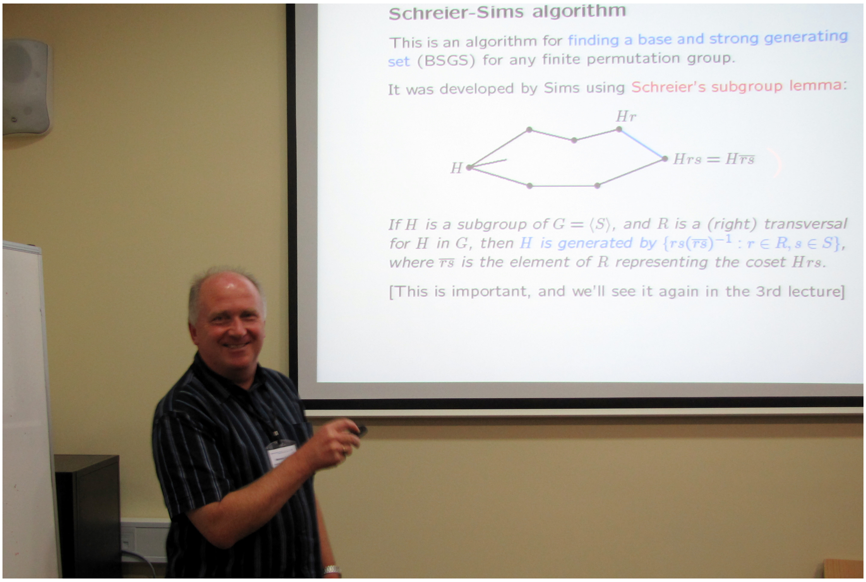
For Marston Conder's 65th Birthday

I have said many times that everything you do is dependent on who you know, and who you know is often a matter of luck. In my case, I must be very lucky to have had the chance to work with Marston. Of course, having a Conder-number of one isn't really a matter or luck, since there are 100 other mathematicians with Conder-number one.



I probably have known Marston as long as any of his many co-authors. I still have a handwritten letter I wrote to Graham Higman in 1982 asking about the work of one of his students on Hurwitz groups $\langle a, b : a^2 = b^3 = (ab)^7 = 1, \ldots \rangle$. I have it because Higman returned it to me saying his student would write me. And I got a typed (!) response from Marston summarizing his results, which I also still have. We arranged to meet at Tübingen on my first mathematical trip to Europe for an Oberwolfach meeting. We missed connecting on the first day, but we did get together the next day for a long walk and talk. It was then 16 years until we saw each other again in Flagstaff for the first SIGMAP workshop. But it was not until Marston's first New Zealand conference at Auckland in December 2000 that we started working together on regular Cayley maps, mostly with Robert Jajcay.

Marston has made up for those 16 years of not traveling by becoming a presence at combinatorics conferences throughout Europe, the US and the East (often bringing two or three bottles of New Zealand wine). Last March I was stunned that he was going to make the 24,000 mile round trip from New Zealand to Boston just for a 20-minute talk at a two-day AMS Sectional conference at Tufts (cancelled of course). His frequent traveler miles are near 2 million. When I ask him how he'll use them, he talks about a round-the-world trip with Jenny, but then I point out he actually needs to take ten such trips.

Working with Marston will always be one of the joys of my mathematical life. First, if it comes to permutation groups, graphs, polytopes, maps, dessins, Riemann surfaces, he's been there. And if he hasn't been there before, he picks it up immediately. Second, he also brings with him all the mathematical machinery from those fields. I remember his amusement when he managed to use most of the group theoretic techniques he had learned in graduate school in one paper: the 2011 *JEMS* paper with Jozef, which to this day is the paper I am proudest of (with no training in group theory, I was the bare-hands co-author).

Of course, when it comes to machinery, there is MAGMA. His files of all regular maps, hypermaps, symmetric and semi-symmetric cubic graphs, up to some given order, are a foundation for work in these fields. At one of Marston's Queenstown conferences, there was a short course on MAGMA. I got blown away in the first 10 minutes. Then I realized I didn't need MAGMA; I had Marston. So much easier to just email him. But there is something much more important about Marston's work with computers. In mathematics, it is always examples first, theorems second. I remember him saying once: "All-but-finitely-many results are nice, but I am interested in those finitely many." That is where the real work and beauty are. And that is Marston.

Tom Tucker

THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.01
https://doi.org/10.26493/2590-9770.1531.46a
(Also available at http://adam-journal.eu)

Hole operations on Hurwitz maps*

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Dedicated to Marston Conder in celebration of his 65th birthday

Received 14 October 2021, accepted 19 March 2022, published online 27 June 2022

Abstract

For a given group G the orientably regular maps with orientation-preserving automorphism group G are used as the vertices of a graph $\mathcal{O}(G)$, with undirected and directed edges showing the effect of duality and hole operations on these maps. Some examples of these graphs are given, including several for small Hurwitz groups. For some G, such as the affine groups $\mathrm{AGL}_1(2^e)$, the graph $\mathcal{O}(G)$ is connected, whereas for some other infinite families, such as the alternating and symmetric groups, the number of connected components is unbounded.

Keywords: Regular map, orientably regular, duality, hole operation, Hurwitz group.

Math. Subj. Class.: 05C10 (primary), 20B25 (secondary)

1 Introduction

For any group G let $\mathcal{O}(G)$ denote the set of (isomorphism classes of) orientably regular maps $\mathcal{M}$ with orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{M}$ isomorphic to G. For a given finite group G this set is finite since its elements correspond bijectively to the orbits of $\mathrm{Aut}\, G$ on certain generating sets for G. These maps are related to each other

*The authors thank the referees for a number of valuable comments.

†Corresponding author. Supported by the Hungarian National Research, Development and Innovation Office, OTKA grant No. SNN 132625.

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by the application of certain operations, described by Wilson in [40], such as duality D and the hole operations H_j for j coprime to the valency of the vertices, since all of these preserve the group G. One way of understanding how the various maps associated with G are related to each other is to regard this set $\mathcal{O}(G)$ as the vertex set of a graph, with directed or undirected edges indicating the actions of these operations. For example, one can consider whether or not this graph is connected, and if not, what invariants might be used to distinguish maps in different components. We will describe the maps $\mathcal{M}$ and graphs $\mathcal{O}(G)$ arising for some particularly interesting groups, including several of the smallest Hurwitz groups. We will show that for the symmetric and alternating groups, the number of connected components of $\mathcal{O}(G)$ is unbounded, building on work of Marston Conder [5] on alternating groups as Hurwitz groups in the latter case.

In addition to topological graph theory and the theory of Riemann surfaces, Grothendieck's dessins d'enfants provide motivation for this study, as a preparation for an investigation of the relationship between the actions of these operations and that of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ on finite oriented maps, regarded as algebraic curves defined over number fields (see [26], for example). We also see this paper as a first step towards a study of a wider class of operations, including Petrie duality, on the set $\mathcal{R}(G)$ of all regular maps with a given automorphism group G. Constraints of space and time oblige us to defer these extensions to a later date.

2 Orientably regular maps

If $\mathcal{M}$ is an orientably regular map then its arcs (directed edges) can be identified with the elements of a group G, so that its orientation-preserving monodromy and automorphism groups are identified with the right and left regular representations of G.

Let $\mathcal{M}$ have type $\{p, q\}$, meaning that its faces are p-gons and its vertices have valency q. (The term *Schläfli type*, or *Schläfli symbol*, is also used [10]). We note that later on in this paper, we shall use the same symbols p and q for denoting a prime and a prime power, respectively; to avoid confusion, where it is not clear from the context, the actual meaning of these symbols will be emphasized.)

Then G has generators x, y, z satisfying

$$x^q = y^2 = z^p = xyz = 1,$$

where x and y act (as monodromy permutations) by rotating all arcs around their incident vertices, following the orientation, and by reversing them, so that z rotates them around faces. Conversely any such generating triple for G defines a map $\mathcal{M}$ of this type, with isomorphic maps corresponding to triples equivalent under $\mathrm{Aut}\,G$. Of course the triple, and hence the map, is uniquely determined by any two of its members, so we will often restrict attention to the generating pair x, y.

Map duality D, transposing vertices and faces, replaces the triple (x, y, z) with $(z, y, (zy)^{-1} = x^y)$, corresponding to the dual map $D(\mathcal{M})$. Then D^2 sends (x, y, z) to $(x^y, y, z^y) = (x, y, z)^y$, giving a map isomorphic to $\mathcal{M}$.

If j is coprime to q then another orientably regular map $\mathcal{M}_j = H_j(\mathcal{M})$, of type $\{p', q\}$ for some p' and with the same orientation-preserving automorphism group G as $\mathcal{M}$, can be found by applying the hole operation H_j, described by Wilson in [40], to $\mathcal{M}$. This map embeds the same graph as $\mathcal{M}$, but the rotation x of arcs around each vertex is replaced with x^j, so that the faces, and hence the underlying surface, may be changed. In terms of group theory, $H_j(\mathcal{M})$ corresponds to the generating triple (x^j, y, z_j) for G, where $z_j := (x^j y)^{-1}$. Thus the boundary of each face of $H_j(\mathcal{M})$, given the orientation of the underlying surface, follows a j*-hole* of $\mathcal{M}$, at each vertex v taking the j-th incident edge, following the reverse orientation around v, rather than the first; its length is the order of the element z_j. (Note that a j-hole, taken in the reverse direction, becomes an $(q-j)$-hole.)

Example 2.1. Figure 1 shows a 3-hole $(16, 5, 6, 20, 16)$ of length 4 in Klein's map $\mathcal{K}$ of type $\{3, 7\}$ and genus 3 with $G = \mathrm{PSL}_2(7)$ (see Section 7 for details).

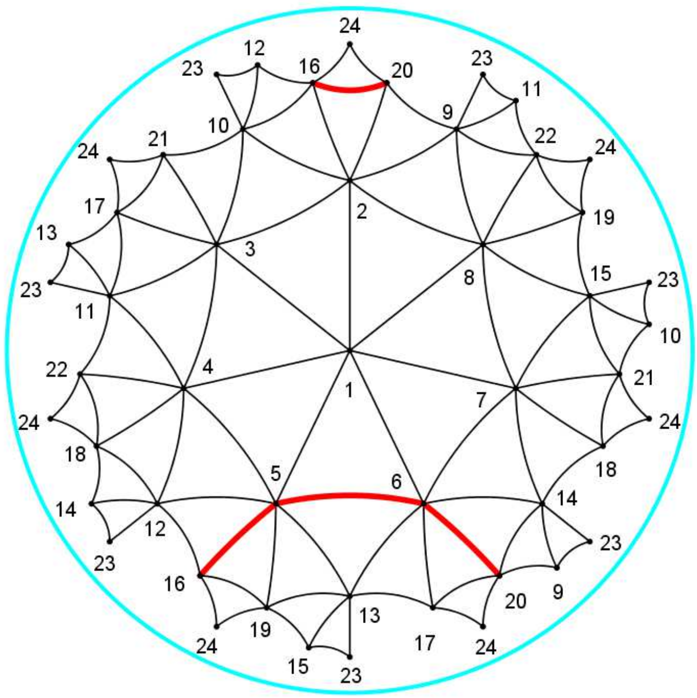

Figure 1: Klein's map $\mathcal{K}$ of type $\{3, 7\}$ and genus 3 represented in Poincaré's disc model of the hyperbolic plane. A 3-hole $(16, 5, 6, 20, 16)$ is highlighted.

Clearly $H_j = H_k$ whenever $j \equiv k \bmod (q)$; since $H_j \circ H_k = H_{jk}$ for all j and k coprime to q, these operations give a representation of the group U_q of units mod (q). The operation H_{-1} sends each map to its mirror-image, so an orientably regular map $\mathcal{M}$ is regular (has a flag-transitive full automorphism group $\mathrm{Aut}\,\mathcal{M}$) if and only if $H_{-1}(\mathcal{M}) \cong \mathcal{M}$; thus the action on regular maps gives a representation of the group $U_q/\{\pm 1\}$.

One can also apply H_j to $\mathcal{M}$ when $k := \gcd(j, q) > 1$. However, in this case the operation is not invertible, and it does not preserve the embedded graph, since each vertex is replaced with k vertices of valency q/k. Moreover, the orientation-preserving automorphism group may also change: if $G_0 := \langle x^j, y \rangle$ is a proper subgroup of index n in G, then $H_j(\mathcal{M})$ is the disjoint union of n isomorphic connected orientably regular maps $\mathcal{M}_0$, each with $\mathrm{Aut}^+\mathcal{M}_0 \cong G_0$, and $\mathrm{Aut}^+ H_j(\mathcal{M})$ is isomorphic to the wreath product $G_0 \wr \mathrm{S}_n$. Since our aim is to study those maps with a given orientation-preserving automorphism group G, we will avoid such complications by concentrating on cases where $k = 1$.

A further reason for doing this is that we eventually hope to compare hole operations with cyclotomic Galois operations, and these act by raising qth roots of unity to their jth powers where $\gcd(j, q) = 1$.

Another useful map operation is Petrie duality. A *Petrie polygon* in a map $\mathcal{M}$ is a closed zig-zag path turning alternately first right and first left at successive vertices. The *Petrie dual* $P(\mathcal{M})$ of $\mathcal{M}$ embeds the same graph as $\mathcal{M}$, but the faces of $\mathcal{M}$ are replaced with new faces bounded by the Petrie polygons. If $\mathcal{M}$ is orientably regular their length r (the *Petrie length* of $\mathcal{M}$) is twice the order of the commutator $[x, y] = x^{-1}y^{-1}xy = x^{-1}yxy$, so that $P(\mathcal{M})$ has type $\{r, q\}$. As in [11, 12], we will often refer to the *extended type* $\{p, q\}_r$ of $\mathcal{M}$. For instance, in Figure 1 the 3-holes $(16, 5, 6, 20, 16)$ and $(24, 17, 13, 19, 24)$ in $\mathcal{K}$ enclose a Petrie polygon of length $r = 8$. (More generally, whenever $p = 3$ each Petrie polygon is enclosed by two 3-holes of length $r/2$ in this way.)

Given any group G, one can regard $\mathcal{O}(G)$ as the vertex set of a graph (also denoted by $\mathcal{O}(G)$) by adding edges showing the actions of the operations D and H_j (for selected j); they will be undirected for cycles of length 2, e.g. for involutions such as D, and directed in the case of longer cycles for hole operations H_j. Loops, corresponding to invariant maps, will be omitted. In particular, we will use dashed and dotted undirected edges for D and H_{-1}, and unbroken edges for other hole operations. Although the operation P preserves the full automorphism group of a map, it has the disadvantage from our current point of view of not always preserving orientability. For example, if $\mathcal{M}$ is the tetrahedral map on the sphere, then $P(\mathcal{M})$ is the antipodal quotient of the cube, on the real projective plane; similarly, $P(\mathcal{K})$ is a non-orientable regular map of genus 41. In this paper we will therefore concentrate mainly on the operations D and H_j, which preserve automorphism groups and orientability.

In general we will not draw edges labelled H_j for all j coprime to the relevant valencies q, but merely enough to identify the connected components of $\mathcal{O}(G)$ by showing how various maps can be tranformed into each other. For example, if U_q (or $U_q/\{\pm 1\}$ in the case of regular maps) is cyclic it is sufficient to take j to be a generator.

Provided its genus g is neither too large nor less than 2, each map $\mathcal{M} \in \mathcal{O}(G)$ can be located in Conder's computer-generated lists of regular or chiral orientable maps in [7]. Each entry there has the form R$g.n$ or C$g.n$ respectively, where g denotes the genus, and n denotes the nth entry in the list of maps of that genus, ordered lexicographically by their type $\{p, q\}$, with $p \leq q$. Within each list, each entry refers either to a dual pair of maps of types $\{p, q\}$ and $\{q, p\}$ or to a single self-dual map with $p = q$. If $p < q$ we will denote the maps of type $\{p, q\}$ and $\{q, p\}$ by X$g.n$a and X$g.n$b respectively, where X is R or C; if $p = q$ and the entry denotes a non-isomorphic dual pair we will assign the labels X$g.n$a and X$g.n$b arbitrarily, whereas a single self-dual map will be denoted simply by X$g.n$. Similar conventions will apply to the list of non-orientable regular maps, denoted by N$g.n$ in [7].

2.1 Some simple examples of graphs $\mathcal{O}(G)$

If G is the alternating group A_4 then $\mathcal{O}(G)$ contains only the tetrahedral map $\{3, 3\}$ of genus 0.

If G is a dihedral group D_m for some $m > 2$ then $\mathcal{O}(G)$ consists of the dual pair of maps $\{2, m\}$ and $\{m, 2\}$ of genus 0, whereas if $m = 2$, so that $G = \mathrm{D}_2 \cong \mathrm{V}_4$, then $\mathcal{O}(G)$ contains only the self-dual map $\{2, 2\}$ of genus 0. In either case the operations H_j for j coprime to m act trivially on $\mathcal{O}(G)$.

If $G = \mathrm{S}_4$ then the involution y must be a transposition: the double transpositions lie in the normal subgroup $K \cong \mathrm{V}_4$ with $G/K \cong \mathrm{S}_3$ non-cyclic, so they cannot be members of generating pairs. One can take x to be any 3-cycle or 4-cycle not inverted by y, giving the cube $\{4,3\}$ or its dual, the octahedron $\{3,4\}$. The operations H_j act trivially on $\mathcal{O}(G)$.

The graphs $\mathcal{O}(G)$ corresponding to these groups G are shown in Figure 2.

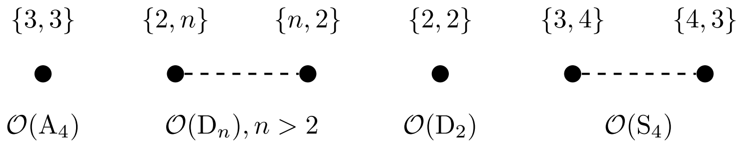

Figure 2: The graphs $\mathcal{O}(\mathrm{A}_4)$, $\mathcal{O}(\mathrm{D}_n)$ and $\mathcal{O}(\mathrm{S}_4)$

If G is the alternating group A_5 (also isomorphic to $\mathrm{PSL}_2(4)$ and $\mathrm{PSL}_2(5)$) then $\mathcal{O}(G)$ consists of three maps, namely the dodecahedron $\{5,3\}$, the icosahedron $\{3,5\}$ and the great dodecahedron $\{5,5\}_6$ of genus 4 (the self-dual map R4.6 in [7]); duality D transposes the first two, while H_2 transposes the last two, so $\mathcal{O}(G)$ is connected (see Figure 3).

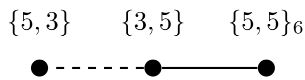

Figure 3: The graph $\mathcal{O}(\mathrm{A}_5)$

3 Isotactic polygons

Let $\mathcal{M}$ be an orientably regular map of type $\{p,q\}$. An *isotactic polygon* P of *type* $(d_1, d_2, \ldots, d_m)^n$ in $\mathcal{M}$ is a closed path in the underlying graph of $\mathcal{M}$ formed by successively taking the d_i-th edge at the i-th vertex v_i visited, following the local orientation around v_i, using the sequence $d_1, d_2, \ldots, d_m \in \mathbb{Z}_q$ repeated n times. We call d_i the *right degree* of P at v_i, and $l = mn$ the *length* of P. By orientable regularity, if such a polygon P exists, then it does so starting at any directed edge in $\mathcal{M}$, since it is equivalent to a relation

$$(x^{d_1} y x^{d_2} y \ldots x^{d_m} y)^n = 1$$

in the monodromy group $G = \langle x, y \rangle$ of $\mathcal{M}$.

This concept is a common generalisation of the classical notions of *face*, *Petrie polygon* and *hole*. In fact,

- an isotactic polygon of type $(1)^p$ or $(-1)^p$ is the boundary of a p-valent face;
- more generally, an isotactic polygon of type $(j)^l$ with j coprime to q is a j-hole of length l;
- an isotactic polygon of type $(1,-1)^{r/2}$ is a Petrie polygon of length r (provided this length is even);
- more generally, an isotactic polygon of type $(j,-j)^{r/2}$ is a Petrie polygon of order j, where r is its (even) length.

These well-known examples of isotactic polygons are distinguished by their roles in the Petrie and hole operations, but other types also occur in various roles. For example, hexagons of type $(2,4)^3$ occur in the polyhedral realisation of Klein's map $\mathcal{K}$ of genus 3 (see Section 7) given by Schulte and Wills in [31]. On the other hand, the heuristic role of hexagons of type $(2,4,4)^2$ is emphasised in the investigation of 7-fold rotational symmetry of the Fricke–Macbeath map $\mathcal{F}$ of genus 7 (see Section 8) in [3, 4]. Moreover, in this map the 3-holes, together with three suitable triangular faces, form generalised Petersen graphs of type $\mathrm{GP}(9,3)$ (see Figure 4); the presence of such subgraphs in the underlying graph of $\mathcal{F}$ confirms that this map has no polyhedral embedding in $\mathbb{E}^3$ with 9-fold symmetry [4].

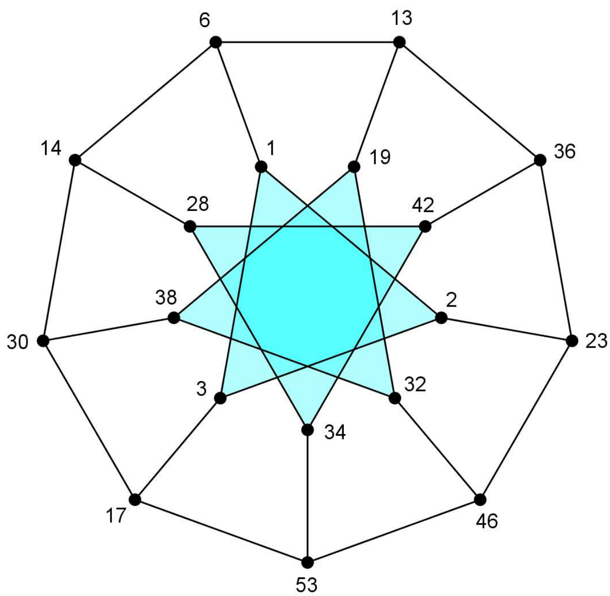

Figure 4: A generalized Petersen graph of type GP(9,3) in the Fricke–Macbeath map $\mathcal{F}$ of type $\{3,7\}$ and genus 7 (vertex labels correspond to those given in Figure 9 below).

4 Regularity

An orientably regular map $\mathcal{M}$, corresponding to a generating pair x, y of G, is regular if and only if there is an automorphism α of G inverting x and y, or equivalently inverting x and centralising y. In this case the full automorphism group $\mathrm{Aut}\,\mathcal{M}$ is a semidirect product $G \rtimes \langle\alpha\rangle$ of G and C_2, with α acting as a reflection of $\mathcal{M}$. For example, the maps in $\mathcal{O}(G)$ for $G = \mathrm{A}_4$, D_n and S_4 are all regular, with $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{S}_4$, $\mathrm{D}_n \times \mathrm{C}_2$ and $\mathrm{S}_4 \times \mathrm{C}_2$ respectively.

We will say that a regular map $\mathcal{M}$ is *inner* or *outer regular* as α is or is not an inner automorphism, that is, conjugation by some element $c \in G$. If α is inner then c^2 is in the centre $Z(G)$ of G, so if $Z(G) = 1$ (as is the case with all the groups considered here, apart from D_n for n even), then $c^2 = 1$ and $\mathrm{Aut}\,\mathcal{M} \cong G \times \mathrm{C}_2$ with $\alpha c = \alpha c^{-1}$ generating the second direct factor. In this case $\mathcal{M}$ has a non-orientable regular quotient map $\overline{\mathcal{M}} = \mathcal{M}/\mathrm{C}_2$, of genus $g+1$ where $\mathcal{M}$ has genus g, with automorphism group G; then $\mathcal{M}$ is the canonical orientable double cover of $\overline{\mathcal{M}}$.

It is easy to check that regularity, inner regularity and $\mathrm{Aut}\,\mathcal{M}$ are preserved by the operations D and H_j, so they are constant throughout each connected component of $\mathcal{O}(G)$. In particular, maps in $\mathcal{O}(G)$ with different regularity properties must lie in different connected components of the graph.

4.1 Non-orientable regular maps

Although the emphasis of this paper is on orientably regular maps $\mathcal{M}$ with a given group $\mathrm{Aut}^+\mathcal{M} \cong G$, non-orientable regular maps $\mathcal{N}$ with $\mathrm{Aut}\,\mathcal{N} \cong G$ will also be detected by the construction of $\mathcal{O}(G)$. (Note that if G has no subgroup of index 2, then every map in $\mathcal{R}(G)$ is non-orientable.) Each such map $\mathcal{N}$ has a canonical orientable double cover $\mathcal{M}$, an inner regular map with $\mathrm{Aut}\,\mathcal{M} \cong G \times Z$ where $Z \cong \mathrm{C}_2$, and with $\mathrm{Aut}^+\mathcal{M} \cong G$, so that $\mathcal{N}$ arises as the central quotient $\overline{\mathcal{M}} = \mathcal{M}/Z$ of a map $\mathcal{M} \in \mathcal{O}(G)$.

Conversely, each inner regular map $\mathcal{M} \in \mathcal{O}(G)$ has full automorphism group $\mathrm{Aut}\,\mathcal{M} = G \times Z$, with $Z \cong \mathrm{C}_2$ reversing orientation, so it has a non-orientable regular quotient $\mathcal{N} = \overline{\mathcal{M}} = \mathcal{M}/Z$ with automorphism group G. This gives a bijection $\mathcal{M} \mapsto \mathcal{N}$ between the subsets of $\mathcal{O}^\times(G) \subseteq \mathcal{O}(G)$ and $\mathcal{R}^-(G) \subseteq \mathcal{R}(G)$ consisting of their inner regular and non-orientable maps; here $\mathcal{N}$ has the same type as $\mathcal{M}$ (but half its Petrie length), and has genus $g+1$ if $\mathcal{M}$ has genus g. Since this bijection commutes with the operations D and H_j, it induces an isomorphism $\mathcal{O}^\times(G) \to \mathcal{R}^-(G)$ of directed graphs.

4.2 $\mathcal{O}(G)$ for $G = \mathrm{S}_5$

The earlier examples were very straightforward, involving familiar maps. This case needs rather more thought.

If $G = \mathrm{S}_5$ then the involution y can be a transposition or a double transposition. In the first case, there are possible generators x of orders 4, 5 and 6, giving rise to two dual pairs of regular maps, the dual pair R4.2 of types $\{4,5\}_6$ and $\{5,4\}_6$ on Bring's curve of genus 4 (see [38], for example), and the dual pair R9.16 of types $\{5,6\}_4$ and $\{6,5\}_4$, with H_2 transposing R4.2a and R9.16b; the component of $\mathcal{O}(G)$ containing these four maps is a path graph, with three edges labelled D, H_2, D.

In the second case x, which must be odd, can have order 4 or 6, giving the dual pair R6.2 of types $\{4,6\}_{10}$ and $\{6,4\}_{10}$, together with the self-dual map R11.5 of type $\{6,6\}_6$ (for this last map one can take $x = (1,5,3)(2,4), y = (1,2)(3,4)$); all operations H_j act trivially on these maps, so the pair R6.2 and the map R11.5 form two more connected components of $\mathcal{O}(G)$. The graph is shown in Figure 5: the vertices are labelled with the extended types and our extension of the notation in [7] for the corresponding maps. This example shows that the involution y (or more precisely its orbit under $\mathrm{Aut}\,G$) is insufficient to characterise a component of $\mathcal{O}(G)$.

The maps $\mathcal{M} \in \mathcal{O}(G)$ are all regular, and since $\mathrm{Out}\,\mathrm{S}_5$ and $Z(\mathrm{S}_5)$ are both trivial they are inner regular, with automorphism group $\mathrm{S}_5 \times \mathrm{C}_2$; their non-orientable quotients $\overline{\mathcal{M}}$, reading Figure 5 from left to right, are the dual pairs N5.1, N10.4, N7.1 and the self-dual map N12.3, all with automorphism group S_5.

We note that R6.2b occurs first as one of Coxeter's regular skew polyhedra [10]. It is realised as a subcomplex of the boundary complex of the dual of the 4-polytope $t_{1,2}\{3,3,3\}$. Its faces form the faces of 10 equal Archimedean truncated tetrahedra (the facets of the polytope). The non-identity element of C_2 is realised in this case as a central inversion (i.e. reflection in a point) in $\mathbb{E}^4$ interchanging two disjoint 10-tuples of hexagons.

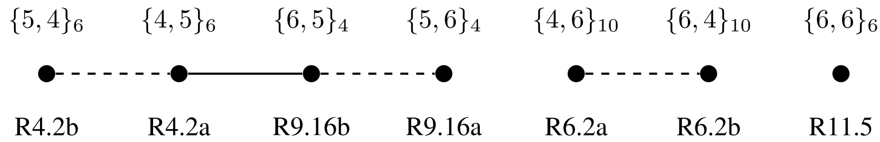

Figure 5: The graph $\mathcal{O}(\mathrm{S}_5)$.

Two triples of hexagons sharing a hole of length 3 belong to the same such 10-tuples; in Figure 6 both the hexagonal faces and the triangular holes can clearly be seen.

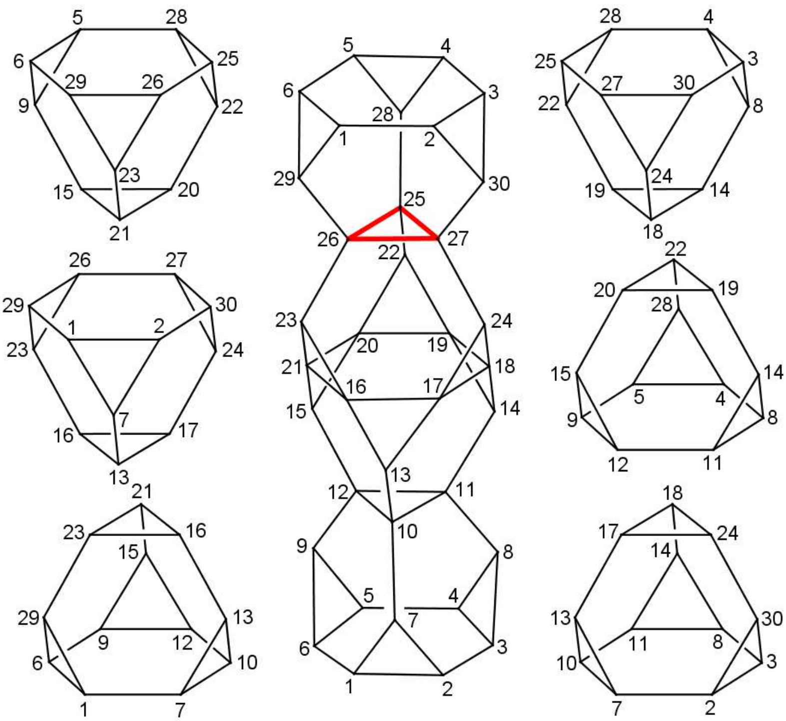

Figure 6: The combinatorial structure of R6.2b [18], with one hole of length 3 highlighted.

R4.2a goes back even earlier, in fact to Gordan's work [19]. Both the dual pairs R4.2a, R4.2b and R6.2a, R6.2b have polyhedral realisations in Euclidean 3-space [32, 33].

5 Hurwitz surfaces, groups and maps

Many of our chosen examples of automorphism groups are Hurwitz groups. This is partly because these groups exhibit interesting phenomena, and partly because they and their associated maps and surfaces have been intensively studied. Here we summarise some of their important properties.

Hurwitz [23] showed that the automorphism group $G = \mathrm{Aut}\,\mathcal{S}$ of a compact Riemann surface $\mathcal{S}$ of genus $g \geq 2$ has order at most $84(g-1)$, attained if and only if $\mathcal{S} \cong \mathbb{H}/K$, where $\mathbb{H}$ is the hyperbolic plane and K is a normal subgroup of finite index in the triangle group

$$\Delta = \Delta(7,2,3) = \langle X, Y, Z \mid X^7 = Y^2 = Z^3 = XYZ = 1\rangle$$

with $G \cong \Delta/K$. Such surfaces and groups attaining this upper bound are called ***Hurwitz surfaces*** and ***Hurwitz groups***. In each case, $\mathcal{S}$ carries an orientably regular map $\mathcal{M}$ of type $\{3,7\}$, called a ***Hurwitz map***, with orientation-preserving automorphism group G. This map is regular if and only if K is normal in the extended triangle group $\Delta[7,2,3]$ which contains Δ with index 2, in which case the full automorphism group of $\mathcal{M}$ is isomorphic to $\Delta[7,2,3]/K$; this is equivalent to G having an automorphism inverting two of its standard generators x, y and z. Conder has written some very useful surveys on Hurwitz groups in [6, 8].

Every Hurwitz group is perfect, since Δ is, so it is a covering of a non-abelian finite simple group, which is itself a Hurwitz group. Two important infinite classes of simple Hurwitz groups are given by the following theorems [30, 5]:

Theorem 5.1 (Macbeath, 1969)**.** *The group* $\mathrm{PSL}_2(\mathrm{q})$ *is a Hurwitz group if and only if*

(1) $q = 7$*, with a unique Hurwitz surface and map, or*

(2) $q = p$ *for some prime* $p \equiv \pm 1 \bmod (7)$*, with three Hurwitz surfaces and maps for each* q*, or*

(3) $q = p^3$ *for some prime* $p \equiv \pm 2$ *or* $\pm 3 \bmod (7)$*, with one Hurwitz surface and map for each* q*.*

(Note that here we use the standard group notation, so that q denotes a prime power.)

Theorem 5.2 (Conder, 1980)**.** *The alternating group* A_n *is a Hurwitz group for each* $n \geq 168$*.*

In [5], Conder also determined which alternating groups of degree $n < 168$ are Hurwitz groups; the smallest is A_{15}.

6 Maps with $G = \mathrm{PSL}_2(\mathrm{q})$

6.1 Properties of $\mathrm{PSL}_2(q)$

Later in this paper we will construct the graphs $\mathcal{O}(G)$ for some specific groups $G = \mathrm{PSL}_2(q)$. Here we briefly summarise a few of the properties of these groups which we will need; see [13, Ch. XII] or [22, II.8] for full details.

The group $G = \mathrm{PSL}_2(\mathrm{q}) = \mathrm{SL}_2(\mathrm{q})/\{\pm\mathrm{I}\}$, $q = p^e$, has order $q(q^2-1)$ or $q(q^2-1)/2$ as $p = 2$ or $p > 2$. It is simple for all $q \geq 4$.

One useful property of $G = \mathrm{PSL}_2(\mathrm{q})$ is that conjugacy classes of non-identity elements $g \in G$ are determined uniquely by their traces (more precisely, trace-pairs) $\pm\mathrm{tr}(g) \in \mathbb{F}_q$. For example, the elements of orders 2, 3 and p are the non-identity elements with traces 0, ± 1 and ± 2 respectively.

6.2 Consistent choice of y

For any group G, since the generator y is invariant (up to automorphisms) under the operations D and H_j, it makes sense to use the same element for each map in a given component of $\mathcal{O}(G)$, or even for each map in $\mathcal{O}(G)$ when this is possible.

For example, if $G = \mathrm{PSL}_2(q)$ then all involutions in G are conjugate to

$$y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(Here we identify each matrix $A \in \mathrm{SL}_2(q)$ with $-A$.) With this as y, its zero entries make it easier to see the traces of words such as $x^j y$ and $[x, y]$, and hence to find their orders, giving the extended types of various maps.

For instance, if y is as above and

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad xy = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}, \quad \text{so} \quad z = (xy)^{-1} = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix},$$

with trace $\pm(c - b)$. We will call $\tau = \pm(a + d)$ and $\tau' = \pm(c - b)$ the *trace* and *cotrace* of the element x and of the triple (x, y, z). If this triple generates G then it corresponds to a map $\mathcal{M} \in \mathcal{O}(G)$ of Schläfli type $\{p, q\}$, where the pair consisting of the trace $\pm(a + d)$ and cotrace $\pm(c - b)$ (unique up to the action of $\mathrm{Gal}\,\mathbb{F}_q$) determine the orders q and p of x and z and thus the type $\{p, q\}$ of $\mathcal{M}$, together with its genus

$$g = 1 + \frac{|G|}{2(\frac{1}{2} - \frac{1}{p} - \frac{1}{q})}$$

by the Riemann–Hurwitz formula. We also have

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}yxy = \begin{pmatrix} -b^2 - d^2 & ab + cd \\ ab + cd & -a^2 - c^2 \end{pmatrix}$$

with trace $\sigma = \pm(a^2 + b^2 + c^2 + d^2)$, giving the Petrie length r (twice the order of $[x, y]$) and hence the extended type $\{p, q\}_r$ of $\mathcal{M}$.

Once y is chosen, for example as above, all of this data for a map $\mathcal{M}$ is uniquely determined by the generating element x, so instead of working with triples (x, y, z) it is more efficient to work with the single elements x when applying operations such as D and H_j.

For example, if $\tau = \pm(a + d)$ is the trace of x then

$$x^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix}$$

has trace $\pm(a^2 + 2bc + d^2) = \pm(a^2 + 2(ad - 1) + d^2) = \pm(\tau^2 - 2)$, and

$$(x^2y)^{-1} = \begin{pmatrix} c(a + d) & -a^2 - bc \\ d^2 + bc & -b(a + d) \end{pmatrix}$$

has trace $\pm(a + d)(c - b) = \pm\tau\tau'$, giving the cotrace and hence the type and genus of $H_2(\mathcal{M})$. Similarly, its Petrie length is determined by the trace

$$\pm((a^2 + bc)^2 + (b(a + d))^2 + (c(a + d))^2 + (d^2 + bc)^2)$$

of $[x^2, y]$. Applying H_{-1} or D is achieved by replacing x with

$$x^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad z = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix}.$$

Of course, these leave r and g invariant, while D transposes p and q.

By an observation of Singerman [35], if $G = \mathrm{PSL}_2(q)$ then all maps $\mathcal{M} \in \mathcal{O}(G)$ are regular, so $H_{-1}(\mathcal{M}) \cong \mathcal{M}$ and there is no need to apply H_{-1}. However, this allows us, if we wish, to replace x with

$$z^{-1} = xy = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$$

instead of z when applying D.

Of course, many of the above expressions for matrices and traces are a little simpler if q is a power of 2.

7 $G = \mathrm{PSL}_2(7)$

There is a unique Hurwitz surface of genus 3, Klein's quartic curve (see [27, 28]) given in projective coordinates by $x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = 0$. The corresponding Hurwitz map $\mathcal{K}$ of type $\{3, 7\}$, *Klein's map* (see Figure 1), has orientation-preserving automorphism group $G \cong \mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$, a simple group of order 168, and has full automorphism group $\mathrm{PGL}_2(7)$. This is the map R7.1 in [7], discussed as a polyhedron by Schulte and Wills in [31].

The non-identity elements of $\mathrm{PSL}_2(7)$ have trace $0, \pm1, \pm2$ or ±3 as they have order 2, 3, 7 or 4 respectively. For a Hurwitz triple we require x and z to have orders 7 and 3, that is, $\tau = \pm(a + d) = \pm2$, $\tau' = \pm(c - b) = \pm1$ and of course $ad - bc = 1$, so we may take $a = 1$, $b = 1$, $c = 0$, $d = 1$; the resulting generating triple for $\mathcal{K}$ is

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since $\sigma = a^2 + b^2 + c^2 + d^2 = 3$, $[x, y]$ has order 4, so $\mathcal{K}$ has Petrie length 8 and extended type $\{3, 7\}_8$.

Squaring x to apply H_2, we obtain

$$x^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

corresponding to the map $H_2(\mathcal{K})$, which has extended type $\{7, 7\}_6$ and genus 19. Squaring again, we obtain

$$x^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},$$

corresponding to the map $H_4(\mathcal{K}) = H_{-3}(\mathcal{K}) = H_3(\mathcal{K})$ of type $\{4, 7\}_8$ and genus 10. Since $2^3 \equiv 1 \bmod (7)$, applying H_2 again simply returns us to $\mathcal{K}$.

For $D(\mathcal{K})$, of type $\{7, 3\}_8$ and genus 3, we can replace x with

$$z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

and for $D(H_4(\mathcal{K}))$, of type $\{7,4\}_8$ and genus 10, we can replace x^4 with

$$\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

However, $H_2(\mathcal{K})$ is self-dual, that is, $DH_2(\mathcal{K}) \cong H_2(\mathcal{K})$; this follows immediately from the following result:

Proposition 7.1. *There is a unique orientably regular map of type* $\{7,7\}$ *with orientation-preserving automorphism group* $G \cong \mathrm{PSL}_2(7)$.

Proof. Any such map $\mathcal{M}$ corresponds to a generating triple (x,y,z) for G of type $(7,2,7)$. We will use the Frobenius triple-counting formula [17], which states that if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are conjugacy classes in any finite group G, then the number of triples $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ with $xyz = 1$ is equal to

$$\frac{|\mathcal{X}|\cdot|\mathcal{Y}|\cdot|\mathcal{Z}|}{|G|}\sum_{\chi}\frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}, \tag{7.1}$$

where the sum is over all irreducible complex characters χ of G.

In ATLAS notation [9], the conjugacy classes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ of $G = \mathrm{PSL}_2(7)$ containing x, y and z are respectively 7A or 7B, 2A, and 7A or 7B. In all cases

$$\frac{|\mathcal{X}|\cdot|\mathcal{Y}|\cdot|\mathcal{Z}|}{|G|} = \frac{(2^3\cdot 3)\cdot(3\cdot 7)\cdot(2^3\cdot 3)}{2^3\cdot 3\cdot 7} = 2^3\cdot 3^2 = 72.$$

If $\mathcal{X} \neq \mathcal{Z}$ then by using the character values in [9] we find that the character sum in (7.1) is

$$1 + 2\cdot\frac{(-1)\cdot(-1+i\sqrt{7})/2\cdot(-1-i\sqrt{7})/2}{3} + \frac{(-1)\cdot 2\cdot(-1)}{6} = 0.$$

Thus there are no such triples in G and hence there are no corresponding maps. If $\mathcal{X} = \mathcal{Z} =$ 7A or 7B then the character sum is

$$1 + \frac{(-1)\cdot((-1+i\sqrt{7})/2)^2}{3} + \frac{(-1)\cdot((-1-i\sqrt{7})/2)^2}{3} + \frac{(-1)\cdot 2\cdot(-1)}{6} = \frac{7}{3},$$

so, adding these contributions, the total number of such triples is

$$2\cdot 72\cdot\frac{7}{3} = 336.$$

These triples all generate G, since no proper subgroup of G has order divisible by 14. The triples are permuted by $\mathrm{Aut}\,G = \mathrm{PGL}_2(7)$, semi-regularly since only the identity automorphism can fix a generating set. Since $|\mathrm{PGL}_2(7)| = 336$, $\mathrm{Aut}\,G$ is transitive on the triples, so $\Delta(7,2,7)$ has one normal subgroup with quotient G, and hence there is one orientably regular map of type $\{7,7\}$ with orientation-preserving automorphism group G. □

By the Riemann–Hurwitz formula, the above map $\mathcal{M}$ has genus 19. By its uniqueness, it is regular and self-dual, and is isomorphic to $H_2(\mathcal{K})$ and to the map R19.23 of extended

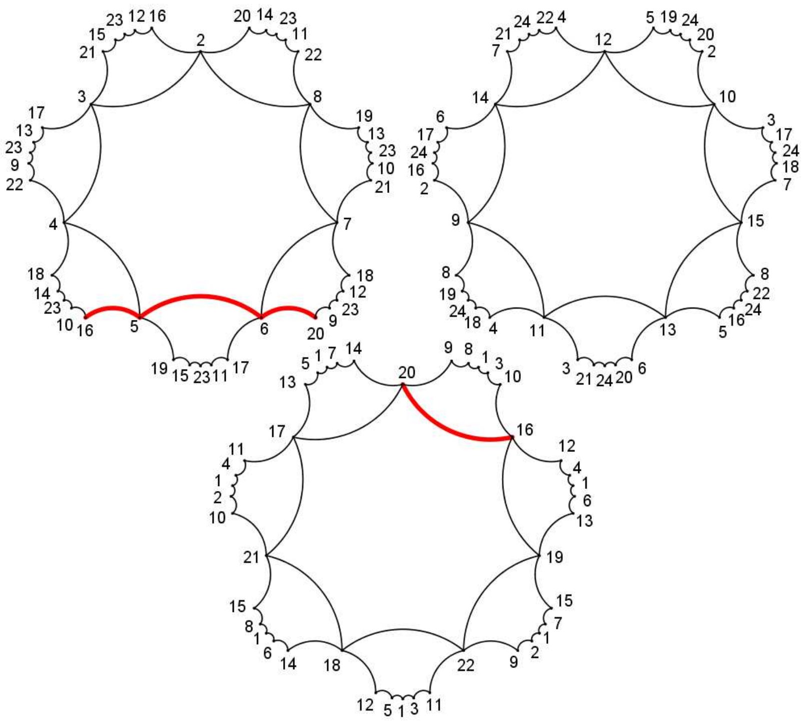

Figure 7: The map $H_2(\mathcal{K})$ of type $\{7,7\}_6$ and genus 19. It is trisected, and each part is represented in the disc model of the hyperbolic plane. A 2-hole is highlighted; this is the same as the 3-hole in $\mathcal{K}$ highlighted in Figure 1.

type $\{7,7\}_6$, the only regular orientable map of this genus and type in [7]. This map is shown in Figure 7.

Our other claims for uniqueness of maps, or enumerations of maps of a given genus and type, can be proved in a similar way. However, in future we will not give such full details unless there are special aspects of the calculation which need to be mentioned.

In fact the maps $\mathcal{K}$, $D(\mathcal{K})$, $H_2(\mathcal{K})$, $H_4(K)$ and $DH_4(\mathcal{K})$ are the only maps in $\mathcal{O}(G)$. To see this, note that the face- and vertex-valencies p and q for any such map must be orders of non-identity elements of G, so they must take the values $2, 3, 4$ or 7. Since $p^{-1} + q^{-1} < 1/2$ the only possible types (up to duality) are $\{3,7\}$, $\{7,3\}$, $\{4,7\}$, $\{7,4\}$ and $\{7,7\}$; as in the case of $\{7,7\}$ it can be verified by using the Frobenius triple-counting formula (or by inspection of [7]) that there is only one map of each type in $\mathcal{O}(G)$.

The graph $\mathcal{O}(G)$ is therefore as shown in Figure 8, where the actions of D and H_2 are represented by undirected broken and directed unbroken edges; that of H_3 is given by reversing the directed edges for H_2.

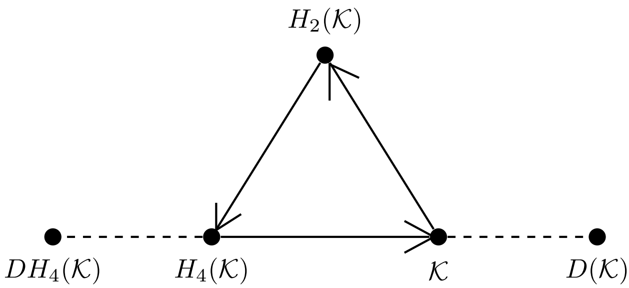

Figure 8: The graph $\mathcal{O}(\mathrm{PSL}_2(7))$

In Table 1 we list the maps $\mathcal{M} \in \mathcal{O}(\mathrm{PSL}_2(7))$. The first column shows the corresponding entry in [7], with letters 'a' and 'b' assigned as explained earlier. The second column gives the extended type $\{p, q\}_r$ of $\mathcal{M}$, with p, q and r denoting the face- and vertex-valencies and the Petrie length. The third column shows how $\mathcal{M}$ may be obtained from Klein's map $\mathcal{K}$ by applying duality and hole operations. The final column gives the effect of applying the hole operation H_2 to $\mathcal{M}$; this is left blank if $\mathcal{M}$ is unchanged (as when $q = 3$, for example) or q is even. Of course, duality D transposes pairs R$g.n$a and R$g.n$b, while leaving maps R$g.n$ invariant. Since the graph $\mathcal{O}(G)$ is connected, and $\mathcal{K}$ is regular with full automorphism group $\mathrm{PGL}_2(7)$, the same applies to every map $\mathcal{M}$ in $\mathcal{O}(G)$. These maps are all outer regular, so we obtain no non-orientable regular maps with automorphism group G.

Entry in [7]	Type	Relationship to $\mathcal{K}$	$H_2(\mathcal{M})$
R3.1a	$\{3,7\}_8$	$\mathcal{K}$	R19.23
R3.1b	$\{7,3\}_8$	$D(\mathcal{K})$	
R10.9a	$\{4,7\}_8$	$H_4(\mathcal{K})$	R3.1a
R10.9b	$\{7,4\}_8$	$DH_4(\mathcal{K})$	
R19.23	$\{7,7\}_6$	$H_2(\mathcal{K})$	R10.9a

Table 1: The maps $\mathcal{M}$ in $\mathcal{O}(\mathrm{PSL}_2(7))$.

Example 7.2. Taking subscripts mod (7) and using regularity, we see that

$$H_3(\mathcal{K}) \cong H_{-3}(\mathcal{K}) \cong H_4(\mathcal{K}) \cong H_2^2(\mathcal{K}) \cong \mathrm{R10.9a},$$

of type $\{4, 7\}$; Figure 1 confirms this, showing that the 3-holes of $\mathcal{K}$, giving the faces of $H_3(\mathcal{K})$, have length 4.

8 $G = \mathbf{PSL_2(8) = SL_2(8)}$

There is a unique Hurwitz surface $\mathcal{S}$ of genus 7, described by Fricke [16] in 1899 and rediscovered by Macbeath [29] in 1965; its automorphism group G, isomorphic to the simple group $\mathrm{PSL}_2(8) = \mathrm{SL}_2(8)$ of order 504, is the orientation-preserving automorphism group of a regular map $\mathcal{F}$ of type $\{3, 7\}$ on $\mathcal{S}$, the *Hurwitz map* of genus 7, with full automorphism group $G \times \mathrm{C}_2$.

The combinatorial structure of this map is shown in Figure 9. We note that because of the large size of the map, this representation is more suitable for studying its geometric properties, in comparison with the examples in Figures 1 and 7 where Poincaré's disc model is used. For other representations, including various topological embeddings by Carlo Sequin and by Jarke van Wijk, as well as for topological embeddings of regular maps of large genus given by Polthier and Razafindrazaka, see [3] and the references therein. For polyhedral realisations of the Hurwitz surface $\mathcal{S}$ of genus 7, see [2, 3, 4]. A detailed summary of all the known polyhedral realizations of regular maps of genus $g \geq 2$ can be found in [4].

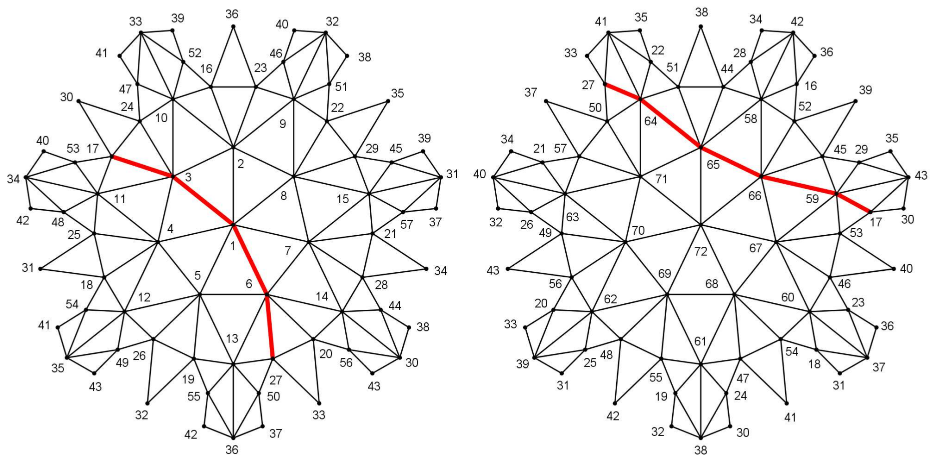

Figure 9: Combinatorial scheme of the Hurwitz map $\mathcal{F}$ of genus 7 (with vertex labels taken from [4]). A 3-hole is highlighted.

8.1 The group G

In order to work with this group G, let us represent the field $\mathbb{F}_8$ of order 8 as $\mathbb{F}_2[t]/(t^3 + t + 1)$, with its elements represented as polynomials in $\mathbb{F}_2[t]$ of degree at most 2. Then the multiplicative group of $\mathbb{F}_8$ consists of

$$t,\ t^2,\ t^3 = t + 1,\ t^4 = t^2 + t,\ t^5 = t^2 + t + 1,\ t^6 = t^2 + 1,\ t^7 = 1.$$

The non-identity elements of G form the following conjugacy classes:

- one class of elements of order 2, with trace 0;
- one class of elements of order 3, with trace 1;
- three classes of elements of order 7, with traces $t + 1, t^2 + 1, t^2 + t + 1$;
- three classes of elements of order 9, with traces $t, t^2, t^2 + t$.

Each class is inverse-closed. The outer automorphism group of G, isomorphic to C_3, is induced by the Galois group of $\mathbb{F}_8$ which is generated by the Frobenius automorphism $t \mapsto t^2$. This group permutes the three classes of elements of order 7 in a single cycle, and likewise for those of order 9.

Since $\mathrm{Out}\, G$ has odd order, and $Z(G)$ is trivial, all maps in $\mathcal{O}(G)$ are inner regular, with full automorphism group $G \times \mathrm{C}_2$.

8.2 Construction of $\mathcal{O}(G)$

Since all involutions in G are conjugate, in forming $\mathcal{O}(G)$ we can take

$$y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

using the fact that $\mathbb{F}_8$ has characteristic 2 to eliminate minus signs. If

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad z = \begin{pmatrix} c & a \\ d & b \end{pmatrix},$$

with the trace $\tau = a + d$ and the cotrace $\tau' = b + c$ giving the type and hence the genus of $\mathcal{M}$. Also the trace $\sigma = a^2 + b^2 + c^2 + d^2$ of $[x, y]$ gives its Petrie length and hence its extended type. Then

$$x^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

has trace $a^2 + d^2 = \tau^2$, and the trace of

$$(x^2 y)^{-1} = \begin{pmatrix} c(a+d) & a^2 + bc \\ d^2 + bc & b(a+d) \end{pmatrix}$$

gives the cotrace $(a+d)(b+c) = \tau\tau'$, giving the type and genus of $H_2(\mathcal{M})$. Similarly, its Petrie length is determined by the trace

$$(a^2 + bc)^2 + (b(a+d))^2 + (c(a+d))^2 + (d^2 + bc)^2$$

of $[x^2, y]$. Applying H_{-1} or D is achieved by replacing x with

$$x^{-1} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \quad \text{or} \quad z = \begin{pmatrix} c & a \\ d & b \end{pmatrix}.$$

Alternatively, one can replace x with

$$z^{-1} = xy = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

instead of z when applying D.

The Frobenius triple-counting formula shows that $\mathcal{F}$ is the only map of type $\{3, 7\}$ in $\mathcal{O}(G)$. For x to correspond to such a map we require $ad + bc = 1$, $a + d = t + 1$, $t^2 + 1$ or $t^2 + t + 1$ and $b + c = 1$. Without loss of generality we can use the solution

$$a = t, \quad b = 1, \quad c = 0, \quad d = t^2 + 1,$$

so that $\mathcal{F}$ is represented by the matrix

$$x = \begin{pmatrix} t & 1 \\ 0 & t^2+1 \end{pmatrix}$$

with trace $\tau = t^2+t+1$ and cotrace $\tau' = 1$, confirming that the type is $\{3,7\}$ and the genus is 7. Moreover, since $\sigma = t^2+1^2+0^2+(t^2+1)^2 = t$, the Petrie length is 18, so the extended type is $\{3,7\}_{18}$. This is therefore the map R7.1a in [7]. The dual map $D(\mathcal{F}) =$ R7.1b of extended type $\{7,3\}_{18}$ is represented by the matrix

$$z = \begin{pmatrix} 0 & t \\ t^2+1 & 1 \end{pmatrix}.$$

The map $H_2(\mathcal{F})$ is represented by the matrix

$$\begin{pmatrix} t^2 & t^2+t+1 \\ 0 & t^2+t+1 \end{pmatrix}$$

with trace $\tau^2 = t+1$, cotrace $\tau\tau' = t^2+t+1$ and sum of squares t^2+t, so it has extended type $\{7,7\}_{18}$ and genus 55. It must therefore be one of the dual pair R55.32 in [7]; we will denote it by R55.32a, and its dual map $DH_2(\mathcal{F})$, corresponding to the matrix

$$\begin{pmatrix} 0 & t^2 \\ t^2+t+1 & t^2+t+1 \end{pmatrix}$$

and also of type $\{7,7\}_{18}$, by R55.32b. (See Proposition 8.1 for a proof that there are just two maps of type $\{7,7\}$ in $\mathcal{O}(G)$; the fact that the matrix x^2 for $H_2(\mathcal{F})$ has distinct trace and cotrace shows that they are not self-dual.)

Iterating this process, we find that $H_2^2(\mathcal{F}) = H_4(\mathcal{F}) = H_3(\mathcal{F})$ is represented by the matrix

$$\begin{pmatrix} t^2+t & t \\ 0 & t+1 \end{pmatrix}$$

with trace t^2+1, cotrace t and sum of squares t^2+1, so it has extended type $\{9,7\}_{14}$ and genus 63. It is therefore either R63.6b or R63.7b, since R63.6 and R63.7 are the only entries of this genus and extended type in [7]. One can verify that it is R63.7b by checking that the matrices x (given above) and z corresponding to this map satisfy the defining relations for the full automorphism group of R63.7 given in [7], but not those for R63.6. Specifically, the defining relations for R63.7, with generators R and S corresponding to our x and z, and a third orientation-reversing generator T inverting them both, are given as

$$R^{-7} = S^{-9} = (RS)^2 = (S^{-1}R)^3 = (RS^{-3}R^2)^2 = 1,$$

$$T^2 = (RT)^2 = (ST)^2 = 1.$$

One can check that the matrices

$$x = \begin{pmatrix} t^2+t & t \\ 0 & t+1 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} 0 & t^2+t \\ t+1 & t \end{pmatrix}$$

corresponding to $H_3(\mathcal{F})$ satisfy these relations when substituted for R and S.

For example,

$$z^{-1}x = \begin{pmatrix} t & t^2+t \\ t+1 & 0 \end{pmatrix} \begin{pmatrix} t^2+t & t \\ 0 & t+1 \end{pmatrix} = \begin{pmatrix} t^2+t+1 & t^2+1 \\ 1 & t^2+t \end{pmatrix}$$

has trace 1, so $(z^{-1}x)^3 = 1$. On the other hand, the relations for R65.6 include $(RS^{-2}R)^2 = 1$, and since

$$xz^{-2}x = \begin{pmatrix} t^2+1 & t \\ t^2+t & t^2+t+1 \end{pmatrix}$$

has trace $t \neq 0$ we have $(xz^{-2}x)^2 \neq 1$. Thus $H_3(\mathcal{F})$ is R63.7b.

Proposition 8.1. *There are two orientably regular maps of type $\{7,7\}$ with orientation-preserving automorphism group $G = \mathrm{SL}_2(8)$. They form a dual pair.*

Proof. We will use the Frobenius triple-counting formula (7.1), as in the proof of Proposition 7.1, to count triples of type $(7,2,7)$ in G. There are three choices for the conjugacy classes $\mathcal{X}$, $\mathcal{Z}$ of elements of order 7 in G, each containing $2^3 \cdot 3^2$ elements, and one class $\mathcal{Y}$ of $3^2 \cdot 7$ involutions. Using the character values in [9], we find that for each of the six choices of $\mathcal{X} \neq \mathcal{Z}$ the number of triples (x,y,z) of type $(7,2,7)$ in G, with $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, is

$$\begin{aligned} &\frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \left(1 + \frac{1}{9} \sum_{j=0}^{2} (\zeta^{2^j} + \zeta^{-2^j})(\zeta^{2^{j+1}} + \zeta^{-2^{j+1}}) \right) \\ &= \frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \cdot \frac{7}{9} = 2^3 \cdot 3^2 \cdot 7, \end{aligned}$$

where $\zeta = \exp(2\pi i/7)$, so the total number of such triples is

$$2^4 \cdot 3^3 \cdot 7 = 2|\mathrm{Aut}\, G|.$$

These triples all generate G, since no maximal subgroup of G contains such a triple with non-conjugate x and z, so there are two corresponding maps.

There are also three choices of classes $\mathcal{X} = \mathcal{Z}$, and the total number of corresponding triples in G is

$$\begin{aligned} &3 \cdot \frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \left(1 + \frac{1}{9} \sum_{j=0}^{2} (\zeta^{2^j} + \zeta^{-2^j})^2 \right) \\ &= 3 \cdot \frac{(2^3 \cdot 3^2) \cdot (3^2 \cdot 7) \cdot (2^3 \cdot 3^2)}{2^3 \cdot 3^2 \cdot 7} \cdot \frac{14}{9} = 2^4 \cdot 3^3 \cdot 7. \end{aligned}$$

Now G has nine Sylow 2-subgroups T, each with normaliser $N_G(T) \cong \mathrm{AGL}_1(8) \cong \mathrm{V}_8 \rtimes \mathrm{C}_7$ generated by $48 \cdot 7$ triples of type $(7,2,7)$, all with conjugate x and z (see Section 10 for details). Since

$$9 \cdot 48 \cdot 7 = 2^4 \cdot 3^3 \cdot 7,$$

these account for all the triples of type $(7,2,7)$ in G with x and z conjugate; thus no such triples generate G, so they do not correspond to maps in $\mathcal{O}(G)$.

We have shown that there are two maps of type $\{7,7\}$ in $\mathcal{O}(G)$. One map corresponds to the pair $(t+1, t^2+1)$ of traces for x and z, together with its Galois conjugates (t^2+1, t^2+t+1) and $(t^2+t+1, t+1)$, while the other map corresponds to the pairs $(t+1, t^2+t+1)$, $(t^2+1, t+1)$ and (t^2+t+1, t^2+1). Since the first three pairs differ from the last three by transposition of x and z, these two maps form a dual pair. □

8.3 Summary of results for $\mathcal{O}(G)$

Continuing in this way, using a similar criterion to identify the other maps of extended type $\{9,7\}_{14}$ or $\{7,9\}_{14}$, one eventually finds that there are fourteen maps $\mathcal{M} \in \mathcal{O}(G)$; they are described in Table 2, where the first four columns are analogues of those in Table 1 for $\mathrm{PSL}_2(7)$, with the Fricke–Macbeath map $\mathcal{F}$ replacing Klein's map $\mathcal{K}$. The final column gives the entries in [7] corresponding to the non-orientable quotients $\mathcal{M}/\mathrm{C}_2$ by the centre C_2 of the full automorphism group $G \times \mathrm{C}_2$ of $\mathcal{M}$ (see Section 8.4).

Entry in [7]	Type	Relationship to $\mathcal{F}$	$H_2(\mathcal{M})$	$\mathcal{M}/\mathrm{C}_2$
R7.1a	$\{3,7\}_{18}$	$\mathcal{F}$	R55.32a	N8.1a
R7.1b	$\{7,3\}_{18}$	$D(\mathcal{F})$		N8.1b
R15.1a	$\{3,9\}_{14}$	$H_2DH_3(\mathcal{F})$	R71.15a	N16.1a
R15.1b	$\{9,3\}_{14}$	$DH_2DH_3(\mathcal{F})$		N16.1b
R55.32a	$\{7,7\}_{18}$	$H_2(\mathcal{F})$	R63.7b	N56.5a
R55.32b	$\{7,7\}_{18}$	$DH_2(\mathcal{F})$	R63.5b	N56.5b
R63.5a	$\{7,9\}_{6}$	$DH_2DH_2(\mathcal{F})$	R71.15b	N64.3a
R63.5b	$\{9,7\}_{6}$	$H_2DH_2(\mathcal{F})$	R63.6b	N64.3b
R63.6a	$\{7,9\}_{14}$	$DH_3DH_2(\mathcal{F})$	R63.5a	N64.4a
R63.6b	$\{9,7\}_{14}$	$H_3DH_2(\mathcal{F})$	R55.32b	N64.4b
R63.7a	$\{7,9\}_{14}$	$DH_3(\mathcal{F})$	R15.1a	N64.5a
R63.7b	$\{9,7\}_{14}$	$H_3(\mathcal{F}) = H_2^2(\mathcal{F})$	R7.1a	N64.5b
R71.15a	$\{9,9\}_{18}$	$H_4DH_3(\mathcal{F})$	R63.7a	N72.9a
R71.15b	$\{9,9\}_{18}$	$DH_4DH_3(\mathcal{F})$	R63.6a	N72.9b

Table 2: The maps in $\mathcal{O}(\mathrm{SL}_2(8))$.

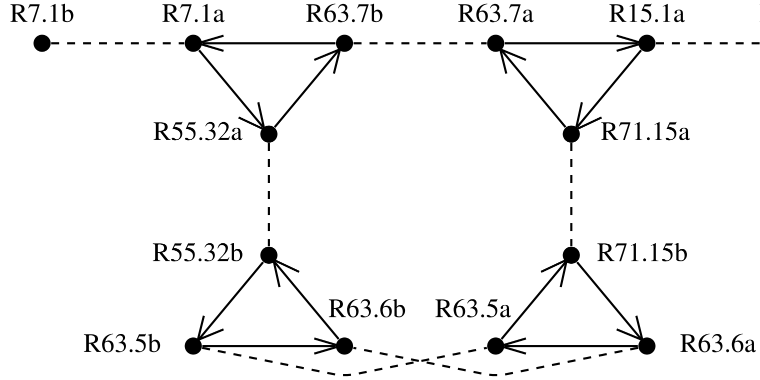

Figure 10: The graph $\mathcal{O}(\mathrm{SL}_2(8))$, with names of maps.

