$

Combining (3.10) and (3.11), we find that

$$(4 - \frac{2}{3}d_2)v_{d_2} + \underbrace{\sum_{i=1}^{Y}(4-s_i)f_{s_i}}_{<0} = 8.$$

From this it follows that

$$4 - \frac{2}{3}d_2 > 0,$$

and so $d_2 = 4$ or $d_2 = 5$. This means that all $2, Y$-alternating plane graphs have degrees 3 and 4 or degrees 3 and 5.

We can find lower bounds for the number of vertices in $2, Y$-alternating plane graphs. We can rewrite (3.10) as

$$v_{d_1} + (4 - d_2)v_{d_2} = 8 + \sum_{i=1}^{Y}(s_i - 4)f_{s_i}. \qquad (3.12)$$

Since $d_1 = 3$, there are at least 3 different face sizes and all face sizes are even, since the graph is bipartite. This means that the left hand side in (3.12) is at least 14 (one face of size 4, one of size 6 and one of size 8). So we find:

$$v_{d_1} + (4 - d_2)v_{d_2} \geq 14. \qquad (3.13)$$

If $d_2 = 4$, then (3.13) implies that $n_1 \geq 14$ and, combined with (3.11), this also implies $n_2 \geq 11$, so we get:

$$V = v_{d_1} + v_{d_2} \geq 25.$$

If $d_2 = 5$, then (3.13) implies that $v_{d_1} - v_{d_2} \geq 14$. Combined with (3.11), this implies that $v_{d_1} \geq 35$ and $v_{d_2} \geq 21$, so we get:

$$V = v_{d_1} + v_{d_2} \geq 56.$$

This shows that the minimum order of $2, Y$-alternating plane graphs lies out of reach of the exhaustive search described in Section 4. However, the restrictions imposed on the relation between the number and size of faces and the number and degree of vertices are quite strong, so we conjecture that there exist no $2, Y$-alternating plane graphs. The situation changes if we relax the definition of alternating plane graph to also allow for vertices of degree 2. This is explained in Section 9.

4 Exhaustive search

In this section we describe the exhaustive search that was used to verify the minimality of the two alternating plane graphs with 17 vertices. The algorithm described here checks each plane graph with a given number of vertices for being an alternating plane graph. The number of plane graphs however increases too fast with increasing number of vertices to be able to verify all graphs up to 17 vertices in an acceptable time span (see Table 1). Therefore we apply several bounding criteria which prune the graphs so that not all plane graphs need to be verified individually.

We use the algorithm described in [2] to generate plane graphs with a given number of vertices. In this algorithm plane graphs are generated by starting from triangulations and removing edges of triangles to obtain the other plane graphs. That each plane graph can be constructed by this algorithm can be realised by looking at the reverse process. We can recursively add an edge between two vertices at distance 2 in a face of size greater than 3. This can be done until we end up with a triangulation. In order to avoid isomorphic graphs the algorithm in [2] uses McKay's canonical construction path method.

A plane graph G is the *parent* of a plane graph G', if in the algorithm described above G' is obtained from G by removing an edge. A plane graph G is an *ancestor* of a plane graph G' if there exist plane graphs $G_1, \ldots, G_n$ such that G is the parent of G_1, G_i is the parent of G_{i+1} for $1 \leq i < n$ and G_n is the parent of G'.

n	Graphs
4	1
5	2
6	9
7	48
8	429
9	4 794
10	64 968
11	954 362
12	14 791 881
13	237 306 720
14	3 910 739 201
15	65 870 458 907
16	1 130 289 662 773
17	19 709 446 129 094

Table 1: The number of 1-connected plane simple graphs with n vertices. This table was constructed using `plantri`.

The next lemmas follow immediately from the fact that a vertex degree can only decrease during the process above and that the minimum degree is 3.

Lemma 4.1. *A plane graph with two adjacent vertices of degree 3 is never an ancestor of an alternating plane graph.*

Lemma 4.2. *A plane graph with a triangle with vertices of degree 3, 4 and 4 is never an ancestor of an alternating plane graph.*

These two lemmas have been used to implement a modification of the program `plantri` in order to only generate alternating plane graphs. As can be seen in Table 2, Lemma 4.1 gives the most restrictions on the generation process. This motivates our choice to implement the modifications to generate alternating plane graphs as follows. After an edge is removed we check whether we can prune the graph based on Lemma 4.1. During an edge removal only two vertices change their degree. If either of these vertices have degree 3, we need to check whether there are any neighbours with degree 3. Then the algorithm from [2] continues and checks whether the edge removal was canonical. If this test also passes, we then check to see whether the graph can be pruned based on Lemma 4.2. When the algorithm finds a graph that it wants to output, we still need to check it for being an alternating plane graph. However, the number of graphs that remain to be checked, is considerably smaller than when just using such a filter on the unmodified algorithm from [2].

The results of the exhaustive search are shown in Table 3. As can be seen, the running time increases greatly near the end of the table. To obtain these results the jobs were split into several parts. These were run on 2.26 GHz Intel Xeon Nehalem processors and 2.6 GHz Intel Xeon Sandy Bridge processors. The time needed per job is not evenly distributed: some jobs were finished in less than a minute, while other jobs still needed more

	Lemma 4.1		Lemma 4.2		Lemma 4.1 and Lemma 4.2	
n	Graphs	Ratio	Graphs	Ratio	Graphs	Ratio
4	1	100.0%	1	100.0%	1	100.0%
5	1	50.0%	1	50.0%	1	50.0%
6	3	33.3%	6	66.7%	3	33.3%
7	14	29.2%	30	62.5%	13	27.1%
8	105	24.5%	273	63.6%	85	19.8%
9	1 039	21.7%	2 901	60.5%	786	16.4%
10	13 073	20.1%	37 549	57.8%	9 164	14.1%
11	179 961	18.9%	533 883	55.9%	119 395	12.5%
12	2 616 640	17.7%	8 034 607	54.3%	1 664 062	11.2%
13	39 229 044	16.5%	125 435 404	52.9%	24 075 368	10.1%
14	601 955 195	15.4%	2 013 603 025	51.5%	358 017 589	9.2%
15	9 410 493 660	14.3%	33 047 399 191	50.2%	5 438 015 472	8.3%
16	149 488 913 702	13.2%	552 519 039 867	48.9%	84 066 660 749	7.4%
17	2 408 166 869 587	12.2%	9 385 351 956 659	47.6%	1 319 262 418 144	6.7%

Table 2: The influence of Lemma 4.1 and Lemma 4.2 on the number of graphs. The ratio shows what percentage of graphs remain to be checked for being an alternating plane graph compared to the total number of plane graphs on n vertices.

than a week. This uneven distribution makes it difficult to also obtain the results for 20 vertices even when we split the generation into many jobs.

5 Heuristic searches

This section describes the implementation of the algorithm, which was used to find (3,4,5)-alternating plane graphs with $17, 20, 21, \dots, 41, 42$ vertices and also some other alternating plane graphs with certain sought properties.

The basic idea of the algorithm is to grow a $(3, \dots, x)$-alternating plane graph with exactly N vertices by starting with a smallest face (e.g., a triangle) as the current graph and then systematically adding one face at at time "at the border" of the current graph, using backtracking when it becomes obvious that no solution (or no better solution than the best solution found so far) can be found.

Note that the border of the faces added so far is also a face of the graph, which we call the exterior face. We call the other faces interior faces.

The degree of already created interior faces never changes during the construction and adjacent interior faces always have different degree, i.e., fulfill the face constraint. Vertices adjacent only to interior faces (no longer at the border) are called interior vertices. Their degree also cannot change and so they always have to have different degree (fulfill the vertex constraint).

The recursive search takes a graph represented by the faces added so far and the current border and then recursively tries all possible ways to add a new face at the border. Listing 1 gives an overview.

The recursive search first checks, if any interior vertices violate the vertex constraint. In that case it backtracks, because the degree of interior vertices does not change by adding faces at the border and so the vertex constraint cannot be fulfilled. Then a lower bound on the number of edges needed to fix vertex constraint violations between border vertices or between border and interior vertices is computed. If the algorithm has already found a

```
recursiveSearch(currentGraph, remaining number of vertices)
{
  return,
     if interior vertices violate the vertex-constraint or
     if no solution with less edges than the best solution
        found so far is possible.

  probe hashtable

  check, if a solution has been found

  generate all feasible branches
  sort them
  for branches
      add face to graph
      recursiveSearch
      remove added face

  remember current graph in hashtable
}
```

Listing 1: Recursive search

```
generateBranches(currentGraph)
{
  for s = start-vertex of the new face
     for e = end-vertex of the new face
         for n=0...N-nVertices new vertices to add
             generate branch n-path between s and e
             generate branch n-path between e and s
             check if the new branches are feasible
}
```

Listing 2: Generating branches

solution, it does not search for solutions with more edges.

Now, the algorithm probes a hashtable to check if an equivalent graph has been visited previously by the recursive search. The hash table stores the current border (number of vertices, degree of each vertex, degree of each (interior) face on border edges) and the number of vertices and edges used so far. However, it does not store the vertex indices of the border vertices or any interior vertices/faces of the current graph. So on the one hand, the hashing only detects some isomorphisms, but on the other hand, it prevents the algorithm from extending on a current graph, which only differs from a previously searched graph (with identical border vertex degrees and faces) at irrelevant interior vertices/faces.

Unlike the exhaustives search in Section 4 the algorithm does not generate all alternating plane graphs but it can be used to find at least one alternating plane graph with sought properties (e.g., a $(3,4,5)$-alternating plane graph with 17 vertices).

If the current graph is a valid alternating plane graph with the sought properties (e.g., a $(3,4,5)$-alternating plane graph with 17 vertices), it is returned. Otherwise the search continues by generating all feasible branches. Each feasible branch adds one face at the border of the current graph by adding a single edge or a n-path between two border vertices.

The program can be run to search the whole searchspace or (when just trying to find a graph) as a beamsearch. In beamsearch mode, successor nodes of the search are ordered heuristically and only the best X successors are searched. Listing 2 shows how branches are generated.

For all pairs of border-vertices s and e and each number n of vertices to be added, generateBranches() tries to generate two branches. The first connects s to e by a n-path and the second connects e to s by a n-path. A branch is feasible if

- the degree of the new face is valid (e.g., $3 \leq$ degree ≤ 5 for a $(3,4,5)$-alternating plane graph), and
- the face-constraint between the new face and interior faces is valid.

At the root of the search tree (when adding the second face to the first face), some extra rules are used to prevent generating (too many) isomorphic graphs.

For $n \leq 17$, the number of nodes needed to fully search (3,4,5)-alternating plane graphs with n vertices is roughly four times that of $n-1$. For $n = 18$ and $n = 19$ that factor increases to about 25. This behaviour is caused by the diminishing effect of the hashtable on the number of nodes searched, when the size of the graph increases.

6 List of constructed alternating plane graphs

As can be seen in Table 3, the exhaustive search shows that the smallest alternating plane graphs have 17 vertices. Both are (3,4,5)-alternating plane graphs, have 8 triangles, 5 quadrangles and 4 pentagons, and are self-dual. Figure 2 on the left shows the graph found by Frank Schneider using the heuristic search and on the right the graph found in Ghent using the exhaustive search. The minimality of the 17-vertex graphs has been confirmed by an independent implementation.

n	Graphs	Time
4	0	0.0 s
5	0	0.0 s
6	0	0.0 s
7	0	0.0 s
8	0	0.0 s
9	0	0.0 s
10	0	0.0 s
11	0	0.2 s
12	0	2.1 s
13	0	31.4 s
14	0	$\approx$ 7.9 min
15	0	$\approx$ 2.0 hours
16	0	$\approx$ 1.3 days
17	2	$\approx$ 16.4 days
18	0	$\approx$ 301.3 days
19	5	$\approx$ 13.1 years

Table 3: The number of 1-connected alternating plane graphs found by the exhaustive search described in Section 4. For the largest orders, the jobs were split into several parts and the cumulated running time is given. These were run on 2.26 GHz Intel Xeon Nehalem processors and 2.6 GHz Intel Xeon Sandy Bridge processors.

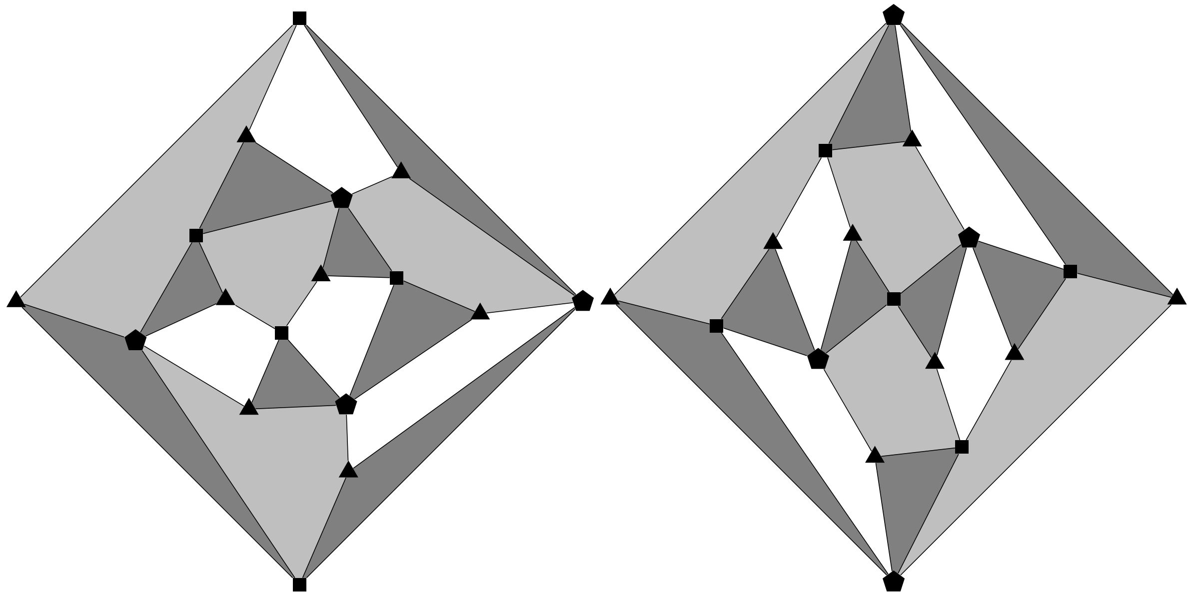

Figure 2: The two smallest alternating plane graphs: on the left Schneider-17 and on the right Ghent-17. The vertices of degree 3, resp. 4 and 5, are shown with triangles, resp. squares and pentagons. The faces of size 3, resp. 4 and 5, are shown in dark gray, resp. white and light gray.

nr	V	F	Γ	dual	author	vertices						faces									
						3	4	5	6	7	8	3	4	5	6	7	8	9	10	11	12
1:	17	17	C_2	S	FS	8	5	4				8	5	4							
2:	17	17	C_2	S	FS, GG	8	5	4				8	5	4							
3:	19	19	C_1	6	GG	9	6	3	1			9	5	5							
4:	19	19	C_3	S	GG	9	6	3	1			9	6	3	1						
5:	19	19	C_1	S	GG	9	6	3	1			9	6	3	1						
6:	19	19	C_1	3	GG	9	5	5				9	6	3	1						
7:	19	19	C_3	S	GG	9	6	3	1			9	6	3	1						
8:	20	20	C_1	S	FS	9	6	5				9	6	5							
9:	21	21	C_2	S	FS	10	5	6				10	5	6							
10:	22	22	C_1	11	FS	10	6	6				10	6	6							
11:	22	22	C_1	10	FS	10	6	6				10	6	6							
12:	22	22	C_2	13	FS	10	6	6				10	6	6							
13:	22	22	C_2	12	FS	10	6	6				10	6	6							
14:	23	23	C_1	15	FS	10	7	6				10	7	6							
15:	23	23	C_1	14	FS	10	7	6				10	7	6							
16:	23	23	C_2	17	FS	10	7	6				10	7	6							
17:	23	23	C_2	16	FS	10	7	6				10	7	6							
18:	24	24	C_1	19	FS	10	8	6				10	8	6							
19:	24	24	C_1	18	FS	10	8	6				10	8	6							
20:	25	25	C_1	21	KNLS	12	6	6	1			12	8	2	3						
21:	25	25	C_1	20	KNLS	12	8	2	3			12	6	6	1						
22:	25	25	C_3	23	KS	12	6	6	1			12	9	0	4						
23:	25	25	C_3	22	KS	12	9	0	4			12	6	6	1						
24:	25	25	C_2	S	FS	10	9	6				10	9	6							
25:	25	25	C_2	26	KS	12	7	4	2			12	7	4	2						
26:	25	25	C_2	25	KS	12	7	4	2			12	7	4	2						
27:	26	26	C_1	28	FS	11	8	7				11	8	7							
28:	26	26	C_1	27	FS	11	8	7				11	8	7							
29:	27	27	C_1	S	FS	11	9	7				11	9	7							
30:	28	28	C_1	S	FS	11	10	7				11	10	7							
31:	29	29	C_1	32	FS	12	9	8				12	9	8							
32:	29	29	C_1	31	FS	12	9	8				12	9	8							
33:	29	31	C_1	37	KS	12	8	6	3			15	9	7							
34:	30	30	C_1	36	FS	12	10	8				12	10	8							
35:	30	33	C_1	46	FS	12	9	5	3	1		16	11	6							
36:	30	30	C_1	34	FS	12	10	8				12	10	8							
37:	31	29	C_1	33	KS	15	9	7				12	8	6	3						
38:	31	31	C_1	S	FS	12	11	8				12	11	8							
39:	32	32	C_1	S	FS	12	12	8				12	12	8							
40:	32	32	C_1	41	FS	12	12	8				12	12	8							
41:	32	32	C_1	40	FS	12	12	8				12	12	8							
42:	32	32	C_2	43	KS	14	10	6	2			16	8	6	1	0	1				
43:	32	32	C_2	42	KS	16	8	6	1	0	1	14	10	6	2						
44:	33	33	C_1	45	FS	13	11	9				13	11	9							
45:	33	33	C_1	44	FS	13	11	9				13	11	9							
46:	33	30	C_1	35	FS	16	11	6				12	9	5	3	1					
47:	33	33	C_2	49	KS	14	10	8	1			16	8	8	0	0	1				
48:	33	33	C_2	50	KS	16	11	2	2	2		16	8	6	3						
49:	33	33	C_2	47	KS	16	8	8	0	0	1	14	10	8	1						
50:	33	33	C_2	48	KS	16	8	6	3			16	11	2	2	2					
51:	34	34	C_1	52	FS	13	12	9				13	12	9							

Continued on next page

Table 4 – *Continued from previous page*

nr	V	F	Γ	dual	author	vertices						faces									
						3	4	5	6	7	8	3	4	5	6	7	8	9	10	11	12
52:	34	34	C_1	51	FS	13	12	9				13	12	9							
53:	34	32	C_2	N	KS	16	10	8				16	8	6	0	0	1	0	1		
54:	34	33	C_1	N	KS	16	10	6	2			15	8	8	1	1					
55:	34	34	C_1	N	KS	16	10	5	2	1		16	9	8	0	0	1				
56:	35	35	C_1	S	FS	13	13	9				13	13	9							
57:	35	35	C_2	N	KS	16	11	4	4			16	10	8	0	0	1				
58:	35	36	C_1	59	KS	17	10	4	2	1	1	17	10	8	0	1					
59:	36	35	C_1	58	KS	17	10	8	0	1		17	10	4	2	1	1				
60:	36	36	C_1	61	FS	14	12	10				14	12	10							
61:	36	36	C_1	60	FS	14	12	10				14	12	10							
62:	36	34	C_1	N	KS	17	11	7	1			15	9	7	2	0	1				
63:	37	37	C_1	S	FS	14	13	10				14	13	10							
64:	37	37	C_2	65	KS	16	11	8	2			16	10	10	1						
65:	37	37	C_2	64	KS	16	10	10	1			16	11	8	2						
66:	37	38	C_1	70	KS	17	10	6	3	1		18	10	9	0	1					
67:	37	38	C_1	77	KS	16	10	8	2	0	1	18	11	10							
68:	38	38	C_1	S	FS	14	14	10				14	14	10							
69:	38	36	C_2	N	KS	18	12	6	2			18	9	6	2	0	0	0	0	0	1
70:	38	37	C_1	66	KS	18	10	9	0	1		17	10	6	3	1					
71:	38	38	C_1	72	KS	18	10	6	4			18	10	7	2	1					
72:	38	38	C_1	71	KS	18	10	7	2	1		18	10	6	4						
73:	38	39	C_1	78	KS	17	10	9	1	0	1	18	10	10	1						
74:	39	39	C_1	75	FS	14	15	10				14	15	10							
75:	39	39	C_1	74	FS	14	15	10				14	15	10							
76:	39	37	C_1	N	KS	18	11	10				18	8	8	2	0	0	0	1		
77:	39	37	C_2	67	KS	18	11	10				16	10	8	2	0	1				
78:	39	38	C_1	73	KS	18	10	10	1			17	10	9	1	0	1				
79:	39	39	C_2	80	KS	18	9	10	2			18	10	10	0	0	1				
80:	39	39	C_2	79	KS	18	10	10	0	0	1	18	9	10	2						
81:	40	40	C_1	82	FS	15	14	11				15	14	11							
82:	40	40	C_1	81	FS	15	14	11				15	14	11							
83:	40	42	D_2	85	KS	16	10	12	2			20	10	12							
84:	41	41	C_1	S	FS	15	15	11				15	15	11							
85:	42	40	D_2	83	KS	20	10	12				16	10	12	2						
86:	42	42	C_1	87	FS	15	16	11				15	16	11							
87:	42	42	C_1	86	FS	15	16	11				15	16	11							
88:	44	42	C_2	N	KS	20	12	12				18	10	12	1	0	1				

Table 4: An overview of all the alternating plane graphs constructed for this paper. The first column gives a number for each alternating plane graph. The second column gives the number of vertices. The third column gives the number of faces. The fourth column gives the symmetry group Γ. If the graph does not have a simple dual, then the fifth column contains the letter N. If the graph is selfdual, then the fifth column contains the letter S, otherwise the fifth column contains the number of the dual of this graph. The sixth column gives the author of the graph: FS = Frank Schneider, GG = Ghent group, KNLS = Katrin Nimczick and Lisa Schreiber, KS = Karl Scherer. The remaining columns give the number of vertices for each degree and the number of faces for each number of sides.

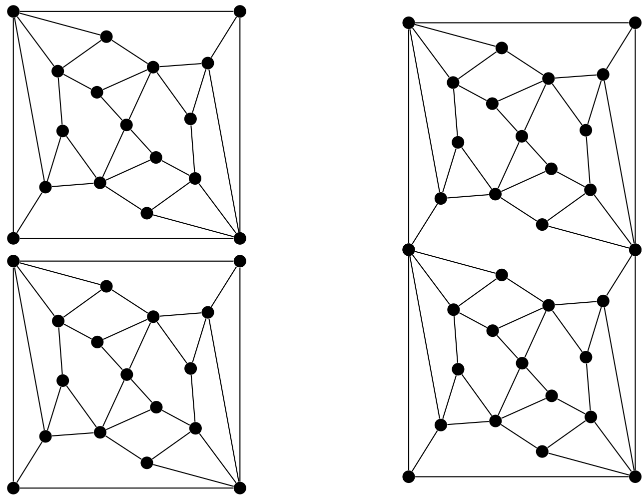

Figure 3: Glueing alternating plane graphs together by identifying vertices in the exterior face.

7 Glueing alternating plane graphs together

The idea for this technique is quite simple: take two alternating plane graphs A and B, place them apart so that they do not intersect, then identify some of the "exterior" vertices and check whether the result is an alternating plane graph. This is best explained with an example.

In Figure 3 two copies of a $(3, 4, 5)$-alternating plane graph with 17 vertices and 17 faces each are displayed. If we identify the vertex of degree 3 at the lower left of the top alternating plane graph with the vertex of degree 5 at the upper left of the bottom alternating plane graph, and the vertex of degree 5 at the lower right of the top alternating plane graph with the vertex of degree 3 at the upper right of the bottom alternating plane graph and remove the identified edge, we obtain the alternating plane graph shown on the right side of Figure 3, which has 32 vertices and 32 faces. Note that the new alternating plane graph is not a $(3, 4, 5)$-alternating plane graph anymore. Two vertices now have degree 6, one face has size 6 (the exterior face) and one face has size 8. The degrees of all other vertices and faces are unchanged.

Let us look at the left of Figure 3 again. If we identify the vertex of degree 3 at the lower left of the top alternating plane graph with the vertex of degree 3 at the upper right of the bottom alternating plane graph, we obtain a new alternating plane graph with 33 vertices and 33 faces, as shown in Figure 4.

Note that the new alternating plane graph is not a $(3, 4, 5)$-alternating plane graph anymore. One of the vertices now has degree 6 and one face has size 8. The degrees of all other vertices and faces are unchanged. This new alternating plane graph is also only 1-connected.

Looking at both examples it is clear how we can concatenate an unlimited number of alternating plane graphs to make larger and large alternating plane graphs. Using the

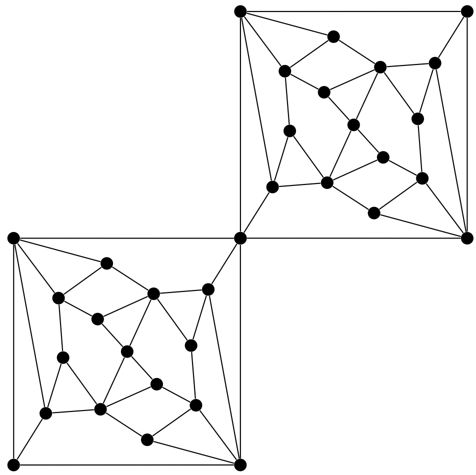

Figure 4: An alternating plane graph obtained by identifying two vertices of degree 3 in the left part of Figure 3

alternating plane graphs that were found by the heuristic and exhaustive search, we obtain the following result.

Theorem 7.1. *For any $n \geq 19$ there exists a 3-edge-connected alternating plane graph on n vertices.*

8 Large (3,4,5)-alternating plane graphs

In the previous section we showed that for any $n \geq 19$ there exists an alternating plane graph on n vertices. The size of the faces and the degree of the vertices can however increase vastly using the construction described there.

In this section we show that for each number $n \geq 111$ there exists an (3,4,5)-alternating plane graph on n vertices.

At the basis of the construction lie the 4 building blocks A, B, C, D shown in Figure 5. We will first describe how to construct a (3,4,5)-alternating plane graph from these building blocks.

All black vertices in the building blocks have degrees 3, 4 or 5. No vertex of degree 3, 4 or 5 is adjacent to a vertex with the same degree. The white vertices are only adjacent to vertices of degree 5. All faces, except the face in the middle and the exterior face, have size 3, 4 or 5. No face of size 3, 4 or 5 is adjacent to a face with the same size. The face in the middle and the exterior face are only adjacent to faces of size 5. The building blocks have 21, 22, 23 and 24 vertices for A, B, C, D respectively.

To combine two building blocks the three white vertices in the middle face of one block need to be identified one by one with a white vertex in the exterior face of the other block. An example is shown in Figure 6. The identified vertices all have degree 4 and are still only adjacent to vertices of degree 5. The newly created faces have size 4 and are only adjacent to faces of size 5. The new graph is again a building block with the same properties, but if the two original building blocks had n_1 and n_2 vertices, then this new building block has

A: B:

C: D:

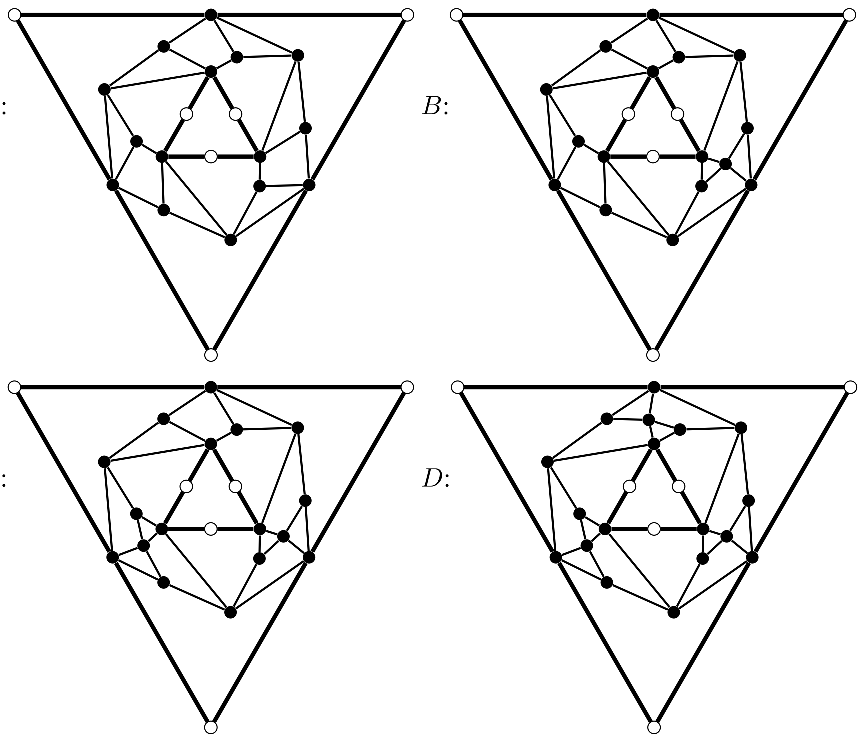

Figure 5: The four building blocks that are used to construct a (3,4,5)-alternating plane graph on n vertices for any $n \geq 111$.

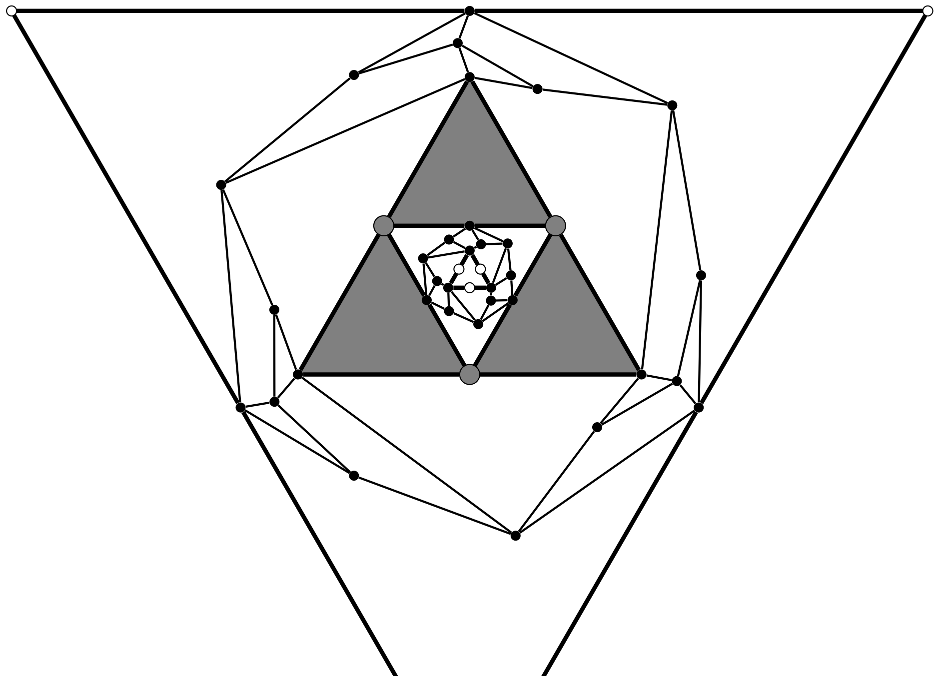

Figure 6: The combination of the A-block (inner level) and the D-block (outer level). The identified vertices are shown in gray. The new faces are also highlighted in gray.

$n_1 + n_2 - 3$ vertices. If you combine a copies of the building block with 21 vertices, b copies of the one with 22 vertices, c copies of the one with 23 vertices and d copies of the one with 24 vertices, then the number of vertices in the new block will be

$$21a + 22b + 23c + 24d - 3(a + b + c + d - 1),$$

which can be rewritten as

$$18a + 19b + 20c + 21d + 3.$$

Once you have a building block of the desired size, you still need to turn it into a (3,4,5)-alternating plane graph. This can be done by connecting one white vertex in the hexagon to the other two in the same hexagon. This replaces the hexagon by two triangles and a quadrangle. The two triangles are not adjacent and the hexagon itself was only adjacent to pentagons. The white vertices now have degrees 3, 3 and 4. The two vertices of degree 3 are not adjacent, and the white vertices were only adjacent to vertices of degree 5. So the graph is still an alternating plane graph. After doing this for both hexagons (the central and the outside one), the graph will be a (3,4,5)-alternating plane graph.

Theorem 8.1. *For any $n \geq 111$ there exists a (3,4,5)-alternating plane graph on n vertices*

Proof. By taking a copies of building block A, b copies of B, c copies of C and d copies of D, you get a (3,4,5)-alternating plane graph with $18a + 19b + 20c + 21d + 3$ vertices. The Frobenius number[1] of 18, 19, 20 and 21 is equal to 107, so we can write each number larger than 110 in the form $18a + 19b + 20c + 21d + 3$. □

For a (3,4,5)-alternating plane graph constructed by the technique above, if we colour each vertex with its degree in the (3,4,5)-alternating plane graph, then the subgraphs isomorphic to the subgraph that corresponds to the gray faces in Figure 6 can only appear between two blocks. This shows that any isomorphism between two (3,4,5)-alternating plane graphs constructed by the technique above will always map building blocks to building blocks. Together with the freedom we have in rotating and mirroring building blocks A through D and interchanging building block D with building block D' (see Figure 7), we get the following corollary.

Corollary 8.2. *The number of (3,4,5)-alternating plane graphs on n vertices grows exponentially with n.*

9 Weak alternating plane graphs

One way we can relax the conditions for alternating plane graphs, is by allowing vertices of degree 2. These graphs are called weak alternating plane graphs.

Definition 9.1. A plane graph is called a *weak alternating plane graph*, when the following conditions are fulfilled:

- There are no adjacent vertices with the same degree.
- There are no adjacent faces with the same size.
- Each vertex has degree at least 2.
- Each face has size at least 3.

A weak alternating plane graph is called an *X, Y-weak alternating plane graph* if there are exactly X different vertex degrees and Y different face sizes.

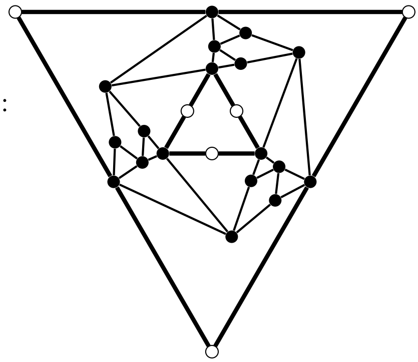

Figure 7: A second building block with 24 vertices.

9.1 Non-existence for vertex degrees 2 and k with $k \geq 11$

For weak alternating plane graphs it is clear that $2, Y$-weak alternating plane graphs do exist, e.g., take a 3-regular plane graph, substitute each edge by a digon and then subdivide each edge by a vertex.

A first result we can prove is that there exists no weak alternating plane graph with degrees 2 and k for $k \geq 11$. We do this in two steps.

Lemma 9.2. *There exists no weak alternating plane graph with vertex degrees 2 and k for* $k \geq 12$.

Proof. Let G be a weak alternating plane graph with degrees 2 and k. Let v, e, f respectively denote the number of vertices, the number of edges and the number of faces. Euler's Formula says:

$$v + f = e + 2. \tag{9.1}$$

Let f_j denote the number of faces with j sides. Since the graph is bipartite, all f_j for j odd are 0. Due to the definition of weak alternating plane graph, f_2 is also equal to 0.

Let $e_{r,s}$ denote the number of edges between r-faces and s-faces.

It is

$$rf_r = \sum_{s>2} e_{r,s}. \tag{9.2}$$

We denote this sum by S_r, so (9.2) is equivalent to

$$f_r = \frac{S_r}{r}.$$

If we look at all the sums, then we see that $e_{r,s}$ occurs two times for each pair (r, s): namely in f_r and in f_s, so we have:

$$f = \sum_{r>2} f_r = \sum_{4<s, 2<r<s} \left(\frac{e_{r,s}}{r} + \frac{e_{r,s}}{s} \right). \tag{9.3}$$

For each such pair (r, s), we have

$$\frac{e_{r,s}}{r} + \frac{e_{r,s}}{s} \leq \left(\frac{1}{4} + \frac{1}{6} \right) e_{r,s} = \frac{5}{12} e_{r,s}. \tag{9.4}$$

Combining (9.3) and (9.4), we find

$$f \leq \frac{5}{12} \sum_{4<s,2<r<s} e_{r,s} = \frac{5}{12} e. \tag{9.5}$$

If we denote the number of vertices with degree i by v_i, then we have

$$v = v_2 + v_k, \tag{9.6}$$

and,

$$v_2 = \frac{e}{2},\ v_k = \frac{e}{k}. \tag{9.7}$$

Combining (9.6) and (9.7), we find

$$v = \left(\frac{1}{2} + \frac{1}{k}\right) e. \tag{9.8}$$

Putting v- and f-values together gives via (9.5) and (9.8)

$$v + f \leq \left(\frac{1}{2} + \frac{1}{k}\right) e + \frac{5}{12} e = \left(\frac{1}{k} + \frac{11}{12}\right) e. \tag{9.9}$$

If we combine (9.1) and (9.9), we get

$$e + 2 \leq \left(\frac{1}{k} + \frac{11}{12}\right) e,$$

which is equivalent to

$$2 + \frac{1}{12} e \leq \frac{1}{k} e. \tag{9.10}$$

For all $k \geq 12$, inequality (9.10) does not hold. □

In order to show that there exist no weak alternating plane graph with vertex degrees 2 and 11, we first need the following lemma.

Lemma 9.3. *A plane multigraph containing no faces of size 2 has a vertex of degree at most 5.*

Proof. Let G be a plane multigraph containing no faces of size 2. Each face contains at least three edges, and each edge is contained in two faces. If we combine this with Euler's Formula, we get

$$e \leq 3v - 6. \tag{9.11}$$

Let δ be the minimum degree of G. Each vertex is incident to at least δ edges. Each edge contains 2 vertices. This gives

$$\delta v \leq 2e.$$

Combining this with (9.11), we find

$$12 \leq (6 - \delta) v.$$

Since v is positive, this means that δ is at most 5. □

Lemma 9.4. *There exists no weak alternating plane graph with vertex degrees 2 and 11.*

Proof. Let G be a weak alternating plane graph with vertex degrees 2 and 11. Let G' be the graph obtained from G by smoothing out the vertices of degree 2, i.e., removing each vertex v of degree 2 and connecting the vertices that were neighbours of v. This graph G' can be a multigraph, but since G was a weak alternating plane graph, there are no neighbouring faces of size 2 in G'. All vertices in G' have degree 11. Let G'' be the graph obtained from G' by replacing each face of size 2 by a single edge. This means that G'' is a plane multigraph containing no faces of size 2 and all vertices have degree at least 6. This is a contradiction with Lemma 9.3. □

Theorem 9.5. *There exists no weak alternating plane graph with vertex degrees 2 and k for $k \geq 11$.*

Proof. This follows immediately from Lemma 9.2 and Lemma 9.4. □

9.2 Existence for vertex degrees 2 and k with $k \leq 10$

If a $2, Y$-weak alternating plane graph exists with degrees 2 and k, then we can show the following result.

Lemma 9.6. *Let $G(V, E)$ be a $2, Y$-weak alternating plane graph with degrees 2 and k. The number of vertices $|V|$ is a multiple of $\frac{k+2}{2}$. So if k is even, then $|V|$ is a multiple of $\frac{k+2}{2}$ and if k is odd, then $|V|$ is a multiple of $k+2$.*

Proof. Denote by V_2 the set of vertices with degree 2 and by V_k the set of vertices with degree k. Since each edge is incident to exactly one vertex of each degree, we have that $|E| = 2|V_2| = k|V_k|$. So we find that $|V| = |V_2| + |V_k| = \frac{k}{2}|V_k| + |V_k| = \frac{k+2}{2}|V_k|$. □

There is a bijection between the weak alternating plane graphs with degrees 2 and k and the vertex-alternating k-angulations with minimum degree 2. Take any weak alternating plane graph with degrees 2 and k. First we smooth out the vertices of degree 2, i.e., we remove the vertex and the two incident edges and connect the two remaining endpoints by a new edge which replaces the two removed edges in the cyclic order for each of the two endpoints. This operations gives a k-regular, plane multigraph that is face-alternating. If we take the dual of this, then we get a vertex-alternating k-angulations with minimum degree 2. The other way around, it is clear to see that applying the inverse of this process to a vertex-alternating k-angulations with minimum degree 2 always leads to a weak alternating plane graph with degrees 2 and k.

We used this bijection to generate weak alternating plane graphs with degrees 2 and k. For $k = 3$ and $k = 4$, we generated k-angulations using the program `plantri` and filtered out those k-angulations that are vertex-alternating. For $5 \leq k \leq 10$, we used the data obtained in [4] and filtered out those k-angulations that are vertex-alternating. For $k = 9$ and $k = 10$, the available data was not sufficient to find any weak alternating plane graphs with degrees 2 and k. The results are shown in Table 5.

Although no weak alternating plane graphs with degrees 2 and 10 were found using the exhaustive method described in the previous paragraphs, it is clear that they exist due to the following construction for weak alternating plane graphs with degrees 2 and k from $\frac{k}{2}$-regular plane graphs for k even. Take a $\frac{k}{2}$-regular plane graph. Replace each of its edges by a digon. This results in a k-regular, face-alternating, plane multigraph. Finally subdivide

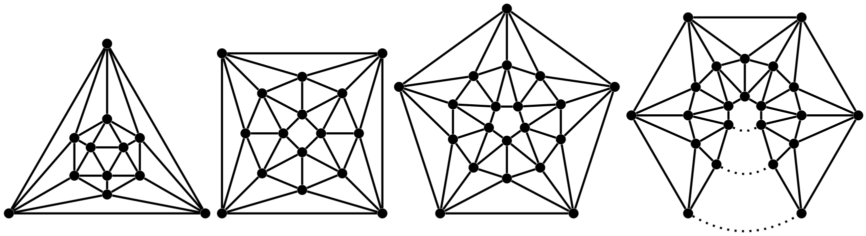

Figure 8: An infinite family of 5-regular plane graphs that can be used to construct weak alternating plane graphs with degrees 2 and 10.

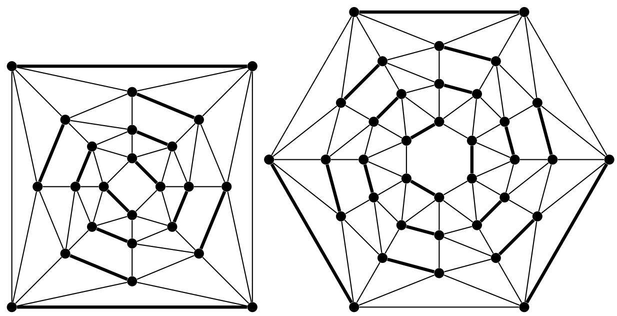

Figure 9: The first two members of an infinite family of 5-regular plane graphs that can be used to construct weak alternating plane graphs with degrees 2 and 9. A face-alternating matching is shown in bold in each graph.

each edge with a vertex to obtain a weak alternating plane graph with degrees 2 and k. Since there exist infinite families of 2-regular plane graphs (the cycles), 3-regular plane graphs (e.g., the prisms), 4-regular plane graphs (e.g., the anti-prisms) and 5-regular plane graphs (e.g., the family shown in Figure 8), this implies that there are infinitely many weak alternating plane graphs with degrees 2 and 4 (respectively 2 and 6, 2 and 8, and 2 and 10).

A similar construction can also be used to find weak alternating plane graphs with degrees 2 and 9. A face-alternating matching is a matching in a plane graph that has the property that for each edge e in the matching, e is incident with two faces with distinct sizes. Take a $\frac{k+1}{2}$-regular plane graph together with a face-alternating matching. Replace each of its edges that is not in the matching by a digon. This results in a k-regular, face-alternating, plane multigraph. Finally subdivide each edge with a vertex to obtain a weak alternating plane graph with degrees 2 and k. Since there exist infinite families of 3-regular plane graphs with face-alternating matchings (e.g., the prisms on $4n$ vertices with $n \geq 3$), 4-regular plane graphs with face-alternating matchings (e.g., the anti-prisms on $4n$ vertices with $n \geq 2$) and 5-regular plane graphs with face-alternating matchings (e.g., the family shown in Figure 9), this implies that there are infinitely many weak alternating plane graphs with degrees 2 and 5, respectively 2 and 7, and 2 and 9.

$n\backslash k$	3	4	5	6	7	8
9		1				
12		1				
15		2				
16				1		
18		4				
20	1			0		
21		7				
24		19		1		
25	6					
27		43				
28			7	1		
30	43	125				1
32				11		
33		368				
35	316		139			0
36		1 264		10	1	
39		4 744				
40	2 420			83		1
42		18 723	4 731			
45	19 648	78 657				1
48		338 945				
50	165 724					
51		1 518 480				
55	1 437 049					

Table 5: The number of weak alternating plane graphs with degrees 2 and k on n vertices found using the technique described in Section 9. Due to Lemma 9.6 the orders are always integers and multiples of $\frac{k+2}{2}$.

10 Conjectures and open problems

- As was explained in Section 3, one conjecture which our intuition suggests is the following.

  Conjecture 10.1. *There are no* $2, Y$*-alternating plane graphs and no* $X, 2$*-alternating plane graphs.*

- What the typical parameters are for large alternating plane graphs is still an open problem. E.g., if we let r be the number of vertices of degree 3 in a (3,4,5)-alternating plane graph, then we know from Theorem 3.2 that the number of vertices of degree 4 is in the interval $[r-5, \frac{3}{2}r-5]$. The question is, given this interval, how are the alternating plane graphs distributed. Is there a density function on the interval $[1, 1.5]$ which gives the asymptotic fractions of (3,4,5)-alternating plane graphs for large vertex numbers n? If so, what does the density function look like?
- The exhaustive search showed that there are no (3,4,5)-alternating plane graphs on less than 17 vertices and on 18 and 19 vertices. The heuristic search found (3,4,5)-alternating plane graphs on all numbers of vertices from 20 to 42. In Section 8 we showed that (3,4,5)-alternating plane graphs exist on all numbers of vertices starting from 111, but the same construction can also construct (3,4,5)-alternating plane graphs on n vertices for $n \in [21, \ldots, 24] \cup [39, \ldots, 45] \cup [57, \ldots, 66] \cup [75, \ldots, 87] \cup [93, \ldots, 108]$. This means that we do not know whether there exists a (3,4,5)-alternating plane graph on n vertices for $n \in [46, \ldots, 56] \cup [67, \ldots, 74] \cup [88, \ldots, 92] \cup \{109, 110\}$.

  Conjecture 10.2. *For all* $n \geq 20$ *there exist (3,4,5)-alternating plane graphs on* n *vertices.*

- In Section 7 we proved that there exist alternating plane graphs on n vertices for any $n \geq 19$. The alternating plane graphs that were constructed in that section are not 3-connected, and some are not 2-connected. The (3,4,5)-alternating plane graphs constructed in Section 8 and most of the alternating plane graphs mentioned in Section 6 are 3-connected. That is why we also pose the following conjecture.

  Conjecture 10.3. *For any* $n \geq 19$ *there exists a 3-connected alternating plane graph on* n *vertices.*

11 Concluding remarks

One central experience of our investigations is that without computer help we would never have come this far. Only the union of machine power and human creativity together let us achieve the findings in this paper.

All the graphs in this paper are also available through the website [8] and can be downloaded from House of Graphs [3] by searching for the keyword apg.

Acknowledgements

Kathrin Nimczick and Lisa Schreiber found the very first alternating plane graph back in February 2008. In those days, a long thread on alternating plane graphs started in the forum of online game server LittleGolem.net. Thanks to the people who contributed there, in particular to: wccanard (UK; he proposed the empty graph as an extremely small example of an alternating plane graph; he also immediately mentioned graphs with vertices of degree 2) and to FatPhil (Finland), Carroll (France), Hjallti (Belgium). The whole thread can be found at [9].

We would like to thank Mohammadreza Jooyandeh for providing us with the data obtained in [4].

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On global location-domination in graphs*

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Abstract

A dominating set S of a graph G is called *locating-dominating*, *LD-set* for short, if every vertex v not in S is uniquely determined by the set of neighbors of v belonging to S. Locating-dominating sets of minimum cardinality are called LD-codes and the cardinality of an LD-code is the *location-domination number* $\lambda(G)$. An LD-set S of a graph G is *global* if it is an LD-set of both G and its complement $\overline{G}$. The *global location-domination number* $\lambda_g(G)$ is introduced as the minimum cardinality of a global LD-set of G.

In this paper, some general relations between LD-codes and the location-domination number in a graph and its complement are presented first. Next, a number of basic properties involving the global location-domination number are showed. Finally, both parameters are studied in-depth for the family of block-cactus graphs.

Keywords: Domination, global domination, locating domination, complement graph, block-cactus.

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1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *close neighborhood* is $N_G[v] = \{u \in V : uv \in E\} \cup \{v\}$. The *complement* of a graph G, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The distance between vertices $v, w \in V$ is denoted by $d_G(v, w)$. We write $N(u)$ or $d(v, w)$ if the graph G is clear from the context. Assume that G and H is a pair of graphs whose vertex sets are disjoint. The *union* $G + H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \vee H$ has $V(G) \cup V(H)$ as vertex set and $E(G) \cup E(H) \cup \{uv : u \in v(G) \text{ and } v \in V(H)\}$ as edge set. For further notation, see [6].

A set $D \subseteq V$ is a *dominating set* if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set is *global* if it is a dominating set of both G and its complement graph, $\overline{G}$. The minimum cardinality of a global dominating set of G is the *global domination number* of G, denoted with $\gamma_g(G)$ [3, 4, 18]. If D is a subset of V and $v \in V \setminus D$, we say that v *dominates* D if $D \subseteq N(v)$.

A set $S \subseteq V$ is a *locating set* if every vertex is uniquely determined by its vector of distances to the vertices in S. The *location number* of G $\beta(G)$ is the minimum cardinality of a locating set of G [10, 12, 20].

A set $S \subseteq V$ is a *locating-dominating set*, *LD-set* for short, if S is a dominating set such that for every two different vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The *location-domination number* of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [21]. Certainly, every LD-set of a non-connected graph G is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. Notice also that a locating-dominating set is both a locating set and a dominating set, and thus $\beta(G) \leq \lambda(G)$ and $\gamma(G) \leq \lambda(G)$. LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1, 2, 5, 8, 13, 15] A complete and regularly updated list of papers on locating dominating codes is to be found in [16].

A *block* of a graph is a maximal connected subgraph with no cut vertices. A graph is a *block graph* if it is connected and each of its blocks is complete. A connected graph G is a *cactus* if all its blocks are cycles or complete graphs of order at most 2. Cactus are characterized as those graphs such that two different cycles share at most one vertex. A *block-cactus* is a connected graph such that each of its blocks is either a cycle or a complete graph. The family of block-cactus graphs is interesting because, among other reasons, it contains all cycles, trees, complete graphs, block graphs, unicyclic graphs and cactus (see Figure 1). Cactus, block graphs, and block-cactus have been studied extensively in different contexts, including the domination one; see [7, 11, 17, 22, 23].

The remaining part of this paper is organized as follows. In Section 2, we deal with the problem of relating the locating-dominating sets and the location-domination number of a graph and its complement. Also, *global LD-sets* and *global LD-codes* are defined. In Section 3, we introduce the so-called *global location-domination number*, and show some basic properties for this new parameter. In Section 4, we are concerned with the study of the sets and parameters considered in the preceding sections for the family of *block-cactus* graphs. Finally, the last section is devoted to address some open problems.

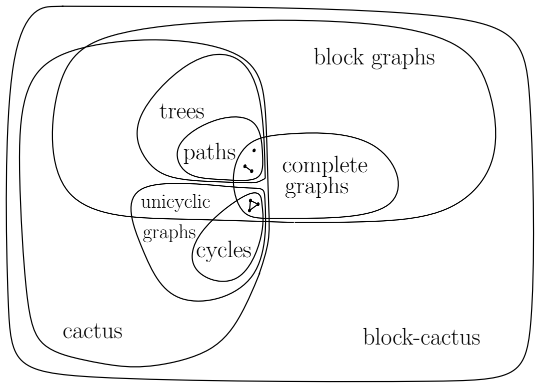

Figure 1: Families of block-cactus.

2 Relating $\lambda(G)$ to $\lambda(\overline{G})$

This section is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph G. Some of the results we present were previously shown in [13] and we include them for the sake of completeness.

Notice that $N_{\overline{G}}(x) \cap S = S \setminus N_G(x)$ for any set $S \subseteq V$ and any vertex $x \in V \setminus S$. A straightforward consequence of this fact is the following lemma.

Lemma 2.1. *Let $G = (V, E)$ be a graph and $S \subseteq V$. If $x, y \in V \setminus S$, then $N_G(x) \cap S \neq N_G(y) \cap S$ if and only if $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$.*

As an immediate consequence of this lemma, the following result is derived.

Proposition 2.2. *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$.*

Proposition 2.3 ([13])**.** *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ dominating S in G.*

Proof. By Proposition 2.2, S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$. But S is a dominating set of $\overline{G}$ if and only if $N_{\overline{G}}(u) \cap S \neq \emptyset$, for any vertex $u \in V \setminus S$. This condition is equivalent to $N_G(u) \cap S \neq S$ for any vertex $u \in V \setminus S$. Therefore, S is an LD-set of $\overline{G}$ if and only if there is no vertex $u \in V \setminus S$ such that $S \subseteq N_G(u)$, that is, there is no vertex in $V \setminus S$ dominating S. □

Proposition 2.4 ([13])**.** *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$ then there is at most one vertex $u \in V \setminus S$ dominating S, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.*

Proof. By definition of LD-set of G, there is at most one vertex adjacent to all vertices of S. Moreover, u is the only vertex not adjacent to any vertex of S in $\overline{G}$. Therefore $S \cup \{u\}$ is an LD-set of G and a dominating set of $\overline{G}$. By Proposition 2.2, it is also an LD-set of $\overline{G}$. □

Theorem 2.5 ([13])**.** *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

Proof. If S has an LD-code of G not containing a vertex dominating S, then S is an LD-set of $\overline{G}$ by 2.3. Consequently, $\lambda(\overline{G}) \leq \lambda(G)$. If S is an LD-code of G with a vertex $u \in V \setminus S$ dominating S, then $S \cup \{u\}$ is an LD-set of $\overline{G}$ by 2.4. Hence, $\lambda(\overline{G}) \leq \lambda(G) + 1$. In any case, $\lambda(\overline{G}) - \lambda(G) \leq 1$. By symmetry, $\lambda(G) - \lambda(\overline{G}) \leq 1$, and thus $|\lambda(G) - \lambda(\overline{G})| \leq 1$. □

According to the preceding result, for every graph G, $\lambda(\overline{G}) \in \{\lambda(G)-1, \lambda(G), \lambda(G)+1\}$, all cases being feasible for some connected graph G. For example, it is easy to check that the star $K_{1,n-1}$ of order $n \geq 2$ satisfies $\lambda(\overline{K_{1,n-1}}) = \lambda(K_{1,n-1})$, and the bi-star $K_2(r, s)$, $r, s \geq 2$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,s}$, satisfies $\lambda(\overline{K_2(r, s)}) = \lambda(K_2(r, s)) - 1$.

We intend to obtain either necessary or sufficient conditions for a graph G to satisfy $\lambda(\overline{G}) > \lambda(G)$, i.e., $\lambda(\overline{G}) = \lambda(G) + 1$. After noticing that this fact is closely related to the existence or not of sets that are simultaneously locating-dominating sets in both G and its complement $\overline{G}$, the following definition is introduced.

Definition 2.6. A set S of vertices of a graph G is a *global LD-set* if S is an LD-set of both G and its complement $\overline{G}$.

Certainly, an LD-set is non-global if and only if there exists a (unique) vertex $u \in V(G) \setminus S$ which dominates S, i.e., such that $S \subseteq N(u)$.

Accordingly, an LD-code S of a graph G is said to be *global* if it is a global LD-set, i.e. if S is both an LD-code of G and an LD-set of $\overline{G}$. In terms of this new definition, a result proved in [13] can be presented as follows.

Proposition 2.7 ([13]). *If G is a graph with a global LD-code, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 2.8. *If G is a graph with a non-global LD-set S and u is the only vertex dominating S, then the following conditions are satisfied:*

1. *The eccentricity of u is $ecc(u) \leq 2$;*
2. *the radius of G is $rad(G) \leq 2$;*
3. *the diameter of G is $diam(G) \leq 4$;*
4. *the maximum degree of G is $\Delta(G) \geq \lambda(G)$.*

Proof. If $x \in N(u)$, then $d(u, x) = 1$. If $x \notin N(u)$, since S is a dominating set of G, then there exists a vertex $y \in S \cap N(x) \subseteq N(u)$. Hence, $ecc(u) \leq 2$. Consequently, $rad(G) \leq 2$ and $diam(G) \leq 4$. By the other hand, $deg_G(u) = |N_G(u)| \geq |S| = \lambda(G)$, implying that $\Delta(G) \geq \lambda(G)$. □

Corollary 2.9. *If G is a graph satisfying $\lambda(\overline{G}) = \lambda(G) + 1$, then G is a connected graph such that $rad(G) \leq 2$, $diam(G) \leq 4$ and $\Delta(G) \geq \lambda(G)$.*

The above result is tight in the sense that there are graphs of diameter 4 and radius 2 (resp. $\Delta(G) = \lambda(G)$), verifying $\lambda(\overline{G}) = \lambda(G) + 1$. The graph displayed in Figure 2 is an example of graph satisfying $rad(G) = 2$, $diam(G) = 4$ and $\lambda(\overline{G}) = \lambda(G) + 1$, and the complete graph K_n is an example of a graph such that $\Delta(G) = \lambda(G)$ and $\lambda(\overline{G}) = \lambda(G) + 1$, since $\lambda(\overline{K_n}) = n$, $\lambda(K_n) = \Delta(K_n) = n - 1$.

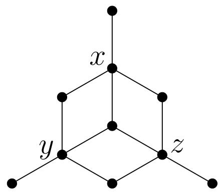

Figure 2: This graph satisfies: $rad(G) = 2$, $diam(G) = 4$, $\lambda(G) = 3$, $\lambda(\overline{G}) = 4$ and $\{x, y, z\}$ is a non-global LD-code.

3 The global location-domination number

Definition 3.1. The *global location-domination number* of a graph G, denoted by $\lambda_g(G)$, is defined as the minimum cardinality of a global LD-set of G.

Notice that, for every graph G, $\lambda_g(\overline{G}) = \lambda_g(G)$, since for every set of vertices $S \subset V(G) = V(\overline{G})$, S is a global LD-set of G if and only if it is a global LD-set of $\overline{G}$.

Proposition 3.2. *For any graph $G = (V, E)$, $\lambda(G) \leq \lambda_g(G) \leq \lambda(G) + 1$.*

Proof. The first inequality is a consequence of the fact that a global LD-set of G is also an LD-set of G. For the second inequality, suppose that S is an LD-code of G, i.e. $|S| = \lambda(G)$. If S is a global LD-set of G, then $\lambda_g(G) = \lambda(G)$. Otherwise, there exists a vertex $u \in V \setminus S$ dominating S and $S \cup \{u\}$ is an LD-set of $\overline{G}$. Therefore, $\lambda_g(G) \leq \lambda(G) + 1$. □

Corollary 3.3. *For any graph $G = (V, E)$, $\max\{\lambda(G), \lambda(\overline{G})\} \leq \lambda_g(G) \leq \min\{\lambda(G) + 1, \lambda(\overline{G}) + 1\}$.*

Corollary 3.4. *Let $G = (V, E)$ be a graph.*

- *If $\lambda(G) \neq \lambda(\overline{G})$, then $\lambda_g(G) = \max\{\lambda(G), \lambda(\overline{G})\}$.*
- *If $\lambda(G) = \lambda(\overline{G})$, then $\lambda_g(G) \in \{\lambda(G), \lambda(G)+1\}$, and both possibilities are feasible.*

Proof. Both statements are consequence of Proposition 3.2. Next, we give some examples to illustrate all possibilities given. It is easy to check that the complete graph K_2 satisfies $1 = \lambda(K_2) \neq \lambda(\overline{K_2}) = 2$ and $\lambda_g(K_2) = \lambda(\overline{K_2})$; the path P_3 satisfies $\lambda(P_3) = \lambda(\overline{P_3}) = \lambda_g(P_3) = 2$ and the cycle C_5, satisfies $\lambda(C_5) = \lambda(\overline{C_5}) = 2$ and $\lambda_g(C_5) = 3$. □

Proposition 3.5. *For any graph $G = (V, E)$, $\lambda_g(G) = \lambda(G) + 1$ if and only if every LD-code of G is non-global.*

Proof. A global LD-code of G is an LD-set of both G and $\overline{G}$. Hence, if G contains at least a global LD-code, then $\lambda_g(G) = \lambda(G)$. Conversely, if every LD-code of G is non-global, then there is no global LD-set of G of size $\lambda(G)$. Then, $\lambda_g(G) = \lambda(G) + 1$. □

As a consequence of Propositions 2.8 and 3.5, the following corollary holds.

Corollary 3.6. *If G is a graph with $diam(G) \geq 5$, then $\lambda_g(G) = \lambda(G)$.*

We finalize this section by determining the exact values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ for some basic graph families.

Lemma 3.7. *If $n \geq 7$, then $\lambda(\overline{C_n}) = \lambda(\overline{P_n}) = \lambda(P_{n-1})$.*

Proof. Firsty, we prove that $\lambda(\overline{C_n}) \leq \lambda(P_{n-1})$ and $\lambda(\overline{P_n}) \leq \lambda(P_{n-1})$. Suppose that $V(P_{n-1}) = \{1, 2, \ldots, n-1\}$ and $E(P_{n-1}) = \{(i, i+1) : i = 1, 2, ..., n-2\}$ are the vertex set and the edge set of P_{n-1}, respectively. Assume that S is an LD-code of P_{n-1} such that S does not contain vertex 1 neither $n-1$ (it is easy to construct such an LD-code from those given in [1]). Since $n-1 \geq 6$, S has at least 3 vertices and there is no vertex in $V(P_{n-1}) \setminus S$ dominating S in P_{n-1}. Hence, S is an LD-set of $\overline{P_{n-1}}$.

Next, consider the graph G^* obtained by adding to the graph $\overline{P_{n-1}}$ a new vertex u adjacent to the vertices $2, 3, \ldots, n-2$, and may be to 1 or $n-1$. Clearly, by construction, u is adjacent to all vertices of S in G^* and there is no vertex in $\overline{P_{n-1}}$ adjacent to all vertices in S. Therefore, S is an LD-set of G^* and $\lambda(G^*) \leq \lambda(P_{n-1})$. Finally, observe that if u is not adjacent to 1, neither to $n-1$, then G^* is the graph $\overline{C_n}$ and if u is adjacent to exactly one of the vertices 1 or $n-1$, then G^* is the graph $\overline{P_n}$, which proves the inequalities before stated.

Lastly, we prove that $\lambda(P_{n-1}) \leq \lambda(\overline{G})$, when $G \in \{P_n, C_n\}$. Consider an LD-code S of $\overline{G}$. Let x be the only vertex dominating S in $\overline{G}$, if it exists, or any vertex not in S, otherwise. By construction, S is an LD-set of $G - x$, hence $\lambda(G - x) \leq \lambda(\overline{G})$. To end the proof, we distinguish two cases.

- If G is the cycle C_n, then $G - x$ is the path P_{n-1}, implying that $\lambda(P_{n-1}) \leq \lambda(\overline{C_n})$.
- If G if the path P_n, then $G-x$ is either the path P_{n-1} or the graph P_r+P_s, with $r, s \geq 1$ and $r+s = n-1 \geq 6$. Since, $\lambda(P_r + P_s) = \lambda(P_r)+\lambda(P_s) = \lceil 2r/5 \rceil + \lceil 2s/5 \rceil \geq \lceil 2(r+s)/5 \rceil = \lambda(P_{n-1})$, we conclude that, in any case, $\lambda(P_{n-1}) \leq \lambda(\overline{P_n})$. □

Proposition 3.8. *Let G be a graph of order $n \geq 1$. If G belongs to the set $\{P_n, C_n, W_n, K_n, K_{1,n-1}, K_{r,n-r}, K_2(r, n-r-2)\}$, then the values of $\lambda(G)$ and $\lambda(\overline{G})$ are known and they are displayed in Tables 1 and 2.*

Proof. The values of the location-domination number of all these families, except the wheels, are already known (see [1, 13, 21]). Next, let us calculate the values of the location-domination number for the wheels and for the complements of all these families and also, from the results previously proved, the global location-domination number of them.

- For paths, cycles and wheels of small order, the values of $\lambda(G)$ and $\lambda_g(G)$ can easily be checked by hand (see Table 1).
- If $n \geq 7$, then $\lambda(W_n) = \lambda(C_{n-1}) = \lceil \frac{2n-2}{5} \rceil$, since (i) $W_n = K_1 \vee C_{n-1}$, (ii) every LD-code S of C_{n-1} is an LD-set of W_n, and (iii) every LD-code of C_{n-1} is global.
- $\lambda(\overline{K_n}) = \lambda(K_1 + \cdots + K_1) = \lambda(K_1) + \cdots + \lambda(K_1) = n$.
- $\lambda(\overline{K_{1,n-1}}) = \lambda(K_1 + K_{n-1}) = \lambda(K_1) + \lambda(K_{n-1}) = 1 + (n-2) = n-1$.
- $\lambda(\overline{K_{r,n-r}}) = \lambda(K_r + K_{n-r}) = \lambda(K_r) + \lambda(K_{n-r}) = (r-1) + (n-r-1) = n-2$, if $2 \leq r \leq n-r$.
- The complement of the bi-star $K_2(r, s)$, with $s = n-r-2$, is the graph obtained by joining a vertex v to exactly r vertices of a complete graph of order $r+s$ and joining a vertex w to the remaining s vertices of the complete graph of order $r+s$. It is immediate to verify that the set containing all vertices except w, a vertex adjacent to v and a vertex adjacent to w is an LD-code of $\overline{K_2(r, s)}$ with $n-3$ vertices. Thus, $\lambda(\overline{K_2(r, s)}) = n-3$.

G	P_1	P_2	P_3	P_4	P_5	P_6	C_4	C_5	C_6	W_5	W_6	W_7
$\lambda(G)$	1	1	2	2	2	3	2	2	3	2	3	3
$\lambda(\overline{G})$	1	2	2	2	2	3	2	2	3	3	3	4
$\lambda_g(G) = \lambda_g(\overline{G})$	1	2	2	2	3	3	2	3	3	3	3	4

Table 1: The values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ of small paths, cycles and wheels.

- For every $n \geq 7$, $\lambda(\overline{P_n}) = \lambda(\overline{C_n}) = \lceil \frac{2n-2}{5} \rceil$. This result is a direct consequence of Lemma 3.7 and the fact that $\lambda(P_n) = \lambda(C_n) = \lceil \frac{2n}{5} \rceil$.
- According to Lemma 3.7, $\lambda(\overline{W_n}) = \lambda(K_1 + \overline{C_{n-1}}) = \lambda(K_1) + \lambda(\overline{C_{n-1}}) = 1 + \lambda(P_{n-2}) = 1 + \lceil 2(n-2)/5 \rceil = \lceil (2n+1)/5 \rceil$. □

Theorem 3.9. *Let G be a graph of order $n \geq 1$. If G belongs to the set $\{P_n, C_n, W_n, K_n, K_{1,n-1}, K_{r,n-r}, K_2(r, n-r-2)\}$, then $\lambda_g(G)$ is known and it is displayed in Tables 1 and 2.*

Proof. All the cases follow from Corollary 3.4, except $K_{1,n-1}$ and $K_{r,n-r}$, which are trivial. □

G	P_n	C_n	W_n	K_n	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r, n-r-2)$
order n	$n \geq 7$	$n \geq 7$	$n \geq 8$	$n \geq 2$	$n \geq 4$	$2 \leq r \leq n-r$	$2 \leq r \leq n-r-2$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$n-1$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-3$
$\lambda_g(G) =$ $= \lambda_g(\overline{G})$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-2$

Table 2: The values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ for some families of graphs.

4 Global location-domination in block-cactus

This section is devoted to characterizing those block-cactus G satisfying $\lambda(\overline{G}) = \lambda(G)+1$. By Proposition 2.7, this equality is feasible only for graphs without global LD-codes.

We will refer in this section to some specific graphs, such as the *paw*, the *bull*; the *banner* P, the *complement of the banner*, $\overline{P}$; the *butterfly* and the *corner* L (see Figure 3).

The block-cactus of order at most 2 are K_1 and K_2. For these graphs we have $\lambda(K_1) = \lambda(\overline{K_1}) = 1$ and $\lambda(K_2) = 1 < 2 = \lambda(\overline{K_2})$.

In [5], all 16 non-isomorphic graphs with $\lambda(G) = 2$ are given. After carefully examining all cases, the following result is obtained (see Figure 4).

Proposition 4.1. *Let $G = (V, E)$ be a block-cactus such that $\lambda(G) = 2$. Then, $\lambda(\overline{G}) \geq \lambda(G)$. Moreover, $\lambda(\overline{G}) = \lambda(G) + 1 = 3$ if and only if G is isomorphic to the cycle of order 3, the paw, the butterfly or the complement of a banner.*

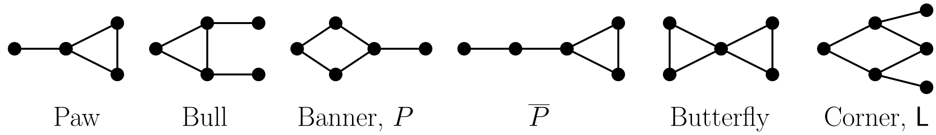

Figure 3: Some special graphs.