Figure 10 shows the graph $\mathcal{O}(G)$, with broken and unbroken edges representing the actions of D and H_2. The bilateral symmetry reveals an interesting duality between the parameters 7 and 9 appearing in the types of the maps in $\mathcal{O}(G)$ (see Figure 11). However, this does not extend consistently to the Petrie lengths: for example, R7.1b of extended type $\{7,3\}_{18}$ is paired with R15.1b of extended type $\{9,3\}_{14}$, whereas R63.7b, of extended type $\{9,7\}_{14}$, is paired with R63.7a, of extended type $\{7,9\}_{14}$.

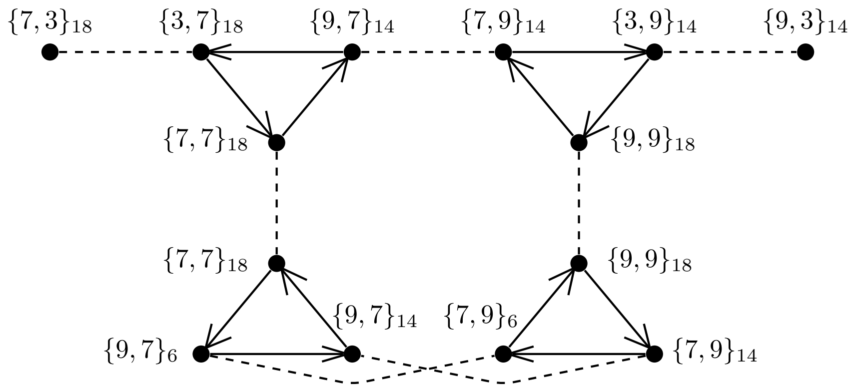

Figure 11: The graph $\mathcal{O}(\mathrm{SL}_2(8))$, with types of maps.

Example 8.2. $H_3(\mathcal{F}) = H_2^2(\mathcal{F}) \cong$ R63.7b, of type $\{9,7\}$; this is confirmed by Figure 9, which shows that the 3-holes of $\mathcal{F}$ have length $4+5=9$.

Note that for this group G an operation H_j with $\gcd(j,q) > 1$ also transforms some maps in $\mathcal{O}(G)$ into others in $\mathcal{O}(G)$, namely H_3 for maps with $q=9$, since one can check that if $x^9 = 1$ and $\langle x, y\rangle = G$ then $\langle x^3, y\rangle = G$ also. However, since $\mathcal{O}(G)$ is already connected, adding the resulting extra directed edges to the graph would not change its connectivity properties.

8.4 Non-orientable quotients

Each map $\mathcal{M} \in \mathcal{O}(G)$ is inner regular, with full automorphism group $G \times \mathrm{C}_2$: this is true for $\mathcal{M} = \mathcal{F}$ since the corresponding generators x and y are both inverted by the matrix

$$\begin{pmatrix} t+1 & t \\ t & t+1 \end{pmatrix} \in G;$$

now the graph $\mathcal{O}(G)$ is connected, and $\operatorname{Aut}\mathcal{M}$ is preserved by D and H_j, so it is true for all $\mathcal{M} \in \mathcal{O}(G)$. Since G, being simple, has no subgroups of index 2, every map in $\mathcal{R}(G)$ is non-orientable, so as explained in Section 4.1, we obtain an isomorphism $\mathcal{O}(G) \to \mathcal{R}(G), \mathcal{M} \mapsto \mathcal{N} = \mathcal{M}/\mathrm{C}_2$ of directed graphs. Using their extended types to identify the maps $\mathcal{N}$ in the list of non-orientable regular maps in [7] gives the final column of Table 2.

9 Some other Hurwitz groups

We have seen that if $G = \mathrm{PSL}_2(7)$ or $\mathrm{SL}_2(8)$ then $\mathcal{O}(G)$ is connected, consisting of maps which are respectively all outer or all inner regular, with automorphism groups isomorphic to $\mathrm{PGL}_2(7)$ or $\mathrm{SL}_2(8) \times \mathrm{C}_2$.

In fact, a group G can have both inner and outer regular maps in $\mathcal{O}(G)$, necessarily in different components. For instance, if $G = \mathrm{PSL}_2(13)$ then $\mathcal{O}(G)$ contains three regular Hurwitz maps R14.1, R14.2 and R14.3, of genus 14 and extended types $\{3,7\}_{12}$, $\{3,7\}_{26}$ and $\{3,7\}_{14}$. Calculations similar to those given earlier show that R14.2 is inner regular, with automorphism group $G \times \mathrm{C}_2$ and non-orientable regular quotient N15.1 of type $\{3,7\}_{13}$, while the other two maps are outer regular, with automorphism group $\mathrm{PGL}_2(13)$. This is interesting because, as shown by Streit [37], for each prime $p \equiv \pm 1 \bmod (7)$ the three Hurwitz dessins in $\mathcal{O}(\mathrm{PSL}_2(p))$ are Galois conjugate, under the Galois group C_3 of the real cyclotomic field $\mathbb{Q}(\cos 2\pi i/7)$, so although the orientation-preserving automorphism group of a dessin is a Galois invariant, the full automorphism group is not.

More generally, Wendy Hall [21] has shown that a Hurwitz map $\mathcal{M}$ with $\mathrm{Aut}^+\mathcal{M} \cong \mathrm{PSL}_2(q)$ is inner regular if and only if $3 - \tau^2$ is a square in $\mathbb{F}_q$, where τ is the trace of the canonical generator x of order 7. For primes $q = p \equiv \pm 1 \bmod (7)$, where there are three Hurwitz maps, her examples $p = 167$, 13, 43 and 181 show that none, one, two or all three of them can be inner regular.

In order to realise alternating groups A_n as Hurwitz groups (Theorem 5.2), Conder [5] constructed a sequence of planar maps $\mathcal{M}_n$ of type $\{7,3\}$ which have Hurwitz maps $\mathcal{H}_n$ with $\mathrm{Aut}^+\mathcal{H}_n \cong \mathrm{A}_n$ as orientably regular covers. (Each $\mathcal{M}_n$ is presented as a coset diagram for a subgroup $\mathrm{A}_{n-1} < \mathrm{A}_n = \langle x, y\rangle$, but by shrinking the triangles for x to points it can be interpreted as a cubic map with monodromy group A_n.) The maps $\mathcal{H}_n$ are all outer regular: indeed, the bilateral symmetry in the construction of the maps $\mathcal{M}_n$ was designed to show that each map $\mathcal{H}_n$ has full automorphism group S_n.

Example 9.1. Among the alternating groups, the smallest Hurwitz group is A_{15}. There are three Hurwitz maps $\mathcal{H}$ with $\mathrm{Aut}^+\mathcal{H} \cong \mathrm{A}_{15}$. Instead of drawing them (they have genus 7 783 776 001), we show their planar quotients $\mathcal{M} = \mathcal{H}/\mathrm{A}_{14}$ in Figure 12. On the left is the map $\mathcal{M}_{15}$, corresponding to Conder's diagram B from his set of basic coset diagrams $A, \ldots, N$ in [5]; it is covered by the outer regular Hurwitz map $\mathcal{H}_{15}$. The other two maps are covered by a chiral pair of Hurwitz maps $\mathcal{H}$.

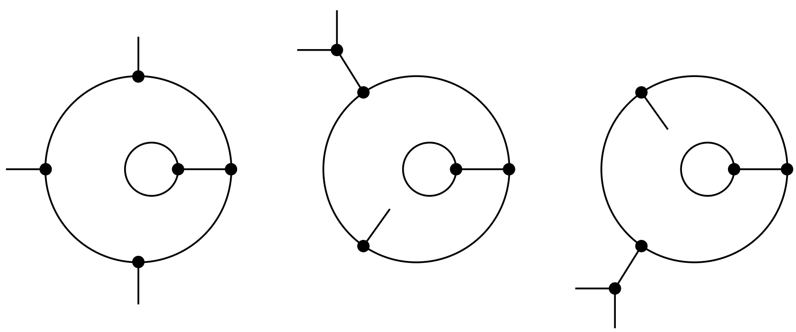

Figure 12: Three maps of type $\{7,3\}$ with monodromy group A_{15}.

By contrast, the simple Ree groups $\mathrm{Re}(q) = {}^2G_2(q)$ ($q = 3^e$ for odd $e \geq 3$) are also Hurwitz groups, but as shown in [25] the Hurwitz maps associated with them are all chiral (the generator x of order 3 is not inverted by any automorphism). In the next section we will consider a more straightforward example of this last phenomenon, but this time not involving Hurwitz maps.

10 $\mathrm{AGL}_1(q), q = 2^e$

In this section, instead of considering individual groups such as $\mathrm{PSL}_2(7)$ and $\mathrm{SL}_2(8)$, we will consider an infinite family of groups G for which $\mathcal{O}(G)$ exhibits uniform behaviour.

Let G be the 1-dimensional affine group $\mathrm{AGL}_1(q)$ for $q = 2^e$, consisting of the affine transformations

$$t \mapsto at + b, \quad (a, b \in \mathbb{F}_q,\ a \neq 0)$$

of the field $\mathbb{F}_q$. This is a semidirect product $T \rtimes S$ of an elementary abelian normal subgroup

$$T \cong (\mathbb{F}_q, +) \cong \mathrm{V}_q = (\mathrm{C}_2)^e$$

consisting of the translations $t \mapsto t + b$, by a complement

$$S \cong (\mathbb{F}_q^*, \times) \cong \mathrm{C}_{q-1},$$

consisting of the transformations $t \mapsto at, a \neq 0$. To avoid trivial cases, we will assume from now on that $e \geq 2$. (Note that these groups G, being solvable, are not perfect, so they are not Hurwitz groups.)

Theorem 10.1. (a) *If $e > 2$ there are $\phi(q-1)/e$ maps in $\mathcal{M} \in \mathcal{O}(G)$, all of extended type $\{q-1, q-1\}_4$ and genus $(q-1)(q-4)/4$; they are all chiral, and satisfy $D(\mathcal{M}) \cong H_{-1}(\mathcal{M})$ and $H_2(\mathcal{M}) \cong \mathcal{M}$.*

(b) *If $e = 2$ there is a single map $\mathcal{M} \in \mathcal{O}(G)$, the tetrahedral map $\{3, 3\}$ of genus 0, which is outer regular with $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{A\Gamma L}_1(4) \cong \mathrm{S}_4$.*

(c) *For each $e \geq 2$ the graph $\mathcal{O}(G)$ is connected.*

Proof. The $q - 1$ involutions in T are all conjugate in G, and the remaining non-identity elements, all of order dividing $q-1$, form $q-2$ conjugacy classes of size q (the non-identity cosets of T in G). These fuse into $(q-2)/e$ orbits of size qe under the action of

$$\mathrm{Aut}\, G = \mathrm{A\Gamma L}_1(q) \cong G \rtimes \mathrm{Gal}\,\mathbb{F}_q \cong G \rtimes \mathrm{C}_e.$$

Any map $\mathcal{M} \in \mathcal{O}(G)$ corresponds to a generating triple (x, y, z) for G, where y has order 2 and xT generates G/T. There are $\phi(q-1)q$ choices for x and $q-1$ choices for y, giving $\phi(q-1)q(q-1) = \phi(q-1)|\mathrm{Aut}\, G|/e$ triples, so there are $\phi(q-1)/e$ maps $\mathcal{M}$ in $\mathcal{O}(G)$. Since x and z have order $q-1$ these maps have type $\{q-1, q-1\}$ and hence have genus $(q-1)(q-4)/4$. Since $[x, y]$ is a non-identity element of T it has order 2, so the Petrie length is 4. In particular, if $e = 2$ there is a single map $\mathcal{M} \in \mathcal{O}(G)$; this is the tetrahedral map $\{3, 3\}$, which is outer regular with $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{A\Gamma L}_1(4) \cong \mathrm{S}_4$. If $e > 2$ then no element of $\mathrm{Gal}\,\mathbb{F}_q$ inverts $\mathbb{F}_q^*$, so the maps $\mathcal{M} \in \mathcal{O}(G)$ are all chiral; since z is conjugate to x^{-1} they satisfy $D(\mathcal{M}) \cong H_{-1}(\mathcal{M})$. Since $x \mapsto x^2$ is an automorphism of $\mathbb{F}_q$ they are all invariant under H_2. Since any two generators of S are powers of each other, any two maps $\mathcal{M}, \mathcal{M}' \in \mathcal{O}(G)$ satisfy $\mathcal{M}' = H_j(\mathcal{M})$ for some j coprime to $q-1$, so $\mathcal{O}(G)$ is connected. (It is, in fact, a quotient of a Cayley graph for the group of units U_{q-1}.) □

The involutions in G all lie in the proper subgroup T, so $\mathcal{R}(G)$ is empty. These maps $\mathcal{M}$ are instances of the orientably regular embeddings of complete graphs K_q constructed by Biggs in [1], where he showed that K_q has such an embedding if and only if q is a prime power; see [24] for the classification of such maps.

10.1 Examples

The maps $\mathcal{M}$ arising for $e = 3$ are the chiral pair of Edmonds maps of type $\{7, 7\}_4$, corresponding to the entry C7.2 in [7]; these both lie on the Fricke–Macbeath surface $\mathcal{S}$ of genus 7 realising the Hurwitz group $\mathrm{SL}_2(8)$. The affine group $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{AGL}_1(8)$ is a subgroup of index 9 in $\mathrm{Aut}\,\mathcal{S} = \mathrm{Aut}^+\mathcal{F} \cong \mathrm{SL}_2(8)$: it is the stabiliser of ∞ in the natural representation of $\mathrm{SL}_2(8)$ on the projective line $\mathbb{P}^1(\mathbb{F}_8)$, and also the normaliser of a Sylow 2-subgroup. This inclusion lifts to an index 9 inclusion $\Delta(7, 2, 7) < \Delta = \Delta(7, 2, 3)$ of triangle groups, namely item (B) in Singerman's list of triangle group inclusions [34].

For $e = 4$ we have the chiral pair C45.2 of type $\{15, 15\}_2$, and for $e = 5$ we have the three chiral pairs C217.45–47 of type $\{31, 31\}_4$. The index 9 inclusions of automorphism groups and triangle groups mentioned above for $e = 3$ do not generalise to higher powers of 2.

When $e = 5$, if we take a to be a generator of $\mathbb{F}_{32}^* = \langle a \mid a^{31} = 1\rangle \cong \mathrm{C}_{31}$, then the chiral pairs of maps $\mathcal{M}_{\pm i}$ $(i = 1, 3, 5)$ correspond to the following mutually inverse pairs of orbits $\Omega_{\pm i}$ of $\mathrm{Gal}\,\mathbb{F}_{32} = \langle t \mapsto t^2\rangle \cong \mathrm{C}_5$ on $\mathbb{F}_{32}^*$:

$$\Omega_1 = \{a^i \mid i = 1, 2, 4, 8, 16\}, \quad \Omega_{-1} = \{a^i \mid i = 15, 30 \equiv -1, 29, 27, 23\};$$
$$\Omega_3 = \{a^i \mid i = 3, 6, 12, 24, 17\}, \quad \Omega_{-3} = \{a^i \mid i = 7, 14, 28 \equiv -3, 25, 19\};$$
$$\Omega_5 = \{a^i \mid i = 5, 10, 20, 9, 18\}, \quad \Omega_{-5} = \{a^i \mid i = 11, 22, 13, 26 \equiv -5, 21\}.$$

Then $D(\mathcal{M}_i) = H_{-1}(\mathcal{M}_i) = \mathcal{M}_{-i}$ and $H_2(\mathcal{M}_i) = \mathcal{M}_i$ for all i, while H_3 induces a 6-cycle $(1, 3, 5, -1, -3, -5)$ on the subscripts i. The graph $\mathcal{O}(\mathrm{AGL}_1(32))$ is shown in Figure 13, with directed edges representing the action of H_3, and undirected dashed and dotted edges the actions of D and H_{-1}. The identification of these maps with the entries C217.45–47 in [7] depends on the choice of a, or more precisely that of its minimal polynomial in the action on the additive group of $\mathbb{F}_{32}$, one of the six irreducible factors of the cyclotomic polynomial $\Phi_{31}(t) = t^{30} + t^{29} + \cdots + t + 1$ in $\mathbb{F}_2[t]$ (see [24]).

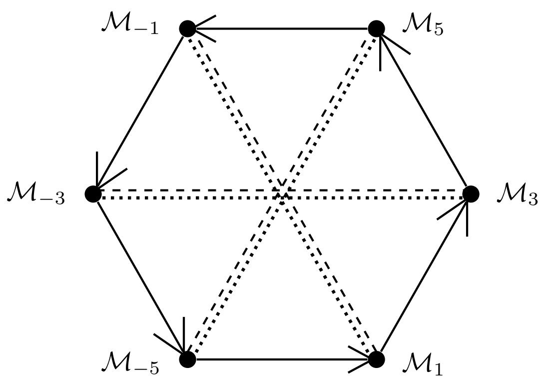

Figure 13: The graph $\mathcal{O}(\mathrm{AGL}_1(32))$

11 Groups G for which $\mathcal{O}(G)$ has many components

In contrast with the groups $G = \mathrm{AGL}_1(2^e)$, for which $\mathcal{O}(G)$ is connected for all e, we will now consider some families of groups G for which $\mathcal{O}(G)$ has an unbounded number of connected components.

The map operations D and H_j on orientably regular maps preserve the involution $y \in G$ in generating triples, up to the action of $\mathrm{Aut}\, G$, so if $\mathrm{Aut}\, G$ has $i = i(G)$ orbits on useful involutions $y \in G$, where 'useful' means 'member of a generating pair for G', then the number $c = c(G)$ of connected components of $\mathcal{O}(G)$ satisfies $c \geq i$. The example $G = \mathrm{S}_5$, with $i = 2$ but $c = 3$ (see Section 4.2), shows that c can exceed i; the following example shows that both can be arbitrarily large.

Lemma 11.1. *If $G = \mathrm{S}_n$ with $n \geq 5$ then each involution $y \in G$ is useful.*

Proof. It is sufficient to show that at least one involution in each conjugacy class of G is useful. The result is straightforward if y is a transposition (i, j), since one can then take x to be an n-cycle with $ix = j$, so we may assume that y consists of t transpositions and $n - 2t$ fixed points for some $t \geq 2$.

First we will deal with the case $n \geq 8$. Let $m = \lfloor n/2 \rfloor$, so that $m > 3$. By the Bertrand–Chebyshev Theorem (see [36], for example) there is a prime p such that $m < p < 2m - 2$, so $n/2 < p \leq n - 3$. Let us take x to have cycles of length p and $n - p$ if n is odd, and p, $n - p - 1$ and 1 if n is even, so that x is odd in either case. Now x has at most three cycles, so given any $t \geq 2$ we can choose the involution y, which moves $2t \geq 4$ points, so that $H := \langle x, y \rangle$ is transitive. It follows that H must be primitive, for otherwise H is contained in a wreath product $\mathrm{S}_a \wr \mathrm{S}_b$ for some proper factorisation $n = ab$ of n, which is impossible since x has order divisible by p whereas $\mathrm{S}_a \wr \mathrm{S}_b$ has order $(a!)^b b!$ coprime to p. Thus H is a primitive group containing a p-cycle (a suitable power of x) with $n - p \geq 3$ fixed points, so by a classic theorem of Jordan (see [39, Theorem 13.9]) $H \geq \mathrm{A}_n$. Since H contains the odd permutation x we must have $H = \mathrm{S}_n$, as required.

The case $n = 5$ was considered in Section 4.2, while the cases $n = 6$ and $n = 7$ can easily be dealt with by hand or by using GAP. □

The restriction $n \geq 5$ is required in this lemma: see Section 2.1.

Corollary 11.2. *For each $m \in \mathbb{N}$ there exists $N_m \in \mathbb{N}$ such that if $n \geq N_m$ then $\mathcal{O}(\mathrm{S}_n)$ has at least m connected components.*

Proof. By Lemma 11.1 the involutions in $G = \mathrm{S}_n$ are all useful for $n \geq 5$. They have $\lfloor n/2 \rfloor$ possible cycle-structures, so if $n \neq 6$ then $\mathrm{Aut}\, G$ $(= G)$ has $\lfloor n/2 \rfloor$ orbits on them. Thus $\mathcal{O}(G)$ has $c(G) \geq i(G) = \lfloor n/2 \rfloor$ connected components for all $n \geq 7$. We may therefore take $N_m = \max\{2m, 7\}$. □

By adapting Conder's proof in [5] of Theorem 5.2 one can also prove:

Theorem 11.3. *For each $m \in \mathbb{N}$ there exists $N'_m \in \mathbb{N}$ such that if $n \geq N'_m$ then $\mathcal{O}(\mathrm{A}_n)$ has at least m connected components containing Hurwitz maps.*

Outline of proof. In [5] Conder proved that A_n is a Hurwitz group for each $n \geq 168$ (and also for some smaller n) by constructing coset diagrams for subgroups of index n in Δ, and showing that the induced permutation group on the cosets is A_n. These diagrams are constructed by joining copies of 14 basic cosets diagrams $A, B, \ldots, N$ of degrees between 14 and 108. An *i-handle* in a coset diagram for Δ is a pair (a, b) of fixed points of y with $a = bz^i$, where $i = 1, 2$ or 3. (In this paper we have transposed Conder's notation in [5] for the generators x and y of Δ.) Two coset diagrams of degrees d and d' with i-handles (a, b) and (a', b') can be joined by an *i-join* to create a diagram of degree $d + d'$ by replacing a and a' with a transposition (a, a') for y, and similarly for b and b'.

Conder's construction (simplified a little here) is as follows. His diagram G of degree 42 has three 1-handles. First use (1)-joins to form a chain of k copies of G for some $k \geq 1$. For each of the 42 congruence classes $[c] = [0], \ldots [41] \bmod (42)$, join two specified combinations of basic diagrams, of total degree $d_c \equiv c \bmod (42)$, to the ends of the chain, to give a coset diagram of degree $n = 42k + d_c \equiv c \bmod (42)$. By taking $k = 1, 2, \ldots$ this realises A_n for each sufficiently large $n \in [c]$ as a quotient of Δ and hence as a Hurwitz group.

Each of the k copies of G in the chain has an unused (1)-handle, giving two fixed points of y. By using these to make i further joins, for any $i \leq \lfloor k/2 \rfloor$, one can reduce the total number of fixed points of y by $4i$; this does not change the fact that the resulting permutation group is A_n since Conder's proof of this still applies. We thus obtain $\lfloor k/2 \rfloor + 1$ Hurwitz maps in $\mathcal{O}(\mathrm{A}_n)$, with mutually distinct cycle-structures for y and hence in distinct components of this graph. Taking k sufficiently large proves the result. □

12 The order of $\mathcal{O}(G)$

One obvious problem which we have not yet addressed is to determine $|\mathcal{O}(G)|$ for any finite group G; however, a method for solving this is already known. The maps in $\mathcal{O}(G)$ correspond bijectively to the orbits of $\mathrm{Aut}\, G$ on generating pairs x, y for G satisfying $y^2 = 1$. Since $\mathrm{Aut}\, G$ acts semiregularly (i.e. fixed-point-freely) on all generating sets for G, we have

$$|\mathcal{O}(G)| = \phi(G)/|\mathrm{Aut}\, G|,$$

where $\phi(G)$ is the number of such generating pairs x, y for G. If $\sigma(G)$ denotes the total number of pairs $x, y \in G$ with $y^2 = 1$, then by applying Philip Hall's technique of Möbius inversion in groups [20] to the obvious equation $\sigma(G) = \sum_{H \leq G} \phi(H)$ we obtain

$$\phi(G) = \sum_{H \leq G} \mu(H)\sigma(H), \tag{12.1}$$

where μ is the Möbius function for the subgroup lattice of G, defined by

$$\mu(G) = 1 \quad \text{and} \quad \sum_{K \geq H} \mu(K) = 0 \quad \text{for all} \quad H < G.$$

This applies to generating sets satisfying any given set of relations, but in our case, with $y^2 = 1$ the only relation, we have

$$\sigma(H) = |H|(|H|_2 + 1),$$

where $|H|_2$ is the number of involutions in H.

In [20] Hall determined the function μ for many groups, including $\mathrm{PSL}_2(p)$ for primes p. Downs [14] extended this to $\mathrm{PSL}_2(q)$ for prime powers $q = p^e$. Their results for $p > 2$ are complicated but $|\mathcal{O}(\mathrm{PSL}_2(13))| = 33$ is a simple example. If $q = 2^e$ then $|\mathrm{Aut}\, G| = e|G|$ and as in [15] we have

$$|\mathcal{O}(G)| = \frac{1}{e} \sum_{f|e} \mu\left(\frac{e}{f}\right) (2^f - 1)(2^f - 2),$$

where μ now denotes the Möbius function of elementary number theory (see Sections 2.1 and 8 for the cases $e = 2$ and $e = 3$). For any q the sum in (12.1) is dominated by the summand with $H = G$, so that for $G = \mathrm{PSL}_2(q)$ we have

$$|\mathcal{O}(G)| \sim \frac{|G|(|G|_2 + 1)}{|\mathrm{Aut}\, G|} \sim \frac{q^2}{e} \quad \text{or} \quad \frac{q^2}{4e} \quad \text{as} \quad q \to \infty,$$

where q is respectively even or odd.

We close with a question: if $G = \mathrm{PSL}_2(q)$, how does the number $c(G)$ of connected components of $\mathcal{O}(G)$ behave as $q \to \infty$?

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.02
https://doi.org/10.26493/2590-9770.1436.0f3
(Also available at http://adam-journal.eu)

Cyclotomic association schemes of broad classes and applications to the construction of combinatorial structures*

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Received 12 August 2021, accepted 31 October 2021, published online 27 June 2022

Abstract

In 2010, G. Fernández, R. Kwashira and L. Martínez gave a new cyclotomy on any product $R = \prod_{i=1}^{n} \mathbb{F}_{q_i}$, where $\mathbb{F}_{q_i}$ is a finite field with q_i elements. They defined a certain subgroup H of the group of units of this product ring R for which the quotient is cyclic. The orbits of the corresponding multiplicative action of the subgroup on the additive group of R are of two types:

(1) The cyclotomic cosets of the quotient of the group of units of R over the subgroup H.

(2) The n-tuples with arbitrary non-zero elements in positions indicated by a proper subset S of $\{1, \dots, n\}$ and zeroes elsewhere.

In this paper, we introduce and study a fusion of a class of association schemes derived from the mentioned cyclotomy. The association schemes that we are proposing correspond with a fusion of orbits associated to subsets S of $\{1, \dots, n\}$ of the same cardinality. We call standard cyclotomic association schemes of broad classes to these association schemes. The fusion corresponds to the operation of adding to the permutation group that determines the original association scheme the permutations of R induced by the permutations of the symmetric group S_n.

*The authors were supported by the UPV/EHU and Basque Center of Applied Mathematics grant US18/21. The authors would like to thank G.A. Fernández-Alcober and the anonymous referee for helpful suggestions.

†Corresponding author. The author was supported by the Basque Government, grant IT974-16.

‡The author was supported by the Basque Government, grant IT974-16 and by the Spanish Government, grant MTM2017-86802-P.

§The author was supported by the Basque Government, grant IT974-16.

We use these association schemes to obtain sporadic examples and infinite families of difference sets and partial difference sets.

Keywords: Cyclotomy, difference sets, partial difference sets, association schemes.

Math. Subj. Class.: 05C25, 05E18, 05B10

1 Standard cyclotomy

Cyclotomy of finite rings is based on considering the action of a subgroup G of the multiplicative group of a finite ring on the additive group of the same ring through multiplication by the elements of G. Cyclotomy is a fundamental tool to build many types of combinatorial structures, in which the objects are formed by taking a union of orbits of the action, so that the group G becomes a group of automorphisms of the combinatorial structure considered (not necessarily equal to the complete group of automorphisms).

Fernández, Kwashira and Martínez introduced in [8] a new type of cyclotomy, which they called standard cyclotomy, in finite Cartesian products of finite fields, that generalized the two most widely used in the literature (namely, the classic one used by Gauss and the one introduced by Whiteman in [19], and they used it to find certain combinatorial structures, including divisible difference sets, relative difference sets, which give rise to groop divisible designs, partial difference sets, which originate strongly regular (undirected) graphs, and also three class association schemes.

Araluze, Kutnar, Martínez and Marusic used this standard cyclotomy in [1] to construct an infinite series of partial sum families that originate strongly regular digraphs with new parameters.

Subsequently, Momihara found in [12] an infinite series of strongly regular graphs with parameters like those of Proposition 4.3 of the article by Fernández, Kwashira and Martínez, as well as another infinite family with new parameters.

Ott used in [13] and [14] the standard cyclotomy to obtain partial difference sets that originate new strongly regular graphs.

Later, Michel and Wang used the standard cyclotomy in [11] to construct partial geometric designs from subgroups of $\mathbb{F}_q^\star \times \mathbb{F}_q$.

Let us recall now the basic definitions and results from the standard cyclotomy. Let $R = \mathbb{F}_{q_1} \times \cdots \times \mathbb{F}_{q_n}$ be a product of finite fields (thus $q_1, \ldots, q_n$ are prime powers) and let $X = \{1, \ldots, n\}$. For every $k \in X$ we choose a primitive root θ_k in $\mathbb{F}_{q_k}$. Let e be a common divisor of all the $q_k - 1$ for $k \in X$, and let us write $q_k - 1 = ef_k$. Then the following subgroup of the multiplicative group $R^\times$ is well-defined:

$$H = \{(\theta_1^{r_1}, \ldots, \theta_n^{r_n}) \mid \textstyle\sum_{k=1}^n r_k \equiv 0 \bmod e\}.$$

The multiplicative subgroup H acts by multiplication on the additive group of R, and our first result describes the corresponding orbits.

Proposition 1.1 ([8, Proposition 2.1]). *The orbits of R under the action of H are the following subsets:*

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(1) $C_i = \{(\theta_1^{r_1}, \ldots, \theta_n^{r_n}) \mid \sum_{k=1}^{n} r_k \equiv i \bmod e\}$, *for all* $0 \leq i \leq e-1$.

(2) $F_S = \{(x_1, \ldots, x_n) \in R \mid x_k \neq 0$ *for* $k \in S$ *and* $x_k = 0$ *for* $k \notin S\}$, *for all* $S \subsetneq X$. *In other words,* $F_S = U_1 \times \cdots \times U_n$, *where* $U_k = \mathbb{F}_{q_k}^{\times}$ *if* $k \in S$ *and* $U_k = \{0\}$ *otherwise.*

Moreover,

$$|C_i| = \frac{\prod_{k \in X} (q_k - 1)}{e} \quad \textit{and} \quad |F_S| = \prod_{k \in S} (q_k - 1).$$

We call the orbits of the form C_i *cyclotomic cosets*.

We define the *standard cyclotomy* of order e with respect to the primitive roots $\theta_1, \ldots, \theta_n$ as the partition of R into orbits of the action of H.

Definition 1.2 ([8, Definition 2.2])**.** Given a standard cyclotomy of order e over a Galois domain corresponding to primitive roots of unity $\theta_1, \ldots, \theta_n$, its *inverse cyclotomy* is the cyclotomy of order e defined by $\theta_1^{-1}, \ldots, \theta_n^{-1}$.

Definition 1.3 ([8, Definition 2.3])**.** Let two standard cyclotomies of the same order e be given over the rings R and R', defined by primitive roots of unity $\theta_1, \ldots, \theta_n$ and $\theta'_1, \ldots, \theta'_m$, respectively. Then the *product cyclotomy* of the two is the cyclotomy over $R \times R'$ of order e and primitive roots of unity $\theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_m$. This generalizes obviously to the product of more than two cyclotomies.

Proposition 1.4 ([8, Proposition 2.5])**.** *Let* A*,* B *and* C *be three orbits of* R*. Then the number of solutions* $a \in A$ *and* $b \in B$ *to the equation* $a + b = c$*, with* $c \in C$ *fixed, is independent of the choice of* c*. The same result holds if we fix* $a \in A$ *or* $b \in B$*.*

Thus, if we we consider the additive group $(R, +)$ and the group ring $\mathbb{Z}R$, and if $*$ denotes the product in this group ring, we can write

$$A * B = \sum_{C \text{ orbit of } R} \alpha(A, B, C)\, C \tag{1.1}$$

for some non-negative integers $\alpha(A, B, C)$.

Proposition 1.5 ([8, Proposition 2.6])**.** *For any three orbits* A*,* B *and* C *of* R*, the following properties hold:*

(1) $\alpha(B, A, C) = \alpha(A, B, C)$.

(2) $\alpha(B, C, A) = \dfrac{|C|}{|A|}\, \alpha(A, -B, C)$.

(3) $\alpha(uA, uB, uC) = \alpha(A, B, C)$ *for all* $u \in R^{\times}$.

(4) *Let* a *be any element of* A*, and denote by* N *the number of* $b \in B$ *such that* $a+b \in C$*. (Note that* N *is independent of the choice of* a *by the previous proposition.) Then*

$$\alpha(A, B, C) = \frac{N\,|A|}{|C|}.$$

As a consequence of these simple properties, in order to determine $\alpha(A,B,C)$ for all orbits A, B and C, it suffices to know the values $\alpha(i,j,k)$, $\alpha(i,j,S)$, $\alpha(S,T,i)$ and $\alpha(S,T,U)$, for $0 \leq i,j,k \leq e-1$ and $S,T,U \subsetneq X$.

Proposition 1.6 ([8, Proposition 2.7]). *Let S, T and U be proper subsets of X and let $i \in \{0,\dots,e-1\}$. Then*

$$\alpha(S,T,i) = \begin{cases} \prod_{k\in S\cap T}(q_k-2), & \textit{if } S\cup T = X, \\ 0, & \textit{otherwise,} \end{cases}$$

and

$$\alpha(S,T,U) = \begin{cases} \prod_{k\in S\cap T\cap U}(q_k-2)\cdot\prod_{k\in(S\cap T)\smallsetminus U}(q_k-1), & \\ & \textit{if } (S\cup T)\smallsetminus(S\cap T)\subseteq U\subseteq S\cup T, \\ 0, & \textit{otherwise.} \end{cases}$$

The δ symbol in the following proposition and in the rest of the paper is defined by

$$\delta_i = \begin{cases} 1, & \text{if } i = r \\ 0, & \text{in other case} \end{cases},$$

where r is defined modulo e by the condition $(-1,\dots,-1) \in C_r$. It was proved in Proposition 2.4 in [8] that, if Y denotes the set of indices $k \in X$ for which q_k is odd, then

$$r = \begin{cases} 0, & \text{if } \sum_{k\in Y} f_k \text{ is even} \\ e/2, & \text{in other case} \end{cases}$$

Proposition 1.7 ([8, Proposition 2.10]). *Let S be a proper subset of X of cardinality s. Then*

$$\alpha(i,j,S) = \frac{\prod_{k\notin S}(q_k-1)}{e}\left\{\frac{\prod_{k\in S}(q_k-2)-(-1)^s}{e} + (-1)^s\delta_{j-i}\right\}.$$

with respect to the integers $\alpha(i,j,k)$, which satisfy $\alpha(i,j,k) = \alpha(0,j-i,k-i)$ for every i,j,k and are denoted by $(j-i,k-i)$ and called *cyclotomic numbers*, there is not a closed algebraic form to express all of them (except for certain especial cases, since for instance the regular cyclotomies and other special cyclotomies similar to the regular one), and they are calculated (with pain and effort!) for each particular value of e.

We will calculate next the values for $e = 2$ and $e = 3$.

We begin by a couple of simple results.

Proposition 1.8 ([8, Proposition 3.1]).

(1) $(i,j) = (-i, j-i)$.

(2) $(i,j) = (j+r, i+r)$.

Proposition 1.9 ([8, Proposition 3.2]). $\sum_{j=0}^{e-1}(i,j) = (-1)^n(\delta_i + \dot{M})$, *where*

$$\dot{M} = \frac{\prod_{k=1}^{n}(2-q_k)-1}{e}.$$

By using the previous two propositions, we can already calculate the cyclotomic numbers for $e = 2$. We only need to give the values of $(0,0)$ and $(0,1)$, since the other cyclotomic numbers can be deduced from these according to Proposition 1.8. Note also that all the q_k are odd for $e = 2$.

Proposition 1.10. *For $e = 2$, we have that:*

(1) *If $f_1 + \cdots + f_n$ is odd, then*

$$(0,0) = (-1)^n \frac{\dot{M}+1}{2} \quad \textit{and} \quad (0,1) = (-1)^n \frac{\dot{M}-1}{2}.$$

(2) *If $f_1 + \cdots + f_n$ is even, then*

$$(0,0) = (-1)^n \Big(\frac{\dot{M}}{2} + 1\Big) \quad \textit{and} \quad (0,1) = (-1)^n \frac{\dot{M}}{2}.$$

As in the classical case, for the calculation of the cyclotomic numbers of order greater than 2, it is convenient to introduce periods and Stickelberger-Gauss sums. For each $k \in \{1,\dots,n\}$, let p_k be the characteristic of the field $\mathbb{F}_{q_k}$ and choose a complex root of unity ξ_k of order p_k. Then we define the *periods* η_i to be

$$\eta_i = \sum_{(x_1,\dots,x_n)\in C_i} \xi_1^{T_1(x_1)} \dots \xi_n^{T_n(x_n)}, \quad \text{for } i = 0,\dots,e-1,$$

where T_k is the trace map corresponding to the extension $\mathbb{F}_{q_k}/\mathbb{F}_{p_k}$. Now, for any complex e-th root of unity β, we define the *Stickelberger-Gauss sum* $H(\beta)$ by means of

$$H(\beta) = \sum_{i_1=0}^{q_1-2} \cdots \sum_{i_n=0}^{q_n-2} \xi_1^{T_1(\theta_1^{i_1})} \dots \xi_n^{T_n(\theta_n^{i_n})} \beta^{i_1+\cdots+i_n} = \sum_{i=0}^{e-1} \eta_i \beta^i.$$

Since for any $d > 1$ the sum of all d-th roots of unity is zero, we have the following inversion formula expressing the periods η_i in terms of the Stickelberger-Gauss sums:

$$e\eta_i = \sum_{\beta} H(\beta)\beta^{-i}, \tag{1.2}$$

where the sum runs over all complex e-th roots of unity.

It is clear that these generalized Stickelberger-Gauss sums satisfy the following multiplicative property: if we decompose the ring R as a cartesian product, say $R = R_1 \times R_2$, and we view the cyclotomy over R as a product of cyclotomies over R_1 and R_2, then $H(\beta) = H_1(\beta)H_2(\beta)$. (Here H, H_1 and H_2 are the Stickelberger-Gauss sums over R, R_1 and R_2, respectively.) In particular, if H_k denotes the classical Stickelberger-Gauss sums over $\mathbb{F}_{q_k}$ for the primitive root of unity θ_k, then $H(\beta) = H_1(\beta)\dots H_n(\beta)$. As a consequence, $H(1) = (-1)^n$.

Lemma 1.11. *For every i and j,*

$$\eta_i\eta_j = \sum_{S \subsetneq X} \alpha(i,j,S)(-1)^{|S|} + \sum_{k=0}^{e-1} (j-i, k-i)\eta_k.$$

Proof. We have that

$$\eta_i\eta_j = \sum_{(x_1,\dots,x_n)\in C_i} \sum_{(y_1,\dots,y_n)\in C_j} \xi_1^{T_1(x_1+y_1)} \dots \xi_n^{T_n(x_n+y_n)}$$
$$= \sum_A \{\alpha(i,j,A) \sum_{(z_1,\dots,z_n)\in A} \xi_1^{T_1(z_1)} \dots \xi_n^{T_n(z_n)}\},$$

where the sum runs over all orbits A of R. Since the sum corresponding to an orbit of the type F_S is clearly equal to $(-1)^{|S|}$, the result follows. □

Proposition 1.12. *Let $\beta \neq 1$ be a complex e-th root of unity. Then*

$$H(\beta)H(\beta^{-1}) = q_1 \cdots q_n \beta^r.$$

Proof. By using the multiplicative property of H, we have that this a direct consequence of the case $n = 1$, which is well known (see, for instance, Equation (5) in [4]). □

Proposition 1.13. *Let β be a complex primitive e-th root of unity, and let u, v be integers such that $u \not\equiv 0 \pmod e$, $v \not\equiv 0 \pmod e$ and $u + v \not\equiv 0 \pmod e$. Then:*

(1) $H(\beta^u)\, H(\beta^v) = R(u, v)\, H(\beta^{u+v})$, *where*

$$R(u,v) = \sum_{i=0}^{e-1}\sum_{j=0}^{e-1} (i,j)\beta^{vi}\beta^{-(u+v)j}.$$

(2) $R(u, v)\, R(-u, -v) = q_1 \cdots q_n$.

Proof. The product $H(\beta^u)H(\beta^v)$ can be developed with the help of Lemma 1.11. By using the expression given in Proposition 1.7 for the $\alpha(i, j, S)$ and the identity

$$\prod_{k\in X} (x_k + y_k) = \sum_{T\subseteq X} \prod_{k\in T} x_k \prod_{k\notin T} y_k,$$

which is valid in any commutative ring and is a direct consequence of the distributive law, we obtain easily the result in (1).

In order to prove (2), multiply together the expression in (1) corresponding to β^u, β^v with the one corresponding to β^{-u}, β^{-v}, and then use Proposition 1.12. □

By combining these two propositions we get the following result.

Corollary 1.14. *In the conditions of the previous proposition,*

$$R(u, v) = R(v, u) = \beta^{ur} R(-v - u, u).$$

Now, by using Propositions 1.8, 1.9, and 1.13, we can derive the cyclotomic numbers for $e = 3$. All cyclotomic numbers can be easily obtained from the ones in the following proposition.

Proposition 1.15. *For $e = 3$, we have that*

$$\begin{array}{rcl} 9(0,0) & = & (-1)^n(7 - x + 3\dot{M}), \\ 18(0,1) & = & (-1)^n(2 + x + 9y + 6\dot{M}), \\ 18(0,2) & = & (-1)^n(2 + x - 9y + 6\dot{M}), \\ 9(1,2) & = & (-1)^n(-2 - x + 3\dot{M}), \end{array}$$

for some integers x, y satisfying the Diophantine equation $x^2 + 27y^2 = 4q_1 \dots q_n$, and where $x \equiv 1 \pmod 3$.

Proof. Take $u = v = 1$ in Proposition 1.13. □

2 Broad classes

It is evident from (1.1) that, given an standard cyclotomy, if we define for each orbit A the relation

$$\mathfrak{R}_A = \{(x, y) \in R \times R \mid x - y \in A\}$$

then these relations form an association scheme. When $n = 1$, that is, when we have a single field, we obtain the classical cyclotomic association scheme (see, for instance, [3]). This is why we will call to this association scheme the *standard cyclotomic association scheme*.

Now we will introduce the main object in this work. We will assume that all the fields have the same cardinality (therefore we can assume, without losing generality that they are the same field. Nonetheless, we are not necessarily taking the same primitive root in all of them).

Definition 2.1. Let us suppose that $q_i = q\ \forall i \in \{1, \dots, n\}$. For $i \in \{0, \dots, n-1\}$, we define

$$B_i = \bigcup_{|S|=i} F_S.$$

We will call *broad orbits* to these sets B_i. It is obvious that the cyclotomic cosets C_i and the broad orbits B_j determine an association scheme with the relations $\mathfrak{C}_i = \{(x, y) \in R \times R \mid x - y \in C_i\}$ and $\mathfrak{B}_j = \{(x, y) \in R \times R \mid x - y \in B_j\}$. It is a fusion of the standard cyclotomic association scheme defined before. We will call it the *standard cyclotomic association scheme of broad classes*, and the associated cyclotomy, *standard cyclotomy of broad classes*. Besides of the already mentioned connection with the classical cyclotomic association schemes, when $e = 1$ we obtain the classical Hamming association schemes (see [2]).

When we identify the orbits C_i and B_j with the corresponding sum of its elements in the group ring $\mathbb{Z}R$, we have a decomposition for the product of orbits similar to the one in (1.1). To simplify the notation, when we consider an orbit C_i we will put simply $\bar{i}$ in the $\alpha(A, B, C)$, and when we take an orbit B_j we will put $\underline{j}$. Thus, for instance, $\alpha(B_i, C_j, B_k)$ will be denoted simply by $\alpha(\underline{i}, \bar{j}, \underline{k})$.

An analysis similar to that done in the first section shows that to calculate the coefficients $\alpha(A, B, C)$ it is sufficient to know the values $\alpha(\underline{i}, \underline{j}, \underline{k}), \alpha(\underline{i}, \underline{j}, \bar{k}), \alpha(\bar{i}, \bar{j}, \underline{k})$, and $\alpha(\bar{i}, \bar{j}, \bar{k})$. Obviously, the numbers $\alpha(\bar{i}, \bar{j}, \bar{k})$ are the $\alpha(i, j, k)$ introduced in the previous section. With respect to the other three types of coefficients we have:

Proposition 2.2.

(1) *The* $\alpha(\underline{i}, \underline{j}, \underline{k})$ *are the corresponding structure constants* $p_{i,j}^k$ *of the Hamming association scheme.*

(2) *The* $\alpha(\underline{i}, \underline{j}, \overline{k})$ *are the corresponding structure constants* $p_{i,j}^n$ *of the Hamming association scheme.*

(3)
$$\alpha(\bar{i}, \bar{j}, \underline{k}) = \frac{(q-1)^{n-k}}{e} \left\{ \frac{(q-2)^k - (-1)^k}{e} + (-1)^k \delta_{j-i} \right\}.$$

Proof. The first two parts of the proposition are evident.

The third part is a consequence of Proposition 1.7, because the number in that proposition depends on the cardinality of S, but not on the specific set S. □

As a consequence of what we told in the first section, the standard cyclotomic association scheme admits the group H (when viewed as a group of permutations) as a group of automorphisms. When we consider the broad cyclotomy we have a richer group of automorphisms, because any permutation of the symbols in an n-tuple of R is an automorphism of the association scheme, and these permutations form a group K isomorphic to the symmetric group S_n, and therefore the permutation group generated by $H \cup K$ is a group of automorphisms.

3 Combinatorial applications

The computational search of combinatorial structures with the standard cyclotomy of broad classes is easier than with the standard cyclotomy, because the number of orbits is much less, and hence we have to consider unions of fewer orbits.

The following constructions given in [8] can be interpreted in terms of broad classes:

Proposition 3.1 ([8, Proposition 4.1])**.** *Let* q *be an odd prime power such that* $3q-2$ *is also a prime power, and consider a standard cyclotomy of order* $e = q-1$ *over* $R = \mathbb{F}_{3q-2} \times \mathbb{F}_{3q-2} \times \mathbb{F}_q$ *where the cyclotomies in the first two factors are inverse of each other. Then* $D = B_1 \cup C_0$ *is a divisible difference set in* R *with respect to the subgroup* $\{0\} \cup F_{\{3\}}$*, with* $\lambda = 9q - 10$ *and* $\mu = q - 2$*.*

Proposition 3.2 ([8, Proposition 4.5])**.** *Let* $q = 2^n$*, and consider a cyclotomy over* $R = \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$ *of order* $e = q-1$*, where the cyclotomies in the first two factors are inverse of each other. Then* $D = B_1 \cup C_0$ *is a partial difference set with* $v = 2^{3n}$*,* $k = (2^n - 1)(2^n + 2)$*,* $\lambda = 2^n - 2$ *and* $\mu = 2^n + 2$*.*

Proposition 3.3 ([8, Proposition 4.6])**.** *Let* q *be a prime power. Then, for any standard cyclotomy of order* $e = q-1$ *over* $R = \mathbb{F}_{q^2} \times \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$*, the union* $D = B_1 \cup C_0$ *is a partial difference set if and only if* $q = 2$ *or if* $q \geq 3$ *and the following conditions hold:*

$$(0,0) = q^4 + q^3 - 7q - 2, \qquad (0,i) = q^4 + 2q^3 - 2q^2 - 9q - 4, \quad \text{for all } i = 1, \ldots, e-1.$$

In this case, $v = q^6$*,* $k = (q+2)(q^2 - q + 1)(q^2 - 1)$*,* $\lambda = q^4 + q^3 - q - 2$ *and* $\mu = (q-1)(q+2)(q^2 + q - 1)$*.*

In the following two propositions we consider the product of a cyclotomy over $\mathbb{F}_q \times \mathbb{F}_q$ and its inverse cyclotomy.

Proposition 3.4 ([8, Proposition 4.10]). *If $f = 1$, then $B_2 \cup C_0 \cup C_1 \cup C_2$ is a $(q^4, 3(q+1)(q-1)^2, (5q-6)(2q-3), 3(q-1)(3q-4))$-partial difference set.*

Proposition 3.5 ([8, Proposition 4.11]). *If $f = 3$, then $B_2 \cup C_0$ is a $(q^4, 3(q+1)(q-1)^2, (5q-6)(2q-3), 3(q-1)(3q-4))$-partial difference set.*

Proposition 3.6 ([8, Proposition 4.15]). *Consider a cyclotomy over $R = \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_4$ of order 3, where the cyclotomies of the first two factors are inverse of each other. Then $D_1 = B_2$, $D_2 = B_1 \cup C_0$ is a basic blueprint of a three-class association scheme with intersection matrices*

$$L_1 = \begin{bmatrix} 10 & 8 & 8 \\ 12 & 9 & 6 \\ 12 & 6 & 9 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 8 & 6 & 4 \\ 9 & 2 & 6 \\ 6 & 6 & 6 \end{bmatrix}, \qquad \text{and} \qquad L_3 = \begin{bmatrix} 8 & 4 & 6 \\ 6 & 6 & 6 \\ 9 & 6 & 2 \end{bmatrix}.$$

Here we will present some new constructions not appearing in [8]:

Lemma 3.7. *Let U be a non-empty finite set. Then:*

(1) *For any $z \in \mathbb{C}$,*

$$\sum_{V \subseteq U} z^{|V|} = (z+1)^{|U|}.$$

(2) *For any $z \in \mathbb{C}$,*

$$\sum_{\substack{V \subseteq U \\ |V| \equiv |U| \bmod 2}} z^{|V|} = \frac{(z+1)^{|U|} + (z-1)^{|U|}}{2}$$

and

$$\sum_{\substack{V \subseteq U \\ |V| \not\equiv |U| \bmod 2}} z^{|V|} = \frac{(z+1)^{|U|} - (z-1)^{|U|}}{2}$$

(3) $\#\{V \subseteq U \mid |V| \text{ odd}\} = \#\{V \subseteq U \mid |V| \text{ even}\} = 2^{|U|-1}$.

Proof. Note first that (1) is simply the binomial theorem. In order to get (2) it suffices to sum and subtract the two equalities $\sum_{V \subseteq U} z^{|V|} = (z+1)^{|U|}$ and

$$\sum_{V \subseteq U} (-1)^{|U|+|V|} z^{|V|} = (-1)^{|U|}(-z+1)^{|U|} = (z-1)^{|U|}.$$

Finally, (3) follows from (2) by setting $z = 1$. □

Theorem 3.8. *For all $n \geq 1$, the set D of tuples in $\mathbb{F}_4^n$ with an odd number of components equal to zero is a difference set.*

Proof. Let us write $R = \mathbb{F}_4^n$ and $X = \{1, \ldots, n\}$, and consider the standard cyclotomy over R of order 1. Then

$$D = \cup\{F_S \mid |S| \not\equiv |X| \bmod 2\}.$$

We will write F_X to denote $R^\times$. In this case $F_X = C_0$ is an orbit of the cyclotomy, and the formula for $\alpha(S,T,0)$ coincides with the formula for $\alpha(S,T,U)$ if it is allowed to take $U = X$.

We want to prove that for any $x \in R$, $x \neq (0,\dots,0)$, the number σ of different ways of writing $x = a - b$ with $a, b \in D$ is independent of x. There exists a non-empty subset U of X such that $x \in F_U$ and then

$$\sigma = \sum_{\substack{S,T\subseteq X\\ |S|,|T|\not\equiv|X| \bmod 2}} \alpha(S,T,U).$$

Here we have used that $-F_T = F_T$.

By Proposition 1.6, we know that $\alpha(S,T,U) = 0$ unless $(S\cup T)\setminus(S\cap T) \subseteq U \subseteq S\cup T$, in which case

$$\alpha(S,T,U) = 2^{|S\cap T\cap U|}\, 3^{|(S\cap T)\setminus U|}.$$

Thus σ is a sum of a number of products of the type $2^{|V|}3^{|W|}$, with $V \subseteq U$ and $W \subseteq U^c$. More precisely,

$$\sigma = \sum_{V\subseteq U}\ \sum_{W\subseteq U^c} \tau(V,W)\, 2^{|V|}\, 3^{|W|}, \tag{3.1}$$

where $\tau(V,W)$ is the number of subsets S, T of X satisfying the following conditions:

(1) $|S|, |T| \not\equiv |X| \bmod 2$.

(2) $(S\cup T)\setminus(S\cap T) \subseteq U \subseteq S\cup T$.

(3) $S\cap T\cap U = V$ and $(S\cap T)\setminus U = W$.

Given $U \subseteq X$, $V \subseteq U$ and $W \subseteq U^c$, it is easy to determine all the subsets S and T fulfilling these conditions: it suffices to choose $P \subseteq U \setminus V$ and put $S = P\cup V\cup W$ and $T = (U\setminus P)\cup W$, as shown in Figure 1 below.

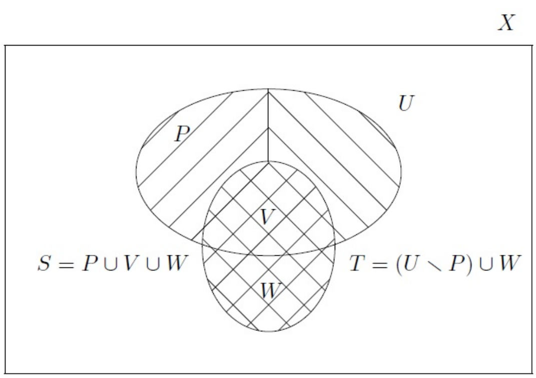

Figure 1: Relative positions of S and T.

Since all the unions in the definition of S and T are disjoint, it follows that $|S| = |P| + |V| + |W|$ and $|T| = |U| - |P| + |W|$. Hence $|S| \not\equiv |X| \bmod 2$ if and only if $|P| \not\equiv |X| + |V| + |W| \bmod 2$, and once this holds, we get that $|T| \not\equiv |X| \bmod 2$ if and only if $|V| \equiv |U| \bmod 2$.

In particular, in the sum (3.1) we only need to consider the subsets V of U satisfying $|V| \equiv |U| \bmod 2$. In that case, Lemma 3.7 yields that

$$\begin{aligned}\tau(V,W) &= \#\{P \subseteq U \setminus V \mid |P| \not\equiv |X| + |V| + |W| \bmod 2\} \\ &= \begin{cases} 2^{|U|-|V|-1}, & \text{if } V \subsetneq U, \\ 0, & \text{if } V = U \text{ and } |W| \equiv |X| + |U| \bmod 2, \\ 1, & \text{if } V = U \text{ and } |W| \not\equiv |X| + |U| \bmod 2. \end{cases}\end{aligned}$$

Hence, we split the sum in (3.1) according as $V = U$ or $V \subsetneq U$, and by using repeatedly Lemma 3.7 we get that

$$\begin{aligned}\sigma &= 2^{|U|} \sum_{\substack{W \subseteq U^c \\ |W| \not\equiv |X| + |U| \bmod 2}} 3^{|W|} + 2^{|U|-1} \sum_{\substack{V \subsetneq U \\ |V| \equiv |U| \bmod 2}} \sum_{W \subseteq U^c} 3^{|W|} \\ &= 2^{2|X|-|U|-1} - 2^{|X|-1} + 2^{|U|-1} \sum_{\substack{V \subsetneq U \\ |V| \equiv |U| \bmod 2}} 4^{|X|-|U|} \\ &= 2^{2|X|-|U|-1} - 2^{|X|-1} + 2^{2|X|-|U|-1} \left(\#\{V \subseteq U \mid |V| \equiv |U| \bmod 2\} - 1\right) \\ &= 2^{2|X|-|U|-1} - 2^{|X|-1} + 2^{2|X|-|U|-1}(2^{|U|-1} - 1) \\ &= 2^{2|X|-2} - 2^{|X|-1}.\end{aligned}$$

Thus σ is independent of x, as desired. □

The underlying additive groups are 2-groups, and it is well known that the parameters of these difference sets must be of the form $v = 4u^2$, $k = 2u^2 - u$ and $\lambda = u^2 - u$, in other words the parameters of a Hadamard difference set. Hadamard difference sets are known to exist over any elementary abelian 2-group whose order is a square (see [5]). In fact, our construction provides an alternative proof of this result, since it applies to arbitrary elementary abelian 2-groups of square order.

An sporadic Hadamard difference set with $v = 256, k = 120, \lambda = 56$ can be obtained for $e = 3$ and $n = 4$, and is

$$D = B_1 \cup B_2 \cup C_0 \cup C_1$$

Although the parameters for this sporadic difference set are the same that the ones in Theorem 3.8 with $n = 4$, they are not isomorphic. In fact, calculations in GAP show that the order of the automorphisms group of the associated graph in this case is 331776, and hence has the minimum possible size, that is, the group is the sum $G = G_1 + G_2 + G_3$, where G_1 is the group derived from the regular action of the additive group of $\mathbb{F}_4^4$, G_2 is the group derived from the group H associated to the cyclotomy, and G_3 is a group of order 8 associated to the ways to choose a primitive root in each one of the four factors of $\mathbb{F}_4^4$. On the other hand, the automorphisms group for the graph associated to the difference set constructed in Theorem 3.8 contains strictly to G and has order 89181388800.

By using a similar way of reasoning to the one in Theorem 3.8, the following result can be obtained:

Theorem 3.9. *For all $n \geq 1$, the set D of tuples in $\mathbb{F}_2^n$ with an odd number of components equal to zero is a partial difference set.*

By using the fact that, if D is a partial difference set over a group G (and, in particular, if D is a difference set with $0 \notin D$ and $D = -D$), then $G - \{0\} - D$ is also a partial difference set, we obtain the following two corollaries from the two previous theorems, respectively:

Corollary 3.10. *For all $n \geq 1$, the set D of non-zero tuples in $\mathbb{F}_4^n$ with an even number of components equal to zero is a partial difference set.*

Corollary 3.11. *For all $n \geq 1$, the set D of non-zero tuples in $\mathbb{F}_2^n$ with an even number of components equal to zero is a partial difference set.*

Partial difference sets with parameters as in Corollary 3.10 are known by constructions using projective 4-ary codes, affine polar graphs and 2-graphs.

Now we will give some sporadic constructions of partial difference sets obtained for e from 1 to 3 and n from 3 to 5. They will be described in the form $(v, k, \lambda, \mu, S, T)$, where v, k, λ, μ are the parameters of the strongly regular graph and $S \subseteq \{0, \dots, n-1\}$ and $T \subseteq \{0, \dots, e-1\}$, and where the corresponding partial difference set is

$$D = \bigcup_{i \in S} B_i \cup \bigcup_{j \in T} C_j.$$

Thus, for instance, $(\{1, 3\}, \{0, 1\})$ would represent the union of the broad orbits B_1 and B_3 and the cyclotomic cosets C_0 and C_1. When searching the partial difference sets we can assume without losing generality by taking the complement if necessary that $0 \in T$ whenever $T \neq \emptyset$.

The proof that the sets D are in fact partial difference set is easy to do and will be omitted. It is only necessary to check that $0 \notin D$ and $D = -D$ and that the identity $D \star D = \lambda D + \mu(R - D - \{0\})$ holds in the group ring $\mathbb{Z}R$, by using Propositions 1.10, 1.15 and 2.2.

For $e = 1, n = 4$
$(16, 5, 0, 2, \{1\}, \{0\}), (81, 24, 9, 6, \{1\}, \{0\}), (81, 24, 9, 6, \{2\}, \{\}),$
$(16, 5, 0, 2, \{3\}, \{0\}), (81, 32, 13, 12, \{3\}, \{\})$

For $e = 2, n = 4$
$(625, 240, 95, 90, \{1, 2\}, \{0\}), (2401, 864, 319, 306, \{2\}, \{0\})$

For $e = 3, n = 3$
$(64, 18, 2, 6, \{\}, \{0, 1\})$

For $e = 3, n = 4$
$(2401, 672, 203, 182, \{1, 2\}, \{0\}), (10000, 2673, 748, 702, \{2\}, \{0\})$

For $e = 3, n = 5$
$(1024, 186, 50, 30, \{1, 2\}, \{0\})$

For the parameters of the associated strongly regular graphs with v up to 1024, several constructions appear for each one of the parameters set in the table of Andries E. Brouwer [6].

With respect to the parameters set $(2401, 864, 319, 306)$, it is known that strongly regular graphs with these parameters exist coming from two-weight codes over $\mathbb{F}_7$ of length 144 and dimension 4, with weights 119 and 126 [7], and that they belong to the families of $SU2, CY1$ and $CY4$ codes (and one that can be constructed as a complement of a [256, 4; 217:1536, 224:864] 7-code), and with respect to the parameters set $(2401, 672, 203, 182)$, also several strongly regular graphs with these parameters exist coming from two-weight codes over $\mathbb{F}_7$ of length 112 and dimension 4, with weights 91 and 98 [7], and they belong to the families of $SU2, CY1$ and $CY4$ codes (and one that can be constructed as a complement of a [288, 4; 245:1728, 252:672] 7-code).

To finish, we would like to note that when considering cyclotomic classes of order e in $\mathbb{F}_{q_1} \times \cdots \times \mathbb{F}_{q_n}$ such that $e | q_i - 1\ \forall i$, if each cyclotomic class of order e in each field is a partial difference set then it can be viewed as a product construction. For instance, the difference set in Theorem 3.8 (note that difference sets are special cases of partial difference sets) is a union of products of trivial partial difference sets consisting of a single element or all the elements in $\mathbb{F}_{2^2}$. Some works in which product constructions of partial difference sets appear in the mathematical literature are [9, 10, 15, 16, 17, 18]. For instance, the parameters v, k, λ, μ of the 5 previously shown sporadic partial difference sets corresponding to $e = 1, n = 4$ can be obtained with a product construction in the following two results, that can be found in [10] and also in [16]:

Result 1 Let p be a prime, and let u and $s_2, s_4, \ldots, s_{2v}$ be nonnegative integers. Let $G = \mathbb{Z}_p^{2u} \times \prod_{i=1}^{v} \mathbb{Z}_{p^{2i}}^{2s_{2i}}$ and let $n = \sqrt{|G|}$. Then there is a partition of $G - 1_G$ into p regular partial difference sets in G, of which $p-1$ are of $(n, \frac{n}{p})$ Latin square type and one is of $(n, \frac{n}{p}+1)$ Latin square type.

Result 2

(1) Let u, w and $s_4, s_6, \ldots, s_{2v}$ be nonnegative integers satisfying $u + w \geq 1$. Let $G = \mathbb{Z}_2^{4u} \times \mathbb{Z}_4^{2w} \times \prod_{i=2}^{v} \mathbb{Z}_{2^{2i}}^{4s_{2i}}$ and let $n = \sqrt{|G|}$. Then there is a partition of $G - 1_G$ into 4 regular partial difference sets in G, of which three are of $(n, \frac{n}{4})$ negative Latin square type and one is of $(n, \frac{n}{4} - 1)$ negative Latin square type.

(2) Let u and $s_2, s_4, \ldots, s_{2v}$ be nonnegative integers. Let $G = \mathbb{Z}_3^{2u+2} \times \prod_{i=1}^{v} \mathbb{Z}_{3^{2i}}^{2s_{2i}}$ and let $n = \sqrt{|G|}$. Then there is a partition of $G - 1_G$ into 3 regular partial difference sets in G, of which two are of $(n, \frac{n}{3})$ negative Latin square type and one is of $(n, \frac{n}{3} - 1)$ negative Latin square type.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.03
https://doi.org/10.26493/2590-9770.1464.7cd
(Also available at http://adam-journal.eu)

The graphs with a symmetrical Euler cycle*

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Dedicated to our friend and colleague Marston Conder on the occasion of his 65th birthday.

Received 17 September 2021, accepted 1 December 2021, published online 27 June 2022

Abstract

The graphs in this paper are finite, undirected, and without loops, but may have more than one edge between a pair of vertices. If such a graph has ℓ edges, then an Euler cycle is a sequence $(e_1, e_2, \ldots, e_\ell)$ of these ℓ edges, each occurring exactly once, such that e_i, e_{i+1} are incident with a common vertex for each i (reading subscripts modulo ℓ). An Euler cycle is symmetrical if there exists an automorphism of the graph such that $e_i \to e_{i+2}$ for each i. The cyclic group generated by this automorphism has one orbit on edges if ℓ is odd, or two orbits of length $\ell/2$ if ℓ is even: that is to say, the group is regular or bi-regular on edges, respectively. Symmetrical Euler cycles arise naturally from arc-transitive embeddings of graphs in surfaces since, for each face of the embedded graph, the sequence of edges on the boundary of the face forms a symmetrical Euler cycle for the induced subgraph on this edge-set.