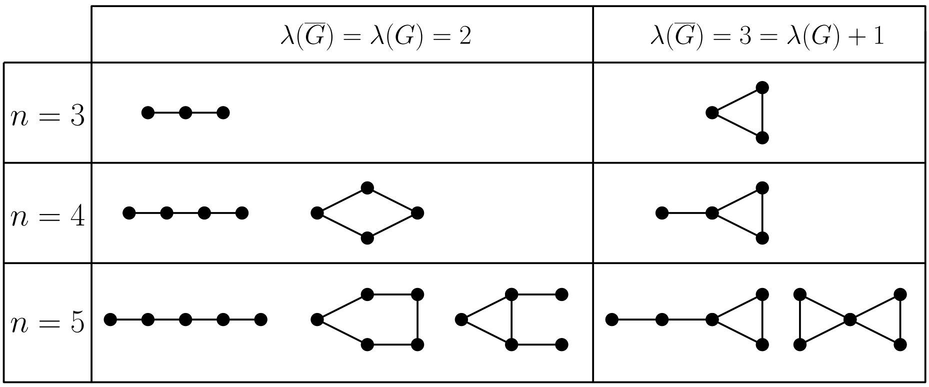

Figure 4: All block-cactus with $\lambda(G) = 2$.

Next, we approach the case $\lambda(G) \geq 3$. First of all, let us present some lemmas, providing a number of necessary conditions for a given block-cactus to have at least a non-global LD-set.

Lemma 4.2. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S, then $G[N(u)]$ is a disjoint union of cliques.*

Proof. Let x, y be a pair of vertices belonging to the same component H of $G[N(u)]$. Suppose that $xy \notin E$ and take an $x - y$ path P in H. Let z be an inner vertex of P. Notice that the set $\{u, x, y, z\}$ is contained in the same block B of G. As B is not a clique, it must be a cycle, a contradiction, since $deg_B(u) \geq 3$. □

Lemma 4.3. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S and $W = V \setminus N[u]$, then, for every vertex $w \in W$, the following properties hold.*

i) $1 \leq \mid N(u) \cap N(w) \mid \leq 2$.

ii) *If $N(u) \cap N(w) = \{x\}$, then $x \in S$.*

iii) *If $N(u) \cap N(w) = \{x, y\}$, then $xy \notin E$.*

iv) *If $w' \in W$ and $N(u) \cap N(w) = N(u) \cap N(w') = \{x\}$, then $w' = w$.*

v) *If $w' \in W$, $w' \neq w$ and $|N(u) \cap N(w)| = |N(u) \cap N(w')| = 2$, then $N[w] \cap N[w'] = \emptyset$.*

Proof. i),ii),iii): $| N(u) \cap N(w) | \geq 1$ as $S \subset N(u)$ and S dominates vertex w. If $N(u) \cap N(w) = \{x\}$, then necessarily $x \in S$. Assume that $| N(u) \cap N(w) | > 1$. Observe that the set $N[u] \cap N[w]$ is contained in the same block B of G. Certainly, B must be a cycle since $uw \notin E$. Hence, $| N(u) \cap N(w) | = 2$. Moreover, in this case B is isomorphic to the cycle C_4, which means that, if $V(B) = \{u, x, y, w\}$, then $xy \notin E$.
iv): If $w' \neq w$, then $S \cap N(w) \neq S \cap N(w')$, as S is an LD-set.
v): Suppose that $w \neq w'$, $N(u) \cap N(w) = \{x, y\}$ and $N(u) \cap N(w') = \{z, t\}$. Notice that $\{x, y\} \neq \{z, t\}$, since S is an LD-set. If $y = z$, then the set $\{u, w, w', x, y, t\}$ is contained in the same block B of G, a contradiction, because B is neither a clique, since $uw \notin E$, nor a cycle, as $deg_G(u) \geq 3$. Assume thus that $\{x, y\} \cap \{z, t\} = \emptyset$. If either $ww' \in E$ or $N(w) \cap N(w') \neq \emptyset$, then the set $\{u, w, w', x, y, z, t\}$ is contained in the same block B of G, again a contradiction, because B is neither a clique, since $uw \notin E$, nor a cycle, as $deg_G(u) \geq 4$. □

Lemma 4.4. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S and $W = V \setminus N[u]$, then*

- *Every component of $G[W]$ is isomorphic either to K_1 or to K_2.*
- *If $w, w' \in W$ and $ww' \in E$, then the set $\{w, w'\}$ is contained in the same block, which is isomorphic to C_5.*

Proof. Let w, w' such that $ww' \in E$. According to Lemma 4.3, the set $\{u\} \cup N[w] \cup N[w']$ forms a block B of G, which is isomorphic to the cycle C_5. In particular, no vertex of $W \setminus \{w, w'\}$ is adjacent to w or to w'. □

As a corollary of the previous three lemmas the following proposition is obtained.

Proposition 4.5. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S, then every maximal connected subgraph of G such that u is not a cut-vertex is isomorphic to one of the following graphs (see Figure 5):*

a) u is adjacent to every vertex of a complete graph K_r, $r \geq 1$, and each one of the vertices of K_r is adjacent to at most one new vertex of degree 1;

b) u is a vertex of a cycle of order 4, and each neighbor of u is adjacent to at most one new vertex of degree 1;

c) u is a vertex of a cycle of order 5.

In the next theorem, we characterize those block-cactus not containing any global LD-code of order at least 3.

Theorem 4.6. *Let $G = (V, E)$ be a block-cactus such that $\lambda(G) \geq 3$. Then, every LD-code of G is non-global if and only if G is isomorphic to one of the following graphs (see Figure 6):*

a) $K_1 \vee (K_1 + K_r)$, $r \geq 3$;

b) the graph obtained by joining one vertex of K_2 with a vertex of a complete graph of order $r + 1$, $r \geq 3$;

c) K_{r+1}, $r \geq 3$;

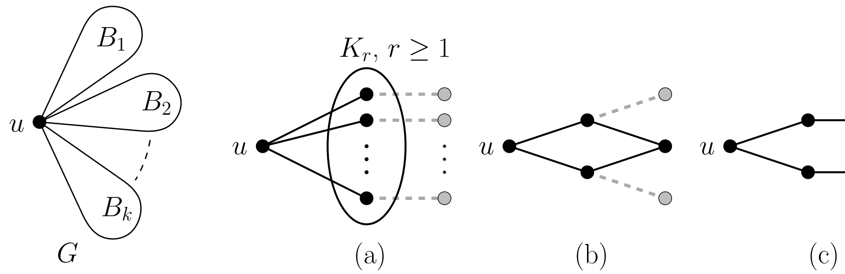

Figure 5: If $B_1, \dots, B_k$ are the maximal connected subgraphs of G with vertex u not being a cut-vertex, each subgraph B_i is isomorphic to one of the graphs displayed in (a), (b), (c). Gray vertices are optional.

d) the graph obtained by joining a vertex of K_2 with one of the vertices of degree 2 of a corner;

e) if we consider the graph $K_1 \vee (K_{r_1} + \cdots + K_{r_t})$ and t' copies of a corner, with $t + t' \geq 2$ and $r_1, \dots, r_t \geq 2$, the graph obtained by identifying the vertex u of K_1 with one of the vertices of degree 2 of each copy of the corner.

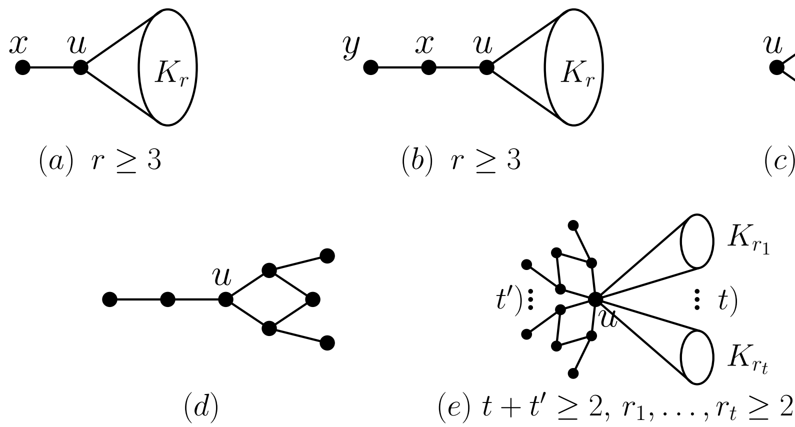

Figure 6: Block-cactus with $\lambda(G) \geq 3$ not containing any global LD-code.

Proof. Firstly, let us show that none of these graphs contains a global LD-code.

a) Let G be the graph showed in Figure 6(a). Observe that $\lambda(G) = r$ and, for every LD-code S, $|S \cap \{x, u\}| = 1$ and $|S \cap K_r| = r - 1$. Let w be the vertex of K_r not in S. If $x \in S$, then $S \subset N(u)$. Otherwise, if $u \in S$, then $S \subset N(w)$.

b) Let G be the graph showed in Figure 6(b). Notice that $\lambda(G) = r$ and, for every LD-code S, $x \in S$ and $|S \cap K_r| = r - 1$. Hence , if S is an LD-code of G, then $S \subset N(u)$.

c) If $G = K_n$ (Figure 6(c)), then G contains no global LD-code.

d) Let G be the graph showed in Figure 6(d). Clearly, the unique LD-code of G is $S = N(u)$.

e) Let G be the graph showed in Figure 6(e). In this graph, every LD-code contains both vertices adjacent to vertex u in each copy of the corner and, for every $i \in \{1,\dots,t\}$, $r_i - 1$ vertices of K_{r_i}. Thus, for every LD-code S of G, $S \subset N(u)$.

In order to prove that these are the only graphs not containing any global LD-code, we previously need to show the following lemmas.

Lemma 4.7. *Let $G = (V,E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S, then, for every component H of $G[N(u)]$ of cardinality r, $|V(H) \cap S)| = \max\{1, r-1\}$.*

Proof. This result is an immediate consequence of Lemma 4.2 ($G[N(u)]$ is a disjoint union of cliques), along with the fact that S is an LD-set. □

Given a cut vertex u of a connected graph G, let Λ_u be the set of all maximal connected subgraphs H of G such that (i) $u \in V(H)$ and (ii) u is not a cut vertex of H. Observe that any subgraph of Λ_u can be obtained from a certain component of the graph $G - u$, by adding the vertex u according to the structure of G.

Lemma 4.8. *Let $G = (V,E)$ be a block-cactus with $\lambda(G) \geq 3$ and let $S \subseteq V$ be a non-global LD-set of G. If $u \in V \setminus S$ dominates S and the set Λ_u contains a graph isomorphic to one of the graphs displayed in Figure 7, then G has a global LD-code.*

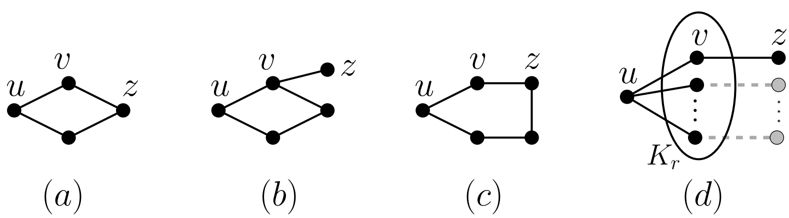

Figure 7: Some possible elements of Λ_u.

Proof. Let v, z the pair of vertices shown in Figure 7. Then, by to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{v\}) \cup \{z\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G. □

Lemma 4.9. *Let $G = (V,E)$ be a block-cactus with $\lambda(G) \geq 3$ and let $S \subseteq V$ be a non-global LD-set of G. If $u \in V \setminus S$ dominates S and the set Λ_u contains a pair of graphs H_1 and H_2 such that $H_1, H_2 \in \{P_2, P_3\}$, then G has a global LD-code.*

Proof. If H_1 is isomorphic to P_3, with $V(H_1) = \{u, v, z\}$ and $E(H_1) = \{uv, vz\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{v\}) \cup \{z\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G.

If both H_1 and H_2 are isomorphic to P_2, and $V(H_1) = \{u, t\}$ and $E(H_1) = \{ut\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{t\}) \cup \{u\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G. □

Lemma 4.10. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G whose dominating vertex is u. If Λ_u contains three graphs H_1, H_2 and H_3 such that $H_1 \in \{P_2, P_3\}$ and $H_2, H_3 \in \{K_r, \mathsf{L}\}$, where L denotes the corner graph displayed in Figure 3, then G has a global LD-code.*

Proof. If H_1 is isomorphic to P_2, with $V(H_1) = \{u, t\}$ and $E(H_1) = \{ut\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{t\}) \cup \{u\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G.

If H_1 is isomorphic to P_2, $V(H_1) = \{u, v, z\}$ and $E(H_1) = \{uv, vz\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{v\}) \cup \{z\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G. □

We are now ready to end the proof of the Theorem 4.6. Suppose that G is a block-cactus such that every LD-code of G is non-global. Let $S \subseteq V$ be an LD-code of G and let $u \in V \setminus S$ be a vertex dominating S. Notice that, according to Proposition 4.5, every graph of Λ_u is isomorphic to one of the graphs displayed in Figure 5. Moreover, having into account the results obtained in Lemma 4.8, Lemma 4.9 and Lemma 4.10, the set Λ_u is one the following sets:

- $\{P_2, K_r\}$. In this case, G is the graph shown in Figure 6(a).
- $\{P_3, K_r\}$. In this case, G is the graph shown in Figure 6(b).
- $\{P_2, \mathsf{L}\}$. Let u, t be the vertices of P_2. Then, according to Lemma 4.7, $t \in S$, and $S' = (S \setminus \{t\}) \cup \{u\}$ is a global LD-code of G.
- $\{P_3, \mathsf{L}\}$. In this case, G is the graph shown in Figure 6(d).
- $\{K_r\}$. In this case, G is the graph shown in Figure 6(c).
- A set of cardinality at least two, being every graph isomorphic either to a clique or to a corner. In this case, G is a graph as shown in Figure 6(e).

This completes the proof of Theorem 4.6. □

As an immediate consequence of Propositions 3.5 and 4.1 and Theorem 4.6, the following corollaries are obtained.

Corollary 4.11. *A block-cactus G satisfies $\lambda_g(G) = \lambda(G)+1$ if and only if G is isomorphic either to one of the graphs described in Figure 6 or it belongs to the set* $\{P_2, P_5, C_3, C_5, \overline{P}$, paw, bull, butterfly$\}$.

Corollary 4.12. *Every tree T other than P_2 and P_5 satisfies $\lambda(T) = \lambda_g(T)$.*

Corollary 4.13. *Every unicyclic graph G different from the one displayed in Figure 6(d) and not belonging to the set* $\{C_3, C_5, \overline{P}$, paw, bull$\}$ *satisfies $\lambda(G) = \lambda_g(G)$.*

If G is a block-cactus of order at least 2, we have obtained the following characterization.

Theorem 4.14. *If $G = (V, E)$ is a block-cactus of order at least 2, then $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if G is isomorphic to one of the following graphs (see Figure 8):*

(a) $K_1 \vee (K_1 + K_r)$, $r \geq 2$;

(b) the graph obtained by joining one vertex of K_2 with a vertex of a complete graph of order $r + 1$, $r \geq 2$;

(c) K_{r+1}, $r \geq 1$;

(d) $K_1 \vee (K_{r_1} + \cdots + K_{r_t})$, $t \geq 2$, $r_1, \ldots, r_t \geq 2$.

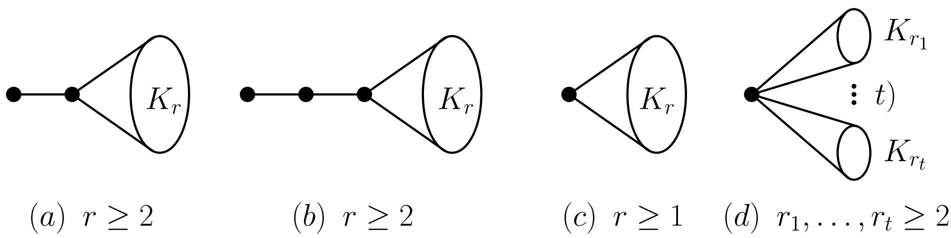

Figure 8: Block-cactus satisfying $\lambda(\overline{G}) = \lambda(G) + 1$.

Proof. Let us see first that all graphs described above satisfy $\lambda(G) < \lambda(\overline{G})$. Recall that if W is a set of twin vertices of a graph G, then every LD-set must contain at least all but one of the vertices of W. Consider one of the graphs described in (a), $G \cong K_1 \vee (K_1 + K_r)$, $r \geq 2$. The complement of G is the graph $K_1 + K_{1,r}$. It is easy to verify that $\lambda(G) = r < r + 1 = \lambda(\overline{G})$. If G is one of the graphs described in b), then $\lambda(G) = r < r + 1 = \lambda(\overline{G})$. Finally, if $G \cong K_1 \vee +(K_{r_1} + \cdots + K_{r_t})$ is a graph of order n, with $t \geq 1$ and $r_1, \ldots, r_t \geq 2$, then we have $\lambda(G) = n - t - 1 < n - t = \lambda(\overline{G})$.

Now, suppose that $G = (V, E)$ is a block-cactus of order at least 3 satisfying $\lambda(\overline{G}) = \lambda(G) + 1$.

If $\lambda(G) = 1$, as the order of G is at least 2, then G is the 2-path P_2, which satisfies $2 = \lambda(\overline{P_2}) = \lambda(P_2) + 1$. This case is described under (c) when r=1 (see Figure 8).

If $\lambda(G) = 2$, then by Proposition 4.1 the graph G is the paw, the complement of the banner, the 3-cycle C_3 or the butterfly, and these graphs are described, respectively, under (a) when $r = 2$; (b) when $r = 2$; (c) when $t = 1$ and $r_1 = 2$ and (d) when $t = r_1 = r_2 = 2$ (see Figure 8).

If $\lambda(G) \geq 3$, by Proposition 2.7, G does not contain a global LD-code, and therefore it must be one of those graphs described in Theorem 4.6. Hence, it suffices to prove that the graphs described under items d) or e) with $t' > 0$, in Theorem 4.6, do not satisfy $\lambda(\overline{G}) = \lambda(G) + 1$. The graph G described in item d) satisfies $\lambda(G) = \lambda(\overline{G}) = 3$, since an LD-code of G is the set containing the three vertices adjacent to the three vertices of degree 1 in G and an LD-code of $\overline{G}$ is the set containing the three vertices adjacent to the three vertices of degree 3 in G. Finally, if G is one of the graphs described in item e) obtained from t copies of complete graphs and t' copies of corners, $t' \geq 1$, then the set of vertices including all but one vertex of each complete graph and the two vertices of degree 3 of each copy of the corner, is an LD-code of G. If we change exactly one of the vertices of degree

3 of a copy of the corner by the vertex of degree 2 in this copy, then we obtain an LD-code of $\overline{G}$. Thus, $\lambda(G) = \lambda(\overline{G}) = 2t' + (r_1 - 1) + \cdots + (r_t - 1)$. □

Corollary 4.15. *Every tree T other than P_2 satisfies $\lambda(\overline{T}) \leq \lambda(T)$.*

Corollary 4.16. *Every unicyclic graph G not belonging to the set $\{C_3, \overline{P}, \mathrm{paw}\}$ satisfies $\lambda(\overline{G}) \leq \lambda(G)$.*

5 Further research

This work can be continued in several directions. Next, we propose a few of them.

- We have completely solved the equality $\lambda(\overline{G}) = \lambda(G) + 1$ for the block-cactus family. In [14], a similar study has been done for the family of bipartite graphs. We suggest to approach this problem for other families of graphs, such as outerplanar graphs, chordal graphs and cographs.
- Characterizing those trees T satisfying $\lambda(\overline{T}) = \lambda(T) = \lambda_g(T)$.
- We have proved that every tree other than P_2 and P_5, every cycle other than C_3 and C_5, and every complete bipartite graph satisfies the equality $\lambda(G) = \lambda_g(G)$. We propose to find other families of graphs with this same behaviour.

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Quartic integral Cayley graphs*

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Abstract

We give exhaustive lists of connected 4-regular integral Cayley graphs and connected 4-regular integral arc-transitive graphs. An *integral* graph is a graph for which all eigenvalues are integers. A *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ for a given group Γ and *connection set* $S \subset \Gamma$ is the graph with vertex set Γ and with a connected to b if and only if $ba^{-1} \in S$. Up to isomorphism, we find that there are 32 connected quartic integral Cayley graphs, 17 of which are bipartite. Many of these can be realized in a number of different ways by using non-isomorphic choices for Γ and/or different choices for S. A graph is *arc-transitive* if its automorphism group acts transitively on the ordered pairs of adjacent vertices. Up to isomorphism, there are 27 quartic integral graphs that are arc-transitive. Of these 27 graphs, 16 are bipartite and 16 are Cayley graphs. By taking quotients of our Cayley or arc-transitive graphs we also find a number of other quartic integral graphs. Overall, we find 9 new spectra that can be realised by bipartite quartic integral graphs.

Keywords: Graph spectrum, integral graph, Cayley graph, arc-transitive, vertex-transitive bipartite double cover, voltage assignment, graph homomorphism.

Math. Subj. Class.: 05C50, 05C25

1 Introduction

We give exhaustive lists of connected 4-regular integral Cayley graphs and connected 4-regular integral arc-transitive graphs. For reasons which will become apparent, we first restrict our attention to the bipartite case.

An *integral* graph is a graph for which all eigenvalues of the adjacency matrix are integers. The *spectrum* of a graph is the eigenvalues with their multiplicity. Bipartite graphs have eigenvalues that are symmetric with respect to 0 and r-regular graphs have largest eigenvalue r with multiplicity equal to the number of connected components. For

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details we refer to [10]. Therefore for connected 4-regular bipartite integral graphs, the spectrum has the form $\{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$; which we abbreviate by simply specifying the quadruple $[x, y, z, w]$.

There are only finitely many connected 4-regular bipartite integral graphs. Cvetković [6] proved that the diameter D of a connected graph satisfies $D \leqslant s-1$, where s is the number of distinct eigenvalues. For connected r-regular integral graphs, it follows that $R \leqslant D \leqslant 2r$ where R is the radius of the graph. Cvetković *et al.* [7] showed that the number of vertices in a connected r-regular bipartite graph is bounded above by $(2(r-1)^R - 2)/(r-2)$ if $r \geqslant 3$. Therefore, connected 4-regular bipartite integral graphs have at most 6560 vertices.

All graphs in this paper are simple, undirected, and have n vertices. Since a 4-regular graph is integral if and only if each of its components is integral, from this point on we will assume that all graphs are connected. We use the acronym QIG as shorthand for a connected quartic integral graph. Cvetković *et al.* [7] found quadruples $[x, y, z, w]$ that are candidates for the spectrum of a bipartite QIG. They called these *possible spectra*. Research activities regarding the set of possible spectra fall into two streams: eliminate possible spectra based on new information and/or techniques, or find graphs that realize a possible spectrum. Useful tools include an identity by Hoffman [11] and equations relating the spectral moments to the closed walks of length $\ell \leqslant 6$. All QIGs that avoid eigenvalues of ± 3 and realize a possible spectrum are found in [24]. Stevanović [23] eliminates spectra using equations arising from graph angles. In the same paper he determines that the possible values for n are between 8 and 1260, except for 5 identified spectra.

Stevanović *et al.* [25] extend the equations for the ℓ-th spectral moment to an inequality for $\ell = 8$. They make use of a correspondence between closed walks in an r-regular graph and walks in an infinite r-regular tree and find recurrence relations for the number of closed walks. The upper bound for n is improved to give $8 \leqslant n \leqslant 560$. Equations for $\ell \geqslant 8$ are found in [16] by counting a certain type of closed walk in terms of the counts of small subgraphs of the graph. All of the bipartite QIGs with $n \leqslant 24$ that realize one of the possible spectra were found and are listed with drawings in [25]. We give 12 new graphs that realize possible spectra from the set given in [25]. Of these graphs, 3 are co-spectral to an integral graph listed in [7]. Their spectra are $[4, 6, 4, 5]$ and $[6, 16, 10, 3]$, and $[9, 16, 19, 0]$. The spectra not previously known to be realized by a graph are $[3, 4, 1, 6]$, $[3, 5, 9, 0]$, $[5, 4, 7, 4]$, $[6, 12, 2, 9]$, $[8, 10, 16, 1]$, $[10, 14, 18, 2]$, $[12, 28, 4, 15]$, $[22, 28, 34, 5]$, and $[27, 28, 49, 0]$. Of the 12 graphs, 3 appear in the census of Potočnik *et al.* [17, 18] but were not recognized as integral.

We also list 49 new non-bipartite QIGs that, to our knowledge, do not appear anywhere in the literature. Of these graphs, only 3 appear in the census of Potočnik *et al.* [17, 18] but were not tested for integrality.

A *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ for a group Γ and *connection set* $S \subset \Gamma$ is the graph with vertex set Γ and with a connected to b if and only if $ba^{-1} \in S$. Let $\mathbb{Z}_t$, D_t, and Q_t denote the cyclic, dihedral, and quaternion groups of order t respectively, and S_t, A_t the symmetric and alternating groups of degree t.

Klotz and Sander [12] showed that if every Cayley graph $\mathrm{Cay}(\Gamma, S)$ over a finite Abelian group Γ is integral then $\Gamma \in \{\mathbb{Z}_2^s, \mathbb{Z}_3^s, \mathbb{Z}_4^s, \mathbb{Z}_2^s \times \mathbb{Z}_3^t, \mathbb{Z}_2^s \times \mathbb{Z}_4^t\}$, where $s \geqslant 1$, $t \geqslant 1$. The analogous result for non-Abelian Γ was determined independently by Abdollahi and Jazaeri [1] and Ahmady *et al.* [4]: if every Cayley graph $\mathrm{Cay}(\Gamma, S)$ over a finite non-Abelian group Γ is integral then $\Gamma \in \{S_3, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, Q_8 \times \mathbb{Z}_2^r\}$, where $r \geqslant 0$.

Estélyi and Kovács [8] considered the groups for which all Cayley graphs $\mathrm{Cay}(\Gamma, S)$ over a group Γ are integral if $|S| \leqslant k$. The authors proved that for $k \geqslant 6$, Γ consists only of the groups above: $\{\mathbb{Z}_2^s, \mathbb{Z}_3^s, \mathbb{Z}_4^s, \mathbb{Z}_2^s \times \mathbb{Z}_3^t, \mathbb{Z}_2^s \times \mathbb{Z}_4^t\} \cup \{S_3, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, Q_8 \times \mathbb{Z}_2^r\}$, $s \geqslant 1$, $t \geqslant 1$, $r \geqslant 0$. Moreover, for $k \in \{4, 5\}$ there is only one extra possibility, namely that Γ is the generalised dicyclic group with $\mathbb{Z}_{3^q} \times \mathbb{Z}_6$ as a subgroup of index 2, where $q \geqslant 1$.

Abdollahi and Vatandoost [2] showed that there are exactly 7 connected cubic integral Cayley graphs. They found that $\mathrm{Cay}(\Gamma, S)$ is integral for some S with $|S| = 3$ if and only if Γ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, S_3, $\mathbb{Z}_2^3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12}, A_4, S_4, $D_8 \times \mathbb{Z}_3$, $D_6 \times \mathbb{Z}_4$ or $A_4 \times \mathbb{Z}_2$.

A set of possible orders for Cayley QIGs on finite Abelian groups have been determined by Abdollahi *et al.* [3]. They showed that for an Abelian group, Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\},$$

but they did not establish whether Cayley QIGs of these orders exist. We find that the precise set of orders of Cayley QIGs on Abelian groups is $\{5, 6, 8, 9, 10, 12, 16, 18, 24, 36\}$. More generally, we consider all groups and find that many Cayley QIGs are on non-Abelian groups. Thus, we show that for any group Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 72, 120\}.$$

Furthermore, for each of these orders Cayley QIGs exist.

For a given Cayley graph G, there may exist many different pairs (Γ, S) of groups Γ and connection sets S such that $G \cong \mathrm{Cay}(\Gamma, S)$. We call isomorphic Cayley graphs on the same group Γ *equivalent* if their connection sets are from the same orbit of the automorphism group of Γ (see for example [13]):

Definition 1.1. Let Γ be a group and $\mathrm{Aut}(\Gamma)$ be the automorphism group of Γ. If Cayley graph $\mathrm{Cay}(\Gamma, S) \cong \mathrm{Cay}(\Gamma, T)$ and $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(\Gamma)$ then $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, T)$ are *equivalent*.

Any other connection sets give non-equivalent Cayley Graphs. Cayley graphs from different groups are non-equivalent. There are, up to isomorphism, only 32 connected quartic integral Cayley graphs; but each graph is realized in up to 18 non-equivalent ways. Of the 32 graphs, 17 are bipartite.

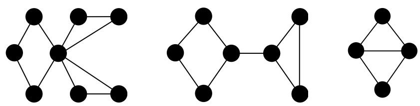
A graph is *arc-transitive* if its automorphism group acts transitively on the ordered pairs of adjacent vertices. There are, up to isomorphism, only 27 connected quartic integral graphs that are arc-transitive. Of the 27 graphs, 16 are bipartite, 5 of which are not Cayley graphs.

In Section 2 we find that most of the feasible spectra from [25] cannot be realized by vertex-transitive QIGs. Section 3 summarises the algorithm used for finding all of the bipartite Cayley QIGs. Section 4 gives our main results. It includes tables giving the details of the Cayley QIGs and the bipartite arc-transitive QIGs, some drawings, and some non-bipartite QIGs that result from finding quotients of our bipartite graphs.

2 Vertex-transitive quartic integral graphs

A graph is *vertex-transitive* if its automorphism group acts transitively on its vertices. In this section, our aim is to compile a set Ξ that includes all possible spectra that might be realized by a vertex-transitive QIG, but is otherwise as small as we can make it. Initially we take Ξ to be all possible spectra from [25], and candidates will be progressively removed from the set as we work through this section.

We will need some notation for (unlabelled) subgraphs. We let C_i denote the i-cycle, $C_{i_1 \cdot i_2 \cdots i_h}$ denote i_j-cycles sharing a single vertex for $j = 1, \dots, h$, $C_{i_1 - i_2}$ an i_1-cycle joined to an i_2-cycle by an edge, and $\Theta_{i_1, i_2, \dots, i_h}$ two vertices joined by internally disjoint paths of lengths i_j for $j = 1, \dots, h$. Examples of this notation for subgraphs appear in Figure 1.



(a) $C_{4\cdot3\cdot3}$ (b) C_{4-3} (c) $\Theta_{2,2,1}$

Figure 1: Subgraph notation

If at any point we encounter subgraphs that cannot be described by our notation, we draw a picture of the subgraph like those in Figure 1. For any graph H, let $[H]$ denote the number of subgraphs of G that are isomorphic to H, where the parent graph G will be implicitly specified by the context.

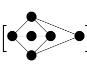
In [7], Equations (2.1) and (2.2) are used to determine $[C_4]$ and $[C_6]$ for a given $[x, y, z, w]$.

$$2(4^4 + 3^4 x + 2^4 y + z) = 28n + 8[C_4], \tag{2.1}$$

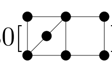
$$2(4^6 + 3^6 x + 2^6 y + z) = 232n + 144[C_4] + 12[C_6]. \tag{2.2}$$

In [16], these equations were extended to higher spectral moments of general regular graphs. By specialising to 4-regular bipartite graphs, we obtain the following equations:

$$\begin{aligned} 2(4^8 + 3^8 x + 2^8 y + z) &= 2092n + 2024[C_4] + 288[C_6] + 16[C_8] + 32[C_{4\cdot4}] \\ &\quad + 96[\Theta_{2,2,2,2}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,3,1}], \\ 2(4^{10} + 3^{10} x + 2^{10} y + z) &= 19864n + 26160[C_4] + 4860[C_6] + 480[C_8] + 20[C_{10}] \\ &\quad + 960[C_{4\cdot4}] + 40[C_{4-4}] + 40[C_{6\cdot4}] + 1440[\Theta_{2,2,2}] \\ &\quad + 520[\Theta_{3,3,1}] + 2880[\Theta_{2,2,2,2}] + 40[\Theta_{4,2,2}] + 20[\Theta_{5,3,1}] \\ &\quad + 120[\Theta_{3,3,3,1}] + 120[\Theta_{4,2,2,2}] + 120[\end{aligned}$$



$$] + 80[$$



$$]. \tag{2.3}$$

The *girth* of a graph is the length of the shortest cycle contained in the graph. We use Equations (2.3) to determine the girth where $[C_4] = [C_6] = 0$ for a given $[x, y, z, w]$ and also to determine the values for $[C_8]$ and $[C_{10}]$ where possible. Vertex-transitive graphs have the same number of i-cycles incident with each vertex, so the number of vertices divides $i[C_i]$. We apply this observation for $i \in \{4, 6, 8, 10\}$ to the possible spectra for which the value of $[C_i]$ can be deduced. We eliminate those quadruples that cannot be realized by a vertex-transitive QIG from Ξ.

For example, if we consider $[5, 6, 11, 1]$ with $n = 48$, $[C_4] = 24$, and $[C_6] = 140$ then

$$\frac{4[C_4]}{n} = \frac{4(24)}{48} = 2 \in \mathbb{N} \text{ but } \frac{6[C_6]}{n} = \frac{6(140)}{48} = \frac{35}{2} \notin \mathbb{N},$$

where $\mathbb{N}$ denotes the set of non-negative integers. Thus $[5, 6, 11, 1]$ is eliminated from Ξ. In contrast, for $[12, 12, 20, 3]$ with $n = 96$, $[C_4] = 24$, $[C_6] = 128$, and $[C_8] = 528$. We are able to find $[C_8]$ from (2.3) by deducing that $[C_{4\cdot 4}] = [\Theta_{2,2,2,2}] = [\Theta_{2,2,2}] = [\Theta_{3,3,1}] = 0$ because there is only one 4-cycle incident with each vertex. In fact,

$$\frac{4[C_4]}{n} = \frac{4(24)}{96} = 1 \in \mathbb{N}, \quad \frac{6[C_6]}{n} = \frac{6(128)}{96} = 8 \in \mathbb{N}, \text{ and } \frac{8[C_8]}{n} = \frac{8(528)}{96} = 44 \in \mathbb{N}.$$

In this case, $[C_{10}]$ cannot be determined from (2.3), so we consider it unknown. Thus $[12, 12, 20, 3]$ remains in Ξ.

It is also plausible to eliminate quadruples from Ξ using arguments specific to particular cases. We give one example to demonstrate the possibility. Consider $[24, 4, 40, 3]$ with $[C_4] = 72$ and $[C_6] = 0$. There are $4(72)/144 = 2$ copies of C_4 incident at each vertex. Since $[C_6] = 0$, we know $[\Theta_{3,3,1}] = 0$. Also, with only two 4-cycles at each vertex, $[\Theta_{2,2,2,2}] = [\Theta_{2,2,2}] = 0$. Since two 4-cycles meet at exactly one vertex of a $C_{4\cdot 4}$, $[C_{4\cdot 4}] = 144$. From Equation (2.3) we get that,

$$2(4^8 + 3^8(24) + 2^8(4) + 40) = 2092(144) + 2024(72) + 16[C_8] + 32(144),$$

which gives the contradiction $[C_8] = -216$. Thus we remove $[24, 4, 40, 3]$ from Ξ. This entry is underlined in Table 1.

We eliminate two quadruples from Ξ using the following Lemma [5, Prop. 16.6]:

Lemma 2.1. *Let G be a vertex-transitive graph which has degree r and an even number of vertices. If λ is a simple eigenvalue of G, then λ is one of the integers $2\alpha - r$ for $0 \leqslant \alpha \leqslant r$.*

The orders associated with the eliminated quadruples are 36 and 72. Both entries have 1 as a simple eigenvalue. These entries are underlined and highlighted in bold in Table 1.

Using the above methods, we reduced the set Ξ from the initial 828 possible spectra to 59 quadruples in the final version. Henceforth Ξ will refer to this final set of 59 quadruples (see Appendix A).

n	Girth	n	Girth	n	Girth	n	Girth
8	4	36	**4**,4,4,q^7	96	4,q^{27},h^3	240	6,8,q^{30},h^2
10	4	40	4,q^{10}	112	q^{34},h^2	252	q^{28},h^3
12	4,4	42	4,q^{14}	120	4,4,4,6,6,q^{28}	280	8,q^{23},h^2
14	q^1	48	4,q^{12},h^2	126	4,6,q^{38},h^1	288	6,q^{21},h^1
16	4,q^1	56	q^{16},h^2	140	q^{40},h^2	336	q^{14},h^2
18	4,q^1	60	4,4,4,4,6,q^{15}	144	4,4,6,q^{31},h^1	360	6,6,8,q^{11}
20	4,q^3	70	6,q^{23}	160	q^{33},h^2	420	8,q^5,h^1
24	4,4,4,q^3	72	**4**,4,4,4,6,q^{18}	168	6,q^{35},h^2	480	8,q^2
28	q^8	80	q^{22},h^2	180	4,6,6,6,q^{38}	504	h^1
30	4,4,6,q^6	84	q^{23},h^7	210	6,q^{35},h^2	560	10
32	6,q^8	90	4,4,6,6,q^{27}	224	q^{32},h^3		

Table 1: Finding the set Ξ

Table 1 summarizes the process of finding Ξ. For every order, we consider each $[x, y, z, w]$ and check whether we get integer counts at each vertex for each C_i where $[C_i]$ is known and $i \in \{4, 6, 8, 10\}$. A 'q^j' in the table denotes that for the given n there were j possible spectra eliminated because $4[C_4]/n \notin \mathbb{N}$. An '$h^j$' in the table denotes that for the given n there were j possible spectra which satisfied $4[C_4]/n \in \mathbb{N}$ that were eliminated because $6[C_6]/n \notin \mathbb{N}$. If $i[C_i]/n \in \mathbb{N}$ for all i where $[C_i]$ is known for a possible spectra, then the girth is recorded. Thus an entry of 4,4,6,q^6 indicates that there are three possible spectra in Ξ associated with that order. If those quadruples are all realized by graphs (where a graph in this case may actually be a set of cospectral graphs) then two graphs will have girth 4 and the other will have girth 6. It also indicates that 6 possible spectra with $4[C_4]/n \in \mathbb{N}$ were eliminated because $6[C_6]/n \notin \mathbb{N}$.

3 The algorithm

In this section we outline our method for finding bipartite Cayley QIGs, using the set Ξ compiled in Section 2.

Define Ω to be the set of orders associated with the spectra in Ξ. Cayley graphs are vertex-transitive, so we only consider groups Γ of order $n \in \Omega$. To reduce the number of groups to be considered, we use a result similar to one in [18]. Let Γ' denote the commutator subgroup of a group Γ.

Lemma 3.1. *Let Γ be a finite group and let $\mathrm{Cay}(\Gamma, S)$ be a connected Cayley graph of degree at most 4. Then Γ/Γ' is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$ with $a \geqslant 2$; $\mathbb{Z}_a \times \mathbb{Z}_b$ with $a, b \geqslant 2$; or $\mathbb{Z}_a$ with $a \geqslant 1$.*

Proof. Since $\mathrm{Cay}(\Gamma, S)$ is connected and has degree at most 4, Γ is generated by an inverse-closed set of at most 4 elements. This must also be true of the quotient group Γ/Γ'. Now since Γ/Γ' is Abelian, the result follows. □

By Lemma 3.1, we need only consider groups Γ with Γ/Γ' isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$, $\mathbb{Z}_a \times \mathbb{Z}_b$, or $\mathbb{Z}_a$. We denote the set of groups that satisfy this property by Φ.

To construct connected simple undirected 4-regular Cayley graphs $\mathrm{Cay}(\Gamma, S)$, we considered inverse-closed sets S of four non-identity elements of Γ that generate Γ. The search was pruned by placing additional restrictions on S. Let g denote the girth of the graph $\mathrm{Cay}(\Gamma, S)$.

- Since $\mathrm{Cay}(\Gamma, S)$ is bipartite, the order of s is even for each $s \in S$.
- If $s_1, s_2 \in S$ and $s_1 \neq s_2^{-1}$, then the order of $s_1 s_2$ is at least $g/2$ (in particular non-involutions have order no smaller than the girth).
- For any set of connection sets that result in equivalent Cayley graphs (in the sense of Definition 1.1), only one representative is chosen.

We note that the minimum girth possible for $\mathrm{Cay}(\Gamma, S)$ is given by Table 1.

We summarize the results of our computations in Table 2. The values for $n \in \Omega$ appear as the first column and in the second column the number of groups of order n is given. (We reiterate that Ω does not include orders eliminated by the vertex-transitive tests of Section 2). The number of groups in Φ of order n are listed in column three. Column 4 contains the number of connection sets S among the groups counted by column 3, subject to the restrictions on S given above. The graphs $\mathrm{Cay}(\Gamma, S)$ that are bipartite are counted in column 5. The number of isomorphism classes of these graphs appears in column 6. The number of isomorphism classes of integral graphs is recorded in column 7. The last column gives the number of isomorphism classes of arc-transitive integral graphs. A '-' indicates that there are no integral graphs to consider.

n	#Groups Γ	#$\Gamma \in \Phi$	#Sets S	#Bipartite $\mathrm{Cay}(\Gamma, S)$	#Isomorphism Classes	#Integral	#Arc-Transitive
8	5	5	13	7	1	1	1
10	2	2	2	2	1	1	1
12	5	5	19	11	3	2	1
16	14	14	66	44	5	1	1
18	5	5	12	12	5	1	1
20	5	5	34	20	8	0	-
24	15	15	151	98	23	3	1
30	4	4	31	31	17	1	1
32	51	48	58	51	16	1	1
36	14	14	149	105	48	1	1
40	14	14	201	146	54	1	0
42	6	6	55	55	36	0	-
48	52	51	840	616	177	1	0
60	13	13	385	281	161	0	-
70	4	4	96	96	73	0	-
72	50	49	1014	765	338	2	1

90	10	10	236	236	175	0	-
96	231	218	4434	3545	1292	0	-
120	47	47	2833	1968	1123	1	1
126	16	16	427	427	346	0	-
144	197	190	6563	5350	2722	0	-
168	57	57	2388	2212	1601	0	-
180	37	37	2927	2497	1883	0	-
210	12	12	1172	1172	1017	0	-
240	208	205	10884	9885	6791	0	-
280	40	40	4080	3929	3223	0	-
288	1045	968	26391	24815	15695	0	-
360	162	160	15928	14703	11524	0	-
420	41	41	10558	10204	9271	0	-
480	1213	1148	68179	63804	48322	0	-
560	180	177	21764	21433	18704	0	-

Table 2: Results at each algorithm step

4 Quartic integral graphs

In this section we present the graphs that our computations discovered, starting with the bipartite Cayley case.

4.1 Bipartite Cayley integral graphs

As a result of the computation described in Section 3, we have:

Theorem 4.1. *There are precisely* 17 *isomorphism classes of connected* 4*-regular bipartite integral Cayley graphs, as detailed in Table* 3*.*

For each bipartite Cayley QIG in Table 3 we give n and the spectrum $[x, y, z, w]$. Graphs appearing in the paper by Cvetković *et al.* [7] are labelled $I_{n,index}$ as in that paper. If the graph is in the census of Potočnik *et al.* [17, 18] then we give the index in their notation: AT4Val[n][index]. In two columns, we give the groups and connection sets that give rise to each Cayley graph. The first column contains the group, Γ, with a presentation of that group. We stick as close as possible to the convention of using generators in $\{a, b, c, d, e\}$ for cyclic groups, $\{s, t, u, v\}$ for symmetric or alternating groups, and $\{r, f\}$ for the quaternion group, the dihedral group, or the quasidihedral group. The last column contains the number of involutions in the connection set, S, followed by the connection set itself in terms of the generators from the previous column.

Group	Connection Sets (#involutions S)
$\mathbf{G_1:\ n=8\quad [0,0,0,3]\quad I_{8,1}\quad AT4Val[8][1]}$	
$\mathbb{Z}_8$ $< a \mid a^8 >$	$0\ \{a, a^3, a^5, a^7\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 >$	$2\ \{a, b, a^3, a^2b\}$ $0\ \{a, ab, a^3, a^3b\}$
D_8 $< r, f \mid r^4, f^2, (rf)^2 >$	$4\ \{f, fr, fr^2, rf\}$ $2\ \{f, r, fr^2, frf\}$
Q_8 $< r, f \mid r^4, f^4, r^2f^2, rfrf^{-1} >$	$0\ \{r, f, r^3, r^2f\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 >$	$4\ \{a, b, c, abc\}$
$\mathbf{G_2:\ n=10\quad [0,0,4,0]\quad I_{10,1}\quad AT4Val[10][2]}$	
D_{10} $< r, f \mid r^4, f^2, (rf)^2 >$	$4\ \{f, fr, fr^2, r^2f\}$
$\mathbb{Z}_{10}$ $< a \mid a^{10} >$	$0\ \{a, a^3, a^7, a^9\}$
$\mathbf{G_3:\ n=12\quad [0,2,0,3]\quad I_{12,4}\quad AT4Val[12][2]}$	
$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1} >$	$0\ \{b, b^3, ba, b^3a\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0\ \{a, a^5, a^7, a^{11}\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{r^2f, f, r^5, r\}$ $4\ \{r^4f, rf, r^2f, r^5f\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$0\ \{a^5, a^2b, a, a^4b\}$
$\mathbf{G_4:\ n=12\quad [0,1,4,0]\quad I_{12,2}}$	
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{rf, r^3, r, r^5\}$ $4\ \{rf, r^3, r^5f, r^3f\}$ $4\ \{rf, r^4f, r^5f, r^3f\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2\ \{a^3, b, a^5, a\}$

$\mathbf{G_5}$: $\mathbf{n = 16}$ $\mathbf{[0,4,0,3]}$ $\mathbf{I_{16,1}}$ AT4Val[16][1]

$\mathbb{Z}_4 \times \mathbb{Z}_4$ $< a \mid a^4 > \times < b \mid b^4 >$	$0\ \{a, b, a^3, b^3\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, (bc)^2, baca^{-1}c >$	$2\ \{ac, a^2bc, a^3bc, a^2c\}$ $2\ \{bc, a^3b, a^2c, ab\}$ $0\ \{a, a^3c, a^3, abc\}$
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ $< a, b \mid a^4, b^4, aba^{-1}b >$	$0\ \{a, a^3ba, a^3, b\}$
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ $< a, b \mid a^8, b^2, aba^3b >$	$0\ \{a, ab, a^3b, a^7\}$
QD_{16} $< r, f \mid r^8, f^2, rfr^5f >$	$2\ \{r, r^4f, r^6f, r^7\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 > \times < c \mid c^2 >$	$2\ \{a, b, c, a^3\}$
$\mathbb{Z}_2 \times D_8$ $< a \mid a^2 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$4\ \{a, f, r^3f, r^2f\}$ $4\ \{f, r^3f, af, rf\}$ $4\ \{f, r^3f, af, arf\}$ $2\ \{a, r, f, r^3\}$ $2\ \{a, r, af, r^3\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 > \times < d \mid d^2 >$	$4\ \{a, b, c, d\}$

$\mathbf{G_6}$: $\mathbf{n = 18}$ $\mathbf{[0,4,4,0]}$ $\mathbf{I_{18,1}}$ AT4Val[18][2]

$\mathbb{Z}_3 \times S_3$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{s, st, ats, a^2ts\}$ $0\ \{sa, sa^2, sat, sa^2t\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^3, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >$	$4\ \{c, ca, cb, cab\}$
$\mathbb{Z}_6 \times \mathbb{Z}_3$ $< a \mid a^6 > \times < b \mid b^3 >$	$0\ \{a, a^5, a^3b, a^3b^2\}$

$\mathbf{G_7}$: $\mathbf{n = 24}$ $\mathbf{[0,8,0,3]}$ $\mathbf{I_{24,2}}$ AT4Val[24][1]

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{s, st, at, a^3sts\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, a^3(cb)^2 >$	$2\ \{a^3c, a^2c, ab, a^5b\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$0\ \{ar^3f, a^2r^3f, ar^3, a^2r\}$
S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$4\ \{st^2sts, t^2st, stst^2s, tst^2\}$ $0\ \{ts, st^2, ststs, tst\}$ $2\ \{st^2sts, tst^2, ts, st^2\}$

$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s,t \mid s^2, t^3, (st)^3 >$	$0\ \{ast, astst, asts, atst\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	$4\ \{s, bs, st, ast\}$

$\mathbf{G_8}$: $\mathbf{n = 24}$ $\mathbf{[2,2,6,1]}$ $\mathbf{I_{24,3}}$

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a, a^3, s, st\}$ $2\ \{s, st, ats, a^3ts\}$
D_{24} $< r,f \mid r^{12}, f^2, (rf)^2 >$	$4\ \{f, rf, r^5f, r^6f\}$ $2\ \{f, r^3, r^9, r^8f\}$
$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b,c \mid b^3, c^4, bcbc^{-1} >$	$0\ \{c, c^3, ab, ac^3bc\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a,b,c \mid a^3, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^4 >$	$4\ \{c, b, ca, cbc\}$ $2\ \{c, bcb, ba, bcac\}$ $2\ \{c, ca, cbcac, bac\}$ $0\ \{cb, bc, ba, bcac\}$
$\mathbb{Z}_{12} \times \mathbb{Z}_2$ $< a \mid a^{12} > \times < b \mid b^2 >$	$0\ \{a^3, a^9, a^4b, a^8b\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r,f \mid r^4, f^2, (rf)^2 >$	$2\ \{r^3f, rf, a^2f, af\}$ $0\ \{r, r^3, ar^3f, a^2r^3f\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	$4\ \{s, b, a, st\}$ $4\ \{s, b, st, ats\}$ $4\ \{s, sb, ast, ats\}$ $2\ \{s, b, at, asts\}$ $2\ \{s, sb, at, asts\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 > \times < c \mid c^2 >$	$2\ \{a^3, b, a^2c, a^4c\}$

$\mathbf{G_9}$: $\mathbf{n = 24}$ $\mathbf{[3,0,5,3]}$ $\mathbf{I_{24,4}}$

S_4 $< s,t \mid s^2, t^3, (st)^4 >$	$4\ \{s, t^2st, st^2sts, stst^2st\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s,t \mid s^2, t^3, (st)^3 >$	$2\ \{a, as, at^2s, ast\}$

$\mathbf{G_{10}}$: $\mathbf{n = 30}$ $\mathbf{[0,10,4,0]}$ $\mathbf{I_{30,1}}$ AT4Val[30][4]

$\mathbb{Z}_5 \times S_3$ $< a \mid a^5 > \times < s,t \mid s^2, t^3, (st)^2 >$	$0\ \{as, a^2st, a^4s, a^3st\}$
D_{30} $< r,f \mid r^{15}, f^2, (rf)^2 >$	$4\ \{f, r^2f, r^3f, r^{11}f\}$

G_{11} : $n = 32$ $[0, 12, 0, 3]$ $I_{32,1}$ AT4Val[32][4]

$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$ $< a, b \mid a^8, b^4, ab^2a^{-1}b^2, aba^3b^{-1} >$	$0\ \{a, a^7, ab, a^3b^3\}$
$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, a^2ba^6b, (aac)^2, (bc)^2, ba^{-1}cac >$	$2\ \{a^4c, a^2c, a^7bc, a^5c\}$
$\mathbb{Z}_2.((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) = (\mathbb{Z}_2 \times \mathbb{Z}_2).(\mathbb{Z}_4 \times \mathbb{Z}_2)$ $< a, b \mid a^8, b^4, ab^2a^{-1}b^2, a^4b^2, aba^{-1}b^{-1}ab^{-1}a^{-1}b, aba^6ba >$	$0\ \{ba, a^3b, a^3, a^5\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^4, c^2, aba^{-1}b^{-1}, aca^3c, (bbc)^2(bc)^4, a^3(bc)^2 >$	$2\ \{ab^2c, c, bc, a^3bc\}$
$\mathbb{Z}_4.D_8 = \mathbb{Z}_4.(\mathbb{Z}_4 \times \mathbb{Z}_2)$ $< a, b \mid a^8, b^8, aba^3b, ab^{-1}a^3b^{-1}, ab^{-1}a^{-1}b^3 >$	$0\ \{a, a^7, a^7ba, a^4b\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^4, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >$	$4\ \{c, cb, ca, abc\}$
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, aba^{-1}b, aca^{-1}c, a^4bcbc, (bc)^4 >$	$2\ \{b, c, abc, a^3bc\}$
$\mathbb{Z}_2 \times QD_{16}$ $< a \mid a^2 > \times < r, f \mid r^8, f^2, rfr^5f >$	$2\ \{ar, r^3, r^5, r^2f\}$
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, a^4(bc)^2 >$	$4\ \{a^7c, a^2c, ac, a^4b\}$
$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^2, b^2, aba^{-1}b^{-1}, r(fa)^2, r^2(bf)^2 >$	$4\ \{r^2f, ar^2, rf, rab\}$
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^4, b^2, ara^{-1}r^{-1}, afa^{-1}f^{-1}, r^2f^2, rfrf^{-1},$ $(rrb)^2, r^2(ab)^2, brbr^{-1}f >$	$2\ \{r^2b, a, ar^3fb, ar^3b\}$
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^4, b^2, ara^{-1}r^{-1}, afa^{-1}f^{-1}, r^2f^2, rfrf^{-1},$ $(rb)^2, fbf^{-1}b^{-1}, arbar^{-1}b >$	$4\ \{b, a, br, brf\}$

G_{12} : $n = 36$ $[4, 4, 4, 5]$ $I_{36,3}$ AT4Val[36][3]

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^3 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0\ \{ac, a^2c^3, a^2cb, ac^3b\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, (acc)^2, acac^{-1}b^{-1} >$	$0\ \{a^2b^2c^3, b^2c, a^2bc^3, ac\}$
$S_3 \times S_3$ $< s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^3, (uv)^2 >$	$4\ \{u, s, uv, st\}$ $4\ \{u, uv, svu, stvu\}$ $2\ \{su, stu, tsv, tsuvu\}$ $0\ \{tu, stsu, sv, suvu\}$

$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	2 $\{a, a^5, a^3s, a^3st\}$ 2 $\{a^3s, a^3st, a^2ts, a^4ts\}$ 0 $\{as, a^3t, a^5s, a^3sts\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$ $< a \mid a^2 > \times < b,c,d \mid b^3, c^3, d^2, bcb^{-1}c^{-1}, (bd)^2, (cd)^2 >$	4 $\{d, dc, adb, adcdbd\}$ 2 $\{d, dc, ab, adbd\}$
$\mathbb{Z}_6 \times \mathbb{Z}_6$ $< a \mid a^6 > \times < b \mid b^6 >$	0 $\{a, a^5, b, b^5\}$

$\mathbf{G_{13}}$: $\mathbf{n = 40}$ $[\mathbf{4, 6, 4, 5}]$

$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b,c \mid b^5, c^4, cbc^{-1}b^2, cb^2c^{-1}b^{-1} >$	2 $\{ac^2, ac^2b, c, c^3\}$

$\mathbf{G_{14}}$: $\mathbf{n = 48}$ $[\mathbf{6, 4, 10, 3}]$ $\mathbf{I_{48,1}}$

$\mathbb{Z}_2 \times \mathbb{Z}_4 \times S_3$ $< a \mid a^2 > \times < b \mid b^4 > \times < s,t \mid s^2, t^3, (st)^2 >$	2 $\{s, a, bt, b^3sts\}$
$D_8 \times S_3$ $< r,f \mid r^4, f^2, (rf)^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	4 $\{s, rfs, fts, r^2fst\}$ 4 $\{rf, rfs, fts, r^2fst\}$ 2 $\{s, rfs, r^3t, rsts\}$ 2 $\{rf, rfs, r^3t, rsts\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $< a \mid a^2 > \times < b,c,d \mid b^6, c^2, d^2, bcb^{-1}c^{-1}, (bbd)^2, b^3(dc)^2 >$	4 $\{a, c, b^4d, b^3d\}$
$\mathbb{Z}_6 \times D_8$ $< a \mid a^6 > \times < r,f \mid r^4, f^2, (rf)^2 >$	2 $\{a^3, a^3r^3f, ar, a^5r^3\}$
$\mathbb{Z}_2 \times S_4$ $< a \mid a^2 > \times < s,t \mid s^2, t^3, (st)^4 >$	4 $\{a, s, stst^2s, st^2sts\}$ 4 $\{as, at^2st, atst^2, stst^2st\}$ 2 $\{s, astst^2, atst^2s, atst^2st\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < b \mid b^2 > \times < s,t \mid s^2, t^3, (st)^3 >$	2 $\{a, abtst^2, abtst, abt^2st^2\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	4 $\{s, b, cst, ats\}$

$\mathbf{G_{15}}$: $\mathbf{n = 72}$ $[\mathbf{6, 16, 10, 3}]$ $\mathbf{AT4Val[72][12]}$

$\mathbb{Z}_3 \times S_4$ $< a \mid a^3 > \times < s,t \mid s^2, t^3, (st)^4 >$	0 $\{ast, a^2tst, a^2t^2s, aststs\}$
$(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ $< a,s,t,b \mid a^3, s^2, t^3, b^2, asa^{-1}s^{-1}, ata^{-1}t^{-1}, stbsbt^{-1}, (ab)^2, (tb)^2, (st)^3 >$	4 $\{atb, ab, tsbt, tbs\}$
$A_4 \times S_3$ $< s,t \mid s^2, t^3, (st)^3 > \times < u,v \mid u^2, v^3, (uv)^2 >$	0 $\{tu, t^2u, tsuv, st^2uv\}$

$\mathbb{Z}_6 \times A_4$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^3 >$	$0\ \{ast, a^3t, a^3t^2, a^5stst\}$

$\mathbf{G_{16}}$: $\mathbf{n = 72}$ $[\mathbf{8, 10, 16, 1}]$

$(\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)) \rtimes \mathbb{Z}_2$ $< a,b,c,d \mid a^3, b^3, c^4, d^2, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bdb^{-1}d^{-1},$ $adad^{-1}, bcbc^{-1}, c^2d^2 >$	$2\ \{dc, dacb, ad, d^2ad\}$
$(\mathbb{Z}_6 \times S_3) \rtimes \mathbb{Z}_2$ $< a,b,c,d \mid a^2, b^4, c^3, d^3, (ab^{-1})^2, acac^{-1}, (ad)^2, cbcb^{-1},$ $bdb^{-1}d^{-1}, cdc^{-1}d^{-1} >$	$4\ \{a, ab^2, abd, abacda\}$ $2\ \{ab, abcd, cb, b^2cb\}$
$\mathbb{Z}_6 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^6 > \times < b,c \mid b^3, c^4, bcbc^{-1} >$	$0\ \{ab^2, a^5b, a^3b^2c, a^3b^2c^3\}$
$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $< a \mid a^3 > \times < b,c,d \mid$ $b^6, c^2, d^2, bcb^{-1}c^{-1}, (bbd)^2, b^3(dc)^2 >$	$2\ \{b^5d, b^2d, a^2b^4c, ab^2c\}$ $0\ \{b^2cd, b^5cd, a^2b^4c, ab^2c\}$
$(S_3 \times S_3) \rtimes \mathbb{Z}_2$ $< s,t,u,v,a \mid s^2, t^3, u^2, v^3, a^2, tvt^{-1}v^{-1}, (uv)^2, (av)^2,$ $svst^{-1}, asasu >$	$4\ \{a, sastsat, s, sast\}$ $2\ \{sas, stsa, atsa, asat^2\}$ $2\ \{asa, atsat^2, sastst, asastsat\}$ $0\ \{sa, as, atsat, asastst^2\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b,c,d \mid$ $b^3, c^3, d^4, bcb^{-1}c^{-1}, (bdd)^2, bdbd^{-1}c^{-1} >$	$2\ \{ad^2, ab^2c^2d^2, ab^2d^3, ab^2cd\}$ $0\ \{ab^2cd, ab^2d^3, abc^2, ab^2c\}$
$\mathbb{Z}_2 \times S_3 \times S_3$ $< a \mid a^2 > \times < s,t \mid s^2, t^3, (st)^2 > \times < u,v \mid$ $u^2, v^3, (uv)^2 >$	$4\ \{u, s, auvst, atsvu\}$ $4\ \{u, au, suv, stvu\}$ $2\ \{u, s, atv, astsuvu\}$
$\mathbb{Z}_2 \times \mathbb{Z}_6 \times S_3$ $< a \mid a^2 > \times < b \mid b^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{ts, ab^3ts, bsts, b^5t\}$

$\mathbf{G_{17}}$: $\mathbf{n = 120}$ $[\mathbf{12, 28, 4, 15}]$ $\mathbf{AT4Val[120][4]}$

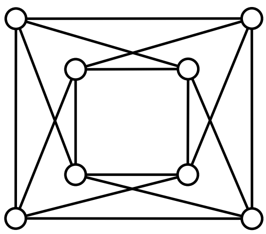
S_5 $< s,t \mid s^2, t^5, (st)^4, (st^2st^3)^2 >$	$0\ \{t^2st^3, tst^2st^2st, st^2stst, tst^4\}$ $4\ \{t(st)^2tst^4, st^2(st)^2t, (t^2s)^2ts,$ $(st^2)^2st\}$
$\mathbb{Z}_2 \times A_5$ $< a \mid a^2 > \times < s,t \mid s^2, t^3, (st)^5 >$	$2\ \{a(tst^2s)^2t, ast(ts)^2, ast^2(st)^2,$ $a(st)^3ts\}$
$S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ $< s,t \mid s^2, t^3, (st)^2 > \times < a,b \mid$ $a^5, b^4, ab^{-1}a^2b, a^2b^{-1}a^{-1}b >$	$2\ \{sb^2, stb^2a, tb^3, stsb\}$
$\mathbb{Z}_5 \times S_4$ $< a \mid a^5 > \times < s,t \mid s^2, t^3, (st)^4 >$	$0\ \{a^2st, a^3t^2s, aststs, a^4tst\}$

$(\mathbb{Z}_5 \times A_4) \rtimes \mathbb{Z}_2$ $< a, s, t, b \mid a^5, s^2, t^3, b^2, asa^{-1}s^{-1}, ata^{-1}t^{-1}, bsbt^{-1}st,$ $(st)^3, (tb)^2, (ab)^2 >$	$4\ \{tba^2, bta, btbsb, tabs\}$

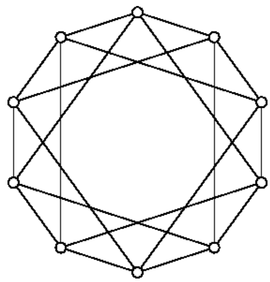
Table 3: Bipartite Cayley QIGs

Drawings for all but the three largest bipartite Cayley QIGs appear below. With over 70 vertices, it is difficult to present G_{15}, G_{16}, and G_{17} clearly.

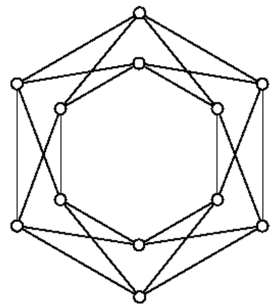
$\mathbf{G_1}$



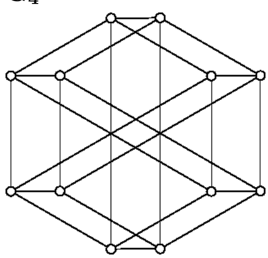
$\mathbf{G_2}$



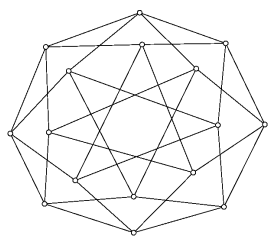
$\mathbf{G_3}$



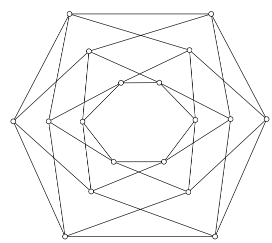
$\mathbf{G_4}$



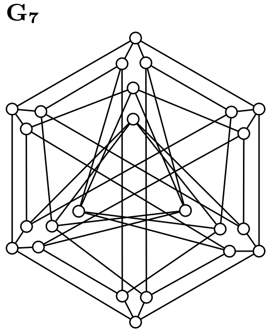
$\mathbf{G_5}$



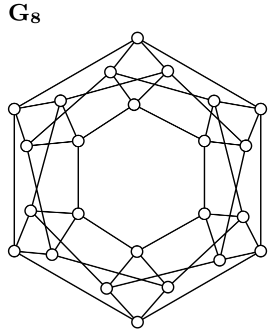
$\mathbf{G_6}$



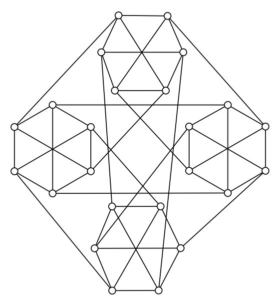
$\mathbf{G_7}$



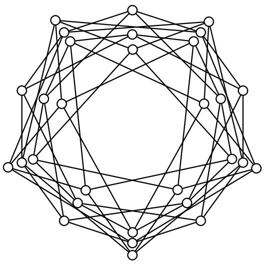
$\mathbf{G_8}$



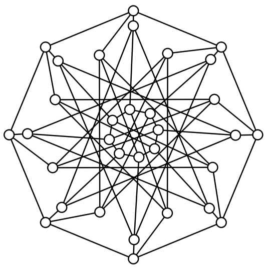
$\mathbf{G_9}$



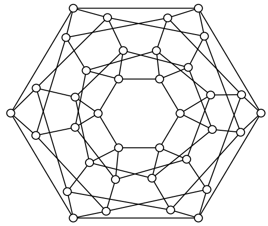
$\mathbf{G_{10}}$



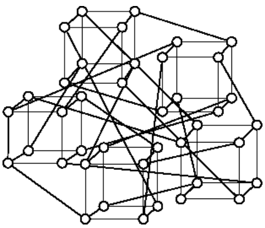
$\mathbf{G_{11}}$



$\mathbf{G_{12}}$



$\mathbf{G_{13}}$



$\mathbf{G_{14}}$

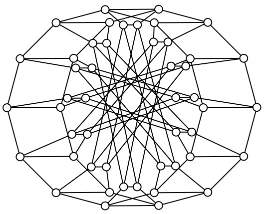


Table 4: Drawings of quartic bipartite integral Cayley graphs G_1 to G_{14}

4.2 Bipartite arc-transitive integral graphs

We considered all arc-transitive 4-regular graphs from the census of Potočnik *et al.* [17, 18] and tested them for integrality. The only arc-transitive bipartite QIGs that are not Cayley and thus not accounted for in Table 3 are the five that appear in Table 5. We let $[\Gamma : H] = \{Ha : a \in \Gamma\}$ denote the set of right cosets of $H \in \Gamma$. A *Schreier coset graph* $\mathrm{Sch}(\Gamma, H, HSH)$ for a group Γ, subgroup $H \leqslant \Gamma$, and connection set $S \subset \Gamma$ is the graph with vertex set $[\Gamma : H]$ and with Ha connected to Hb if and only if $ba^{-1} \in HSH$. We represent these 5 graphs as Schreier coset graphs. We give the order n and the spectrum $[x, y, z, w]$ followed by the graph index from [17, 18]. Graphs appearing in the paper by Cvetković *et al.* are labelled with the notation of [7]: $I_{n,index}$. The first line consists of the group Γ, with a presentation of that group. The second line consists of the subgroup H and its generators in terms of the generators of Γ followed by the connection set S in terms of the generators of Γ.

Group Subgroup, Subset
$\mathbf{F_1} : \mathbf{n = 60} \quad [\mathbf{4, 16, 4, 5}] \quad \mathbf{I_{60,1}} \quad \mathbf{AT4Val[60][4]}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_5 : \quad < a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^5, (st)^4, (st^2st^3)^2 >$ $D_8 : \quad < bstst^2st^{-1}, abstst >, \quad \{s, bt^2, s^t, bt^{-2}\}$

$\mathbf{F_2}:\ \mathbf{n = 70}\quad [\mathbf{6, 14, 14, 0}]\quad \mathbf{I_{70,1}}\quad \mathbf{AT4Val[70][4]}$

$\mathbb{Z}_2 \times S_7: \quad < a \mid a^2 > \times < s, t \mid s^2, t^7, (st)^6, (st^2st^5)^2, (stst^{-1})^3 >$
$S_3 \times S_4: \quad < t^2st^{-2}, t^{-2}st^{-1}(st)^2t, t^2(st)^2(ts)^3t^{-1}, t(st)^2(ts)^3, stst^{-1}s >,$
$\{ast^4, atstst^{-1}, at, at^{-1}\}$

$\mathbf{F_3}:\ \mathbf{n = 90}\quad [\mathbf{9, 16, 19, 0}]\quad \mathbf{I_{90,1}}\quad \mathbf{AT4Val[90][1]}$

$\mathbb{Z}_2 \times \mathrm{P\Gamma L}(2,9): \quad < a \mid a^2 > \times < x, y, z \mid$
$x^8, y^3, z^2, xzx^5z, yzy^{-1}z^{-1}, xyxy^{-1}x^6yx^6y^{-1},$
$(xyx^2y)^2, xyx^{-2}y^{-1}x^4yx^{-1}y^{-1} >$
$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2: \quad < yzxy^{-1}x, x^{-1}yxzx^{-1}y, x^2zy^{-1}x^{-1}y^{-1}xy >,$
$\{ayx^{-1}y^{-1}x, az, ayxy^{-1}x,$
$axy^{-1}x^{-1}y\}$

$\mathbf{F_4}:\ \mathbf{n = 180}\quad [\mathbf{22, 28, 34, 5}]\quad \mathbf{AT4Val[180][12]}$

$\mathbb{Z}_2 \times S_3 \times S_5: \quad < a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^5, (uv)^4, (uv^2uv^3)^2 >$
$D_8: \quad < v^{-2}uv^2, vuv^2uv^2, astuvuv^2uv^{-1} >, \quad \{at^{-1}v^{-2}, atuv^2, su, atv^2\}$

$\mathbf{F_5}:\ \mathbf{n = 210}\quad [\mathbf{27, 28, 49, 0}]\quad \mathbf{AT4Val[210][10]}$

$S_7: \quad < s, t \mid s^2, t^7, (st)^6, (st^2st^5)^2, (stst^{-1})^3 >$
$S_4: \quad < tst^3st^3, tst^{-2}st, (st)^2t^2st^{-1}(st)^2tst >, \quad \{t^3s, st^4, (st)^3, (ts)^3\}$

Table 5: Bipartite arc-transitive non-Cayley QIGs

The census [17, 18] of arc-transitive graphs contains all arc-transitive graphs with at most 640 vertices. Thus, the upper bound of 560 given in [25] for the order of a bipartite QIG, ensures that Table 3 and Table 5 contain all bipartite arc-transitive QIGs. The non-bipartite arc-transitive QIGs will be given in Sections 4.4 and 4.5. However, first we describe our method for finding all Cayley QIGs.