We first classify all finite connected graphs which admit a cyclic subgroup of automorphisms that is regular or bi-regular on edges, and identify more than a dozen infinite

*This work was partially supported by Australian Research Council Discovery project DP160102323, and NSFC projects nos. 11771200, 11931005, 61771019, 12101518, and NSFS no. ZR2020MA044. The authors sincerely thank the reviewer for carefully reading their manuscript and for several very helpful comments.

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families of examples. We then prove that exactly six of these families consist of graphs with symmetrical Euler cycles. These are the (only) candidates for the induced subgraphs of the boundary cycles of the faces of arc-transitive maps.

Keywords: edge-transitive graphs, graphs with multiple edges, graph embeddings, arc-transitive maps.

Math. Subj. Class.: 20B25, 05C25, 05C35

1 Introduction

The graphs studied in this paper are finite, undirected, without loops, but may have multiple edges. Thus a graph $\Gamma = (V, E, \mathbf{I})$ consists of finite sets V of vertices, and E of edges, together with an incidence relation $\mathbf{I} \subseteq V \times E$ such that each edge e is incident with exactly two distinct vertices. We often suppress $\mathbf{I}$ in the notation and write simply $\Gamma = (V, E)$. An edge e of Γ incident with the two vertices α and β is sometimes denoted by $[\alpha, e, \beta]$. Many graphs of this type admit natural embeddings as maps into closed surfaces: perhaps the simplest being the graph $\mathbf{K}_2^{(\lambda)}$ with exactly two vertices $V = \{\alpha, \beta\}$ and λ edges forming $E = \{[\alpha, e_i, \beta] \mid 1 \leq i \leq \lambda\}$, and embedded into a sphere with α, β at the poles, and the edges e_i arranged as λ lines of 'longitude' joining the two poles.

Throughout the paper all graphs will be of the kind just described. Motivating our investigation was our wish to study embeddings of graphs as arc-transitive maps in surfaces. If such a map has at least two faces then the edge-sequence C obtained by moving around the boundary of a face forms a 'cycle', as defined in (1.1). We emphasise that we are avoiding certain degeneracies by making the assumption that each edge in a map is incident with exactly two faces, and as a consequence each 'boundary cycle' of a face has pairwise distinct edges. We show in Lemma 3.2 that this cycle C is a 'symmetrical Euler cycle' for the induced subgraph $[C]$: this is a cycle which admits a large subgroup of the corresponding dihedral group acting with at most two orbits on edges, see Subsection 1.2 and Section 3 for more details. The most natural example for the cycle C is the sequence obtained by traversing the edges around a simple n-cycle several times, say λ times, and we call the induced subgraph $[C]$ in this example $\mathbf{C}_n^{(\lambda)}$, see Section 3. Maps for which all boundary cycles are of this form with $\lambda = 1$ (simple cycles) have been studied in [10, 11, 14]. On the other hand, quite different subgraphs $[C]$ have been identified for maps with a single face in [2, 12]. The problem, which we address in the paper, is to determine the kinds of subgraphs $[C]$ that arise, induced by symmetrical Euler cycles C, and to describe the possible groups induced on these cycles by the subgroup of automorphisms leaving $[C]$ invariant.

Note that the subgraph induced by the edges of a boundary cycle of a map is connected, and for an arc-transitive map, the group induced on the cycle contains a cyclic subgroup having at most two edge-orbits which acts faithfully on each of these edge-orbits (Lemma 3.2). In our first main result, Theorem 1.1, we broaden the scope of this study slightly, and classify all connected graphs admitting a cyclic group with at most two edge-orbits and acting faithfully on each of its edge-orbits. We find a dozen infinite families of examples. Then, in Theorem 1.2, we show that only six of these families contain graphs for which the edge set can be sequenced into a cycle preserved by a cyclic group with at most

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two edge-orbits (of equal size). To assist with our analysis we develop, in Section 5, the theory of coset graphs which may have multiple edges and which admit an edge-transitive group with two vertex-orbits.

1.1 Graphs admitting cyclic edge regular or bi-regular groups

A graph $\Gamma = (V, E)$ is a *simple graph* if, for all distinct $\alpha, \beta \in V$, the number of edges incident with both α and β is 0 or 1. Given a graph $\Gamma = (V, E)$ and a positive integer λ, the λ-*extender* of Γ is the graph $\Gamma^{(\lambda)}$ with vertex set V such that each edge $[\alpha, e, \beta]$ of Γ is replaced by λ edges $[\alpha, e_i, \beta]$ of $\Gamma^{(\lambda)}$, for $1 \leq i \leq \lambda$. If Γ is simple, then Γ is said to be the *base graph* of $\Gamma^{(\lambda)}$; in this case if α, β are adjacent in Γ, that is, if there exists an edge $[\alpha, e, \beta]$ in Γ, then there are exactly λ edges of $\Gamma^{(\lambda)}$ incident with α and β, and we say that Γ has *edge-multiplicity* λ. For each edge $[\alpha, e, \beta]$, we have $[\alpha, e, \beta] = [\beta, e, \alpha]$ and the edge e corresponds to two arcs (α, e, β) and (β, e, α).

Let a group G act on a set Ω. Then G is called *transitive* or *bi-transitive* if G has a single orbit or exactly two orbits in Ω, respectively. Further, if G is finite, then G is said to be *regular* or *bi-regular* on Ω if the permutation group G^{Ω} induced by G on Ω is transitive and $|G^{\Omega}| = |\Omega|$, or G^{Ω} is bi-transitive and $|G^{\Omega}| = \frac{1}{2}|\Omega|$, respectively; in other words, G^{Ω} has at most two orbits in Ω and is faithful and regular on each.

An *automorphism* of a graph $\Gamma = (V, E, \mathbf{I})$ is a permutation of $V \cup E$ which preserves V, E and the incidence relation $\mathbf{I}$. The set of automorphisms forms the automorphism group $\mathrm{Aut}\Gamma$. Usually $\mathrm{Aut}\Gamma$ acts faithfully on E, see Lemma 2.1 for details: the unique exceptions among connected graphs are the graphs $\mathbf{K}_2^{(\lambda)}$ mentioned above. Our first result Theorem 1.1 presents a classification of connected graphs admitting a cyclic subgroup of automorphisms that is regular or bi-regular on edges. The exceptional graphs $\mathbf{K}_2^{(\lambda)}$ are treated separately in detail in Proposition 2.2. The families of examples, apart from $\mathbf{K}_2^{(\lambda)}$, are defined in Section 4. Note that for graphs $\Gamma = (V, E)$ with an edge partition $E = E_1 \cup E_2$, we sometimes write $\Gamma = [E_1] + [E_2]$ to give a rough description of the graph structure, even though this notation does not uniquely define the graph in general, see Subsection 2.1. We give a precise description in Section 4 of all the graphs in the tables for Theorem 1.1.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected graph with $|V| \geq 3$ such that a cyclic subgroup $G \leq \mathrm{Aut}\Gamma$ is regular or bi-regular on E, and has N_V orbits on V. Then $\Gamma = \Gamma_0^{(\lambda)}$ and $|G| = \lambda N$, for some λ, N, and either*

(i) *G is regular on E and Γ_0, N, N_V are as in one of the lines of* Table 1*, or*

(ii) *G is bi-regular on E and Γ_0, N are as in one of the lines of* Table 2 *if $N_V = 1$, and Γ_0, N, N_V are as in one of the lines of* Table 3 *if $N_V \geq 2$. In particular $N_V \leq 3$.*

1.2 Cycles in graphs and boundary cycles of maps

A *cycle of length* ℓ in a graph $\Gamma = (V, E)$, sometimes called an ℓ-*cycle*, is a sequence

$$C = (e_1, e_2, \ldots, e_\ell) \tag{1.1}$$

of ℓ pairwise distinct edges each of the form $[\alpha_{i-1}, e_i, \alpha_i]$, for $1 \leq i \leq \ell$, and we read the subscripts modulo ℓ so that, in particular, $\alpha_\ell = \alpha_0$. The *edge induced subgraph* $[C]$ of a

Γ_0	N	N_V	Conditions	Reference
$\mathbf{C}_n$	n	1	$n \geq 3$	Definition 4.1, Lemma 4.2(a)
$\mathbf{K}_{s,t}$	st	2	$\gcd(s,t) = 1$, $st > 1$	Definition 4.8, Lemma 4.9

Table 1: Table for Theorem 1.1 with G regular on E.

Γ_0	N	Conditions	Reference
$\mathbf{C}_n^{(2)}$	n	$n \geq 3$	Definition 4.1, Lemma 4.2(b)
$\mathbf{C}_{2n} + n\mathbf{K}_2^{(2)}$	$2n$	$n \geq 2$	Definition 4.3, Corollary 4.5(a)
$2\mathbf{C}_n + n\mathbf{K}_2^{(2)}$	$2n$	$n \geq 3$ odd	Definition 4.3, Corollary 4.5(b)
$\mathrm{Circ}(n, S)$	n	$n \geq 5$, $\lvert S\rvert = 4$, $S = \{a, -a, b, -b\}$, $\gcd(n, a, b) = 1$	Definition 4.3, Lemma 4.4

Table 2: Table for Theorem 1.1 with G bi-regular on E and transitive on V.

Γ_0	N	N_V	Conditions	Reference
$\mathbf{C}_n = \frac{n}{2}\mathbf{K}_2 + \frac{n}{2}\mathbf{K}_2$	$\frac{1}{2}n$	2	$n \geq 4$ even	Def. 4.1, Lemma 4.2(c)
$\mathbf{K}_{s,t}^{(2)}$	st	2	$\gcd(s,t) = 1$, $st > 1$	Def. 4.8, Lemma 4.9
$\mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$	rst	2	$r \geq 2$, $\gcd(s,t) = 1$, $st > 1$	Def. 4.6, Lemma 4.7
$r\mathbf{K}_2^{(2t)} + \mathbf{K}_{2r,t}$	$2rt$	2	$rt \geq 1$, $\gcd(2r,t) = 1$	Def. 4.10, Lemma 4.11
$r\mathbf{K}_2^{(2t)} + 2\mathbf{K}_{r,t}$	$2rt$	2	r odd, $rt \geq 1$, $\gcd(r,t) = 1$	Def. 4.12, Lemma 4.13
$r\mathbf{C}_n^{(t)} + \mathbf{K}_{nr,t}$	nrt	2	$n \geq 3$, $rt \geq 1$, $\gcd(nr,t) = 1$	Def. 4.10, Lemma 4.11
$r\mathbf{C}_{su}^{(t)} + u\mathbf{K}_{sr,t}$	$srut$	2	$su \geq 3$, $u \geq 2$, $\gcd(r,u) = 1$ and $\gcd(sr,t) = 1$	Def. 4.12, Lemma 4.13
$r\mathbf{K}_{sr',ut'}^{(t)} + r'\mathbf{K}_{sr,ut}^{(t')}$	$rr'stt'u$	3	$\gcd(r,r') = \gcd(t,t') = 1$, $\gcd(sr,ut) = \gcd(sr',ut') = 1$	Def. 4.15, Lemma 4.16

Table 3: Table for Theorem 1.1 with G bi-regular on E and intransitive on V.

cycle C is the graph with vertex set $V(C) := \{\alpha_i \mid 1 \leqslant i \leqslant \ell\}$, edge set $E(C) := \{e_i \mid 1 \leqslant i \leqslant \ell\}$, and incidence as in Γ. We call C an *Euler cycle* of Γ if $E(C) = E$, that is, the cycle 'passes through each edge of Γ exactly once'. If Γ possesses an Euler cycle then, in particular, Γ is connected. Further, if C as in (1.1) is an Euler cycle of Γ, then since each edge of E occurs exactly once in C, and since whenever a vertex $\alpha = \alpha_i$ then both e_i and e_{i+1} are incident with α, it follows that *for each vertex $\alpha \in V$, the number of edges (of C, and hence of Γ) incident with α is even.*

For any cycle C, the following bijections on $E(C)$ form a dihedral group of order 2ℓ:

$$\varphi\colon e_i \to e_{i+1} \quad \text{and} \quad \tau\colon e_i \to e_{\ell+1-i} \quad (\text{for each } i, \text{reading subscripts modulo } \ell). \tag{1.2}$$

The reflection $C\tau = (e_\ell, e_{\ell-1}, \ldots, e_1)$, and each shift $C\varphi^i = (e_{i+1}, \ldots, e_\ell, e_1, \ldots, e_i)$ are also ℓ-cycles of Γ, and

$$\mathrm{D}(C) := \langle \varphi, \tau \rangle = \langle \varphi, \tau \mid \varphi^\ell = \tau^2 = 1, \varphi^\tau = \varphi^{-1} \rangle \cong \mathrm{D}_{2\ell} \tag{1.3}$$

is the subgroup of all permutations of $E(C)$ which preserve the edge-sequencing of C, up to rotations and reflections. Thus the subgroup of $\mathrm{Aut}\Gamma$ leaving the cycle C invariant induces on $[C]$ a subgroup $H(C)$ of $\mathrm{Aut}[C]$ contained in $\mathrm{D}(C)$. If this subgroup contains φ^2, that is to say, if there is an element $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma|_{[C]} = \varphi^2$, then we say that C is *symmetrical in* Γ. In particular, for an Euler cycle C of a graph $\Gamma = (V, E)$, C is symmetrical if and only if $\mathrm{Aut}\Gamma$ contains a cyclic subgroup which preserves C and is regular or bi-regular on E. For future reference, we note the definitions of the elements φ^2 and $\varphi\tau$ of $D(C)$:

$$\varphi^2\colon e_i \to e_{i+2} \quad \text{and} \quad \varphi\tau\colon e_i \to e_{\ell-i} \quad (\text{for each } i, \text{reading subscripts modulo } \ell). \tag{1.4}$$

For the exceptional graphs $\mathbf{K}_2^{(\lambda)}$, we show in Proposition 2.2 that there is a symmetrical Euler cycle if and only if λ is even. For all other (connected) graphs we apply Theorem 1.1 to determine whether or not they have a symmetrical Euler cycle.

Theorem 1.2. *Let $\Gamma = (V, E)$ be a graph with $|V| \geq 3$ which has a symmetrical Euler cycle C, and suppose that the subgroup of $\mathrm{Aut}\Gamma$ which preserves C induces the subgroup $H(C)$ of $D(C)$, as in* (1.3).

(a) *Then $(\Gamma, H(C))$ are as in one of the lines of* Table 4.

(b) *Conversely, if Γ is one of the graphs in lines $1-5$ of* Table 4*, then Γ has a symmetrical Euler cycle C with the given $H(C)$; and also if Γ is as in line 6 of* Table 4 *with at least one of $\gcd(n, a+b) = 1$ or $\gcd(n, a-b) = 1$, then Γ has a symmetrical Euler cycle.*

As we discuss in Remark 7.5, if $\Gamma = \Gamma_0^{(\lambda)}$ with $\Gamma_0 = \mathrm{Circ}(n, S)$ as in line 6 of Table 4, and if Γ_0 is a *normal Cayley graph* (that is the translation subgroup $\mathbb{Z}_n$ is normal in $\mathrm{Aut}\Gamma_0$), then the the condition '$\gcd(n, a+b) = 1$ or $\gcd(n, a-b) = 1$' is necessary and sufficient for existence of a symmetrical Euler cycle C. A complete analysis of these graphs would need a better understanding of any non-normal, edge-transitive, Cayley graphs in this family.

The principle motivation for our work was the study of boundary cycles of faces of arc-transitive maps. The link between arc-transitive maps and symmetrical Euler cycles is

Γ	Conditions for C to exist	$H(C)$	Reference
$\mathbf{C}_n^{(\lambda)}$	all $n \geq 3, \lambda \geq 1$	$H(C) = D(C)$	Lemma 7.1(a)
$\mathbf{K}_{s,t}^{(\lambda)}$	$st > 1, \gcd(s,t) = 1, \lambda$ even	$H(C) = \langle \varphi^2, \varphi\tau \rangle$	Lemma 7.1(b)
$(\mathbf{C}_{2n} + n\mathbf{K}_2^{(2)})^{(\lambda)}$	$n \geq 2, n$ even, $\lambda \geq 1$	$H(C) = \langle \varphi^2, \varphi\tau \rangle$	Lemma 7.3(a)
$(2\mathbf{C}_n + n\mathbf{K}_2^{(2)})^{(\lambda)}$	$n \geq 3, n$ odd, $\lambda \geq 1$	$H(C) = \langle \varphi^2, \varphi\tau \rangle$	Lemma 7.3(b)
$(\mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1])^{(\lambda)}$	$r \geq 2, st > 1, \gcd(s,t) = 1, \lambda \geq 1$	$H(C) = \langle \varphi^2, \tau \rangle$	Lemma 7.6
$(\mathrm{Circ}(n,S))^{(\lambda)}$	$n \geq 5, S = \{a, -a, b, -b\}$, $\lvert S \rvert = 4, \gcd(n,a,b) = 1, \lambda \geq 1$	$H(C) \geq \langle \varphi^2 \rangle$	Lemma 7.4 see Remark 7.5

Table 4: Table for Theorem 1.2 with C a symmetrical Euler cycle for Γ.

made explicit in Lemma 3.2. A natural problem which we plan to explore in further work is to understand which of these graphs from Theorem 1.2 actually arise in arc-transitive maps.

Problem 1.3. Determine which of the graphs in Table 4 arise as the induced subgraph of the boundary cycle of a face in an arc-transitive map.

We were able to find a partial answer in our study of vertex-rotary maps, that is, arc-transitive maps for which a vertex-stabiliser is cyclic and regular on the edges incident with it. We prove in [8, Theorem 1.9] that each face boundary cycle C for a vertex-rotary map has induced subgraph $[C] = \mathbf{C}_n^{(\lambda)}$ as in line 1 of Table 4, and moreover the cycle C is of the form $C_0^{(\lambda)}$, with C_0 a simple n-cycle, as in Proposition 3.4(b). In addition, for $n \equiv 2 \pmod 4$, we construct in [8, Section 6] infinitely many examples of such maps from extenders of complete bipartite graphs, namely, if $n = 2m$ with m odd, and if λ is such that $\gcd(n, \lambda) = 2$, we construct an embedding of $\mathbf{K}_{m,m}^{(2\lambda)}$ with all face boundary cycles of the form $C_0^{(\lambda)}$ for a simple n-cycle C_0. In particular, it would be interesting to know precisely which values of n, λ are possible for vertex-rotary maps.

2 Preliminaries and the exceptional graph $\mathbf{K}_2^{(\lambda)}$

2.1 Notation

For a graph $\Gamma = (V, E, \mathbf{I})$, and any subset of edges $E' \subseteq E$, the *edge-induced subgraph* is the graph $[E'] := (V(E'), E', \mathbf{I}')$, where

$$V(E') = \{\alpha \mid (\alpha, e) \in \mathbf{I},\ \text{for some } e \in E'\}, \text{ and } \mathbf{I}' = \mathbf{I} \cap (V(E') \times E'.$$

An *isolated vertex* of a graph Γ, is a vertex which is not incident to any edge of Γ. Thus Γ has no isolated vertices if and only if $V(E) = V$. The graph Γ is *connected* if, for each pair of vertices α, β there exists a sequence of edges $e_1, \ldots, e_r$ such that $e_i = [\alpha_{i-1}, e_i, \alpha_i]$ for each i, and $\alpha_0 = \alpha$ and $\alpha_r = \beta$. A *connected component of* Γ is either (i) a one-vertex graph $(\{\alpha\}, \emptyset)$ where α is an isolated vertex, or (ii) a connected edge-induced subgraph $[E']$ for some non-empty $E' \subseteq E$ such that, for each edge $e \in E$, either $e \in E'$, or e is incident with no vertex of $V(E')$.

For some of the graphs in Theorem 1.1, the group considered has two edge-orbits, and we view the graph as an edge-disjoint union of two smaller graphs. We use the following notation: given graphs $\Gamma_i = (V_i, E_i, \mathbf{I}_i)$, for $i = 1, 2$, where the vertex sets V_1, V_2 may overlap but $E_1 \cap E_2 = \emptyset$, the *edge-disjoint union* of Γ_1 and Γ_2 is the graph $\Gamma_1 + \Gamma_2 = (V, E, \mathbf{I})$, such that

$$V = V_1 \cup V_2,\ E = E_1 \cup E_2, \text{ and } \mathbf{I} = \mathbf{I}_1 \cup \mathbf{I}_2 \subset V \times E.$$

2.2 The graph $\mathbf{K}_2^{(\lambda)}$

First we prove the assertion, mentioned in the introduction, that essentially the only graphs for which the automorphism group is not faithful on edges are those with at least one component $\mathbf{K}_2^{(\lambda)}$.

Lemma 2.1. *Let $\Gamma = (V, E, \mathbf{I})$ be a graph with no isolated vertices and $G \leq \mathrm{Aut}\Gamma$. Then either G acts faithfully on E, or some connected component of Γ is isomorphic to $\mathbf{K}_2^{(\lambda)}$, for some λ.*

Proof. Let $K = G_{(E)}$ be the kernel of the action of G on E. By the definition of an automorphism, K acts faithfully on V. If K is trivial on V then $K = 1$ and there is nothing to prove, so assume that $x \in K$ and $\alpha \in V$ such that $\alpha^x \neq \alpha$. Since Γ has no isolated vertices there exists an edge $e = [\alpha, e, \beta]$, and since x fixes e it follows that x interchanges α and β. Since x fixes all edges, it follows that each edge incident with α is also incident with β, and conversely, each edge incident with β is also incident with α. Hence Γ has a connected component with vertex set $\{\alpha, \beta\}$, and it is isomorphic to $K_2^{(\lambda)}$, for some λ. □

The graph $\Gamma = \mathbf{K}_2^{(\lambda)} = (V, E)$ has automorphism group $\mathrm{Aut}\Gamma = \mathrm{Sym}(E) \times \mathrm{Sym}(V) \cong \mathrm{Sym}(\lambda) \times \mathbb{Z}_2$, with $\mathrm{Sym}(E) = \mathrm{Sym}(\lambda)$ acting naturally on E and fixing V pointwise, and $\mathrm{Sym}(V)$ acting naturally on V and fixing E pointwise. The additional automorphisms, since $\mathrm{Aut}(\Gamma)$ is not faithful on edges, give rise to extra examples of edge-regular and bi-regular actions, and it is instructive to consider these graphs separately here. Of the six edge regular or bi-regular groups G identified in Table 5, for precisely two of them (lines 1 and 4) the induced regular or bi-regular action on the edge set is unfaithful.

Proposition 2.2. *Let $\Gamma = \mathbf{K}_2^{(\lambda)} = (V, E)$ for some $\lambda \geq 1$, and let $G = \langle g \rangle \leq \mathrm{Aut}\Gamma$ be regular or bi-regular on E. Then*

(a) *$G \leq L \times \langle y \rangle$, where $\langle y \rangle = \mathrm{Sym}(V) \cong \mathbb{Z}_2$, $L = \langle x \rangle \cong \mathbb{Z}_\lambda$ is a cyclic edge-regular subgroup of* $\mathrm{Sym}(E)$*, and G, λ are as in one of the lines of* Table 5.

(b) *Moreover, Γ has a symmetrical Euler cycle C if and only if λ is even, and in this case $H(C) = \langle \varphi, \tau \rangle = D(C)$, with $\varphi, \tau, D(C)$ as in* (1.2) *and* (1.3).

Proof. (a) The projection $\pi\colon G \to \mathrm{Sym}(E)$ has image a cyclic edge-regular or bi-regular subgroup, and in either case there exists a cyclic edge-regular subgroup $L \leq \mathrm{Sym}(E)$ containing $\pi(G)$. Thus $G \leq L \times \langle y \rangle$ with $L = \langle x \rangle \cong \mathbb{Z}_\lambda$. Suppose first that G is edge-regular, so $\pi(G) = L$ and $|G|$ is a multiple of λ. One possibility is that $G = L \times \langle y \rangle$, and since G is cyclic this implies that λ is odd and line 1 of Table 5 holds. So suppose that G is a proper subgroup of $L \times \langle y \rangle$. Since $|G|$ is a multiple of λ, it follows that $|G| = \lambda$. If G is vertex-transitive then $G = \langle xy \rangle \cong \mathbb{Z}_\lambda$ and this implies that λ is even, and line 2 of Table 5

G	$\lvert G\rvert$	Action on V	Action on E	Conditions/comments
$\langle x\rangle \times \langle y\rangle$	2λ	transitive	regular	λ odd, $G_{(E)} \neq 1$
$\langle xy\rangle$	λ	transitive	regular	λ even
$\langle x\rangle$	λ	trivial	regular	λ arbitrary
$\langle x^2\rangle \times \langle y\rangle$	λ	transitive	bi-regular	$\lambda = 2\lambda_0$, λ_0 odd, $G_{(E)} \neq 1$
$\langle x^2 y\rangle$	λ_0	transitive	bi-regular	$\lambda = 2\lambda_0$, λ_0 even
$\langle x^2\rangle$	λ_0	trivial	bi-regular	$\lambda = 2\lambda_0$, λ_0 arbitrary

Table 5: Edge regular and bi-regular actions on $\mathbf{K}_2^{(\lambda)}$.

holds. On the other hand, if G is intransitive, then $G = \langle x\rangle \cong \mathbb{Z}_\lambda$ and line 3 of Table 5 holds (and here λ is arbitrary).

Suppose now that G is edge-bi-regular, so $\lambda = 2\lambda_0$ is even and $\pi(G) = \langle x^2\rangle \cong \mathbb{Z}_{\lambda_0}$, and $G \leq \langle x^2\rangle \times \langle y\rangle$. If $G = \langle x^2\rangle \times \langle y\rangle$, then $\lambda_0 = |x^2|$ is odd since G is cyclic, and line 4 of Table 5 holds. Suppose now that G is a proper subgroup of $\langle x^2\rangle \times \langle y\rangle$, so $|G| = \lambda_0$. If G is vertex-transitive then $G = \langle x^2 y\rangle \cong \mathbb{Z}_{\lambda_0}$ and this implies that λ_0 is even, and line 5 of Table 5 holds. On the other hand, if G is vertex-intransitive, then $G = \langle x^2\rangle \cong \mathbb{Z}_{\lambda_0}$ and line 6 of Table 5 holds (and here λ_0 is arbitrary).

(b) By our comments in Subsection 1.2, in an Euler cycle each vertex is incident with an even number of edges. Hence there is no Euler cycle if λ is odd. Suppose that λ is even. Then it is easy to sequence the edges into a symmetrical Euler cycle using, say, the edge-regular automorphism x: choose $e \in E$ and define $e_0 = e$ and $e_i = e^{x^i}$ for $1 \leq i \leq \lambda - 1$. Then $C = (e_0, \ldots, e_{\lambda-1})$ is an Euler cycle and x induces $\varphi \in D(C)$ and fixes each of the vertices. Also there is an involution $z \in N_{\mathrm{Sym}(E)}(\langle x\rangle)$ given by $z\colon e_i \to e_{\lambda-i}$ for each i, and hence z induces $\varphi\tau$ in $H(C)$, and it follows that $H(C) = D(C)$. □

3 Cycles, symmetrical Euler cycles, and maps

Let C be an ℓ-cycle in a graph $\Gamma = (V, E, \mathbf{I})$, as in (1.1), with edges $[\alpha_{i-1}, e_i, \alpha_i]$, reading the subscripts modulo ℓ. If the vertices α_i are pairwise distinct, then $[C]$ is a simple graph of valency 2, and hence $[C] = \mathbf{C}_\ell$. In this case we call C a *simple ℓ-cycle*, and since the vertices α_i are pairwise distinct, we can identify C with the vertex sequence $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$.

Other examples of ℓ-cycles C arise naturally with $[C] = \mathbf{C}_n^{(\lambda)}$ where $\ell = n\lambda$, with the edges sequenced in any way so that $\alpha_i = \alpha_j$ whenever $i \equiv j \pmod{n}$. The set of all ℓ-cycles with this property forms a single orbit under the automorphism group of $\mathbf{C}_n^{(\lambda)}$, and we call each such ℓ-cycle *standard*, or a *standard edge-sequencing*. However not all cycles in graphs are of this kind. Even for $[C] = \mathbf{C}_n^{(\lambda)}$ with $\lambda > 1$, there are many non-standard ways to seqence the edges to form a cycle. For example, if e_{ij}^1, e_{ij}^2 denote the two edges of $\mathbf{C}_3^{(2)}$ incident with vertices i, j, then $(e_{12}^1, e_{12}^2, e_{13}^1, e_{33}^1, e_{23}^2, e_{13}^2)$ is a non-standard 6-cycle. Moreover the induced subgraph $[C]$ of a cycle C may be quite different from $\mathbf{C}_n^{(\lambda)}$, as the next examples show (some of which appear again in our main theorems).

Example 3.1. Let $\Gamma = \Sigma^{(\lambda)}$, with $\lambda = 2$ and with Σ one of the following simple graphs. Then Γ has constant edge-multiplicity 2, and its edge set can be sequenced to form an Euler cycle. (Finding these sequencings is left as an easy exercise.)

(a) Σ is the complete bipartite graph $\mathbf{K}_{2,3}$, so Γ has $\ell = 12$ edges and $n = 5$ vertices;

(b) Σ is the cartesian product $\mathbf{C}_r \,\square\, \mathbf{K}_2$, so Γ has $\ell = 6r$ edges and $n = 2r$ vertices, and Γ is regular of valency 3 (each vertex is adjacent to 3 vertices);

(c) $\Sigma = (V, E, I)$ with $V = \mathbb{Z}_r \times \mathbb{Z}_2$ and $E = \{e_i, f_i \mid i \in \mathbb{Z}_r\}$ such that, for each $i \in \mathbb{Z}_r$, e_i is incident with $(i, 0)$ and $(i + 1, 0)$, and f_i is incident with $(i, 0)$ and $(i, 1)$. Here $\Gamma = \Sigma^{(2)}$ has $\ell = 4r$ edges, $n = 2r$ vertices, and we note that $\ell = n\lambda$.

These examples illustrate that, for an ℓ-cycle C, the induced graph $[C]$ need not be regular (examples (a) and (c)), or $[C]$ may be regular of valency greater than 2 (example (b)), and even if $[C]$ has constant edge-multiplicity λ and $n\lambda$ edges, where n is the number of vertices, $[C]$ need not be $\mathbf{C}_n^{(\lambda)}$ (example (c) again). These observations hold in particular for the Euler cycles of the graphs in Example 3.1.

3.1 Symmetrical cycles and arc-transitive maps

Let $C = (e_1, e_2, \ldots, e_\ell)$ be an ℓ-cycle in a graph Γ. As we noted in Section 1.2, each rotation (shift) $C\varphi^i$ and reflection (reversal followed by a shift) $C\tau\varphi^i$ of C is again a cycle involving the same edge-set $E(C)$ as C. This collection of cycles is called the *sequence class* of cycles containing C. We say that an automorphism $g \in \mathrm{Aut}\Gamma$ *preserves* C if g leaves invariant the sequence class of C. Thus the subgroup $(\mathrm{Aut}\Gamma)_C$ of $\mathrm{Aut}\Gamma$ preserving C induces a subgroup $H(C)$ of automorphisms of the induced subgraph $[C]$ such that $H(C) \leq \mathrm{D}(C)$ with $\mathrm{D}(C)$ as in (1.3). As discussed above, C is called symmetrical if $H(C)$ contains $\langle\varphi^2\rangle$, and for such cycles, Theorem 1.2 determines all possible induced subgraphs $[C]$. Apart from $\langle\varphi^2\rangle$ itself, the subgroups $H(C)$ of $\mathrm{D}(C)$ containing $\langle\varphi^2\rangle$ are listed in Table 6. These subgroups are relevant for studying edge-transitive or face-transitive embeddings of graphs in Riemann surfaces, see for example, [12] or [4, Chapter 3, especially Theorem 3.5].

A *map* $\mathcal{M} = (V, E, F)$ is a 2-cell embedding of a graph $\Gamma = (V, E)$ in a closed surface with face set F, and the graph Γ is called the *underlying graph* of $\mathcal{M}$. The *automorphism group* of the map is the subgroup of $\mathrm{Aut}(\Gamma)$ consisting of those graph automorphisms which extend to self-homeomorphisms of the supporting surface [7, Section 6]. It is also sometimes defined as the group of permutations of the *flags* (mutually incident vertex-edge-face triples) that 'preserve vertex-vertex and face-face adjacencies' [12, Section 2]. An *arc* of a map, or a graph, is an incident vertex–edge pair, and a map $\mathcal{M}$ is called *arc-transitive* if its automorphism group $\mathrm{Aut}\mathcal{M}$ is transitive on the arc-set of $\mathcal{M}$. The next result describes the the kinds of groups $H(C)$ which may arise for boundary cycles C of arc-transitive maps. In [9] we discuss finite maps $\mathcal{M} = (V, E, F)$ with arc-transitive automorphism groups, where the underlying graph $\Gamma = (V, E)$ may have multiple edges, as in this paper. The only facts about maps we use in the proof of Lemma 3.2 are that each edge in E is incident with exactly two faces in F, together with the arc-transitivity of the map group $\mathrm{Aut}\mathcal{M} \leq \mathrm{Aut}\Gamma$.

Lemma 3.2. *Let $\mathcal{M} = (V, E, F)$ and $G \leqslant \mathrm{Aut}\mathcal{M}$ such that G is arc-transitive on $\mathcal{M}$, and each edge in E is incident with exactly two faces in F. Let $f \in F$ be a face, and $C = (e_1, \ldots, e_\ell)$ the boundary cycle of f. Then the stabiliser G_f induces a subgroup $H(C)$ of* $\mathrm{D}(C)$ *as in one of the lines of* Table 6*, which gives also the $H(C)$-orbits on edges $E(C)$ and vertices $V(C)$ of C. In each case, $H(C)$ contains the cyclic subgroup*

$H(C)$	orbits in $E(C)$	orbits in $V(C)$	Comments
$\mathrm{D}(C)$	$E(C)$	$V(C)$	
$\langle\varphi\rangle$	$E(C)$	$V(C)$	equals $\langle\varphi^2\rangle$ if ℓ odd
$\langle\varphi^2, \varphi\tau\rangle$	two orbits	$V(C)$	ℓ even
$\langle\varphi^2, \tau\rangle$	$E(C)$	two orbits if $\lambda = 1$	ℓ even

Table 6: Groups $H(C)$ for a boundary cycle C of an arc-transitive map $\mathcal{M}$.

$\langle\varphi^2\rangle$ which is regular (if ℓ is odd) or bi-regular (if ℓ is even) on $E(C)$, and $\langle\varphi^2\rangle$ has at most two orbits in $V(C)$. In particular C is a symmetrical Euler cycle of $[C]$.

Proof. For $C = (e_1, \ldots, e_\ell)$ we have $e_i = [\alpha_{i-1}, e_i, \alpha_i]$ for each i, reading subscripts modulo ℓ. Let f_i be the second face incident with e_i, for each i. The setwise stabiliser G_f preserves C and induces a subgroup $H(C)$ of $\mathrm{D}(C)$. As $\mathcal{M}$ is G-arc-transitive, there exist $g, h \in G$ such that $(\alpha_0, e_1, \alpha_1)^h = (\alpha_1, e_1, \alpha_0)$ and $(\alpha_0, e_1, \alpha_1)^g = (\alpha_1, e_2, \alpha_2)$. Suppose first that $f^h = f$. Then also $f_1^h = f_1$. For each i there exists $g_i \in \mathrm{Aut}\mathcal{M}$ such that $e_1^{g_i} = e_i$ and so g_i maps the pair $\{f, f_1\}$ to the pair of faces $\{f, f_i\}$ incident with e_i. In particular $f^{g_i^{-1}}$ is one of f, f_1 and so is fixed by h and hence $f^{g_i^{-1}hg_i} = f^{g_i^{-1}g_i} = f$. It follows that $H(C)$ contains each edge-reflection, and hence contains φ^2 as well as the edge-reflection induced by h. Thus in this case $H(C)$ is either $\langle\varphi^2, \varphi\tau\rangle$ with ℓ even and has two orbits in $E(C)$, or is $\mathrm{D}(C)$, and in either case $H(C)$ is transitive on $V(C)$. Thus we may assume that each element h such that $(\alpha_0, e_1, \alpha_1)^h = (\alpha_1, e_1, \alpha_0)$ interchanges f and f_1, and hence that $g_i^{-1}hg_i$ interchanges f and f_i, for each i. If g leaves the face f invariant then $H(C)$ contains φ so $H(C) = \langle\varphi\rangle$ (by our assumption on the elements h) and $H(C)$ is transitive on both $E(C)$ and $V(C)$.

So assume also that g does not fix f for any such g mapping $(\alpha_0, e_1, \alpha_1)$ to $(\alpha_1, e_2, \alpha_2)$. Then $f_1^g = f$, $f^g = f_2$, and hence hg leaves f invariant. Now hg induces a reflection of C in the vertex α_1. Similar arguments, with the edges e_1, e_2 replaced by e_2, e_3, either yield one of the groups above for $H(C)$, or show that $H(C)$ contains also the reflection of C in the vertex α_2. In the latter case the product of the reflection in α_1 and the reflection in α_2 is a generator of the group $\langle\varphi^2\rangle$, and hence these two reflections generate the group $H(C) = \langle\varphi^2, \tau\rangle$, which is regular on $E(C)$ and has two vertex-orbits when ℓ is even. (If ℓ is odd the group generated is $\mathrm{D}(C)$.) □

Remark 3.3. We note that the subgroup $\langle\varphi^2, \tau\rangle$ contains all reflections of C 'through a vertex': τ reflects C through the vertex α_0 (between the edges e_ℓ and e_1), while $\tau\varphi^2$ reflects C through the vertex α_1 (between the edges e_1 and e_2), etc. Thus, for a symmetrical cycle C as in (1.1), to prove that $H(C) = \langle\varphi^2, \tau\rangle$ we need only prove that $H(C)$ contains one of these reflections. Similar comments apply to the subgroup $\langle\varphi^2, \varphi\tau\rangle$, which contains all reflections of C 'through an edge', where $\varphi\tau$ reflects C through the edge e_ℓ, etc. To prove that $H(C) = \langle\varphi^2, \varphi\tau\rangle$, for a symmetrical cycle C, we need only prove that $H(C)$ contains one reflection through an edge. We note also that it is sometimes more convenient in our analysis to label the edges of an ℓ-cycle as $(e_0, e_1, \ldots, e_{\ell-1})$ for working with the action of the cyclic group $\mathbb{Z}_\ell$.

ψ induces in $H(C)$	$\psi^{(\lambda)}$ on edges, for i mod ℓ, j mod λ
φ	$e_i^j \to \begin{cases} e_{i+1}^j & \text{if } 1 \le i \le \ell-1 \\ e_{i+1}^{j+1} & \text{if } i = \ell \end{cases}$
φ^2	$e_i^j \to \begin{cases} e_{i+2}^j & \text{if } 1 \le i \le \ell-2 \\ e_{i+2}^{j+1} & \text{if } i \in \{\ell-1, \ell\} \end{cases}$
τ	$e_i^j \to \begin{cases} e_{\ell+1-i}^{\lambda+1-j} & \text{if } 1 \le i \le \ell-1 \\ e_{\ell+1-i}^{\lambda-j} & \text{if } i = \ell \end{cases}$
$\varphi\tau$	$e_i^j \to \begin{cases} e_{\ell-i}^{\lambda+1-j} & \text{if } 1 \le i \le \ell-2 \\ e_{\ell-i}^{\lambda-j} & \text{if } i = \ell-1 \\ e_{\ell}^{\lambda-j} & \text{if } i = \ell \end{cases}$

Table 7: Elements in $H(C^{(\lambda)})$ for an Euler cycle C of Γ; in all cases $\psi^{(\lambda)}|_V = \psi|_V$.

3.2 Symmetrical Euler cycles and λ-extenders

It turns out that if a graph Γ (not necessarily a simple graph) has a symmetrical Euler cycle, then so does its λ-extender for each λ. We prove this as a consequence of a more general result about cyclic edge regular or bi-regular subgroups of automorphisms. It is useful to use the following notation: for $\Gamma = (V, E)$, we have $\Gamma^{(\lambda)} = (V, E^{(\lambda)})$, and we label the λ edges in $E^{(\lambda)}$ corresponding to the edge $e = [\beta, e, \alpha] \in E$ by

$$e^j = [\beta, e^j, \alpha], \text{ for } 1 \leqslant j \leqslant \lambda. \tag{3.1}$$

Proposition 3.4. *Suppose that, for a graph $\Gamma = (V, E)$, $\mathrm{Aut}\Gamma$ has a cyclic subgroup that is regular or bi-regular on E with t orbits on V, and let λ be a positive integer. Then, for the λ-extender $\Gamma^{(\lambda)}$,*

(a) $\mathrm{Aut}\Gamma^{(\lambda)}$ *has a cyclic regular or bi-regular subgroup, respectively, with t orbits on V;*

(b) *if $C = (e_1, \ldots, e_\ell)$ is an Euler cycle of Γ, and if $\psi \in \mathrm{Aut}\Gamma$ induces one of $\varphi, \varphi^2, \tau, \varphi\tau$ in $H(C)$, as in (1.2) and (1.4), then*

$$C^{(\lambda)} = (e_1^1, \ldots, e_\ell^1, e_1^2, \ldots, e_\ell^2, \ldots, e_1^\lambda, \ldots, e_\ell^\lambda)$$

is an Euler cycle of $\Gamma^{(\lambda)}$, and the map $\psi^{(\lambda)}$ as in Table 7 *lies in $\mathrm{Aut}\,\Gamma^{(\lambda)}$ and induces the element $\varphi, \varphi^2, \tau, \varphi\tau$ in $H(C^{(\lambda)})$, respectively. In particular, if Γ has a symmetrical Euler cycle, then also $\Gamma^{(\lambda)}$ has a symmetrical Euler cycle.*

Proof. (a) By assumption, there is an element $\psi \in \mathrm{Aut}\Gamma$ such that $G = \langle\psi\rangle$ is regular or bi-regular on E. Thus we may label the edge-set $E = \{e_1, \ldots, e_\ell\}$, such that $e_i = [\beta_i, e_i, \alpha_i]$ for each i, and for $1 \leqslant i \leqslant \ell$, reading subscripts modulo ℓ,

$$\psi\colon\ e_i \to e_{i+1},\ \text{if } \psi \text{ is regular on } E, \text{ or } \psi\colon\ e_i \to e_{i+2},\ \text{if } \psi \text{ is bi-regular on } E,$$

and hence $\psi\colon \{\beta_i, \alpha_i\} \to \{\beta_{i+1}, \alpha_{i+1}\}$ or $\{\beta_i, \alpha_i\} \to \{\beta_{i+2}, \alpha_{i+2}\}$, respectively, for each i. For each i, we label the λ edges of $\Gamma^{(\lambda)} = (V, E^{(\lambda)})$ which correspond to the edge

$e_i = [\beta_i, e_i, \alpha_i]$ by e_i^j, for $1 \leqslant j \leqslant \lambda$, as in (3.1), and we define the map $\psi^{(\lambda)} \colon V \cup E^{(\lambda)} \to V \cup E^{(\lambda)}$ (reading subscripts modulo ℓ, and superscripts modulo λ) by $\psi^{(\lambda)}|_V = \psi|_V$ and

$$\psi^{(\lambda)} \colon e_i^j \to \begin{cases} e_{i+1}^j & \text{if } 1 \le i \le \ell - 1 \\ e_{i+1}^{j+1} & \text{if } i = \ell \end{cases}$$

if $\langle \psi \rangle$ is regular, or

$$\psi^{(\lambda)} \colon e_i^j \to \begin{cases} e_{i+2}^j & \text{if } 1 \le i \le \ell - 2 \\ e_{i+2}^{j+1} & \text{if } i \in \{\ell - 1, \ell\} \end{cases}$$

if $\langle \psi \rangle$ is bi-regular. Then $\psi^{(\lambda)}$ is a bijection and preserves incidence in $\Gamma^{(\lambda)}$, and hence $\psi^{(\lambda)} \in \mathrm{Aut}\Gamma^{(\lambda)}$. Further $\langle \psi^{(\lambda)} \rangle$ is regular or bi-regular on $E^{(\lambda)}$, according as G is regular or bi-regular on E, respectively, and as $\psi^{(\lambda)}|_V = \psi|_V$, the groups $\langle \psi \rangle, \langle \psi^{(\lambda)} \rangle$ have the same number of vertex-orbits. This proves part (a).

(b) Suppose now that Γ has an Euler cycle $C = (e_1, \ldots, e_\ell)$ with each $e_i = [\alpha_{i-1}, e_i, \alpha_i]$, and that ψ induces one of the maps $\varphi, \varphi^2, \tau, \varphi\tau$ in $H(C)$. Let $C^{(\lambda)}$ and $\psi^{(\lambda)}$ be as in the statement and Table 7. From the definition of $E^{(\lambda)}$ it is clear that $C^{(\lambda)}$ is a cycle, and hence an Euler cycle of $\Gamma^{(\lambda)}$. Clearly $\psi^{(\lambda)}$ is a bijection on $V \cup E^{(\lambda)}$ and preserves incidence and hence lies in $\mathrm{Aut}\,\Gamma^{(\lambda)}$. A somewhat tedious checking shows that the map $\psi^{(\lambda)}$ is equal to the element $\varphi, \varphi^2, \tau, \varphi\tau$ in $H(C^{(\lambda)})$, respectively. In particular if C is a symmetrical Euler cycle of Γ, so there is an element $\psi \in \mathrm{Aut}\Gamma$ inducing φ^2 in $H(C)$, then we have just shown that $\psi^{(\lambda)}$ induces φ^2 in $H(C^{(\lambda)})$. Hence $C^{(\lambda)}$ is a symmetrical Euler cycle in $\Gamma^{(\lambda)}$. □

4 Examples of graphs

Here we introduce the families of graphs occurring in the tables for Theorem 1.1. In the light of Proposition 3.4, to show that a λ-extender $\Gamma = \Gamma_0^{(\lambda)}$ satisfies the hypotheses of Theorem 1.1, it is sufficient to show that the graph Γ_0 satisfies them. Thus we will in general consider graphs which are not λ-extenders.

4.1 Circulants

View $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ as a cyclic group of order n under addition. The circulants we construct may have multiple edges. Let S be a multiset of elements from $\mathbb{Z}_n \setminus \{0\}$ such that, if an element $z \in \mathbb{Z}_n$ appears exactly λ times in S, we write $z^{(\lambda)} \in S$. Assume further that S is *self-inverse*, that is, if $z^{(\lambda)} \in S$ then $(-z)^{(\lambda)} \in S$. For such a self-inverse multiset S, $\mathrm{Circ}(n, S)$ denotes the (Cayley) graph with vertex set $\mathbb{Z}_n$ such that, for each $v \in \mathbb{Z}_n$ and each $z^{(\lambda)} \in S$, there are exactly λ edges between v and $v + z$. Graphs of this form are called *circulants of order* n. If each element of a multiset has the same multiplicity λ, then we sometimes denote the multiset by $S^{(\lambda)}$ for some subset $S \subseteq \mathbb{Z}_n$, that is, $S^{(\lambda)} = \{s^{(\lambda)} \mid s \in S\}$. First we construct the graphs occurring in Theorem 1.1 which are essentially cycles.

Definition 4.1. Let n be an integer, $n \geq 3$, and let $S = \{1, -1\}$, regarded as a multiset in $\mathbb{Z}_n$. Let $\Gamma = \mathrm{Circ}(n, S) = (V, E)$ with $V = \mathbb{Z}_n$ and $E = \{e_i \mid i \in \mathbb{Z}_n\}$, where $e_i = [i, e_i, i+1]$, for $i \in \mathbb{Z}_n$. Define the map $g \colon V \cup E \to V \cup E$ by

$$g \colon i \to i + 1, \text{ and } g \colon e_i \to e_{i+1} \text{ for } i \in \mathbb{Z}_n.$$

Define the edge sequence $C = (e_0, \ldots, e_{n-1})$ of length n.

Lemma 4.2. *With the notation of* Definition 4.1*, the graph* $\Gamma \cong \mathbf{C}_n$*,* Γ *is connected,* $g \in \mathrm{Aut}\Gamma$*, and*

(a) *the group* $\langle g \rangle$ *is transitive on* V *and regular on* E *and line* 1 *of* Table 1 *holds. Moreover,* C *is a symmetrical Euler cycle for* Γ*,* g *induces the map* φ *of* (1.2)*, and* $H(C) = D(C) \cong \mathrm{D}_{2n}$*.*

(b) *For the* 2*-extender* $\mathbf{C}_n^{(2)}$*, the group* $\langle g \rangle$ *is transitive on* V *and bi-regular on* $E^{(2)}$ *and line* 1 *of* Table 2 *holds.*

(c) *If* n *is even,* $n \geq 4$*, then* $\langle g^2 \rangle \cong \mathbb{Z}_{n/2}$ *is bi-transitive on* V*, and bi-regular on* E *with orbits* E_0 *and* E_1 *such that* $[E_0] \cong [E_1] \cong \frac{n}{2}\mathbf{K}_2$*, and line* 1 *of* Table 3 *holds.*

Proof. Clearly $\Gamma \cong \mathbf{C}_n$, Γ is connected, and $g \in \mathrm{Aut}\Gamma$, and most of part (a) follows immediately from the definitions of g and a symmetrical Euler cycle for Γ, with g inducing the map φ in $H(C)$. Further $\mathrm{Aut}\Gamma$ contains an automorphism τ as in (1.2), and $\langle g, \tau \rangle \cong \mathrm{D}_{2n}$, so $H(C) = D(C) = \mathrm{D}_{2n}$. For part (b), we note that $\langle g \rangle$ has two orbits in $E^{(2)}$, namely, for $j \in \{1, 2\}$, $E_j := \{e_i^j \mid i \in \mathbb{Z}_n\}$, where $e_i^j = [i, e_i^j, i+1]$ for $i \in \mathbb{Z}_n$. Finally for part (c), for n even, $\langle g^2 \rangle \cong \mathbb{Z}_{n/2}$ is bi-transitive on V, and bi-regular on E with edge-orbits $E_0 := \{e_{2i} \mid 0 \leq i < n/2\}$ and $E_1 := \{e_{2i+1} \mid 0 \leq i < n/2\}$, and the induced subgraphs $[E_0] \cong [E_1] \cong \frac{n}{2}\mathbf{K}_2$. □

Next we consider the remaining circulants occurring in Table 2 for Theorem 1.1. For $a \in \mathbb{Z}_n$, by $|a|$ we mean the additive order of a, that is, the least positive integer m such that $ma \equiv 1 \pmod{n}$.

Definition 4.3. Let n be a positive integer with $n \geq 3$, and let $a, b \in \mathbb{Z}_n \setminus \{0\}$ such that $\gcd(n, a, b) = 1$. Let $\Gamma = \Gamma(n, a, b) = \mathrm{Circ}(n, S)$ where

$$S = \begin{cases} \{a, -a, b, -b\} & \text{if } |a| \geq 3,\, |b| \geq 3,\, a \neq \pm b \\ \{a, -a, b^{(2)}\} & \text{if } |a| \geq 3,\, 2b = n \end{cases}$$

so Γ has $2n$ edges, namely $e_{i,a} = [i, e_{i,a}, i+a]$, and $e_{i,b} = [i, e_{i,b}, i+b]$, for $i \in \mathbb{Z}_n$, and we note that, if $2b = n$ then $e_{i,b}$ and $e_{i+b,b}$ are both incident with the same pair of vertices $\{i, i+b\}$. Let $E_a = \{e_{i,a} \mid i \in \mathbb{Z}_n\}$, $E_b = \{e_{i,b} \mid i \in \mathbb{Z}_n\}$. Thus Γ is not always simple, but in all cases the natural generator g of $\mathbb{Z}_n$ acts as follows:

$$g\colon i \to i+1, \quad e_{i,a} \to e_{i+1,a}, \quad e_{i,b} \to e_{i+1,b}, \text{ for } i \in \mathbb{Z}_n.$$

Lemma 4.4. *With the notation of* Definition 4.3*, the graph* $\Gamma = \Gamma(n, a, b)$ *is connected, and the group* $\langle g \rangle = \mathbb{Z}_n$ *induces a cyclic subgroup of* $\mathrm{Aut}\Gamma$ *which is regular on vertices and bi-regular on edges, with edge-orbits* E_a *and* E_b*. Moreover, if* $2b \neq n$ *then* Γ *is simple of valency* 4 *as in line* 4 *of* Table 2*.*

Proof. The vertices reached by sequences of edges beginning at 0 are precisely those of the form $ia + jb$ for some integers i, j, and these vertices are the multiples of $\gcd(a, b)$, modulo n. Since $\gcd(n, a, b) = 1$ it follows that all vertices occur so Γ is connected and $\mathbb{Z}_n = \langle a, b \rangle$. By definition of circulants, $\mathbb{Z}_n$, acting naturally by 'right multiplication' induces a cyclic subgroup of $\mathrm{Aut}\Gamma$ which is regular on vertices, and it has E_a, E_b as its edge-orbits. If $2b \neq n$ then Γ is simple, has valency $|S| = 4$, and is a graph in line 4 of Table 2. $\square$

Note that if $|b| = 2$ in Definition 4.3 then n is even and $b = n/2$. The following corollary describes two special cases of the construction in Definition 4.3 where $|b| = 2$. We note that in case (b) below, the graph $\Gamma(2r, 2, r)$ is isomorphic to the cartesian product $\mathbf{C}_r \square \mathbf{K}_2^{(2)}$.

Corollary 4.5. *Let $n = 2r \geq 4$. Then the following hold, with the notation of* Definition 4.3*,*

(a) $\Gamma = \Gamma(2r, 1, r) = \mathrm{Circ}(2r, \{1, -1\}) + \mathrm{Circ}(2r, \{r^{(2)}\}) = \mathbf{C}_{2r} + r\mathbf{K}_2^{(2)}$*; and*

(b) *if r is odd, then* $\Gamma = \Gamma(2r, 2, r) = \mathrm{Circ}(2r, \{2, -2\}) + \mathrm{Circ}(2r, \{r^{(2)}\}) = 2\mathbf{C}_r + r\mathbf{K}_2^{(2)}$.

In either case, Γ is connected, and $\mathrm{Aut}\Gamma$ has a cyclic subgroup that is transitive on vertices and bi-regular on edges, as in line 2 or 3 of Table 2.

Proof. This follows immediately from Lemma 4.4, noting that $\gcd(2r, 1, r) = 1$, and when r is odd also $\gcd(2r, 2, r) = 1$. $\square$

4.2 Modified Kronecker product graphs

Our next construction is a modification of the Kronecker product construction for graphs, so we use similar notation. It produces the graphs in line 3 of Table 3.

Definition 4.6. Let r, s, t be positive integers such that $r \geq 2$, $st \geq 2$ and $\gcd(s, t) = 1$, and let $S = \mathbb{Z}_s, T = \mathbb{Z}_t$. Then $\Gamma = \mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1] = (V, E)$ is the graph defined as follows. The vertex set $V = V_S \cup V_T$, where

$V_S = \{(2k, i) \mid 0 \leq k \leq r-1,\ i \in S\}$, and $V_T = \{(2k+1, j) \mid 0 \leq k \leq r-1,\ j \in T\}$,

and edge set $E = E_S \cup E_T$ where

$$E_S = \{[(2k, i), e_{i,j}^{2k}, (2k+1, j)] \mid 0 \leq k \leq r-1, i \in S, j \in T\}, \text{and}$$
$$E_T = \{[(2k+1, j), e_{j,i}^{2k+1}, (2k+2, i)] \mid 0 \leq k \leq r-1, i \in S, j \in T\},$$

and we read the entry $2k+2$ modulo $2r$. Define the maps $g, y\colon V \cup E \to V \cup E$ as follows, for $0 \leq k \leq r-1, i \in S, j \in T$:

$$\begin{array}{lcccccl}
g\colon & (2k,i) & \to & (2k+2,i), & (2k+1,j) & \to & (2k+3,j) \quad \text{if } k \leq r-2,\\
& (2r-2,i) & \to & (0,i+1), & (2r-1,j) & \to & (1,j+1)\\
& e_{i,j}^{2k} & \to & e_{i,j}^{2k+2}, & e_{j,i}^{2k+1} & \to & e_{j,i}^{2k+3} \quad \text{if } k \leq r-3,\\
& e_{i,j}^{2r-4} & \to & e_{i,j}^{2r-2}, & e_{j,i}^{2r-3} & \to & e_{j,i+1}^{2r-1}\\
& e_{i,j}^{2r-2} & \to & e_{i+1,j+1}^{0}, & e_{j,i}^{2r-1} & \to & e_{j+1,i}^{1}
\end{array}$$

$$\begin{array}{lcccccc}
y\colon & (0,i) & \to & (0,-i), & (1,j) & \to & (2r-1,-j-1)\\
& (2k,i) & \to & (2(r-k),-i-1), & (2k+1,j) & \to & (2(r-k-1)+1,-j-1)\\
& e_{i,j}^{0} & \to & e_{-j-1,-i}^{2r-1}, & e_{j,i}^{2r-1} & \to & e_{-i,-j-1}^{0}\\
& e_{i,j}^{2k} & \to & e_{-j-1,-i-1}^{2(r-k-1)+1}, & e_{j,i}^{2k-1} & \to & e_{-i-1,-j-1}^{2(r-k)}
\end{array}$$

where $i \in S, j \in T$, and in lines 2 and 4 in the definition of y we have $1 \leq k \leq r-1$.

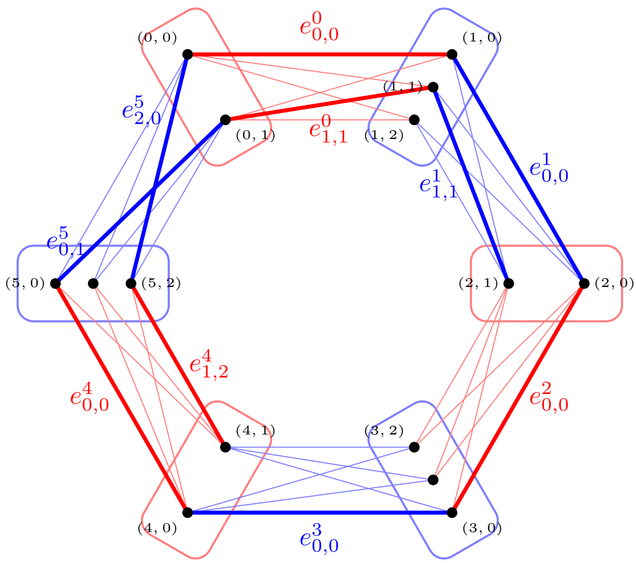

Figure 1: Graph $\Gamma = \mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$ from Definition 4.6, with $r = 3$, $s = 2$, $t = 3$.

We could have defined the graph $\mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$ with $s = t = 1$, but then it is just the cycle $\mathbf{C}_{2r}$ so we avoid this degenerate case. It is not difficult to see that the automorphism group of $\mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$ is isomorphic to $(\mathrm{Sym}(s) \times \mathrm{Sym}(t)) \wr \mathrm{D}_{2r}$. We show that the map g generates a vertex-bi-transitive, edge-bi-regular group of automorphisms, and moreover that $\langle g, y\rangle$ is an edge-regular dihedral group.

Lemma 4.7. *With the notation of* Definition 4.6*,* $\Gamma = \mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$ *is connected and is the graph in line* 3 *of* Table 3*, the maps* g, y *are automorphisms of* Γ*, and*

(a) $\langle g\rangle$ *is cyclic of order* rst*, bi-transitive on* V *with orbits* V_S, V_T*, and bi-regular on* E *with orbits* E_S, E_T*; also* $\langle g, y\rangle \cong \mathrm{D}_{2rst}$ *and is regular on* E*;*

(b) *further, for each positive integer* λ*, the graph* $\Gamma^{(\lambda)}$*, admits a cyclic subgroup* $\mathbb{Z}_{rst\lambda}$*, and a dihedral subgroup* $\mathrm{D}_{2rst\lambda}$ *of automorphisms that are both bi-transitive on vertices, and are bi-regular or regular on edges, respectively.*

Proof. It is straightforward to check that Γ is connected, and that Γ is the graph in line 3 of Table 3. Also g, y are bijections and it is straightforward, if tedious, to check that each of g, y preserves incidence. Hence $g, y \in \mathrm{Aut}\Gamma$. First we consider repeated applications of g on the edge $e = e^0_{0,0} = [(0,0), e^0_{0,0}, (1,0)] \in E_S$. Suppose that $e^{g^m} = e$. Now $e^{g^m} = e^{2m}_{i,j}$ for some $i \in S, j \in T$, reading the superscript $2m$ modulo $2r$, and this means, since $e^{g^m} = e$, that $m = r\ell$ for some integer ℓ. Now $e^{g^r} = [(0,1), e^0_{1,1}, (1,1)]$, and repeating this we find that $e^{g^{r\ell}} = [(0,\ell), e^0_{\ell,\ell}, (1,\ell)]$, where in the first entry we read ℓ modulo s, and in the third entry we read ℓ modulo t. Thus, again since $e^{g^{r\ell}} = e^{g^m} = e$, we conclude that ℓ is divisible by both s and t, and hence by st since $\gcd(s,t) = 1$. Thus m is a multiple of rst, and since $|E_S| = rst$ and g leaves E_S invariant it follows that the $\langle g\rangle$-orbit containing

e is E_S. A similar proof shows that the $\langle g\rangle$-orbit containing $e^1_{0,0}$ is E_T, and it follows that $\langle g\rangle$ is bi-regular on E. Also the vertex-orbits of $\langle g\rangle$ are clearly V_S and V_T and we conclude that $\langle g\rangle$ has order rst, and is bi-transitive on V. This proves the first assertion.

Now we consider y. It follows from the previous paragraph that $y^2, (yg)^2 \in \mathrm{Aut}\Gamma$. Again it is tedious, but not too difficult, to verify that both y^2 and $(yg)^2$ act trivially on the vertex set V, and hence each of them leaves invariant the edge subsets $E_{S,k} = \{e^{2k}_{i,j} \mid i \in S, j \in T\}$ and $E_{T,k} = \{e^{2k+1}_{j,i} \mid i \in S, j \in T\}$, for each $k = 0, \ldots, r-1$. For a fixed k, since each vertex pair $\{(2k, i), (2k+1, j)\}$ (for $i \in S, j \in T$) is incident with a unique edge $e^{2k}_{i,j} \in E_{S,k}$, it follows that y^2 and $(yg)^2$ act trivially on $E_{S,k}$. Similarly y^2 and $(yg)^2$ act trivially on $E_{T,k}$, and since this holds for each k, it follows that $y^2 = (yg)^2 = 1$. Thus $ygy = g^{-1}$ and so $\langle g, y\rangle \cong \mathrm{D}_{2rst}$. Moreover, since y interchanges E_S and E_T it follows that $\langle g, y\rangle$ is regular on the edge set E. This completes the proof of part (a). Part (b) now follows immediately from Proposition 3.4(b). □

4.3 Complete bipartite graphs

Next we study complete bipartite graphs, their λ-extenders, and constructions which combine complete bipartite graphs with circulants. First we establish the properties of these graphs occurring in Tables 1 and 3.

Definition 4.8. Let s, t be positive integers such that $st \geq 2$ and $\gcd(s,t) = 1$, and let $S = \mathbb{Z}_s, T = \mathbb{Z}_t$. Then $\Gamma = \mathbf{K}_{s,t} = (V, E)$ with $V = S \cup T$ and

$$E = \{[i, e_{i,j}, j] \mid i \in S, j \in T\}.$$

Define the map $g : V \cup E \to V \cup E$ as follows, for $i \in S, j \in T$:

$$g\colon i \to i+1 \pmod{s}, \quad j \to j+1 \pmod{t}, \quad e_{i,j} \to e_{i+1,j+1}.$$

Lemma 4.9. *With the notation of* Definition 4.8 *so in particular* $\gcd(s,t) = 1$*, the map* g *is an automorphism of* $\Gamma = \mathbf{K}_{s,t}$*, and* $\langle g\rangle \cong \mathbb{Z}_{st}$ *is cyclic, bi-transitive on* V *with orbits* S, T*, and regular on* E*, and* Γ *is as in line* 2 *of* Table 1*. Further, the graph* $\mathbf{K}^{(2)}_{s,t}$ *admits a cyclic subgroup of automorphisms that is bi-transitive on vertices, and bi-regular on edges, as in line* 2 *of* Table 3.

Proof. By definition both $g|_V$ and $g|_E$ are bijections, and g preserves incidence, so $g \in \mathrm{Aut}\Gamma$. Also, since $\gcd(s,t) = 1$, $|g| = st$, and it follows that the cyclic group $\langle g\rangle$ is bi-transitive on V with orbits S, T, and regular on E. The fact that $\mathrm{Aut}\mathbf{K}^{(2)}_{s,t}$ has a cyclic subgroup bi-transitive on vertices, and bi-regular on edges follows on considering $\mathbf{K}^{(2)}_{s,t}$ as an edge-disjoint union of two copies of $\mathbf{K}_{s,t}$ on the same vertex set. □

The next construction starts with a complete bipartite graph and adds a second set of edges incident with the vertices of one bipart which form a perfect matching or an edge-disjoint union of cycles (with multiple edges). These are the graphs in lines 4 and 6 of Table 3.