4.3 Integral graphs as quotients

Let $V(G)$ denote the vertices of a graph G, and $E(G)$ the unordered pairs of vertices which are edges of G. A homomorphism from a graph G to a graph H is a map $V(G) \to V(H)$ which preserves adjacency. Each homomorphism induces an edge map $E(G) \to E(H)$. If the vertex and edge maps of the homomorphism are both surjective then we say that H is a *quotient* of G. In this section we find new integral graphs that are quotients of the integral graphs found in Table 3 and Table 5. To specify a quotient of a graph G it suffices to know G and the vertex map (the edges of the quotient are implied by the surjectivity of the edge map).

We start by considering special classes of possible homomorphisms. A *voltage assignment* α for a graph G is a function from the arcs of G to a group Γ such that $\alpha((u,v)) = \alpha((v,u))^{-1}$ for all $\{u,v\} \in E(G)$. The *derived graph* $\mathrm{Vol}(G, \Gamma, \alpha)$ is the graph with vertex set the Cartesian product $V(G) \times \Gamma$ with (u,x) connected to (v,y) whenever $\{u,v\} \in E(G)$ and $y = x * \alpha((u,v))$, where $*$ is the group operation of Γ. Projection onto the first coordinate, by definition, maps the derived graph of $\mathrm{Vol}(G, \Gamma, \alpha)$ onto G, and this map is a surjective homomorphism. Hence G is a quotient of the derived graph.

As an interesting example for quartic integral graphs, we found a voltage assignment α for which the derived graph $\mathrm{Vol}(F_1, \mathbb{Z}_3, \alpha)$ is isomorphic to F_4. Thus, F_1 is a quotient of F_4.

Given two graphs G_1, G_2 with vertex sets $V(G_1), V(G_2)$, let $G_1 \times G_2$ be the graph with vertex set the Cartesian product $V(G_1) \times V(G_2)$ with (u_1, u_2) adjacent to (w_1, w_2) whenever both u_1 is adjacent to w_1 in G_1 and u_2 is adjacent to w_2 in G_2. The *bipartite double cover* of G is the bipartite graph $G \times K_2$ where K_2 denotes the complete graph on two vertices. Equivalently, $G \times K_2$ is the derived graph $\mathrm{Vol}(G, \mathbb{Z}_2, \alpha)$, where α is the constant function assigning 1 to every arc of G.

We give an example for quartic integral graphs that was also noted in [7]. An *odd graph* O_i is the graph with one vertex for each of the $(i-1)$-element subsets of a $(2i-1)$-element set and with edges joining disjoint subsets. The graph F_2 is the bipartite double cover of the integral graph O_4.

Similar to the result by Schwenk [20] used in [24] and [25], we have that if G is a QIG, then the bipartite double cover of G is a bipartite QIG. If G is a bipartite QIG then the bipartite double cover consists of two disjoint copies of G. For this reason, we have restricted our search to integral graphs that are bipartite up to this point. However, we now want to find all graphs which have their bipartite double cover among the bipartite graphs that we have discovered. This requires us to find quotients of our bipartite graphs. Since it is computationally easy to do, we will actually consider a more general class of homomorphisms than what is required for the task just described. This will increase the number of quartic integral graphs that we find. However, we make no effort to be exhaustive in finding all possible quotients.

A graph automorphism is *k-semiregular* if all its orbits have the same size, k. Note that if $G = H \times K_2$ then the natural homomorphism from G onto H maps orbits of a 2-semiregular automorphism of G to single vertices of H. With this as motivation, the class of homomorphisms that we consider is the following. We identify any k-semiregular automorphism, ϑ of a target graph G. Our homomorphism is to collapse each orbit of ϑ to a single point.

We wrote a routine in Magma [21] to find such quotients of a target graph G, as follows. For one representative, ϑ, of each conjugacy class of (nontrivial) semiregular automorphisms of G, we collapsed the orbits of ϑ to single vertices to obtain a quotient H. If H was a 4-regular graph we checked to see if it was integral. If it was, then we printed it out and called the routine recursively on H.

In some cases we were only interested in finding those H for which G is a bipartite double cover. In such instances, it suffices to only consider 2-semiregular automorphisms and we do not need to make recursive calls to the routine.

We applied our Magma routine to all target graphs G_i for $i \in 1, \ldots, 17$ and to most of the arc-transitive graphs from the census of Potočnik *et al.* [17, 18]. There are graphs in the census with extremely large automorphism groups, and they were impractical for our simple routine. So we decided to only include target graphs from the census if their automorphism group had order no more than 2^{20}. The results of our Magma routine will be given in the following subsections.

4.4 Non-bipartite Cayley integral graphs

In this section we report all quartic Cayley integral graphs that are not bipartite. We rely on this Lemma:

Lemma 4.2. *If G is a 4-regular Cayley graph then $G \times K_2$, the bipartite double cover of G, is isomorphic to a 4-regular Cayley graph.*

Proof. If $G = \mathrm{Cay}(\Gamma, S)$ then we define $G' = \mathrm{Cay}(\Gamma \times \mathbb{Z}_2, \{(s, 1) : s \in S\})$. This graph G' is an undirected Cayley graph. It is not hard to verify that G' is isomorphic to $G \times K_2$ which gives the desired result. □

Hence we can find all the graphs we seek by applying the Magma routine of Section 4.3 to our graphs G_i where $i = 1, \ldots, 17$. We use the following result by Sabidussi [19] to decide which of the graphs that we find are Cayley graphs:

Lemma 4.3. *A graph G is a Cayley graph if and only if $\mathrm{Aut}(G)$ contains a regular subgroup.*

Initial Graph	#Non-bipartite	#Cayley	#Vertex-transitive	#Arc-transitive
G_1	0	0	0	0
G_2	1	1	1	1
G_3	1	1	1	1
G_4	0	0	0	0
G_5	1	1	1	0
G_6	2	1	1	1
G_7	2	1	1	1
G_8	4	2	2	0
G_9	0	0	0	0
G_{10}	1	0	1	1
G_{11}	0	0	0	0
G_{12}	2	1	1	0
G_{13}	1	1	1	0
G_{14}	2	2	2	0
G_{15}	5	1	1	1
G_{16}	13	2	2	0
G_{17}	2	1	1	0

Table 6: Non-bipartite graphs found for G_i

Table 6 summarizes our results using the Magma routine of Section 4.3 when restricted to the 2-semiregular automorphisms for each given G_i. We give the number of non-bipartite graphs found, followed by the numbers of those that are Cayley, vertex-transitive, and arc-transitive.

The non-bipartite graphs counted in column 2 up to row 8 of Table 6 were previously found by Stevanović *et al.* [25]. All graphs counted by column 2 from rows 9 to 17 were previously unknown with the exception of the graph with bipartite double cover G_{10} and one of the two graphs with bipartite double cover G_{14}. In Table 7, we expand upon the counts of non-bipartite Cayley graphs in column three of Table 6 by producing a breakdown of the groups and the connection sets of the underlying graphs. We follow the same conventions as in Table 3 except that we use different notation for the spectrum, since there is no longer symmetry about the origin.

Group	Connection Sets (#involutions S)
$\mathbf{H_1: n=5 \quad -1^4, 4^1 \quad I_{5,1} \quad AT4Val[5][1]}$	
$\mathbb{Z}_5$ $< a \mid a^5 >$	$0\ \{a^3, a^2, a^4, a\}$
$\mathbf{H_2: n=6 \quad -2^2, 0^3, 4^1 \quad I_{6,1} \quad AT4Val[6][1]}$	
S_3 $< s, t \mid s^2, t^3, (st)^2 >$	$2\ \{st, t, t^2, s\}$
$\mathbb{Z}_6$ $< a \mid a^6 >$	$0\ \{a^5, a^2, a, a^4\}$
$\mathbf{H_3: n=8 \quad -2^3, 0^3, 2^1, 4^1 \quad I_{8,2}}$	
$\mathbb{Z}_4 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 >$	$2\ \{a, a^2, a^3, b\}$
D_8 $< r, f \mid r^4, f^2, (rf)^2 >$	$2\ \{r, r^2, r^3, fr^2\}$ $4\ \{f, r^2, rf, fr\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 >$	$4\ \{b, a, abc, ac\}$
$\mathbf{H_4: n=9 \quad -2^4, 1^4, 4^1 \quad I_{9,2} \quad AT4Val[9][1]}$	
$\mathbb{Z}_3 \times \mathbb{Z}_3$ $< a \mid a^3 > \times < b \mid b^3 >$	$0\ \{a^2b, ab^2, a^2, a\}$
$\mathbf{H_5: n=12 \quad -2^5, 0^3, 2^3, 4^1 \quad I_{12,7} \quad AT4Val[12][1]}$	
A_4 $< s, t \mid s^2, t^3, (st)^3 >$	$0\ \{t^2s, ts, st^2, st\}$

$\mathbf{H_6}: \ \mathbf{n = 12} \quad \mathbf{-3^2, -1^4, 0^1, 1^2, 2^2, 4^1} \quad \mathbf{I_{12,5}}$

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b^2 >$	$0 \ \{a, ba^2, b^3a^2, a^2\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0 \ \{a^3, a^4, a^8, a^9\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2 \ \{r^4, fr^4, r^2, fr\}$ $2 \ \{r^3, f, r^4, r^2\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2 \ \{a^2, b, a^3, a^4\}$

$\mathbf{H_7}: \ \mathbf{n = 12} \quad \mathbf{-3^2, -2^2, 0^1, 1^6, 4^1} \quad \mathbf{I_{12,1}}$

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b^2 >$	$0 \ \{b^3, b, b^2a, b^2a^2\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0 \ \{a^3, a^{10}, a^2, a^9\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2 \ \{f, fr^3, r, r^5\}$ $4 \ \{r^3, f, fr, fr^5\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2 \ \{a^3b, a^3, a^2b, a^4b\}$

$\mathbf{H_8}: \ \mathbf{n = 18} \quad \mathbf{-3^2, -2^4, 0^5, 1^4, 3^2, 4^1} \quad \mathbf{I_{18,4}}$

$\mathbb{Z}_3 \times S_3$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \ \{a, s, a^2, st^2\}$ $0 \ \{t, t^2, sat^2, sa^2t^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^3, c^2, cac^{-1}a^{-2}, bc^{-1}b^{-2}c, aba^{-1}b^{-1} >$	$2 \ \{a, c, a^2, cb^2\}$
$\mathbb{Z}_6 \times \mathbb{Z}_3$ $< a \mid a^6 > \times < b \mid b^3 >$	$0 \ \{a^2b, a^5b^2, ab, a^4b^2\}$

$\mathbf{H_9}: \ \mathbf{n = 20} \quad \mathbf{-2^6, -1^4, 0^5, 3^4, 4^1}$

$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ $< a, b \mid a^5, b^4, ab^3a^3b, ab^2ab^2 >$	$2 \ \{a^2b^2, ab^2, a^2b, a^4b^3\}$

$\mathbf{H_{10}}: \ \mathbf{n = 24} \quad \mathbf{-3^3, -2^3, -1^5, 0^3, 1^5, 2^1, 3^3, 4^1} \quad \mathbf{I_{24,5}}$

S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$2 \ \{s, st^2st, (tst)^2, t(ts)^2\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$2 \ \{s, st^2, a, ts\}$

$\mathbf{H_{11}}:\ \mathbf{n = 24}\quad -\mathbf{3^4}, -\mathbf{2^3}, -\mathbf{1^2}, \mathbf{0^3}, \mathbf{1^8}, \mathbf{2^1}, \mathbf{3^2}, \mathbf{4^1}$

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a^3t, at^2, sa^2, a^2\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a,b,c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (a^3c)^2b, cbc^{-1}b^{-1}, a^2ca^2c >$	$4\ \{ca^2, cb, b, a^3\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r,f \mid r^4, f^2, (rf)^2 >$	$2\ \{f, r^2, ar, a^2r^{-1}\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	$4\ \{sabt, sbt^2, ab, sa\}$

$\mathbf{H_{12}}:\ \mathbf{n = 36}\quad -\mathbf{2^{13}}, -\mathbf{1^6}, \mathbf{0^3}, \mathbf{1^4}, \mathbf{2^3}, \mathbf{3^6}, \mathbf{4^1}\quad \mathbf{AT4Val[36][6]}$

$\mathbb{Z}_3 \times A_4$ $< a \mid a^3 > \times < s,t \mid s^2, t^3, (st)^3 >$	$0\ \{ta, tst, t^2a^2, t^2st^2\}$

$\mathbf{H_{13}}:\ \mathbf{n = 36}\quad -\mathbf{3^4}, -\mathbf{2^{10}}, \mathbf{0^1}, \mathbf{1^{16}}, \mathbf{3^4}, \mathbf{4^1}$

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bc^{-1}b^{-2}c >$	$0\ \{ac^2b, cb^2, a^2c^2b^2, c^3b^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, ac^{-1}a^{-1}bc, ac^2ac^2, acbc^{-1}b, c^2bc^2b >$	$2\ \{a^2bc^3, a^2c, ac^2, ab^2c^2\}$
$S_3 \times S_3$ $< s,t \mid s^2, t^3, (st)^2 > \times < u,v \mid u^2, v^3, (uv)^2 >$	$4\ \{suvt, s, sut^2, uv^2\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a^5t^2, at, sa^3t, st\}$

$\mathbf{H_{14}}:\ \mathbf{n = 36}\quad -\mathbf{3^4}, -\mathbf{2^4}, -\mathbf{1^{12}}, \mathbf{0^1}, \mathbf{1^4}, \mathbf{2^6}, \mathbf{3^4}, \mathbf{4^1}$

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bc^{-1}b^{-2}c >$	$0\ \{c^3b, cb, a^2b, ab^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, ac^{-1}a^{-1}bc, ac^2ac^2, acbc^{-1}b, c^2bc^2b >$	$0\ \{a^2bc^3, a^2c, b^2, b\}$
$S_3 \times S_3$ $< s,t \mid s^2, t^3, (st)^2 > \times < u,v \mid u^2, v^3, (uv)^2 >$	$2\ \{v^2t, s, uv, vt^2\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a^2t, sa^3t, a^4t^2, st\}$

$\mathbf{H_{15}}:\ \mathbf{n = 60}\quad -\mathbf{3^4}, -\mathbf{2^{17}}, -\mathbf{1^4}, \mathbf{0^{15}}, \mathbf{2^{11}}, \mathbf{3^8}, \mathbf{4^1}$

A_5 $< s,t \mid s^2, t^3, (st)^5 >$	$2\ \{st^2(st)^2, tst^2(st)^2t,$ $(st)^3ts, ((st)^2t)^2st\}$

Table 7: Non-bipartite Cayley QIGs

Thus, by Theorem 4.1 and Lemma 4.2 we have that $\{G_i : 1 \leqslant i \leqslant 17\} \cup \{H_j : 1 \leqslant j \leqslant 15\}$ is the complete set of Cayley QIGs.

4.5 Non-bipartite arc-transitive integral graphs

In Section 4.2, we listed all bipartite arc-transitive QIGs from the census of Potočnik *et al.* [17, 18]. When searching this census for integral graphs, we also found arc-transitive QIGs that are not bipartite. There are 6 such graphs that are not Cayley and thus not already accounted for in Table 7. By [25], the bipartite double cover of any QIG has order at most 560, so we can be sure that the census contains all the arc-transitive QIGs. In fact, the following folklore result tells us more:

Lemma 4.4. *The bipartite double cover of an arc-transitive graph is arc-transitive.*

Proof. Let G be an arc-transitive graph. Then $H = G \times K_2$ has vertices (a, x) for all $a \in G$ and $x \in \mathbb{Z}_2$ and arcs $((a, x), (b, x + 1))$ and $((b, x), (a, x + 1))$ whenever a is adjacent to b in G. It is not hard to show that the following maps are automorphisms of H:

- $(a, x) \to (\sigma(a), x + 1)$ for all $a \in G$ and $x \in \mathbb{Z}_2$ where $\sigma \in \mathrm{Aut}(G)$.
- $(a, x) \to (a, x + 1)$ for all $a \in G$ and $x \in \mathbb{Z}_2$.

Given these automorphisms, it is routine to check that H is arc-transitive. □

This last result provides a useful cross-check of our results and of the Magma routine from Section 4.3. It tells us that by applying the routine (restricted to 2-semiregular automorphisms) to the bipartite arc-transitive QIGs from Tables 3 and 5, we should find all the arc-transitive integral non-bipartite graphs. This list should tally with the list obtained by directly screening the census for integral graphs, which is what happened in practice.

We now list the spectrum of the non-bipartite arc-transitive QIGs that are not Cayley and whose bipartite double cover is one of the G_i for $i = 1, \ldots, 17$ or F_i for $i = 1, \ldots, 5$. We denote these graphs by J_i for $i = 1, \ldots, 6$. Graphs appearing in the paper by Cvetković *et al.* are included using the notation of [7]: $I_{n,index}$. We give the graph index from the census of Potočnik *et al.* [17, 18] in their notation: AT4Val[n][index].

- From G_{10}, $J_1 \cong I_{15,2} \cong$ AT4Val[15][1] : $[-2^5, -1^4, 2^5, 4^1]$,
- From F_1, $J_2 \cong$ AT4Val[30][2] : $[-3^4, -2^5, -1^4, 0^5, 2^{11}, 4^1]$,
  $J_3 \cong I_{30,4} \cong$ AT4Val[30][3] : $[-2^{11}, -1^4, 0^5, 2^5, 3^4, 4^1]$,
- From F_2, $J_4 \cong I_{35,1} \cong$ AT4Val[35][2] $\cong O_4$: $[-3^6, -1^{14}, 2^{14}, 4^1]$,
- From F_3, $J_5 \cong I_{45,1} \cong$ AT4Val[45][1] : $[-2^{16}, -1^9, 1^{10}, 3^9, 4^1]$,
- From F_4, $J_6 \cong$ AT4Val[90][8] : $[-3^{14}, -2^7, -1^{24}, 0^5, 1^{10}, 2^{21}, 3^8, 4^1]$.

Of the arc-transitive non-bipartite non-Cayley graphs, only J_2 and J_6 were not previously known to be integral. Thus, the arc-transitive QIGs from the census are as follows: G_1, G_2, G_3, G_5, G_6, G_7, G_{10}, G_{11}, G_{12}, G_{15}, G_{17}, F_1, F_2, F_3, F_4, F_5, H_1, H_2, H_4, H_5, H_{12}, J_1, J_2, J_3, J_4, J_5, and J_6. We summarize these results by the following Lemma:

Lemma 4.5. *There are exactly* 27 *quartic integral graphs that are arc-transitive;* 16 *of which are bipartite.*

4.6 Other quartic integral graphs

Finally, we list the spectra of the remaining QIGs which we found using the Magma routine of Section 4.3 in its full generality. These are graphs that are neither Cayley nor arc-transitive, but are quotients of the graphs G_i for $i = 1, \dots, 17$ and/or of the graphs AT4Val[n][index] for $n \leqslant 640$ with automorphism groups of order less than 2^{20}. We note that many of these graphs were obtained from multiple starting graphs, but we only list each graph once.

We list the spectrum of the bipartite QIGs first. We denote these graphs by M_i for $i = 1, \dots, 9$ and follow the same conventions as in the list for J_i where $i = 1, \dots, 6$ except that we use the quadruple form for the spectrum of a bipartite graph.

- From AT4Val[60][4] we have $M_1 \cong I_{30,3} : [1, 8, 3, 2]$,
- From $G_{15} \cong$ AT4Val[72][12] we have $M_2 \cong I_{36,1} : [2, 8, 6, 1]$, $M_3 \cong I_{36,2} : [3, 6, 5, 3]$,
- From $G_{17} \cong$ AT4Val[120][4] we have $M_4 : [3, 4, 1, 6]$, $M_5 : [6, 12, 2, 9]$,
- From AT4Val[180][12] we have $M_6 : [9, 16, 19, 0]$, $M_7 : [10, 14, 18, 2]$,
- From AT4Val[216][12] we have $M_8 : [3, 5, 9, 0]$,
- From AT4Val[546][48] we have $M_9 : [5, 4, 7, 4]$.

We do not list graphs with at most 24 vertices since all bipartite QIGs on 24 or fewer vertices are known [25]. The 6 graphs $M_4, \dots, M_9$ were not previously known to be bipartite QIGs. We find that M_6 is co-spectral to F_3, but 5 of the above spectra were not previously known to be realized by any graph.

Next, we list the spectrum of the non-bipartite QIGs. We denote these graphs by L_i where $i \in 1, \dots, 44$.

- From AT4Val[30][3], $L_1 \cong I_{15,4} : [-2^5, -1^3, 0^2, 2^3, 3^1, 4^1]$.
- From $G_{12} \cong$ AT4Val[36][3], $L_2 : [-3^3, -2^2, -1^1, 0^5, 1^3, 2^2, 3^1, 4^1]$.
- From AT4Val[36][6], $L_3 \cong I_{18,5} : [-2^7, -1^2, 0^1, 1^4, 2^1, 3^2, 4^1]$,
  $L_4 \cong I_{18,6} : [-2^6, -1^3, 0^3, 1^2, 3^3, 4^1]$.
- From AT4Val[60][4], $L_5 : [-3^3, -2^7, -1^3, 0^5, 1^1, 2^9, 3^1, 4^1]$, and
  $L_6 : [-3^2, -2^9, -1^2, 0^5, 1^2, 2^7, 3^2, 4^1]$.
- From AT4Val[70][4], $L_7 : [-3^5, -2^4, -1^9, 1^5, 2^{10}, 3^1, 4^1]$, and
  $L_8 : [-3^4, -2^6, -1^8, 1^6, 2^8, 3^2, 4^1]$.
- From $G_{15} \cong$ AT4Val[72][12], $L_9 : [-3^1, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1]$,
  $L_{10} : [-3^2, -2^3, -1^4, 0^1, 1^2, 2^5, 4^1]$, $L_{11} : [-3^2, -2^5, 0^1, 1^6, 2^3, 4^1]$,
  $L_{12} : [-3^2, -2^4, -1^2, 0^1, 1^4, 2^4, 4^1]$, $L_{13} : [-3^1, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1]$,
  $L_{14} : [-3^2, -2^4, -1^1, 0^3, 1^4, 2^2, 3^1, 4^1]$, $L_{15} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1]$,
  $L_{16} : [-3^4, 2^9, -1^2, 0^3, 1^8, 2^7, 3^2, 4^1]$, $L_{17} : [-3^2, -2^{11}, -1^4, 0^3, 1^6, 2^5, 3^4, 4^1]$, and
  $L_{18} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1]$.
- From G_{16}, $L_{19} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1]$,
  $L_{20} : [-3^4, -2^9, -1^2, 0^1, 1^{14}, 2^1, 3^4, 4^1]$,
  $L_{21} : [-3^4, -2^5, -1^{10}, 0^1, 1^6, 2^5, 3^4, 4^1]$, $L_{22} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$,

$L_{23} : [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1]$, $L_{24} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$, $L_{25} : [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1]$, $L_{26} : [-3^3, -2^7, -1^9, 0^1, 1^7, 2^3, 3^5, 4^1]$, $L_{27} : [-3^3, -2^8, -1^7, 0^1, 1^9, 2^2, 3^5, 4^1]$, $L_{28} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1]$, and
$L_{29} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$.

- From AT4Val[90][1], $L_{30} : [-3^4, -2^{10}, -1^9, 1^{10}, 2^6, 3^5, 4^1]$.
- From AT4Val[90][8], $L_{31} : [-3^5, -2^6, -1^{14}, 1^5, 2^{10}, 3^4, 4^1]$.
- From $G_{17} \cong$ AT4Val[120][4], $L_{32} : [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1]$, $L_{33} : [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1]$, $L_{34} : [-3^4, -2^5, -1^2, 0^9, 2^7, 3^2, 4^1]$, $L_{35} : [-3^7, -2^{13}, -1^3, 0^{15}, 1^1, 2^{15}, 3^5, 4^1]$.
- From AT4Val[180][12], $L_{36} : [-3^4, -2^8, -1^{12}, 0^2, 1^6, 2^6, 3^6, 4^1]$, $L_{37} : [-3^9, -2^{17}, -1^{19}, 0^5, 1^{15}, 2^{11}, 3^{13}, 4^1]$, $L_{38} : [-3^{11}, -2^{13}, -1^{21}, 0^5, 1^{13}, 2^{15}, 3^{11}, 4^1]$, and $L_{39} : [-3^{12}, -2^{15}, -1^{14}, 0^5, 1^{20}, 2^{13}, 3^{10}, 4^1]$.
- From AT4Val[210][10], $L_{40} : [-3^{16}, -2^9, -1^{29}, 1^{20}, 2^{19}, 3^{11}, 4^1]$.
- From AT4Val[273][4], $L_{41} : [-3^1, -2^4, -1^6, 0^4, 1^1, 3^4, 4^1]$.
- From AT4Val[546][48], $L_{42} : [-3^2, -2^3, -1^5, 0^4, 1^2, 2^1, 3^3, 4^1]$, $L_{43} : [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1]$, $L_{44} : [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1]$.

We do not list graphs with at most 12 vertices since all non-bipartite QIGs on 12 or fewer vertices are known [25]. Of the 44 graphs given above, only L_1, L_3 and L_4 previously appear in the literature about integral graphs. The remaining 41 non-bipartite QIGs are new.

5 Concluding remarks

There are precisely 32 connected 4-regular integral Cayley graphs up to isomorphism. Table 3 lists the 17 graphs of the 32 which are bipartite and Table 7 gives the details of the 15 non-bipartite graphs.

There are exactly 27 quartic integral graphs that are arc-transitive. We found that 16 of the 27 graphs are bipartite; these appear in Table 3 and Table 5. We found that 16 of the 27 graphs are Cayley graphs; these appear in Table 3 and Table 7.

There are integral Cayley bipartite graphs that can be decomposed into $H \times K_2$ where H is Cayley and arc-transitive, Cayley but not arc-transitive, or arc-transitive but not Cayley. The graph G_{10} is our only example of this last possibility; refer to Table 6.

The new 4-regular integral graphs that we found that are co-spectral to other graphs are as follows: G_{13} co-spectral to $I_{40,1}$ and $I_{40,2}$, G_{15} to $I_{72,1}$, H_9 to $I_{20,8}$, H_{12} to $I_{36,4}$, and F_3 to M_6. We also mention the co-spectral graphs among the known integral graphs: G_5 is co-spectral to $I_{16,2}$ and another graph appearing in [25], G_6 to $I_{18,2}$ and $I_{18,3}$, G_7 to $I_{24,1}$, F_1 to $I_{60,2}$, H_5 to $I_{12,6}$, and J_3 to $I_{30,5}$.

We find that some integral Cayley graphs are co-spectral to integral non-Cayley graphs and that some integral arc-transitive graphs are co-spectral to integral graphs that are not arc-transitive. For example, the arc-transitive Cayley graph H_5 has a co-spectral mate $I_{12,6}$, that is neither arc-transitive nor Cayley.

As can also be seen in Table 3, there are isomorphic integral graphs that are non-equivalent Cayley graphs $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma^*, S^*)$ in the sense of Definition 1.1. This can occur for $\Gamma \neq \Gamma^*$ as well as $\Gamma = \Gamma^*$ with $S \neq S^*$. Consider G_{12}, which has 12 non-equivalent Cayley Graphs on 6 different groups. For $\Gamma = S_3 \times S_3$, there are 4 non-equivalent Cayley graphs with connection sets occurring for each of the three possible numbers of involutions. There is only one Cayley graph up to equivalence for the graph of order 40. For all other orders the bipartite integral Cayley graphs are not unique up to equivalence. In the non-bipartite case; $H_1, H_4, H_5, H_9, H_{12}$, and H_{15} are all unique up to equivalence.

There are non-isomorphic integral Cayley graphs with the same number of vertices. As can be seen in Table 3 for the bipartite case, there are two graphs on 12 vertices, three graphs on 24 vertices, and two graphs on 72 vertices up to isomorphism. For all other orders there is at most one graph up to isomorphism. There are many more examples in the non-bipartite case (refer to Table 7).

There exist non-isomorphic integral Cayley graphs for the same group Γ. Consider G_i for $i = 7, 8, 9$ in Table 3. The following 6 groups are examples of this: $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_3 \times D_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$, S_4, $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, and $\mathbb{Z}_4 \times S_3$.

We began with the 828 possible spectra from [25], and then narrowed our focus to a set Ξ of 59 candidates for vertex transitive graphs; refer to Table 1 and Appendix A. Of these, we found 22 which are realised by Cayley graphs or arc-transitive graphs. In Section 4.6, by taking quotients, we found 6 new bipartite integral graphs that are neither arc-transitive nor Cayley, but realize a possible spectrum.

Overall, we found 9 bipartite quartic integral graphs (namely, G_{16}, G_{17}, F_4, F_5, M_4, M_5, M_7, M_8, M_9) that realise spectra not previously known to be achieved. It remains open whether the remaining possible spectra are realized by any 4-regular bipartite integral graphs.

All integral graphs discovered in this paper are available in Magma format from:

`http://users.monash.edu.au/~iwanless/data/graphs/IntegralGraphs.html`.

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This research formed part of the first author's PhD thesis [15].

A Feasible vertex-transitive spectra

The following is the set Ξ of possible spectra that might be realized by a connected 4-regular bipartite integral graph G that is vertex-transitive. This set was determined in Section 2. The entries are given as $n\ x\ y\ z\ w\ [C_4]\ [C_6]$ where $|V(G)| = n$ and $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$.

8	0	0	0	3	36	96
10	0	0	4	0	30	130
12	0	1	4	0	27	138
12	0	2	0	3	30	112
16	0	4	0	3	24	128
18	0	4	4	0	18	162
20	0	5	4	0	15	170
24	0	8	0	3	12	160
24	2	2	6	1	30	124
24	3	0	5	3	42	80
30	0	10	4	0	0	210
30	3	2	9	0	30	130
30	4	1	4	5	45	60
32	0	12	0	3	0	192
36	4	4	4	5	36	84
36	5	1	7	4	45	66
40	4	6	4	5	30	100
42	6	0	14	0	42	98
48	6	4	10	3	36	96
60	4	16	4	5	0	180
60	6	9	14	0	15	170
60	7	8	9	5	30	100
60	8	7	4	10	45	30
60	9	1	19	0	45	90
70	6	14	14	0	0	210
72	11	4	13	7	54	24
72	6	16	10	3	0	192
72	8	10	16	1	18	156
72	9	10	7	9	36	60
90	12	13	4	15	45	0
90	13	7	19	5	45	60
90	8	22	4	10	0	150
90	9	16	19	0	0	210
96	12	12	20	3	24	128
120	12	28	4	15	0	120
120	13	22	19	5	0	180
120	15	20	9	15	30	40
120	16	14	24	5	30	100
120	19	6	29	5	60	20
126	13	28	7	14	0	126
126	20	7	28	7	63	0
144	16	28	16	11	0	144
144	20	16	28	7	36	72
168	20	28	28	7	0	168
180	20	40	4	25	0	60
180	21	34	19	15	0	120
180	22	28	34	5	0	180
180	26	19	34	10	45	30
210	27	28	49	0	0	210
240	28	52	4	35	0	0
240	30	40	34	15	0	120
280	34	56	14	35	0	0
288	36	52	28	27	0	48
360	46	64	34	35	0	0
360	47	58	49	25	0	60
360	48	52	64	15	0	120
420	55	70	49	35	0	0
480	64	76	64	35	0	0
560	76	84	84	35	0	0

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Strongly light subgraphs in the 1-planar graphs with minimum degree 7

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Abstract

A graph is 1-*planar* if it can be drawn in the plane such that every edge crosses at most one other edge. A connected graph H is *strongly light* in a family of graphs $\mathfrak{G}$, if there exists a constant λ, such that every graph G in $\mathfrak{G}$ contains a subgraph K isomorphic to H with $\deg_G(v) \leq \lambda$ for all $v \in V(K)$. In this paper, we present some strongly light subgraphs in the family of 1-planar graphs with minimum degree 7.

Keywords: Strongly light subgraph, 1-planar graph.

Math. Subj. Class.: 05C10

1 Introduction

All graphs considered are finite, simple and undirected unless otherwise stated. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G. We shall denote by $F(G)$ the set of faces of an embedded graph G. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the number of edges of G incident with v. We denote the minimum and maximum degrees of vertices in G by $\delta(G)$ and $\Delta(G)$, respectively. A *wheel* W_n is a graph obtained by taking the join of a cycle C_n and a single vertex. In an embedded graph G, the *degree* of a face f, denoted by $\deg_G(f)$, is the number of edges with which it is incident, each cut edge being counted twice. A k-vertex, k^+-vertex and k^--vertex is a vertex of degree k, at least k and at most k, respectively. Similarly, we can define k-face, k^+-face and k^--face.

A graph is 1-*embeddable* in a surface S if it can be drawn in S such that every edge crosses at most one other edge. In particular, a graph is 1-*planar* if it can be drawn in

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the plane such that every edge crosses at most one other edge. The concept of 1-planar graph was introduced by Ringel [9] in 1965, while he simultaneously colors the vertices and faces of a plane graph such that any pair of adjacent/incident elements receive different colors. Ringel [9] proved that every 1-planar graph is 7-colorable, and conjectured that every 1-planar graph is 6-colorable. In 1984, Borodin [1] confirmed this conjecture, and later Borodin [2] found a better proof for it. Recently, various coloring problems of 1-planar graphs are considered, see [4, 13, 10].

A connected graph H is *strongly light* in a family of graphs $\mathfrak{G}$, if there exists an integer λ, such that every graph G in $\mathfrak{G}$ contains a subgraph K isomorphic to H with $\deg_G(v) \leq \lambda$ for all $v \in V(K)$. A graph H is said to be *light* in a family $\mathfrak{G}$ of graphs if at least one member of $\mathfrak{G}$ contains a copy of H and there is an integer $\lambda(H, \mathfrak{G})$ such that each member G of $\mathfrak{G}$ with a copy of H also has a copy K of H such that $\deg_G(v) \leq \lambda(H, \mathfrak{G})$ for all $v \in V(K)$. Note that a light subgraph may be not strongly light, for example, the graph K_5 is light in the family of graphs $\mathfrak{G} = \{\text{planar graphs}\} \cup \{K_6\}$, but K_5 is not strongly light in $\mathfrak{G}$ since not every graph in $\mathfrak{G}$ contains a subgraph K_5. The light subgraphs are well studied when $\mathfrak{G}$ is a subclass of planar graphs, and we refer the reader to a good survey [8].

Fabrici and Madaras [5] studied the structure of 1-planar graphs, mainly on the light subgraphs of 1-planar graphs. They showed that every 3-connected 1-planar graph contains an edge with each end having degree at most 20, and this bound is the best possible. Hudák and Madaras [6] proved that each 1-planar graph of minimum degree 5 and girth 4 contains (i) a 5-vertex adjacent to a vertex of degree at most 6, (ii) a 4-cycle whose vertices all have degree at most 9 (the upper bound was further improved to 8 by Borodin, Dmitriev and Ivanova [3]), (iii) a star $K_{1,4}$ with all vertices having degree at most 11.

In 1965, Ringel [9] found that each 1-planar graph has a vertex of degree at most 7 and the bound is tight. Hudák and Madaras [7] considered strongly light subgraphs in the family of 1-planar graphs with minimum degree 7, and proved the following theorem.

Theorem 1.1 (Hudák and Madaras [7])**.** Each 1-planar graph with minimum degree 7 contains

(a) two adjacent 7-vertices,

(b) a K_4 whose vertices all have degree at most 13,

(c) a $K_{2,3}^*$ whose vertices all have degree at most 13, where $K_{2,3}^*$ is a graph $K_{2,3}$ with an extra edge between two vertices of the smaller bipartition,

(d) a W_4 whose vertices all have degree at most 11.

In this paper, we also consider strongly light subgraphs in the family of 1-planar graphs with minimum degree 7.

2 Strongly light subgraphs

Let G be a graph having been drawn in a surface; if we treat all the crossing points as vertices, then we obtain an embedded graph $G^\dagger$, and call it *the associated graph of* G, call the vertices of G *true vertices* and the crossing points *crossing vertices*. In the associated graph, a 3-face is called a *false* 3*-face* if it is incident with a crossing vertex; otherwise, it is a *true* 3*-face*. Clearly, a false 3-face is incident with exactly one crossing vertex. Note that the set of crossing vertices in the associated graph is independent. In the figures of this paper, the solid dots denote true vertices and the hollow dots denote crossing vertices, and some degree restrictions are beside the vertices.

Zhang et al. presented two strongly light subgraphs on four vertices in the family of 1-planar graphs with minimum degree 7.

Theorem 2.1 (Zhang et al. [14])**.** Each 1-planar graph with minimum degree 7 contains a K_4 with all vertices of degree at most 11.

Theorem 2.2 (Zhang et al. [11])**.** Each 1-planar graph with minimum degree 7 contains a 4-cycle $C = [x_1x_2x_3x_4]$ with a chord x_1x_3, where $\deg(x_1) = 7$, $\deg(x_2) \le 10$, $\deg(x_3) \le 8$ and $\deg(x_4) \le 10$.

We improve the above two results to the following. A K_4 is of type (d_1, d_2, d_3, d_4) if its degrees are d_1, d_2, d_3 and d_4, respectively. Similarly, we can define a K_4 of type $(d_1^+, d_2^+, d_3^+, d_4^+)$, etc.

Theorem 2.3. If G is a 1-planar graph with minimum degree 7, then it contains a K_4 of the type $(7, 8^-, 8^-, 10^-)$.

Proof. Suppose that G is a connected counterexample to the theorem, which implies that G contains no K_4 or every copy of K_4 is of the type $(8^+, 8^+, 8^+, 8^+)$ or $(7, 9^+, 9^+, 9^+)$ or $(7, 8^-, 9^+, 9^+)$ or $(7, 8^-, 8^-, 11^+)$.

Furthermore, we may assume that G has been 1-embedded in the plane. Clearly, every face of its associated graph is homeomorphic to an open disk. Let $K^\dagger$ be the associated graph of G. By Euler's formula, we have

$$\sum_{v\in V(K^\dagger)} (\deg_{K^\dagger}(v) - 6) + \sum_{f\in F(K^\dagger)} (2\deg_{K^\dagger}(f) - 6) = -12. \tag{2.1}$$

We will use the discharging method to complete the proof. The initial charge of every vertex v is $\deg_{K^\dagger}(v) - 6$, and the initial charge of every face f is $2\deg_{K^\dagger}(f) - 6$. By (2.1), the sum of all the elements' charge is -12. We then transfer some charge from the 4^+-faces and the 7^+-vertices to crossing vertices, such that the final charge of every crossing vertex becomes nonnegative and the final charge of every 4^+-face and every 7^+-vertex remains nonnegative, and thus the sum of the final charge of vertices and faces is nonnegative, which leads to a contradiction.

The Discharging Rules:

(R1) every 4^+-face donates its redundant charge equally to incident crossing vertices;

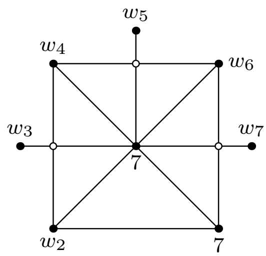
(R2) every 7^+-vertex donates its redundant charge equally to incident false 3-faces;

(R3) after applying (R2), every false 3-face donates its redundant charge to the incident crossing vertex.

By the discharging rules, the final charge of every face and every 7^+-vertex is nonnegative. So it suffices to consider the final charge of crossing vertices in $K^\dagger$.

By the construction of $K^\dagger$, every face is incident with at most $\left\lfloor \frac{\deg_{K^\dagger}(v)}{2} \right\rfloor$ crossing vertices. Thus, every 4^+-face sends at least 1 to each incident crossing vertex.

Note that every 7^+-vertex v is incident with at most $2\left\lfloor \frac{\deg_{K^\dagger}(v)}{2} \right\rfloor$ false 3-faces. More formally, every 7-vertex sends at least $\frac{1}{6}$ to each incident false 3-face; every 8-vertex sends



(a)

at least $\frac{1}{4}$ to each incident false 3-face; every 9-vertex sends at least $\frac{3}{8}$ to each incident false 3-face; every 10-vertex sends at least $\frac{2}{5}$ to each incident false 3-face; every 11^+-vertex sends at least $\frac{1}{2}$ to each incident false 3-face.

Let w be an arbitrary crossing vertex in $K^\dagger$. Notice that the four neighbors of w are 7^+-vertices.

If w is incident with at least two 4^+-faces, then its final charge is greater than $4 - 6 + 2 \times 1 = 0$. If w is incident with exactly one 4^+-face, then its final charge is at least $4 - 6 + 1 + 6 \times \frac{1}{6} = 0$.

If there is no crossing vertex which is incident with four 3-faces, then the sum of the final charge is nonnegative, which leads to a contradiction. So we may assume that w is incident with four 3-faces. It is obvious that the four neighbors of w induce a K_4 in G. If this K_4 is of the type $(8^+, 8^+, 8^+, 8^+)$, then the final charge of w is at least $4 - 6 + 8 \times \frac{1}{4} = 0$; if this K_4 is of the type $(7, 9^+, 9^+, 9^+)$, then the final charge of w is at least $4 - 6 + 2 \times \frac{1}{6} + 6 \times \frac{3}{8} > 0$; if this K_4 is of the type $(7, 8^-, 9^+, 9^+)$, then the final charge of w is at least $4 - 6 + 4 \times \frac{1}{6} + 4 \times \frac{3}{8} > 0$; if this K_4 is of the type $(7, 8^-, 8^-, 11^+)$, then the final charge of w is at least $4 - 6 + 6 \times \frac{1}{6} + 2 \times \frac{1}{2} = 0$.

Finally, all the faces and vertices have nonnegative charge, which leads to a contradiction. □

To the author's knowledge, all the known strongly light graphs have at most five vertices. Now, we give a strongly light graph on 8 vertices in the family of 1-planar graphs with minimum degree 7.

Theorem 2.4. If G is a 1-planar graph with minimum degree 7, then G contains a subgraph as illustrated in Fig. (a). Moreover, (i) every vertex in $\{w_2, w_3, \ldots, w_7\}$ has degree at most 23; (ii) at most one vertex in $\{w_2, w_3, \ldots, w_7\}$ is a 12^+-vertex; (iii) if no vertex in $\{w_2, w_3, w_5, w_7\}$ is a 7-vertex, then $w_2w_3, w_3w_4, w_4w_5, w_5w_6, w_6w_7, w_7w_1 \in E(G)$.

Proof. Suppose that G is a connected 1-planar graph with minimum degree 7, and it has been 1-embedded in the plane. Clearly, every face of its associated graph is homeomorphic to an open disk. Let $K^\dagger$ be the associated graph of G.

By Euler's formula, we have

$$\sum_{v \in V(K^\dagger)} (\deg_{K^\dagger}(v) - 4) + \sum_{f \in F(K^\dagger)} (\deg_{K^\dagger}(f) - 4) = -8. \tag{2.2}$$

We will use the discharging method to complete the proof. The initial charge of every vertex v is $\deg_{K^\dagger}(v) - 4$, and the initial charge of every face f is $\deg_{K^\dagger}(f) - 4$. By (2.2), the sum of all the elements' charge is -8. We then transfer some charge from the 7^+-vertices to the 3-faces, such that the final charge of every face and every 8^+-vertex is nonnegative, thus there exists a 7-vertex such that its final charge is negative and the local structure is desired.

The Discharging Rules:

(R1) every 7^+-vertex sends $\frac{1}{2}$ to each incident false 3-face and sends $\frac{1}{3}$ to each incident true 3-face;

(R2) let f be a face with a face angle w_1ww_2 and $\deg(w) = k \geq 8$,

(a) if f is a 3-face with $\deg(w_1) = 7$ and $\deg(w_2) \geq 8$, then w sends $\frac{k-4}{k} - \frac{1}{3}$ to w_1 through f;

(b) if f is a 3-face with $\deg(w_1) = \deg(w_2) = 7$, then each of w_1 and w_2 receives $\frac{k-4}{2k} - \frac{1}{6}$ from w through f;

(c) if f is a false 3-face with crossing vertex w_1 and w_1 is on the edge uw of G, then w sends $\frac{k-4}{2k} - \frac{1}{4}$ to w_2 through f, and additionally w sends $\frac{k-4}{2k} - \frac{1}{4}$ to u through f;

(d) if f is a 4^+-face with crossing vertex w_1 and w_1 is on the edge uw of G, then w sends $\frac{k-4}{2k}$ to u through f.

By the discharging rules, the final charge of every face and every 8^+-vertex is nonnegative. Hence, there exists a 7-vertex w_0 such that its final charge is negative.

If w_0 is incident with at least one 4^+-face, then its final charge is at least $7-4-6\times\frac{1}{2} = 0$. So we may assume that w_0 is incident with seven 3-faces. Notice that the number of incident false 3-faces is even. If w_0 is incident with at most four false 3-faces, then its final charge is at least $7 - 4 - 4 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$. Hence, the vertex w_0 must be incident with six false 3-faces and one true 3-face. We also notice that w_0 receives less than $\frac{1}{3}$ from all the other vertices; otherwise, its final charge is at least $7 - 4 + \frac{1}{3} - 6 \times \frac{1}{2} - \frac{1}{3} = 0$.

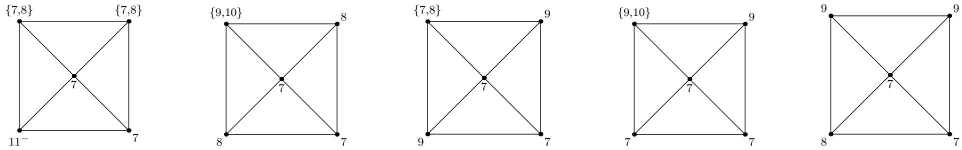
Let $w_1w_0w_2$ be the true 3-face. If both w_1 and w_2 are 8^+-vertices, then w_0 receives at least $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ from each of w_1 and w_2 by (R2-a), thus w_0 receives at least $\frac{1}{3}$ from all the other vertices, a contradiction. Hence, at least one of w_1 and w_2 must be a 7-vertex, so we may assume that w_1 is a 7-vertex, see Fig. (a).

(i) Suppose that w_0 is adjacent to a 24^+-vertex w in G. By the discharging rules, the vertex w_0 receives at least $2 \times (\frac{5}{12} - \frac{1}{4}) = \frac{1}{3}$ from w, which leads to a contradiction. Hence, every vertex in $\{w_2, w_3, \ldots, w_7\}$ has degree at most 23.

(ii) If at least two vertices in $\{w_2, w_3, \ldots, w_7\}$ are 12^+-vertices, then w_0 will receive at least $4 \times (\frac{1}{3} - \frac{1}{4}) = \frac{1}{3}$, which leads to a contradiction. Hence, at most one vertex in $\{w_2, w_3, \ldots, w_7\}$ is a 12^+-vertex.

(iii) Suppose, to derive a contradiction, that $w_2w_3, w_3w_4, w_4w_5, w_5w_6, w_6w_7, w_7w_1 \in E(G)$ does not hold. Thus, at least one crossing vertex in Fig. (a) is incident with a 4^+-face. By (R2-d), the vertex w_0 receives at least $\frac{1}{4}$ from a 8^+-vertex through a 4^+-face. By (R2-b), the vertex w_0 receives at least $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ from w_2 through the true 3-face $w_0w_1w_2$, thus it receives at least $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$ from all the other vertices, which derives a contradiction. □

Corollary 2.5 (Hudák and Madaras [7]). If G is a 1-planar graph with minimum degree 7, then it contains an edge such that each end has degree exactly 7.



(b) (c) (d) (e) (f)

Corollary 2.6. Every 1-planar graph with minimum degree 7 contains a $K_{1,7}$ with the center of degree 7 and the other vertices of degree at most 23.

By Theorem 2.4, the wheel W_4 is strongly light in the family of 1-planar graphs with minimum degree 7. In the next theorem, we further improve the degree restriction on each vertex in W_4.

Theorem 2.7. If G is a 1-planar graph with minimum degree 7, then G contains at least one subgraph as illustrated in Fig. (b)–(f).

Proof. Suppose that G is a connected 1-planar graph with minimum degree 7, and it has been 1-embedded in the plane. Clearly, every face of its associated graph is homeomorphic to an open disk. Let $K^\dagger$ be the associated graph of G.

By Euler's formula, we have

$$\sum_{v\in V(K^\dagger)} (\deg_{K^\dagger}(v) - 4) + \sum_{f\in F(K^\dagger)} (\deg_{K^\dagger}(f) - 4) = -8. \tag{2.3}$$

We will use the discharging method to complete the proof. The initial charge of every vertex v is $\deg_{K^\dagger}(v) - 4$, and the initial charge of every face f is $\deg_{K^\dagger}(f) - 4$. By (2.3), the sum of all the elements' charge is -8. We then transfer some charge from the 7^+-vertices to the 3-faces, such that the final charge of every face and every 8^+-vertex is nonnegative, thus there exists a 7-vertex such that its final charge is negative and the local structure is desired.

The Discharging Rules:

(R1) every 7^+-vertex sends $\frac{1}{2}$ to each incident false 3-face and sends $\frac{1}{3}$ to each incident true 3-face;

(R2) let f be a face with a face angle w_1ww_2 and $\deg(w) = k \geq 8$,

- (a) if f is a 3-face with $\deg(w_1) = 7$ and $\deg(w_2) \geq 8$, then w sends $\frac{k-4}{k} - \frac{1}{3}$ to w_1 through f;
- (b) if f is a 3-face with $\deg(w_1) = \deg(w_2) = 7$, then each of w_1 and w_2 receives $\frac{k-4}{2k} - \frac{1}{6}$ from w through f;
- (c) if f is a false 3-face with crossing vertex w_1, then w sends $\frac{k-4}{k} - \frac{1}{2}$ to w_2 through f.

By the discharging rules, the final charge of every face and every 8^+-vertex is nonnegative. Hence, there exists a 7-vertex w_0 such that its final charge is negative.

If w_0 is incident with at least one 4^+-face, then its final charge is at least $7-4-6\times\frac{1}{2} = 0$. So we may assume that w_0 is incident with seven 3-faces. Notice that the number of incident false 3-faces is even. If w_0 is incident with at most four false 3-faces, then its final charge is at least $7 - 4 - 4 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$. Hence, the vertex w_0 must be incident with six false 3-faces and one true 3-face. We also notice that w_0 receives less than $\frac{1}{3}$ from all the other vertices; otherwise, its final charge is at least $7 - 4 + \frac{1}{3} - 6 \times \frac{1}{2} - \frac{1}{3} = 0$.

Let $w_1w_0w_2$ be the true 3-face. If both w_1 and w_2 are 8^+-vertices, then w_0 receives at least $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ from each of w_1 and w_2 by (R2-a), thus w_0 receives at least $\frac{1}{3}$ from all the other vertices, a contradiction. Hence, at least one of w_1 and w_2 must be a 7-vertex, so we may assume that w_1 is a 7-vertex, see Fig. (a).

Case 1. Both $\deg(w_4)$ and $\deg(w_6)$ belong to $\{7, 8\}$.

Since the vertex w_0 receives less than $\frac{1}{3}$ from the vertex w_2, it follows that $(\frac{\deg(w_2)-4}{\deg(w_2)} - \frac{1}{2}) + (\frac{\deg(w_2)-4}{2\deg(w_2)} - \frac{1}{6}) < \frac{1}{3}$ and $\deg(w_2) < 12$, see Fig. (b).

Case 2. Exactly one of $\deg(w_4)$ and $\deg(w_6)$ belongs to $\{7, 8\}$.

Note that $\max\{\deg(w_4), \deg(w_6)\} \geq 9$, if w_2 is a 10^+-vertex, then w_0 receives at least $2\times(\frac{5}{9}-\frac{1}{2})+(\frac{3}{5}-\frac{1}{2})+(\frac{3}{10}-\frac{1}{6}) > \frac{1}{3}$, a contradiction. So we may assume that w_2 is a 9^--vertex. If w_2 is a 7-vertex and $\max\{\deg(w_4), \deg(w_6)\} \geq 12$, then w_0 will receive at least $2\times(\frac{2}{3}-\frac{1}{2}) = \frac{1}{3}$, which is a contradiction. If w_2 is a 8-vertex and $\max\{\deg(w_4), \deg(w_6)\} \geq 11$, then w_0 will receive at least $2 \times (\frac{7}{11} - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{6}) > \frac{1}{3}$, a contradiction. If w_2 is a 9-vertex and $\max\{\deg(w_4), \deg(w_6)\} \geq 10$, then w_0 receives at least $2 \times (\frac{3}{5} - \frac{1}{2}) + (\frac{5}{9} - \frac{1}{2}) + (\frac{5}{18} - \frac{1}{6}) = \frac{11}{30} > \frac{1}{3}$, which leads to a contradiction. In summary, if w_2 is a 7-vertex, then $\max\{\deg(w_4), \deg(w_6)\} \in \{9, 10, 11\}$, and thus G contains a subgraph isomorphic to that in Fig. (b); if w_2 is a 8-vertex, then $\max\{\deg(w_4), \deg(w_6)\} \in \{9, 10\}$, and thus G contains a subgraph isomorphic to that in Fig. (b) or Fig. (c); if w_2 is a 9-vertex, then $\max\{\deg(w_4), \deg(w_6)\} = 9$, and thus G contains a subgraph isomorphic to that in Fig. (d), Fig. (e) or Fig. (f).

Case 3. Both $\deg(w_4)$ and $\deg(w_6)$ are at least 9.

If w_2 is a 9^+-vertex, then the vertex w_0 will receive at least $(\frac{5}{18} - \frac{1}{6}) + 5 \times (\frac{5}{9} - \frac{1}{2}) > \frac{1}{3}$, a contradiction. So we may assume that w_2 is a 7- or 8-vertex. If $\min\{\deg(w_4), \deg(w_6)\} \geq 10$, then the vertex w_0 will receive at least $4 \times (\frac{3}{5} - \frac{1}{2}) = \frac{2}{5} > \frac{1}{3}$, a contradiction. Hence, we have that $\min\{\deg(w_4), \deg(w_6)\} = 9$. If w_2 is a 7-vertex and $\max\{\deg(w_4), \deg(w_6)\} \geq 11$, then the vertex w_0 will receive at least $2\times(\frac{7}{11} - \frac{1}{2}) + 2\times(\frac{5}{9} - \frac{1}{2}) > \frac{1}{3}$, a contradiction. If w_2 is a 8-vertex and $\max\{\deg(w_4), \deg(w_6)\} \geq 10$, then w_0 will receive at least $2\times(\frac{3}{5}-\frac{1}{2})+2\times(\frac{5}{9}-\frac{1}{2})+(\frac{1}{4}-\frac{1}{6}) > \frac{1}{3}$, which is a contradiction. In summary, if w_2 is a 7-vertex, then G contains a subgraph as illustrated in Fig. (e); if w_2 is a 8-vertex, then G contains a subgraph as illustrated in Fig. (f). □

Corollary 2.8. If G is a 1-planar graph with minimum degree 7, then G contains a triangle having vertex degree 7, 7 and at most 9, respectively.

As an immediate consequence of Theorem 2.7, the following corollary is an improvement of Theorem 2.2.

Corollary 2.9. If G is a 1-planar graph with minimum degree 7, then G contains a 4-cycle $C = [x_1x_2x_3x_4]$ with a chord x_1x_3, where $\deg(x_1) = 7$, $\deg(x_2) \leq 9$, $\deg(x_3) \leq 8$ and $\deg(x_4) \leq 9$.

Corollary 2.10 ([12])**.** If G is a 1-planar graph with minimum degree 7, then G contains a copy of $K_1 \vee (K_1 \cup K_2)$ with all the vertices of degree at most 9.

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The multisubset sum problem for finite abelian groups

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Abstract

We use a similar techique as in [2] to derive a formula for the number of multisubsets of a finite abelian group G with any given size and any given multiplicity such that the sum is equal to a given element $g \in G$. This also gives the number of partitions of g into a given number of parts over a finite abelian group.

Keywords: Composition, partition, subset sum, polynomials, finite fields, character, finite abelian groups.

Math. Subj. Class.: 11B30, 05A15, 20K01, 11T06

1 Introduction

Let G be a finite abelian group of size n and D be a subset of G. The well known subset sum problem in combinatorics is to decide whether there exists a subset S of D which sums to a given element in G. This problem is an important problem in complexity theory and cryptography and it is NP-complete (see for example [3]). For any $g \in G$ and i a positive integer, we let the number of subsets S of D of size i which sum up to g be denoted by

$$N(D, i, g) = \#\{S \subseteq D : \#S = i, \sum_{s \in S} s = g\}.$$

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When D has more structure, Li and Wan made some important progress in counting these subset sums by a sieve technique [3, 4]. Recently Kosters [2] gives a shorter proof of the formula obtained by Li and Wan earlier, using character theory.

$$N(G,i,g)=\frac{1}{n}\sum_{s|\gcd(exp(G),i)}(-1)^{i+i/s}\binom{n/s}{i/s}\sum_{d|\gcd(e(g),s)}\mu(s/d)\#G[d],$$

where $exp(G)$ is the exponent of G, $e(g)=\max\{d : d \mid exp(G), g\in dG\}$, μ is the Möbius function, and $G[d]=\{h\in G : dh=0\}$ is the d-torsion of G.

More generally, we consider a multisubset M of D. The number of times an element belongs to M is the *multiplicity* of that member. We define the *multiplicity* of a multisubset M is the largest multiplicity among all the members in M. We denote

$$M(D,i,j,g)=\#\{\text{multisubset } M \text{ of } D : \text{multiplicity}(M)\le j, \#M=i, \sum_{s\in M}s=g\}.$$

It is an interesting question by its own to count $M(D,i,j,g)$, the number of multisubsets of D of cardinality i which sum to g where every element is repeated at most j times. If $j=1$, then $M(D,i,j,g)=N(D,i,g)$. If $j\ge i$, this problem is also equivalent to counting partitions of g with at most i parts over D, which is $M(D,i,i,g)$. In this case we use a simpler notation $M(D,i,g)$ because the second i does not give any restriction.

Another motivation to study the enumeration of multisubset sums is due to a recent study of polynomials of prescribed ranges over a finite field. Indeed, through the study of enumeration of multisubset sums over finite fields [5], we were able to disprove a conjecture of polynomials of prescribed ranges ove a finite field proposed in [1]. Let $\mathbb{F}_q$ be a finite field of q elements and $\mathbb{F}_q^*$ be the cyclic multiplicative group. When D is $\mathbb{F}_q$ (the additive group) or $\mathbb{F}_q^*$, counting the multisubset sum problem is the same as counting partitions over finite fields, which has been studied earlier in [6].

In this note, we use the similar method as in [2] to obtain $M(D,i,j,g)$ when $D=G$. However, we work in a power series ring instead of a polynomial ring.

Theorem 1. Let G be a finite abelian group of size n and let $g\in G$, $i,j\in\mathbb{Z}$ with $i\ge 0$ and $j\ge 1$. For any $s\mid n$, we define

$$C(n,i,j,s)=\sum_{\substack{k\ge 0,\,0\le t\le \frac{n\gcd(s,j+1)}{s},\\ sk+t\cdot\mathrm{lcm}(s,j+1)=i}}(-1)^t\binom{n/s+k-1}{k}\binom{\frac{n\gcd(s,j+1)}{s}}{t}.$$

Then we have

$$M(G,i,j,g)=\frac{1}{n}\sum_{s|\gcd(exp(G),i)}C(n,i,j,s)\sum_{d|\gcd(s,e(g))}\mu(s/d)\#G[d].$$

where $exp(G)$ is the exponent of G, $e(g)=\max\{d : d\mid exp(G), g\in dG\}$, μ is the Möbius function, and $G[d]=\{h\in G : dh=0\}$ is the d-torsion of G.

As a corollary, we obtain the main theorem in [2] when $j=1$.

Corollary 1. (Theorem 1.1 in [2]) Let G be a finite abelian group of size n and let $g \in G$ and $i \in \mathbb{Z}$. Then we have

$$N(G,i,g) = \frac{1}{n} \sum_{s|\gcd(exp(G),i)} (-1)^{i+i/s} \binom{n/s}{i/s} \sum_{d|gcd(s,e(g))} \mu(s/d)\#G[d].$$

where $exp(G)$ is the exponent of G, $e(g) = \max\{d : d \mid exp(G), g \in dG\}$, μ is the Möbius function, and $G[d] = \{h \in G : dh = 0\}$ is the d-torsion of G.

Moreover, when $j \geq i$, the formula gives the number of partitions of g with at most i parts over a finite abelian group. To avoid confusion the multiset consisting of $a_1, \ldots, a_n$ is denoted by $\{\{a_1, \ldots, a_n\}\}$, with possibly repeated elements, and by $\{a_1, \ldots, a_n\}$ the usual sets. We define a partition of the element $g \in G$ with exactly i parts in D as a multiset $\{\{a_1, a_2, \ldots, a_i\}\}$ such that all a_k's are nonzero elements in D and

$$a_1 + a_2 + \ldots + a_i = g.$$

Then the number of these partitions is denoted by $P_D(i,g)$, i.e.,

$$P_D(i,g) = \left|\left\{\{\{a_1, a_2, \ldots, a_i\}\} \subseteq D : a_1 + a_2 + \ldots + a_i = g, a_1, \ldots, a_i \neq 0\right\}\right|.$$

It turns out $M(D,i,g) = \sum_{k=0}^{i} P_D(k,g)$ is the number of partitions of $g \in G$ with at most i parts in D.

Corollary 2. Let G be a finite abelian group of size n and let $g \in G$. Then the number of partitions of g over G with at most i parts is

$$\frac{1}{n} \sum_{s|\gcd(exp(G),i)} \binom{n/s + i/s - 1}{i/s} \sum_{d|gcd(s,e(g))} \mu(s/d)\#G[d].$$

where $exp(G)$ is the exponent of G, $e(g) = \max\{d : d \mid exp(G), g \in dG\}$, μ is the Möbius function, and $G[d] = \{h \in G : dh = 0\}$ is the d-torsion of G.

Proof. The number is $M(G,i,j,g)$ when $j \geq i \geq 0$. If $j \geq i$, then the linear Diophantine equation $sk + t \cdot \mathrm{lcm}(s, j+1) = i$ reduces to $sk = i$ and $t = 0$. The rest of proof follows immediately. □

In Section 2, we prove our main theorem and derive Corollary 1 as a consequence. In Section 3, we extend our study to a subset of a finite abelian group and make a few remarks on how to obtain the number of partitions over any subset of a finite abelian group.

2 Proof of Theorem 1

To make this paper self-contained, we recall the following lemmas (see Lemmas 2.1-2.4 in [2]). Let G be a finite abelian group of size n. Let $\mathbb{C}$ be the field of complex numbers and $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$ be the group of characters of G. Let $\chi \in \hat{G}$ and $\bar{\chi}$ be the conjugate character which satisfies $\bar{\chi}(g) = \overline{\chi(g)} = \chi(-g)$ for all $g \in G$. We note that a character χ can be naturally extended to a $\mathbb{C}$-algebra morphism $\chi : \mathbb{C}[G] \to \mathbb{C}$ on the group ring $\mathbb{C}[G]$.

Lemma 1. Let $\alpha = \sum_{g \in G} \alpha_g g \in \mathbb{C}[G]$. Then we have $\alpha_g = \frac{1}{n} \sum_{\chi \in \hat{G}} \bar{\chi}(g)\chi(\alpha)$.

Lemma 2. Let m be a positive integer and $g \in G$. Then

$$\sum_{\chi \in \hat{G}, \chi^m = 1} \chi(g) = \delta_{g \in mG} \# G[m],$$

where $\delta_{g \in mG}$ is 1 if $g \in mG$ and it is zero otherwise.

Lemma 3. Let $\chi \in \hat{G}$ be a character and m be its order. Then we have

$$\prod_{\sigma \in G} (1 - \chi(\sigma) Y) = (1 - Y^m)^{n/m}.$$

Lemma 4. Let $g \in G$. The number $e(g)$ is equal to $lcm\{d : d \mid exp(G),\ g \in dG\}$. For $d \mid exp(G)$ we have $g \in dG$ if and only if $d \mid e(g)$.

Let us present the proof of Theorem 1. We use the multiplicative notation for the group.

Proof. Fix $j \geq 1$. Working in the power series ring $\mathbb{C}[G][[X]]$ over the group ring, the generating function of $\sum_{g \in G} M(G, i, j, g) g$ is

$$\sum_{i=0}^{\infty} \sum_{g \in G} M(G, i, j, g) g X^i = \prod_{\sigma \in G} (1 + \sigma X + \cdots + \sigma^j X^j) = \prod_{\sigma \in G} \frac{1 - \sigma^{j+1} X^{j+1}}{1 - \sigma X} \in \mathbb{C}[G][[X]].$$

Using Lemma 1, we write

$$\sum_{i=0}^{\infty} M(G, i, j, g) X^i = \frac{1}{n} \sum_{\chi \in \hat{G}} \bar{\chi}(g) \prod_{\sigma \in G} \frac{1 - \chi^{j+1}(\sigma) X^{j+1}}{1 - \chi(\sigma) X}.$$

Separating the first sum on the right hand side, we obtain

$$\sum_{i=0}^{\infty} M(G, i, j, g) X^i = \frac{1}{n} \sum_{s \mid exp(G)} \sum_{\chi \in \hat{G}, ord(\chi) = s} \bar{\chi}(g) \prod_{\sigma \in G} \frac{1 - \chi^{j+1}(\sigma) X^{j+1}}{1 - \chi(\sigma) X}.$$

For each fixed χ of the order s, we know that χ^{j+1} has the order $\frac{s}{\gcd(s, j+1)}$. Therefore by Lemma 3, we simplify the above as follows:

$$\sum_{i=0}^{\infty} M(G, i, j, g) X^i = \frac{1}{n} \sum_{s \mid exp(G)} \sum_{\chi \in \hat{G}, ord(\chi) = s} \bar{\chi}(g) \frac{\left(1 - X^{\mathrm{lcm}(s, j+1)}\right)^{\frac{n \gcd(s, j+1)}{s}}}{(1 - X^s)^{n/s}}. \quad (2.1)$$

Note that

$$\sum_{\chi \in \hat{G}, \chi^s = 1} \bar{\chi}(g) = \sum_{d \mid s} \sum_{\chi \in \hat{G}, ord(\chi) = d} \bar{\chi}(g).$$

By Lemma 2 and the Möbius inversion formula, we obtain

$$\sum_{\chi \in \hat{G}, ord(\chi) = s} \bar{\chi}(g) = \sum_{d \mid s} \mu(s/d) \sum_{\chi \in \hat{G}, \bar{\chi}^d = 1} \bar{\chi}(g) = \sum_{d \mid s} \mu(s/d) \delta_{g \in dG} \# G[d].$$

Because $d \mid s \mid exp(G)$, by Lemma 4, $g \in dG$ if and only if $d \mid e(g)$. Hence

$$\sum_{\chi \in \hat{G}, ord(\chi)=s} \bar{\chi}(g) = \sum_{d \mid s} \mu(s/d)\delta_{g \in dG} \# G[d] = \sum_{d \mid \gcd(s, e(g))} \mu(s/d) \# G[d].$$

Plugging this into Equation (2.1), we get

$$\sum_{i=0}^{\infty} M(G,i,j,g)X^i = \frac{1}{n} \sum_{s \mid exp(G)} \sum_{d \mid gcd(s,e(g))} \mu(s/d)\#G[d] \frac{\left(1 - X^{\mathrm{lcm}(s,j+1)}\right)^{\frac{n \gcd(s,j+1)}{s}}}{(1-X^s)^{n/s}}.$$

By applying the binomial theorem to the right hand side and comparing coefficients of X^i in both sides, we single out $M(G,i,j,g)$ and obtain

$$M(G,i,j,g) = \frac{1}{n} \sum_{s \mid exp(G)} \sum_{d \mid gcd(s,e(g))} \mu(s/d)\#G[d]C(n,i,j,s).$$

After bringing $C(n,i,j,s)$ out of the inner sum we complete the proof. □

Finally we remark that we can derive Corollary 1 using $N(G,i,g) = M(G,i,1,g)$. When $j = 1$, let us consider $sk + t \cdot \mathrm{lcm}(s, j+1) = sk + t \cdot \mathrm{lcm}(s, 2) = i$. If s is even, we obtain $sk + st = i$ and thus $k + t = i/s$. Note that we have the following power series expansions

$$\frac{1}{(1-x)^{n/s}} = \sum_{k=0}^{\infty} \binom{n/s+k-1}{k} x^k,$$

$$(1-x)^{2n/s} = \sum_{t=0}^{2n/s} (-1)^t \binom{2n/s}{t} x^t,$$

and

$$(1-x)^{n/s} = \sum_{j=0}^{n/s} \binom{n/s}{j} (-1)^j x^j.$$

Now we compare the coefficients of the term $x^{i/s}$ in both sides of

$$\frac{1}{(1-x^s)^{n/s}}(1-x^s)^{2n/s} = (1-x^s)^{n/s},$$

after expanding these power series. Hence we obtain

$$C(n,i,1,s) = \sum_{\substack{k+t=i/s \\ k \geq 0, 0 \leq t \leq 2n/s}} (-1)^t \binom{n/s+k-1}{k} \binom{2n/s}{t} = (-1)^{i/s} \binom{n/s}{i/s}.$$

Moreover, $C(n,i,1,s) = (-1)^{i+i/s} \binom{n/s}{i/s}$ because i is even.