Definition 4.10. Let r, n, t be positive integers such that $rt \geq 1$, $n \geq 2$, and $\gcd(nr, t) = 1$, and let $V_1 = \mathbb{Z}_{rn}, V_2 = \mathbb{Z}_t$. Define a graph $\Gamma = \mathbf{CK}(r, n, t) = (V, E)$

with vertex-set $V = V_1 \cup V_2$, and edge-set $E = E_0 \cup E_1$ where

$$E_0 = \{[k, e_{k,j}, j] \mid k \in V_1, j \in V_2\}, \quad \text{and}$$
$$E_1 = \{[k, e_k^j, k+r] \mid k \in V_1, j \in V_2\}.$$

Define the map $g\colon V \cup E \to V \cup E$ as follows, for $k \in V_1, j \in V_2$:

$$g\colon \quad \begin{array}{ccccccc} k & \to & k+1, & j & \to & j+1 \\ e_{k,j} & \to & e_{k+1,j+1}, & e_k^j & \to & e_{k+1}^{j+1} \end{array}$$

Lemma 4.11. *With the notation of* Definition 4.10 *so in particular* $\gcd(rn,t) = 1$*, the graph* $\Gamma = \mathbf{CK}(r,n,t)$ *is connected and the map* g *is an automorphism of* Γ*, and* $\langle g\rangle \cong \mathbb{Z}_{rnt}$ *is cyclic, bi-transitive on* V *with orbits* V_1, V_2*, and bi-regular on* E *with orbits* E_0, E_1*. Further* $\Gamma = [E_0] + [E_1]$*, where* $[E_0] = \mathbf{K}_{rn,t}$ *and, if* $n = 2$ *then* $[E_1] = r\mathbf{K}_2^{(2t)}$ *as in line* 4 *of* Table 3*, while if* $n \geq 3$ *then* $[E_1] = r\mathbf{C}_n^{(t)}$ *as in line* 6 *of* Table 3.

Proof. It is clear that $\Gamma = [E_0] + [E_1]$ is connected and $[E_0] = \mathbf{K}_{rn,t}$. Further, if $n = 2$ then for each $k \in V_1, j \in V_2$, both e_k^j and e_{k+r}^j are incident with k and $k+r$, and hence $[E_1] = r\mathbf{K}_2^{(2t)}$. On the other hand if $n \geq 3$ then $[E_1] = r\mathbf{C}_n^{(t)}$, so the last assertions hold.

By definition, both $g|_V$ and $g|_E$ are bijections, and g preserves incidence, so $g \in \mathrm{Aut}\Gamma$. Consider the action of g on E. First let $e = e_0^0 \in E_1$, and suppose that m is the least positive integer such that $e^{g^m} = e$. Now, by definition of g, $e^{g^m} = [m, e_m^m, m]$, where the first entry $m \in \mathbb{Z}_{rn}$ and the last entry $m \in \mathbb{Z}_t$. Thus, since $e^{g^m} = e$ the integer m is divisible by rn and t, and hence by rnt since $\gcd(rn,t) = 1$. On the other hand g^{rnt} fixes e and hence $m = rnt$ and as $|E_1| = rnt$ we conclude that $\langle g\rangle$ acts regularly on E_1. A similar argument shows that $\langle g\rangle$ acts regularly on E_0, and so $|g| = rnt$ and $\langle g\rangle$ is bi-transitive on V and bi-regular on E. □

In a more general but similar construction, we start with an edge-disjoint union of complete bipartite graphs $u\mathbf{K}_{sr,t}$, and add a second set of edges incident with the vertices of the bipart of size usr, which form either a perfect matching of this bipart, or an edge-disjoint union of cycles. In this case no edge of the second edge-set is incident with two vertices of the same component $\mathbf{K}_{sr,t}$. These are the graphs in lines 5 and 7 of Table 3, and are depicted in Figure 2.

Definition 4.12. Let r, s, t, u be positive integers such that $u \geq 2$, and $\gcd(r,u) = \gcd(sr,t) = 1$, and let $V_1 = \mathbb{Z}_{sru}, V_2 = \mathbb{Z}_{ut}$. Define a graph $\Gamma = \mathbf{CK}_{(2)}(r,s,t,u) = (V,E)$ with vertex-set $V = V_1 \cup V_2$, and edge-set $E = E_0 \cup E_1$ where

$$E_0 = \{[i+uk, e_{i,k,j}, i+uj] \mid 0 \leq i \leq u-1, 0 \leq k \leq sr-1, 0 \leq j \leq t-1\}, \quad \text{and}$$
$$E_1 = \{[\ell + rk, e_{\ell,k}^j, \ell + r(k+1)] \mid 0 \leq \ell \leq r-1, 0 \leq k \leq su-1, 0 \leq j \leq t-1\}.$$

Note that for edges in E_0, we read the first entry as an element of $\mathbb{Z}_{sru}$ and the third entry as an element of $\mathbb{Z}_{ut}$, while for edges in E_1, the first and third entries are elements of $\mathbb{Z}_{sru}$. Define the map $g\colon V \cup E \to V \cup E$ as follows, for $k \in V_1, j \in V_2$:

$$g\colon \quad \begin{array}{lll} k & \to & k+1, \text{ and } j \to j+1, \\ e_{i,k,j} & \to & e_{i+1,k,j+1}, \\ e_{\ell,k}^j & \to & e_{\ell+1,k}^{j+1}, \text{ if } 0 \leq \ell \leq r-2, \text{ and } \quad e_{r-1,k}^j \to e_{0,k+1}^{j+1} \end{array}$$

where in the edge-image $e_{i+1,k,j+1}$ we read $i+1$ modulo u and $j+1$ modulo t, and for the edge-image $e^{j+1}_{\ell+1,k}$ we read $\ell+1$ modulo r and $j+1$ modulo t.

Lemma 4.13. *With the notation of* Definition 4.12*, the graph* $\Gamma = \mathbf{CK}_{(2)}(r,s,t,u)$ *is connected and the map* g *is an automorphism of* Γ*, and* $\langle g\rangle \cong \mathbb{Z}_{srut}$ *is cyclic, bi-transitive on* V *with orbits* V_1, V_2*, and bi-regular on* E *with orbits* E_0, E_1*. Further* $\Gamma = [E_0] + [E_1]$*, where* $[E_0] = u\mathbf{K}_{sr,t}$ *and, if* $(s,u) = (1,2)$ *then* $[E_1] = r\mathbf{K}_2^{(2t)}$ *as in line* 5 *of* Table 3*, and otherwise* $su \geq 3$ *and* $[E_1] = r\mathbf{C}_{su}^{(t)}$ *as in line* 7 *of* Table 3.

Proof. By definition $\Gamma = [E_0] + [E_1]$ and $[E_0] = u\mathbf{K}_{sr,t}$. Further, if $(s,u) = (1,2)$, then for each ℓ, j, both $e^j_{\ell,0}$ and $e^j_{\ell,1}$ are incident with ℓ and $\ell + r$ in V_1 and hence $[E_1] = r\mathbf{K}_2^{(2t)}$. On the other hand if $(s,u) \neq (1,2)$, then $su \geq 3$ and $[E_1] = r\mathbf{C}_{su}^{(t)}$, so the last assertions hold.

By definition, both $g|_V$ and $g|_E$ are bijections, and g preserves incidence, so $g \in \mathrm{Aut}\Gamma$. Consider the action of g on E. First let $e = e_{0,0,0} \in E_0$, and suppose that m is the least positive integer such that $e^{g^m} = e$. By definition of g, $e^{g^m} = [m, e_{m,0,m}, m]$, where the first entry $m \in \mathbb{Z}_{sru}$ and the last entry $m \in \mathbb{Z}_{ut}$. Thus, since $e^{g^m} = e$ the integer m is divisible by sru and ut, and hence by $\mathrm{lcm}\{sru, ut\} = srut$ since $\gcd(sr,t) = 1$. On the other hand g^{srut} fixes e and hence $m = srut$ and as $|E_0| = srut$ we conclude that $\langle g\rangle$ acts regularly on E_0. Now let $e = [0, e^0_{0,0}, 0] \in E_1$, and suppose that $e^{g^m} = e$ with $m \geq 1$ minimal. By definition of g, $e^{g^m} = [m, e^m_{m,k}, m]$, for some k, where the first entry $m \in \mathbb{Z}_{sru}$ and the last entry $m \in \mathbb{Z}_{ut}$, and as before we deduce that $m = srut = |E_1|$ and that $\langle g\rangle$ acts regularly on E_1. Thus $|g| = srut$ and $\langle g\rangle$ is bi-transitive on V and bi-regular on E. □

Remark 4.14. We give some details about Figure 2, for the graph $\Gamma = \mathbf{CK}_{(2)}(r,s,t,u) = r\mathbf{C}_{su}^{(t)} + u\mathbf{K}_{sr,t}$ with $su \geq 3$, from Definition 4.12. The subset V_1 of vertices admits two $\langle g\rangle$-invariant partitions: first $\{B_1, \ldots, B_r\}$ where the B_i are the components of the edge-induced subgraph $[E_1] = r\mathbf{C}_{su}^{(t)}$, and second $\{C_1, \ldots, C_u\}$ where the C_i are the intersections with V_1 of the components of $[E_0] = u\mathbf{K}_{sr,t}$. The vertex-subset V_2 admits the $\langle g\rangle$-invariant partition $\{D_1, \ldots, D_u\}$, where the D_i are the intersections with V_2 of the components of $[E_0] = u\mathbf{K}_{sr,t}$.

Each of the final family of graphs admitting a cyclic edge-bi-regular action involves three vertex-orbits. They are the graphs in the last line of Table 3, and Figure 3 gives a broad description of their structure, see Remark 4.17.

Definition 4.15. Let r, r', s, t, t', u be positive integers such that

$$\gcd(r,r') = \gcd(t,t') = \gcd(sr,ut) = \gcd(sr',ut') = 1,$$

and let $V_1 = \mathbb{Z}_{rut'}$, $V_2 = \mathbb{Z}_{srr'}$ and $V_3 = \mathbb{Z}_{r'ut}$. Define a graph $\Gamma = \mathbf{KK}(r,r',s,t,t',u) = (V,E)$ with vertex-set $V = V_1 \cup V_2 \cup V_3$, and edge-set $E = E_0 \cup E_1$ where

$$\begin{aligned} E_0 &= \{[i + r\ell, e^j_{i,\ell,k}, i + rk] \mid 0 \leq i \leq r-1, 0 \leq \ell \leq ut'-1, 0 \leq k \leq sr'-1, \\ &\qquad 0 \leq j \leq t-1\}, \\ E_1 &= \{[i' + r'\ell', e^{j'}_{i',\ell',k'}, i' + r'k'] \mid 0 \leq i' \leq r'-1, 0 \leq \ell' \leq ut-1, 0 \leq k' \leq sr-1, \\ &\qquad 0 \leq j' \leq t'-1\}. \end{aligned}$$

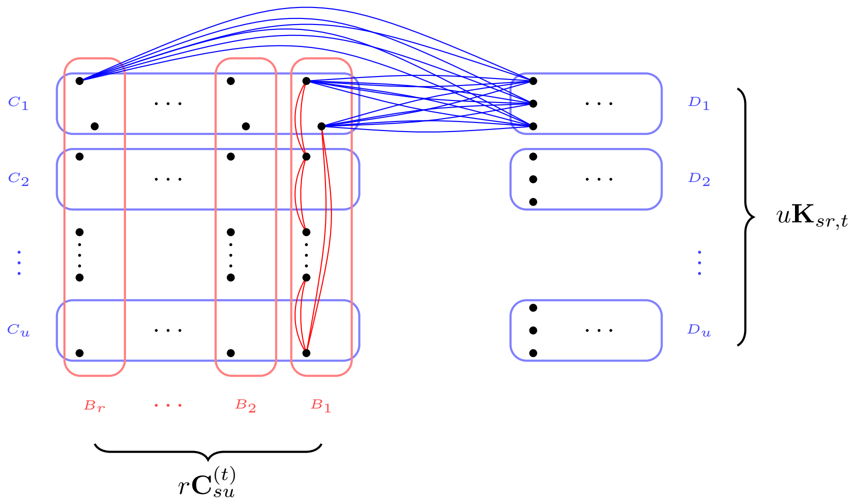

Figure 2: $\Gamma = r\mathbf{C}_{su}^{(t)} + u\mathbf{K}_{sr,t}$, where $[B_i] = \mathbf{C}_{su}^{(t)}$ and $[C_j \cup D_j] = \mathbf{K}_{sr,t}$.

Note that the last entry in each edge lies in $V_2 = \mathbb{Z}_{srr'}$, while the first edge-entries lie in $V_1 = \mathbb{Z}_{rut'}$ for edges in E_0, and in $V_3 = \mathbb{Z}_{r'ut}$ for edges in E_1. For $c = 1, 2, 3$, each $v_c \in V_c$ can be written uniquely as follows:

$$\begin{array}{llll} v_1 & = i + r\ell & \text{with} & 0 \le i \le r-1, \quad 0 \le \ell \le ut'-1 \\ v_2 & = i + rk & \text{with} & 0 \le i \le r-1, \quad 0 \le k \le sr'-1 \\ & = i' + r'k' & \text{with} & 0 \le i' \le r'-1, \quad 0 \le k' \le sr-1 \\ v_3 & = i' + r'\ell' & \text{with} & 0 \le i' \le r'-1, \quad 0 \le \ell' \le ut-1 \end{array}$$

We will define a map $g\colon V \cup E \to V \cup E$ so that the vertex action is given by $v_c \to v_c + 1$ where we evaluate the vertex image modulo $rut', srr', r'ut$ for $c = 1, 2, 3$ respectively. To define the g-action on edges, set $x := rr'sut'$ and $x' := rr'sut$, so that $|E_0| = xt$ and $|E_1| = x't'$. Then for $e^j_{i,\ell,k} \in E_0$ with i, ℓ, k, j in the ranges above, we have

$$1 \le (i + r\ell + 1)(k+1)(j+1) \le (rut')(sr')t = xt = |E_0|,$$

and similarly, for $e^{j'}_{i',\ell',k'} \in E_1$, we have

$$1 \le (i' + r'\ell' + 1)(k'+1)(j'+1) \le (r'ut)(sr)t' = x't' = |E_1|.$$

Then, writing $(i+r\ell+1)(k+1)(j+1) = ax+b$ and $(i'+r'\ell'+1)(k'+1)(j'+1) = a'x'+b'$, with $0 \le a \le t-1$, $1 \le b \le x$, $0 \le a' \le t'-1$ and $1 \le b' \le x'$, we define the g-action by

$$g\colon \quad e^j_{i,\ell,k} \ \to\ e^J_{I,L,K} \quad \text{and} \quad e^{j'}_{i',\ell',k'} \ \to\ e^{J'}_{I',L',K'}$$

where I, J, K, L and I', J', K', L' are as in Table 8.

a	i	J	I	L	K
$\leq x-2$	$\leq r-2$	j	$i+1$	ℓ	k
$\leq x-2$	$r-1$	j	$i+1$	$\ell+1$	$k+1$
$x-1$	$\leq r-2$	$j+1$	$i+1$	ℓ	k
$x-1$	$r-1$	$j+1$	$i+1$	$\ell+1$	$k+1$
a'	i'	J'	I'	L'	K'
$\leq x'-2$	$\leq r'-2$	j'	$i'+1$	ℓ'	k'
$\leq x'-2$	$r'-1$	j'	$i'+1$	$\ell'+1$	$k'+1$
$x'-1$	$\leq r'-2$	$j'+1$	$i'+1$	ℓ'	k'
$x'-1$	$r'-1$	$j'+1$	$i'+1$	$\ell'+1$	$k'+1$

Table 8: Edge action for g in Definition 4.15.

Lemma 4.16. *With the notation of* Definition 4.15*, the graph* $\Gamma = \mathbf{KK}(r, r', s, t, t', u)$ *is connected, and* $\Gamma = [E_0] + [E_1]$*, where* $[E_0] = r\mathbf{K}^{(t)}_{sr',ut'}$ *and* $[E_1] = r'\mathbf{K}^{(t')}_{sr,ut}$ *as in the last line of* Table 3*. Further the map* $g \in \mathrm{Aut}\Gamma$*, and* $\langle g\rangle$ *is cyclic of order* $srr'utt'$*, with three vertex-orbits* V_1, V_2, V_3*, and is bi-regular on* E *with orbits* E_0, E_1*.*

Proof. By definition $\Gamma = [E_0] + [E_1]$ and $[E_0] = r\mathbf{K}^{(t)}_{sr',ut'}$ and $[E_1] = r'\mathbf{K}^{(t')}_{sr,ut}$, so the first assertion holds. Further, it follows from the definition of g that both $g|_V$ and $g|_E$ are bijections, and g preserves incidence, so $g \in \mathrm{Aut}\Gamma$. Consider the action of g on E. First let $e = e^0_{0,0,0} \in E_0$, and suppose that m is the least positive integer such that $e^{g^m} = e$. It follows from Table 8, that after xt repeated applications of g, we have $e^{g^{xt}} = e^t_{xt,y,y}$ where $y = sr'ut'$, and as y is divisible by both ut' and sr', this edge-image is equal to e. Thus g^{xt} fixes e and so m divides xt. Suppose for a contradiction that $0 < m < xt$. Then m is of the form $m = cx + d$ where $0 \leq c \leq t-1$ and $0 \leq d \leq x-1$, and $e = e^{g^m} = e^c_{m,v,w}$, for some v, w. This implies that c is a multiple of t and hence $c = 0$, so $0 < m < x$. Now, with $y = sr'ut'$ so $x = ry$, we write $m = c'r + d'$ with $0 \leq c' \leq y-1$ and $0 \leq d' \leq r-1$. Then $e = e^{g^m} = e^0_{d',c',c'}$, and this implies that $d' = 0$ and that c' is divisible by both ut' and sr'. Since $\gcd(sr', ut') = 1$, we deduce that c' is divisible by $\mathrm{lcm}\{sr', ut'\} = sr'ut' = y$, and hence $c' = 0$ so $m = 0$, a contradiction. Thus $m = xt$ and $\langle g\rangle$ acts regularly on E_0.

An analogous argument for the edge $e^0_{0,0,0} \in E_1$ shows that $\langle g\rangle$ also acts regularly on E_1, and as $|E_0| = |E_1| = xt = x't'$, the subgroup $\langle g^{xt}\rangle$ fixes $V \cup E$ pointwise, and hence is trivial. Thus $\langle g\rangle$ is edge-bi-regular, and the proof is complete. □

Remark 4.17. Figure 3 gives a description of the structure of the graph

$$\Gamma = \mathbf{KK}(r, r', s, t, t', u) = r\mathbf{K}^{(t)}_{sr',ut'} + r'\mathbf{K}^{(t')}_{sr,ut}$$

from Definition 4.15. The vertex-subset V_2 admits two $\langle g\rangle$-invariant partitions: first the partition $\{B_1, \ldots, B_r\}$ where the B_i are the intersections with V_2 of the components of $[E_0] = r\mathbf{K}^{(t)}_{sr',ut'}$, and second $\{C_1, \ldots, C_{r'}\}$ where the C_i are the intersections with V_2 of the components of $[E_1] = r'\mathbf{K}^{(t')}_{sr,ut}$. The vertex-subsets V_1, V_3 admit the $\langle g\rangle$-invariant partitions $\{U_1, \ldots, U_r\}$ and $\{W_1, \ldots, W_{r'}\}$, where the U_i and W_i are the intersections with V_1, V_3 of the components of $[E_0], [E_1]$, respectively.

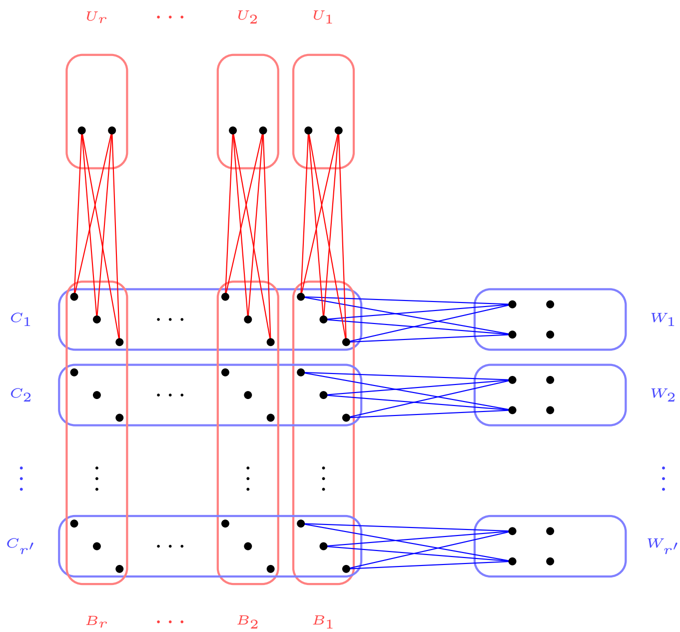

Figure 3: $\Gamma = r\mathbf{K}^{(t)}_{sr',ut'} + r'\mathbf{K}^{(t')}_{sr,ut}$, where $[B_i \cup U_i] = \mathbf{K}^{(t)}_{sr',ut'}$ and $[C_j \cup W_j] = \mathbf{K}^{(t')}_{sr,ut}$.

5 Coset graphs and cyclic edge-regular actions

In [8, Section 1.1], a new coset graph construction was given for arc-transitive, not necessarily simple, graphs, which proved helpful for analysing such graphs. Here we develop a similar theory of coset graph representations for edge-transitive bipartite graphs with two vertex-orbits. It extends the theory in [1, Section 2] and [3, Section 3.2] for simple graphs of this kind. We use this theory in Subsection 5.1 to prove Theorem 1.1 in the case of edge-regular actions.

Construction 5.1. Let G be a group with proper subgroups L, R, J, such that $L \neq R$, $J \leq L \cap R$ and J is core-free in G, and let $\lambda := |L \cap R : J|$. Define an incidence structure $\mathrm{BiCos}(G, L, R, J) = (V, E, \mathbf{I})$, called a *bi-coset graph*, by setting

$$V = [G : L] \cup [G : R] = \{Lx \mid x \in G\} \cup \{Rx \mid x \in G\}$$
$$E = [G : J] = \{Jx \mid x \in G\}$$
$$\mathbf{I} = \{(Lx, Jy) \mid Lx \cap Jy \neq \emptyset\} \cup \{(Rx, Jy) \mid Rx \cap Jy \neq \emptyset\} \subseteq V \times E.$$

Also set $s := |L : L \cap R|$ and $t := |R : L \cap R|$.

We prove that $\mathrm{BiCos}(G, L, R, J)$ is a G-edge-transitive bipartite graph and obtain various of its other properties.

Proposition 5.2. *Let $G, L, R, J, \lambda, s, t$ be as in* Construction 5.1*, and denote the graph* $\mathrm{BiCos}(G, L, R, J)$ *by* Γ.

(a) *For $x, y, z \in G$, $Jy = [Lx, Jy, Rz]$ is an edge if and only if $Jy = [Ly, Jy, Ry]$, (that is, $Lx = Ly$ and $Rz = Ry$). Thus the edges of Γ are precisely $Jy = [Ly, Jy, Ry]$, for $y \in G$.*

(b) *Then Γ is a bipartite graph which is bi-regular with valencies s, t, and constant edge-multiplicity λ.*

(c) *The group G, acting by right-multiplication, is an edge-transitive group of automorphisms of Γ with vertex-orbits $[G : L]$ and $[G : R]$, and L, R, J are the stabilisers of the vertices L, R and edge J, respectively.*

(d) *The base graph of Γ is $\Gamma_0 := \mathrm{BiCos}(G, L, R, L \cap R)$, and $\Gamma = \Gamma_0^{(\lambda)}$. In particular Γ is simple if and only if $L \cap R = J$.*

(e) *The graph Γ is connected if and only if $G = \langle L, R\rangle$; and $\Gamma \cong K_{s,t}^{(\lambda)}$ if and only if $G = LR$.*

Proof. (a) Let $x, y, z \in G$. Then $Jy = [Lx, Jy, Rz]$ is an edge if and only if $(Lx, Jy) \in \mathbf{I}$ and $(Rz, Jy) \in \mathbf{I}$, that is, $Lx \cap Jy \neq \emptyset$ and $Rz \cap Jy \neq \emptyset$. This in turn is equivalent to $Lxy^{-1} \cap J \neq \emptyset$ and $Rzy^{-1} \cap J \neq \emptyset$, and since $J \leq L \cap R$, this is equivalent to $xy^{-1} \in L$ and $zy^{-1} \in R$, that is to say $Lx = Ly$ and $Rz = Ry$ so $Jy = [Ly, Jy, Ry]$.

(b) and (c) It is clear from the definition of Γ that $[G : L]$ and $[G : R]$ form the parts of a bipartition of Γ and each edge is incident with one vertex from each of these sets. Also, the group G, acting by right-multiplication, preserves the incidence relation $\mathbf{I}$ so induces a group of automorphisms of Γ. The biparts $[G : L]$ and $[G : R]$ are the two G-vertex-orbits, and by part (a), G is transitive on edges. The subgroups L, R, J are the stabilisers of vertices L, R and edge J respectively, and since J is core-free in G, G acts faithfully on Γ. These transitivity properties imply that Γ is biregular with each vertex of $[G : L]$ incident with $|L : L \cap R| = s$ vertices and each vertex of $[G : R]$ incident with $|R : L \cap R| = t$ vertices, and that Γ has constant edge-multiplicity $\lambda = |L \cap R : J|$ (the number of edges incident with L and R).

(d) From the discussion in the previous paragraph it is clear that the base graph Γ_0 of Γ has edge set identified with $[G : L \cap R]$ and is precisely the graph $\mathrm{BiCos}(G, L, R, L \cap R)$, and that Γ is a λ-extender of Γ_0.

(e) Now Γ is connected if and only if Γ_0 is connected, and the latter is true if and only if $G = \langle L, R\rangle$, by [3, Lemma 3.7(1)]. Assume now that $G = LR$. Then as R is the stabiliser of the vertex $R \in [G : R]$ it follows that L is transitive on $[G : R]$ and similarly R is transitive on $[G : L]$. Thus L, R is adjacent to each vertex of $[G : R], [G : L]$, respectively and it follows that $\Gamma_0 \cong K_{s,t}$ and $\Gamma \cong K_{s,t}^{(\lambda)}$. Conversely suppose that $\Gamma \cong K_{s,t}^{(\lambda)}$. Then $\Gamma_0 \cong K_{s,t}$, and as G is edge-transitive, the vertex-stabiliser L is transitive on the set $[G : R]$ of vertices adjacent to L. Then, since R is the stabiliser of $R \in [G : R]$ this means that $G = RL = LR$. □

Next we show that essentially all edge-transitive graphs with two vertex-orbits arise from Construction 5.1.

Γ	$\lvert G\rvert$	G^V	G^E	G^A	Conditions
$\mathbf{C}_n^{(\lambda)}$	$n\lambda$	$\surd$	$\surd$	$\times$	$n \geq 3$
$\mathbf{K}_{s,t}^{(\lambda)}$	$st\lambda$	$\times$	$\surd$	$\times$	$\gcd(s,t) = 1$, $st > 1$

Table 9: Table for Theorem 5.4.

Proposition 5.3. *Let $\Gamma = (V, E, \mathbf{I})$ be a graph and $G \leq \mathrm{Aut}\Gamma$ such that G is transitive on E and has two orbits V_1, V_2 in V, and Γ has no isolated vertices, and constant edge-multuplicity λ. Then either*

(a) *$\Gamma \cong \mathrm{BiCos}(G, L, R, J)$ where for some edge $e = [\alpha, e, \beta]$ of Γ, the subgroups L, R, J are the stabilisers of α, β, e respectively, and $|L \cap R : J| = \lambda$; or*

(b) *$\Gamma \cong rK_2^{(\lambda)}$, where $r = |V_1| = |V_2|$.*

Proof. If some edge e of Γ is incident with two distinct vertices of V_i, for some i, then this is true for all edges since Γ is G-edge transitive and G leaves V_i invariant. This is a contradiction since Γ has no isolated vertices. Hence each edge is incident with one vertex from V_1 and one from V_2. Let $e = [\alpha, e, \beta]$ be an edge with $\alpha \in V_1$ and $\beta \in V_2$, and let $L = G_\alpha, R = G_\beta$ and $J = G_e$. Then we may identify the sets V_1, V_2, E with $[G : L], [G : R], [G : J]$, respectively, with G acting by right multiplication. Note that G_e fixes the unique vertex of V_i with which it is incident, for each i, and so $J = G_e \leq G_\alpha \cap G_\beta = L \cap R$. With this identification, the edge $e = [L, J, R]$ and each edge can be expressed as $e^y = [Ly, Jy, Ry]$ for some $y \in G$ (since G is transitive on E). Thus Lx is incident with Jz if and only if there exists y such that $Lx = Ly$ and $Jz = Jy$, and hence $Lx \cap Jz$ contains y so is non-empty. Conversely, if $Lx \cap Jz$ contains an element y, then $Lx = Ly$ and $Jz = Jy$. Similar statements hold for incidences between Jx and Rz. Thus, provided $L \neq R$, we see that, under these identifications we have $\Gamma = \mathrm{BiCos}(G, L, R, J)$ and part (a) holds.

Finally suppose that $L = R$. Since G is transitive on E, L is transitive on the edges incident with α, and since $L = R$ fixes β this means that all of these edges are incident with β. Thus the connected component of Γ induced on $\{\alpha, \beta\}$ is $K_2^{(\lambda)}$, and we have $\Gamma \cong rK_2^{(\lambda)}$, where $r = |V_1| = |V_2|$, as in (b). □

5.1 Graphs admitting cyclic edge-transitive groups

We use the theory in the previous subsection, to determine all graphs admitting a cyclic subgroup of automorphisms that is regular on edges. First we consider connected graphs. We assume that $|V| \geq 3$, in the light of Proposition 2.2.

Theorem 5.4. *Let $\Gamma = (V, E, \mathbf{I})$ be a connected graph with $|V| \geq 3$, let $G \leq \mathrm{Aut}\Gamma$ be a cyclic subgroup acting regularly on E, and let Γ have edge-multiplicity λ. Then Γ and $|G|$ are as in one of the lines of* Table 9*, and the induced permutation groups G^X on vertices ($X = V$), edges ($X = E$) and arcs ($X = A$) are transitive if the entry in the column headed G^X is $\surd$ and intransitive if the entry is $\times$.*

Proof. Since $|V| \geq 3$ we have $\Gamma \ncong \mathbf{K}_2^{(\lambda)}$, and since Γ is connected and G-edge-transitive, G has at most two orbits in V. Suppose first that G has two vertex-orbits, V_1 and V_2. Then by Proposition 5.3, $\Gamma = \mathrm{BiCos}(G, L, R, J)$ with $L = G_\alpha, R = G_\beta, J = G_e$ for some edge $e = [\alpha, e, \beta]$. It follows from Proposition 5.2(e) that $G = \langle L, R\rangle$, and since G is cyclic this means that $G = LR$ and hence that $\Gamma = \mathbf{K}_{s,t}^{(\lambda)}$, where $s = |L : L \cap R|$ and $t = |R : L \cap R|$, and $st > 1$ since $\Gamma \ncong \mathbf{K}_2^{(\lambda)}$. Also $L \cap R$ acts trivially on V and the edge-stabiliser J acts trivially on E (since G is abelian), and hence $J = 1$ by Lemma 2.1. Thus $|G| = |E| = st\lambda$, and $|L \cap R| = \lambda$. Now we have $|L| = s\lambda$ and $|R| = t\lambda$ and so $|G| = |LR| = \mathrm{lcm}\{s\lambda, t\lambda\} = \lambda\,\mathrm{lcm}\{s, t\}$, and hence $st = \mathrm{lcm}\{s, t\}$, that is, $\gcd(s, t) = 1$. Thus all the entries of line 2 of Table 9 hold.

We may now assume that G is transitive on V as well as on E. Suppose that G is not transitive on arcs. Then the stabiliser of an edge $e = [\alpha, e, \beta]$ satisfies $G_e = 1$, and $|G| = |E|$. Also, the stabiliser G_α is normal in G since G is abelian, and so G_α acts trivially on V. This implies that $G_\alpha = G_\beta$ and so G_α is transitive on the λ edges incident with α and β. Since $G_e = 1$ and $G_e \leq G_{\alpha,\beta} = G_\alpha$, we have $|G_\alpha| = \lambda$, and $n := |V| = |G : G_\alpha|$ so $|G| = n\lambda$, and note that $n = |V| \geq 3$. Let Γ_0 be the base graph of Γ so $\Gamma = \Gamma_0^{(\lambda)}$. Then G/G_α acts transitively on the vertices and edges of Γ_0, and Γ_0 has $|E|/\lambda = n = |V|$ edges and also n vertices. These properties imply that Γ_0 has valency 2 and hence is a cycle of length n. Hence $\Gamma = \mathbf{C}_n^{(\lambda)}$ and $n \geq 3$, so all the entries of line 1 of Table 9 hold.

Suppose finally that G is transitive (hence regular) on the arcs of Γ. Then the stabiliser G_e of an edge $e = [\alpha, e, \beta]$ contains an element g which interchanges α and β. Now $G_e \cap G_\alpha$ acts trivially on both V and E (since G is abelian and transitive on V and E), and hence $G_e \cap G_\alpha = 1$. Thus $G_e = \langle g\rangle \cong Z_2$. Moreover, $g \notin G_\alpha$, but g normalises G_α (since G is abelian) and $G = \langle G_\alpha, g\rangle$ (since Γ is connected). Thus $|V| = |G : G_\alpha| = 2$, which is a contradiction. □

Now we use this classification to determine all graphs with no isolated vertices which admit a cyclic subgroup of automorphisms regular on edges. Such graphs have constant edge-multiplicity.

Corollary 5.5. *Let $\Gamma = (V, E, \mathbf{I})$ be a graph with no isolated vertices, and constant edge-multiplicity λ, and let $G \leq \mathrm{Aut}\Gamma$ be a cyclic subgroup transitive on E. Then*

(a) *$\Gamma = r\Gamma_0$ where Γ has r connected components, each isomorphic to Γ_0, and*

(b) *for G_0 the setwise stabiliser in G of a component Γ_0, $|G| = r|G_0|$, and either $\Gamma_0 = \mathbf{K}_2^{(\lambda)}$, or $(\Gamma_0, |G_0|)$ is as in one of the lines of* Table 9, *and Γ_0 is as in one of the lines of* Table 1.

Proof. The result follows from Theorem 5.4 if Γ is connected, so suppose that Γ has $r \geq 2$ connected components $\Sigma_1, \ldots, \Sigma_r$, and let E_i be the edge-set of Σ_i, for $1 \leqslant i \leqslant r$. Since G is edge-transitive it follows that $\Gamma_0 := \Sigma_1 \cong \ldots \cong \Sigma_r$, so $\Gamma = r\Gamma_0$, and $\{E_1, \ldots, E_r\}$ forms a G-invariant partition of E. Then since G is cyclic, it induces a regular action on $\{E_1, \ldots, E_r\}$, and the unique subgroup G_0 of index r in G is the setwise stabiliser of E_i for each i. Also G_0 is transitive on each E_i and G_0 is cyclic, and hence the unique subgroup J of G_0 of index $|E_i|$ is the stabiliser of each edge of each of the E_i. That is to say, J is the kernel of the action of G on E. It follows from Lemma 2.1 that either $J = 1$, or $\Gamma_0 = \mathbf{K}_2^{(\lambda)}$. Thus we may assume that $J = 1$. In particular G_0 acts edge-regularly and faithfully as a cyclic group of automorphisms of Σ_i, for each i. Then by Theorem 5.4, $(\Gamma_0, |G_0|)$ is as in one of the lines of Table 9, and so the base graph Γ_0 is as in one of the lines of Table 1. □

6 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. Let $\Gamma = (V, E)$ be a connected graph with $|V| \geq 3$, and assume that $G \leqslant \mathrm{Aut}\Gamma$ is a cyclic subgroup which is regular or bi-regular on the edge set E. In particular Γ has no isolated vertices. If G is regular on E then the possibilities for (Γ, G) are determined in Theorem 5.4 and (see Corollary 5.5(b)) the simple base graph Γ_0 of Γ is as in one of the lines of Table 1. So we may assume that G is bi-regular on E, with two edge orbits E_0, E_1 of equal size. Let

$$\Pi_i = [E_i] = (V(E_i), E_i), \text{ where } i = 0 \text{ or } 1$$

be the induced subgraphs. Then Π_0, Π_1 are edge disjoint graphs, and by our convention, $\Gamma = \Pi_0 + \Pi_1$. By the definition of an induced subgraph, Π_i has no isolated vertices, and by the definition of bi-regular, G induces an edge-regular action on Π_i for each i. Hence by Corollary 5.5, the following holds for each of $i = 0$ or 1. The graph $\Pi_i = r_i\Sigma_i$, where Π_i has r_i connected components, each isomorphic to Σ_i, and for H_i the setwise stabiliser in G of a component Σ_i, $|G| = r_i|H_i|$, and one of

(1) $\Sigma_i = \mathbf{C}_{n_i}^{(\lambda_i)}$, and $H_i = \mathbb{Z}_{n_i\lambda_i}$ is transitive on $V(E_i)$, for some $n_i \geqslant 3, \lambda_i \geq 1$; or

(2) $\Sigma_i = \mathbf{K}_{s_i,t_i}^{(\lambda_i)}$, and $H_i = \mathbb{Z}_{s_it_i\lambda_i}$ is bi-transitive on $V(E_i)$, with $s_it_i > 1$ and $\gcd(s_i, t_i) = 1, \lambda_i \geq 1$; or

(3) $\Sigma_i = \mathbf{K}_2^{(\lambda_i)}$, and by Proposition 2.2, one of lines 1–3 of Table 5 holds for H_i.

It follows from the above discussion (and Corollary 5.5) that G has at most two orbits in $V(E_i)$ for each i, and since Γ is connected, the sets $V(E_0)$ and $V(E_1)$ are not disjoint. Thus $V(E_0)$ and $V(E_1)$ share at least one G-orbit, and hence G^V has at most three orbits. We show that Γ and $|G|$ are as in one of the cases of Theorem 1.1 in the following three subsections, according to the number of G-orbits in V.

6.1 The case where G^V is transitive

Suppose in this subsection that G^V is transitive, so $V = V(E_i)$ for each i.

Assume first that $\Pi_0 = r_0\mathbf{K}_2^{(\lambda_0)}$. Then, by Proposition 2.2, line 1 or 2 of Table 5 holds, so either (i) $|G| = 2|E_0| = 2r_0\lambda_0$ with λ_0 odd, and the G-action on Π_0 is arc-transitive, or (ii) $|G| = |E_0| = r_0\lambda_0$ with λ_0 even and G is faithful on E_0 and not arc-transitive on Π_0. In either case $|G|$ is even, so G has a unique subgroup X of order 2, and the X-orbits in V are the components of Π_0. Suppose to start with that $\Pi_1 = r_1\mathbf{K}_2^{(\lambda_1)}$. Then $|V| = 2r_0 = 2r_1$ so $r_0 = r_1$, and $|E_i| = r_0\lambda_0 = r_1\lambda_1$ (by definition, since G is biregular on E), so also $\lambda_0 = \lambda_1 = \lambda$, say, and $|G| = 2r_1\lambda$ or $r_1\lambda$ according as λ is odd or even, respectively. It follows that the X vertex-orbits are also the connected components of Π_1, so a connected component of Γ has size 2. Since Γ is connected this implies that $|V| = 2$, which is a contradiction. Hence $\Pi_1 \neq r_1\mathbf{K}_2^{(\lambda_1)}$. This implies in particular that G acts faithfully on E_1 (by Lemma 2.1). Thus $|G| = |E_1|$ since G is abelian, and therefore, since $|E_0| = |E_1|$ as G is bi-regular on E, we have $|G| = |E_0|$ so (ii) above holds, that is, $|G| = r_0\lambda_0$, λ_0 is even, and G is faithful on E_0. Also, since $\Pi_1 \neq r_1\mathbf{K}_2^{(\lambda_1)}$ and G^V is transitive, it follows from Corollary 5.5 that $\Pi_1 = r_1\mathbf{C}_{n_1}^{(\lambda_1)}$ for some $n_1 \geq 3$. Then $|V| = 2r_0 = r_1n_1$, and $|E_i| = r_0\lambda_0 = r_1n_1\lambda_1$, so $\lambda_0 = 2\lambda_1$.

Suppose that X fixes a Π_1-component Σ setwise. Then $V(\Sigma)$ is a union of (the vertex-sets of) some components of Π_0, and since Γ is connected it follows that $\Pi_1 = \Sigma$, so $V = V(\Sigma)$, $r_1 = 1$ and $n_1 = 2r_0$ is even. Moreover the two vertices of each component $\mathbf{K}_2^{(\lambda_0)}$ of Π_0 form an antipodal pair of vertices of $\Pi_1 = \mathbf{C}_{n_1}^{(\lambda_1)}$, and X interchanges this vertex-pair and interchanges in pairs the $\lambda_0 = 2\lambda_1$ edges of Π_0 incident with them. Thus, setting $\lambda := \lambda_1$ and $n := n_1/2$, we have $\Gamma = \Gamma_0^{(\lambda)}$ with $|G| = 2n\lambda$ and Γ_0 the Cayley graph $\Gamma(2n, 1, n) = \mathbf{C}_{2n} + n\mathbf{K}_2^{(2)}$ of Corollary 4.5(a), and line 2 of Table 2 holds.

On the other hand, suppose that X interchanges the components of Π_1 in pairs. Then r_1 is even and the two vertices of a component $\mathbf{K}_2^{(\lambda_0)}$ of Π_0 lie in two distinct components of Π_1. Further, the union of these two distinct components of Π_1 is also a union of n_1 components of Π_0 and hence forms a component of Γ. Since Γ is connected we conclude that $r_1 = 2$ and $r_0 = n_1 = n$, say. Again the group X interchanges the two Π_1-components $\mathbf{C}_n^{(\lambda)}$, where $\lambda := \lambda_1$, and for each component $\mathbf{K}_2^{(2\lambda)}$ of Π_0, X interchanges the 2λ edges in pairs. Since G is cyclic and transitive on the $2n = |V|$ vertices of Γ, and since the index 2 subgroup of G stabilising a component of Π_1 is transitive on the n vertices of that component, it follows that n is odd, and hence $|G| = 2n\lambda$ and $\Gamma = \Gamma_0^{(\lambda)}$ with Γ_0 the Cayley graph $\Gamma(2n, 2, n) = 2\mathbf{C}_n + n\mathbf{K}_2^{(2)}$ of Corollary 4.5(b), and line 3 of Table 2 holds.

Thus we may assume from now on that no component of either Π_0 or Π_1 is $\mathbf{K}_2^{(\lambda)}$ for any λ. Then by Corollary 5.5, since G^V is transitive, each $\Pi_i = r_i\mathbf{C}_{n_i}^{(\lambda_i)}$ for some $n_i \geq 3$. Thus $|V| = r_1n_1 = r_0n_0$ and $|E_1| = |E_0| = r_1n_1\lambda_1 = r_0n_0\lambda_0$, and hence in particular $\lambda_1 = \lambda_0$. This means that $\Gamma = \Gamma_0^{(\lambda_0)}$ and $\Pi_i = \Phi_i^{(\lambda_0)}$, for graphs Φ_i with edge-multiplicity 1 admitting a cyclic vertex-transitive group H (induced by G) of order $n := r_1n_1 = r_0n_0$. Thus H is bi-regular on the edge-set of Γ_0 with orbits the edge-sets of Φ_0 and Φ_1. This means that each Φ_i may be regarded as a simple Cayley graph for the cyclic group H. If as simple graphs, the edges sets of Φ_0 and Φ_1 coincide then, since Γ is connected, $r_1 = r_0 = 1$ and $n = n_0 = n_1$, and setting $\lambda := \lambda_0$, $\Gamma = \mathrm{Circ}(n, \{1, -1\})^{(2\lambda)} = \mathbf{C}_n^{(2\lambda)}$, with $G = \mathbb{Z}_{n\lambda}$ acting with two edge orbits E_0 and E_1. Thus $\Gamma = \Gamma_0^{(\lambda)}$, with $\Gamma_0 = \mathbf{C}_n^{(2)}$ and $|G| = n\lambda$, and line 1 of Table 2 holds.

If this is not the case then the edge sets of Φ_0 and Φ_1 are disjoint (as they are G-orbits), and hence $\Gamma_0 = \mathrm{Circ}(n, S)$, where $S = \{a, -a, b, -b\} \subset \mathbb{Z}_n \setminus \{0\}$ with $\Phi_0 = \mathrm{Circ}(n, \{a, -a\})$ and $\Phi_1 = \mathrm{Circ}(n, \{b, -b\})$, for some a, b with $|a| \geq 3$, $|b| \geq 3$, $a \neq \pm b$, (see Section 4.1). Since Γ, and hence also Γ_0 is connected it follows (as in the proof of Lemma 4.4) that $\gcd(n, a, b) = 1$, and hence, setting $\lambda := \lambda_1 = \lambda_0$, $\Gamma = \Gamma_0^{(\lambda)}$, where $\Gamma_0 = \mathrm{Circ}(n, \{a, -a, b, -b\})$ and $|G| = n\lambda$ as in line 4 of Table 2 (see Lemma 4.4).

This proves Theorem 1.1 for cyclic vertex-transitive, edge-bi-regular actions.

6.2 The case where G^V has two orbits

In this subsection we assume that G has two orbits V_1, V_2 on vertices. Suppose first that $V = V(E_i)$ for each i. Then G acts faithfully on each E_i. Assume that $\Pi_0 = r_0\mathbf{K}_2^{(\lambda_0)}$, so by Proposition 2.2, $|V| = 2r_0$, $|G| = |E_0| = |E_1| = r_0\lambda_0$, and each of the two G-orbits in V has size r_0. We claim that also $\Pi_1 = r_1\mathbf{K}_2^{(\lambda_1)}$. If this is not the case then, by Corollary 5.5, $\Pi_1 = r_1\mathbf{K}_{s,t}^{(\lambda_1)}$ with $\gcd(s, t) = 1$ and $st > 1$. However this means that the G-vertex-orbits have unequal sizes r_1s and r_1t, which is a contradiction. Hence $\Pi_1 = r_1\mathbf{K}_2^{(\lambda_1)}$, and so $|V| = 2r_1$. Thus $r_0 = r_1 = r$, say, and $|E_1| = r_1\lambda_1$

so $\lambda_0 = \lambda_1 = \lambda$, say. If some Π_0-component is also a Π_1-component, then $r = 1$ since Γ is connected, and $|V| = 2$, which is a contradiction. Thus no Π_0-component is equal to a Π_1-component. Hence $\Gamma = \Gamma_0^{(\lambda)}$, and the graph Γ_0 is connected (since Γ is connected) with edge-multiplicity 1, and valency 2 (since each vertex is adjacent to exactly one vertex by an edge from a Π_i-component, for each i). Thus $\Gamma_0 = \mathbf{C}_n$, where $n = r_0 + r_1 = 2r$, and the group G induces a cyclic subgroup $H \leq \mathrm{Aut}\Gamma_0$ of order $n/2 = r$ with two vertex-orbits and two edge-orbits. Thus, as in Lemma 4.2, $\Gamma = \mathbf{C}_n^{(\lambda)}$, with n even, $\Pi_0 \cong \Pi_1 \cong (n/2)\,\mathbf{K}_2^{(\lambda)}$ and $|G| = n\lambda/2$, and line 1 of Table 3 holds for Γ_0.

Therefore, in the case where $V = V(E_i)$ for each i, we may assume that neither of the Π_i has a component $\mathbf{K}_2^{(\lambda)}$ for any λ. It follows from Corollary 5.5 that, for each i, $\Pi_i = r_i\mathbf{K}_{s_i,t_i}^{(\lambda_i)}$ with $\gcd(s_i, t_i) = 1$ and $s_it_i > 1$. Without loss of generality we may assume that the G-vertex-orbits V_1, V_2 have sizes $|V_1| = r_0s_0 = r_1s_1$ and $|V_2| = r_0t_0 = r_1t_1$. Also we have $r_0s_0t_0\lambda_0 = |E_0| = |E_1| = r_1s_1t_1\lambda_1$. Let Σ_1 be a Π_1-component. Suppose first that, for some vertices $u_i \in V_i \cap V(\Sigma_1)$, for $i = 1, 2$, there exists an edge $e_0 = [u_1, e_0, u_2] \in E_0$. Since $\Sigma_1 = \mathbf{K}_{s_1,t_1}^{(\lambda_1)}$, we also have an edge $e_1 = [u_1, e_1, u_2] \in E_1$, and by the transitivity of G on E_0 and E_1, it follows that $V(\Sigma_1)$ is contained in the vertex-set of a Π_0-component Σ_0, and the same argument gives $V(\Sigma_0) \subseteq V(\Sigma_1)$, yielding equality $V(\Sigma_1) = V(\Sigma_0)$. This implies that $s := s_1 = s_0, t := t_1 = t_0$, and since Γ is connected, also $r_1 = r_0 = 1$ and hence $\lambda_1 = \lambda_0 = \lambda$, say. Thus $\Gamma = \Gamma_0^{(\lambda)}$, with $\Gamma_0 = \mathbf{K}_{s,t}^{(2)}$ and $|G| = st\lambda$, and line 2 of Table 3 holds for Γ_0.

We may therefore assume that, for $u \in V_1 \cap V(\Sigma_1)$, the set $\Pi_1(u)$ of t_1 vertices adjacent to u in Π_1 (that is to say, adjacent in Σ_1) is disjoint from the set $\Pi_0(u)$ of t_0 vertices adjacent to u in Π_0. Since G is transitive on E_1 and E_0, it follows that G_u is transitive on each of the disjoint sets $\Pi_1(u)$ and $\Pi_0(u)$. Moreover since G is cyclic, G_u is normal in G and hence all of its orbits in V_2 have the same size. Hence $t_1 = t_0 = t$, say, and the number of G_u-orbits in V_2 is $r_0 = |V_2|/t_0 = |V_2|/t_1 = r_1 = r$, say, and $r \geq 2$. An analogous argument with a vertex $w \in V_2 \cap V(\Sigma_1)$ yields that $s_0 = s_1 = s$, say, and G_w has all orbits in V_1 of length s. Thus $\lambda_0 = |E_0|/rst = |E_1|/rst = \lambda_1 = \lambda$, say, and it follows that $\Pi_0 \cong \Pi_1 \cong r\,\mathbf{K}_{s,t}^{(\lambda)}$. Since E_0, E_1 are disjoint, it follows that $\Gamma = \Gamma_0^{(\lambda)}$, and each $\Pi_i = \Phi_i^{(\lambda)}$, with $\Gamma_0 = (V, F)$ connected and $\Phi_i = (V, F_i) \cong r\,\mathbf{K}_{s,t}$ such that $E = F^{(\lambda)}$, each $E_i = F_i^{(\lambda)}$, and F is the disjoint union $F_0 \cup F_1$. Also G induces a cyclic bi-transitive, edge-bi-regular group H on Γ_0. Note that the H-action on V is equivalent to the G-action on V, and $|H| = rst$.

Let $L = H_u$ and $K = H_w$ so the K-orbits in V_1 and the L-orbits in V_2 are the vertex-subsets, in V_1, V_2 respectively, of the components of the Π_i, or equivalently the Φ_i. Since $H = \langle h \rangle$ permutes these two families of r subsets cyclically, we may label the subsets as U_ℓ, with $\ell \in \mathbb{Z}_{2r}$, such that $V_1 = \cup\{U_{2k} \mid 0 \leq k \leq r-1\}$, $V_2 = \cup\{U_{2k+1} \mid 0 \leq k \leq r-1\}$, with each $|U_{2k}| = s, |U_{2k+1}| = t$, and $U_{2k} \cup U_{2k+1}$ the vertex set of a Φ_1-component, and such that $U_\ell^h = U_{\ell+2}$ for each $\ell \in \mathbb{Z}_{2r}$. Now the vertex-set of the Φ_0-component containing U_1 is $U_{2u} \cup U_1$, for some $2u \in \mathbb{Z}_{2r}$, and the H-action implies that the Φ_0-components have vertex-sets $U_{2(k+u)} \cup U_{2k+1}$, for $0 \leq k \leq r-1$. It follows that there are paths in Γ_0 from vertices in U_0 (with edges alternately in F_1 and F_0) to vertices in $U_{2u\ell}$ for all ℓ, and to no other vertices in V_1. Since Γ_0 is connected, this implies that $\gcd(r, u) = 1$, and hence there exists $v \in \mathbb{Z}_r$ such that $uv \equiv 1 \pmod{r}$. Let $\varphi : V \to V$ be any map which induces bijections $U_{2k} \to U_{2kv}$ and $U_{2k+1} \to U_{2kv+1}$ for each

k. Then φ permutes the Φ_1-components among themselves, and maps the Φ_0-component with vertex-set $U_{2(k+u)} \cup U_{2k+1}$ to a graph $\mathbf{K}_{s,t}$ with vertex set $U_{2kv+2} \cup U_{2kv+1}$. Thus φ induces a graph isomorphism from Γ_0 to the graph $\mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$ in Definition 4.6. Therefore $\Gamma = \Gamma_0^{(\lambda)}$, with $\Gamma_0 \cong \mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$, $|G| = rst\lambda$, and line 3 of Table 3 holds for Γ_0.

This completes our analysis of the case where $V = V(E_i)$ for each i, so we may without loss of generality assume from now on that $V(E_1) = V_1$ and (since Γ is connected) that $V(E_0) = V = V_1 \cup V_2$. In particular $|V_1| > 1$ since $E_1 \neq \emptyset$. Then, by Corollary 5.5, we have (i) $\Pi_1 = r_1\,\Sigma_1^{(\lambda_1)}$ with $\Sigma_1 = \mathbf{K}_2$ or $\mathbf{C}_n$ and $|V_1| = 2r_1$ or nr_1 respectively, and (ii) $\Pi_0 = r_0\Sigma_0^{(\lambda_0)}$ with $\Sigma_0 = \mathbf{K}_2$ or $\mathbf{K}_{s,t}$ where $\gcd(s,t) = 1$ and $st > 1$. For case (ii) we remove the constraint $st > 1$ and consider the two possibilities for Σ_0 together, and further, we assume that each Π_0-component has $s \geq 1$ vertices in V_1 and $t \geq 1$ vertices in V_2, where $\gcd(s,t) = 1$, so $|V_1| = r_0 s$ and $|V_2| = r_0 t$.

Suppose first that some Π_1-edge (that is, an edge in E_1) is incident with two distinct vertices from the same Π_0-component, so in particular $s > 1$. Since G is transitive on E_0 and acts as automorphisms of Π_1, this holds for all edges in E_1, and hence there are no E_1-edges between distinct Π_0-components. Since Γ is connected this implies that Π_0 is connected, so $r_0 = 1$, $|V_2| = t$, and $|V_1| = s = 2r_1$ or nr_1 according as $\Sigma_1 = \mathbf{K}_2$ or $\mathbf{C}_n$, respectively. Consider first $\Sigma_1 = \mathbf{K}_2$, so $s = 2r_1$ and hence t is odd and $\gcd(t, r_1) = 1$. Then $|E_1| = r_1\lambda_1$ and $|E_0| = st\lambda_0 = 2r_1t\lambda_0$, so with $\lambda := \lambda_0$ and $r := r_1$ we have $\lambda_1 = 2t\lambda$. Thus $\Pi_0 \cong \mathbf{K}_{2r,t}^{(\lambda)}$ and $\Pi_1 = r\,\mathbf{K}_2^{(2t\lambda)}$, and as in Definition 4.10, $\Gamma = \Gamma_0^{(\lambda)}$ with $\Gamma_0 = r\,\mathbf{K}_2^{(2t)} + \mathbf{K}_{2r,t}$, $|G| = 2rt\lambda$, and line 4 of Table 3 holds for Γ_0.

Now consider $\Sigma_1 = \mathbf{C}_n$, so $s = nr_1$ and $\gcd(t, nr_1) = 1$. Then $|E_1| = nr_1\lambda_1$ and $|E_0| = nr_1t\lambda_0$, so with $\lambda := \lambda_0$ and $r := r_1$ we have $\lambda_1 = t\lambda$, $\Pi_0 \cong \mathbf{K}_{nr,t}^{(\lambda)}$ and $\Pi_1 = r\,\mathbf{C}_n^{(t\lambda)}$, and as in Definition 4.10, $\Gamma = \Gamma_0^{(\lambda)}$ with $\Gamma_0 = r\,\mathbf{C}_n^{(t)} + \mathbf{K}_{nr,t}$, $|G| = nrt\lambda$, and line 6 of Table 3 holds for Γ_0.

Finally suppose that each edge of E_1 is incident with vertices from two different components of Π_0, so $r_0 \geq 2$. Thus $e \in E_1$ satisfies $e = [\alpha_1, e, \alpha_2]$ with α_i in component $\Sigma_{0,i}$ of Π_0 for $i = 1, 2$, and the subgroup H of index r_0 in G fixes each of the Π_0-components setwise (since H is normal in G), and the H-orbits in V_1 are the s-subsets of V_1 lying in the Π_0-components. Thus each of the s vertices of $\Sigma_{0,1}$ in V_1 is joined by an edge of E_1 to a vertex in $\Sigma_{0,2}$. Further since G permutes the Π_0-components transitively and cyclically, it follows that the component, $\mathbf{K}_2^{(\lambda_1)}$ or $\mathbf{C}_n^{(\lambda_1)}$, of Π_1 containing e meets each of the Π_0-components. Consider first $\Sigma_1 = \mathbf{K}_2$. Then there are exactly $r_0 = 2$ components of Π_0, and $r_1 = s$ components of Π_1. Thus $|V_1| = 2s$ and for the transitive cyclic group $\mathbb{Z}_{2s}$ induced on V_1, the stabiliser $\mathbb{Z}_2$ of a Π_1-component interchanges its two vertices and hence interchanges the two Π_0-components. This implies that s is odd. Also $s\lambda_1 = |E_1| = |E_0| = 2st\lambda_0$, so setting $\lambda := \lambda_0$ we have $\lambda_1 = 2t\lambda$ and so $\Gamma = \Gamma_0^{(\lambda)}$, with $\Gamma_0 = s\,\mathbf{K}_2^{(2t)} + 2\mathbf{K}_{s,t}$, $|G| = 2st\lambda$, and line 5 of Table 3 holds for Γ_0.

Now consider $\Sigma_1 = \mathbf{C}_n$. Then Σ_1 meets each of the Π_0-components in a constant number u of points, so we have $n = ur_0$ and $s = ur_1$, so $\gcd(ur_1, t) = 1$. Now the stabiliser of a Π_1-component is still transitive on the Π_0-components, and conversely, and this holds if and only if $\gcd(r_0, r_1) = 1$. Also $|E_1| = nr_1\lambda_1 = ur_0r_1\lambda_1$ and $|E_0| = r_0st\lambda_0 = r_0ur_1t\lambda_0$, so with $\lambda := \lambda_0$ we have $\lambda_1 = t\lambda$, and so $\Pi_1 = r_1\,\mathbf{C}_{ur_0}^{(t\lambda)}$, $\Pi_0 = r_0\mathbf{K}_{ur_1,t}^{(\lambda)}$, $|G| = ur_0r_1t\lambda$, and $\Gamma = \Gamma_0^{(\lambda)}$ with $\Gamma_0 = r_1\,\mathbf{C}_{ur_0}^{(t)} + r_0\mathbf{K}_{ur_1,t}$, $ur_0 \geq 3$, $r_0 \geq 2$, and $\gcd(r_0, r_1) = \gcd(ur_1, t) = 1$. So line 7 of Table 3 holds for Γ_0.

This completes the analysis of the case where there are two G-vertex-orbits.

6.3 The case where G^V has three orbits

In this final subsection we assume that G has three orbits V_1, V_2, V_3 on vertices with, say, $V(E_0) = V_1 \cup V_2$ and $V(E_1) = V_2 \cup V_3$. Then G acts faithfully on each E_i, and by Corollary 5.5, we have $\Pi_0 = r_0\mathbf{K}^{(\lambda_0)}_{s_0,t_0}$ and $\Pi_1 = r_1\mathbf{K}^{(\lambda_1)}_{s_1,t_1}$, with $\gcd(s_0, t_0) = \gcd(s_1, t_1) = 1$, and to simplify our analysis we assume $s_0t_0 \geq 1, s_1t_1 \geq 1$, (identifying $\mathbf{K}_2$ with $\mathbf{K}_{1,1}$). Also we assume, for each i, that each component of Π_i has s_i vertices in V_2, so $|V_2| = r_0s_0 = r_1s_1$, and $|G| = r_0s_0t_0\lambda_0 = r_1s_1t_1\lambda_1$, which gives $t_0\lambda_0 = t_1\lambda_1$.

For each i, let H_i be the subgroup of G of index r_i. Then the H_i-orbits in V_2 are the s_i-subsets of vertices in the Π_i-components. Let $d = \gcd(r_0, r_1)$ and let H be the index d subgroup of G. Then for each i, $H_i \leq H$ and hence the H-orbits in V_2 are unions of H_i-orbits, for each i. Let Δ be one such H-orbit and let $\delta \in \Delta$. Then all paths in Γ starting from δ and ending in V_2 (using edges from E_0 or E_1 or both) must end at a vertex of Δ. Since Γ is connected, it follows that $\Delta = V_2$, and hence $d = 1$, that is, $\gcd(r_0, r_1) = 1$. Therefore $s := s_0/r_1 = s_1/r_0$ is an integer (since $r_0s_0 = r_1s_1$); and each Π_0-component meets each Π_1-component in exactly s vertices of V_2.

Let $\lambda := \gcd(\lambda_0, \lambda_1)$ and set $t = \lambda_0/\lambda$ and $t' = \lambda_1/\lambda$, so $t_0t = t_1t'$ (since $t_0\lambda_0 = t_1\lambda_1$). Since $\gcd(t, t') = 1$, $u := t_0/t' = t_1/t$ is an integer. Set $r := r_0, r' := r_1$. Then $\Gamma = \Gamma_0^{(\lambda)}$ and each $\Pi_i = \Phi_i^{(\lambda)}$, where $\Phi_0 = r\mathbf{K}^{(t)}_{sr',ut'}$ and $\Phi_1 = r'\mathbf{K}^{(t')}_{sr,ut}$, and $|G| = rr'su\lambda tt'$; and we have $\gcd(r, r') = \gcd(t, t') = 1$, and $\gcd(sr, ut) = \gcd(sr', ut') = 1$. Thus line 8 of Table 3 holds for Γ_0.

This graph family covers some special cases: for example if, say, $s = r = 1$ then $\Gamma_0 = \mathbf{K}^{(t)}_{r',ut'} + r'\mathbf{K}^{(t')}_{1,ut}$; and if in addition $u = t = 1$ then $\Gamma_0 = \mathbf{K}_{r',t'} + r'\mathbf{K}^{(t')}_2$.

This completes the proof that all graphs Γ with at least three vertices, and no isolated vertices, and admitting a cyclic edge-regular or edge-bi-regular group of automorphisms, are of the form $\Gamma = \Gamma_0^{(\lambda)}$ with Γ_0 listed in one of the Tables 1, 2, and 3. Conversely it follows from Lemmas 4.2, 4.4, 4.7, 4.9, 4.11, 4.13, 4.16, and Corollary 4.5, that all the graphs in these tables admit such groups.

7 Proof of Theorem 1.2

The aim of this section is to complete the proof of Theorem 1.2. So let $\Gamma = (V, E)$ be a graph which has a symmetrical Euler cycle

$$C = (e_0, e_1, \ldots, e_{\ell-1})$$

where $\ell = |E|$. Thus each $e_i = [\alpha_{i-1}, e_i, \alpha_i]$ where we write $\alpha_\ell = \alpha_0$. In particular Γ is connected, and there exists $x \in \mathrm{Aut}\Gamma$ such that

$$x\colon\ e_i \to e_{i+2},\ \text{for each } i, \text{reading subscripts modulo } \ell.$$

Then $\langle x\rangle$ is edge-regular if ℓ is odd, or edge-bi-regular if ℓ is even, and hence, by Theorem 1.1, Γ is one of the graphs in Tables 1, 2, or 3. Our task is to decide which of these graphs has a symmetrical Euler cycle, and for each of these graphs, to determine the largest subgroup $H(C)$ of the group $D(C)$ in (1.3) induced by $(\mathrm{Aut}\Gamma)_{[C]}$. Note that x induces the element φ^2 of (1.4) so $H(C)$ contains φ^2. Recall that, for each vertex α, the number of edges of C incident with α must be even. First we consider some of the examples listed in Theorem 1.2.

Lemma 7.1. (a) *If* $\Gamma = \mathbf{C}_n^{(\lambda)}$ *for some* $n \geq 3, \lambda \geq 1$, *then* Γ *has a symmetrical Euler cycle* C *and* $H(C) = \langle \varphi, \tau \rangle$.

(b) *If* $\Gamma = \mathbf{K}_{s,t}^{(\lambda)}$ *for some* $\lambda \geq 1$ *and* $st > 1$ *with* $\gcd(s,t) = 1$, *then* Γ *has a symmetrical Euler cycle* C *if and only if* λ *is even, and in this case* $H(C) = \langle \varphi^2, \varphi\tau \rangle$.

Proof. (a) Let $\Gamma = \mathbf{C}_n^{(\lambda)}$. By Proposition 3.4, it is sufficient to assume that $\lambda = 1$, and in this case the assertion follows from Lemma 4.2(a).

(b) Now let $\Gamma = \mathbf{K}_{s,t}^{(\lambda)}$. By our comments above, for an Euler cycle to exist each vertex must be incident with an even number of edges, so both $s\lambda$ and $t\lambda$ are even. Since $\gcd(s,t) = 1$, this implies that λ is even. Therefore, by Proposition 3.4, it is sufficient to assume that $\lambda = 2$ and to prove that $\Gamma = \mathbf{K}_{s,t}^{(2)}$ has a symmetrical Euler cycle C with $H(C)$ as claimed. We use the notation from Definition 4.8 for the vertices and edges of $\mathbf{K}_{s,t}$ and the map g, and the convention in (3.1) for edges of Γ. Thus the vertex set is $V = V_1 \cup V_2$ with $V_1 = \mathbb{Z}_s$ and $V_2 = \mathbb{Z}_t$. Let $e_0 = e_{0,0}^1$ and $e_1 = e_{1,0}^2$, and for each i such that $1 \leq i \leq st-1$, define

$$e_{2i} := e_{i,i}^1 = e_0^{g^i} \text{ and } e_{2i+1} := e_{i+1,i}^2 = e_1^{g^i}.$$

Note that $e_{2i} = [i, e_{2i}, i]$ and $e_{2i+1} = [i+1, e_{2i}, i]$, where we read the first entries (elements of V_1) modulo s and the last entries (elements of V_2) modulo t. Thus e_{2i}, e_{2i+1} are both incident with $i \in V_2$, and similarly e_{2i-1}, e_{2i} are both incident with $i \in V_1$. Further, since $\gcd(s,t) = 1$, for each edge e of $\mathbf{K}_{s,t}$, the edge e^1 occurs as e_{2i} for some i, and e^2 occurs as e_{2i+1} for some i. Also, since $|g| = st$ and $\langle g \rangle$ is regular on the edge-set of $\mathbf{K}_{st}$, it follows that

$$C := (e_0, e_1, \ldots, e_{2st-1})$$

is an Euler cycle for Γ, that C is preserved by $\langle g \rangle$, and that g induces the map φ^2 in $D(C)$. Thus C is a symmetrical Euler cycle. Finally, $\mathrm{Aut}\Gamma$ contains the following involution y, where $y|_{V_1} : i \to -i$ for all $i \in V_1 = \mathbb{Z}_s$, and $y|_{V_2} : i \to -i$ for all $i \in V_2 = \mathbb{Z}_t$; and

$$y \colon e_{i,i}^1 \to e_{-i,-i}^1, \text{ and } y \colon e_{i+1,i}^2 \to e_{-i,-i-1}^2, \text{ for all } i.$$

It is straightforward to check that y preserves C, namely $y \colon e_{2i} \to e_{2j}$ and $y \colon e_{2i+1} \to e_{2j-1}$ where $i + j = st$; and y induces the map $\varphi\tau$ of $D(C)$ as in (1.4). Thus $H(C)$ contains $\langle \varphi^2, \varphi\tau \rangle$, and since Γ is not vertex-transitive, equality holds. □

Lemma 7.1 deals with all the graphs which admit a cyclic edge-regular subgroup, see Table 1. So we may assume that Γ admits no cyclic edge-regular subgroup. Hence the length ℓ of C is even, and $\langle x \rangle$ is bi-regular on E, say with orbits E_0, E_1. This means that, in the symmetrical Euler cycle C the edges occur alternately in E_0 and E_1. Replacing C by a shift if necessary, we may assume then that $E_0 = \{e_{2i} \mid 0 \leq i \leq \ell - 1\}$ and $E_1 = \{e_{2i+1} \mid 0 \leq i \leq \ell - 1\}$.

Lemma 7.2. (a) *The group $\langle x \rangle$ has at most two orbits in V.*

(b) *If $\langle x \rangle$ has two orbits in V, say V_1 and V_2, then each edge is incident with exactly one vertex of V_1 and one vertex of V_2.*

(c) *Thus Γ is one of the graphs in lines 2 – 4 of* Table 2 *or line 3 of* Table 3.

Proof. (a) Since $E_0 = \{e_{2i} \mid 0 \le i \le \ell - 1\}$ and $E_1 = \{e_{2i+1} \mid 0 \le i \le \ell - 1\}$, it follows from the definition of C that every vertex is incident with at least one edge in E_0 and at least one edge in E_1. Then, since $\langle x \rangle$ is transitive on E_0 and E_1 we conclude that $\langle x \rangle$ has at most two orbits in V.

(b) Suppose $\langle x \rangle$ has two orbits V_1 and V_2 in V. Then as Γ is connected, at least one of the edge-orbits, say E_0, has edges incident with vertices from each of V_1 and V_2, so $e_0 = [\alpha, e_0, \beta]$ with, say, $\alpha \in V_1$ and $\beta \in V_2$. Thus e_{2i-1} is incident with $\alpha \in V_1$ and e_{2i+1} is incident with $\beta \in V_2$, and hence also the edges of E_1 are incident with vertices from both V_1 and V_2.

(c) The last line of Table 3 is not possible by part (a), graphs in line 1 of Table 2 or lines 1–2 of Table 3 are excluded because they admit cyclic edge-regular subgroups, and graphs in lines 4–7 of Table 3 are not possible by part (b). □

The next two lemmas deal with the remaining graphs from Table 2.

Lemma 7.3. (a) *If $\Gamma = \Gamma_0^{(\lambda)}$ with $\Gamma_0 = \Gamma(2r, 1, r) = \mathbf{C}_{2r} + r\mathbf{K}_2^{(2)}$ as in* Corollary 4.5(a) *(line 2 of* Table 2*), where $r \ge 2$ and $\lambda \ge 1$, then Γ has a symmetrical Euler cycle C if and only if r is even, and in this case $H(C) = \langle \varphi^2, \varphi\tau \rangle$.*

(b) *If $\Gamma = \Gamma_0^{(\lambda)}$ with $\Gamma_0 = \Gamma(2r, 2, r) = 2\mathbf{C}_r + r\mathbf{K}_2^{(2)}$ as in* Corollary 4.5(b) *(line 3 of* Table 2*), where r is odd, $r \ge 3$, and $\lambda \ge 1$, then Γ has a symmetrical Euler cycle C and $H(C) = \langle \varphi^2, \varphi\tau \rangle$.*

Proof. We use the notation from Definition 4.3. In both cases (a) and (b), $\Gamma_0 = \mathrm{Circ}(n, S) = (V, E)$ with $S = \{a, -a, r^{(2)}\}$ as in Definition 4.3 (with $a = 1$ or 2), and we have $V = \mathbb{Z}_{2r}$ and $4r$ edges $e_{i,a} = [i, e_{i,a}, i + a]$ and $e_{i,r} = [i, e_{i,r}, i + r]$, for $i \in \mathbb{Z}_{2r}$. Also the map $g\colon i \to i + 1, e_{i,a} \to e_{i+1,a}, e_{i,r} \to e_{i+1,r}$ lies in $\mathrm{Aut}\Gamma_0$, by Lemma 4.4, and is bi-regular on E with $\langle g \rangle$-orbits $E_a = \{e_{i,a} \mid i \in \mathbb{Z}_{2r}\}$ and $E_r = \{e_{i,r} \mid i \in \mathbb{Z}_{2r}\}$. We also use the notation from (3.1) for edges of $\Gamma = (V, E^{(\lambda)})$ and from the proof of Proposition 3.4(a) for the edge-bi-regular automorphism $g^{(\lambda)}$ of Γ corresponding to the edge-bi-regular map g on Γ_0. The $\langle g^{(\lambda)} \rangle$-orbits in $E^{(\lambda)}$ are $E_a^{(\lambda)}$ and $E_r^{(\lambda)}$.

(a) Consider first case (a), so $a = 1$. Then the vertex-action $(\mathrm{Aut}\Gamma)^V$ is contained in $(\mathrm{Aut}[E_1])^V \cong \mathrm{D}_{4r}$ and contains g^V of order $2r$. Suppose that $C = (e_0, e_1, \ldots, e_{4r\lambda-1})$ is an Euler cycle for Γ with element x inducing $\varphi^2 \in H(C)$, as above. Then $\langle x \rangle$ induces $\mathbb{Z}_{2r}$ on V, and hence $x^V = (g^i)^V$ for some i such that $\gcd(i, 2r) = 1$. We show first that r is even. We may assume that the 'even' edges e_{2i} of C lie in $E_a^{(\lambda)}$ and the 'odd' edges $e_{2i+1}^{(\lambda)}$ lie in $E_r^{(\lambda)}$. Moreover, replacing C by a cycle in its sequence class, if necessary, we may assume further that $e_0 = e_{0,a}^1$ and e_1 is $e_{1,r}^u$ or $e_{r+1,r}^u$, for some u, and hence that e_2 is $e_{r+1,1}^v$ or $e_{r,1}^v$, for some v, and $e_{4r\lambda-1}$ is $e_{0,r}^w$ or $e_{r,r}^w$, for some w. Considering the vertices incident with these edges, it follows from $e_0^x = e_2$ that $\{0, 1\}^{g^i} = \{r+1, r+2\}$ or $\{r, r+1\}$, and from $e_{4r\lambda-1}^x = e_1$ that $\{0, r\}^{g^i} = \{1, r+1\}$. These conditions together imply that $i = r + 1$, and then from $\gcd(i, 2r) = 1$ we conclude that r is even. To complete the proof of part (a) we exhibit a symmetrical Euler cycle C for Γ such that $H(C) = \langle\varphi^2, \varphi\tau\rangle$ (noting that $H(C) \neq D(C)$ since Γ is not edge-transitive). By Proposition 3.4 we may assume that $\lambda = 1$. As noted above, in a symmetrical Euler cycle C preserved by $x \in \mathrm{Aut}\Gamma$ which induces the map φ^2 in $D(C)$, we may assume that $e_0 = e_{0,1}$, that $e_1 = e_{1,r}$ or $e_{r+1,r}$, and that $x = g^{r+1}$. So for our construction let us choose $e_1 = e_{1,r}$, so that, for all $i = 1, \dots, 2r-1$,

$$e_{2i} := e_{0,1}^{g^{i(r+1)}} = e_{i(r+1),1} \text{ and } e_{2i+1} := e_{1,r}^{g^{i(r+1)}} = e_{i(r+1)+1,r}.$$

Then e_{2i}, e_{2i+1} are both incident with $i(r+1)+1$, and e_{2i+1}, e_{2i+2} are both incident with $(i+1)(r+1)$. Hence $C := (e_0, e_1, \dots, e_{4r-1})$ is an Euler cycle of Γ. The element g^{r+1} preserves the cycle C and acts as $g^{r+1}\colon e_j \to e_{j+2}$, that is g^{r+1} induces the map φ^2 of $D(C)$. The cycle C is also preserved by the automorphism $y \in \mathrm{Aut}\Gamma$ defined by $y\colon -i \leftrightarrow i+1$ on V and $y\colon e_{-i,1} \leftrightarrow e_{i,1}, e_{-i,r} \leftrightarrow e_{i+1,r}$ on E. This map induces $\varphi\tau \in D(C)$ as in (1.4), and hence $H(C)$ contains $\langle\varphi^2, \varphi\tau\rangle$, and equality holds since Γ is not edge-transitive. An exactly similar argument works for the case where $e_1 = e_{r+1,r}$. This completes the proof of part (a).

(b) Now consider case (b), so $a = 2$ and r is odd. By Proposition 3.4 we may assume that $\lambda = 1$. Thus, for the edge-bi-regular automorphism g in the first paragraph of this proof, $\gcd(r+2, |g|) = \gcd(r+2, 2r) = 1$, and hence $x := g^{r+2}$ generates $\langle g\rangle$ and $\langle x\rangle$ is edge-bi-regular on Γ. We construct a symmetrical Euler cycle $C = (e_0, \dots, e_{4r-1})$ preserved by x and on which x induces φ^2 in $H(C)$. Let $e_0 := e_{0,r}$ and $e_1 := e_{r,2}$, and for $1 \leq i \leq 2r-1$, let

$$e_{2i} := e_0^{x^i} = e_{0,r}^{g^{i(r+2)}} = e_{i(r+2),r} \text{ and } e_{2i+1} := e_1^{x^i} = e_{r,2}^{g^{i(r+2)}} = e_{i(r+2)+r,2}.$$

Then e_{2i}, e_{2i+1} are both incident with $i(r+2)+r$, and e_{2i+1}, e_{2i+2} are both incident with $(i+1)(r+2)$. Hence C is an Euler cycle of Γ, and by definition x preserves C and induces φ^2. In addition $\mathrm{Aut}\Gamma$ contains an involution y inducing $\varphi\tau$ on C, namely $y\colon i \to r-i$ on $V = \mathbb{Z}_{2r}$, and $y\colon e_i \to e_{4r-i}$. This is straightforward to check. Hence $H(C)$ contains, and so is equal to $\langle\varphi^2, \varphi\tau\rangle$. □

Lemma 7.4. *Suppose that* $\Gamma = \Gamma_0^{(\lambda)}$ *with* $\Gamma_0 = \mathrm{Circ}(n, S) = (V, E)$ *with* $S = \{a, -a, b, -b\}$ *as in* Definition 4.3 *with* $V = \mathbb{Z}_n$, $|S| = 4$ *and* $\gcd(n, a, b) = 1$, *and that* $\lambda \geq 1$.

(a) *If either* $\gcd(n, a+b) = 1$ *or* $\gcd(n, a-b) = 1$, *and then* Γ *has a symmetrical Euler cycle, and* $H(C) = \langle\varphi^2, \varphi\tau\rangle$ *for all such cycles* C.

(b) *Moreover, if* $\mathrm{Aut}\Gamma_0$ *is not edge-transitive, then* Γ *has a symmetrical Euler cycle* C *if and only if either* $\gcd(n, a+b) = 1$ *or* $\gcd(n, a-b) = 1$.

Remark 7.5. We comment on part (b) of Lemma 7.4. The proof depends on the fact that, for a symmetrical Euler cycle C of $\Gamma = \Gamma_0^{(\lambda)}$ (with $\Gamma_0 = \mathrm{Circ}(n, S) = (V, E)$ as in line 4 of Table 2), an automorphism x inducing $\varphi^2 \in H(C)$ acts on V as a power of the map g^V from Definition 4.3. The assumption that Γ_0 is not edge-transitive is sufficient to deduce this fact. However it is possible for Γ_0 to be edge-transitive. For example if $n = 5, a = 1, b = 2$ then $\Gamma_0 \cong \mathbf{K}_5$. Edge-transitivity occurs more generally, for example, when n is odd and there exists $d \in \mathbb{Z}_n$ with $d^2 = -1$, and $b = \pm da$ so $S = \{a, da, d^2a, d^3a\}$.

In this case the 'multiplicative' automorphism $y\colon i \to di$ interchanges the two $\langle g\rangle$ edge-orbits E_a and E_b defined in Definition 4.3. If Γ_0 is a so-called *normal Cayley graph*, that is, if $\langle g, y\rangle = \mathbb{Z}_n \rtimes \mathbb{Z}_4$ is the full automorphism group $\mathrm{Aut}\Gamma_0$, then the assertion in case (b) can be proved by a slightly modified argument. However we do not know if there are parameters n, a, b for which Γ_0 is a non-normal Cayley graph (and hence is edge-transitive) and the necessary and sufficient condition in part (b) does not hold.

Proof. Here $\Gamma_0 = \mathrm{Circ}(n, S) = (V, E)$ with $S = \{a, -a, b, -b\}$ as in Definition 4.3 with $|S| = 4$, $V = \mathbb{Z}_n$, and $2n$ edges $e_{i,a} = [i, e_{i,a}, i + a]$ and $e_{i,b} = [i, e_{i,b}, i + b]$, for $i \in \mathbb{Z}_n$. The map $g\colon i \to i + 1, e_{i,a} \to e_{i+1,a}, e_{i,b} \to e_{i+1,b}$ lies in $\mathrm{Aut}\Gamma_0$, by Lemma 4.4, and is bi-regular on E with $\langle g\rangle$-orbits $E_a = \{e_{i,a} \mid i \in \mathbb{Z}_n\}$ and $E_b = \{e_{i,b} \mid i \in \mathbb{Z}_n\}$. We also use the notation from (3.1) for edges of $\Gamma = (V, E^{(\lambda)})$.

(a) Suppose first that $\gcd(a + b, n) = 1$ or $\gcd(a - b, n) = 1$. We construct a symmetrical Euler cycle for Γ. By Proposition 3.4 it is sufficient to do this for $\lambda = 1$, that is, for $\Gamma = \Gamma_0 = \mathrm{Circ}(n, S)$. So let $j = a \pm b$ and assume that $\gcd(j, n) = 1$. First let $j = a + b$, let $e_0 = e_{0,a}, e_1 = e_{a,b}$ and for $1 \le i \le n - 1$, let

$$e_{2i} := e_0^{g^{ij}} = e_{ij,a} \text{ and } e_{2i+1} := e_1^{g^{ij}} = e_{ij+a,b}.$$

Then $C = (e_0, \ldots, e_{2n-1})$ is a symmetrical Euler cycle and g^j acts as $\varphi^2 \in H(C)$. In addition $y \in \Gamma$ defined by $y : i \to a - i$ on V, and $y\colon e_i \to e_{2n-i}$, induces $\varphi\tau$ on C so $H(C) \ge \langle \varphi^2, \varphi\tau\rangle$. Similarly, for $j = a - b$, let $e_0 = e_{0,a}, e_1 = e_{a-b,b} = e_{j,b}$ (with e_1 incident with $\{a, a - b\}$) and for $1 \le i \le n - 1$, let

$$e_{2i} := e_0^{g^{ij}} = e_{ij,a} \text{ and } e_{2i+1} := e_1^{g^{ij}} = e_{a-b+ij,b} = e_{(i+1)j,b}.$$

Then again $C = (e_0, \ldots, e_{2n-1})$ is a symmetrical Euler cycle and g^j acts as $\varphi^2 \in H(C)$. Again the element $y \in \Gamma$ defined by $y\colon i \to a-i$ on V and $y\colon e_i \to e_{2n-i}$ induces $\varphi\tau$ on C so $H(C) \ge \langle\varphi^2, \varphi\tau\rangle$. If, for some symmetrical Euler cycle C, $H(C)$ is strictly larger than $\langle\varphi^2, \varphi\tau\rangle$, then $H(C) = D(C)$ and so some element $x \in \mathrm{Aut}\Gamma$ induces φ on C, and hence $\langle x\rangle$ is a cyclic edge-regular subgroup of $\mathrm{Aut}\Gamma$. This however contradicts Theorem 1.1 as $\Gamma = \Gamma_0$ does not appear in Table 1. Thus part (a) is proved.

(b) Suppose now that $\mathrm{Aut}\Gamma_0$ is not edge-transitive and that Γ has a symmetrical Euler cycle $C = (e_0, \ldots, e_{2n\lambda-1})$ and $x \in \mathrm{Aut}\Gamma$ preserves C and acts by $x\colon e_i \to e_{i+2}$ for each i. Since Γ_0 is not edge-transitive, $\mathrm{Aut}\Gamma_0 \le \mathrm{Aut}[E_a]$ and hence the groups induced on V by $\mathrm{Aut}\Gamma$ and $\mathrm{Aut}\Gamma_0$ are both isomorphic to D_{2n} with index 2 subgroup $\langle g^V\rangle$ generated by $g^V : i \to i + 1$, with g as in the first paragraph of the proof. Since $\langle x\rangle$ is cyclic of order $n\lambda$ it follows that x^V has order n and hence $x^V = (g^V)^j$ for some j coprime to n. Relabelling the elements of S if necessary, we may assume that $e_0 = e_{0,a}^u$ for some u. Then, the next edge e_1 lies in $E_b^{(\lambda)}$ and hence is $e_{a,b}^v$ or $e_{a-b,b}^v$ for some v. Assume first that $e_1 = e_{a,b}^v$. Then e_2 is incident with $a+b$ while, since $e_2 = e_0^x$, the pair of vertices incident with e_2 are $\{0, a\}^x = \{0, a\}^{g^j} = \{j, a + j\}$. Hence $j = a + b$ or $j = b$ and $e_2 = e_{a+b,a}^w$ or $e_{b,a}^w$, for some w, and e_3 is incident with $2a + b$ or b, respectively. Moreover since $e_3 = e_1^x$, the pair of vertices incident with e_3 are $\{a, a + b\}^x = \{a, a + b\}^{g^j}$ which is $\{2a + b, 2a + 2b\}$ if $j = a + b$ or $\{a + b, a + 2b\}$ if $j = b$. However, if $j = b$ then e_3 should be incident with b, and $b \notin \{a + b, a + 2b\}$ since $a \ne 0, -b$. Hence $j = a + b$ so $\gcd(a + b, n) = 1$. A similar argument in the case where $e_1 = e_{a-b,b}^v$ leads to the condition $\gcd(a - b, n) = 1$. This proves the asserted congruence conditions, and the converse follows from part (a). This completes the proof. □

Finally we consider the remaining graphs from Table 3.

Lemma 7.6. *If* $\Gamma = \Gamma_0^{(\lambda)}$ *with* $\Gamma_0 = \mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1]$ *as in* Definition 4.6 *(line* 3 *of* Table 3*), with* $r \geq 2$*,* $st \geq 2$*,* $\gcd(s,t) = 1$*, and* $\lambda \geq 1$*, then* Γ *has a symmetrical Euler cycle* C *and* $H(C) = \langle \varphi^2, \tau \rangle$*.*

Proof. We use the notation from Definition 4.6. Thus $\Gamma_0 = \mathbf{C}_{2r}[s\mathbf{K}_1, t\mathbf{K}_1] = (V, E)$, and $V = V_S \cup V_T$ and $E = E_S \cup E_T$, where $S = \mathbb{Z}_s, T = \mathbb{Z}_t$. Also we have $g, y \in \mathrm{Aut}\Gamma_0$ such that $\langle g \rangle$ is edge-bi-regular with edge-orbits E_S, E_T and $\langle g, y \rangle \cong \mathrm{D}_{2rst}$ is bi-transitive on V with vertex-orbits V_S, V_T, and regular on E (see Lemma 4.7). By Proposition 3.4 it is sufficient to construct a cycle C with the required properties in the case $\lambda = 1$. So we assume now that $\lambda = 1$ and $\Gamma = \Gamma_0$, and we construct $C = (e_0, \ldots, e_{2rst-1})$.

From Definition 4.6 the edges in E_S, E_T are labelled $e_{i,j}^{2k}$ and $e_{j,i}^{2k+1}$, respectively, where $0 \leq k \leq r-1, i \in S, j \in T$. Each integer i such that $0 \leq i \leq 2rst - 1$ can be represented uniquely as $i = \ell + 2rm$, where $0 \leq \ell \leq 2r-1$ and $0 \leq m \leq st-1$. For such a representation we define $e_i = e_{m,m}^{\ell}$ if $\ell \leq 2r-2$ and $e_i = e_{m,m+1}^{\ell}$ if $\ell = 2r-1$. If $\ell \leq 2r-2$, then e_i, e_{i+1} are both incident with the vertex (ℓ, m) where $m \in T$ if ℓ is even or $m \in S$ if ℓ is odd. If $\ell = 2r-1$, then e_i, e_{i+1} are both incident with the vertex $(0, m+1)$ with $m+1 \in S$. This proves that C is a cycle of length $2rst$, and from its definition C involves all edges and hence is an Euler cycle. Moreover C is preserved by g and y, with g inducing $\varphi^2 \in H(C)$ and y inducing $\tau \in H(C)$. If $H(C)$ is strictly larger than $\langle \varphi^2, \tau \rangle$, then $H(C) = D(C)$, which implies that Γ_0 is vertex-transitive, and hence that $s = t$, which is a contradiction. This completes the proof. □

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.04
https://doi.org/10.26493/2590-9770.1458.4b9
(Also available at http://adam-journal.eu)

Groups of automorphisms of Riemann and Klein surfaces, our joint work with Marston Conder

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Received 1 September 2021, accepted 1 December 2021, published online 27 June 2022

Abstract

This work is a survey on the research that we have carried out together with Professor Marston Conder on groups of automorphisms of Riemann and Klein surfaces over the last twenty years.

Keywords: Groups of automorphisms, Riemann surfaces, Klein surfaces

Math. Subj. Class.: 30F, 14H

Introduction

The first author met Marston Conder at the St Andrews 1989 Conference on Groups. In the first conversation we had, Marston pointed out that we lived in antipodal cities, Madrid and Auckland, a detail that the authors had not realized but which did not stop him from visiting our university. Our collaboration began ten years later. This collaboration has focused on the study, from a combinatorial point of view, of the groups of automorphisms of Riemann and Klein surfaces.

The paper is divided into five sections. In Section 1 we introduce the notation and the results that are necessary in the rest of the work. In Section 2 we deal with the problem of deciding whether a group G acting on genus g also acts as a full group on genus g. Another problem on which many results have been obtained is the determination of the automorphism groups of the surfaces of a given genus. The amount of combinatorial data involved increases rapidly with the genus. In Section 3 we show some results where the computational methods implemented in programs as MAGMA, in which Marston is an expert, turn out to be very useful to determine automorphism groups of Klein surfaces. In Section 4 we

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consider the relation between the automorphism groups of a Klein surface and that of its Riemann double cover in some special cases. Finally, in Section 5 we study bounds of the automorphism groups of pseudo-real Riemann surfaces.

1 Preliminaries

The analytic structure of a compact Riemann surface makes it orientable and unbordered. Non-orientable surfaces and surfaces with boundary also admit a *dianalytic structure*, which behaves in many aspects as the analytic structure of a classical Riemann surface, see [2]. Endowed with this structure, these surfaces are known as *Klein surfaces.* Groups of automorphisms of both types of surfaces (either Riemann or Klein) can be studied from a combinatorial point of view due to the Uniformization Theorem. Indeed, every such compact surface S of algebraic genus $g > 1$ can be represented as the orbit space $\mathcal{H}/\Gamma$ where $\mathcal{H}$ is the hyperbolic plane and Γ is a discrete subgroup of isometries of the hyperbolic plane without elliptic elements. Groups as Γ are called *surface groups.* In this situation, each (finite) group G of automorphisms of S is described as the factor group Λ/Γ where Λ is a *non-euclidean crystallographic (NEC) group,* that is, a discrete subgroup of isometries of the hyperbolic plane such that the orbit space $\mathcal{H}/\Lambda$ is also compact. Observe that Λ may contain orientation reversing isometries and elliptic elements. Thus, the action of a group G on a surface $\mathcal{H}/\Gamma$ is described by an epimorphism $\theta\colon \Lambda \to G$ with $\ker\theta = \Gamma$. These are called *surface-kernel epimorphisms.*

The first presentations for NEC groups appeared in [43] and their structure was clarified by the introduction of signatures in [33]. This is a collection of symbols and non-negative integers of the form

$$\sigma = (\gamma; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

The non-negative integer γ is called the orbit genus of σ. If the sign "+" appears, then we write sign(σ) = "+"; otherwise sign(σ) = "−". The integers $m_1, \ldots, m_r$ are called the proper periods of σ and the n_{ij} are called the link periods of the period cycle $(n_{i1}, \ldots, n_{is_i})$. An empty set of proper periods, (*i.e.* $r = 0$), will be denoted by $[-]$, an empty period cycle (*i.e.,* $s_i = 0$) by $(-)$, and the fact that σ has no period cycles (*i.e.,* $k = 0$) by $\{-\}$.

The signature of an NEC group provides a presentation by generators and relations. We omit it here since it is not necessary for the purposes of this paper. The interested reader may find it in [43] or [33]. We just point out that since a Fuchsian group contains no orientation reversing elements, its signature has no period cycles and its sign is always "+". Hence we may drop such data and represent its signature simply by $(\gamma; m_1, \ldots, m_r)$.

A symmetry on a Riemann surface S is an anticonformal involution $\tau\colon S \to S$. The orbit space $S/\langle\tau\rangle$ can be endowed with a dianalytic structure, making it a Klein surface. Actually, all Klein surfaces can be obtained in this way, that is, for every Klein surface X there exist a Riemann surface S and a symmetry τ on it such that $X = S/\langle\tau\rangle$. In terms of algebraic geometry, X is a real algebraic curve and S is its complexification. The pair (S, τ) is called the double cover of X.

2 On extendability of group actions

An abstract finite group G is said to act on genus $g > 1$ if it is (isomorphic to) a group of automorphisms of some compact surface of genus g. We say that G acts as a full group on genus g if G is (isomorphic to) the full automorphism group of some compact surface of genus g. An interesting problem in Riemann and Klein surface theory is to investigate whether a group G acting on genus g also acts as a full group on genus g. A key point in the solution are the concepts of maximal NEC group and maximal signature. An NEC group Λ is said to be maximal if it is not properly contained in another NEC group. In such a case, for every surface Fuchsian group Γ contained in Λ as a normal subgroup, the group Λ/Γ acting on the Riemann surface $\mathcal{H}/\Lambda$ is precisely its full automorphism group.

However, if Λ is non-maximal then the action of Λ/Γ on $\mathcal{H}/\Lambda$ could possibly be extended to the action of a larger group. Indeed, if Λ' is another NEC group containing Λ with finite index then Λ' could possibly contain Γ as a normal subgroup, in which case Λ'/Γ would be a larger group acting on $\mathcal{H}/\Lambda$. The determination of maximal and non-maximal NEC groups is therefore a key point in this topic. This leads to the more cumbersome concepts of maximal and non-maximal signatures. We avoid their definitions here, which can be found, for instance, in [21, Chapter 5]. The maximality of a signature σ does not imply the maximality of every NEC group having signature σ. However, if σ is a maximal signature then there exists a maximal NEC group Λ with signature σ. The study of maximal signatures was initiated by Greenberg [28] in the case of Fuchsian signatures. He obtained a list of those which fail to be maximal. The list was completed by Singerman [42]. The analogous list for NEC signatures was obtained by Bujalance [4] for the so called normal pairs, and by Estévez and Izquierdo [24] for the non-normal pairs.

In this section we present the results we have obtained with Marston on the topic of extendability of group actions. A natural distinction is made depending on the type of surface: Riemann surfaces in Subsection 2.1 and Klein surfaces in Subsection 2.2. We would like to point out the interest of other authors on this topic. It is worth mentioning the paper [23] by Costa and Parlier, where the extendability of group actions on Riemann surfaces is considered using a somewhat different approach.

2.1 Extendability of group actions on Riemann surfaces

The first case to study is when G is cyclic and the surface is unbordered and orientable, that is, a Riemann surface. This case was solved in Marston's first visit to our university UNED in Madrid, which took place in 1999. The results were published in [14], the two main ones being the following.

Theorem 2.1. *Suppose that the cyclic group C_n acts as a group of automorphisms of a compact Riemann surface S of genus $g>1$, corresponding to a surface-kernel epimorphism from the Fuchsian group Γ with signature $\sigma(\Gamma)=(\gamma; m_1,\dots,m_r)$ onto C_n. Let $z_1,\dots,z_r$ be the images in G of the canonical elliptic generators $x_1\dots,x_r$, of orders $m_1,\dots,m_r$, of Γ. Then in each of the following cases, the action of C_n on S can always be extended to the action of a larger group, so that C_n is not the full group $\mathrm{Aut}(S)$ of all automorphisms of S.*

(i) $\sigma(\Gamma)=(2;-)$;

(ii) $\sigma(\Gamma)=(1;t,t)$, *whenever* $t\geq 2$;

(iii) $\sigma(\Gamma)=(0;t_1,t_1,t_2,t_2)$, *whenever* $t_1+t_2\geq 5$;

(iv) $\sigma(\Gamma)=(0;t,t,t)$, *whenever* $t\geq 4$, *provided that C_n has an automorphism of order 3 permuting z_1,z_2,z_3 in a 3-cycle;*

(v) $\sigma(\Gamma)=(0;t_1,t_1,t_2)$, *whenever* $t_1\geq 3$ *and* $t_1+t_2\geq 7$, *provided that either* $z_1=z_2$ *or C_n has an automorphism of order 2 interchanging z_1 and z_2;*

(vi) $\sigma(\Gamma) = (0; 3, 4, 12)$.

Theorem 2.2. *Suppose that the cyclic group C_n acts as a group of automorphisms of a compact Riemann surface S of genus $g > 1$, corresponding to a surface-kernel epimorphism onto C_n from a Fuchsian triangle group Γ with signature $(0; m_1, m_2, m_3)$. Then, with the exception of cases (iv), (v) and (vi) in the above theorem, the action of C_n on S can never be extended to the action of a larger group, so that $C_n = \mathrm{Aut}(S)$.*

The general case of an arbitrary group G was considered shortly after, when the second author of this survey joined the UNED. The question of extendability was completed and the results published in [6], where special attention was paid to non-cyclic abelian groups. The main results are the following, where we distinguish whether the group G acts with triangular signature or not.

Theorem 2.3. *Let G be a finite group acting with a non-maximal and non-triangular Fuchsian signature on a compact Riemann surface S of genus $g > 1$.*

(i) *If G acts with signature $(2; —)$, and for a corresponding presentation $G = \langle a, b, c, d \mid [a, b][c, d] = \cdots = 1 \rangle$ the assignment $a \mapsto a^{-1}$, $b \mapsto ab^{-1}a^{-1}$, $c \mapsto (b^{-1}cd)c^{-1}(b^{-1}cd)^{-1}$ and $d \mapsto (b^{-1}c)d^{-1}(b^{-1}c)^{-1}$ is an automorphism of G, then G is not the full automorphism group of S.*

(ii) *If G acts with signature $(1; t, t)$, and for a corresponding presentation $G = \langle\, a, b, x \mid x^t = ([a, b]x)^t = \cdots = 1 \,\rangle$ the assignment $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ and $x \mapsto (ab)^{-1}x^{-1}(ba)$ is an automorphism of G, then G is not the full automorphism group of S.*

(iii) *If G acts with signature $(1; t)$ and for a corresponding presentation $G = \langle\, a, b \mid [a, b]^t = \cdots = 1 \,\rangle$ the assignment $a \mapsto a^{-1}$ and $b \mapsto b^{-1}$ is an automorphism of G, then G is not the full automorphism group of S.*

(iv) *If G acts with signature $(0; t, t, u, u)$ where $t + u \geq 5$, and for a corresponding presentation $G = \langle\, a, b, c, d \mid a^t = b^t = c^u = d^u = abcd = \cdots = 1 \,\rangle$ the assignment $a \mapsto b$, $b \mapsto a$, $c \mapsto a^{-1}da$ and $d \mapsto bcb^{-1}$ is an automorphism of G, then G is not the full automorphism group of S.*

Theorem 2.4. *Let G be a finite group acting on a compact Riemann surface S of genus $g > 1$ with a triangular signature $(0; m_1, m_2, m_3)$, corresponding to a presentation of the form $G = \langle\, a, b, c \mid a^{m_1} = b^{m_2} = c^{m_3} = abc = \cdots = 1 \,\rangle$. Then G is the full automorphism group of S unless at least one of the following conditions is satisfied (up to permutation of the periods m_1, m_2, m_3), in which case $G \neq \mathrm{Aut}(S)$:*

(i) *G acts with signature $(0; t, t, t)$ where $t \geq 4$, and the assignment $a \mapsto b$, $b \mapsto c$ and $c \mapsto a$ induces an automorphism of G;*

(ii) *G acts with signature $(0; t, t, u)$ where $t \geq 3$ and $t + u \geq 7$, and the assignment $a \mapsto b$, $b \mapsto a$ and $c \mapsto bcb^{-1}$ induces an automorphism of G;*

(iii) *G acts with signature $(0; 2, 7, 7)$, the conjugates of $bc^{-1}bac^3$ generate a normal subgroup K of index 56 in G, and G is extendable to a group G' containing G as a subgroup of index 9 such that G' is generated by a and an element α which normalises K and satisfies $\alpha^3 = 1$, $(a\alpha)^7 = 1$, $b = (\alpha a\alpha)^{-1}a\alpha(\alpha a\alpha)$ and $c = (\alpha a\alpha^{-1})a\alpha(\alpha a\alpha^{-1})$;*

(iv) *G acts with signature $(0;3,3,7)$, the conjugates of bac^2 generate a normal subgroup K of index 21 in G, and G is extendable to a group G' containing G as a subgroup of index 8 such that G' is generated by c and an element α which normalises K and satisfies $\alpha^3=1$, $(\alpha c)^2=1$, $a=\alpha c^{-2}\alpha c^2\alpha^{-1}$ and $b=\alpha^{-1}c^2\alpha c^{-2}\alpha$;*

(v) *G acts with signature $(0;3,8,8)$, conjugates of b^2ac^2 and $c^{-1}ba^{-1}b^{-1}ab^{-1}$ generate a normal subgroup K of index 72 in G, and G is extendable to a group G' containing G as a subgroup of index 10 such that G' is generated by c and an element α which normalises K and satisfies $\alpha^3=1$, $(\alpha c)^2=1$, $a=\alpha c^{-2}\alpha c^2\alpha^{-1}$ and $b=\alpha^{-1}c^2\alpha^{-1}c\alpha c^{-2}\alpha$;*

(vi) *G acts with signature $(0;4,4,5)$, the conjugates of $a^{-1}b^{-1}c^2$ generate a normal subgroup K of index 20 in G, and G is extendable to a group G' containing G as a subgroup of index 6 such that G' is generated by c and an element α which normalises K and satisfies $\alpha^4=1$, $(\alpha c)^2=1$, $a=\alpha^2c\alpha c^{-1}\alpha^2$ and $b=\alpha^{-1}c\alpha c^{-1}\alpha$;*

(vii) *G acts with signature $(0;3,n,3n)$ where $n\geq 3$, the conjugates of b generate a normal subgroup K of index 3 in G, and G is extendable to a group G' containing G as a subgroup of index 4 such that G' is generated by c and an element α which normalises K and satisfies $\alpha^2=1$, $(c\alpha)^3=1$, $a=\alpha c(c\alpha)^{-1}c^{-1}\alpha$ and $b=\alpha c^3\alpha$;*

(viii) *G acts with signature $(0;2,n,2n)$ where $n\geq 4$, the conjugates of b generate a normal subgroup K of index 2 in G, and G is extendable to a group G' containing G as a subgroup of index 3 such that G' is generated by c and an element α which normalises K and satisfies $\alpha^3=1$, $(\alpha c)^2=1$, $a=\alpha(\alpha c)\alpha^{-1}$ and $b=\alpha^{-1}c^2\alpha$.*

As an example of application of the above theorems, we have the following, which deals with non-cyclic abelian actions.

Theorem 2.5. *Let G be a non-cyclic finite abelian group acting on a compact Riemann surface S of genus $g>1$ with non-maximal Fuchsian signature σ. Then the following hold:*

(a) *The signature σ cannot be one of $(1;t)$, $(0;2,7,7)$, $(0;3,3,7)$, $(0;3,8,8)$, $(0;4,4,5)$, $(0;3,n,3n)$ or $(0;2,n,2n)$;*

(b) *If $\sigma=(2;—)$ or $(1;t,t)$ then $G\neq \mathrm{Aut}(S)$;*

(c) *If $\sigma=(0;t,t,u,u)$ then G is a factor group of $C_t\oplus C_u\oplus C_m$, where $m=\gcd(t,u)$, and further, if the action of G corresponds to a (partial) monodromy presentation in terms of commuting generators a,b,c,d subject to relations $a^t=b^t=c^u=d^u=1$, then if the assignment $a\mapsto b\mapsto a$, $c\mapsto d\mapsto c$ is an automorphism of G then $G\neq \mathrm{Aut}(S)$;*

(d) *If $\sigma=(0;t,t,t)$ then $G\cong C_t\oplus C_k$ with monodromy presentation of the form $G=\langle\, a,b \mid a^t=b^t=(ab)^t=[a,b]=1,\ b^k=a^{ks}\,\rangle$ for some k dividing t and some s coprime to t/k, and in this case $G\neq \mathrm{Aut}(S)$ if and only if $s^2\equiv 1 \pmod{t/k}$ or $s^2+s+1\equiv 0 \pmod{t/k}$;*

(e) *If $\sigma=(0;t,t,u)$ with $t\neq u$ then u divides t, and $G\cong C_t\oplus C_k$ with monodromy presentation $G=\langle\, a,b\mid a^t=b^t=(ab)^u=[a,b]=1,\ b^k=a^{ks}\,\rangle$ for some k dividing u and some s coprime to t/k, and in this case $G\neq \mathrm{Aut}(S)$ if and only if $s^2\equiv 1 \pmod{t/k}$.*

2.2 Extendability of group actions on Klein surfaces

The next step in the problem of extendability of group actions is to consider other types of surfaces. In this setting, the analysis of NEC group inclusions carried out by Marston in 2013 was the seminal work to study this problem on Klein surfaces.

Recall from Theorems 2.1 and 2.2 that, in the case of Riemann surfaces, the extendability of a cyclic action depends on the signature and on the surface-kernel epimorphism. In the case of cyclic group actions on non-orientable unbordered surfaces, the situation is surprisingly quite different, as the next theorem, proved in [8], shows.

Theorem 2.6. *A cyclic group acting with non-maximal signature on a non-orientable surface always extends to the action of a larger group.*

We also describe some applications concerning non-orientable surfaces whose full automorphism group is cyclic of largest possible order. For instance, we show that this order is $g+1$, g or $g-1$ depending on whether $g \equiv 1 \pmod 4$, g is even or $g \equiv 3 \pmod 4$ respectively, where g is the algebraic genus of the surface. As a further application, we determine for each n the smallest algebraic genus of a non-orientable surface on which the cyclic group of order n acts as the full automorphism group of the surface We call this the *full cross-cap genus* of the cyclic group, following the definition by May [38] of the symmetric cross-cap number as the smallest topological genus of a non-orientable surface on which the cyclic group acts effectively. We also show that for each integer $n > 1$, there exists some g_0 such that C_n is the full automorphism group of some non-orientable surface of algebraic genus g for every $g \geq g_0$. Finally, we give a table showing the spectrum of all algebraic genera of the non-orientable Riemann surfaces on which C_n acts as the full automorphism group, for $2 \leq n \leq 10$.

An analogous study for bordered surfaces was carried out in 2015, obtaining the following result in [9].

Theorem 2.7. *A cyclic group acting with non-maximal signature on a bordered surface always extends to the action of a larger group.*

Some applications are also obtained. We calculate, for a given $g > 1$ the order of the largest cyclic group that acts as the full automorphism group of a bordered surface of algebraic genus g. The topological type of the surfaces attaining this bound is also determined. This bound turns out to be the same as for unbordered non-orientable surfaces, and we show that, in some cases, surfaces of both types attaining the bound come together as the Klein surfaces associated to two symmetries on the same Riemann surface. We also determine for each n the smallest algebraic genus of a bordered surface on which the cyclic group of order n acts as the full automorphism group of the surface. We call this the *full real genus* of the cyclic group, following the definition by May [37] of the real genus as the smallest algebraic genus of a bordered surface on which the cyclic group acts effectively. We deal separately with orientable and non-orientable surfaces, which leads to the concepts of full real orientable and full real non-orientable genus of a group.

Let us mention that the concept analogous to the real genus for Riemann surfaces is called the symmetric genus of a group G (if we allow only orientation preserving automorphisms then we are led to the concept of strong symmetric genus). These are classic topics in Riemann and Klein surface theories which have catalysed a large amount of research. Let us mention, for Riemann surfaces, the seminal paper [31] by Harvey on cyclic groups, and [27, 32, 40], among many others. For the real genus, see [26, 39], for instance.

3 Automorphism groups of Klein surfaces

It was Klein the first who realized that a bordered surface may be seen as the orbit space of a Riemann surface under the action of a symmetry with fixed points. This initiated the study of groups of automorphisms of bordered Klein surfaces at the end of the 19th century. Interest in this topic was resumed seven decades later when the work of Macbeath opened the door to the combinatorial study of groups of automorphisms. Many results have been obtained since then.

Lists of all finite groups of automorphisms of bordered surfaces are known for algebraic genus 2 and 3; see [20, 22]. The amount of combinatorial data involved increases rapidly with the genus, and this makes it very difficult to deal with higher genus. The main goal of the paper [13] is to determine, up to topological equivalence, all the finite group actions on compact (orientable or non-orientable) bordered surfaces of algebraic genus p for $2 \leq p \leq 6$. Computational methods using the MAGMA program, in which Marston is an expert, have been used for some of the results.

The lists of groups given in [20, 22] are lists of non-isomorphic abstract groups. In [13] we are interested in lists of inequivalent topological group actions. This is a finer classification, as the same abstract group may act on the same surface in different topological ways.

Instead of looking for all groups acting on low genus, we may look for *large* group actions. The largest order of a group acting on bordered surfaces of algebraic genus g is $12(g-1)$. Groups attaining this bound are known as M*-groups. They play the same role on bordered surfaces as Hurwitz groups play on Riemann surfaces. In [7] we determine, up to topological equivalence, all the finite group actions of order at least $6(g-1)$ on compact bordered surfaces of algebraic genus g for $2 \leq g \leq 101$. The topological type of the surfaces where these actions occur are also given.

Let S be a compact Riemann surface of genus $g > 1$ and let $\tau\colon S \to S$ be a symmetry. The topological type of the conjugacy classes of all symmetries of S constitute what is known as the symmetry type of S. The surface S is said to have maximal real symmetry if it admits a symmetry τ such that the compact Klein surface $S/\langle\tau\rangle$ has maximal symmetry, which means that $S/\langle\tau\rangle$ has the largest possible number of automorphisms with respect to its genus. If τ has fixed points then this maximum number is $12(g-1)$. In the first part of [10] we develop a computational procedure to calculate the symmetry type of every Riemann surface of genus g with maximal real symmetry, for given small values of $g > 1$. We use this to find all of them for $1 < g \leq 101$, and give details for $1 < g \leq 25$ (in an appendix). In the second part, we determine the symmetry types of four infinite families of Riemann surfaces with maximal real symmetry.

4 Double covers of Klein surfaces

Let X be a compact Klein surface of algebraic genus $g > 1$ and let (S, τ) be its Riemann double cover. As explained in the preliminaries, this means that S is a compact Riemann surface of genus g which admits an anticonformal involution $\tau\colon S \to S$ such that the orbit space $S/\langle\tau\rangle$ is a compact Klein surface isomorphic to X. It is well known that the full group $\mathrm{Aut}(X)$ of automorphisms of X is isomorphic to the group of all conformal automorphisms of S that commute with τ. It follows that the full group $\mathrm{Aut}(S)$ of all automorphisms of S (conformal or anticonformal) contains the direct product $\mathrm{Aut}(X) \times C_2$ where C_2 is the cyclic group generated by τ. The following question arises naturally: is $\mathrm{Aut}(S)$ equal to

Aut$(X) \times C_2$, or does S admit additional automorphisms? This problem was firstly studied by May in 1991. We proposed to Marston to work on this topic during our visit to Auckland in 2016. The results were published in [11].

This question has been considered for bordered Klein surfaces with the largest possible number of automorphisms, namely $12(g-1)$, by May in [36], and for those with the second largest possible number of automorphisms, namely $8(g-1)$, by Bujalance, Costa, Gromadzki and Singerman in [19]. In both cases the authors showed that the equality Aut$(S) = \mathrm{Aut}(X) \times C_2$ holds for almost all surfaces, with one single exception when $|\mathrm{Aut}(X)| = 12(g-1)$ and five exceptions when $|\mathrm{Aut}(X)| = 8(g-1)$. Subsequently, Costa and Porto showed in [41] that equality also holds except in a finite number of cases when $|\mathrm{Aut}(X)| > 6(g-1)$, while if $|\mathrm{Aut}(X)| = 6(g-1)$ then there are infinitely many Klein surfaces X (with different topological types) such that Aut$(S) \neq \mathrm{Aut}(X) \times C_2$.

If $|\mathrm{Aut}(X)| > 6(g-1)$, then it follows from the Riemann-Hurwitz formula that Aut(X) is uniformised by an NEC group with quadrangular signature $(0; +; [-]; \{(2,2,2,n)\})$ with $n = 3$, 4 or 5. Using techniques of hyperbolic geometry, Costa and Porto investigated the case of signature $(0; +; [-]; \{(2,2,2,n)\})$ where n is an odd prime, and showed the equality Aut$(S) = \mathrm{Aut}(X) \times C_2$ holds with one single exception (for each odd prime n).

In [11] we consider the general case where a group G of automorphisms of X is uniformised by some NEC group with quadrangular signature $(0; +; [-]; \{(2,2,2,n)\})$ for an arbitrary value of $n > 2$. We investigate the relationship between G and the full automorphism groups Aut(X) and Aut(S). Furthermore, we consider actions not only on bordered Klein surfaces, but also on unbordered non-orientable surfaces.

5 Automorphism groups of pseudo-real surfaces

Marston co-organized the workshop "Symmetries of Surfaces, Maps and Dessins", held in the Banff International Research Station from September 24th to September 29th, 2017. It was there that the idea arose to study with Marston upper bounds for the order of the automorphism groups of pseudo-real surfaces.

A compact Riemann surface is called pseudo-real if it admits anticonformal automorphisms, but none of them has order two. Another term used for such surfaces is asymmetric. Their importance stems from the fact that in the moduli space of compact Riemann surfaces of given genus, pseudo-real surfaces represent the points that have real moduli but are not definable over the reals.

There are no pseudo-real surfaces of genus 0 or 1, since in those cases every reflexible surface admits an anticonformal automorphism of order two. On the other hand, it was shown in [15] that there exists at least one pseudo-real surface of genus g for every $g > 1$. The pseudo-real surfaces of genus 2 to 4 were classified in [15, 18], and this was extended for genus 5 to 10 in [3], except that five of the entries in the tables in [3] should be deleted.

Upper bounds on the order of a group of automorphisms of a pseudo-real surface of given genus $g > 1$ are studied in [12]. This is motivated by a number of theorems about the orders of groups of automorphisms of other kinds of surfaces. The most famous of these are the theorems of Hurwitz (1893) and Wiman (1896) that give an upper bound of $84(g-1)$ on the number of orientation-preserving automorphisms of a compact Riemann surface of genus $g > 1$, and an upper bound of $4g + 2$ on the order of any such automorphism, respectively. Other such theorems deal with special cases where the group is abelian, or the surface is non-orientable, or the automorphism reverses orientation; see [5, 25, 29, 30, 34].

For every integer $g > 1$, let $M(g)$ be the order of the largest group of automorphisms of a pseudo-real surface of genus g. It was shown in [15] that $M(g) \leq 12(g-1)$. This upper bound is attained for infinitely many g, but certainly not for all $g > 1$, and so it makes sense to look for more refined bounds on the group order, in general and special cases. One particular question of interest is to find a sharp lower bound for $M(g)$, akin to the Accola-Maclachlan bound for general compact Riemann surfaces (see [1, 35]). In [12] we show that $M(g) \geq 2g$ for every even $g \geq 2$, while $M(g) \geq 4(g-1)$ for every odd $g \geq 3$, and we prove that the latter bound is sharp for a very large and possibly infinite set of odd values of $g \geq 3$. Unfortunately we are not yet able to determine the level of sharpness of the former bound (for even g), so this is left as an open question. We also give the precise values of $M(g)$ for all g between 2 and 128, together with the signatures for the actions of the corresponding groups of largest order.

When we restrict to the cases where the group is cyclic or abelian we do find sharp bounds for every genus $g > 1$. Let $M_{\mathrm{ab}}(g)$ and $M_{\mathrm{cyc}}(g)$ be respectively the orders of the largest abelian and largest cyclic group of automorphisms of a pseudo-real surface of genus g, such that the group contains orientation-reversing elements. In [12] we show that $M_{\mathrm{ab}}(g) = M_{\mathrm{cyc}}(g) = 2g$ for every even $g \geq 2$, and that $M_{\mathrm{cyc}}(g) = 2g - 2$ for every odd $g \geq 3$. On the other hand, for odd $g > 1$ we find that $M_{\mathrm{ab}}(g) = 2g + 6$ when $g \equiv 1 \bmod 4$, while $M_{\mathrm{ab}}(g) = 2g + 2$ when $g \equiv 3 \bmod 4$. In all these cases we give specific details about the surfaces and groups attaining the bound. It is worth mentioning that we also find similar bounds when the groups (cyclic or abelian) contain no orientation-reversing elements.

6 A final note

The first author of this survey also has other papers with Marston on related topics of automorphisms of Riemann or Klein surfaces, see [15, 16, 17] for instance.

Finally, we want to thank Marston for working with us over the years, as we have learned a lot from him. We are really impressed by his speed in solving the problems we pose to him. Many times we have asked him about a problem in group theory we had been thinking about for several days, and the next morning (Spanish time) we had his solution in the email. Marston has indeed a great capacity for work and a vast knowledge of group theory and their computational methods. But above all, he has many other human qualities that make it enjoyable to work with him.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.05
https://doi.org/10.26493/2590-9770.1450.f39
(Also available at http://adam-journal.eu)

Some remarks on group actions on hyperbolic 3-manifolds

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Dedicated to Marston Conder on the occasion of his 65th birthday

Received 30 August 2021, accepted 28 March 2022, published online 18 July 2022

Abstract

We prove that there are infinitely many non-commensurable closed orientable hyperbolic 3-manifolds X, with the property that there are finite groups G_1 and G_2 acting freely by orientation-preserving isometries on X with X/G_1 and X/G_2 isometric, but G_1 and G_2 are not conjugate in $\mathrm{Isom}(X)$. We provide examples where G_1 and G_2 are non-isomorphic, and prove analogous results when G_1 and G_2 act with fixed-points.

Keywords: Hyperbolic manifold, finite group action, arithmetic group.

Math. Subj. Class.: 57M60

*Corresponding author. The author was partially supported by the N. S. F. The author is grateful to Emilio Bujalence and Javier Cirre for sending him a detailed list of references in connection with actions of p-groups on Riemann surfaces.

†The author was partially supported by the U.T. Austin - Portugal Mathematics CoLab program (FCT) and the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

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1 Introduction

The study of finite group actions, both free and with fixed points, on closed Riemannian manifolds has a long and rich history. In the context of low-dimensional geometry and topology, two notable examples of this are Kerckhoff's solution to the Nielsen Realization Problem for surfaces [19] and in the setting of geometric 3-manifolds, it is known that any such action is always conjugate to an isometric action (see [7, 12, 14], and [25]) which formed part of Thurston's geometrization program for 3-manifolds and 3-orbifolds. In this paper we will be concerned with (isometric) finite group actions on hyperbolic manifolds and in particular in dimension 3.

More specifically we will be interested in the following situation: *two finite groups G_1 and G_2 acting freely or with fixed points by (orientation-preserving) isometries on a closed orientable hyperbolic 3-manifold X with $X/G_1 \cong X/G_2$* (here and throughout the symbol $\cong$ in the context of manifolds or orbifolds will denote *isometric*). By volume considerations, it is clear that in this setting, G_1 and G_2 must have the same order, and it is also clear that if G_1 and G_2 are conjugate in $\mathrm{Isom}(X)$ then the quotients will be isometric. The aim of this article is to provide constructions, both general and explicit, of examples of closed orientable hyperbolic 3-manifolds X, and groups G_1 and G_2 *that are not conjugate* in $\mathrm{Isom}(X)$ with $X/G_1 \cong X/G_2$. In fact, our methods provide examples of groups that are not even isomorphic.

Our first result deals with free actions.

Theorem 1.1. *There are infinitely many non-commensurable closed orientable hyperbolic 3-manifolds X, with the property that there are finite groups G_1 and G_2 satisfying:*

(1) *G_1 and G_2 act freely on X by orientation-preserving isometries on X with $X/G_1 \cong X/G_2$.*

(2) *$|G_1| = |G_2|$, but G_1 and G_2 are not conjugate in $\mathrm{Isom}(X)$.*

As mentioned above, since $X/G_1 \cong X/G_2$, it is immediate that $|G_1| = |G_2|$. However, by making the construction explicit we can actually exhibit examples of manifolds X as in Theorem 1.1 for which G_1 is an elementary abelian group of order p^3 (for certain primes p), and G_2 is the non-abelian group of order p^3 containing an element of order p^2 (see §4.1).

Examples of closed orientable hyperbolic 3-manifolds X that are fibered over S^1 admitting free actions by finite groups G_1 and G_2 with $X/G_1 \cong X/G_2$ and for which G_1 and G_2 are not conjugate in $\mathrm{Isom}(X)$ are given [21] (although they are unable to determine whether these examples fall into infinitely many commensurability classes). By focusing on very explicit examples, we can also impose topological conditions on the manifold X and the quotients $X/G_1 \cong X/G_2$, namely we prove the following

Corollary 1.2. (1) *There are infinitely many examples of (commensurable) closed orientable hyperbolic 3-manifolds X that fiber over the circle with the property that there are finite groups G_1 and G_2 as in the conclusion of* Theorem 1.1 *such that $X/G_1 \cong X/G_2$ also fiber over the circle.*

(2) *There is a hyperbolic 3-manifold X which is a rational homology 3-sphere with the property that there are finite groups G_1 and G_2 as in the conclusion of* Theorem 1.1 *such that $X/G_1 \cong X/G_2$ is also a rational homology 3-sphere.*

The finite groups G_1 and G_2 in both cases of Corollary 1.2 are of the type described before Corollary 1.2 (i.e. elementary abelian p-groups and non-abelian p-groups of order p^3 for certain primes p). As far as the authors are aware, the example of Corollary 1.2(2) is the first such example of a hyperbolic rational homology 3-sphere as in the conclusion of Corollary 1.2(2).

By way of contrast, results in [28] and [29] consider the question to what extent G_1 and G_2 acting *with fixed points* must be conjugate in $\mathrm{Isom}(X)$ (for certain closed Riemannian 3-manifolds X not necessarily hyperbolic), and for example [28, Theorem 8 and Proposition 13] provides a uniqueness statement in certain settings (e.g. rational homology 3–spheres and most cyclic group actions).

Our methods also provide a construction when the action is not non-free.

Theorem 1.3. *There are infinitely many non-commensurable closed orientable hyperbolic 3-manifolds X, with the property that there are finite groups G_1 and G_2 satisfying:*

(1) *G_1 and G_2 act by orientation-preserving isometries on X, have non-empty fixed-point set, and with $X/G_1 \cong X/G_2$.*

(2) $|G_1| = |G_2|$, *but G_1 and G_2 are not conjugate in* $\mathrm{Isom}(X)$.

The examples constructed in the proofs of Theorems 1.1, 1.3 and Corollary 1.2 come from the class of arithmetic hyperbolic 3-manifolds (see Section 3 and [23] for further details), and exploit the fact that such manifolds have fundamental groups with large commensurator. The advantages of the arithmetic nature of the construction are first, it provides infinitely many commensurability classes of examples, and second, the groups G_1 and G_2 can be made explicit.

As will be clear, the method of proof of Theorem 1.1 (see Section 2) is very general for arithmetic groups (given a description of maximal groups in the commensurability class), and although our main focus is hyperbolic 3-manifolds, we sketch some variations of Theorem 1.1 in other dimensions; for example, we provide examples of Riemann surfaces which admit actions of distinct finite p-groups with conformally equivalent quotients. Although there is a vast literature on (p-)group actions on Riemann surfaces, we were unable to find results which have precise overlap with ours, although questions of a similar nature have been addressed (see [17, 18] and [20] to name a few). However, we do note that the method of proof of [21] also provides examples of Riemann surfaces with finite groups G_1 and G_2 acting freely on X with X/G_1 conformally equivalent to X/G_2.

More care is needed in generalizing the proof of Theorem 1.3, but we expect that this can also be done.

This paper has its origins in a visit of the third author to the second in the Spring of 2011 whilst the second author held a position at U.T. Austin. The first author sadly passed away in November 2012, and the paper remained stubbornly unfinished since that time. Because of Marston's close personal and mathematical connection to Colin Maclachlan through his visits to the U.K., and Colin's to New Zealand, it seemed an appropriate opportunity to finish the paper, and submit as part of the celebration of Marston's 65th birthday. The second author has no doubts that Colin would have been very pleased with this arrangement, and would have very much enjoyed raising a glass of Scotland's national drink in Marston's honor!

As was remarked upon above, results similar to Theorems 1.1, 1.3 have since appeared in the literature (see [21], [22]), but our methods are very different, and so still seem worthy of publication. The reader will likely note the influence of the first author in this work. However, since the first author was neither able to verify nor influence the final version of the paper, any errors in the paper should be attributed to the other two authors.

2 A basic construction

The basic idea of our construction is contained in Proposition 2.1 below. Although our main focus is in dimension 3, we state it for hyperbolic manifolds of arbitrary dimension. In Section 3 we provide a detailed discussion how to construct examples of hyperbolic 3-manifolds satisfying the hypothesis of Proposition 2.1 using arithmetic techniques in dimension 3, and in Section 7 construct examples in dimension 2. In Section 8 we discuss applying Proposition 2.1 to higher dimensional arithmetic hyperbolic manifolds, but we have decided to only offer a sketch of a proof. The complete proof requires a detailed discussion of maximal arithmetic lattices in this setting.

Recall that if $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is a lattice, the commensurator of Γ is the group

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}^+(\mathbb{H}^n) \mid g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}. \tag{2.1}$$

It was proved by Margulis [24], that Γ is an arithmetic lattice if and only if $\mathrm{Comm}(\Gamma)$ is dense in $\mathrm{Isom}^+(\mathbb{H}^n)$. Furthermore, it is known in this case (see [8], [23, Theorem 11.4], and Theorem 3.2 below for dimension 3 and [9] more generally) that there are infinitely many distinct maximal arithmetic lattices commensurable with Γ. On the other hand, if $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is a non-arithmetic lattice, then $[\mathrm{Comm}(\Gamma) : \Gamma] < \infty$. Moreover, in this case, $\mathrm{Comm}(\Gamma)$ is the unique maximal discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ containing Γ.

Notation: For $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$, we denote by Γ^+ the subgroup of index at most 2 obtained as $\Gamma \cap \mathrm{Isom}^+(\mathbb{H}^n)$.

Proposition 2.1. *Let $\Gamma_0 \subset \mathrm{Isom}(\mathbb{H}^n)$ be a maximal arithmetic lattice, and let Γ_1 be a normal torsion-free subgroup of finite index in Γ_0 which is contained in Γ_0^+. Assume that there exists $g \in \mathrm{Comm}(\Gamma_0^+) \setminus \Gamma_0^+$ such that $g\Gamma_1 g^{-1} \subset \Gamma_0^+$.*

Then there exists $\Delta \subset \Gamma_0^+$ for which $X = \mathbb{H}^n/\Delta$ satisfies the conclusion of Theorem 1.1.

Proof. Since $g\Gamma_1 g^{-1} \subset \Gamma_0^+$, we deduce, by volume considerations, that $g\Gamma_1 g^{-1}$ has finite index in Γ_0^+ equal to $[\Gamma_0^+ : \Gamma_1]$. By maximality, the normalizer of Γ_1 in $\mathrm{Isom}(\mathbb{H}^n)$ is Γ_0. Note also that $g \notin \Gamma_0$, otherwise $g \in \Gamma_0 \cap \mathrm{Comm}(\Gamma_0^+) \subset \Gamma_0 \cap \mathrm{Isom}^+(\mathbb{H}^n) = \Gamma_0^+$ contradicting the hypothesis on g. It follows that $g\Gamma_1 g^{-1} \neq \Gamma_1$.

Since the subgroup $\Gamma_1 \cap g\Gamma_1 g^{-1}$ has finite index in Γ_0, it contains a finite index subgroup Δ, normal in Γ_0, namely the core (i.e. the intersection of all conjugates of $\Gamma_1 \cap g\Gamma_1 g^{-1}$ in Γ_0). Then $X = \mathbb{H}^n/\Delta$ is a hyperbolic n-manifold which admits free finite group actions by the groups $G_1 = \Gamma_1/\Delta$ and $G_2 = g\Gamma_1 g^{-1}/\Delta$ with quotients $\mathbb{H}^n/\Gamma_1 \cong M \cong \mathbb{H}^n/g\Gamma_1 g^{-1}$. Clearly $|G_1| = |G_2|$ and G_1 and G_2 act by orientation-preserving isometries. Furthermore, $\mathrm{Isom}(X) = \Gamma_0/\Delta$ since Γ_0 is maximal. Now $G_1 \neq G_2$ and G_1 is a normal subgroup of $\mathrm{Isom}(X)$. It follows that G_1 and G_2 cannot be conjugate in $\mathrm{Isom}(X)$. □

Remark 2.2. A version of this Proposition still holds if we assume that Γ_1 is not torsion-free. In this case, to obtain the manifold X as in the conclusion of Theorem 1.3 we must further require that Δ as in the proof, is torsion-free. However, standard methods makes this easy to arrange.

Remark 2.3. In the non-arithmetic setting, the discussion of the commensurator above shows that there is a unique maximal lattice in the commensurability class; i.e. Γ_0 in this case. Hence, if in Proposition 2.1 we were to assume that Γ_0 is non-arithmetic, then there could be no element g as stated. However, a variation of Proposition 2.1 can be formulated:

Let $\Gamma_0 \subset \mathrm{Isom}(\mathbb{H}^n)$ be a maximal non-arithmetic lattice, and $\Gamma < \Gamma_0^+$ a proper torsion-free subgroup of finite index containing a normal subgroup Δ of finite index that is not normal in Γ_0. Suppose that there exists a subgroup Γ_1 with $\Delta \subset \Gamma_1 \subset \Gamma$ and $g \in \Gamma_0$ such that $\Delta \subset g\Gamma_1 g^{-1} \subset \Gamma$ with $G_1 = \Gamma_1/\Delta$ and $G_2 = g\Gamma_1 g^{-1}/\Delta$ not isomorphic. Then the conclusion of Theorem 1.1 holds.