Similarly, if s is odd, we obtain $sk + 2st = i$ and thus $k + 2t = i/s$. Moreover, $i + i/s$ is even. Using

$$(1-x^{2s})^{n/s} \frac{1}{(1-x^s)^{n/s}} = (1+x^s)^{n/s},$$

we obtain

$$C(n,i,1,s) = \sum_{\substack{k+2t=i/s \\ k\geq 0, 0\leq t\leq n/s}} (-1)^t \binom{n/s+k-1}{k}\binom{n/s}{t} = (-1)^{i+i/s}\binom{n/s}{i/s}.$$

3 A few remarks

In this section we study $M(D,i,j,g)$ where $j \geq i$ and D is a subset of G. We recall that in this case we use the notation $M(D,i,g)$ because j does not really put any restriction. First of all, we note that

$$\sum_{i=0}^{\infty}\sum_{g\in G} M(G\setminus\{0\},i,g)gX^i = \prod_{\sigma\in G,\sigma\neq 0}\frac{1}{1-\sigma X} = (1-X)\sum_{i=0}^{\infty}\sum_{g\in G} M(G,i,g)gX^i.$$

By Corollary 2, we obtain

$$\begin{aligned} & M(G\setminus\{0\},i,g) \\ = \; & \frac{1}{n}\left(\sum_{s|\gcd(exp(G),i)}\binom{n/s+i/s-1}{i/s}\sum_{d|gcd(s,e(g))}\mu(s/d)\#G[d]\right. \\ & \left. -\sum_{s|\gcd(exp(G),i-1)}\binom{n/s+(i-1)/s-1}{(i-1)/s}\sum_{d|gcd(s,e(g))}\mu(s/d)\#G[d]\right). \end{aligned}$$

We note $M(G\setminus\{0\},i,g) = P_G(i,g)$. Therefore we obtain an explicit formula for the number of partitions of g into i parts over G. More generally, let $D = G\setminus S$, where $S = \{u_1,u_2,\ldots,u_{|S|}\} \neq \emptyset$. Denote by $M_S(G,i,g)$ the number of multisubsets of G of sizes i that contain at least one element from S. Then the number of multisubsets of $D = G\setminus S$ with i parts which sum up to g is equal to

$$M(G\setminus S,i,g) = M(G,i,g) - M_S(G,i,g).$$

Note that $M(G,0,0) = 1$ and $M(G,0,s) = 0$ for any $s \in G\setminus\{0\}$. The principle of inclusion-exclusion immediately implies that $M_S(G,i,g)$ is given in the following formula. We note that the formula is quite useful when the size of S is small in order to compute $M(G\setminus S,i,g)$.

Proposition 1. For all $i = 1,2,\ldots$ and $g\in G$ we have

$$\begin{aligned} M_S(G,i,g) = \sum_{u\in S} M(G,i-1,g-u) - \ldots \\ +(-1)^{t-1}\sum_{\{u_1,u_2,\ldots,u_t\}\subseteq S} M(G,i-t,g-(u_1+u_2+\ldots+u_t)) + \ldots \\ +(-1)^{i-2}\sum_{\{u_1,u_2,\ldots,u_{i-1}\}\subseteq S} M(G,1,g-(u_1+u_2+\ldots+u_{i-1})) + \\ (-1)^{i-1}\sum_{\{u_1,u_2,\ldots,u_i\}\subseteq S} M(G,1,g-(u_1+u_2+\ldots+u_i)). \end{aligned}$$

Proof. Fix an element $g \in G$. Denote by $\mathcal{A}_u$ the family of all the multisubsets of G with i parts which sum up to g and each multisubset also contains the element u. The principle of the inclusion-exclusion implies that

$$| \cup_{u \in S} \mathcal{A}_u| = \sum_{u \in S} |\mathcal{A}_u| - \sum_{\{u_1, u_2\} \subseteq S} |\mathcal{A}_{u_1} \cap \mathcal{A}_{u_2}| + \dots \tag{3.1}$$

It is obvious to see $|\mathcal{A}_{u_1} \cap \mathcal{A}_{u_2}| = M(G, i-2, g-(u_1+u_2))$ etc. by definition and the result follows directly. □

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Chiral covers of hypermaps

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Abstract

Generalising a conjecture of Singerman, it is shown that there are orientably regular chiral hypermaps (equivalently regular chiral *dessins d'enfants*) of every non-spherical type. The proof uses the representation theory of automorphism groups of Riemann surfaces acting on homology and on various spaces of differentials. Some examples are given.

Keywords: Chiral hypermap, chiral map, harmonic differential, homology group.

Math. Subj. Class.: 05C10, 05E18, 14H57, 20B25, 20C15, 30F10, 30F30

1 Introduction

In 1992, in an unpublished preprint [17], Singerman conjectured that if $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{2}$ there is an orientably regular chiral map of type $\{m, n\}$. Conder, Hucíková, Nedela and Širáň have announced a proof of this in [4] (see also [19, §3.4]). The aim of this note is to prove a similar but more general result for hypermaps (equivalently *dessins d'enfants* in Grothendieck's terminology [10]) of non-spherical type. Whereas most constructions of maps or hypermaps, including those in [4], involve group theory or combinatorics, this construction is mainly based on analysis (specifically, spaces of differentials on Riemann surfaces) and representation theory (the action of automorphism groups on such spaces and on associated homology groups). For background on the first topic see [9, 11], for the second see [7], and for applications of the second topic to the first see [16]. It is hoped that such techniques may find further applications in this area.

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2 Chiral covers

Orientably regular hypermaps $\mathcal{H}$ of type (l,m,n) correspond to torsion-free normal subgroups N of the triangle group $\Delta = \Delta(l,m,n)$; here N is an orientable surface group of genus g equal to that of $\mathcal{H}$, and $\mathcal{H}$ has *spherical type* if $g = 0$, or equivalently $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$. The orientation-preserving automorphism group $G := \mathrm{Aut}^{+}\mathcal{H}$ of $\mathcal{H}$ is isomorphic to Δ/N. We say that $\mathcal{H}$ is *reflexible* if N is normal in the extended triangle group Δ^* of type (l,m,n), in which case $\mathcal{H}$ has full automorphism group $A := \mathrm{Aut}\,\mathcal{H} \cong \Delta^*/N$; otherwise $\mathcal{H}$ and its mirror image $\overline{\mathcal{H}}$ form a *chiral pair*, corresponding to N and its other conjugate in Δ^*.

Theorem 2.1. *If $\mathcal{H}$ is a finite orientably regular hypermap of non-spherical type, then $\mathcal{H}$ is covered by infinitely many finite orientably regular chiral hypermaps of that same type.*

Proof. Let Δ and Δ^* be the triangle group and extended triangle group of the given type (l,m,n), so $\mathcal{H}$, which has genus $g > 0$, corresponds to a torsion-free normal subgroup N of finite index in Δ with $G := \mathrm{Aut}^{+}\mathcal{H} \cong \Delta/N$.

Case 1. Suppose that $\mathcal{H}$ is chiral, so N is not normal in Δ^*. Choose any prime p not dividing $|G|$: by a theorem in Euclid's *Elements*, there are infinitely many such primes. Let $N_p = N'N^p$, the group generated by the commutators and p-th powers in N; this is a torsion-free subgroup which is characteristic in N and hence normal in Δ, so it corresponds to a finite orientably regular hypermap $\mathcal{H}_p$ of type (l,m,n), which covers $\mathcal{H}$ since $N_p \leq N$. Now $|N : N_p| = p^{2g}$, which is coprime to $|\Delta : N|$, so N/N_p is a Sylow p-subgroup of Δ/N_p; being normal, it is the unique Sylow p-subgroup, so it is a characteristic subgroup of Δ/N_p. If N_p were normal in Δ^* then N/N_p would be normal in Δ^*/N_p, and hence N would be normal in Δ^*, against our assumption. Thus N_p is not normal in Δ^*, so $\mathcal{H}_p$ is chiral. As a smooth p^{2g}-sheeted covering of $\mathcal{H}$ it has genus $p^{2g}(g-1)+1$.

Case 2. Suppose that $\mathcal{H}$ is reflexible, so that N is normal in Δ^*. Let $\mathcal{S}$ be the compact Riemann surface $\mathcal{U}/N$ canonically associated with $\mathcal{H}$, where $\mathcal{U} = \mathbb{C}$ or $\mathbb{H}$ as $g = 1$ or $g > 1$, so that the elements of G and of $A \setminus G$ induce conformal and anticonformal automorphisms of $\mathcal{S}$. The harmonic differentials on $\mathcal{S}$ form a complex vector space $V = H^1(\mathcal{S},\mathbb{C})$ of dimension $2g$ affording a representation ρ of A. This space admits a G-invariant direct sum decomposition $V^+ \oplus V^-$, where V^+ and V^- are the g-dimensional spaces of holomorphic and antiholomorphic differentials on $\mathcal{S}$, affording complex conjugate representations ρ^+ and ρ^- of G. The elements of $A \setminus G$ transpose holomorphic and antiholomorphic differentials, and hence transpose the subspaces V^+ and V^- of V.

Integration around closed paths allows one to identify the first homology group

$$H_1(\mathcal{S},\mathbb{C}) = H_1(\mathcal{S},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

of $\mathcal{S}$ with the dual space V^* of V, so it affords the dual (or contragredient) representation ρ^* of A, which is equivalent to the complex conjugate representation $\overline{\rho}$ since A is finite. Since $\rho\,|_G$ is the sum of two complex conjugate representations, $\overline{\rho}\,|_G$ is equivalent to $\rho\,|_G$. Thus $H_1(\mathcal{S},\mathbb{C})$ also decomposes as a direct sum $H_1^+ \oplus H_1^-$ of two g-dimensional G-invariant subspaces affording complex conjugate representations ρ^+ and ρ^- of G, and these are transposed by elements of $A \setminus G$.

For any prime p, let $N_p := N'N^p$; this is a characteristic subgroup of N and hence a normal subgroup of Δ^*. Let $M_p := N/N_p$. Since $N \cong \pi_1\mathcal{S}$ we have $N/N' \cong H_1(\mathcal{S},\mathbb{Z}) \cong$

$\mathbb{Z}^{2g}$ and hence

$$M_p \cong H_1(\mathcal{S}, \mathbb{Z})/pH_1(\mathcal{S}, \mathbb{Z}) \cong H_1(\mathcal{S}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong H_1(\mathcal{S}, \mathbb{F}_p) \cong \mathbb{F}_p^{2g}$$

where $\mathbb{F}_p$ is the field of p elements. Indeed, these are isomorphisms of $\mathbb{F}_pA$-modules, with the natural action of A on homology corresponding to the induced action of Δ^*/N by conjugation on M_p.

Let e be the exponent of A. By a theorem of Dirichlet, there are infinitely many primes $p \equiv 1 \bmod (e)$. Each such prime p is coprime to $|A|$, so Maschke's Theorem holds for representations of A over $\mathbb{F}_p$; moreover, since $\mathbb{F}_p$ contains a full set of e-th roots of unity, $\mathbb{F}_p$ is a splitting field for A [7, Corollary 70.24], so the representation theory of A over $\mathbb{F}_p$ is 'the same' as that over $\mathbb{C}$, and this also applies to any subgroup of A. Specifically, the representation ρ_p of A on M_p can be regarded as the reduction mod (p) of its representation $\overline{\rho} \sim \rho$ on $V^* = H_1(\mathcal{S}, \mathbb{C})$ with respect to a suitable basis for V^*, with the same decomposition as a direct sum of absolutely irreducible subspaces. In particular, M_p has a G-invariant decomposition $M_p^+ \oplus M_p^-$, with M_p^+ and M_p^- affording g-dimensional representations of G, and the elements of $A \setminus G$ transpose the two direct factors. (See [16, §3] for full details concerning these representations of G; the extension to A is straightforward.)

The inverse images N_p^+ and N_p^- of M_p^+ and M_p^- in N are torsion-free normal subgroups of finite index p^g in Δ, so let $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ be the corresponding finite orientably regular hypermaps of type (l, m, n). These cover $\mathcal{H}$, and are non-isomorphic as oriented hypermaps since $N_p^+ \neq N_p^-$. Elements of $\Delta^* \setminus \Delta$ transpose N_p^+ and N_p^- by conjugation, so $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ form a chiral pair. As smooth p^g-sheeted coverings of $\mathcal{H}$, they have genus $(g-1)p^g + 1$. □

Corollary 2.2. *There exist infinitely many orientably regular chiral hypermaps of each non-spherical type.*

Proof. Being residually finite, the triangle group of the given type has a normal subgroup of finite index which contains no non-identity powers of the canonical generators, and is therefore torsion-free. Applying Theorem 2.1 to the corresponding orientably regular hypermap gives the required chiral hypermaps. □

Remark 2.3. In many cases, the condition $p \equiv 1 \bmod (e)$ in case 2 is unduly restrictive. It guarantees that every representation of A or G over $\mathbb{C}$ decomposes in the same way when regarded as a representation over $\mathbb{F}_p$, whereas we are interested in just one representation, namely ρ. There are cases, illustrated in the following examples, where ρ decomposes in the required way, yielding a chiral pair, for certain primes $p \not\equiv 1 \bmod (e)$.

Remark 2.4. Theorem 2.1 and Corollary 2.2 can be reinterpreted as statements about dessins d'enfants by replacing the phrase 'orientably regular hypermap' with the equivalent 'regular dessin'. However, their proofs make no essential use of the fact that Δ and Δ^* are triangle groups, or equivalently that the Riemann surface $\mathcal{S}$ is quasiplatonic, so they yield more general results concerning coverings of compact Riemann surfaces. It is hoped to explore these in a later paper.

Remark 2.5. The method used in case 1 of the proof of Theorem 2.1 is sometimes called the 'Macbeath trick', since it was used by Macbeath [15] to produce an infinite sequence of Hurwitz groups of the form $\Delta/N'N^m$, with Δ of type $(2, 3, 7)$ and $G = \Delta/N \cong L_2(7)$. (See §4 for background, and for more details concerning this example.) The resulting hypermaps (maps of type $\{3,7\}$) are all regular, whereas here, in cases 1 and 2 of Theorem 2.1, Macbeath's method is modified to produce chiral hypermaps.

Remark 2.6. It is *not* claimed that there are chiral hypermaps of every non-spherical genus. Indeed, Conder's lists of chiral maps and hypermaps [3] show that there are none of genus 2, 5 or 23. It would be interesting to characterise the genera with this property. Conder, Širáň and Tucker [5] have shown that there are no chiral maps of genus $g = p+1$ for primes p such that $p-1$ is not divisible by $3, 5$ or 8, but finding a similar result for hypermaps would seem to be more difficult.

3 An example in genus 2

There is a unique orientably regular hypermap $\mathcal{H}$ of genus 2 and type $(8,2,8)$. This is a map of type $\{8,8\}$, arising from an epimorphism $\Delta = \Delta(8,2,8) \to G = C_8$ with kernel $N = \Delta'$. By its uniqueness $\mathcal{H}$ is reflexible, with $A \cong \Delta^*/N \cong D_8$; it is R2.6 in Conder's list of regular maps [3], and is the first entry in [6, Table 9].

One can construct $\mathcal{H}$ as a map by identifying opposite sides of an octagon, so that $\mathcal{H}$ has one vertex, four edges and one face. Going around the boundary of the octagon gives a presentation

$$N = \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} = 1\rangle.$$

The images of a, b, c, d in N/N' form a basis for the homology group $V^* = H_1(\mathcal{S}, \mathbb{C})$ of the underlying surface $\mathcal{S}$. The automorphism group A of $\mathcal{H}$ is induced by the isometries of the octagon: thus $G = \langle r\rangle$ and $A = \langle r, s\rangle$ where r is a rotation through $\pi/4$ and s is a reflection; these are represented on homology by the matrices

$$\begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

The Riemann surface $\mathcal{S}$ underlying $\mathcal{H}$ is the hyperelliptic curve

$$w^2 = z(z^4 - 1),$$

a quasiplatonic curve with Belyĭ function $\beta : (w, z) \mapsto z^4$. The vertex is at $(0,0)$, the face-centre is at (∞, ∞), and the edge-centres are at the four points $(0, z)$ with $z^4 = 1$, where $\beta = 0, 1, \infty$ respectively. The automorphisms r and s send points $(w, z) \in \mathcal{S}$ to $(\zeta w, iz)$ and $(\overline{w}, \overline{z})$ respectively, where $\zeta = \exp(2\pi i/8)$.

The differentials

$$\omega_1 = \frac{dz}{w} \quad \text{and} \quad \omega_2 = \frac{z\,dz}{w}$$

form a basis for V^+ [9, §III.7.5], and their conjugates form a basis for V^-. The rotation r sends ω_1 and ω_2 to $i\omega_1/\zeta = \zeta\omega_1$ and $i^2\omega_2/\zeta = \zeta^3\omega_2$, so these span 1-dimensional G-invariant subspaces E_λ on which r has eigenvalues $\lambda = \zeta$ and ζ^3; there is a similar decomposition for V^-, except that here the eigenvalues are ζ^{-1} and ζ^{-3}. The action of s is to transpose E_ζ with $E_{\zeta^{-1}}$, and E_{ζ^3} with $E_{\zeta^{-3}}$.

Taking the dual space and then reducing mod (p), we see that the same applies to the A-module $M_p = H_1(\mathcal{S}, \mathbb{F}_p)$ for any prime $p \equiv 1 \bmod (8)$, with ζ now interpreted as a primitive 8th root of 1 in the splitting field $\mathbb{F}_p$. This gives a decomposition $M_p =$

$M_p^+ \oplus M_p^-$ with G-submodules $M_p^+ = E_\zeta \oplus E_{\zeta^3}$ and $M_p^- = E_{\zeta^{-1}} \oplus E_{\zeta^{-3}}$ transposed by s. Lifting these back to N gives subgroups N_p^+ and N_p^- corresponding to a chiral pair of maps $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ of type $\{8, 8\}$ and genus p^2+1, as in case 2 of the proof of Theorem 2.1.

Remark 3.1. In this example, A has exponent 8, so the smallest prime for which this construction applies is $p = 17$, giving a chiral pair of orientably regular hypermaps of type $\{8, 8\}$ and genus 290. They correspond to entry C290.4 in Conder's list of chiral maps [3].

Remark 3.2. This particular example also yields two chiral pairs of p-sheeted coverings of $\mathcal{H}$, corresponding to the four maximal submodules of M_p, each omitting one of the four 1-dimensional eigenspaces E_λ. These are orientably regular maps of type $\{8, 8\}$ and genus $p + 1$, and each pair consists of the duals of the other (see also [1, Theorem 1(a)(iii) and §3, Example (iii)] for the associated groups and Riemann surfaces). When $p = 17$ they correspond to entry C18.1 in [3].

Remark 3.3. It is interesting to see what happens if we use primes $p \not\equiv 1 \bmod (8)$ in this example, so that $\mathbb{F}_p$ is not a splitting field for A. If $p \equiv 3 \bmod (8)$ then M_p is a direct sum of two irreducible G-submodules, with r having eigenvalues ζ and ζ^3 on one, and their inverses on the other, where ζ is now a primitive 8th root of 1 in $\mathbb{F}_{p^2}$; these submodules are transposed by s, so we obtain a chiral pair of self-dual maps of type $\{8, 8\}$ and genus $p^2 + 1$; see C10.3 in [3] for the case $p = 3$, and C122.7 for $p = 11$.

The same applies if $p \equiv -3 \bmod (8)$, except that r now has eigenvalues ζ and ζ^{-3} on one submodule, and their inverses on the other; again these submodules are transposed by s, so we obtain a chiral pair of maps of type $\{8, 8\}$ and genus $p^2 + 1$, now duals of each other; see C26.1 and C170.7 in [3] for $p = 5$ and 13.

If $p \equiv -1 \bmod (8)$ then M_p is a direct sum of two irreducible G-submodules, with r having eigenvalues $\zeta^{\pm 1}$ on one and $\zeta^{\pm 3}$ on the other; these submodules are both invariant under s, so we obtain a dual pair of regular maps of type $\{8, 8\}$, rather than a chiral pair; see R50.7 in [3] for the case $p = 7$.

Finally, if $p = 2$ we again obtain no chiral maps, but there is a unique series

$$M_2 > (r-1)M_2 > (r-1)^2 M_2 > (r-1)^3 M_2 > (r-1)^4 M_2 = 0$$

of A-submodules, giving regular maps of type $\{8, 8\}$ and genus $2, 3, 5, 9$ and 17.

Remark 3.4. Further examples of chiral and regular hypermaps, arising as elementary abelian coverings of regular hypermaps of genus 2, have been found by Kazaz in [12]; this example is based on his methods (see also [13]).

4 Chiral covers of Klein's quartic curve

Klein's quartic curve $x^3y + y^3z + z^3x = 0$ is a compact Riemann surface $\mathcal{S}$ of genus 3, which carries a regular map $\mathcal{H}$ of type $\{3, 7\}$ with $G \cong L_2(7)$ and $A \cong PGL_2(7)$; see [14] for a comprehensive study of this curve. The method of proof of case 2 of Theorem 2.1, with Δ of type $(2, 3, 7)$, yields chiral pairs of maps of type $\{3, 7\}$ and genus $2p^3 + 1$ for all primes $p \equiv 1 \bmod (168)$. These are of interest because their automorphism groups, as finite quotients of Δ, are all Hurwitz groups, attaining Hurwitz's upper bound of $84(g - 1)$ for the number of automorphisms of a compact Riemann surface of a given genus $g > 1$.

In fact, such chiral pairs exist for all primes $p \equiv 1, 2$ or $4 \bmod (7)$. These can be obtained from a classification by Cohen [2] of those Hurwitz groups which arise as abelian coverings of G; his methods of construction are purely algebraic, using 6×6 matrices which can be interpreted as representing generators of Δ on various homology modules. (These covers were also obtained in an earlier paper of Wohlfahrt [20], using ideals in the ring of integers of $\mathbb{Q}(\sqrt{-7})$.) The case $p = 2$ is due to Sinkov [18], giving the chiral pair C17.1 of genus 17 in [3].

In this example, V^+ and V^- are irreducible G-submodules of V, affording complex conjugate representations with characters taking the values $3, -1, 0$ and 1 on elements of orders $1, 2, 3$ and 4, and $(-1 \pm \sqrt{-7})/2$ on the two classes of elements of order 7. Elkies has studied these representations, and their reduction modulo various primes, in [8, §1]. They give irreducible representations over $\mathbb{F}_p$ for any prime p such that -7 is a quadratic residue, that is, for $p \equiv 1, 2$ or $4 \bmod (7)$; for such primes we have a G-module decomposition $M_p = M_p^+ \oplus M_p^-$, giving a chiral pair $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$. However, no other primes give chiral pairs. The module M_7 is indecomposable but reducible, with a submodule of dimension 3 yielding a regular map of genus 687, and the zero submodule yielding one of genus 235299. For primes $p \equiv 3, 5$ or $6 \bmod (7)$, M_p is irreducible and again no chiral coverings arise. (Sah's statement in [16, §3(b)] is incorrect for such primes: for instance, M_3 yields only a regular map of genus 1459, corresponding to its zero submodule, and not the chiral pair of genus 55 claimed there.)

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The Cayley isomorphism property for groups of order $8p$

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Abstract

For every prime $p > 3$ we prove that $Q \times \mathbb{Z}_p$ is a DCI-group, where Q denotes the quaternion group of order 8. Using the same method we reprove the fact that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group for every prime $p > 3$, which was obtained in [3]. This result completes the description of CI-groups of order $8p$.

Keywords: Cayley graphs, CI-groups.

Math. Subj. Class.: 05C25

1 Introduction

Let G be a finite group and S a subset of G. The Cayley graph $Cay(G, S)$ is defined by having the vertex set G and g is adjacent to h if and only if $g^{-1}h \in S$. The set S is called the connection set of the Cayley graph $Cay(G, S)$. A Cayley graph $Cay(G, S)$ is undirected if and only if $S = S^{-1}$, where $S^{-1} = \{ s^{-1} \in G \mid s \in S \}$. Every left multiplication via elements of G is an automorphism of $Cay(G, S)$, so the automorphism group of every Cayley graph on G contains a regular subgroup isomorphic to G. Moreover, this property characterises the Cayley graphs of G.

Similarly to Cayley graphs one can also define ternary Cayley relational structures. $(V, E_1, E_2, \ldots, E_l)$ is a colour ternary relational structure if $E_i \subset V^3$ for $i = 1, \ldots, l$. We say that a colour ternary relational structure $(V, E_1, \ldots, E_l)$ is a Cayley ternary relational structure of the group G if the automorphism group of $(V, E_1, \ldots, E_l)$ contains a regular subgroup isomorphic to G.

It is clear that $Cay(G, S) \cong Cay(G, \mu(S))$ for every $\mu \in Aut(G)$. A Cayley graph $Cay(G, S)$ is said to be a CI-graph if, for each $T \subset G$, the Cayley graphs $Cay(G, S)$

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and $Cay(G,T)$ are isomorphic if and only if there is an automorphism μ of G such that $\mu(S)=T$. Furthermore, a group G is called a DCI-group if every Cayley graph of G is a CI-graph and it is called a CI-group if every undirected Cayley graph of G is a CI-graph.

Similarly, a group G is called a CI-group with respect to colour ternary relational structures, if for any pair of isomorphic colour ternary relational structures of G there exists an isomorphism induced by an automorphism of G.

Let G be a CI-group of order $8p$, where p is an odd prime. It is easy to verify that $\mathbb{Z}_2 \times \mathbb{Z}_4$ and the dihedral group of order 8 are not CI-groups. It can easily be seen that every subgroup of a CI-group is also a CI-group. Therefore the Sylow 2-subgroup of G can only be $\mathbb{Z}_8$, $\mathbb{Z}_2^3$ or the quaternion group Q of order 8.

It was proved by Li, Lu and Pálfy [5, Theorem 1.2 (b)] that a finite CI-group of order $8p$ containing an element of order 8 can only be

$$H=\left\langle a,z \mid a^p=1, z^8=1, z^{-1}az=a^{-1}\right\rangle.$$

It was also shown in [5, Theorem 1.3] that H is a CI-group, though not a DCI-group. In view of these results, for the rest of the discussion, we assume that the Sylow 2-subgroup of G is isomorphic to Q or $\mathbb{Z}_2^3$.

It was proved by Dobson [2] that $\mathbb{Z}_2^3\times\mathbb{Z}_p$ is a CI-group with respect to ternary relational structures if $p \geq 11$. Moreover, Dobson and Spiga [3] proved that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a DCI-group with respect to colour ternary relational structures if and only if $p \neq 3$ and 7. As a consequence of this result it was proved in [3] that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a DCI-group for all primes p.

If $p > 8$ or $p = 5$, then by Sylow's Theorem the Sylow p-subgroup of G is a normal subgroup, therefore G is isomorphic to one of the following groups: $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, $Q \times \mathbb{Z}_p$, $\mathbb{Z}_2^3 \ltimes \mathbb{Z}_p$ or $Q \ltimes \mathbb{Z}_p$. It can also be seen from [5, Theorem 1.2] that neither $Q \ltimes \mathbb{Z}_p$ nor $\mathbb{Z}_2^3 \ltimes \mathbb{Z}_p$ is a CI-group.

If $p = 7$, then either the Sylow 7-subgroup is normal, in which case G is as before, or G has 8 Sylow 7-subgroups and the Sylow 2-subgroup of G is normal. Then the Sylow 7-subgroup of G acts transitively by conjugation on the the non-identity elements of the Sylow 2-subgroup. Hence $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, which is not a CI-group by [5, Theorem 1.2.(b)].

If $p = 3$, then the order of G is 24. A complete list of CI-groups of order at most 31 was given in the Ph.D. thesis of Royle, see [7]. The CI-groups of order 24 are the following: $Q \times \mathbb{Z}_3$, $\mathbb{Z}_8 \ltimes \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_3$.

Spiga [6] proved that $Q \times \mathbb{Z}_3$ is not a CI-group with respect to colour ternary relational structures.

Using different methods depending on whether $p > 8$, or $p = 5, 7$ we show that the other groups are DCI-groups. By extending our result with the fact that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ is a CI-group we get that $Q \times \mathbb{Z}_p$ is a CI-group for every odd prime p.

Theorem 1.1. For every prime $p \geq 3$ the group $Q \times \mathbb{Z}_p$ is a DCI-group.

We also prove the following result which was first obtained in [3].

Theorem 1.2 (Dobson, Spiga [3])**.** For every prime $p \geq 3$ the group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a DCI-group.

Our paper is organized as follows. In Section 2 we introduce the notation that will be used throughout this paper. In Section 3 we collect important ideas which are useful in the proof of Theorem 1.1 and 1.2. Section 4 contains the proof of Theorem 1.1 and 1.2 for primes $p > 8$ and Section 5 contains the proof of Theorem 1.1 and 1.2 for $p = 5$ and 7.

2 Technical details

In this section we introduce some notation. Let G be a group. We use $H \leq G$ to denote that H is a subgroup of G and by $N_G(H)$ and $C_G(H)$ we denote the normalizer and the centralizer of H in G, respectively.

Let us assume that the group H acts on the set Ω and let G be an arbitrary group. Then by $G \wr_\Omega H$ we denote the wreath product of G and H. Every element $g \in G \wr_\Omega H$ can be uniquely written as hk, where $k \in K = \prod_{\omega \in \Omega} G_\omega$ and $h \in H$. The group $K = \prod_{\omega \in \Omega} G_\omega$ is called the base group of $G \wr_\Omega H$ and the elements of K can be treated as functions from Ω to G. If $g \in G \wr_\Omega H$ and $g = hk$ we denote k by $(g)_b$. In order to simplify the notation Ω will be omitted if it is clear from the definition of H and we will write $G \wr H$.

The symmetric group on the set Ω will be denoted by $Sym(\Omega)$. Let G be a permutation group on the set Ω. For a G-invariant partition $\mathcal{B}$ of the set Ω we use $G^{\mathcal{B}}$ to denote the permutation group on $\mathcal{B}$ induced by the action of G and similarly, for every $g \in G$ we denote by $g^{\mathcal{B}}$ the action of g on the partition $\mathcal{B}$.

For a group G, let $\hat{G}$ denote the subgroup of the symmetric group $Sym(G)$ formed by the elements of G acting by right multiplication on G. For every Cayley graph $\Gamma = Cay(G, S)$ the subgroup $\hat{G}$ of $Sym(G)$ is contained in $Aut(\Gamma)$.

Definition 2.1. Let $G \leq Sym(\Omega)$ be a permutation group. Let

$$G^{(2)} = \left\{ \pi \in Sym(\Omega) \middle| \forall a, b \in \Omega \ \exists g_{a,b} \in G \text{ with } \begin{array}{r} \pi(a) = g_{a,b}(a) \text{ and} \\ \pi(b) = g_{a,b}(b) \end{array} \right\}.$$

We say that $G^{(2)}$ is the 2-closure of the permutation group G.

The following lemma is well-known and follows directly from the definition of $G^{(2)}$.

Lemma 2.2. *Let Γ be a graph. If $G \leq Aut(\Gamma)$, then $G^{(2)} \leq Aut(\Gamma)$.*

3 Basic ideas

In this section we collect some results and some important ideas that we will use in the proof of Theorem 1.1 and Theorem 1.2.

We begin with a fundamental lemma that we will use all along this paper.

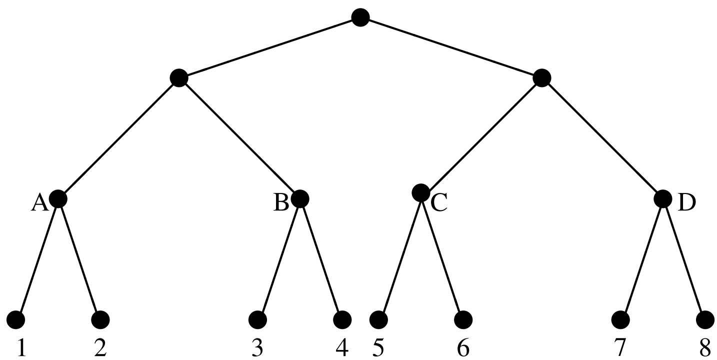
Lemma 3.1 (Babai [1]). *The Cayley graph $Cay(G, S)$ is a CI-graph if and only if for every regular subgroup $\mathring{G}$ of $Aut(Cay(G, S))$ isomorphic to G there is a $\mu \in Aut(Cay(G, S))$ such that $\mathring{G}^\mu = \hat{G}$.*

We introduce the following definition.

Definition 3.2. (a) We say that a Cayley graph $Cay(G, S)$ is a CI$^{(2)}$-graph if and only if for every regular subgroup $\mathring{G}$ of $Aut(Cay(G, S))$ isomorphic to G there is a $\sigma \in \langle \mathring{G}, \hat{G} \rangle^{(2)}$ such that $\mathring{G}^\sigma = \hat{G}$.

(b) A group G is called a DCI$^{(2)}$-group if for every $S \subset G$ the Cayley graph $Cay(G, S)$ is a CI$^{(2)}$-graph.

Let R be either Q or $\mathbb{Z}_2^3$. Let us assume that $A = Aut(Cay(G, S)) \leq Sym(8p)$ contains two copies of regular subgroups, $\mathring{R} \times \mathring{\mathbb{Z}}_p$ and $\hat{R} \times \hat{\mathbb{Z}}_p$. By Sylow's theorem we may assume that $\mathring{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$ are in the same Sylow p-subgroup P of $Sym(8p)$. If $p > 8$, then P is isomorphic to $\mathbb{Z}_p^8$. Moreover, P is generated by 8 disjoint p-cycles. It follows that both $\mathring{R}$ and $\hat{R}$ normalize P so we may assume that $\mathring{R}$ and $\hat{R}$ lie in the same Sylow 2-subgroup of $N_A(P)$. Let P_2 denote a Sylow 2-subgroup of $Sym(8)$. It is also well known that P_2 is isomorphic to the automorphism group of the following graph Δ:



Every automorphism of Δ permutes the leaves of the graph and the permutation of the leaves determines the automorphism, therefore $Aut(\Delta)$ can naturally be embedded into $Sym(8)$.

Lemma 3.3. *(a) There are exactly two regular subgroups of P_2 which are isomorphic to Q.*

(b) There are exactly two regular subgroups of P_2 which are isomorphic to $\mathbb{Z}_2^3$.

Proof. (a) Let Q be a regular subgroup of $Aut(\Delta)$ isomorphic to the quaternion group with generators i and j. Since Q is regular, for every $1 \leq m \leq 4$ there is a $q_m \in Q$ (not necessarily distinct) such that $q_m(2m-1) = 2m$. These are automorphisms of Δ so $q_m(2m) = 2m-1$ and hence since Q is regular the order of q_m is 2. There is only one involution in Q so $q_m = i^2$ for every $1 \leq m \leq 4$ and this fact determines completely the action of i^2 on Δ. Note that the automorphisms q_m are all equal.

We can assume that $i(1) = 3$. Such an isomorphism of Δ fixes setwise $\{1, 2, 3, 4\}$ so we have that $i(3) = 2$, $i(2) = 4$ and $i(4) = 1$ since i is of order 4. Using again the fact that Q is regular on Δ and $i^2(5) = 6$, we get that there are two choices for the action of i: $i = (1324)(5768)$ or $i = (1324)(5867)$.

We can also assume that $j(1) = 5$. This implies that $j(5) = j^2(1) = i^2(1) = 2$, and $j(2) = 6$ since $j \in Aut(\Delta)$ and $j(6) = 1$. The action of i determines the action of j on Δ since $iji = j$. Applying this to the leaf 3 we get that $j(3) = 8$ if $i = (1324)(5768)$ and $j(3) = 7$ if $i = (1324)(5867)$ so there is no more choice for the action of j. Finally, i and j generate Q and this gives the result.