This construction is very much in the spirit of that given in [21]. Indeed, it is likely that "many" of the examples in [21] are non-arithmetic, so the above statement would cover the construction of their examples.

Remark 2.4. We could have defined the commensurator of Γ in $\mathrm{Isom}(\mathbb{H}^n)$. The group $\mathrm{Comm}(\Gamma)$ defined at (2.1) is a subgroup of index at most 2 in this larger group. However, we did not want to constantly keep distinguishing the "orientation-preserving" commensurator and so we retain $\mathrm{Comm}(\Gamma)$ to be as defined at (2.1).

3 Preliminaries on arithmetic hyperbolic 3-manifolds

As mentioned in Section 1, our examples are built using arithmetic hyperbolic manifolds. Here we focus on the case of arithmetic Kleinian groups and arithmetic hyperbolic 3-orbifolds, and recall some relevant results and facts that will be needed (see [23] for further details).

3.1 Arithmetic Kleinian groups

Arithmetic Kleinian groups are obtained as follows. Let k be a number field having exactly one complex place, R_k its ring of integers and B a quaternion algebra over k which ramifies at all real places of k. Let $\rho\colon\ B \to M(2,\mathbb{C})$ be an embedding, $\mathcal{O}$ an order of B, and $\mathcal{O}^1$ the elements of norm one in $\mathcal{O}$. Then $\mathrm{P}\rho(\mathcal{O}^1) \subset \mathrm{PSL}(2,\mathbb{C})$ is a finite co-volume Kleinian group, which is co-compact if and only if B is a division algebra, which in turn is equivalent to B not being isomorphic to $M(2,\mathbb{Q}(\sqrt{-d}))$, where d is a square-free positive integer. Following [8], we denote the group $\mathrm{P}\rho(\mathcal{O}^1)$ by $\Gamma^1_{\mathcal{O}}$.

An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ commensurable with a group $\Gamma^1_{\mathcal{O}}$. In addition, we call $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ arithmetic if it is commensurable with an arithmetic Kleinian group. We call $Q = \mathbb{H}^3/\Gamma$ *arithmetic* if Γ is arithmetic.

The wide (i.e. up to conjugacy) commensurability class of an arithmetic Kleinian group is determined by the isomorphism class of B (see e.g. [23, Theorem 8.4.1]). We can refine this further by noting that if $\mathrm{Ram}_f(B)$ denotes the finite set of prime ideals $\mathcal{P}$ of k where B is ramified, i.e. $B_{\mathcal{P}} = B \otimes_k k_{\mathcal{P}}$ is a division algebra, then the isomorphism class of B (as in the definition of an arithmetic Kleinian group given above) is determined by $\mathrm{Ram}_f(B)$. In particular, using the previous remark, to construct infinitely many commensurability classes of arithmetic hyperbolic 3-manifolds, it is sufficient to fix the field k and vary $\mathrm{Ram}_f(B)$.

Our arguments crucially depend on a fine understanding of maximal arithmetic Kleinian groups defined using maximal and Eichler orders (intersections of maximal orders) in the quaternion algebra B (see [8] or [23, Chapter 11] for more details). We will discuss this further below, however, for convenience, we first provide a "warm-up" version of the general construction that may be useful as a template for the reader to bear in mind. This will construct finite volume non-compact examples that satisfy the conclusion of Theorem 1.1.

3.2 Warm-up construction

The group $\Gamma = \mathrm{PGL}(2, \mathbb{Z}[i])$ is a maximal arithmetic Kleinian group, although it is not maximal in $\mathrm{Isom}(\mathbb{H}^3)$. The maximal group is obtained as the group generated by $< \tau, \mathrm{PGL}(2, \mathbb{Z}[i]) >$ where τ is the reflection on $\mathbb{H}^3$ obtained by extension of complex conjugation on $\mathbb{C}$. We will let this group be denoted by Γ_0. Let $A \subset \mathbb{Z}[i]$ be an ideal, and let

$$\Gamma(A) = \ker\{\phi_A \colon \mathrm{PGL}(2, \mathbb{Z}[i]) \to \mathrm{PGL}(2, \mathbb{Z}[i]/A)\},$$

which will be a subgroup of the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ for most ideals $A \subset \mathbb{Z}[i]$, e.g. those of odd norm.

We will focus on ideals A of the form $p\mathbb{Z}[i]$ where $p \in \mathbb{Z}$ is a prime congruent to $3 \mod 4$, and so the ideal $p\mathbb{Z}[i]$ remains prime in $\mathbb{Z}[i]$. It is easy to check that $\Gamma(p) = \Gamma(p\mathbb{Z}[i])$ is torsion-free. Note that since complex conjugation preserves the ideal $p\mathbb{Z}[i]$, $\Gamma(p)$ is also a normal subgroup of Γ_0.

Now it is a simple matter to check that the element $\sigma_p = \mathrm{P}\begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$ normalizes the subgroup $\Gamma_0(p) = \phi_p^{-1}(B_p)$ where B_p is the group of upper triangular matrices in $\mathrm{PSL}(2, \mathbb{Z}[i]/p\mathbb{Z}[i])$. In particular, $\sigma_p \in \mathrm{Comm}(\mathrm{PGL}(2, \mathbb{Z}[\mathrm{i}]))$. Moreover, and most importantly in our situation, observe that

$$\sigma_p \Gamma(p) \sigma_p^{-1} = \mathrm{P}\{\begin{pmatrix} 1 + ap & b \\ cp^2 & 1 + dp \end{pmatrix} : a, b, c, d \in \mathbb{Z}[i], \text{ and determinant } 1\}.$$

With this in hand, we can now apply Proposition 2.1 to the groups Γ_0 and $\Gamma_1 = \Gamma(p)$, with $g = \sigma_p$.

From above, the group $g\Gamma_1 g^{-1}$ contains the group $\Delta = \Gamma(p^2)$ which is also normal in Γ_0. Now take $X = \mathbb{H}^3/\Delta$, and the groups G_1 and G_2 to be given by Γ_1/Δ and $g\Gamma_1 g^{-1}/\Delta$ respectively. This finishes the construction of the manifold X as in Theorem 1.1 in this setting.

To get a version of Theorem 1.3, we continue to use the group Γ_0 as above, and tweak the above construction as follows. In this case we choose the ideal $< 1 + i >$ and construct the group $\Gamma(1 + i)$. Again, since complex conjugation preserves the ideal $< 1 + i >$, $\Gamma(1 + i)$ is a normal subgroup of Γ_0. However, $\Gamma(1 + i)$ is not torsion-free since the element $\mathrm{P}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \Gamma(1 + i)$. Thus we cannot take this group to be Γ_1. However, setting $g = \mathrm{P}\begin{pmatrix} 0 & -1/\sqrt{1+i} \\ \sqrt{1+i} & 0 \end{pmatrix}$, the group Γ_1 can be constructed by passing to a torsion-free subgroup of $\Gamma(1 + i) \cap g\Gamma(1 + i)g^{-1}$, and taking its core in Γ_0. This is the group defining the manifold X in Theorem 1.3. We refer the reader to the end of the proof of Theorem 1.3 in Subsection 6.2 for details of why the groups G_1 and G_2 are not conjugate in this case.

The discussion that follows in Subsections 3.3 and 3.4 provides the necessary generalization of the framework described above that will allow us to pass to the closed case, and thereby prove Theorem 1.1.

Remark 3.1. In the argument above in the case of $\Gamma(1+i)$ and

$$g = \mathrm{P}\begin{pmatrix} 0 & -1/\sqrt{1+i} \\ \sqrt{1+i} & 0 \end{pmatrix},$$

then as in the first case of $\Gamma(p)$, the intersection $\Gamma(1+i) \cap g\Gamma(1+i)g^{-1}$ contains the principal congruence subgroup $\Gamma((1+i)^2) = \Gamma(2)$ which is a normal torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{Z}[i])$ (being the fundamental group of a link complement in S^3 [5]), so in fact, in this case one can take $\Delta = \Gamma(2)$ and so there is no need to pass to any further subgroup of finite index.

3.3 Orders and maximal groups

The argument of Subsection 3.2 provides the template, and so to arrange that the manifolds are closed we use the discussions in Subsection 3.1 to replace $M(2,\mathbb{Q}(i))$ with quaternion division algebras over a number field with one complex place, generalize the groups $\Gamma(p)$ and $\Gamma(1+i)$ using the principal congruence subgroups described in Subsection 3.4 below, and generalize the group $\Gamma_0(p)$ and element σ_p using the theory of Eichler orders and their normalizers as we now describe.

Thus let $\mathcal{O} \subset B$ be a maximal order and $\mathcal{E} \subset \mathcal{O}$ an Eichler order, i.e. the intersection $\mathcal{O} \cap \mathcal{O}'$, for some maximal order $\mathcal{O}' \neq \mathcal{O}$. The Eichler order $\mathcal{E}$ is said to be of *square-free level* S, where S is a finite set of prime ideals of k, disjoint from $\mathrm{Ram}_f(B)$, if, locally at each finite place, $\mathcal{E}_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}} = \mathcal{O}'_{\mathcal{P}}$ if $\mathcal{P} \notin S$ and, if $\mathcal{P} \in S$, then $\mathcal{E}_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}} \cap \mathcal{O}'_{\mathcal{P}}$ has level $\mathcal{P}$ so that $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{O}'_{\mathcal{P}}$ are adjacent maximal orders in the tree of maximal orders in $B_{\mathcal{P}} \cong M_2(k_{\mathcal{P}})$ (see [23, Chapter 6.5]). When $S = \emptyset$, we simply get the maximal order $\mathcal{O}$.

Indeed, it can always be arranged that for square-free level S and $\mathcal{P} \in S$ we have $\mathcal{E}_{\mathcal{P}} = \mathcal{T}_{\mathcal{P}}$ where

$$\mathcal{T}_{\mathcal{P}} = \left\{ \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix} \mid a,b,c,d \in R_{\mathcal{P}} \right\}, \tag{3.1}$$

and $R_{\mathcal{P}} \subset k_{\mathcal{P}}$ is the valuation ring with uniformizer π (see [23, Chapter 6.5]).

Let $N(\mathcal{E})$ and $N(\mathcal{O})$ denote the normalizers of $\mathcal{E}$ and $\mathcal{O}$ respectively in B^*. Their images, $\mathrm{P}\rho(N(\mathcal{E}))$ and $\mathrm{P}\rho(N(\mathcal{O}))$ in $\mathrm{PGL}(2,\mathbb{C}) \cong \mathrm{PSL}(2,\mathbb{C})$, denoted by $\Gamma_{\mathcal{E}}$ and $\Gamma_{\mathcal{O}}$ respectively, are arithmetic Kleinian groups and any arithmetic Kleinian group is conjugate to a subgroup of some such $\Gamma_{\mathcal{E}}$ or $\Gamma_{\mathcal{O}}$ (see [8] and [23, Chapter 11.4]).

Note that since conjugation preserves the norm, the groups $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{E}}$ normalize the groups $\Gamma^1_{\mathcal{O}}$ and $\Gamma^1_{\mathcal{E}}$ (the image in $\mathrm{PSL}(2,\mathbb{C})$ of $\mathcal{E}^1$ the elements of norm one in $\mathcal{E}$). For convenience we state the following result of Borel [8](see also [23, Chapter 11.4]).

Theorem 3.2. *Fix a maximal order $\mathcal{O} \subset B$. Then there exist infinitely many distinct sets of prime ideals $S_i \subset R_k$ and Eichler orders $\mathcal{E}_i \subset \mathcal{O}$ of level S_i such that $\Gamma_{\mathcal{E}_i}$ are distinct maximal arithmetic Kleinian groups.*

Remark 3.3. We single out two cases of Theorem 3.2 that we will make use of: if k is quadratic imaginary or cubic with one complex place and has class number 1, $\mathcal{P}$ a k-prime ideal and $S = \{\mathcal{P}\}$ then the group $\Gamma_{\mathcal{E}}$ can be proved to be maximal (see [8] and [23, Chapters 6.7 and 11.4]).

3.4 Principal congruence subgroups

For $\mathcal{O}$ a maximal order, we now describe a distinguished class of subgroups of $\Gamma^1_{\mathcal{O}}$, known as *principal congruence subgroups*. To that end, let $I \subset B$ be a 2-sided integral ideal (see [23, Chapter 6] for further details). Then $I \subset \mathcal{O}$ and $\mathcal{O}/I$ is a finite ring. Define:

$$\mathcal{O}^1(I) = \{\alpha \in \mathcal{O}^1 \mid \alpha - 1 \in I\}.$$

The corresponding *principal congruence subgroup* of $\Gamma^1_{\mathcal{O}}$ is $\Gamma(\mathcal{O}(I)) = \mathrm{P}\rho(\mathcal{O}^1(I))$.

Since I is a 2-sided ideal, it follows that $\Gamma(\mathcal{O}(I))$ is a normal subgroup of finite index in $\Gamma^1_{\mathcal{O}}$. Indeed, we can say more.

Lemma 3.4. *The principal congruence subgroups $\Gamma(\mathcal{O}(I))$ are normal subgroups of $\Gamma_{\mathcal{O}}$. Thus the normalizer of $\Gamma(\mathcal{O}(I))$ in $\mathrm{PSL}(2,\mathbb{C})$ is $\Gamma_{\mathcal{O}}$.*

Proof. The second statement follows from the first since, as noted above, $\Gamma_{\mathcal{O}}$ is maximal. Now let $\alpha \in \mathcal{O}^1(I)$ and $x \in N(\mathcal{O})$. Then $\alpha \in 1 + I$ and $x(1+I)x^{-1} = 1 + xIx^{-1}$. Now I and $x\mathcal{O}$ are elements of the set of two-sided integral ideals, which is an abelian group (see [23, Chapter 6.7]). This, together with integrality of I gives:

$$xIx^{-1} = (x\mathcal{O})I(\mathcal{O}x^{-1}) = (x\mathcal{O})I(x\mathcal{O})^{-1} = I,$$

and we deduce that $x\alpha x^{-1} \in \mathcal{O}^1(I)$. □

In addition, for most ideals I, the groups $\Gamma(\mathcal{O}(I))$ are torsion-free. We record the following for convenience which needs some additional notation (see [23, Chapters 6.5, 6.6] for details). Apart from a finite set of k-primes $T(I)$, $I_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}$. For $\mathcal{P} \in T(I)$, we have $I_{\mathcal{P}} = \pi^{n_{\mathcal{P}}}\mathcal{O}_{\mathcal{P}}$ where π is a uniformizer for $k_{\mathcal{P}}$.

Proposition 3.5. *In the notation above, suppose for $\mathcal{P} \in T(I)$ the following holds:*

(1) $\mathcal{P} \notin \mathrm{Ram}_f(B)$;

(2) *$\mathcal{P}$ does not ramify in $k \mid \mathbb{Q}$;*

(3) *$\mathcal{P}$ is not dyadic (i.e. $\mathcal{P}$ does not divide 2).*

Then $\Gamma(\mathcal{O}(I))$ is torsion-free.

A group $\Gamma \subset \Gamma^1_{\mathcal{O}}$ is called *a congruence subgroup* if there is some ideal I such that $\Gamma(\mathcal{O}(I)) \subset \Gamma$.

As an example of a congruence subgroup that we will exploit (and the analogue of $\Gamma_0(p)$ in Subsection 3.2), it is shown in [1] that if $\mathcal{E} \subset \mathcal{O}$ is an Eichler order of square-free level S, then $\Gamma^1_{\mathcal{E}} \subset \Gamma^1_{\mathcal{O}}$ is a congruence subgroup. Using (3.1), we can be more explicit. If $\Gamma_{\mathcal{E}}$ is of square-free level S, then for each $\mathcal{P} \notin S$, we have $\mathcal{E}_p = \mathcal{O}_{\mathcal{P}} = M_2(R_{\mathcal{P}})$ and for $\mathcal{P} \in S$ we have $\mathcal{E}_{\mathcal{P}} = \mathcal{T}_{\mathcal{P}}$ where $\mathcal{T}_{\mathcal{P}}$ is as defined at (3.1).

Then in the notation above, defining I locally by $I_{\mathcal{P}} = \pi\mathcal{O}_{\mathcal{P}}$ for each $\mathcal{P} \in S$ and $I_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}$ otherwise, it follows that, $T(I) = S$. Hence, for $\mathcal{P} \in S$ and $\alpha \in \mathrm{SL}(2, R_{\mathcal{P}})$ is such that $\alpha - 1 \in I_{\mathcal{P}}$ then $\alpha \in \mathcal{T}^1_{\mathcal{P}}$. It now follows that $\Gamma(\mathcal{O}(I)) \subset \Gamma^1_{\mathcal{E}}$.

We will next exploit the feature of having a "large" commensurator which requires an additional piece of terminology.

For a maximal order $\mathcal{O}$ in B, choose an Eichler order $\mathcal{E}$ of square-free level S such that $\Gamma_{\mathcal{E}}$ is maximal (using Theorem 3.2). By maximality, we can find elements $g \in \Gamma_{\mathcal{E}}$ such that $g \notin \Gamma_{\mathcal{O}}$. Call such an element *admissible* and note that $g \in \mathrm{Comm}(\Gamma_{\mathcal{O}})$.

Lemma 3.6. *Suppose that* $\Gamma^1_{\mathcal{E}}$ *contains a principal congruence subgroup* $\Gamma(\mathcal{O}(I))$ *and that* g *is an admissible element of* $\Gamma_{\mathcal{E}}$*. Then* $g\Gamma(\mathcal{O}(I))g^{-1}$ *is commensurable with, but distinct from,* $\Gamma(\mathcal{O}(I))$.

Proof. Commensurability is straightforward since $g \in \Gamma_{\mathcal{E}}$. If $g\Gamma(\mathcal{O}(I))g^{-1} = \Gamma(\mathcal{O}(I))$, then $g \in \Gamma_{\mathcal{O}}$ by Lemma 3.4. But this contradicts the admissibility of g. □

4 Proof of Theorem 1.1

We now focus on constructing explicit examples in dimension 3 using the material from Section 3.

4.1 General construction

Let k be a number field with exactly one complex place. It will be convenient in the construction that follows to show that, for certain k, maximal arithmetic subgroups Γ of $\mathrm{Isom}(\mathbb{H}^3)$ are Kleinian groups, i.e. $\Gamma = \Gamma^+$.

Lemma 4.1. *If* $[k : \mathbb{Q}]$ *is odd or* $[k : \mathbb{Q}]$ *is even but* k *has no subfield of index* 2*, then any maximal arithmetic Kleinian group defined over* k *will be maximal in* $\mathrm{Isom}(\mathbb{H}^3)$.

Proof. Suppose that Γ is a maximal arithmetic Kleinian group which is properly contained in Γ_0, a maximal discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. Note that it follows that $[\Gamma_0 : \Gamma] = 2$, otherwise Γ is properly contained in Γ_0^+ which contradicts maximality.

Let $\Gamma^{(2)}$ denote the subgroup of Γ generated by squares of elements of Γ. Then $\Gamma^{(2)}$ is a finite index characteristic subgroup of Γ, and so is normal in Γ_0. Hence the orbifold $\mathbb{H}^3/\Gamma^{(2)}$ admits an orientation-reversing isometry.

By [23, Theorem 8.3.1], the field k can be identified with the field $\mathbb{Q}(\mathrm{tr}\gamma : \gamma \in \Gamma^{(2)})$, and by [27, Proposition 3.4], the existence of the orientation-reversing isometry on $\mathbb{H}^3/\Gamma^{(2)}$ implies that k is stable under complex conjugation. As k is a field with one complex place, it follows that $[k : k \cap \mathbb{R}] = 2$. This contradicts the choice of k. □

Proof of Theorem 1.1: In the light of Lemma 4.1, we now assume that $[k : \mathbb{Q}]$ is odd, and let B be a quaternion algebra defined over k, ramified at all real places. The discussion in Subsection 3.1 immediately implies that all arithmetic hyperbolic 3-manifolds arising from B/k are closed. We make an additional assumption about $\mathrm{Ram}_f(B)$ to ensure torsion-freeness in principal congruence subgroups.

Thus let R denote the finite set of prime ideals $\mathcal{P}$ of k such that, either $\mathcal{P} \in R$ is a dyadic prime of k or, if $p\mathbb{Z} = \mathcal{P} \cap \mathbb{Z}$, then p is ramified in k. We assume that $\mathrm{Ram}_f(B)$ contains R.

Let $\mathcal{O}$ be a maximal order in B so that $\Gamma_{\mathcal{O}}$ is a maximal arithmetic Kleinian group and, by Lemma 4.1, also maximal in $\mathrm{Isom}(\mathbb{H}^3)$. Let $\mathcal{E} \subset \mathcal{O}$ be an Eichler order of square-free level S, chosen so that $\Gamma_{\mathcal{E}}$ is maximal (e.g. as in Theorem 3.2). Then as noted in Subsection 3.4, $\Gamma^1_{\mathcal{E}}$ contains a principal congruence subgroup $\Gamma(\mathcal{O}(I))$ where I is defined by S. By definition S is disjoint from $\mathrm{Ram}_f(B)$ (and so from R), so it follows from Proposition 3.5 that $\Gamma(\mathcal{O}(I))$ is torsion-free.

Choose an admissible element g in $\Gamma_{\mathcal{E}}$ (which as noted in Subsection 3.4 exists and is an element of $\mathrm{Comm}(\Gamma_{\mathcal{O}})$). By Lemma 3.4, $\Gamma(\mathcal{O}(I))$ is a normal subgroup of finite index in $\Gamma_{\mathcal{O}}$. Using the fact (recall Subsection 3.3) that $\Gamma^1_{\mathcal{E}}$ is a normal subgroup of $\Gamma_{\mathcal{E}}$ we deduce:

$$g\Gamma(\mathcal{O}(I))g^{-1} \subset g\Gamma^1_{\mathcal{E}}g^{-1} = \Gamma^1_{\mathcal{E}} \subset \Gamma^1_{\mathcal{O}} \subset \Gamma_{\mathcal{O}}.$$

Furthermore, by Lemma 3.6, $g\Gamma(\mathcal{O}(I))g^{-1} \neq \Gamma(\mathcal{O}(I))$. Now apply Proposition 2.1 with $\Gamma_0 = \Gamma_{\mathcal{O}}$ and $\Gamma_1 = \Gamma(\mathcal{O}(I))$.

By Theorem 3.2 we can choose infinitely many sets S, giving infinitely many examples where Theorem 1.1 holds in a *fixed* commensurability class. From Subsection 3.1, the commensurability class of $\Gamma_{\mathcal{O}}$ is determined by the isomorphism class of B which, in turn, is determined by its ramification set. Hence we have an infinite number of choices of $\mathrm{Ram}_f(B)$ subject to the restriction that $R \subset \mathrm{Ram}_f(B)$, and Theorem 1.1 now follows. □

4.2 Specific examples

We now refine the construction in Subsection 4.1 to provide more specific examples of finite groups. In particular we will be able to gain extra control in the construction of a normal subgroup Δ as in the proof of Proposition 2.1, and this will allow us to get control of G_1 and G_2.

We fix $k = \mathbb{Q}(x)$ where $x^3+x-1 = 0$ so that k has one complex place, has discriminant -31, and has class number 1. Indeed, in what follows, k can be any cubic number field k with one complex place having class number 1. Let B be a quaternion algebra defined over k with $\mathrm{Ram}_f(B) = \{\mathcal{Q}\}$ where the prime ideal $\mathcal{Q}$ does not divide any prime p that splits completely to k (e.g. in this case we can take $\mathcal{Q}$ to be the unique prime dividing 5 of norm 5^3).

Let $p \in \mathbb{Z}$ be an odd prime that splits completely to k: that there are infinitely many such primes p is a well-known consequence of Cebotarev Density theorem (see for example [23, Chapter 0]). Let $\mathcal{P} \subset R_k$ be a prime dividing such a p, and so in particular $N\mathcal{P} = p$.

Define the two-sided integral ideal $I = \mathcal{P}\mathcal{O}$. This can be done locally as follows: For all k-prime ideals $\mathcal{Q}$ let $\mathcal{O}_{\mathcal{Q}} = M_2(R_{\mathcal{Q}})$, and then set $I_J = \mathcal{O}_J$ for all primes $J \neq \mathcal{P}$ and $I_{\mathcal{P}} = \pi\mathcal{O}_{\mathcal{P}}$. The Eichler order $\mathcal{E}$ is then defined locally by $\mathcal{E}_J = \mathcal{O}_J$ for $J \neq \mathcal{P}$ and $\mathcal{E}_{\mathcal{P}} = \mathcal{T}_{\mathcal{P}}$ as at (3.1) so that $S = \{\mathcal{P}\}$ and $\Gamma^1_{\mathcal{E}}$ contains $\Gamma(\mathcal{O}(I))$. Since p is odd and $p \neq 31$, $\mathcal{P}$ is unramified in $k \mid \mathbb{Q}$, and Proposition 3.5 applies to show that $\Gamma(\mathcal{O}(I))$ is torsion-free.

Applying Remark 3.3, the group $\Gamma_{\mathcal{E}}$ is maximal. Indeed, an examination of Borel's construction [8] and [23, Chapters 6.7 and 11.4] provides an element $\alpha \in N(\mathcal{E})$ with $g = \mathrm{P}\rho(\alpha)$ so that locally at $\mathcal{P}$, g acts as conjugation by the element $\begin{pmatrix} 0 & 1 \\ \pi_{\mathcal{P}} & 0 \end{pmatrix}$, that is to say, it acts locally on $\Gamma(\mathcal{O}(I))$ by (cf. the discussion in Subsection 3.2)

$$g\begin{pmatrix} 1+a\pi_{\mathcal{P}} & b\pi_{\mathcal{P}} \\ c\pi_{\mathcal{P}} & 1+d\pi_{\mathcal{P}} \end{pmatrix}g^{-1} = \begin{pmatrix} 1+d\pi_{\mathcal{P}} & c \\ b\pi^2_{\mathcal{P}} & 1+a\pi_{\mathcal{P}} \end{pmatrix}. \tag{4.1}$$

With this in place, we now define a two-sided $\mathcal{O}$-ideal I' by $I'_J = \mathcal{O}_J$ for $J \neq \mathcal{P}$ and $I'_{\mathcal{P}} = \pi^2_{\mathcal{P}}\mathcal{O}_{\mathcal{P}}$ so that $\Gamma(\mathcal{O}(I)) \cap g\Gamma(\mathcal{O}(I))g^{-1} \supset \Gamma(\mathcal{O}(I'))$. Now by Lemma 3.4, $\Gamma(\mathcal{O}(I'))$ is a normal subgroup of $\Gamma_{\mathcal{O}}$. Thus the manifold $X = \mathbb{H}^3/\Gamma(\mathcal{O}(I'))$ admits free actions by the groups $G_1 = \Gamma(\mathcal{O}(I))/\Gamma(\mathcal{O}(I'))$ and $G_2 = g\Gamma(\mathcal{O}(I))g^{-1}/\Gamma(\mathcal{O}(I'))$.

We now identify the groups G_1 and G_2 explicitly. Note first that since B is unramified at $\mathcal{P}$ we have:

$$\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I')) \cong \mathrm{PSL}(2, R_{\mathcal{P}}/\pi^2_{\mathcal{P}}R_{\mathcal{P}})$$

and G_1 is the kernel of the reduction homomorphism $\mathrm{PSL}(2, R_{\mathcal{P}}/\pi_{\mathcal{P}}^2 R_{\mathcal{P}}) \to \mathrm{PSL}(2, R_{\mathcal{P}}/\pi_{\mathcal{P}} R_{\mathcal{P}})$. Since $N\mathcal{P} = p$, G_1 is an elementary abelian group of order p^3 generated by the images of the matrices

$$\begin{pmatrix} 1 & \pi_{\mathcal{P}} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pi_{\mathcal{P}} & 1 \end{pmatrix}, \begin{pmatrix} 1+\pi_{\mathcal{P}} & 0 \\ 0 & 1-\pi_{\mathcal{P}} \end{pmatrix}.$$

The group G_2 has the same order p^3 as G_1. To identify G_2, we consider the reduction of $g\Gamma(\mathcal{O}(I))g^{-1}$ locally modulo $\pi_{\mathcal{P}}^2$. From (4.1), the image modulo the ideal $\pi_{\mathcal{P}}^2 R_{\mathcal{P}}$ consists of matrices of the form $\begin{pmatrix} 1+d\pi_{\mathcal{P}} & c \\ 0 & 1+a\pi_{\mathcal{P}} \end{pmatrix}$. From this description it is easy to check that G_2 is non-abelian and contains the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of order p^2. We summarize this discussion as follows.

Corollary 4.2. *For infinitely primes p, we can find a closed hyperbolic 3-manifold X_p such that X_p admits free, orientation-preserving actions by finite groups of isometries, G_1 and G_2 such that*

(1) *G_1 and G_2 are finite groups of order p^3 with $X_p/G_1 \cong X_p/G_2$.*

(2) *G_1 is isomorphic to an elementary abelian group of order p^3, and G_2 is the unique non-abelian group of order p^3 which contains an element of order p^2.*

By varying the prime $\mathcal{Q}$ ramifying the quaternion algebra B/k we also obtain infinitely many commensurability classes of manifolds as in Corollary 4.2. This again follows from an application of the Cebotarev Density theorem which provides infinitely many rational primes whose inertial degrees are greater than 1.

5 A rational homology 3-sphere and fibered examples

In the construction of the examples stated in Corollary 1.2, we will make use of the cubic number field $k = \mathbb{Q}(x)$ where $x^3 - x^2 + 1 = 0$. This has one complex place, discriminant -23, and class number 1. Let B be the quaternion algebra defined over k with $\mathrm{Ram}_f(B) = \{\mathcal{Q}\}$ where $\mathcal{Q}$ is the unique k-prime ideal of norm 5. This determines the commensurability class of the Weeks manifold (see [23, Chapters 4.8.3 and 9.8.2]), which is the smallest volume closed orientable hyperbolic 3-manifold [16].

As we describe below, what is important for us is that if $\mathcal{O} \subset B$ is the unique (up to B^*-conjugacy) maximal order, the group $\Gamma^1_{\mathcal{O}}$ contains subgroups of index 24 that are congruence subgroups of certain levels which are the fundamental groups of a fibered hyperbolic 3-manifold or of a rational homology 3-sphere. In Section 9 we include some Magma [10] computations, which shows, amongst other things, that $\Gamma^1_{\mathcal{O}}$ has 11 conjugacy classes of subgroups of index 24, and unique conjugacy classes with abelianization $\mathbb{Z}\oplus\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z}$. We will make use of these in what follows. Note that the presentation for $\Gamma^1_{\mathcal{O}}$ that is used in the Magma calculations comes from a description of the orbifold $\mathbb{H}^3/\Gamma^1_{\mathcal{O}}$ as $(3, 0)$ Dehn surgery on the knot 5_2 (see [11, Subsection 5.4]).

5.1 Fibered examples

From [11, Proof of Lemma 9.3], the manifold M arising as the double cover of 0-surgery on the knot 6_2 has fundamental group that is a subgroup of $\Gamma^1_{\mathcal{O}}$ of index 24 and has

$H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z}$. As we remarked upon above, $\Gamma^1_{\mathcal{O}}$ has a unique such subgroup (up to conjugacy), which we denote by Γ. As shown in Section 9, Magma computes the core C of Γ in $\Gamma^1_{\mathcal{O}}$ and is the kernel of a homomorphism onto $\mathrm{PSL}(2, \mathbb{F}_{23})$, the finite simple group of order 6072. As is also discussed in [11, Subsection 9.1] there are two k-primes of norm 23, one of which is ramified in $k \mid \mathbb{Q}$ (recall the discriminant is -23) and one unramified. Denote these primes by $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, and these give rise to the 2-sided integral ideals $I_1 = \mathcal{P}_1\mathcal{O}$ and $I_2 = \mathcal{P}_2\mathcal{O}$. Since B is unramified at both of $\mathcal{P}_1$ and $\mathcal{P}_2$, it follows that $\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I_j)) \cong \mathrm{PSL}(2, \mathbb{F}_{23})$ for $j = 1, 2$. Putting all of this together we may deduce that $C = \Gamma(\mathcal{O}(I_j))$ for one of $j = 1, 2$. We will not need to explicitly identify which one, and simply denote the relevant subgroup as $\Gamma(\mathcal{O}(I))$. In particular, since $\mathbb{H}^3/\Gamma(\mathcal{O}(I)) \to M = \mathbb{H}^3/\Gamma$ it is fibered over the circle. We now apply Proposition 2.1 together with Lemma 4.1, using an admissible element from $\Gamma_{\mathcal{E}}$ where $\mathcal{E}$ is the Eichler order of level $S = \{\mathcal{P}_j\}$ for $j = 1$ or 2 (using Remark 3.3).

To pass to infinitely many examples, we can use the principal congruence subgroups $\Gamma(\mathcal{O}(I^n))$ for $n \geq 2$ an integer. Arguing as in Subsection 4.2, in particular the local conjugation given by (4.1), shows that $\Gamma(\mathcal{O}(I^{n+1})) \subset \Gamma(\mathcal{O}(I^n)) \cap g\Gamma(\mathcal{O}(I^n))g^{-1}$ and so the existence of infinitely many fibered examples follows since $\mathbb{H}^3/\Gamma(\mathcal{O}(I^n))$ is fibered for all $n \geq 1$.

Remark 5.1. At present we do not know how to produce infinitely many commensurability classes. Given [2] we know that every closed hyperbolic 3-manifold has a finite cover that fibers over the circle. What is needed is that the finite covers can be identified as congruence subgroups (as in the example above). In principal this should be possible, and note the work of Agol and Stover [4] in this direction.

5.2 A rational homology 3-sphere

As can be checked there is a unique k-prime ideal $\mathcal{P}$ of norm 7, which is unramified in B. Taking I to be the two-sided integral ideal $\mathcal{P}\mathcal{O}$ it follows (as in Subsection 4.2) that $\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I)) \cong \mathrm{PSL}(2, \mathbb{F}_7)$. In particular, the Magma computations in Section 9 shows that there is a unique index 24 subgroup, `l[9]`, which has core a normal subgroup of index 168 with quotient group being simple. Hence the group `l[9]` is a congruence subgroup of $\Gamma^1_{\mathcal{O}}$ containing $\Gamma(\mathcal{O}(I))$ of index 7. From Section 9, we also find that the abelianization of $\Gamma(\mathcal{O}(I))$ is $\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z}$. Hence there is a unique homomorphism $\phi : \Gamma(\mathcal{O}(I)) \to \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$. Now arguing as in Subsection 4.2, it follows that $\Gamma(\mathcal{O}(I))/\Gamma(\mathcal{O}(I^2)) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$ and so we take as the manifold $X = \mathbb{H}^3/\Gamma(\mathcal{O}(I^2))$. Choosing an admissible element $g \in \Gamma_{\mathcal{E}}$ where $\mathcal{E}$ is the Eichler order of level $S = \{\mathcal{P}\}$ (using Remark 3.3), the local conjugation argument given by (4.1) again shows that $\Gamma(\mathcal{O}(I^2)) \subset \Gamma(\mathcal{O}(I)) \cap g\Gamma(\mathcal{O}(I))g^{-1}$. Applying Proposition 2.1 together with Lemma 4.1 and note from Section 9 that X is a rational homology 3-sphere since the abelianization of $\Gamma(\mathcal{O}(I^2))$ (viewed as $\ker \phi$) is still finite (albeit gigantic!). This completes the proof.

6 Proof of Theorem 1.3

Proposition 3.5 shows that the groups $\Gamma(\mathcal{O}(I))$ are typically torsion-free. The proof of Theorem 1.3 will exploit a situation where these groups are *not torsion-free*, and apply Proposition 2.1 (and see Remark 2.2 following it). To construct examples in this setting, we begin with some preliminary discussion that will help locate elements of order 2 in principal congruence subgroups.

6.1 Elements of order 2 in $\Gamma^1_{\mathcal{O}}$

First we recall the following (see [23, Theorem 12.5.4] for details). Let B be a quaternion algebra over a number field k, let $L = k(i)$ and let $\mathcal{O}$ be a maximal order in B. Then $\mathrm{P}(B^1) < \mathrm{PSL}(2, \mathbb{C})$ contains an element of order 2 if and only if $i \notin k$ and no finite place at which B is ramified, splits in $L \mid k$. Moreover, $\Gamma^1_{\mathcal{O}}$ will contain an element of order 2 unless B is unramified at all finite places, and every dyadic place of k splits in $L \mid k$.

Indeed, if we consider $L \hookrightarrow B$ being given as the subfield $k(u) \subset B$ where u is the image of i, then for a suitable choice of maximal order $\mathcal{O}$, u (or simply abusing notation i) will project to an element of order 2 in $\Gamma^1_{\mathcal{O}}$.

6.2 Explicit commensurability classes of examples

We now fix $k = \mathbb{Q}(\sqrt{-2})$, with ring of integers R_k and p a rational prime with $p \equiv 3 \mod 8$. Hence, $pR_k = \mathcal{P}_p\mathcal{P}'_p$. Let B_p/k be the quaternion algebra ramified at exactly the places corresponding to $\mathcal{P}_p$ and $\mathcal{P}'_p$. This can be described explicitly by the Hilbert Symbol (see [23, Chapter 2.1])

$$\left(\frac{-1\,,\,-p}{k}\right).$$

In addition, it can be checked that a maximal order $\mathcal{O}_p$ can be explicitly described as the R_k-submodule of B_p given by $R_k[1, i, (i + j)/2, (1 + ij)/2]$.

Given the discussion in Subsection 6.1 with $L = k(i) = \mathbb{Q}(\mathrm{e}^{\pi i/4})$ we see that L embeds in B_p. In addition, R_L embeds in $\mathcal{O}_p$. The reason for this is that it embeds in some maximal order by [15], and since k has class number 1 there is a unique type of maximal order in all of the quaternion algebras B_p.

For convenience we suppress the subscript p in what follows. By construction, $\Gamma^1_{\mathcal{O}}$ has an element of order 2 given as the image of i. Let J_L (resp. J) denote the principal L-ideal (resp. k-ideal) generated by $i - 1$ (resp. $\sqrt{-2}$). Notice that $\sqrt{-2} = (1 - i)v$ where $v = \frac{(-\sqrt{2}+\sqrt{-2})}{2} \in R^*_L$, so $\sqrt{-2} \in J_L$, from which it follows that $J_L \cap R_k = J$. Note also that 2 is totally ramified in $L \mid k$ (i.e. $2R_L = \mathcal{P}_2^4$ for some ideal $\mathcal{P}_2 \subset R_L$ of norm 2), and $(i - 1)R_L = JR_L$. Now take $I \subset \mathcal{O}$ to be the two-sided ideal $J\mathcal{O}$. The previous discussion shows that $i - 1 \in I$, and so i determines an element of order 2 in $\Gamma(\mathcal{O}(I))$.

We also note that as in the discussion in Subsection 3.2, there is an orientation-reversing involution (an extension of complex conjugation in $\mathbb{C}$ to $\mathbb{H}^3$) that normalizes $\Gamma^1_{\mathcal{O}}$. This follows since, by the explicit nature of $\mathcal{O}$, the involution on B given by the extension of complex conjugation preserves $\mathcal{O}$.

Proof of Theorem 1.3: Using the notation above, we see that B is a division algebra, and so $\Gamma^1_{\mathcal{O}}$ is cocompact. Since there are infinitely many primes $p \equiv 3 \mod 8$, there are infinitely many isomorphism classes of quaternion algebras and so as before, Subsection 3.1 provides infinitely many commensurability classes of arithmetic hyperbolic 3-manifolds.

As discussed above, $\Gamma^1_{\mathcal{O}}$ is normalized by an orientation-reversing involution, so that unlike the proof of Theorem 1.1, the group $\Gamma_{\mathcal{O}}$ is maximal in $\mathrm{PSL}(2, \mathbb{C})$ but is not maximal in $\mathrm{Isom}(\mathbb{H}^3)$. Denote the maximal group in $\mathrm{Isom}(\mathbb{H}^3)$ containing $\Gamma_{\mathcal{O}}$ by $G_{\mathcal{O}}$ (so that

$G^+_{\mathcal{O}} = \Gamma_{\mathcal{O}}$). With I as above, the group $\Gamma(\mathcal{O}(I))$ will again play the role of Γ_1. The explicit description given of I and $\mathcal{O}$ implies that $\Gamma(\mathcal{O}(I))$ is also normal in $G_{\mathcal{O}}$.

We now argue as in Subsection 4.2: apply Lemma 3.6 for a suitable choice of Eichler order $\mathcal{E}$; namely the Eichler order of square-free level $\{J\}$, and just as in Subsection 3.4, $\Gamma^1_{\mathcal{E}}$ contains $\Gamma(\mathcal{O}(I))$. Remark 3.3 shows that $\Gamma_{\mathcal{E}}$ is a maximal arithmetic Kleinian group. Choose an admissible element $g \in \Gamma_{\mathcal{E}}$ and apply Proposition 2.1. As in Subsection 4.2, $\Gamma(\mathcal{O}(I^2)) \subset \Gamma(\mathcal{O}(I)) \cap g\Gamma(\mathcal{O}(I))g^{-1}$. This remains normal in $G_{\mathcal{O}}$ since $I^2 = 2\mathcal{O}$. If $\Gamma(\mathcal{O}(I^2))$ is torsion-free we use this group as we did previously. Otherwise we can pass to a torsion-free subgroup and then to its core Δ in $G_{\mathcal{O}}$, as in the proofs of Proposition 2.1 and Theorem 1.1.

With this we have constructed a hyperbolic 3-manifold $X = \mathbb{H}^3/\Delta$ that admits actions by groups of orientation-preserving isometries G_1 and G_2 acting on X with fixed points and with X/G_1 isometric to X/G_2. That G_1 and G_2 are not conjugate subgroups of $\mathrm{Isom}(X)$, or equivalently $\Gamma(\mathcal{O}(I))$ and $g\Gamma(\mathcal{O}(I))g^{-1}$ are not conjugate in $G_{\mathcal{O}}$, follows as in the proof of Theorem 1.1 since $\Gamma(\mathcal{O}(I))$ is normal in $G_{\mathcal{O}}$. This completes the proof. □

Remark 6.1. With reference to the proof of Theorem 1.3, it seems likely that the group $\Gamma(\mathcal{O}(I^2))$ is torsion-free, but we will not pursue this here.

7 Dimension 2

We now discuss a specific analogue of Theorem 1.1 and the arithmetic constructions given in Subsection 4.2 in the context of hyperbolic surfaces, or equivalently, on emphasizing complex structures, Riemann surfaces. We prove the following.

Theorem 7.1. *For infinitely many primes p, there exist compact hyperbolic (Riemann) surfaces X_p with the property that there are finite p-groups $G_{1,p}$ and $G_{2,p}$ which are non-isomorphic, which act freely on X_p with $X/G_{1,p} \cong X/G_{2,p}$ (or equivalently $X/G_{1,p}$ and $X/G_{2,p}$ are conformally equivalent Riemann surfaces).*

Borel's work [8] applies in this setting, and the structure of maximal arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^2)$ is entirely analogous to what is described in Subsections 3.3, 3.4. So, although the proof given below can be carried out in more generality, we believe it is instructive to simply focus on one commensurability class: that given by the indefinite quaternion algebra $B/\mathbb{Q}$ ramified at the primes 2 and 3. If $\mathcal{O} \subset B$ is a maximal order, then it is known that $\Gamma^+_{\mathcal{O}} = \Gamma_{\mathcal{O}} \cap \mathrm{PSL}(2,\mathbb{R})$ is the $(2,4,6)$ Fuchsian triangle group and $\Gamma_{\mathcal{O}}$ is the group generated by reflections in the faces of this triangle (see [23, Chapter 13.3]). Note the difference with the Fuchsian case in comparison to the case of arithmetic Kleinian groups is that $N(\mathcal{O})$ can contain elements of determinant -1, thereby giving elements in $\mathrm{PGL}(2,\mathbb{R})$. However, a version of Remark 3.3 continues to hold in this setting.

Proof. In the notation established above, we let $\Gamma_0 = \Gamma_{\mathcal{O}}$, and for $p \in \mathbb{Z}$ a prime different from $2, 3$ we construct the principal congruence subgroup $\Gamma(\mathcal{O}(I(p)))$ where $I(p)$ is the two-sided integral ideal defined as $p\mathcal{O}$. We now follow the argument in Subsection 4.2: using a version of Remark 3.3, we may build an Eichler order $\mathcal{E}_p$ of level $S = \{p\}$, a maximal group $\Gamma_{\mathcal{E}_p}$, an admissible element g_p, and then follow the argument in Subsection 4.2 to build the required Riemann surface X_p with the free actions of groups G_1 and G_2. That the covering groups G_1 and G_2 are finite p-groups (of order p^3) follows as in Subsection 4.2. □

Example 7.2. We take the case of $p = 5$ in Theorem 7.1. In this case, it can be shown that $\Gamma^1_{\mathcal{O}}$ is a Fuchsian group of signature $(0; 2, 2, 3, 3)$ (see for example [30]) which has co-area $2\pi/3$. By construction, $\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I(5))) \cong \mathrm{PSL}(2,5)$, and using the co-area computed above, determines a 60-fold Riemann surface cover $\Sigma = \mathbb{H}^2/\Gamma(\mathcal{O}(I(5)))$ of $\mathbb{H}^2/\Gamma^1_{\mathcal{O}}$. Hence Σ has genus 7. The surface X_5 we want is then a $(\mathbb{Z}/5\mathbb{Z})^3$ cover of Σ, so of genus 751, which also admits a free action of a non-abelian 5 group of order 5^3 with quotient isometric to Σ.

8 Higher dimensions

As mentioned in Section 2 we will only sketch the proof of an analogous result to Theorem 1.1 in higher dimensions. This will make use of so-called *arithmetic groups of simplest type* (we refer the reader to [26, Subsection 6.8] for a fuller discussion of these arithmetic lattices). For convenience, we restrict to one particular family in each dimension $n \geq 4$, the generalizations will be clear. We emphasize that this is a *sketch of a proof*, and so we will not designate with "Theorem" as some additional discussion of maximal groups is required to give a complete proof:

For each $n \geq 4$, there are infinitely many non-commensurable closed orientable hyperbolic n-manifolds X, with the property that there are finite groups G_1 and G_2 satisfying:

(1) *G_1 and G_2 act freely by orientation-preserving isometries on X with $X/G_1 \cong X/G_2$.*

(2) *$|G_1| = |G_2|$, but G_1 and G_2 are not conjugate in* $\mathrm{Isom}(X)$.

Sketch Proof: We recall some background on certain arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. Let d be a square-free positive integer, and let $f_{n,d}$ denote the quadratic form:

$$f_{n,d} = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - \sqrt{d}x_n^2.$$

This has signature $(n, 1)$, and after applying the non-trivial Galois automorphism σ given by $\sqrt{d} \mapsto -\sqrt{d}$ the resultant quadratic form $f^\sigma_{n,d}$ has signature $(n+1, 0)$; i.e. the quadratic form $f_{n,d}$ is equivalent over $\mathbb{R}$ to the quadratic form $J_n = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - x_n^2$, and the quadratic form $f^\sigma_{n,d}$ is equivalent over $\mathbb{R}$ to $x_0^2 + x_1^2 + \ldots + x_{n-1}^2 + x_n^2$.

Let $F_{n,d}$ be the symmetric matrix associated to the quadratic form $f_{n,d}$ and let $\mathrm{O}(f_{n,d})$ (resp. $\mathrm{SO}(f_{n,d})$) denote the linear algebraic groups defined over k described as:

$$\mathrm{O}(f_{n,d}) = \{X \in \mathrm{GL}(n+1, \mathbb{C}) : X^t F_{n,d} X = F_{n,d}\} \text{ and}$$
$$\mathrm{SO}(f_{n,d}) = \{X \in \mathrm{SL}(n+1, \mathbb{C}) : X^t F_{n,d} X = F_{n,d}\}.$$

For a subring $L \subset \mathbb{C}$, we denote the L-points of $\mathrm{O}(f_{n,d})$ (resp. $\mathrm{SO}(f_{n,d})$) by $\mathrm{O}(f_{n,d}, L)$ (resp. $\mathrm{SO}(f_{n,d}, L)$). Let $R_d \subset \mathbb{Q}(\sqrt{d})$ denote the ring of integers.

Note that, given this set-up, there exists $T \in \mathrm{GL}(n+1, \mathbb{R})$ such that $T^{-1}\mathrm{O}(f_{n,d}, \mathbb{R})T = \mathrm{O}(n, 1)$, in which case, $\mathrm{Isom}(\mathbb{H}^n)$ can be identified with the group $\mathrm{O}^+(J_n, \mathbb{R}) = \mathrm{O}^+(n, 1)$, which is the subgroup of $\mathrm{O}(n, 1)$ preserving the upper-half sheet of the hyperboloid $J_n = -1$. A similar discussion holds for $T^{-1}\mathrm{SO}(f_{n,d}, \mathbb{R})T = \mathrm{SO}(n, 1)$ and groups of orientation-preserving isometries. In particular this conjugation provides subgroups $\Lambda \subset \mathrm{O}(f_{n,d}, R_d)$ and $\Lambda^+ \subset \mathrm{SO}(f_{n,d}, R_d)$ whose images lie in $\mathrm{O}^+(n, 1)$ and $\mathrm{SO}^+(n, 1)$ respectively.

A subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ commensurable with the image in $\mathrm{Isom}(\mathbb{H}^n)$ of the subgroup of $\mathrm{O}(f_{n,d}, R_d)$ (under the conjugation map described above) is an example of an arithmetic lattice of simplest type. The corresponding arithmetic hyperbolic n-manifold $M = \mathbb{H}^n/\Gamma$ is also called arithmetic of simplest type. By construction, all the arithmetic lattices of simplest type we have described above are cocompact.

For an ideal $I \subset R_d$, let $\Gamma(I)$ denote the principal congruence subgroup of $\mathrm{O}(f_{n,d}, R_d)$ obtained as the kernel of the homomorphism:

$$\pi_I \colon \mathrm{O}(f_{n,d}, R_d) \to \mathrm{O}(f_{n,d}, R_d/I),$$

and note that so long as I is not a dyadic prime ideal, $\Gamma(I) \subset \mathrm{SO}(f_{n,d}, R_d)$.

We now fix the ideal I to be considered, namely let $p \in \mathbb{Z}$ be an odd prime that is inert to R_d; i.e. pR_d remains a prime ideal which we denote by $\mathcal{P}$. In this case, $\Gamma(\mathcal{P})$ is torsion-free (see [26, Subsection 4.8]), and the groups Λ and $\Gamma(\mathcal{P}) \cap \Lambda^+$ will play the roles of Γ_0 and Γ_1 in Proposition 2.1.

As before, the key point now is a detailed understanding of maximal arithmetic lattices in this setting. As in the case of arithmetic Kleinian groups, these maximal arithmetic groups arise as normalizers of certain number theoretically defined arithmetic lattices (using Bruhat-Tits theory), and are again congruence subgroups (see [9] and also [3] and [6] that deal explicitly with the case of $\mathrm{O}^+(n,1)$). In particular our group Λ is a maximal discrete subgroup of $\mathrm{O}^+(n,1)$ and using the description of maximal discrete groups in the commensurability class of Λ given in [9] it is possible to find an element g as required by Proposition 2.1.

9 Magma calculations

In what follows `g` is the group $\Gamma^1_{\mathcal{O}}$ of Section 5. The presentation was computed using SnapPy [13]. The group Γ from Subsection 5.1 is `l[2]`, and the index 24 subgroup from Subsection 5.2 is `l[9]`. The routine `CosetAction` produces a finite image group `i` and a kernel `k` which is the core of the relevant subgroup. The routine `pQuotient` is used to compute the kernel of a homomorphism a to the elementary abelian 7-group $(\mathbb{Z}/7\mathbb{Z})^3$.

```
> g<a,b>:=Group<a,b|b^3,a^2*b^-1*a^-2*b^-1*a^2*b*a^-1*b>;
> print AbelianQuotientInvariants(g);
[ 3 ]
> l:=LowIndexSubgroups(g,<24,24>);
> print #l;
11
> print AbelianQuotientInvariants(l[1]);
[ 30 ]
> print AbelianQuotientInvariants(l[2]);
[ 11, 0 ]
> print AbelianQuotientInvariants(l[3]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(l[4]);
[ 2, 2, 6 ]
> print AbelianQuotientInvariants(l[5]);
[ 5, 30 ]
```

```
> print AbelianQuotientInvariants(l[6]);
[ 3, 6, 6 ]
> print AbelianQuotientInvariants(l[7]);
[ 66 ]
> print AbelianQuotientInvariants(l[8]);
[ 3, 6, 6 ]
> print AbelianQuotientInvariants(l[9]);
[ 7, 42 ]
> print AbelianQuotientInvariants(l[10]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(l[11]);
[ 2, 2, 6 ]
> f,i,k:=CosetAction(g,l[2]);
> print Order(i);
6072
> IsSimple(i);
true
> f,i,k:=CosetAction(g,l[9]);
> print Order(i);
168
> IsSimple(i);
true
> print AbelianQuotientInvariants(k);
[ 7, 7, 42 ]
> K:=Rewrite(g,k);
> F,a,b:=pQuotient(K,7,1:Print:=1);
Lower exponent-7 central series for K

Group: K to lower exponent-7 central class 1 has order 7^3
> K1:=Kernel(a);
> print AbelianQuotientInvariants(K1);
[ 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 2743, 2743, 2743, 2743,
2743, 2743, 2743, 2743, 2743, 2743, 2743, 2743, 2743,
2743, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
3897803, 3897803, 23386818 ]
> f,i,k:=CosetAction(g,l[1]);
> print Order(i);
1320
```

```
> f,i,k:=CosetAction(g,l[3]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[4]);
> print Order(i);
504
> f,i,k:=CosetAction(g,l[5]);
> print Order(i);
504
> f,i,k:=CosetAction(g,l[6]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[7]);
> print Order(i);
6072
> IsSimple(i);
true
> a:=AbelianQuotientInvariants(k);
> print a;
[ 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 161,
161, 161, 161, 966, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
> print Multiplicity(a,0);
44
> f,i,k:=CosetAction(g,l[8]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[10]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[11]);
> print Order(i);
504
```

Referring to Subsection 5.1, there is an index 24 subgroup of $\Gamma^1_{\mathcal{O}}$ whose core is the principal congruence subgroup arising from "the other" k-prime of norm 23. This corresponds to the subgroup `l[7]` of the Magma output above. Although this gives rise to a manifold that is also a rational homology 3-sphere, the manifold with fundamental group which is the principal congruence subgroup (i.e. the core) has first Betti number equal to 44 (as shown in the Magma output above), and so does not provide rational homology 3-sphere examples as in Corollary 1.2.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.06
https://doi.org/10.26493/2590-9770.1391.f46
(Also available at http://adam-journal.eu)

On automorphisms of Haar graphs of abelian groups

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Received 21 October 2020, accepted 16 February 2022, published online 18 July 2022

Abstract

Let G be a group and $S \subseteq G$. In this paper, a Haar graph of G with connection set S has vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0,g)(1,gs) : g \in G \text{ and } s \in S\}$. Haar graphs are then natural bipartite analogues of Cayley digraphs, and are also called BiCayley graphs. We first examine the relationship between the automorphism group of the Cayley digraph of G with connection set S and the Haar graph of G with connection set S. We establish that the automorphism group of a Haar graph contains a natural subgroup isomorphic to the automorphism group of the corresponding Cayley digraph. In the case where G is abelian, we show there are exactly four situations in which the automorphism group of the Haar graph can be larger than the natural subgroup corresponding to the automorphism group of the Cayley digraph together with a specific involution, and analyze the full automorphism group in each of these cases. As an application, we show that all s-transitive Cayley graphs of generalized dihedral groups have a quasiprimitive automorphism group, can be constructed from digraphs of smaller order, or are Haar graphs of abelian groups whose automorphism groups have a particular permutation group theoretic property.

Keywords: Groups, graphs.

Math. Subj. Class.: 05C15, 05C10

A Haar graph of a group G with connection set S has vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0,g)(1,gs) : g \in G \text{ and } s \in S\}$, where $S \subseteq G$. Haar graphs are natural bipartite analogues of Cayley digraphs, and these graphs have appeared in a variety of contexts and under a variety of names. To the author's knowledge, Haar graphs were introduced in [18], where some of their elementary properties were studied, including some results on isomorphic Haar graphs. Recently there has been a fair amount of work on the isomorphism problem for Haar graphs [4, 5, 21–23, 25, 44], some of it motivated by applications (see

*The author is indebted to the referees for their careful reading of the paper, and their suggestions for improvements

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[7, 27, 41]). Most of this work falls into the two categories of considering the isomorphism problem for graphs of small valency or determining the structure of groups which are BCI-groups. Intuitively, these are groups where isomorphism is determined in the "nicest" possible way.

Our work in this paper was motivated by the isomorphism problem, but in the end we will not consider this problem here. It is well known that the isomorphism problem for a Cayley digraph Γ of a group G depends upon the conjugacy classes of natural subgroups of $\mathrm{Aut}(\Gamma)$ (see [6, Lemma 3.1]). A similar result also holds for Haar graphs [26, Lemma 2.2] or [4, Theorem C]. Thus, information about a Cayley digraph's automorphism group is crucial in determining which other Cayley digraphs of G are isomorphic to it, and similarly, for Haar graphs. Typically, more information is known about the automorphism group of Cayley digraphs of a group G (see for example [9, 11, 13, 17]) than of Haar graphs of G, and for every Haar graph there is a corresponding Cayley digraph with the same connection set. It is thus of natural interest to determine the relationship, if any, between the automorphism group of a Haar graph and the automorphism group of its corresponding Cayley digraph (again, much more is known about the automorphism groups of Cayley digraphs). This is the main focus of the work in this paper.

We will show in Corollary 2.16 that the automorphism group of a Haar graph of an abelian group A falls into four natural families. In two of these families, the automorphism groups are a wreath product and so can be found provided one knows the automorphism groups of the graphs involved in the wreath products (which are always of smaller order and so presumably easier to find). In the third family, the automorphism group of the Haar graph of A is determined, up to conjugacy by $|A|$ natural and explicitly defined permutations, by the automorphism group of the corresponding Cayley digraph. For the fourth and final family the situation is more interesting in that there does not seem to be a natural or obvious relationship between the automorphism group of the Haar graph and its corresponding Cayley digraph. We do, though, give a group theoretic construction for all of these graphs, but unfortunately the group theoretic information needed for the construction does not seem easy to obtain. We should also mention that there is some related work on finding automorphism groups of Haar graphs - see [46].

As an application, we next consider the implications of the above automorphism group results to s-arc-transitive Haar graphs. In particular, we characterize s-arc-transitive Cayley graphs of generalized dihedral groups with abelian subgroup of odd order. In all cases except one (which corresponds to the case in the preceding paragraph where there was no obvious relationship between the automorphism group of the Haar graph and its corresponding Cayley digraph), such s-arc-transitive graphs can be constructed from other highly symmetric graphs and digraphs of smaller order without the use of graph covers.

There is another problem in the literature related to this work. For a graph Γ, its *canonical double cover* is the graph $K_2 \times \Gamma$ and is denoted $B(\Gamma)$. So $V(B(\Gamma)) = \mathbb{Z}_2 \times V(\Gamma)$ and $E(B(\Gamma)) = \{(0,x)(1,y) : xy \in E(\Gamma)\}$. If $\Gamma = \mathrm{Cay}(G,S)$ is a Cayley *graph*, then $B(\Gamma) = \mathrm{Haar}(G,S)$. Automorphism groups of $B(\Gamma)$ for Γ a Cayley graph were first studied in [34] and subsequently by several other authors [28, 35, 39, 43]. The main question is, for a graph Γ (not necessarily a Cayley graph), is $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathrm{Aut}(\Gamma)$? If so, such a graph is *stable*, and if not, *unstable*. Corollary 2.16, for example, refines [28, Theorem 3.2] in the case where Γ is a Cayley graph of an abelian group. Our results also hold for digraphs, so one can consider the Haar graph construction for Cayley digraphs of a group G to be a natural generalization of the canonical double cover of graphs to digraphs.

Some words about notation should be mentioned. Haar graphs are special cases of bi-Cayley graphs of G, which are usually defined as graphs that contain a semiregular subgroup with two orbits that is isomorphic to G [2, 19, 33, 45]. Bi-Cayley graphs need not be bipartite, with the prefix *bi* referring to the two orbits of the semiregular subgroup isomorphic to G, not to the graph being bipartite, while Haar graphs are always bipartite. Additionally, some authors refer to Haar graphs as bi-Cayley graphs with the prefix *bi* presumably referring to the fact that they are bipartite. To make the terminology issues more complex, some authors refer to bi-Cayley graphs as defined here as semi-Cayley graphs. We prefer the term Haar graph to bi-Cayley graph as defined here (this definition will be formally stated below to hopefully eliminate all ambiguity and henceforth we will not use the term bi-Cayley graph), simply because this choice of terminology causes less confusion in that there is only one use of the term "Haar graph" in the literature, at least as far as the author knows!

1 Basic definitions and results on automorphism groups

All groups and graphs are finite. In this section, we define Cayley digraphs and Haar graphs and determine a relationship between their automorphism groups. This section is mainly concerned with Haar graphs of general finite groups G, not necessarily abelian.

Definition 1.1. Let G be a group and $S \subseteq G$. Define a *Cayley digraph of* G, denoted $\mathrm{Cay}(G, S)$, to be the digraph with $V(\mathrm{Cay}(G, S)) = G$ and $A(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call S the *connection set of* $\mathrm{Cay}(G, S)$.

Note that the map $g_L\colon G \mapsto G$ given by $g_L(x) = gx$ is always an automorphism of $\mathrm{Cay}(G, S)$ for every group G and connection set S. Thus the group $G_L = \{g_L : g \in G\} \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$ for every group G and connection set S. Here, if Γ is a graph or digraph, then $\mathrm{Aut}(\Gamma)$ denotes its automorphism group. Also note that by [6, Proposition 2.1] if $\alpha \in \mathrm{Aut}(G)$, then $\alpha(\mathrm{Cay}(G, S)) = \mathrm{Cay}(G, \alpha(S))$. In Figure 1, we give an example of a Cayley digraph of the cyclic group $\mathbb{Z}_7$.

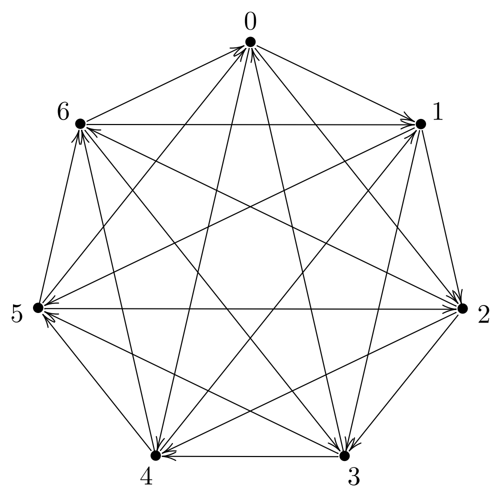

Figure 1: The Cayley digraph $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.

Definition 1.2. Let G be a group and $S \subseteq G$. Define the *Haar graph* $\mathrm{Haar}(G, S)$ with *connection set* S to be the graph with vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0, g)(1, gs) : g \in G \text{ and } s \in S\}$.

Some authors use $H(G, S)$ for $\mathrm{Haar}(G, S)$. We prefer the somewhat longer but more descriptive notation. In Figure 2 we give $\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$. We call $\mathrm{Haar}(G, S)$ the *Haar graph corresponding to* $\mathrm{Cay}(G, S)$. The graph in Figure 2 is the Haar graph corresponding to the Cayley digraph in Figure 1, and is the Heawood graph.

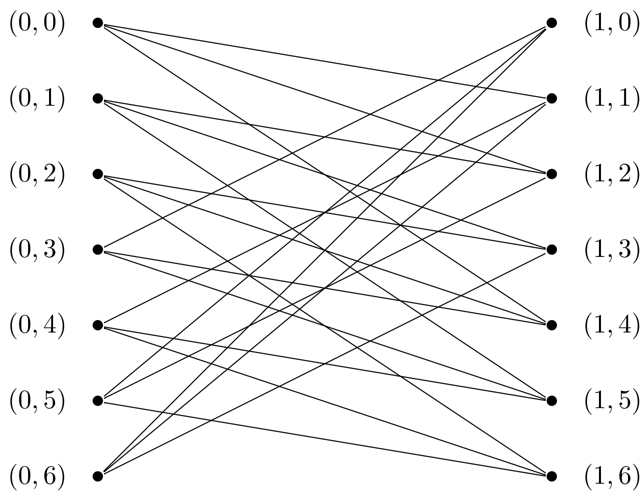

Figure 2: The Heawood graph as $\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$.

Clearly every Haar graph is a bipartite graph. Notice also that it is very much allowed that $1_G \in S$, while for Cayley digraphs this is usually not allowed. This is because the effect of including 1_G in the connection set of a Cayley digraph is to put a loop at each vertex, and doing this does not usually affect the symmetry properties of Cayley digraphs (e.g. adding a loop at each vertex does not change the automorphism group of a Cayley digraph). In some situations, though, allowing $1_G \in S$ for a Cayley digraph is not only advantageous, but crucial (see for example [3]). In this paper we *allow loops in Cayley digraphs*.

Definition 1.3. Throughout this paper, if $\Gamma = \mathrm{Haar}(G, S)$, then the natural bipartition of $V(\Gamma)$ will be denoted by $\mathcal{B}$, where $\mathcal{B} = \{B_0, B_1\}$, $B_0 = \{(0, g) : g \in G\}$, and $B_1 = \{(1, g) : g \in G\}$.

Notice that the map $\widehat{g_L} \colon \mathbb{Z}_2 \times G \mapsto \mathbb{Z}_2 \times G$ given by $\widehat{g_L}(i, j) = (i, gj)$ is an automorphism of $\mathrm{Haar}(G, S)$ for every group G and connection set S, and corresponds to the subgroup $G_L \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$.

It is easily determined using results in [1] that $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ has automorphism group $\{x \mapsto ax + b : a = 1, 2, 4, b \in \mathbb{Z}_7\}$ which is a metacyclic group of order 21. The Heawood graph is a Haar graph of $\mathbb{Z}_7$, as shown in Figure 2. While the Heawood graph does have a metacyclic subgroup of order 21 corresponding to $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\}))$, the automorphism group of the Heawood graph is actually $\mathbb{Z}_2 \ltimes \mathrm{PGL}(3,2) \cong \mathrm{PGL}(2,7)$ which has order 336 and is an almost simple group. Our first result shows that the automorphism group of a Haar graph always has a natural subgroup isomorphic to the automorphism group of its corresponding Cayley digraph. The Heawood graph example, though, shows the automorphism group of the Haar graph may be much larger.

Lemma 1.4. *Let G be a group, and $\gamma \in S_G$. The map $\hat{\gamma}\colon \mathbb{Z}_2 \times G \mapsto \mathbb{Z}_2 \times G$ given by $\hat{\gamma}(i,j) = (i,\gamma(j))$ is an automorphism of $\mathrm{Haar}(G,S)$ if and only if γ is an automorphism of $\mathrm{Cay}(G,S)$.*

Proof. The permutation $\gamma \in S_G$ is in $\mathrm{Aut}(\mathrm{Cay}(G,S))$ if and only if whenever $g \in G$ and $s \in S$, $\gamma(g,gs) = (\gamma(g),\gamma(gs)) \in A(\mathrm{Cay}(G,S))$. This occurs if and only if there exists $s' \in S$ such that $\gamma(gs) = \gamma(g)s'$ which is true if and only if $(0,\gamma(g))(1,\gamma(g)s') = (0,\gamma(g))(1,\gamma(gs)) \in E(\mathrm{Haar}(G,S))$. This last statement is true if and only if $\hat{\gamma} \in \mathrm{Aut}(\mathrm{Haar}(G,S))$. □

We now give a standard notation for the automorphism of $\mathrm{Haar}(G,S)$ induced by an automorphism of $\mathrm{Cay}(G,S)$ which will be used henceforth.

Definition 1.5. Let G be a group and $S \subseteq G$. Let $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$. The automorphism of $\mathrm{Haar}(G,S)$ induced by γ as in Lemma 1.4 will be denoted by $\hat{\gamma}$. That is, if $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$ then the *automorphism of* $\mathrm{Haar}(G,S)$ *corresponding to* γ is $\hat{\gamma}\colon \mathbb{Z}_2 \times G \mapsto \mathbb{Z}_2 \times G$ given by $\hat{\gamma}(i,j) = (i,\gamma(j))$. If $H \leq \mathrm{Aut}(\mathrm{Cay}(G,S))$, then we define $\widehat{H} = \{\hat{h} : h \in H\}$. In particular, the natural semiregular subgroup of $\mathrm{Haar}(G,S)$ isomorphic to G is denoted $\widehat{G_L}$.

In this paper we focus on $\mathrm{Aut}(\mathrm{Haar}(G,S))$ in the special case when G is an abelian group. The reason to restrict our attention to abelian groups is that the automorphism groups of Haar graphs of abelian groups and nonabelian groups are different. As implicitly used in [18], for an abelian group A, $\mathrm{Aut}(\mathrm{Haar}(A,S))$ contains the element $\iota : \mathbb{Z}_2 \times A \mapsto \mathbb{Z}_2 \times A$ given by $\iota(i,j) = (i+1,-j)$. The group $\langle \iota, \widehat{A_L}\rangle$ is transitive and so Haar graphs of abelian groups have transitive automorphism group while Haar graphs of nonabelian groups need not. See for example [15, Proposition 11]. For Haar graphs of abelian groups we may use the automorphism ι defined above to say a little more.

Lemma 1.6. *Let A be an abelian group, and $S \subseteq A$. Then*

$$\mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A,S)) \leq \mathrm{Aut}(\mathrm{Haar}(A,S)).$$

Proof. As A is an abelian group, straightforward computations will show that the map $\iota\colon \mathbb{Z}_2 \times A \mapsto \mathbb{Z}_2 \times A$ given by $\iota(i,j) = (i+1,-j)$ is an automorphism of $\mathrm{Haar}(A,S)$. Set $K = \mathrm{Aut}(\mathrm{Cay}(A,S))$. By Lemma 1.4, $\hat{K} \leq \mathrm{Aut}(\mathrm{Haar}(A,S))$.

Let $a \in A$. Then $(a,a+s) \in A(\mathrm{Cay}(A,S))$ if and only if $(a,a-s) \in A(\mathrm{Cay}(A,-S))$, which occurs if and only if $(a+s,a) \in A(\mathrm{Cay}(A,-S))$. So $\mathrm{Cay}(A,-S)$ can be obtained from $\mathrm{Cay}(A,S)$ by reversing the direction of each arc in $\mathrm{Cay}(A,S)$. Now, $(a,b) \in A(\mathrm{Cay}(A,S))$ if and only if $(\gamma(a),\gamma(b)) \in A(\mathrm{Cay}(A,S))$ for every $\gamma \in \mathrm{Aut}(\mathrm{Cay}(A,S))$. As $(a,b) \in A(\mathrm{Cay}(A,S))$ if and only if $(b,a) \in A(\mathrm{Cay}(A,-S))$, we see $(\gamma(a),\gamma(b)) = \gamma(a,b) \in A(\mathrm{Cay}(A,S))$ if and only if $(\gamma(b),\gamma(a)) = \gamma(b,a) \in A(\mathrm{Cay}(A,-S))$. We conclude $\mathrm{Aut}(\mathrm{Cay}(A,S)) = \mathrm{Aut}(\mathrm{Cay}(A,-S))$.

Define $r\colon A \to A$ by $r(j) = -j$. Then $r \in \mathrm{Aut}(A)$ and $r(\mathrm{Cay}(A, S)) = \mathrm{Cay}(A, -S)$. Also $r^{-1} = r$, and $r(\mathrm{Cay}(A, -S)) = \mathrm{Cay}(A, S)$. Hence

$$\mathrm{Aut}(\mathrm{Cay}(A, S)) = r\mathrm{Aut}(\mathrm{Cay}(A, -S))r.$$

Thus $\gamma \in \mathrm{Aut}(\mathrm{Cay}(A, S))$ if and only if the map $j \mapsto -\gamma(-j)$ is also contained in $\mathrm{Aut}(\mathrm{Cay}(A, S))$. As $\iota\hat{\gamma}\iota(i, j) = (i, -\gamma(-j))$, we see ι normalizes $\hat{K}$. Then the group $\langle \iota, \hat{K}\rangle = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G, S)) \le \mathrm{Aut}(\mathrm{Haar}(A, S))$ as $|\iota| = 2$ and ι normalizes $\hat{K}$. □

In the case where $S = -S$ and $\mathrm{Cay}(A, S)$ is a graph, we have a slightly nicer result, which is contained in the significantly stronger result [31, Lemma 4.2].

Lemma 1.7. *Let A be an abelian group and $S \subseteq A$ such that $S = -S$. Then $\mathbb{Z}_2 \times \mathrm{Aut}(\mathrm{Cay}(A, S)) \le \mathrm{Aut}(\mathrm{Haar}(A, S))$.*

Proof. Simply observe that if $S = -S$ the map $i \mapsto -i$ is an automorphism of $\mathrm{Cay}(A, S)$ and so the map $(i, j) \mapsto (i + 1, j)$ is an automorphism of $\mathrm{Haar}(A, S)$. □

One circumstance in which $\mathrm{Aut}(\mathrm{Haar}(G, S))$ is bigger than $\mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G, S))$ is if $\mathrm{Cay}(G, S)$ is connected but $\mathrm{Haar}(G, S)$ is not. For example, for $n \ge 2$, $\mathrm{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ is a $2n$-cycle with automorphism group D_{2n}, while $\mathrm{Haar}(\mathbb{Z}_{2n}, \{\pm 1\})$ is a disjoint union of two $2n$-cycles, with automorphism group isomorphic to $\mathbb{Z}_2 \wr D_{2n}$ (the group wreath product is given in Definition 2.3; here it is enough to observe that this group is bigger). Notice that the vertex sets of these two $2n$-cycles are **not** the sets B_0 and B_1. A perhaps more extreme example is $\mathrm{Cay}(\mathbb{Z}_{2n}, S)$, where S is all odd elements of $\mathbb{Z}_{2n}$. The graph $\mathrm{Cay}(\mathbb{Z}_{2n}, S) = K_{n,n}$ is connected, but $\mathrm{Haar}(\mathbb{Z}_{2n}, S)$ consists of two disjoint copies of $K_{n,n}$. The necessary and sufficient condition for $\mathrm{Haar}(G, S)$ to be connected is $SS^{-1} = \{st^{-1} : s, t \in S\}$ generates G and is given in [14, Lemma 2.3(iii)]. A more appealing formulation for Haar graphs of abelian groups is the following:

Lemma 1.8. *Let A be an abelian group. $\mathrm{Haar}(A, S)$ is disconnected if and only if $S \subseteq a + H$ for some subgroup $H < A$ and $a \in A$.*

Proof. If $S \subseteq a + H$ for some $H < A$ and $a \in A$ then for $s, t \in S$ we have $s - t = a + h_1 - (a + h_2) \in H < A$ for some $h_1, h_2 \in H$, and so $\mathrm{Haar}(A, S)$ is disconnected by [14, Lemma 2.3(iii)]. If $\mathrm{Haar}(A, S)$ is disconnected, then $\langle s - t : s, t \in S\rangle = H < A$ by [14, Lemma 2.3(iii)]. Fix $a \in S$, and let $s \in S$. Then $a - s = -h \in H$ and $s = a + h$. Hence $S \subseteq a + H$. □

Before proceeding, we will need some permutation group theoretic terms.

Definition 1.9. Let X be a set and $K \le S_X$ be transitive. A subset $C \subseteq X$ is a *block* of K if whenever $k \in K$, then $k(C) \cap C = \emptyset$ or C. If $C = \{x\}$ for some $x \in X$ or $C = X$, then C is a *trivial block*. Any other block is nontrivial. The set $\mathcal{C} = \{k(C) : k \in K\}$ is a partition of X, called a *block system* of K, and is *nontrivial* if C is nontrivial.

A basic fact about a graph Γ is that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\bar{\Gamma})$, where $\bar{\Gamma}$ is the complement of Γ. But the complement of a Haar graph is not a Haar graph. One could consider "bipartite complements" (defined below) to avoid this problem, but it still need not be the case that the automorphism group of the bipartite complement of a Haar graph Γ is the automorphism group of Γ (although if $\mathcal{B}$ is a block system of the automorphism group of the bipartite complement of Γ we do have equality of automorphism groups under bipartite complements [10, Corollary 4]). Our next result is really an exercise and has certainly appeared as a comment in the literature [38]. We state and prove this result due to its importance to the work in this paper.

Lemma 1.10. *Let Γ be a vertex-transitive bipartite graph with bipartition $\mathcal{B} = \{B_0, B_1\}$. If Γ is connected then $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$.*

Proof. We prove the contrapositive, and so suppose there exists $\gamma \in \mathrm{Aut}(\Gamma)$ such that $\gamma(\mathcal{B}) = \mathcal{B}' = \{B_0', B_1'\} \neq \mathcal{B}$. As $\mathcal{B}$ is a bipartition of Γ and $\gamma \in \mathrm{Aut}(\Gamma)$, $\gamma(\mathcal{B}) = \mathcal{B}'$ is also a bipartition of Γ. Let $C_0 = B_0 \cap B_0'$, $C_1 = B_0 \cap B_1'$, $C_2 = B_1 \cap B_0'$, and $C_3 = B_1 \cap B_1'$. As $\mathcal{B} \neq \mathcal{B}'$, none of the sets C_i, $i \in \mathbb{Z}_4$, are empty. If Γ is connected, some vertex v of $C_0 \subset B_0$ is adjacent to some vertex w of B_1. As C_0 is a subset of B_0 and B_0', w must be in both B_1 and B_1', so in C_3. But by a symmetrical argument, any vertex of C_3 can only be adjacent to vertices in C_0, and so there is no path in Γ from v to any vertex of C_2. Hence Γ is disconnected. □

Definition 1.11. Let Γ be a bipartite graph with bipartition $\mathcal{B} = \{B_0, B_1\}$. The *bipartite complement of* Γ is the graph with vertex set $V(\Gamma)$ and two vertices are adjacent if they are in different bipartition classes and are not adjacent in Γ.

Corollary 1.12. *Let G be a group and $S \subseteq G$. If $\mathrm{Haar}(G, S)$ and $\mathrm{Haar}(G, G \setminus S)$ are both connected then $\mathrm{Aut}(\mathrm{Haar}(G, S)) = \mathrm{Aut}(\mathrm{Haar}(G, G \setminus S))$.*

Proof. By Lemma 1.10, $\mathcal{B}$ is a block system of both $\mathrm{Aut}(\mathrm{Haar}(G, S))$ and $\mathrm{Haar}(G, G \setminus S))$, and so by [10, Corollary 4] $\mathrm{Aut}(\mathrm{Haar}(G, S)) = \mathrm{Aut}(\mathrm{Haar}(G, G \setminus S))$. □

It is not true that if both $\mathrm{Haar}(A, S)$ and $\mathrm{Haar}(A, A - S)$ are disconnected then their automorphism groups are the same. Let $A = \mathbb{Z}_{2n}$, and $S = \langle 2\rangle$ so $A - S = 1 + \langle 2\rangle$. Then both $S + (-S)$ and $(A - S) + [-(A - S)] = \langle 2\rangle$ and so by Lemma 1.8 neither are connected. Both of these graphs are isomorphic to two copies of $K_{n,n}$, but the vertex sets of the $K_{n,n}$'s are different (in one it is even vertices adjacent to even vertices and odd vertices adjacent to odd vertices while in the other it is even vertices adjacent to odd vertices and odd vertices adjacent to even vertices), so their automorphism groups are different, but permutation isomorphic!

2 Characterization of automorphism groups of Haar graphs of abelian groups

We now, with some exceptions, focus on abelian groups. We will use the symbol A when the group under consideration is abelian, and G when a nonabelian group is allowed. We first consider disconnected Haar graphs. We begin with more permutation group terms.

Definition 2.1. Let X be a set and suppose $K \leq S_X$ is a transitive group which has a block system $\mathcal{C}$. Then K has an *induced action on* $\mathcal{C}$, denoted $K/\mathcal{C}$. Namely, for $k \in K$, define $k/\mathcal{C}\colon \mathcal{C} \mapsto \mathcal{C}$ by $k/\mathcal{C}(C) = C'$ if and only if $k(C) = C'$, and set $K/\mathcal{C} = \{k/\mathcal{C} : k \in K\}$. We also define the *fixer of* $\mathcal{C}$ *in* K, denoted $\mathrm{fix}_K(\mathcal{C})$, to be $\{k \in K : k/\mathcal{C} = 1\}$. That is, $\mathrm{fix}_K(\mathcal{C})$ is the subgroup of K which fixes each block of $\mathcal{C}$ set-wise, and is the kernel of the induced action of K on $\mathcal{C}$.

We observe that for an abelian group A, $H = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A,S))$ has $\mathcal{B}$ as a block system. Here $H/\mathcal{B} \cong \mathbb{Z}_2$ and $\mathrm{fix}_H(\mathcal{B}) = \widehat{K}$, where $K = \mathrm{Aut}(\mathrm{Cay}(A,S))$. We shall also need definitions of wreath products of digraphs and groups.

Definition 2.2. Let Γ_1 and Γ_2 be digraphs. The *wreath product of* Γ_1 *and* Γ_2, denoted $\Gamma_1 \wr \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and arcs $((u,v),(u,v'))$ for $u \in V(\Gamma_1)$ and $(v,v') \in A(\Gamma_2)$ or $((u,v),(u',v'))$ where $(u,u') \in A(\Gamma_1)$ and $v,v' \in V(\Gamma_2)$.

Definition 2.3. Let $G \leq S_X$ and $H \leq S_Y$. Define the *wreath product of* G *and* H, denoted $G \wr H$, to be the set of all permutations of $X \times Y$ of the form $(x,y) \mapsto (g(x), h_x(y))$, where $g \in G$ and each $h_x \in H$.

It is not hard to show that for vertex-transitive digraphs, $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma_1 \wr \Gamma_2)$. See [12] for more information regarding wreath products.

Let $\Gamma = \mathrm{Haar}(A,S)$ be connected, so by Lemma 1.10 $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$. Then $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ has induced actions on B_0 and B_1. Let $f \in F$ with $f(i,j) = (i, \gamma_i(j))$. The induced action of F on B_0 is given by $f \cdot (0,j) = (0, \gamma_0(j))$, and on B_1 by $f * (1,j) = (1, \gamma_1(j))$. We will be considering these induced actions frequently, and will abuse notation by considering the induced action of F on B_0 as an action simply on A, in which case $f \cdot (0,j) = \gamma_0(j)$ (i.e. we just delete the first coordinate if it is clear from context), and similarly for the induced action of F on B_1: $f * (1,j) = \gamma_1(j)$. We will also not write the actions formally, and not use the $\cdot$ and $*$ notation, but instead write F^{B_i}, $i \in \mathbb{Z}_2$ or analogous notation for a subgroup of F. For example, we simply say that A_L is contained in the image of the actions of F on B_0 and B_1 for the induced action of $\widehat{A_L}$ on B_0 and B_1. Or more simply, $\widehat{A_L}^{B_i} = A_L$ as with the above abuse of notation, $\hat{a}_L \cdot (0,j) = a_L(0)$ and $\hat{a}_L(1,j) = a_L(0)$ for every $a \in A$.

Definition 2.4. Let G be a group. We will use the notation $\bar{g}_R$ for the permutations of $\mathbb{Z}_2 \times G$ given by $\bar{g}_R(0,j) = (0,j)$ and $\bar{g}_R(1,j) = (1,jg)$ in what follows.

It is shown in the proof of [32, Lemma 2.2] that for any group G, $S \subseteq G$, and $g \in G$, $\bar{g}_R(\mathrm{Haar}(G,S)) \cong \mathrm{Haar}(G,Sg) \cong \mathrm{Haar}(G,S)$.

Theorem 2.5. *Let A be an abelian group, and $S \subseteq A$. If $\Gamma = \mathrm{Haar}(A,S)$ is disconnected, then there is $a \in A$ and $H < A$ such that $\Gamma = \bar{a}_R^{-1}(\mathrm{Haar}(A, a+S))$ and $\mathrm{Aut}(\Gamma) \cong \bar{a}_R^{-1}(S_{A/H} \wr \mathrm{Aut}(\mathrm{Haar}(H, a+S)))\bar{a}_R$.*

Proof. If Γ is disconnected, then by Lemma 1.8 $S \subseteq -a + H$ for some $a \in A$ and $H < A$. Set $H = \langle S + (-S) \rangle < A$ (this is written additively as A is abelian). Then $\mathrm{Haar}(H, a+S)$ is a connected graph which is a component of $\mathrm{Haar}(A, a+S) = \bar{a}_R(\Gamma)$. Thus $\Gamma = \bar{a}_R^{-1}(\mathrm{Haar}(A, a+S))$. Also, there are $[A:H]$ components of $\mathrm{Haar}(A, a+S)$, and $\mathrm{Haar}(A, a+S) \cong \bar{K}_{A/H} \wr \mathrm{Haar}(H, a+S)$, where $\bar{K}_{A/H}$ is the complement of the complete graph with vertices the cosets of H in A. Then $\mathrm{Aut}(\mathrm{Haar}(A, a+S)) \cong S_{A/H} \wr \mathrm{Aut}(\mathrm{Haar}(H, a+S))$, and $\mathrm{Aut}(\mathrm{Haar}(A,S)) = \bar{a}_R^{-1}\mathrm{Aut}(\mathrm{Haar}(A, a+S))\bar{a}_R$ as $\bar{a}_R^{-1}(\mathrm{Haar}(A, a+S)) = \mathrm{Haar}(A,S)$. □

Another way in which the automorphism group of a Haar graph Γ of an abelian group is bigger than expected is if the graph is connected and the action of $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is not faithful in its action on a block of $\mathcal{B}$. The next result shows that this situation is easy to spot by examining the connection set of Γ.

Lemma 2.6. *Let G be a group, and $\Gamma = \mathrm{Haar}(G, S)$ be connected and vertex-transitive. Then $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ in its action on B_0 or B_1 is unfaithful if and only if S is a union of left cosets of some nontrivial subgroup of G.*

Proof. Set $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$. First observe that as Γ is connected, $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$. As $\mathrm{Aut}(\Gamma)$ is transitive, the action of F on B_0 is unfaithful if and only if the action of F on B_1 is unfaithful.

Suppose S is a union of left cosets of some subgroup $H \neq 1$ of A. Clearly for $h \in H$, the map $\bar{h}_R$ is an automorphism of Γ as it fixes left cosets of H. Then $\bar{h}_R$ is contained in the kernel of the action of F on B_0.

Suppose F in its action on B_0 is unfaithful with $K \neq 1$ the kernel of the action. Then $K \triangleleft F$ and so the orbits of K form a block system $\mathcal{C}$ of F^{B_1}. As F^{B_1} contains G_L acting regularly, the blocks of $\mathcal{C}$ are left cosets of some subgroup H of G. As K stabilizes $(0, 0)$, it is clear that if $(0, 0)$ is adjacent to some element $(1, gh)$ for $g \in G$ and $h \in H$, then $(0, 0)$ is adjacent to $(1, gh)$ for every $h \in H$. So S is a union of left cosets of H. As K is nontrivial, $H \neq 1$. □

We next explicitly determine the automorphism groups of such Haar graphs of abelian groups. We will need some additional notation.

Definition 2.7. Let K be a transitive group with block system $\mathcal{C}$. If $\mathcal{D}$ is also a block system of K and each block of $\mathcal{D}$ is a union of blocks of $\mathcal{C}$, then we write $D/\mathcal{C}$ for the set of blocks of $\mathcal{C}$ whose union is D. Let Γ be a vertex-transitive digraph with $K \leq \mathrm{Aut}(\Gamma)$. Define the *block quotient digraph of* Γ *with respect to* $\mathcal{C}$, denoted $\Gamma/\mathcal{C}$, to be the digraph with vertex set $\mathcal{C}$ and arc set $\{(C, C') : C \neq C' \in \mathcal{C} \text{ and } (u, v) \in A(\Gamma) \text{ for some } u \in C \text{ and } v \in C'\}$.

Theorem 2.8. *Let A be an abelian group, $S \subseteq A$, and $\Gamma = \mathrm{Haar}(A, S)$. If Γ is connected, and the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is unfaithful on B_1, then there exists a subgroup $1 < H \leq A$ such that $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_\beta$ where $\beta = |H|$ and S is interpreted as a set of cosets of H in A. Also, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/H, S)) \wr S_\beta$, and denoting the natural bipartition of $\mathrm{Haar}(A/H, S)$ by $\mathcal{D}$, the action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(A/H,S))}(\mathcal{D})$ on $D \in \mathcal{D}$ is faithful.*

Proof. By Lemma 2.6 S is a union of cosets of some subgroup $1 < H \leq A$. We choose H to be maximal with respect to this property - clearly such a maximal subgroup exists. Let $\mathcal{C}$ be the block system of $\langle \iota, \widehat{A_L} \rangle$ formed by the subgroup $\langle \hat{h}_L : h \in H \rangle$. Notice that $\mathcal{C} = \{\{(i, a + h) : h \in H\} : i \in \mathbb{Z}_2, a \in A\}$.

Claim 1: In Γ, for any two distinct blocks $C, C' \in \mathcal{C}$, either every vertex of C is adjacent to every vertex of C' or no vertex of C is adjacent to any vertex of C'.

Clearly if $C, C' \subseteq B_0$ or B_1 then this statement is true as there are no edges with both endpoints inside any B_i, $i = 0, 1$. If, say, $C \subset B_0$ and $C' \subset B_1$ and some vertex $(0, v)$ of C is adjacent to some vertex $(1, v')$ of C', then by definition $v' = v + s$ for some $s \in S$. Then $C = \{(0, w) : w \in v + H\}$ and $C' = \{(1, w') : w' \in v + s + H\}$. Let $(0, w) \in C$ and $(1, w') \in C'$ with $w = v + h$ and $w' = v + s + h'$, $h, h' \in H$. Then

$$(0, w)(1, w') = (0, v + h)(1, v + s + h') = (0, v + h)(1 + v + h + s + (h' - h)) \in E(\Gamma),$$

as $s + (h' - h) \in s + H \subseteq S$, and the claim follows.

It is now straightforward to verify using the definition of the wreath product that $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_\beta$.

Claim 2: $\mathrm{Haar}(A/H,S) \not\cong \Gamma_2 \wr \bar{K}_r$ with $r \geq 2$, where Γ_2 is a connected vertex-transitive graph.

Suppose not, so $\mathrm{Haar}(A/H,S) \cong \Gamma_2 \wr \bar{K}_r$ for some vertex-transitive graph Γ_2 and $r \geq 2$ is maximum. By [12, Theorem 5.7] and the maximality of r, we have $\mathrm{Aut}(\mathrm{Haar}(A,S)) \cong \mathrm{Aut}(\Gamma_2) \wr S_{r\beta}$, so $\mathrm{Aut}(\Gamma)$ has a block system $\mathcal{E}$ with blocks of size $r\beta$ such that $\mathrm{Aut}(\Gamma)/\mathcal{E} = \Gamma_2$. Now, if Γ_2 has an odd cycle, then $\Gamma_2 \wr \bar{K}_r$ has an odd cycle. However, $\Gamma_2 \wr \bar{K}_r$ is a Haar graph and bipartite. So Γ_2 is bipartite, with bipartition $\{F_0, F_1\}$. Observe that both F_0 and F_1 are a set of cosets of H in A, and so $\cup_{a+H\in F_0}(a+H)$ and $\cup_{a+H\in F_1}(a+H)$ are subsets of A.

We now show that $\{\cup_{a+H\in F_0}(a+H), \cup_{a+H\in F_1}(a+H)\} = \mathcal{B}$. It suffices to show that if $E, E' \in \mathcal{E}$ and $E, E' \in F_0$, then $E \cup E' \subseteq B_j$, for some fixed $j = 0, 1$. As Γ is connected, Γ_2 is connected. Thus there is an EE' path $E_0, \ldots, E_t$ in Γ_2. Then $E_i \subseteq F_j$ if i is even and $E_i \subseteq F_{j+1}$ if i is odd. Similarly, the E_i with even subscripts are all contained in B_j while the E_i with odd subscripts are contained in B_{j+1}. As both E and E' are contained in F_0, t is even and $E \cup E' \subseteq B_j$, so $\{\cup_{a+H\in F_0}(a+H), \cup_{a+H\in F_1}(a+H)\} = \mathcal{B}$.

As ι interchanges the blocks of $\mathcal{B}$, ι also interchanges the union of the bipartition sets $\cup F_0$ and $\cup F_1$ of Γ_2. Then $\iota/\mathcal{E}$ is well defined and is not 1 i.e. ι permutes the blocks of $\mathcal{E}$ nontrivially as well. Finally, by [8, Theorem 1.5A] there exists a subgroup $K \leq \langle \iota, \widehat{A_L} \rangle$ such that the block of $\mathcal{E}$ that contains $(0,0)$ is the orbit of K that contains $(0,0)$. As $\iota/\mathcal{E} \neq 1$, $K \leq \widehat{A_L}$ is of order $r\beta$. This implies the blocks of $\mathcal{E}$ are orbits of K, and $K = \{\hat{\ell} : \ell \in L \leq A\}$. Also, it should be clear that $H \leq L$ as every block of $\mathcal{C}$ is contained in a block of $\mathcal{E}$. As $r \geq 2$, $H < L$. But this implies that S is a union of cosets of L, contradicting the maximality of H. The claim is established.

By [12, Theorem 5.7] and Claim 2, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/H,S)) \wr \bar{K}_\beta$. That the action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(A/H,S))}(\mathcal{D})$ on $D \in \mathcal{D}$ is faithful follows from Claim 2 and Theorem 2.6. □

We now turn to Haar graphs Γ of abelian groups A that are connected and whose connection set is not a union of cosets of some subgroup of A. These two hypotheses imply that $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$ and that the actions of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_0 and B_1 are faithful. To investigate further, we will need the terminology and some results concerning inequivalent permutation representations.

Definition 2.9. A *permutation representation* of a group K is a homomorphism $\phi\colon K \to S_n$ for some n.

Definition 2.10. Let K be a group, and X and Y sets. Let $\alpha\colon K \mapsto S_X$ and $\omega\colon K \mapsto S_Y$ be permutation representations of K. We say α and ω are *equivalent permutation representations* of K if there exists a bijection $\lambda\colon X \mapsto Y$ such that $\lambda(\alpha(k)(x)) = \omega(k)(\lambda(x))$ for all $x \in X$ and $k \in K$. In this case, we will say that $\alpha(K)$ and $\omega(K)$ are *permutation equivalent.*

The examples of $\mathrm{Cay}(\mathbb{Z}_7, \{1,2,4\})$ and its corresponding Haar graph, the Heawood graph, provide the next way in which the automorphism group of a Haar graph of an abelian group can be larger than its expected automorphism group. Namely, the full automorphism group of the Heawood graph is $\mathbb{Z}_2 \ltimes \mathrm{PGL}(3,2)$, and $\mathrm{PGL}(3,2)$ acts permutation inequivalently on the two blocks B_0 and B_1 of $\mathcal{B}$. It turns out that this is not a coincidence, as we will show. Before turning to that result, we prove a preliminary result which will be crucial.

It does not depend upon A being abelian or even a group, so we will use the symbol X in place of A or G. We keep the notation $B_i = \{(i,x) : x \in X\}$, $i = 0,1$.

Lemma 2.11. *Let X be a set, and let $F \leq S_{\mathbb{Z}_2\times X}$ have B_0 and B_1 as orbits. Additionally, assume that the actions of F on B_0 and B_1 are faithful and the action of F on B_0 is permutation equivalent to the action of F on B_1. Then there exists $\bar{\epsilon} \in S_{\mathbb{Z}_2\times X}$ such that $\bar{\epsilon}(B_0) = B_0$, $\bar{\epsilon}^{B_0} = 1$ and every element of $\bar{\epsilon}^{-1}F\bar{\epsilon}$ has the form $(i,j) \mapsto (i,g(j))$, where $g \in F^{B_0}$.*

Proof. Let $\alpha\colon F \mapsto F^{B_0}$ and $\omega\colon F \mapsto F^{B_1}$ be permutation representations of F^{B_0} and F^{B_1}, respectively. As F is faithful on B_0 and B_1, both α and ω are faithful permutation representations of F. As the action of F on B_0 is permutation equivalent to the action of F on B_1, there exists a bijection $\lambda\colon B_0 \mapsto B_1$ such that $\lambda(\alpha(g)(0,j)) = \omega(g)(\lambda(0,j))$ for every $j \in X$ and $g \in F$. Let $\beta\colon B_0 \mapsto B_1$ be defined by $\beta(0,j) = (1,j)$ for every $j \in X$, so that β is a bijection. Set $\epsilon = \lambda\beta^{-1}$ so $\epsilon\beta = \lambda$ and $\epsilon \in S_{B_1}$. Substituting $\epsilon\beta$ for λ we have $\epsilon\beta(\alpha(g)(0,j)) = \omega(g)(\epsilon\beta(0,j))$ or equivalently, $\beta(\alpha(g)(0,j)) = \epsilon^{-1}\omega(g)\epsilon(\beta(0,j))$ for every $j \in X$. As ϵ is a bijection and ω a faithful permutation representation of F, it is straightforward to verify that the map $\delta\colon F \mapsto S_{B_1}$ given by $\delta(g) = \epsilon^{-1}\omega(g)\epsilon$ is a faithful permutation representation of F.

Now, as $\alpha(g)(0,j) \in B_0$ and similarly, $\delta(g)(1,j) \in B_1$ for every $j \in X$, $\alpha(g)$ and $\delta(g)$ induce permutations $\bar{\alpha}(g)$ and $\bar{\delta}(g)$ in S_X defined by $\alpha(g)(0,j) = (0,\bar{\alpha}(g)(j))$ and $\delta(g)(1,j) = (1,\bar{\delta}(g)(j))$ for all $j \in X$. Then

$$\beta(\alpha(g)(0,j)) = \beta(0,\bar{\alpha}(g)(j)) = (1,\bar{\alpha}(g)(j))$$

and

$$\epsilon^{-1}\omega(g)\epsilon(\beta(0,j)) = \delta(g)(1,j) = (1,\bar{\delta}(g)(j))$$

for all $j \in X$. As $\beta(\alpha(g)(0,j)) = \epsilon^{-1}\omega(g)\epsilon(\beta(0,j))$ for every $j \in X$, we see that $\bar{\alpha}(g)(j) = \bar{\delta}(g)(j)$ for all $j \in X$, and so $\bar{\alpha}(g) = \bar{\delta}(g)$ for all $g \in F$. Define $\bar{\epsilon}\colon \mathbb{Z}_2 \times X \mapsto \mathbb{Z}_2 \times X$ by $\bar{\epsilon}(0,j) = (0,j)$ and $\bar{\epsilon}(1,j) = \epsilon(1,j)$. Let $g \in F$. Then

$$\bar{\epsilon}^{-1}g\bar{\epsilon}(0,j) = \alpha(g)(0,j) = (0,\bar{\alpha}(g)(j))$$

and

$$\bar{\epsilon}^{-1}g\bar{\epsilon}(1,j) = \epsilon^{-1}\omega(g)\epsilon(1,j) = (1,\bar{\delta}(g)(j)) = (1,\bar{\alpha}(g)(j)),$$

and the result follows. □

Theorem 2.12. *Let G be a group, $S \subseteq G$, $\Gamma = \mathrm{Haar}(G,S)$, and let $F \leq \mathrm{Aut}(\Gamma)$ be the largest subgroup of $\mathrm{Aut}(\Gamma)$ that fixes B_0 and B_1 set-wise. Suppose F satisfies the following conditions:*

1. *F acts faithfully on both B_0 and B_1, and*
2. *the action of F on B_0 is permutation equivalent to the action of F on B_1.*

Let L be the group of all elements of $S_{\mathbb{Z}_2\times G}$ of the form $(i,j) \mapsto (i,\ell(j))$ where $\ell \in F^{B_0}$, and $g \in G$ such that $\mathrm{Stab}_F(1,g) = \mathrm{Stab}_F(0,1_G)$. Then $L = \bar{g}_R^{-1}F\bar{g}_R$.

Proof. Clearly every element of F can be written as $(i,j)\mapsto(i,\ell_i(j))$, where $\ell_0,\ell_1\in S_G$. By Lemma 2.11 there exists $\bar{\epsilon}\in S_{\mathbb{Z}_2\times G}$ such that $\bar{\epsilon}^{B_0}=1$ and every element of $\bar{\epsilon}^{-1}F\bar{\epsilon}$ has the form $(i,j)\mapsto(i,\ell(j))$, where $\ell\in F^{B_0}$. Let $\epsilon\in S_G$ such that $\bar{\epsilon}(1,j)=(1,\epsilon(j))$. Define $g_R\colon G\to G$ by $g_R(x)=xg$, and set $G_R=\{g_R : g\in G\}$. We next show that $\epsilon\in G_R$.

Of course, $\widehat{G_L}\leq F$, and $\widehat{G_L}$ has the form $(i,j)\mapsto(i,\ell(j))$, where $\ell\in\widehat{G_L}^{B_0}$. As $\bar{\epsilon}^{B_0}=1$, we have $\bar{\epsilon}^{-1}\widehat{g_L}\bar{\epsilon}=\widehat{g_L}$ for every $g\in G$. Hence $\bar{\epsilon}$ centralizes $\widehat{G_L}$, and so ϵ centralizes G_L as $\bar{\epsilon}^{B_0}=1$. As the centralizer of G_L in S_G is G_R (this well known fact can be deduced from [8, Theorem 4.2A(ii)]), we have $\epsilon\in G_R$. As $\epsilon\in G_R$, there exists $g\in G$ such that $L=\bar{g}_R^{-1}F\bar{g}_R$. Finally, as $F=\bar{g}_RL\bar{g}_R^{-1}$ and $\mathrm{Stab}_L(0,1_G)=\mathrm{Stab}_L(1,1_G)$, $F=\bar{g}_RL\bar{g}_R^{-1}$ stabilizes $(1,g)$. □

Corollary 2.13. *Let A be an abelian group with $S\subseteq A$ such that $\Gamma=\mathrm{Haar}(A,S)$ is connected, and consequently $\mathrm{Aut}(\mathrm{Haar}(A,S))$ has $\mathcal{B}$ as a block system. Set $F=\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$. Then one of the following is true:*

(1) *the induced action of F on B_1 is not faithful,*

(2) *the induced action of F on B_0 is permutation inequivalent to the induced action of F on B_1, or*

(3) $\mathrm{Aut}(\mathrm{Haar}(A,S))=\bar{a}_R^{-1}[\mathbb{Z}_2\ltimes\mathrm{Aut}(\mathrm{Cay}(A,a+S))]\bar{a}_R$ *for some $a\in A$.*

Proof. We may assume without loss of generality that the action of F on B_1 is faithful. As $\mathrm{Aut}(\Gamma)$ is transitive as A is abelian, we have F is faithful on B_0. With that assumption made, we may then assume without loss of generality that the actions of F on B_0 and B_1 are permutation equivalent. Set $L=\{(i,j)\mapsto(i,g(j)) : g\in F^{B_0}\}$ so $\mathrm{Stab}_L(0,0)=\mathrm{Stab}_L(1,0)$. Applying Theorem 2.12, there is $a\in A$ such that $L=\bar{a}_RF\bar{a}_R^{-1}$. By [32, Lemma 2.2], $\bar{a}_R(\Gamma)=\mathrm{Haar}(A,a+S)$. By Lemma 1.4, $\mathrm{Aut}(\mathrm{Cay}(A,a+S))\cong L$, so $\mathrm{Aut}(\mathrm{Haar}(A,a+S))=\mathbb{Z}_2\ltimes\mathrm{Aut}(\mathrm{Cay}(A,-a+S)$. The result follows as $\mathrm{Aut}(\Gamma)=\bar{a}_R^{-1}\mathrm{Aut}(\mathrm{Haar}(A,a+S))\bar{a}_R$. □

Remark 2.14. Suppose $\mathrm{Haar}(A,S)$ is connected and the induced action of F on B_1 is faithful. If $a,b\in A$ with $a\neq b$, then $\bar{a}_R(\mathrm{Haar}(A,S))\neq\bar{b}_R(\mathrm{Haar}(A,S))$ as otherwise $\bar{b}_R^{-1}\bar{a}_R\in\mathrm{Aut}(\mathrm{Haar}(A,S))$ and the induced action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Cay}(A,S))}(\mathcal{B})$ is not faithful on B_0.

Remark 2.15. While $\mathrm{Haar}(A,S)\cong\mathrm{Haar}(A,a+S)$ for every $a\in A$, $\mathrm{Cay}(A,S)$ is not necessarily isomorphic to $\mathrm{Cay}(A,a+S)$. Indeed, let n be an odd positive integer, $A=\mathbb{Z}_n$, $S=\{\pm1\}$, and $a=1$. Then $\mathrm{Cay}(A,S)$ is a cycle, while $\mathrm{Cay}(A,a+S)=\mathrm{Cay}(\mathbb{Z}_n,\{0,2\})$ is a directed cycle together with a loop at every vertex.

The following result is a combination of Theorems 2.5 and 2.8, and Corollary 2.13, and summarizes the possibilities for $\mathrm{Aut}(\mathrm{Haar}(A,S))$ with A abelian.

Corollary 2.16. *Let A be an abelian group, $S\subseteq A$, and $\Gamma=\mathrm{Haar}(A,S)$. Then one of the following is true:*

(1) *Γ is disconnected, then there is $a\in A$ and $H<A$ such that $\Gamma=\bar{a}_R^{-1}(\mathrm{Haar}(A,a+S))$ and $\mathrm{Aut}(\Gamma)\cong\bar{a}_R^{-1}(S_{A/H}\wr\mathrm{Aut}(\mathrm{Haar}(H,a+S)))\bar{a}_R$,*

(2) *Γ is connected, and the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is unfaithful on B_1. There exists a subgroup $1 < H \leq A$ such that $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_\beta$ where $\beta = |H|$ and S is interpreted as a set of cosets of H in A, and $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/H, S)) \wr S_\beta$. Additionally, denoting the natural bipartition of $\mathrm{Haar}(A/H, S)$ by $\mathcal{D}$, the action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(A/H,S))}(\mathcal{D})$ on $D \in \mathcal{D}$ is faithful,*

(3) *$\mathrm{Aut}(\Gamma) \cong \bar{a}_R^{-1}\mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A, a + S))\bar{a}_R$ for some $a \in A$, or*

(4) *the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_1 is faithful but the actions on B_0 and B_1 are not equivalent permutation groups.*

We now give a group theoretic description of the graphs in (4) of Corollary 2.16.

Theorem 2.17. *Let A be an abelian group and $S \subseteq A$ such that $\Gamma = \mathrm{Haar}(A, S)$ is connected and S is not a union of cosets of some subgroup of A. Let $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$, $H = F^{B_0}$, and $L = \mathrm{Stab}_H(b)$, where $b \in B_0$. If the actions of F on B_0 and B_1 are inequivalent, then there exists $\sigma \in \mathrm{Aut}(H)$ which is an involution and maps L to a subgroup R of H which is not conjugate in H to L.*

Proof. The hypothesis implies that the action of F on B_0 and B_1 is faithful by Lemma 2.6. As $\iota \in \mathrm{Aut}(\Gamma)$, conjugation of F by ι induces automorphisms σ and δ which map F^{B_0} to F^{B_1}, and F^{B_1} to F^{B_0}, so we may view F^{B_1} as being contained in H as the image of δ. As ι has order 2, so does σ. Also, as the action of F on B_0 and B_1 are not equivalent, by [8, Lemma 1.6B] $\sigma(L) = R$ is not the stabilizer of a point in B_0, so R is not conjugate in H to L. □

We next give a partial converse of the previous result. We begin with a more general construction of bipartite graphs than that of the Haar graphs.

Definition 2.18. Let G be a group, let $L, R \leq G$, and let D be a union of double cosets of R and L in G, that is $D = \cup_i R d_i L$. Define a bipartite graph $\Gamma = B(G, L, R; D)$ with bipartition $V(\Gamma) = G/L \cup G/R$ (here G/L and G/R are the sets of left cosets of L and R in G) and edge set $E(\Gamma) = \{\{gL, gdR\} : g \in G, d \in D\}$. We call Γ the *bi-coset graph* of G with respect to L, R and D.

The bi-coset graphs $B(G, L, R; D)$ were introduced in [14], where they were shown to be well-defined bipartite graphs whose automorphism group contain a natural subgroup isomorphic to G. Haar graphs are bi-coset graphs with $L = R = 1$. In [14, Lemma 2.6] a sufficient condition to ensure that a bi-coset graph is vertex-transitive is given, and then several more specific circumstances were given where this condition was satisfied. We will need another such more specific circumstance.

Lemma 2.19. *Let G be a group with $L \leq G$ and $\sigma \in \mathrm{Aut}(G)$ an involution such that $\sigma(L) = R$. The bi-coset graph $\Gamma = B(G, L, R; S)$ is vertex-transitive with $\sigma \in \mathrm{Aut}(\Gamma)$ provided $S = LDR$ with $\sigma(D) = D^{-1} = D$.*

Proof. By [14, Lemma 2.6] we need only show that $S^{-1} = \sigma(S)$. Then

$$\sigma(S) = \sigma(LDR) = \sigma(L)\sigma(D)\sigma(R) = RD^{-1}L = R^{-1}D^{-1}L^{-1} = (LDR)^{-1} = S^{-1}.$$

□

Definition 2.20. Let K be a group and $H \leq K$. The largest normal subgroup of K contained in H is called the *core* of H in K. If the core of H in K is trivial, then we say H is *core-free* in K.

It is well known that the left action of a group K on a subgroup $H \leq K$ is faithful if and only if H is core-free in K.

Theorem 2.21. *Let A be an abelian group. Suppose $A_L \leq K \leq S_A$ with $L = \mathrm{Stab}_K(a)$, $a \in A$, $\sigma \in \mathrm{Aut}(K)$ an involution, and $R = \sigma(L) \leq K$ which is not conjugate in K to L and satisfies $R \cap A_L = 1$. Then the bi-coset graph $\Gamma = B(K, L, R; LR) \cong \mathrm{Haar}(A, S)$ for some $S \subseteq A$, and there is a subgroup $H \leq \mathrm{Aut}(\Gamma)$ such that H fixes $\mathcal{B}$, $H \cong K$, the action of H on B_0 is faithful, and the action of H on $B_0 = K/L$ is inequivalent to the action of H on $B_1 = K/R$.*

Proof. By [14, Lemma 2.3] the left multiplication action of K on $V(\Gamma)$ is contained in $\mathrm{Aut}(\Gamma)$, fixes $\mathcal{B}$, and is transitive on B_0 and B_1. Denote by H the corresponding subgroup of $\mathrm{Aut}(\Gamma)$. Then H^{B_0} is permutation equivalent to $K \leq S_A$ as their stabilizers are the same, namely L. It is clear that the left multiplication action of K on B_0 is faithful as L is core-free in K as L is the stabilizer of a point in $K \leq S_A$. Similarly, as $\sigma \in \mathrm{Aut}(K)$ and $\sigma(L) = R$, R is core-free in K (as it is the isomorphic image of a core-free subgroup of K) so the left multiplication action of K on K/R is faithful as well. Note that as left multiplication of R by an element of K fixes R if and only if it is contained in R, $\mathrm{Stab}_{H^{B_1}}(R) = R$. We see that the action of H on B_1 is faithful, and the action of H on B_0 is inequivalent to the action of H on B_1. Setting $D = \{1_K\}$ we see by Lemma 2.19 that $\sigma \in \mathrm{Aut}(\Gamma)$ as $D^{-1} = D$, and so Γ is vertex-transitive.

To see Γ is isomorphic to a Haar graph of A, let $\tilde{A} \leq \mathrm{Aut}(\Gamma)$ be the subgroup of $\mathrm{Aut}(\Gamma)$ induced by left multiplication by elements of A_L in K. Of course, $\tilde{A}$ is transitive on B_0 as $A_L L = K$. Suppose $\tilde{a}_1 R = \tilde{a}_2 R$ for some $\tilde{a}_1, \tilde{a}_2 \in \tilde{A}$. This occurs if and only if there are $a_1, a_2 \in A$ such that $(a_1)_L R = (a_2)_L R$, which occurs if and only if $(a_2^{-1} a_1)_L R = R$. As $A_L \cap R = 1$, we see $(a_1)_L = (a_2)_L$. Then $\tilde{A}$ is faithful on B_1. As $\tilde{A} \cap R = 1$, $\tilde{A}^{B_1}$ is semiregular. By [42, Proposition 4.1] $\tilde{A}$ has one orbit of size $|A|$, and so is transitive. The result follows by [14, Lemma 2.5]. □

Theorem 2.21 is only a partial converse to Theorem 2.17 as it is possible that the graph constructed in the result (or any graph constructed in a similar way) will have an automorphism group larger than the group K in the statement. For example, if $LR = K$ is transitive, the graph $B(K, L, R; LR)$ constructed will be a complete bipartite graph with automorphism group $K_2 \wr S_n$ with $n = |A|$. The next example shows that this can occur.

Example 2.22. Let $K = S_6$, $L = \mathrm{Stab}_{S_6}(x) \cong S_5$ and σ be the outer automorphism of S_6 of order 2. Set $R = \sigma(L)$. Then $B(K, L, R; LR) \cong K_{6,6}$.

Proof. We show that L is transitive on R. That is, we will show that $\sigma(S_5)$ is transitive, where we view S_5 as the stabilizer of the point 0 in the set $\mathbb{Z}_6$ on which we view S_6 permuting. Now, the three cycle $(3, 4, 5)$ is mapped by σ to a product of two disjoint 3-cycles (see for example [20]). So $\sigma(3, 4, 5)$ has two orbits of size 3. This implies that $\sigma(S_5)$ is transitive, or $\sigma(S_5)$ has two orbits of size 3, in which case its maximum order is $|S_3| \cdot |S_3| = 36$. But S_5 has order 120, so $\sigma(S_5)$ is transitive and the result follows. □

Note that the graph $B(K, L, R; LR)$ as constructed in the previous example does not satisfy the hypothesis of Theorem 2.21 as $R \cap A_L \neq 1$.

Definition 2.23. Let $n \geq 3$ be an integer, q a prime-power, and set

$$L = \mathrm{PG}(n-1,q) = \{r\vec{v} : r \in \mathbb{F}_q \text{ and } \vec{v} \text{ is in the vector space } \mathbb{F}_q^n\},$$

the set of lines (or projective points) in $\mathbb{F}_q^n$. Let H be the set of hyperplanes in $\mathbb{F}_q^n$. Define a graph $B(\mathrm{PG}(n-1,q))$ to have vertex set $L \cup H$ and edges $\{L, H\}$ where $L \subseteq H$.

Note that the 'B' used in defining the above graphs has an entirely different meaning from the 'B' used in the previous example. No confusion can arise though as the parameters of the families are completely different. Also, it is known that $B(2,2)$ is isomorphic to the Heawood graph. Our next example, which is also fairly well-known, shows it is a member of a much larger family of graphs with many of the same properties.

Example 2.24. The graph $B(\mathrm{PG}(n-1,q))$ is isomorphic to a Haar graph of the cyclic group of order $(q^n-1)/(q-1)$. Additionally, if $n \geq 3$, $F = \mathrm{fix}_{\mathrm{Aut}(B(\mathrm{PG}(n-1,q)))}(\mathcal{B})$ in its induced actions on B_0 and B_1 are inequivalent representations of $\mathrm{PGL}(n,q)$ with the actions being on points and hyperplanes.

Proof. It is well-known that $\mathrm{PGL}(n,q)$ contains a regular cyclic subgroup of order $(q^n-1)/(q-1)$ - see for example [24, Theorem 3] or [29, Theorem 1.1]. It is easy to see that $F^{B_0} \cong F^{B_1}$ contains $\mathrm{PGL}(n,q)$ as a point contained in a hyperplane is mapped to a point contained in a hyperplane by an element of $\mathrm{PGL}(n,q)$. That F is not larger than $\mathrm{PGL}(n,q)$ basically follows from the Fundamental Theorem of Finite Geometry. By [18, Lemma 4.2] we see $B(\mathrm{PG}(n-1,q))$ is isomorphic to a Haar graph, and, as $n \geq 3$, points and hyperplanes have different dimensions. It is then not hard to see that the stabilizer in $\mathrm{PGL}(n,q)$ of a line does not stabilize any hyperplane. Thus the induced actions of F on B_0 and B_1 are inequivalent by [8, Lemma 1.6B]. □

3 Applications to arc-transitive graphs

The study of s-arc-transitive graphs was initiated in a celebrated paper by Tutte [40]. There has been strong and consistent interest in s-arc-transitive graphs for several decades. Perhaps the most important tool in this area is Praeger's Normal Quotient Lemma [36]. This lemma shows how to reduce an s-arc-transitive graph Γ to an s-arc-transitive quotient of Γ provided one can find $N \triangleleft \mathrm{Aut}(\Gamma)$ that has at least three orbits. If $\mathrm{Aut}(\Gamma)$ is quasiprimitive, then one can study such groups and graphs using the O'Nan-Scott Theorem [8, Theorem 4.1A] and Praeger's quasiprimitive counterpart [37]. So other techniques are necessary to deal with the case when Γ only has normal subgroups with exactly two orbits. In this case, if Γ is disconnected, then there is an obvious reduction to s-arc-transitive graphs of smaller order. If Γ is connected, then there are edges between the two orbits and so no edges inside the orbits. Hence Γ is bipartite, and if the subgroup N of $\mathrm{Aut}(\Gamma)$ fixing the parts of the bipartition set-wise contains a semiregular subgroup isomorphic to G, then Γ is a Haar graph of G by [14, Lemma 2.4]. Thus the study of Haar graphs is essential to the study of s-arc-transitive graphs.

Definition 3.1. Let $s \geq 1$, and Γ a digraph. An *s-arc* of Γ is a sequence $x_0, x_1, \ldots, x_s$ of vertices of Γ such that $(x_i, x_{i+1}) \in A(\Gamma)$, $0 \leq i \leq s-1$, and $x_i \neq x_{i+2}$, $0 \leq i \leq s-2$. The digraph Γ is *s-arc-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs of Γ.

As an arc-transitive graph without isolated vertices is vertex-transitive, we restrict our attention to vertex-transitive Haar graphs.

Definition 3.2. Let Γ be a digraph, and $s \geq 1$. A sequence of arcs $a_1, \ldots, a_s \in A(\Gamma)$ is an *alternating s-arc* if there exists vertices $x_0, \ldots, x_s \in V(\Gamma)$, $x_i \neq x_{i+2}$, and for $1 \leq m \leq s$ the arc $a_m = (x_{m-1}, x_m)$ if m is odd while $a_m = (x_m, x_{m-1})$ if m is even. An *alternating s-arc-transitive digraph* is a digraph whose automorphism group is transitive on the set of alternating s-arcs. The vertices $x_0, \ldots, x_s$ are the *vertex-sequence* of the alternating s-arc $a_1, \ldots, a_s$.

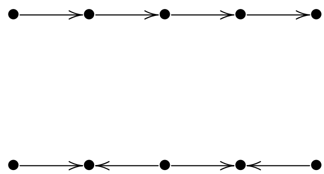

Figure 3: A 4-arc (top) versus an alternating 4-arc (bottom).

Clearly if $s = 1$, then an alternating s-arc is simply an s-arc. If $s \geq 2$ then an alternating s-arc can be obtained from an s-arc by reversing the direction of every other arc - see Figure 3. Now choose an s-arc, and fix an orientation of this s-arc. An s-arc-transitive *graph* is transitive on the set of all s-arcs in Γ with this fixed reorientation, so an s-arc-transitive graph is trivially alternating s-arc-transitive. The next result gives a relationship between alternating s-arc-transitive Cayley digraphs and s-arc-transitive Haar graphs.

Theorem 3.3. *Let G be a group, $S \subseteq G$, and $s \geq 1$. If $\mathrm{Cay}(G,S)$ is an alternating s-arc-transitive digraph and $\mathrm{Haar}(G,S)$ is vertex-transitive, then $\mathrm{Haar}(G,S)$ is s-arc-transitive. Conversely, if $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$ and $\mathrm{Haar}(G,S)$ is s-arc-transitive, then $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive.*

Proof. Suppose $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive and $\mathrm{Haar}(G,S)$ is vertex-transitive. Let $P_1 = (i, x_0), (i+1, x_1), \ldots, (i+s, x_s)$ and $P_2 = (j, y_0), (j+1, y_1), \ldots, (j+s, y_s)$ be two s-arcs in $\mathrm{Haar}(G,S)$. As $\mathrm{Aut}(\mathrm{Haar}(G,S))$ is transitive, replacing P_1 or P_2 by $\iota(P_1)$ or $\iota(P_2)$, for appropriate $\iota \in \mathrm{Aut}(\mathrm{Haar}(G,S))$, if necessary, we may assume without loss of generality that $i = j = 0$. Then there exist $t_1, \ldots, t_s \in S$ such that for $1 \leq m \leq s$

$$x_m = \begin{cases} x_0 t_1 t_2^{-1} \ldots t_{m-1}^{-1} t_m, & \text{if } m \text{ is odd,} \\ x_0 t_1 t_2^{-1} \ldots t_{m-1} t_m^{-1}, & \text{if } m \text{ is even.} \end{cases}$$

Similarly, there exist $u_1, \ldots, u_m \in S$ such that for $1 \leq m \leq s$

$$y_m = \begin{cases} y_0 u_1 u_2^{-1} \ldots u_{m-1}^{-1} u_m, & \text{if } m \text{ is odd,} \\ y_0 u_1 u_2^{-1} \ldots u_{m-1} u_m^{-1}, & \text{if } m \text{ is even.} \end{cases}$$

The arcs $a_m = (x_m, x_m t_m)$ if m is even and $a_m = (x_{m-1} t_m^{-1}, x_{m-1})$ if m is odd, $0 \leq m \leq s-1$, are then contained in $A(\mathrm{Cay}(G,S))$, and $Q_1 = a_0, \ldots, a_{s-1}$ is an alternating s-arc in $\mathrm{Cay}(G,S)$. Similarly, the arcs $b_m = (y_m, x_y u_m)$ if m is even and $b_m = (y_{m-1} u_m^{-1}, y_{m-1})$ if m is odd, $0 \leq m \leq s-1$, are then contained in $A(\mathrm{Cay}(G,S))$, and $Q_2 = b_0, \ldots, b_{s-1}$ is an alternating s-arc in $\mathrm{Cay}(G,S)$. As $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive, there exists $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$ such that $\gamma(Q_1) = Q_2$. Then $\hat{\gamma} \in \mathrm{Aut}(\mathrm{Haar}(G,S))$, and $\hat{\gamma}(P_1) = P_2$. The result follows.

Conversely, suppose $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$ and $\mathrm{Haar}(G,S)$ is s-arc-transitive. Let $P_1 = a_1,\ldots,a_s$ and $P_2 = b_1,\ldots,b_s$ be alternating s-arcs in $\mathrm{Cay}(G,S)$ with vertex-sequences $x_0,x_1,\ldots,x_s$ and $y_0,y_1,\ldots,y_s$, respectively. Then there exist $t_1,\ldots,t_s \in S$ such that for $1 \le m \le s$ we have

$$x_m = \begin{cases} x_0t_1t_2^{-1}\ldots t_{m-1}^{-1}t_m, & \text{if } m \text{ is odd,}\\ x_0t_1t_2^{-1}\ldots t_{m-1}t_m^{-1}, & \text{if } m \text{ is even.}\end{cases}$$

Similarly, there exist $u_1,\ldots,u_m \in S$ such that for $1 \le m \le s$

$$y_m = \begin{cases} y_0u_1u_2^{-1}\ldots u_{m-1}^{-1}u_m, & \text{if } m \text{ is odd,}\\ y_0u_1u_2^{-1}\ldots u_{m-1}u_m^{-1}, & \text{if } m \text{ is even.}\end{cases}$$

Corresponding to these two alternating s-arcs, there are two s-arcs $(0,x_0),(1,x_1),\ldots,(s,x_s)$ and $(0,y_0),(1,y_1),\ldots,(s,y_s)$ in $\mathrm{Haar}(G,S)$. As $\mathrm{Haar}(G,S)$ is s-arc-transitive, there exists $\delta \in \mathrm{Aut}(\mathrm{Haar}(G,S))$ such that

$$\delta((0,x_0),(1,x_1),\ldots,(s,x_s)) = (0,y_0),(1,y_1),\ldots,(s,y_s).$$

Then $\delta(0,x_0) = (0,y_0)$ so $\delta \in \mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(G,S))}(\mathcal{B})$. As $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$, $\delta = \hat{\gamma}$ for some $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$. Then $\gamma(x_i) = y_i$ and so $\gamma(P_1) = P_2$. So $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive. □

We next consider alternating s-arc-transitivity when $\mathrm{Cay}(G,S)$ contains arcs and edges. We will need a preliminary lemma.

Lemma 3.4. *Let Γ be a digraph such that every vertex has in- and out-degree at least two, and $s \ge 2$. If Γ is alternating s-arc-transitive, then Γ is alternating $(s-1)$-arc-transitive.*

Proof. Let $a_1,\ldots,a_{s-1}$ and $b_1,\ldots,b_{s-1}$ be alternating $(s-1)$-arcs in Γ with vertex-sequences $x_0,\ldots,x_{s-1}$ and $y_0,\ldots,y_{s-1}$. As x_{s-1} and y_{s-1} have in- and out-degree at least two, $a_1,\ldots,a_{s-1}$ and $b_1\ldots,b_{s-1}$ can be extended to alternating s-arcs $a_1,\ldots,a_s$ and $b_1,\ldots,b_s$. As Γ is alternating s-arc-transitive, there is $\gamma \in \mathrm{Aut}(\Gamma)$ such that

$$\gamma(a_1,\ldots,a_s) = b_1,\ldots,b_s.$$

So $\gamma(a_1,\ldots,a_{s-1}) = b_1,\ldots,b_{s-1}$ and Γ is alternating $(s-1)$-arc-transitive. □

Corollary 3.5. *Let G be a group, $S \subseteq G$ such that $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$. $\mathrm{Haar}(G,S)$ is s-arc-transitive if and only if*

(1) *$S = S^{-1}$ and $\mathrm{Cay}(G,S)$ is s-arc-transitive, or*

(2) *$S \cap S^{-1} = \emptyset$ and $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive.*

Proof. Suppose $\mathrm{Haar}(G,S)$ is s-arc-transitive. By Theorem 3.3, $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive. Also, $|S| \ge 2$ as $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$, so by Lemma 3.4 we have that $\mathrm{Cay}(G,S)$ is arc-transitive. This implies $\mathrm{Cay}(G,S)$ cannot contain both edges and arcs, so $S = S^{-1}$ or $S \cap S^{-1} = \emptyset$. If $S \cap S^{-1} = \emptyset$, then (2) follows. Otherwise, $S = S^{-1}$ so $\mathrm{Cay}(G,S)$ is a graph, and is alternating s-arc-transitive if and only if it is s-arc-transitive and (1) follows.

For the converse, we have already observed that an s-arc-transitive graph is alternating s-arc-transitive, and so the result holds if $S = S^{-1}$. If $S \cap S^{-1} = \emptyset$, then the result follows by Theorem 3.3. □

Definition 3.6. Let Γ be a digraph. we say that Γ is a *strict digraph* if for every arc $(a, b) \in A(\Gamma)$, the arc $(b, a) \notin A(\Gamma)$.

Note that if G is a group and $S \subseteq G$ such that $S \cap S^{-1} = \emptyset$, then $\mathrm{Cay}(G, S)$ is a strict digraph.

Definition 3.7. Let A be an abelian group. The group $\mathbb{Z}_2 \ltimes A \cong \langle \iota, \widehat{A_L} \rangle$ is a *generalized dihedral group*.

Definition 3.8. A transitive permutation group $G \leq S_n$ is *quasiprimitive* if every nontrivial normal subgroup of G is transitive.

We now characterize s-arc-transitive Cayley graphs of generalized dihedral groups with abelian normal subgroup of odd order.

Theorem 3.9. *Let $s \geq 2$ and Γ be an s-arc-transitive Cayley graph of a generalized dihedral group G with a normal abelian subgroup A of odd order n and index 2 in G. Then one of the following is true:*

(1) Γ *is disconnected,*

(2) $\mathrm{Aut}(\Gamma)$ *is quasiprimitive or primitive,*

(3) $\Gamma \cong K_{n,n}$*,*

(4) Γ *is isomorphic to a Haar graph corresponding to an s-arc-transitive Cayley graph of A,*

(5) Γ *is isomorphic to a Haar graph corresponding to an alternating s-arc-transitive Cayley strict digraph of A, or*

(6) Γ *is isomorphic to a Haar graph of A and its corresponding Cayley digraph need not be s-arc-transitive. In this case, $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$ and the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_1 is faithful and inequivalent to the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_0.*

Proof. We assume (1) - (5) do not hold, and show (6) holds. As $\mathrm{Aut}(\Gamma)$ is neither quasiprimitive or primitive, there exists $1 < N \triangleleft \mathrm{Aut}(\Gamma)$ that is not transitive. Let $\mathcal{C}$ be the nontrivial block system of $\mathrm{Aut}(\Gamma)$ formed by the orbits of N. As Γ is connected, there is some edge in Γ between C_1 and $C_2 \in \mathcal{C}$ with $C_1 \neq C_2$. As Γ is edge-transitive, there can be no edges inside any block of $\mathcal{C}$.

Case 1: N has two orbits. As there are no edges inside any block of $\mathcal{C}$, Γ is a connected bipartite graph with bipartition $\mathcal{C}$. So $\mathrm{Aut}(\Gamma)/\mathcal{C} \cong \mathbb{Z}_2$ has order 2. As G_L has order $2n = |G|$, we see $\mathrm{fix}_{G_L}(\mathcal{C})$ has order n. As the only proper normal subgroups of G of odd order are contained in A as A is of odd order, $\mathrm{fix}_{G_L}(\mathcal{C})$ contains a semiregular subgroup with two orbits isomorphic to A. Then Γ is isomorphic to a Haar graph of A by [14, Lemma 2.5]. So we may assume without loss of generality that $\Gamma = \mathrm{Haar}(A, S)$ (and $\mathcal{C} = \mathcal{B}$).