(b) One can prove this using an argument similar to the previous case.

□

The previous proof also gives the following.

Lemma 3.4. *(a) The following two pairs of permutations generate the two regular subgroups of $Aut(\Delta) \leq Sym(8)$ isomorphic to Q:*

$$i_1 = (1324)(5768),\ j_1 = (1526)(3748) \text{ and}$$
$$i_2 = (1324)(5867),\ j_2 = (1526)(3847).$$

(b) The elements of these regular subgroups of $Aut(\Delta)$ are the following:

Q_l:		Q_r:	
id	(12)(34)(56)(78)	id	(12)(34)(56)(78)
(1324)(5768)	(1423)(5867)	(1324)(5867)	(1423)(5768)
(1526)(3847)	(1625)(3748)	(1526)(3748)	(1625)(3847)
(1728)(3546)	(1827)(3645)	(1728)(3645)	(1827)(3546)

Using the identification given in the following table, Q_l and Q_r act on Q by left-and right-multiplication with the elements of Q, respectively:

$\{1,\dots,8\}$	1	2	3	4	5	6	7	8
Q	1	-1	i	$-i$	j	$-j$	k	$-k$

(c) $A_1 = \langle x_1, x_2, x_3\rangle$ and $A_2 = \langle y_1, y_2, y_3\rangle$ are subgroups of $Aut(\Delta) \leq Sym(8)$ isomorphic to $\mathbb{Z}_2^3$, where

$$x_1 = (12)(34)(56)(78),\ x_2 = (13)(24)(57)(68),\ x_3 = (15)(26)(37)(48) \text{ and}$$
$$y_1 = (12)(34)(56)(78),\ y_2 = (13)(24)(58)(67),\ y_3 = (15)(26)(38)(47).$$

Lemma 3.5. *Let us assume that $G_1 \leq P_2$ is generated by two different regular subgroups Q_a and Q_b of $Aut(\Delta)$ which are isomorphic to Q and $G_2 \leq P_2$ is generated by two different regular subgroups A_1 and A_2 of $Aut(\Delta)$ which are isomorphic to $\mathbb{Z}_2^3$. Then $G_1 = G_2$.*

Proof. It is clear that $|P_2| = |Aut(\Delta)| = 2^7$. One can see using Lemma 3.4 (a) and (c) that G_1 and G_2 are generated by even permutations. Both G_1 and G_2 induce an action on the set $V = \{A,\ B,\ C,\ D\}$ which is a set of vertices of Δ and it is easy to verify that every permutation of V induced by G_1 and G_2 is even. This shows that G_1 and G_2 are contained in a subgroup of P_2 of cardinality 2^5.

Lemma 3.4 (b) shows that $|Q_a \cap Q_b| = 2$ and one can also check using Lemma 3.4 (c) that $|A_1 \cap A_2| = 2$. This gives $|G_1| \geq 2^5$ and $|G_2| \geq 2^5$, finishing the proof of Lemma 3.5. □

Proposition 3.6. *(a) The quaternion group Q is a $DCI^{(2)}$-group.*

(b) The elementary abelian group $\mathbb{Z}_2^3$ is a $DCI^{(2)}$-group.

Proof. (a) Let Q_a and Q_b be two regular subgroups of $Sym(8)$ isomorphic to the quaternion group Q. By Sylow's theorem we may assume that Q_a and Q_b lie in the same Sylow 2-subgroup of $H = \langle Q_a, Q_b\rangle$. Since every Sylow 2-subgroup of H is contained in a Sylow 2-subgroup of $Sym(8)$, we may assume that Q_a and Q_b are subgroups of $Aut(\Delta)$.

Our aim is to find an element $\pi \in \langle Q_a, Q_b\rangle^{(2)}$ such that $Q_a^\pi = Q_b$. Let us assume that $Q_a \neq Q_b$. Using Lemma 3.4 (a) we may also assume that Q_a and Q_b are generated by the permutations $(1324)(5768), (1526)(3748)$ and $(1324)(5867), (1526)(3847)$, respectively. Lemma 3.4 (b) shows that H contains the following three permutations:

$$(12)(34) = (1324)(5768)(1324)(5867)$$
$$(12)(56) = (1526)(3748)(1526)(3847)$$
$$(12)(78) = (1728)(3546)(1728)(3645).$$

Now one can easily see that the permutation (12) is in $H^{(2)}$. Finally, it is also easy to check using Lemma 3.4 (b) that $Q_a^{(12)} = Q_b$.

(b) One can prove this statement using Lemma 3.4 and Lemma 3.5.

Definition 3.7. Let Γ be an arbitrary graph and $A, B \subset V(\Gamma)$ such that $A \cap B = \emptyset$. We write $A \sim B$ if one of the following four possibilities holds:

(a) For every $a \in A$ and $b \in B$ there is an edge from a to b but there is no edge from b to a.

(b) For every $a \in A$ and $b \in B$ there is an edge from b to a but there is no edge from a to b.

(c) For every $a \in A$ and $b \in B$ the vertices a and b are connected with an undirected edge.

(d) There is no edge between A and B.

We also write $A \nsim B$ if none of the previous four possibilities holds.

The following lemma follows easily:

Lemma 3.8. *Let A, B be two disjoint subsets of cardinality p of a graph. We write $A \cup B = \mathbb{Z}_p \cup \mathbb{Z}_p$. Let us assume that a generator $\hat{g}$ of $\hat{\mathbb{Z}}_p$ acts by $\hat{g}(a_1, a_2) = (a_1 + 1, a_2 + 1)$ on $A \cup B$ and for a generator $\mathring{a}$ of the cyclic group $\mathring{\mathbb{Z}}_p$ the action of $\mathring{a}$ is defined by $\mathring{a}(a_1, a_2) = (a_1 + b, a_2 + c)$ for some $b, c \in \mathbb{Z}_p$.*

(a) If $b = c$, then the action of $\hat{\mathbb{Z}}_p$ and $\mathring{Z}_p$ on $A \cup B$ are the same.

(b) If $A \nsim B$, then $b = c$.

(c) If $A \sim B$, then every $\pi \in Sym(A \cup B)$ which fixes A and B setwise is an automorphism of the graph defined on $A \cup B$ as long as $\pi|_A \in Aut(A)$ and $\pi|_B \in Aut(B)$.

4 Main result for $p > 8$

In this section, we will prove that $R \times \mathbb{Z}_p$ is a DCI-group if $p > 8$, where R is either Q or $\mathbb{Z}_2^3$.

Proposition 4.1. *For every prime $p > 8$, the group $R \times \mathbb{Z}_p$ is a DCI-group.*

Our technique is based on Lemma 3.1 so we have to fix a Cayley graph $\Gamma = Cay(R \times \mathbb{Z}_p, S)$. Let $A = Aut(\Gamma)$ and $\mathring{G} = \mathring{R} \times \mathring{\mathbb{Z}}_p$ be a regular subgroup of A isomorphic to $R \times \mathbb{Z}_p$. In order to prove Proposition 4.1 we have to find an $\alpha \in A$ such that $\mathring{G}^\alpha = \hat{G} = \hat{R} \times \hat{\mathbb{Z}}_p$. We will achieve this in three steps.

4.1 Step 1

Since $p > 8$, the Sylow p-subgroup of Sym(8p) is generated by 8 disjoint p-cycles. We may assume $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$ lie in the same Sylow p-subgroup P of $Sym(8p)$. Then both $\mathring{R}$ and $\hat{R}$ are subgroups of $N_{Sym(8p)}(P) \cap A$ so we may assume that $\mathring{R}$ and $\hat{R}$ lie in the same Sylow 2-subgroup of $N_{Sym(8p)}(P) \cap A$ which is contained in a Sylow 2-subgroup of A.

Since $p > 8$, the Sylow p-subgroup P gives a partition $\mathcal{B} = \{B_1, B_2, \ldots, B_8\}$ of the vertices of Γ, where $|B_i| = p$ for every $i = 1, \ldots, 8$ and $\mathcal{B}$ is P-invariant. It is easy to see that $\mathcal{B}$ is invariant under the action of $\hat{R}$ and $\mathring{R}$ and hence $\langle \hat{G}, \mathring{G} \rangle \leq Sym(p) \wr Sym(8)$. Moreover, both $\mathring{G}$ and $\hat{G}$ are regular so $\mathring{R}$ and $\hat{R}$ induce regular action on $\mathcal{B}$ which we denote by R_1 and R_2, respectively. The assumption that $\mathring{R}$ and $\hat{R}$ lie in the same Sylow 2-subgroup of A implies that R_1 and R_2 are in the same Sylow 2-subgroup of $Sym(8)$.

4.2 Step 2

Let us assume that $R_1 \neq R_2$. We intend to find an element $\alpha \in A$ such that $\left(\mathring{R}^{\alpha}\right)^{\mathcal{B}} = R_2$.

We define a graph Γ_0 on $\mathcal{B}$ such that B_m is adjacent to B_n if and only if $B_m \not\sim B_n$. This is an undirected graph with vertex set $\mathcal{B}$ and both R_1 and R_2 are regular subgroups of $Aut(\Gamma_0)$. It follows that Γ_0 is a Cayley graph of R.

Observation 4.1. Since $R_1 \leq Aut(\Gamma_0)$ acts transitively on $\mathcal{B}$ we have that the order of eah connected component of Γ_0 divides 8.

We can also define a coloured graph Γ_1 on $\mathcal{B}$ by colouring the edges of the complete directed graph on 8 vertices. The vertex B_m is adjacent to the vertex B_n with the same coloured edge as $B_{m'}$ is adjacent to $B_{n'}$ in Γ_1 if and only if there exists a graph isomorphism ϕ from the induced subgraph of Γ on $B_m \cup B_n$ to the induced subgraph of Γ on $B_{m'} \cup B_{n'}$ such that $\phi(B_m) = B_{m'}$ and $\phi(B_n) = B_{n'}$. The graph Γ_1 is a coloured Cayley graph of R. Moreover, both R_1 and R_2 act regularly on Γ_1. Using the fact that R has property $DCI^{(2)}$, it is clear that there exists an $\alpha' \in \langle R_1, R_2 \rangle^{(2)} \leq Aut(\Gamma_1)$ such that $R_2^{\alpha'} = R_1$. We would like to lift α' to an automorphism α of Γ such that $\alpha^{\mathcal{B}} = \alpha'$.

(a) Let us assume first that Γ_0 is a connected graph.

Lemma 4.2. *(a)* $\mathring{R} \times \mathring{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$.

(b) If $\mathring{R} \times \mathring{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$, *then for every* $\mathring{r} \in \mathring{R}$ *we have* $(\mathring{r})_b = id$.

Proof. (a) We first prove that $\hat{\mathbb{Z}}_p = \mathring{\mathbb{Z}}_p$. Let x and y generate $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$, respectively. Since x and y lie in the same Sylow p-subgroup and $|B_1| = p$, we can assume that $x|_{B_1} = y|_{B_1}$. Using Lemma 3.8(b) we get that $x|_{B_m} = y|_{B_n}$ if there exists a path in Γ_0 from B_m to B_n. This shows that $x = y$ since Γ_0 is connected. Moreover, $\mathring{R} \times \hat{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$ since the elements of $\mathring{\mathbb{Z}}_p$ and the elements of $\mathring{R}$ commute.

(b) Let $A' = A \cap \left(\hat{\mathbb{Z}}_p \wr Sym(8)\right)$. We have already assumed that $\mathring{R}$ and $\hat{R}$ lie in the same Sylow 2-subgroup of A'. Let $\mathring{r}$ be an arbitrary element of $\mathring{R}$. For every $(a, u) \in R \times \mathbb{Z}_p$ we have $\mathring{r}(a, u) = (b, u + t)$ for some $b \in R$ and $t \in \mathbb{Z}_p$, where t only depends on $\mathring{r}$ and a since $\mathring{r} \leq \hat{\mathbb{Z}}_p \wr Sym(8)$. The permutation group $\hat{G}$ is transitive, hence there exist $\hat{r}_1, \hat{r}_2 \in \hat{R}$ such that $\hat{r}_1(1, u) = (a, u)$

and $\hat{r}_2(b, u+t) = (1, u+t)$. The order of $\hat{r}_2\mathring{r}\hat{r}_1$ is a power of 2 since $\hat{r}_2, \mathring{r}, \hat{r}_1$ lie in a Sylow 2-subgroup. Therefore $t = 0$ and hence $(\mathring{r})_b = id$.

□

Lemma 4.2 says that if Γ_0 is connected, then $\langle \mathring{R}, \hat{R} \rangle \leq \hat{\mathbb{Z}}_p \wr Sym(8)$ and $(r)_b = id$ for every $r \in \langle \mathring{R}, \hat{R} \rangle$. Therefore we can define $\alpha = \alpha' id_{\mathcal{B}}$ to be an element of the wreath product $\hat{\mathbb{Z}}_p \wr Sym(8)$ and clearly $\alpha' id_{\mathcal{B}}$ is an element of A with $\alpha^{\mathcal{B}} = \alpha'$.

(b) Let us assume that Γ_0 is the empty graph.

Then Lemma 3.8(c) shows that every permutation in $\langle R_1, R_2 \rangle^{(2)}$ lifts to an automorphism of Γ.

(c) Let us assume that Γ_0 is neither connected nor the empty graph.

Observation 4.2. If $R_1 \neq R_2$, then $\langle \mathring{R}, \hat{R} \rangle \leq A$ contains β_1, β_2, β_3 such that

$$\beta_1^{\mathcal{B}} = (B_1B_2)(B_3B_4),\ \beta_2^{\mathcal{B}} = (B_1B_2)(B_5B_6),\ \beta_3^{\mathcal{B}} = (B_1B_2)(B_7B_8).$$

Proof. Recall from Lemma 3.5 that $\langle \hat{R}, \mathring{R} \rangle$ is the same group whether R is Q or $\mathbb{Z}_2^3$. By Lemma 3.4 the elements β_1, β_2, β_3 can be generated as products of an element of $\mathring{R}$ and $\hat{R}$, as in the proof of Proposition 3.6, if $R = Q$. □

Lemma 4.3. *We claim that B_{2k-1} and B_{2k} are in the same connected component of Γ_0 for $k = 1, 2, 3, 4$.*

Proof. Since Γ_0 is a Cayley graph and R_1 is transitive on the pairs of the form (B_{2k-1}, B_{2k}) it is enough to prove that B_1 and B_2 are in the same connected component of Γ_0. If $B_1 \nsim B_2$, then B_1 is adjacent to B_2 in Γ_0, so we can assume that $B_1 \sim B_2$. Since Γ_0 is not the empty graph B_1 is adjacent to B_l for some $l > 2$, so $B_1 \nsim B_l$. By Observation (4.2) there exists $\beta \in A$ such that $\beta(B_1) = B_2$ and $\beta(B_l) = B_l$. This shows that $B_2 \nsim B_l$ and hence there is a path from B_1 to B_2 in Γ_0. □

Γ_0 is not connected, so the order of the connected components of Γ cannot be bigger than 4. Since B_1 and B_2 are in the same connected component of Γ_0 there exists a partition $H_1 \cup H_2 = \mathcal{B}$ such that $|H_1| = |H_2| = 4$, $B_1, B_2 \in H_1$ and no vertex in H_1 is adjacent to any vertex of H_2 in Γ_0.

Lemma 4.4. *There exists $\alpha \in A$ such that $\alpha^{\mathcal{B}} = \alpha'$.*

Proof. Let us assume first that $H_1 = \{B_1, B_2, B_3, B_4\}$. Then we define α_1 to be equal to β_2 on H_1 and the identity on H_2, where β_2 is defined in Observation 4.2. Using Lemma 3.8(c) we get that α_1 is in $\langle \mathring{R}, \hat{R} \rangle^{(2)}$.

If $H_1 = \{B_1, B_2, B_5, B_6\}$ or $H_1 = \{B_1, B_2, B_7, B_8\}$, then we define α_2 by $\alpha_2|_{H_1} = \beta_1$ and $\alpha_2|_{H_2} = id$, where β_1 is defined in Observation 4.2. Lemma 3.8(c) shows again that $\alpha_2 \in A$.

It is easy to see that $\alpha_1^{\mathcal{B}} = \alpha_2^{\mathcal{B}} = (B_1B_2)$. Therefore A contains an element α such that $R_1^{\alpha^{\mathcal{B}}} = R_2$. □

We conclude that we can assume that $R_1 = R_2$.

4.3 Step 3

Let us now assume that $R_1 = R_2$. We intend to find $\gamma \in A$ such that $\mathring{R}^{\gamma} = \hat{R}$.

Let $\hat{x}$ and $\mathring{x}$ denote the generators of $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$, respectively. We may assume that $\hat{x}|_{B_1} = \mathring{x}|_{B_1}$.

Lemma 4.5. *There exists* $\gamma \in A$ *such that* $\mathring{x}^{\gamma} = \hat{x}$.

Proof. Let us assume first that Γ_0 is connected. It is clear by Lemma 3.8 (b) that $\mathring{x} = \hat{x}$. So, we may take $\gamma = 1$.

Let us assume that Γ_0 is not connected. In this case there are at least two connected components which we denote by $\mathscr{C}_1, \dots, \mathscr{C}_n$. We may assume that $B_1 \in \mathscr{C}_1$. The permutations $\hat{x}$ and $\mathring{x}$ are elements of the base group of $\hat{\mathbb{Z}}_p \wr Sym(8)$ and hence they can be considered as functions on $\mathcal{B}$. We may assume that $\hat{x}(r, u) = (r, u + 1)$ for every $(r, u) \in R \times \mathbb{Z}_p$. By Lemma 3.8 (b), the function $\mathring{x}$ is constant on each equivalence class.

For every $1 \le m \le n$ there exists $\mathring{r}_m \in \mathring{R}$ such that $\mathring{r}_m(\mathscr{C}_1) = \mathscr{C}_m$ and for every $\mathring{r}_m \in \mathring{R}$ there exists $\hat{r}_m \in \hat{R}$ such that $\mathring{r}_m^{\mathcal{B}} = \hat{r}_m^{\mathcal{B}}$. Let γ be defined as follows:

$$\gamma|_{\cup \mathscr{C}_1} = id$$
$$\gamma|_{\cup \mathscr{C}_m} = \mathring{r}_m \hat{r}_m^{-1} \text{ for } 2 \le m \le n.$$

Let $(b, v) \in \mathring{r}_m(B_e)$ with $B_e \in \mathscr{C}_1$ and we denote $\mathring{r}_m^{-1}(b, v)$ by (a, u). Since $\mathring{x}$ is constant on $\mathscr{C}_m$ we have $\mathring{x}^s(b, v) = (b, v + c_m s)$ for some c_m which only depends on $\mathscr{C}_m$. Thus $\mathring{r}_m(a, u + s) = (b, v + c_m s)$ since $\mathring{x}$ and $\mathring{r}_m$ commute and $\mathring{x}|_{B_e} = \hat{x}|_{B_e}$. Therefore we have

$$\gamma(b, w) = \mathring{r}_m(a, w) = \mathring{r}_m(a, u + (w - u)) = (b, v + c_m(w - u))$$

for every $(b, w) \in \mathring{r}_m(B_e)$. It is easy to verify that $\gamma^{-1}(b, w) = (b, \frac{w - v + uc_m}{c_m})$ for every $w \in \mathbb{Z}_p$ which gives

$$\gamma^{-1}\mathring{x}\gamma(b, w) = \gamma^{-1}\mathring{x}(b, wc_m + v - uc_m) = \gamma^{-1}(b, wc_m + v - uc_m + c_m) = (b, w + 1).$$

It follows that $\gamma^{-1}\mathring{x}\gamma = \hat{\gamma}$.

It remains to show that $\gamma \in A$. Let y and z be two elements of $R \times \mathbb{Z}_p$.

We denote by B_y and B_z the elements of $\mathcal{B}$ containing y and z, respectively. If B_y and B_z are in the same connected component of Γ_0, then either γ is defined on B_y and B_z by $\mathring{r}_m \hat{r}_m^{-1}$ which is the element of the group $\langle \mathring{G}, \hat{G} \rangle \le A$ or $\gamma(y) = y$ and $\gamma(z) = z$.

If B_y and B_z are not in the same connected component, then $B_y \sim B_z$. The definition of γ shows that $\gamma^{\mathcal{B}} = id$. Using Lemma 3.8 (c) we get that $\gamma|_{B_y \cup B_z}$ is an automorphism of the induced subgraph of Γ on the set $B_y \cup B_z$, which proves that $\gamma \in A$, finishing the proof of Lemma 4.5. □

Using Lemma 4.5 we may assume that $\mathring{x} = \hat{x}$. Since $\mathring{x}$ and $\mathring{r}$ commute we have $\mathring{R} \times \mathring{\mathbb{Z}}_p \le \hat{\mathbb{Z}}_p \wr Sym(8)$. Now we can apply Lemma 4.2 which gives $(\mathring{r})_b = id$ for every $\mathring{r} \in \mathring{R}$. This proves that $\mathring{R} = \hat{R}$ since $R_1 = R_2$. Therefore $\mathring{G} = \hat{G}$, finishing the proof of Proposition 4.1. □

It is straightforward to check that the proof of Proposition 4.1 only uses the fact that $p > 8$ in the first step of the argument. We can formulate this fact in Proposition 4.6.

Proposition 4.6. *Let Γ be a Cayley graph of $G = Q \times \mathbb{Z}_p$ or $G = \mathbb{Z}_2^3 \times \mathbb{Z}_p$, where p is an odd prime and let $\mathring{G} = \mathring{Q} \times \mathring{\mathbb{Z}}_p$ or $\mathring{G} = \mathring{\mathbb{Z}}_2^3 \times \mathring{\mathbb{Z}}_p$ be a regular subgroup of $Aut(\Gamma)$ isomorphic to G. Let us assume that there exists a $\langle \hat{G}, \mathring{G} \rangle$-invariant partition $\mathcal{B} = \{B_1, B_2, \ldots, B_8\}$ of $V(\Gamma)$, where $|B_i| = p$ for every $i = \{1, \ldots, 8\}$. In addition, we assume that $\mathring{\mathbb{Z}}_p$ is a subgroup of the base group of $\hat{\mathbb{Z}}_p \wr Sym(\mathcal{B})$. Then there is an automorphism α of the graph Γ such that $\hat{G}^\alpha = \mathring{G}$.*

5 Main result for $p = 5$ and 7

In this section we will prove that $Q \times \mathbb{Z}_5$, $Q \times \mathbb{Z}_7$, $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups.

The whole section is based on the paper [5], so we will only modify the proof of Lemma 5.4 of [5].

Proposition 5.1. *Every Cayley graph of $Q \times \mathbb{Z}_5$, $Q \times \mathbb{Z}_7$, $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-graph.*

We let R denote either Q or $\mathbb{Z}_2^3$, and let $p = 5$ or 7. Let Γ be a Cayley graph of $R \times \mathbb{Z}_p$ and let $A = Aut(\Gamma)$. We denote by P a Sylow p-subgroup of A. Let us assume that A contains two copies of regular subgroups which we denote by $\hat{G} = \hat{R} \times \hat{\mathbb{Z}}_p$ and $\mathring{G} = \mathring{R} \times \mathring{\mathbb{Z}}_p$. We can assume that Γ is neither the empty nor the complete graph and both $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$ are contained in P.

If the order of every orbit of P on $V(\Gamma)$ is p, then it is clear from Proposition 4.6 that Γ is a CI-graph. Therefore P has an orbit $\Lambda \subset G$ such that $|\Lambda| = p^2$ since $p^3 > |G|$. The remaining orbits of P have order p since $2p^2 > 8p$.

It was proved in [5] Lemma 5.4 that the action of A on the vertices of the graph Γ cannot be primitive so there is a nontrivial A-invariant partition $\mathcal{B} = \{B_0, B_1, \ldots, B_{t-1}\}$ of $V(\Gamma) = G$. The elements of the partition $\mathcal{B}$ have the same cardinality since the action of A is transitive on $\mathcal{B}$ so $|B_i| \leq 4p < p^2$ for every $i = 0, 1, \ldots, t-1$. The partition $\mathcal{B}$ is P-invariant so P acts on $\mathcal{B}$. Since P is a p-group, the order of every orbit of P is a power of p.

Let $\mathscr{C} = \{C_0, C_1, \ldots, C_{s-1}\}$ be an orbit of P on $\mathcal{B}$ such that $\Lambda \subseteq \cup_{i=0}^{s-1} C_i$. We may assume that $B_i = C_i$ for $i = 0, 1, \ldots, s-1$. It is clear that s is a power of p. If $s \geq p^2$, then $\left|\cup_{i=0}^{s-1} C_i\right| \geq 2p^2 > 8p$ which is a contradiction. Since $|C_0| = |B_0| < p^2$, we cannot have $s = 1$. It follows that $1 < s < p^2$ which implies $s = p$.

For every $i < s$ and every $x \in P$ the following eqalities hold for some $j < s$

$$x(B_i \cap \Lambda) = x(B_i) \cap x(\Lambda) = B_j \cap \Lambda.$$

This implies that

$$|B_0 \cap \Lambda| = |B_i \cap \Lambda|$$

for every $0 \leq i < s$. Therefore

$$p^2 = |\Lambda| = \left|\cup_{i=0}^{s-1} (B_i \cap \Lambda)\right| = s\,|B_0 \cap \Lambda| = p\,|B_0 \cap \Lambda|\,.$$

This gives $|B_0 \cap \Lambda| = p$ so $|B_0|$ can only be p or 8 since $|B_0|\,t = 8p$ and both $|B_0|$ and $t \geq s$ are at least p.

If $|B_0| = p$, then Λ is the union of p elements of the A-invariant partition $\mathcal{B}$ and every orbit Λ' of P is an element of the partition $\mathcal{B}$ if $\Lambda' \neq \Lambda$. For every orbit $\Lambda' \neq \Lambda$ of P and for every $y \in \hat{\mathbb{Z}}_p \cup \mathring{\mathbb{Z}}_p$ we have $y(\Lambda') = \Lambda'$. In particular, $y(B_7) = B_7$. By Proposition 4.6 we may assume that there exists an element x' in $\hat{\mathbb{Z}}_p \cup \mathring{\mathbb{Z}}_p$ such that $x'(B_0) \neq B_j$ for some $j \neq 0, 7$ and clearly $x'(B_7) = B_7$. Since both $\hat{G}$ and $\mathring{G}$ are regular there exists $a \in C_A(x')$ such that $a(B_0) = B_7$. Since a and x' commute we have $a(B_j) = B_7$, which contradicts the fact that $a(B_0) = B_7$.

We must therefore have $|B_0| = 8$. Let $\hat{x}$ and $\mathring{x}$ generate $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$, respectively. Since $\hat{G}$ and $\mathring{G}$ are regular we have that neither $\hat{x}^{\mathcal{B}}$ nor $\mathring{x}^{\mathcal{B}}$ is the identity, so both $\hat{x}$ and $\mathring{x}$ are regular on $\mathcal{B}$. Since both $\hat{x}^{\mathcal{B}}$ and $\mathring{x}^{\mathcal{B}}$ generate a transitive subgroup in $Sym(\mathcal{B})$ of prime order $p > 2$, and every for $r \in \hat{R} \cup \mathring{R}$ the permutation $r^{\mathcal{B}}$ commutes with one of these two elements, we have $r^{\mathcal{B}} = id$. Since $\hat{x}$ and $\mathring{x}$ are in the same Sylow p-subgroup of P we may assume that $\hat{x}(B_i) = \mathring{x}(B_i) = B_{i+1}$ for $i = 0, 1, \ldots, p-1$, where the indices are taken modulo p. By Proposition 4.6 we may also assume that $\hat{x} \neq \mathring{x}$.

For every m there exists an l such that the action of $\hat{x}^l \mathring{x}^{-l}$ is nontrivial on B_m since $\hat{x} \neq \mathring{x}$. Therefore $A_{B_m}|_{B_m}$ contains a regular subgroup and a cycle of length p such that $p > \frac{|B_0|}{2}$. A theorem of Jordan on primitive permutation groups, which can also be found in [8, Theorem 13.1.], says that such a permutation group is 2-transitive and hence the induced subgraph of Γ on B_m is the complete or the empty graph for every m.

Lemma 5.2. $B_m \sim B_n$ *for* $0 \leq m < n \leq p-1$.

Proof. There exists a unique element $\hat{g} \in \hat{\mathbb{Z}}_p \leq P$ such that $\hat{g}(B_m) = B_n$. We also have a unique element $\mathring{g} \in \mathring{\mathbb{Z}}_p \leq P$ with $\hat{g}^{\mathcal{B}} = \mathring{g}^{\mathcal{B}}$. Since $\mathbb{Z}_p$ is a cyclic group of prime order and $\hat{x} \neq \mathring{x}$ we have $\hat{g} \neq \mathring{g}$. Moreover, we may also assume that $\hat{g}|_{B_m} \neq \mathring{g}|_{B_m}$ since $\hat{g} \neq \mathring{g}$ and the induced subgraphs of Γ on $B_{m+c} \cup B_{n+c}$ are all isomorphic, where both $m+c$ and $n+c$ are taken modulo p.

Clearly, $\tilde{g} = \mathring{g}\hat{g}^{-1}$ is a cycle of length p on B_n. The vertices of $V(\Gamma) \setminus \Lambda$ are contained in P-orbits of order p that contain the orbit of the vertex under x, so meet each B_i in a single vertex, so $\tilde{g}$ fixes every vertex of the set $B_m \cup B_n \setminus \Lambda$ since $\tilde{g}^{\mathcal{B}} = id$.

Let $u \in B_m \setminus \Lambda$. It is enough to show that if u is adjacent to some $v \in B_n$, then u is adjacent to every vertex of B_n. We will prove that A is transitive on the following pairs: $\{(u, w) \mid w \in B_n\}$.

A is transitive on $\{(u, w) \mid w \in B_n \cap supp(\tilde{g})\} = \{(u, w) \mid w \in B_n \cap \Lambda\}$ since $\tilde{g}$ fixes u. Therefore we may assume that $v \in B_n \setminus \Lambda$ and we only have to find an element $a \in A$ such that $a(u) = u$ and $a(v) \in B_n \cap \Lambda$.

The restriction of $\tilde{g}$ to B_n is a cycle of length p so $\tilde{g}$ does not commute with $\mathring{r}|_{B_n}$, where $\mathring{r}$ is an involution of $\mathring{R}$. Since $\mathring{r}$ and $\mathring{g}$ commute we have that there is a $u' \in B_m$ such that $\mathring{r}\hat{g}(u') \neq \hat{g}\mathring{r}(u')$. Since the action of $\hat{R}$ is transitive on B_m there exists $\hat{r} \in \hat{R}$ such that $\hat{r}(u) = u'$. Then

$$(\mathring{r}\hat{r})\, \hat{g}(u) = \mathring{r}\hat{g}\hat{r}(u) = \mathring{r}\hat{g}(u') \neq \hat{g}\mathring{r}(u') = \hat{g}\, (\mathring{r}\hat{r})\, (u)$$

so there exists $a' \in A$ such that

$$a'\hat{g}(u) \neq \hat{g}a'(u). \tag{5.1}$$

Let us suppose that $v = \hat{g}(u)$. Notice that $\hat{g}(u)$ is in a P-orbit of order p, so $\hat{g}(u) \notin \Lambda$. Then the inequality (5.1) gives $a'(v) \neq \hat{g}a'(u)$. Since $\hat{R}|_{B_m}$ is regular on B_m there exists $\hat{s} \in \hat{R}$ such that $\hat{s}(u) = a'(u)$ and since $\hat{s}$ and $\hat{g}$ commute we have $\hat{s}(v) = \hat{s}\hat{g}(u) = \hat{g}\hat{s}(u) = \hat{g}a'(u)$. Therefore $\hat{s}(v) \neq a'(v)$ and hence $\hat{s}^{-1}a'$ fixes u and $\hat{s}^{-1}a'(v) \neq v$ so we may assume that $v \neq \hat{g}(u)$.

If $p = 7$, then $v \in B_n \cap \Lambda$.

Let us assume that $p = 5$. We claim that there exists $\hat{t} \in \hat{R}$ such that $\hat{t}(u) \in B_m \setminus \Lambda = B_m \setminus supp(\tilde{g})$ while $\hat{t}(v) \in B_n \cap \Lambda = B_n \cap supp(\tilde{g})$. It is clear that $\hat{g}(B_m \cap supp(\tilde{g})) = B_n \cap supp(\tilde{g})$ and $\hat{g}$ commutes with each element of $\hat{R}$. Therefore it is enough to show that if $u, v \in B_m \setminus supp(\tilde{g})$ with $u \neq v$, then there exists $\hat{t} \in \hat{R}$ such that $\hat{t}(u) \in B_m \setminus supp(\tilde{g})$ and $\hat{t}(v) \in B_m \cap supp(\tilde{g})$. This can easily be seen from the fact that $\gcd(|\hat{R}|, 5) = 1$.

The permutations $\hat{t}^{-1}\tilde{g}^l\hat{t}$ fix the vertex u for every $0 \leq l \leq 4$ and $\hat{t}^{-1}\tilde{g}^{l_1}\hat{t}(v) \neq \hat{t}^{-1}\tilde{g}^{l_2}\hat{t}(v)$ if $l_1 \not\equiv l_2 \ (mod\, p)$. At least one of the the four elements $\hat{t}^{-1}\tilde{g}\hat{t}$, $\hat{t}^{-1}\tilde{g}^2\hat{t}$, $\hat{t}^{-1}\tilde{g}^3\hat{t}$, $\hat{t}^{-1}\tilde{g}^4\hat{t}$ of A fixes u and maps v to an element of $B_n \cap supp(\tilde{g}) = B_n \cap \Lambda$ since $|B_n \setminus supp(\tilde{g})| = 3$, finishing the proof of the fact that $B_m \sim B_n$ for $0 \leq m \neq n \leq 7$. $\square$

Every permutation of $V(\Gamma)$ which fixes B_m setwise for every m is an automorphism of Γ so there is an $a \in A$ such that $\mathring{x}^a = \hat{x}$. Applying Proposition 4.6 we get that there exists $\alpha \in A$ such that $\left(\hat{R} \times \hat{\mathbb{Z}}_p\right)^\alpha = \mathring{R} \times \mathring{\mathbb{Z}}_p$, finishing the proof of Proposition 5.1.

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ANNOUNCEMENT

PhD Fellowship of the National Institute of Chemistry for the studies of Mathematics in Synthetic Biology at the University of Primorska, Slovenia

National Institue of Chemistry and the University of Primorska announce a PhD scholarship in Mathematics in Synthetic Biology sponsored by the ERC grant BioOrigami under the supervision of

Professor Roman Jerala (Research Advisor and PhD Co-Advisor at the National Institue of Chemistry

Professor Tomaž Pisanski (PhD Advisor at the University of Primorska)

Applicants should have a BSc or equivalent training in Mathematics (by September 30, 2015) with interests in computer programming and synthetic biology. They are expected to enroll in the PhD program at the University of Primorska. The position is for 3 and 1/2 years and include a tuition fee waiver. The holder is expected to complete his/her PhD training in 4 years.
The deadline for applications is **August 15, 2015**. Applicants should send a letter of interest with CV and two recommendation letters to

" PhD position - Mathematics in Synthetic Biology "
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Glagoljaška 8,
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The application should also be sent electronically to the address

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