If the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is not faithful, then by Theorem 2.16 we have $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_{|H|}$ for some maximal subgroup $1 < H \leq A$. If $|A/H| = 1$, then $\Gamma \cong K_{n,n} \cong \mathrm{Haar}(G, A)$, contradicting our assumption that (3) does not hold. If $|A/H| \geq 2$, then it is straightforward to show that no such wreath product is 2-arc-transitive as $|V(\mathrm{Haar}(A/H, S))| \geq 4$. To see this, first observe that $\mathbb{Z}_2 \times A/H$ is a block system of

$\mathrm{Aut}(\Gamma)$ as $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_{|H|}$ and H was chosen to be maximal. Then some 2-arcs in Γ are of the form $((0, ah), (1, ah), (0, ah'))$ for some $a \in G$ and $h, h' \in H$ with $h' \neq h$, and some 2-arcs in Γ are of the form $(0, ah), (1, ah), (0, bh)$ where $h \in H$, and $a, b \in G$ with $aH \neq bH$. As $\mathbb{Z}_2 \times A/H$ is a block system of $\mathrm{Aut}(\Gamma)$, no automorphism of Γ can map a 2-arc of the first kind, whose vertices are in two blocks of $\mathbb{Z}_2 \times A/H$, to a 2-arc of the second kind, whose vertices are in three blocks of $\mathbb{Z}_2 \times A/H$. Thus the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is faithful. If the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_1 is equivalent to the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_0, then by Theorem 2.16 we may assume $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A, S))$, in which case (4) or (5) would occur by Corollary 3.5. Then (6) follows from Theorem 2.16.

Case 2: If N has at least three orbits, then, as Γ is connected, Γ is a cover of some s-arc-transitive graph by the Praeger Normal Cover Lemma [36], so N is semiregular. Let M be the largest subgroup of N that is normal in $\mathrm{Aut}(\Gamma)$ and is contained in $\widehat{A_L}$. Let $\mathcal{D}$ be the block system of $\mathrm{Aut}(\Gamma)$ formed by the orbits of M.

Suppose $M = 1$. As $\mathcal{C}$ is the set of left cosets of some subgroup of G, and as the square of every element of G is contained in A, $\mathcal{C}$ consists of blocks of size 2. So N is a semiregular group of order 2. As A has odd order, $\widehat{A_L}/\mathcal{D} \cong A$ is a semiregular subgroup of order $n = |A|$ permutating n blocks, and so is regular. Then $\langle \widehat{A_L}, M\rangle$ is transitive, $\widehat{A_L} \cap M = 1$, and as M has order 2, $\langle \widehat{A_L}, M\rangle$ is abelian. We conclude that $\langle \widehat{A_L}, M\rangle \cong \widehat{A_L} \times M$ is abelian. Thus Γ is an s-arc-transitive Cayley graph of an abelian group, and as n is odd, we have Γ is isomorphic to a Cayley graph of $\mathbb{Z}_{2n}$. By [30, Theorem 1.1], $\Gamma = K_{2n}$, $\Gamma = K_{n,n}$ or Γ is $K_{n,n}$ with a 1-factor deleted. If $\Gamma = K_n$, then $\mathrm{Aut}(\Gamma)$ is primitive, a contradiction. By hypothesis, $\Gamma \neq K_{n,n}$. Finally, $K_{n,n}$ with a 1-factor deleted is 2-arc-transitive but not 3-arc-transitive and is isomorphic to $\mathrm{Haar}(A, A - \{0\})$. So $K_{n,n}$ with a 1-factor deleted is the Haar graph corresponding to the 2-arc-transitive Cayley graph $K_n = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n - \{0\})$, also a contradiction.

Suppose $M \neq 1$. Then M has at least as many orbits as N. If G/M is abelian, then the commutator subgroup G' of G is contained in M, and as $G' \cong A$, $[G : G'] = 2$, M has at most two orbits, a contradiction. Thus G/M is nonabelian. Then A/M is semiregular in G/M with two orbits, so $\Gamma/\mathcal{D}$ is isomorphic to a Haar graph of A/M by [14, Lemma 2.5]. Then $\Gamma/\mathcal{D}$ is connected, and so $\mathcal{B}/\mathcal{D}$ is a block system of $\mathrm{Aut}(\Gamma/\mathcal{D})$, and so of $\mathrm{Aut}(\Gamma)/\mathcal{D}$. Then $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$, and this case reduces to the one above with $N = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ having two orbits. □

Note that $K_{n,n}$ does not satisfy 3.9(4) as $K_{n,n} = \mathrm{Haar}(A, A)$ but $\mathrm{Cay}(A, A)$ has loops, and even if the loops were deleted, K_n is 2-arc-transitive but not 3-arc-transitive while $K_{n,n}$ is 3-arc-transitive. So including $K_{n,n}$ in the conclusion of the above theorem is not superfluous.

Also, the Heawood graph is 4-arc-transitive by the Foster Census [16], while its corresponding Cayley digraph $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ is arc-transitive but not 2-arc-transitive. This follows as by [1, Theorem 2] $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\}))$ has automorphism group $G = \{x \mapsto ax + b : a \in \{1, 2, 4\}, b \in \mathbb{Z}_7\}$, and it is arc-transitive. However, as G has order 21, $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ cannot be 2-arc-transitive as there are 42 2-arcs in $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.

Finally, while we have shown that there are graphs satisfying Theorem 3.9(6) holds, and some of the other possibilities obviously hold, it is not clear that for each of the six possibilities, there are corresponding graphs.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.07
https://doi.org/10.26493/2590-9770.1388.c56
(Also available at http://adam-journal.eu)

On the direct products of skew-morphisms

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Received 25 September 2020, accepted 26 January 2021, published online 18 July 2022

Abstract

A *skew-morphism* φ of a finite group G is a permutation on G fixing the identity element of G and for which there is an integer-valued function π on G such that $\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$ for all $g, h \in G$. For two permutations $\alpha\colon A \to A$ and $\beta\colon B \to B$ on the sets A and B, their direct product $\alpha \times \beta$ is the permutation on the Cartesian product $A \times B$ given by $(\alpha \times \beta)(a, b) = \big(\alpha(a), \beta(b)\big)$ for all $(a, b) \in A \times B$. In this paper, necessary and sufficient conditions for a direct product of two skew-morphisms to still be a skew-morphism are given.

Keywords: Finite group, skew-morphism, direct product.

Math. Subj. Class.: 20B25, 20B05, 05C25.

1 Introduction

All groups considered in this paper are finite. For a permutation α on a set A, we use $|\alpha|$ to denote the order of α. The direct product $\alpha \times \beta$ of two permutations $\alpha\colon A \to A$ and $\beta\colon B \to B$ is defined as the permutation on the Cartesian product $A \times B$ of sets A and B given by $(\alpha \times \beta)(a, b) = \big(\alpha(a), \beta(b)\big)$ for all $(a, b) \in A \times B$.

A *skew-morphism* φ of a group G is a permutation on G such that $\varphi(1) = 1$ and $\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$ for all $g, h \in G$, where π is a function from G to the set $\mathbb{Z}_{|\varphi|}$ of nonnegative integers smaller than $|\varphi|$, called the *power function* of φ. Furthermore, we say φ is of *skew-type* k provided π takes on exactly k values in $\mathbb{Z}_{|\varphi|}$, and call the set $\operatorname{Ker}\varphi = \{g \in G | \pi(g) = 1\}$ the *kernel* of φ. Clearly, a skew-morphism φ of G is of skew-type 1 if and only if it is an automorphism of G. Therefore skew-morphisms of groups can be viewed as a generalization of automorphisms of groups.

*Supported by the Basic Research and Frontier Exploration Project of Chongqing (No. cstc2018jcyjAX0010); the Science and Technology Research Rrogram of Chongqing Municipal Education Commission (No. KJQN 201800512); and the National Natural Science Foundation of China (No. 11671276).

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It is well known that any automorphism of a group G gives rise to a semidirect product of G and a cyclic group. In a similar way, a skew-morphism φ of a group G induces a so called skew-product of G and a cyclic group. To see this, we use $L_G := \{L_g | g \in G\}$ to denote the left regular representation of G. Then for any $g, h \in G$, we have

$$(\varphi L_g)(h) = \varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h) = (L_{\varphi(g)}\varphi^{\pi(g)})(h)$$

and so $\varphi L_g = L_{\varphi(g)}\varphi^{\pi(g)}$. Therefore, $\langle\varphi\rangle L_G \subseteq L_G\langle\varphi\rangle$. Since G is finite and $L(G) \cap \langle\varphi\rangle = \{1\}$, we have $\langle\varphi\rangle L_G = L_G\langle\varphi\rangle$, which implies that $T := L_G\langle\varphi\rangle$ is a subgroup of the group $\mathrm{Sym}(G)$ of all permutations on G, while T is said to be a *skew-product* of L_G and $\langle\varphi\rangle$. Conversely, suppose that T is any group admitting a complementary factorization $T = GY$ with $Y = \langle y\rangle$ cyclic. Then for any $g \in G$, there exist unique $g' \in G$ and $y^i \in \langle Y\rangle$ such that $yg = g'y^i$. Define $\varphi\colon G \to G$ by $\varphi(g) = g'$, and $\pi\colon G \to \mathbb{Z}_{|\varphi|}$ by $\pi(g) = i$. Then φ is a skew-morphism of G with associated power function π.

Based on the above reasons, skew-morphism can be a powerful tool for studying problems which can be reduced to complementary factorizations of a finite group into two groups, where one of them is cyclic. Actually, the definition of skew-morphism just comes from such problems. The concept of a skew morphism of a group was introduced by Jajcay and J. Širáň [10] about two decades ago in their study of the theory of regular Cayley maps, which are embeddings of graphs on surfaces that admit a group of automorphisms acting regularly on the vertices of the embedded graph. Subsequently, several articles using skew-morphism as a tool for study of regular Cayley maps were published [4, 5, 7, 11, 12, 15, 16, 17, 18, 20, 21, 22]. Recently, several papers devoted to the theory of skew-morphisms itself emerged, which focus on cyclic groups [2, 8, 13, 14], dihedral groups [9, 19, 23] and simple groups [1, 3]. In [6], a general theory of skew-morphisms and its connection with complementary products was presented.

Although skew-morphisms of groups are a generalization of group automorphisms, they fail to have some good properties which possessed by the latter. For example, the composition of two skew-morphisms of a group is often not a skew-morphism. In [6, Lemma 7.2], it was shown that a direct product of a skew-morphism of a group A and the identity permutation on a group B is a skew-morphism of the Cartesian product $A \times B$. However, unlike automorphisms, direct products of skew-morphisms are not always skew-morphisms. The aim of this paper is to deduce necessary and sufficient conditions for a direct product of two skew-morphisms to still be a skew-morphism.

The paper is organized as follows. In Section 2, some preliminary results on skew-morphisms are listed. In Section 3, the main results of this paper are presented and proved.

2 Preliminaries

In this section, we introduce some known result for later use. First let us fix a number of notations. We use $\gcd\{n_1, n_2, \ldots, n_t\}$ and $\mathrm{lcm}\{n_1, n_2, \ldots, n_t\}$ to denote the greatest common divisor and least common multiple of the t integers $n_1, n_2, \ldots, n_t$ respectively. For a skew-morphism φ of G, we use $\mathcal{O}_x$ to denote the orbit of any $x \in G$ under the action of φ, and $|\mathcal{O}_x|$ to denote the length of $\mathcal{O}_x$. If φ preserves the subset S of G, then we use $\varphi|_S$ to denote the restriction of φ to S.

The following two propositions give some basic properties of skew-morphisms and their power functions.

Proposition 2.1 ([10]). *Let φ be a skew-morphism of a group G and let π be the power function of φ. Then the following hold:*

(i) *the set* $\mathrm{Ker}\,\varphi = \{a \in G | \pi(a) = 1\}$ *is a subgroup of* G*;*

(ii) $\pi(g) = \pi(h)$ *if and only if* $(\mathrm{Ker}\,\varphi)g = (\mathrm{Ker}\,\varphi)h$*;*

(iii) $\varphi^k(gh) = \varphi^k(g)\varphi^{\sum_{i=0}^{k-1}\pi(\varphi^i(g))}(h)$ *for all positive* k *and all* $g, h \in G$*;*

(iv) $\pi(gh) \equiv \sum_{i=0}^{\pi(g)-1} \pi(\varphi^i(h)) \pmod{|\varphi|}$ *for all* $g, h \in G$.

Proposition 2.2 ([23]). *Suppose that $G = \langle x_i \mid 1 \le i \le t\rangle$ and φ is a skew-morphism of G with associated power function π. Then*

(i) $|\varphi| = \mathrm{lcm}\{|\mathcal{O}_{x_1}|, |\mathcal{O}_{x_2}|, \dots, |\mathcal{O}_{x_t}|\}$*;*

(ii) *for any* $c \in G$*, if* $\pi(c) \equiv 1 \pmod{|\mathcal{O}_{x_i}|}$ *for all* $i = 1, 2, \dots, t$*, then* $c \in \mathrm{Ker}\varphi$.

Proposition 2.2 has a direct corollary as follows.

Corollary 2.3. *Let φ be a skew-morphism of G preserving two subgroups A and B of G and let s and t be the orders of $\varphi|_A$ and $\varphi|_B$ respectively. Then* $|\varphi| = \mathrm{lcm}\{s, t\}$.

The following proposition is straightforward.

Proposition 2.4. *Suppose that H is a subgroup of G. Let φ be a skew-morphism of G preserving H. Then the restriction of φ to H is a skew-morphism of H.*

3 Main results

In this section, we deduce necessary and sufficient conditions for a direct product of two skew-morphisms to still be a skew-morphism. For convenience, the direct (or semidirect) product of groups refers to their internal direct (or semidirect) product. Therefore, if $G = A \times B$ (or $G = A \rtimes B$) is the direct product (or semidirect product) of two groups A and B, then both of these two groups can be seen as subgroups of G and any element of G can be uniquely written as ab for some $a \in A$ and $b \in B$. If α and β are permutations on the groups A and B respectively, then the direct product $\alpha \times \beta$ of α and β is the permutation on $G = A \times B$ given by $(\alpha \times \beta)(ab) = \alpha(a)\beta(b)$ for all $ab \in G$.

The following lemma plays a key role in the proof of our main theorem and it has some interesting corollaries.

Lemma 3.1. *For a semidirect product $G = A \rtimes B$ of groups A and B, let φ be a skew-morphism of G preserving both A and B and let π be the associated power function of φ. Then $\pi(a) \equiv 1 \pmod{|\mathcal{O}_b|}$ for any $a \in A$ and any $b \in B$. In particular, $A \le \mathrm{Ker}\,\varphi$ if $\varphi|_A$ is an automorphism of A.*

Proof. Take arbitrary $a \in A$ and $b \in B$. Then

$$\begin{aligned}\varphi(a)\varphi^{\pi(a)}(b) =& \varphi(ab) = \varphi(bb^{-1}ab) = \varphi(b)\varphi^{\pi(b)}(b^{-1}ab) \\ =& \varphi(b)\varphi^{\pi(b)}(b^{-1}ab)\varphi(b)^{-1}\varphi(b).\end{aligned} \tag{3.1}$$

Since φ preserves A and $A \lhd G$, we have $\varphi(a), \varphi(b)\varphi^{\pi(b)}(b^{-1}ab)\varphi(b)^{-1} \in A$. Since φ preserves B, we get $\varphi(b), \varphi^{\pi(a)}(b) \in B$. Note that $A \cap B = \{1\}$. By (3.1), we have $\varphi(a) = \varphi(b)\varphi^{\pi(b)}(b^{-1}ab)\varphi(b)^{-1}$ and $\varphi^{\pi(a)}(b) = \varphi(b)$. It follows that $\pi(a) \equiv 1 \pmod{|\mathcal{O}_b|}$. In particular, if $\varphi|_A$ is an automorphism of A, then $\pi(a) \equiv 1 \pmod{|\mathcal{O}_x|}$ for all $x \in A$. Then by Proposition 2.2(ii), we obtain $a \in \operatorname{Ker}\varphi$ and thus $A \leq \operatorname{Ker}\varphi$. □

Corollary 3.2. *Let $G = A \rtimes B$ be a semidirect product of groups A and B and let φ_i, $i = 1, 2$ be two skew-morphisms of G preserving both A and B. If $\varphi_1|_A = \varphi_2|_A$ and $\varphi_1|_B = \varphi_2|_B$, then $\varphi_1 = \varphi_2$.*

Proof. Lemma 3.1 implies $\varphi_i(ab) = \varphi_i(a)\varphi_i(b)$ for any $a \in A$ and $b \in B$. Since $\varphi_1|_A = \varphi_2|_A$ and $\varphi_1|_B = \varphi_2|_B$, we have $\varphi_1(a) = \varphi_2(a)$ and $\varphi_1(b) = \varphi_2(b)$. It follows that $\varphi_1(ab) = \varphi_2(ab)$ and hence $\varphi_1 = \varphi_2$. □

Corollary 3.3. *Let $G = A \times B$ be a direct product of groups A and B, and let φ be a skew-morphism of G preserving both A and B. Then $\operatorname{Ker}\varphi = \operatorname{Ker}\varphi|_A \times \operatorname{Ker}\varphi|_B$.*

Proof. Since φ preserves both A and B, it follows from Proposition 2.4 that $\varphi|_A$ is a skew-morphism of A and $\varphi|_B$ is a skew-morphism of B. Let π be the associated power function of φ. Take any $x \in \operatorname{Ker}\varphi|_A$. Then $\pi(x) \equiv 1 \pmod{|\mathcal{O}_a|}$ for all $a \in A$. By Lemma 3.1, $\pi(x) \equiv 1 \pmod{|\mathcal{O}_b|}$ for all $b \in B$. Then by Proposition 2.2(ii), we get $x \in \operatorname{Ker}\varphi$ and therefore $\operatorname{Ker}\varphi|_A \subset \operatorname{Ker}\varphi$. Similarly, $\operatorname{Ker}\varphi|_B \subset \operatorname{Ker}\varphi$. On the other hand, for any $xy \in \operatorname{Ker}\varphi$ with $x \in A$ and $y \in B$ and any $a \in A$, we have

$$\begin{aligned}\varphi(xa) &= \varphi(xyay^{-1}) \\ &= \varphi(xy)\varphi(ay^{-1}) \\ &= \varphi(x)\varphi(y)\varphi(a)\varphi(y^{-1}) \\ &= \varphi(x)\varphi(a)\varphi(y)\varphi(y^{-1}).\end{aligned}$$

Since $xa \in A$ and φ preserves A, we have $\varphi(y)\varphi(y^{-1}) = 1$ and $\varphi(xa) = \varphi(x)\varphi(a)$. Therefore $x \in \operatorname{Ker}\varphi|_A$. Similarly, $y \in \operatorname{Ker}\varphi|_B$. Summarizing all the above we have $\operatorname{Ker}\varphi = \operatorname{Ker}\varphi|_A \times \operatorname{Ker}\varphi|_B$. □

Corollary 3.4. *Let $G = A \times B$ be a direct product of groups A and B, and let φ be a skew-morphism of G with associated power function π. If φ preserves both A and B, then $\pi(ab) \equiv \pi(a)\pi(b) \pmod{|\varphi|}$ for all $a \in A$ and $b \in B$.*

Proof. Take arbitrary $a \in A$ and $b \in B$ and let i be any nonnegative integer. Since φ preserves both A and B, it follows from Lemma 3.1 that $\pi(\varphi^i(a)) \equiv 1 \pmod{t}$ and $\pi(\varphi^i(b)) \equiv 1 \pmod{s}$ where s and t are the orders of the restriction of φ to A and B respectively. Therefore, by Proposition 2.1(iv), we have

$$\pi(ab) \equiv \sum_{i=0}^{\pi(a)-1} \pi(\varphi^i(b)) \equiv \pi(a) \equiv \pi(a)\pi(b) \pmod{s}$$

and

$$\pi(ab) \equiv \sum_{i=0}^{\pi(a)-1} \pi(\varphi^i(b)) = \pi(b) \equiv \pi(a)\pi(b) \pmod{t}.$$

By Corollary 2.3, $|\varphi| = \operatorname{lcm}\{s, t\}$. It follows that $\pi(ab) \equiv \pi(a)\pi(b) \pmod{|\varphi|}$. □

The following is our main theorem.

Theorem 3.5. *Let $G = A \times B$ be a direct product of groups A and B, and let α, β be skew-morphisms of A and B with associated power functions π_α and π_β respectively. Then the direct product $\alpha \times \beta$ of α and β is a skew-morphism of G if and only if $\pi_\alpha(a) \equiv \pi_\beta(b) \equiv 1 \pmod{d}$ for all $a \in A$ and $b \in B$ where $d = \gcd\{|\alpha|, |\beta|\}$.*

Proof. For convenience, write $\varphi = \alpha \times \beta$. Then $\varphi(ab) = \alpha(a)\beta(b)$ for any $ab \in G$. Furthermore, φ preserves both A and B, and $\varphi|_A = \alpha$ and $\varphi|_B = \beta$. Set $|\alpha| = s$, $|\beta| = t$ and $|\varphi| = m$. Then $d = \gcd\{s, t\}$. By Corollary 2.3, we get $m = \text{lcm}\{s, t\}$.

First we prove the necessity. Suppose that φ is a skew-morphism of G with the associated power function π. By Lemma 3.1, we have $\pi(a) \equiv 1 \pmod{|\mathcal{O}_b|}$ and $\pi(b) \equiv 1 \pmod{|\mathcal{O}_a|}$ for any $a \in A$ and $b \in B$. It follows that $\pi(a) \equiv 1 \pmod{t}$ and $\pi(b) \equiv 1 \pmod{s}$. Since $d = \gcd\{s, t\}$, we have $\pi(a) \equiv \pi(b) \equiv 1 \pmod{d}$. Clearly, $\pi(a) \equiv \pi_\alpha(a) \pmod{s}$ and $\pi(b) \equiv \pi_\beta(b) \pmod{t}$. Then noting that $d \mid s$ and $d \mid t$, we have $\pi_\alpha(a) \equiv \pi_\beta(b) \equiv 1 \pmod{d}$.

Next, we prove the sufficiency. Suppose that $\pi_\alpha(a) \equiv \pi_\beta(b) \equiv 1 \pmod{d}$ for all $a \in A$ and $b \in B$. Let u, v be a pair of integers such that $us + vt = d$. Define a function π from G to $\mathbb{Z}_m$ by the rule:

$$\pi(ab) \equiv \Big(\frac{vt}{d}\big(\pi_\alpha(a) - 1\big) + 1\Big)\Big(\frac{us}{d}\big(\pi_\beta(b) - 1\big) + 1\Big) \pmod{m} \tag{3.2}$$

for any $ab \in G$. Since $\pi_\alpha(a) \equiv \pi_\beta(b) \equiv 1 \pmod{d}$, we have

$$\frac{vt}{d}\big(\pi_\alpha(a) - 1\big) + 1 \equiv 1 \pmod{t}, \tag{3.3}$$

$$\frac{vt}{d}\big(\pi_\alpha(a) - 1\big) + 1 = \pi_\alpha(a) - \frac{us}{d}\big(\pi_\alpha(a) - 1\big) \equiv \pi_\alpha(a) \pmod{s}, \tag{3.4}$$

$$\frac{us}{d}\big(\pi_\alpha(b) - 1\big) + 1 \equiv 1 \pmod{s}, \tag{3.5}$$

and

$$\frac{us}{d}\big(\pi_\alpha(b) - 1\big) + 1 \equiv \pi_\beta(b) \pmod{t}. \tag{3.6}$$

By equations (3.2), (3.4) and (3.5), we get $\pi(ab) \equiv \pi_\alpha(a) \pmod{s}$. By equations (3.2), (3.3) and (3.6), we get $\pi(ab) \equiv \pi_\beta(b) \pmod{t}$. It follows that

$$\begin{aligned}
\varphi(a_1b_1a_2b_2) =& \varphi(a_1a_2b_1b_2)\\
=& \alpha(a_1a_2)\beta(b_1b_2)\\
=& \alpha(a_1)\alpha^{\pi_\alpha(a_1)}(a_2)\beta(b_1)\beta^{\pi_\beta(b_1)}(b_2)\\
=& \alpha(a_1)\beta(b_1)\alpha^{\pi_\alpha(a_1)}(a_2)\beta^{\pi_\beta(b_1)}(b_2)\\
=& \alpha(a_1)\beta(b_1)\alpha^{\pi(a_1b_1)}(a_2)\beta^{\pi(a_1b_1)}(b_2)\\
=& \varphi(a_1b_1)\varphi^{\pi(a_1b_1)}(a_2b_2).
\end{aligned}$$

Therefore φ is a skew-morphism of G with associated power function π. □

By Proposition 2.3, Corollary 3.3 and Corollary 3.4, we have the following theorem, the proof for which is trivial and so is omitted.

Theorem 3.6. *Let $G = A \times B$ be a direct product of groups A and B, and let α, β be skew-morphisms of A and B with associated power functions π_α and π_β respectively. If the direct product $\varphi = \alpha \times \beta$ of α and β is a skew-morphism of G, then the following hold:*

(i) $|\varphi| = \mathrm{lcm}\{|\alpha|, |\beta|\}$;

(ii) *the associated power function π of φ are given by $\pi(ab) \equiv \pi_\alpha(a)\pi_\beta(b) \pmod{|\varphi|}$ for all $a \in A$ and $b \in B$;*

(iii) $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\alpha \times \mathrm{Ker}\,\beta$.

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.08
https://doi.org/10.26493/2590-9770.1392.f9a
(Also available at http://adam-journal.eu)

Classifying edge-biregular maps of negative prime Euler characteristic*

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Dedicated to Marston Conder on the occasion of his 65th birthday.

Received 31 October 2020, accepted 29 March 2020, published online 18 July 2022

Abstract

An edge-biregular map arises as a smooth normal quotient of a unique index-two subgroup of a full triangle group acting with two edge-orbits. We give a classification of all finite edge-biregular maps on surfaces of negative prime Euler characteristic.

Keywords: Biregular map, automorphism group, non-orientable surface, Euler characteristic.

Math. Subj. Class.: 20B25, 20F05, 57M60, 05C25

1 Introduction

A *map* is a 2-cell embedding of a connected graph on a surface (which may be orientable or not but without boundary components). As such it is possible to form the barycentric subdivision of a map, and the resulting regions are the *flags* of the map. An *automorphism*

*Both authors attended the Symmetries of Discrete Objects 2020 conference held in Rotorua, New Zealand and would like to thank the organisers Marston Conder and Gabriel Verret for arranging the event and for their generous hospitality. It was at this Symmetries of Discrete Objects workshop that some of the above research was conducted and partial results were presented. Both authors would like to thank the anonymous referee whose comments made us aware of both errors and omissions in our original manuscript, leading us to make significant improvements to this paper.

†Corresponding author. The author is grateful for the travel grants from both the Open University and also from the London Mathematical Society which enabled her to attend the conference.

‡The author gratefully acknowledges support from the APVV Research Grants 17-0428 and 19-0308, as well as from the VEGA Research Grants 1/0567/22 and 1/0206/20.

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of a map is a structure-preserving bijection of the set of flags onto itself, and the set of all automorphisms of a map form its *automorphism group*.

Informally, how much symmetry a map displays may be measured by the size of its automorphism group. If a map automorphism fixes a flag, then, by connectivity, it must fix all flags and hence be the identity; this means that the action of the automorphism group of a map is semi-regular on flags. A largest 'level of symmetry' exhibited by a map therefore arises when the automorphism group is regular, that is semi-regular and transitive, on the set of flags. Such maps are *fully regular*; they are, in some sense, the most symmetric maps [13]. Elements of the automorphism group of a fully regular map can be identified with flags in a one-to-one manner by fixing a flag and assigning to every other flag the unique automorphism taking the fixed flag to the other one.

An obvious consequence of full regularity of a map is that all its vertices have the same valency and all the faces have boundary walks of the same length; we speak about a map of *type* (k, ℓ) in the case of valency k and face length ℓ. As it turns out, every fully regular map of type (k, ℓ) is a smooth normal quotient of a (k, ℓ)-*tessellation* of a simply connected surface by ℓ-gons, k of which meet at every vertex. Equivalently, in a group-theoretic language, the automorphism group of a fully regular map of type (k, ℓ) is a torsion-free normal quotient of the automorphism group of a (k, ℓ)-tessellation; the latter is known as the full $(2, k, \ell)$-triangle group.

There are other classes of 'highly symmetric' maps which are not necessarily fully regular but which have automorphism groups that can still be described as 'large'. Perhaps the most natural candidate for a map to qualify for such a distinction would be when a group of automorphisms of the map has two (necessarily equal-sized) orbits and is regular on each. Following the example of existing graph-theoretic language [7, 11, 15], we use the term *biregular* to describe such maps. As in the case of fully regular maps, automorphism groups of biregular maps of type (k, ℓ) also arise as torsion-free normal quotients, but this time of index-two subgroups of the full $(2, k, \ell)$-triangle groups. As we shall see, a full $(2, k, \ell)$-triangle group may contain up to 7 subgroups of index two (depending on the parity of k and ℓ). This potentially gives rise to 7 different kinds of biregular maps, and we now give two examples.

Perhaps the most well known kind of biregularity arises from the subgroup of a full triangle group formed by orientation-preserving automorphisms, leading to a widely studied class of maps known as orientably-regular [13] or rotary [14]. We note that orientability of a map is equivalent to admitting local orientations at vertices that are consistent when passing between incident vertices along an edge on the carrier surface of the map. An opposite phenomenon, namely, when a map of type (k, ℓ) admits an assignment of local orientations at vertices which disagree when traversing along an arbitrary edge on the surface and, at the same time, exhibits the highest 'level of symmetry' with respect to this property, was studied in [6]. These maps, called bi-rotary in [6], arise as normal torsion-free quotients of a different index-two subgroup of the full $(2, k, \ell)$-triangle group.

In this paper we will focus on still another kind of biregular maps, those admitting an action of the map automorphism group with two orbits on edges, accordingly called *edge-biregular*. It turns out that edge-biregular maps of type (k, ℓ) arise as torsion-free normal quotients from a *unique* index-two subgroup of the full $(2, k, \ell)$-triangle group. In such a map, edges emanating from a vertex in their cyclic order on the carrier surface must necessarily alternate between the two orbits, so that k must be even. Similarly, ℓ must also be even.

Edge-biregular maps have been investigated in great detail in [12], including their classification on surfaces of non-negative Euler characteristic and also a classification of such maps with a dihedral automorphism group. Our aim here is to complement the results of [12] by deriving a classification of edge-biregular maps on surfaces with negative prime Euler characteristic, extending thereby also the existing classification results for fully regular maps [2] and bi-rotary maps [6] on these surfaces.

To this end, in section 2 we introduce biregular maps in the wider context of symmetric maps of a given type and establish notation and some basic facts about edge-biregular maps. Section 3 contains a summary of our results on classification of edge-biregular maps on surfaces of negative prime Euler characteristic and a setup of proofs; these are subsequently given in sections 4 and 5. Finally section 6 contains further observations and remarks.

2 Algebra of edge-biregular maps

It is well known (see for example [10] or [3], or the survey [13]) that fully regular maps of a given type (k, ℓ) can be identified with torsion-free normal quotients of the full $(2, k, \ell)$-triangle group with presentation

$$\langle\, R_0,\, R_1,\, R_2 \,|\, R_0^2,\, R_1^2,\, R_2^2,\, (R_0R_2)^2,\, (R_1R_2)^k,\, (R_0R_1)^\ell \,\rangle \tag{2.1}$$

This group is isomorphic to the full automorphism group of a *universal* (k, ℓ)*-tessellation* of a simply connected surface, where the tiles are regular l-gons, and the valency of the underlying graph is k. One usually considers a universal (k, ℓ)-tessellation also as a geometrically regular tiling of a sphere, a Euclidean plane, or a hyperbolic plane, depending on whether the value of $1/k + 1/\ell$ is greater than, equal to, or smaller than $1/2$.

As alluded to in the introduction, biregular maps arise as smooth, that is torsion-free, normal quotients of subgroups of the full $(2, k, \ell)$-triangle groups of index two. Any such subgroup is completely determined by a non-empty subset of $\{R_0, R_1, R_2\}$, consisting of the involutory generators *not* contained in the subgroup. There are 7 such subsets, and the number of resulting subgroups of index two depends on the parities of k and ℓ. The index two subgroup avoiding each of R_0, R_1, R_2 occurs in every one of the triangle groups, regardless of parity of k and ℓ. If both k and ℓ are odd then this is the only subgroup of index two in the full $(2, k, \ell)$-triangle group. If exactly a given one of k, ℓ is even, there are two further index two subgroups, and so including the aforementioned subgroup, 3 such subgroups in total. If both k and ℓ are even, the full $(2, k, \ell)$-triangle group contains 7 subgroups of index two.

The subgroup of (2.1) of index two containing none of the three involutory generators, informally denoted $\langle \overline{R}_0, \overline{R}_1, \overline{R}_2\rangle$ here, is known as the (ordinary) $(2, k, \ell)$-triangle group comprising all orientation-preserving automorphisms of a universal (k, ℓ)-tessellation. It is generated, for instance, by the products $U = R_2R_1$ and $V = R_1R_0$ representing rotations of the tessellation about a vertex and about the centre of a face incident to the vertex, and has (irrespective of parity of k, ℓ) a presentation of the form

$$\langle \overline{R}_0,\, \overline{R}_1,\, \overline{R}_2 \,\rangle = \langle U,\, V \,|\, U^k,\, V^\ell,\, (UV)^2 \,\rangle \tag{2.2}$$

Smooth normal quotients of this subgroup give rise to orientably-regular maps in the terminology of [13], also known as rotary maps in [14]. The second type of biregular maps mentioned in the introduction, the bi-rotary maps of [6], arise as smooth quotients of the index-two subgroup $\langle R_0, \overline{R}_1, \overline{R}_2 \rangle$ of (2.1) containing R_0 but avoiding R_1 and R_2, for ℓ even. Using the notation $A = R_0$, $Z = R_1R_2$, and the square brackets for a commutator of two elements as usual, this subgroup admits a presentation

$$\langle R_0, \overline{R}_1, \overline{R}_2 \rangle = \langle A, Z \,|\, A^2, Z^k, [A, Z]^{\ell/2} \rangle \tag{2.3}$$

In this paper we will focus on biregular maps corresponding to torsion-free normal quotients of the subgroup $\langle R_0, \overline{R}_1, R_2 \rangle$ of the full $(2, k, \ell)$-triangle group (2.1) for both k and ℓ even, distinguished by inclusion of R_0 and R_2 and avoidance of R_1. It may be checked that the subgroup $\langle R_0, \overline{R}_1, R_2 \rangle$ is the only one among the seven index-two subgroups of (2.1) inducing *two* edge orbits on a universal (k, ℓ)-tessellation for k, ℓ even. With the help of new variables $X = R_0$, $Y = R_2$, $S = R_1R_0R_1$ and $T = R_1R_2R_1$ it can be shown that a complete presentation of this subgroup is

$$\langle R_0, \overline{R}_1, R_2 \rangle = \langle X, Y, S, T \,|\, X^2, Y^2, S^2, T^2, (XY)^2, (ST)^2, (TY)^{k/2}, (SX)^{\ell/2} \rangle \tag{2.4}$$

If N is a torsion-free normal subgroup of $\langle R_0, \overline{R}_1, R_2 \rangle$ of finite index, the corresponding quotient group $H = \langle R_0, \overline{R}_1, R_2 \rangle / N$ has a presentation of the form

$$H = \langle x, y, s, t \,|\, x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^{k/2}, (sx)^{\ell/2}, \dots \rangle \tag{2.5}$$

with $x = XN$, $y = YN$, $s = SN$ and $t = TN$. By the correspondence between maps and groups, any finite group H together with a presentation as in (2.5) is a group of automorphisms of a finite biregular map, the biregularity of which is derived from the index-two subgroup $\langle R_0, \overline{R}_1, R_2 \rangle$ of (2.1). Maps exhibiting this kind of biregularity were referred to in [12] as *edge-biregular*.

The obvious motivation for this terminology comes from the already mentioned two-orbit action of the subgroup $\langle R_0, \overline{R}_1, R_2 \rangle$ on edges of a universal (k, ℓ)-tessellation. By taking a torsion-free normal quotient as in (2.5) this two-orbit action projects onto edges of the quotient map as shown in Figure 1.

In order to introduce a 'reference system' within the map on the diagram one may fix and shade an arbitrarily chosen flag f, let x and y represent the reflections of H in the sides of f as depicted, and call the quadruple of flags f, xf, yf and xyf surrounding an edge 'distinguished'. The pattern of the shaded flags in this diagram demonstrate how the group of automorphisms H of an edge-biregular map 'spreads around' the distinguished quadruple and hence acts transitively on faces and vertices. At the same time, Figure 1 displays two orbits of H on edges, those which are bold in one orbit, with dashed edges indicating the other orbit; notice also the two H-orbits on flags formed by quadruples of shaded and unshaded flags surrounding respectively bold and dashed edges.

Observe that it is a choice, consistent with the choice of notation, which orbit of edges is coloured bold, and which is dashed. As shown in Figure 1, the generators x and y correspond to automorphisms acting locally as reflections respectively along and across the distinguished bold edge. Meanwhile s and t correspond to reflections respectively along and across the distinguished dashed edge, that is the dashed edge which is surrounded by the quadruple of unshaded flags sharing a boundary with f. We call this, and the corresponding presentation (2.5), the *canonical form* of the edge-biregular map M and denote it $M = (H; x, y, s, t)$. A pair of edge-biregular maps $M = (H; x, y, s, t)$ and $M' = (H'; x', y', s', t')$ given in the canonical form are *isomorphic* if there is a group isomorphism $H \to H'$ taking z onto z' for every $z \in \{x, y, s, t\}$.

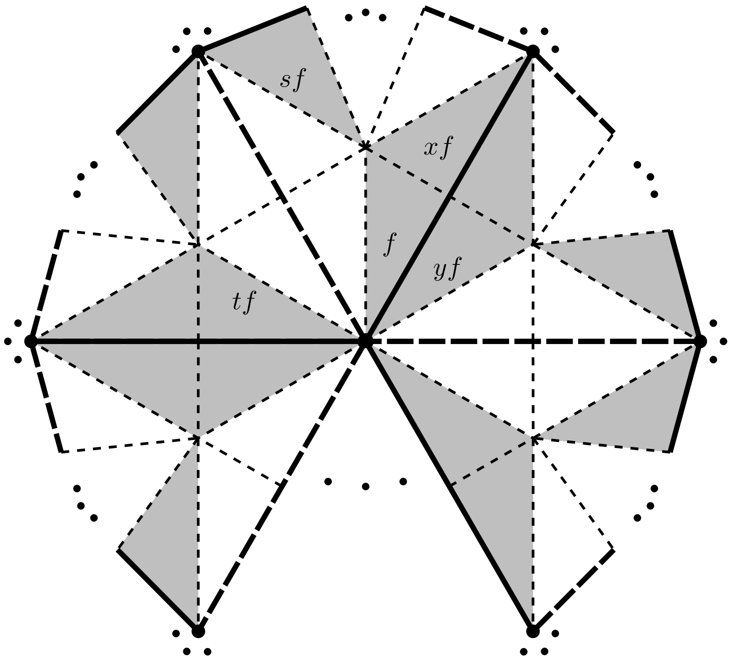

Figure 1: The images under x, y, s, t of the distinguished flag f of an edge-biregular map.

Whether an edge-biregular map $M = (H; x, y, s, t)$ is also fully regular will depend on the existence (or otherwise) of an involutory automorphism of M lying outside H and fusing the two H-orbits of edges together. By the above, this condition is equivalent to the existence of an automorphism of the *group* H which interchanges x with s and y with t. The map $M = (H; x, y, s, t)$ is thus fully regular if and only if M and $M' = (H; s, t, x, y)$ are isomorphic as maps, otherwise we say these maps are *twins*.

To avoid unnecessary work, all our forthcoming results will be up to duality. The dual map M^* of an edge-biregular map $M = (H; x, y, s, t)$ is also an edge-biregular map and is formed by interchanging x with y and s with t in the presentation (2.5) for H, thereby swapping the vertices with the faces and vice versa, to give $M^* = (H; y, x, t, s)$. If M is isomorphic to M^* the map is *self-dual*, which (by the map isomorphism condition) is equivalent to the existence of an automorphism of the *group* H swapping x with y and s with t.

Except for the case of characteristic -2, all our edge-biregular maps $M = (H; x, y, s, t)$ will be carried by non-orientable surfaces. Since each canonical generator of H corresponds to a reflection on the carrier surface, it follows that M is supported by an orientable surface if and only if every relator in the presentation of H has an even length in terms of x, y, s, t. This is easily seen to be equivalent to the statement that the carrier surface of M is non-orientable if and only if H is generated by any three products, of two involutions each, provided that every involution out of x, y, s, t appears in at least one product.

Having assumed that our edge-biregular maps $M = (H; x, y, s, t)$ arise from the index-two subgroup $\langle R_0, \overline{R}_1, R_2 \rangle$ of full triangle groups by torsion-free normal subgroups implies that all the four canonical involutory generators of H are *distinct*. For completeness we address also the situations when some of the generators in the set $\{x, y, s, t\}$ are either trivial or equal to each other. These are commonly known as 'degeneracies' and have been treated in detail in [12]; they give rise to maps on surfaces with boundary or to maps with semi-edges or to maps of spherical type with $2 \in \{k, l\}$. Recall that a semi-edge has one of its endpoints attached to a vertex but the other one is 'dangling' and not incident to any vertex; we will regard edges and semi-edges as different objects.

While we are not considering maps on surfaces with boundary here, edge-biregular maps with semi-edges are relevant for us. Leaving the trivial case of a semi-star (a spherical one-vertex map with some number of attached semi-edges) aside, suppose that an edge-biregular map of type (k, ℓ) with both entries even contains a semi-edge as well as an edge. By [12], edges and semi-edges must then alternate around each vertex on the carrier surface of the map, and consequently the deletion of all semi-edges results in a fully regular map of type $(k/2, \ell/2)$. Conversely, every edge-biregular map containing both an edge and a semi-edge arises from a fully regular map that is not a semi-star by inserting a semi-edge into each corner.

To arrive at an algebraic explanation of this edge/semi-edge phenomenon in edge-biregular maps, consider such a map $M = (H; x, y, s, t)$ in a canonical form and assume that all the dashed edges in Figure 1 would collapse to semi-edges. This is equivalent to identifying the flags marked sf and tf into a single flag, which means regarding s and t as identical automorphisms. It now follows from (2.5) that, up to twinness, an edge-biregular map $M = (H; x, y, s, t)$ of type (k, ℓ) for even k and ℓ, containing both edges and semi-edges, may be identified with its group of automorphisms H presented in the form

$$H = \langle\, x,\, y,\, s,\, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, st,\, (ty)^{k/2},\, (sx)^{\ell/2},\, \ldots \rangle \tag{2.6}$$

Note the difference from (2.5) in the power at the product st, implying $s = t$ in (2.6). Conversely, let a fully regular map M of type $(k/2, \ell/2)$ for k, ℓ even and distinct from a semi-star be given by its (full) automorphism group, that is, by a torsion-free normal quotient of (2.1) presented in the form

$$\langle\, r_0,\, r_1,\, r_2 \,|\, r_0^2,\, r_1^2,\, r_2^2,\, (r_0 r_2)^2,\, (r_1 r_2)^{k/2},\, (r_0 r_1)^{\ell/2}\, \ldots \rangle \tag{2.7}$$

Then, a presentation of the group H as in (2.6) for the corresponding edge-biregular map of type (k, ℓ) arising from M by inserting a semi-edge into every corner of M is obtained from the one in (2.7) simply by letting $x = r_0$, $y = r_2$, and $s = t = r_1$.

3 Results, and proofs setup

In this section we state our classification results for edge-biregular maps on surfaces of negative prime Euler characteristic. We note that a classification of such maps on surfaces with a non-negative Euler characteristic (a sphere, a projective plane, a torus and a Klein bottle) was derived in [12].

As already alluded to, edge-biregular maps on surfaces without boundary split into two families, according to whether they contain semi-edges or not. Also, for the purpose of this section we may disregard the semi-star maps. Let us first consider edge-biregular maps containing both edges as well as semi-edges. By the facts summed up at the end of the previous section, edge-biregular maps of type (k, ℓ) for even k and ℓ containing both edges as well as semi-edges and carried by a particular surface are in a one-to-one correspondence with fully regular maps of type $(k/2, \ell/2)$ on the same surface. It follows that a classification of all fully regular maps on a particular surface automatically implies a classification of edge-biregular maps with semi-edges on the same surface, including presentations of the corresponding groups of automorphisms. And since a classification of fully regular maps on surfaces of negative prime Euler characteristic is available from [2], a corresponding classification of edge-biregular maps with semi-edges on these surfaces follows by the outlined procedure. In the interest of saving space we will not go into any further detail and we also omit a formal statement of the related presentations, referring the reader to [2] or to [13].

We now pass onto classification of edge-biregular maps without semi-edges on surfaces with negative prime Euler characteristic. From the work in [8] and [12], we know that if such a map $M = (H; x, y, s, t)$ is given in a canonical form with the group of automorphisms H presented as in (2.5), then all the four generators are *non-trivial* and *distinct*. We state our main results separately for odd primes p (in which case, of course, all the maps are carried by non-orientable surfaces) and then for $p = 2$.

Theorem 3.1. *A surface of negative odd prime Euler characteristic $\chi = -p$ supports, up to duality and twin maps, only the following pairwise non-isomorphic edge-biregular maps $M = (H; x, y, s, t)$ with no semi-edges, given in their canonical forms:*

(1) *A single-vertex map of type $(4(p+1), 4)$ with $H = H_{p(1)} = \langle y, t\rangle$ isomorphic to the dihedral group of order $4(p+1)$ with canonical presentation*

$$H_{p(1)} = \langle\, x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ xys,\ s(yt)^{p+1}\rangle$$

(2) *A two-vertex map of type $(2(p+2), 4)$ with $H = H_{p(2)} = \langle x, t\rangle$ isomorphic to the dihedral group of order $4(p+2)$ and canonical presentation*

$$H_{p(2)} = \langle\, x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ xys,\ s(xt)^{p+2}\rangle$$

(3) *The maps $M_{p,j}$ of type (k, ℓ) where $k = 4\kappa$ and $\ell = 2\lambda$ for odd and relatively prime κ and λ, such that $p = 2\kappa\lambda - 2\kappa - \lambda$, where $H = H_{p,j}$ is of order $kl/2$ and presented as follows, for any positive integer $j < \lambda$ such that $j^2 \equiv 1 \bmod \lambda$, with $a = (j-1)(\lambda+1)/2$, and $u = sx$, $v = ty$:*

$$H_{p,j} = \langle\, x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ u^\lambda,\ v^{2\kappa},\ [s, v^2],\ [x, v^2],\ tutu^j,\ v^\kappa u^a s\,\rangle$$

(4) *When $p \equiv 5 \bmod 9$, that is when $p = 9m - 4$ we have a map of type $(8, 6m)$ and the presentation of the corresponding group $H = H_p$ which has order $24m$ is of the form*

$$H_p = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (sx)^{3m},\ (ty)^4,\ (sxy)^2 t,\ txty\rangle$$

(5) *When $p = 3$, we have a map of type $(4, 6)$ and the corresponding group $H = H_{(3)} \cong D_6 \times D_6$ has canonical presentation*

$$H_{(3)} = \langle x, y, s, t \,|\, x^2,\ y^2,\ s^2,\ t^2,\ (xy)^2,\ (st)^2,\ (ty)^2,\ (sx)^3,\ (xyt)^3,\ (sty)^3,\ (xyst)^2\rangle$$

Moreover, the only fully regular map in the above list is $H_{(3)}$.

Theorem 3.2. *A surface of Euler characteristic* $\chi = -2$ *supports, up to duality and twin maps, only the following non-isomorphic edge-biregular maps* $M = (H; x, y, s, t)$ *with no semi-edges:*

(1) *One self-dual map of type* $(8, 8)$ *with a single vertex and a single face, with the group* $H = H_{2,1}$ *presented as*

$$\begin{aligned} H_{2,1} = &\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^4, (sx)^4, ysxs, txsx, xtyt, syty \rangle \\ &\cong D_8 \end{aligned}$$

(2) *One map of type* $(4, 12)$ *with three vertices and a single face, with the group* $H = H_{2,2}$ *having canonical presentation*

$$H_{2,2} = \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^6, xyt, t(sx)^3 \rangle \cong D_{12}$$

(3) *One self-dual map of type* $(6, 6)$ *with two vertices and two faces, with the group* $H = H_{2,3}$ *being as follows*

$$H_{2,3} = \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^3, (sx)^3, styx \rangle \cong D_{12}$$

(4) *Six maps of type* $(4, 8)$ *with four vertices and two faces, with groups* $H_{2,i} \cong D_8 \times C_2$ *for* $i = 4, 5, 6, 7, 8, 9$ *presented in the form*

$$\begin{aligned}
H_{2,4} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^4, (tx)^2, ysxs \rangle \\
H_{2,5} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, y(sx)^2 \rangle \\
H_{2,6} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, y(sx)^2 \rangle \\
H_{2.7} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, ty(sx)^2 \rangle \\
H_{2.8} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tyxsx \rangle \\
H_{2.9} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tys \rangle
\end{aligned}$$

(5) *Three maps of type* $(4, 6)$ *with six vertices and four faces, with groups of automorphisms* $H_{2,i}$ *for* $i = 10, 11, 12$ *given by*

$$\begin{aligned}
H_{2,10} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, t(sy)^2, y(xt)^2 \rangle \cong S_4 \\
H_{2,11} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ysx)^2, (tx)^2 \rangle \\
&\cong D_6 \times V_4 \\
H_{2,12} &= \langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ys)^2, (tx)^2 \rangle \\
&\cong D_6 \times V_4
\end{aligned}$$

The maps supported by the groups $H_{2,1}$, $H_{2,3}$, $H_{2,4}$, $H_{2,7}$, $H_{2,11}$ *and* $H_{2,12}$ *are orientable while the other six are not, and the only fully regular ones out of the twelve are those supported by the groups* $H_{2,1}$, $H_{2,3}$, $H_{2,7}$, $H_{2,10}$ *and* $H_{2,12}$.

We make no claim of a lack of redundancy in these presentations - in fact quite the opposite: some of the above presentations have unnecessary relators which have been retained. This is deliberate in order to better demonstrate the interplay between the canonical generators, and also to make evident the presence, or absence, of full regularity or self-duality in the underlying map.

In order to set the stage for proofs in the next three sections, throughout we let $M = (H; x, y, s, t)$ be an edge-biregular map of type (k, ℓ) for even k and ℓ, with no semi-edges, and of characteristic $-p$ (meaning that the carrier surface of M has Euler characteristic $\chi = -p$). All our working will be up to duality, so that we may without loss of generality assume that $k \leq \ell$. We also know by [8] and [12] that in this situation the four generating involutions of the associated group of automorphisms H are mutually distinct. Note that this instantly implies that $k, \ell \geq 4$; moreover, since maps of type $(4, 4)$ necessarily live on surfaces with Euler characteristic 0, we will assume that $k \geq 4$ and $\ell \geq 6$.

Referring to the diagram in Figure 1 which may be assumed to display a fragment of M, the stabiliser in H of the distinguished face is $\langle s, x\rangle$, isomorphic to the dihedral group D_ℓ of order ℓ, while the H-stabiliser of the distinguished vertex is $\langle t, y\rangle$, isomorphic to the dihedral group D_k of order k. The map thus has $\frac{|H|}{\ell}$ faces and $\frac{|H|}{k}$ vertices. The H-stabiliser of the distinguished bold and dashed edge, respectively, is $\langle x, y\rangle$ and $\langle s, t\rangle$, in both cases isomorphic to the Klein four-group $V_4 \cong C_2 \times C_2$; hence the map has $\frac{2|H|}{4}$ edges. The Euler-Poincaré formula applied to the number of vertices, edges and faces of M gives $|H|(\frac{1}{k} - \frac{1}{2} + \frac{1}{\ell}) = \chi = -p$, or, equivalently,

$$|H| = \nu p \quad \text{where} \quad \nu = \nu(k, \ell) = \frac{2k\ell}{k\ell - 2(k + \ell)} \tag{3.1}$$

It may be checked that our lower bounds $k \geq 4$ and $\ell \geq 6$ imply $\nu \leq 12$. Combining this with (3.1) one obtains the upper bound $|H| \leq 12p$ for the group of automorphisms H of the map M under consideration. Also, by (2.5) and distinctness of the involutory generators, the group H contains two distinct subgroups isomorphic to V_4; in particular, the order of H must be divisible by 4.

From this point on we split the proof into four parts. In Section 4 we will consider the case when p divides the order of H, first addressing all odd primes, and then the special case when $p = 2$. Section 5 will deal with the opposite case, and the analysis will be divided according to whether the 2-part Fitting subgroup of H is cyclic or dihedral.

4 The case when p divides the order of H

Let us recall that we are considering an edge-biregular map $M = (H; x, y, s, t)$ of type k, ℓ for even k, ℓ such that $k \leq \ell$, $k \geq 4$, $\ell \geq 6$, carried by a surface of characteristic $-p$ for some prime p. Suppose p is a divisor of $|H|$. This, together with (3.1) and the facts summed up at the end of Section 3, implies that $|H| = \nu p$ where $\nu \in \{4, 6, 8, 12\}$. Moreover, one may check that in our range for the even entries in the type of our biregular map M one has: $\nu(k, \ell) = 12$ only for $(k, \ell) = (4, 6)$; $\nu(k, \ell) = 8$ only for $(k, \ell) = (4, 8)$; $\nu(k, \ell) = 6$ only when $p = 2$ and for types $(6, 6)$ and $(4, 12)$; and finally $\nu(k, \ell) = 4$ only for (k, ℓ) equal to $(6, 12)$ or $(8, 8)$, corresponding to the cases when $p = 3$ and $p = 2$ respectively.

The case when $p = 2$ is treated as a special case in the proof of Theorem 3.2 at the end of this section. Henceforth, unless stated otherwise, we assume that p is an odd prime. We sum up the above observations for odd p in the form of a lemma.

Lemma 4.1. *If p is an odd prime which divides $|H|$, then either $|H| = 12p$ for the type $(4, 6)$, or $|H| = 8p$ for the type $(4, 8)$, or else $p = 3$ and $|H| = 4p$ for type $(6, 12)$.* □

Note that our previous considerations apply also to the situation when $\chi = -1$, that is, Lemma 4.1 can be applied also to the exceptional case when $p = 1$. In fact, a more detailed look into this case will be useful later. Remembering that we need four distinct involutory generators for H if $|H| \in \{4, 8, 12\}$ this leaves us with only two options, namely, $|H| = 8$ when $(k, \ell) = (4, 8)$, and $|H| = 12$ when $(k, \ell) = (4, 6)$. If $|H| = 8$ then $\ell = 8$ and hence $H \cong D_8$. If, on the other hand, $|H| = 12$ then $\ell = 6$, which means that H contains a subgroup isomorphic to D_6. There is only one such group H of order 12, namely D_{12}. By the classification of [12] each of these dihedral groups support an edge-biregular map on a surface with $\chi = -1$. Since we do not need more detailed information about such maps, we just state the following conclusion here:

Lemma 4.2. *If $M = (H; x, y, s, t)$ is an edge-biregular map on a surface of Euler characteristic $\chi = -1$, then H is isomorphic to a dihedral group of order 8 or 12.* □

We continue by showing that assuming $p \mid |H|$ leads to contradictions for $p \geq 13$. To do so we first exclude the existence of maps M with types as above for which the group H would be a semidirect product of a particular form.

Lemma 4.3. *Let p be an odd prime which does not divide κ or λ where $\lambda \geq 3$. Suppose $M = (H; x, y, s, t)$ is an edge-biregular map of type $(2\kappa, 2\lambda)$ such that $H \cong C_p \rtimes D_\nu$. Then $\nu = 2\lambda \equiv 4 \bmod 8$, and $\kappa \in \{2, \lambda\}$ while the supporting surfaces for these maps have $\chi = p(1 - \lambda/2)$ and $\chi = p(2 - \lambda)$ respectively.*

Proof. Suppose that an edge-biregular map of type $(2\kappa, 2\lambda)$ exists with $H \cong C_p \rtimes D_\nu$. With the usual notation, for any edge-biregular map we have $\langle x, y\rangle \cong \langle s, t\rangle \cong V_4$, and hence 4 must divide $|D_\nu|$. In particular this means that D_ν has non-trivial centre, and this leaves us with only two congruence classes to consider for ν, namely 0 mod 8, where the central element of D_ν is a square, and 4 mod 8, where the central element of the dihedral group is not a square.

We use additive notation in the first coordinates to indicate the Abelian cyclic group C_p and multiplicative notation for the dihedral part of the semi-direct product, where $\Phi\colon D_\nu \to Aut(C_p)$, which maps $\alpha \to \phi_\alpha$, is the associated homomorphism. Thus each element of H, and hence each of the canonical generators $\{x, y, s, t\}$, will be written (a, α) for some $a \in C_p$ and some $\alpha \in D_\nu$.

For any involution $\alpha \in D_\nu$, the automorphism ϕ_α must have order dividing two. Since $Aut(C_p) \cong C_{p-1}$ is cyclic there are only two such elements, the identity and the unique involution, so for g, a generating element of C_p, we have ϕ_α must either fix g or invert g. Hence for any $\beta \in D_\nu$ we will have ϕ_β being one of these two automorphisms. Also $(a, \alpha)^2 = (a + \phi_\alpha(a), \alpha^2)$ so the element (a, α) is an involution in H if and only if both $a + \phi_\alpha(a) = 0$ and $\alpha^2 = 1$. Note that if $a \neq 0$ then, in order for (a, α) to be an involution, we must have $\phi_\alpha(a) = -a$, that is ϕ_α must be inverting.

Triples of commuting involutions, that is the non-trivial elements of a copy of V_4 in the group H, have the form: $\{(a, \alpha), (b, \beta), (a + \phi_\alpha(b), \alpha\beta)\}$ such that $a + \phi_\alpha(b) = b + \phi_\beta(a)$, and where α, β and $\alpha\beta$ are all involutions in D_ν, one of which must necessarily be central. So, if both a and b are non-zero in C_p, then $a - b = b - a$, so $2a = 2b$, and hence $a = b$, so the third element of the triple must be $(0, \alpha\beta)$. It is important to note at this point that in this case $\alpha\beta$ would fix any $g \in C_p$. If instead we suppose that $a \neq 0$ and $b = 0$ then $a + \phi_\alpha(b) = b + \phi_\beta(a)$ becomes $a = \phi_\beta(a)$, in other words ϕ_β must fix every element of C_p.

For much of this proof we only need consider the action generated by the dihedral part of the elements in question, rather than the element itself, and so we abuse the notation to denote those in the kernel of Φ, which fix the cyclic elements, by "$+$" and members of the other coset, whose elements invert the cyclic group, by "$-$". Hence any copy of V_4 in H contains three distinct non-identity elements written: $(a, -), (a, -), (0, +)$ for some fixed $a \in C_p$ (which could itself be zero); or, exceptionally, $(0, +), (0, +), (0, +)$.

We now consider what combination of forms can occur within the set of four distinct canonical involutory generators for H, namely $\{x, y, s, t\}$, taking each case in turn.

It will be useful to note that assuming H is generated by any number of involutory elements, each of which has the form $(0, +)$ or $(a, -)$ for a given fixed $a \in C_p$, leads to a contradiction. Indeed since KerΦ must have index two in D_ν, given any two elements α and β which are not in the kernel, we have $\alpha\beta \in$ KerΦ. Hence, in this case, the product of elements $(a, \alpha)(a, \beta)$ must have the form $(0, \alpha\beta)$ that is $(0, +)$. Also then, for $\gamma \in$ KerΦ, the product $(0, \gamma)(a, \alpha)$ must have the form $(0 + a, \gamma\alpha)$ that is $(a, -)$, and similarly $(a, \alpha)(0, \gamma)$ must have the form $(a, -)$. Regardless of the orders of any such products, this restrictive set of forms makes it clear that a is just a place-holder marking when the dihedral part of the element is not in the kernel of Φ, and as such this fixed a cannot contribute towards generating C_p in the first coordinate. So, this is in contradiction to $H \cong C_p \rtimes D_\nu$.

We now highlight another property which will be of use:

Since $\langle x, s\rangle$ is a group isomorphic to $D_{2\lambda}$, henceforth we identify the dihedral parts of $\langle x, s\rangle$ with the corresponding elements for a particular copy of $D_{2\lambda} \leq D_\nu$. Now $2\lambda > 4$ so the central involution of the dihedral group D_ν, which we now call z, will never occur as the second coordinate of x or s (for otherwise we would have $\langle x, s\rangle \cong V_4$).

If λ is odd then $\langle x, s\rangle$ has trivial centre, and every rotation in $D_{2\lambda}$ is a square element in D_ν. This means that either $D_{2\lambda} \leq$ KerΦ or (all of the rotations but) none of the involutions in $D_{2\lambda}$ are in KerΦ. In particular, the canonical involutions generating $\langle s, x\rangle$ must have forms with the same symbol, be that $+$ or $-$, in the second coordinate.

If λ is even then $\langle x, s\rangle$ has non-trivial centre and $z \in D_{2\lambda}$, and, according to the congruence class of ν modulo 8, this z may or may not be a square in D_ν.

Up to twinness of maps, we now divide the argument into two cases.

The first case to consider is when x has the form $(a, -)$, for some a which may be zero:

Suppose s has the form $(b, -)$. But sx, which has order λ, is then denoted $(b - a, +)$. Now p does not divide the face length 2λ, so this forces $b = a$, and hence s is also an $(a, -)$. Now since t commutes with s, and y commutes with x, we know that each of t and y will also have forms within the restrictive set $\{(0, +), (a, -)\}$, contradicting $H \cong C_p \rtimes D_\nu$.

Suppose instead that s has the form $(0, +)$. Thus λ must be even and so $z \in D_{2\lambda}$. This splits into two cases, according to the congruence class of ν.

When $\nu \equiv 0 \bmod 8$, z is a square element in D_ν, and hence $z \in$ KerΦ. Remember that a triple of commuting involutions must include an element $(\dots, z)$ which in this case has the form $(0, +)$. Hence the set $\{s, t, st\}$ must be $\{(0, +), (0, +), (0, +)\}$. In this case, along with y, which is either $(0, +)$ or $(a, -)$, our canonical generators are once again all contained within the restrictive set of forms $\{(0, +), (a, -)\}$, contradicting $H \cong C_p \rtimes D_\nu$.

When $\nu \equiv 4 \bmod 8$, the centre $z \in D_{2\lambda}$ is not a square element in D_ν, and hence there is the possibility of $z \notin$ KerΦ, which we will now explore. (If the central element was in the kernel we would be in the same situation as immediately above, and so the map would not exist.) The group $\langle s, x\rangle \cong \langle r, f | r^\lambda, f^2, (rf)^2\rangle = D_{2\lambda} \leq D_\nu$, can only have an inverting central element if $\lambda = 2(2m+1)$ and $r \notin \mathrm{Ker}\Phi$. Up to the choice of notation for f, a reflection, and r, which generates a cyclic group of order λ, this happens when exactly one of these two generating involutions f, rf is in the kernel, say $rf \in \mathrm{Ker}\Phi$, thereby ensuring $r \notin \mathrm{Ker}\Phi$ and hence $z = r^{2m+1} \notin \mathrm{Ker}\Phi$. With this assumption $x = (a, f)$ and $s = (0, rf)$, which forces $y \in \{(a, z), (0, fz)\}$ and $t \in \{(b, z), (b, rfz)\}$. Notice in particular that t has the form $(b, -)$. If $y = (a, z)$ then the non-divisibility of κ by p forces $a = b$ and thus the canonical generators would be restricted to a set which cannot generate H. So, for such an edge-biregular map to survive, $y = (0, fz)$ and the choices for t with $b \neq a$ would give edge-biregular maps of type $(4, 2\lambda)$ and $(2\lambda, 2\lambda)$ respectively. In particular it is clear that no larger dihedral group than $D_{2\lambda}$ can be found in the second coordinates and so $2\lambda = \nu$. Meanwhile the element xt has the form $(a - b, +)$ and hence the cyclic part C_p of H will also be generated by the set of canonical generators. These maps therefore exist, as do their twins (and indeed duals) while the characteristic for the corresponding supporting surfaces can be found using the Euler-Poincaré formula.

Until now we have been considering the case when x has the form $(a, -)$, for some a which may be zero, and latterly when we also had s being $(0, +)$. The final case to address is when x has the form $(0, +)$:

To avoid being a twin of the maps considered immediately above, we may now assume that s also has the form $(0, +)$. But then, since p does not divide κ, the order of yt, the two canonical generators y and t introduce a maximum of one new form, say $(a, -)$, to the set of canonical generators. Hence we are once again left with an overly restrictive set of generating forms, contradicting $H \cong C_p \rtimes D_\nu$. □

We are now in position to exclude primes $p \geq 13$ from consideration in the case of p dividing $|H|$.

Proposition 4.4. *If $M = (H; x, y, s, t)$ is an edge-biregular map, and p is a divisor of $|H|$, then $p \leq 11$.*

Proof. We suppose that $H = \langle x, y, s, t\rangle$ is the group for an edge-biregular map on surface of Euler characteristic $-p$ and that $p \mid |H|$ where $p \geq 13$. Combining this with Lemma 4.1 we note that the edge-biregular map must have type $(4, 8)$ or $(4, 6)$ forcing $\nu \in \{8, 12\}$ and this implies $|H| = \nu p \leq 12p < p^2$. Hence there is only one Sylow p-subgroup of H and this Sylow subgroup is cyclic, and unique (hence normal) in H.

Since we are restricted to maps of type $(4, 8)$ or $(4, 6)$, and $p \geq 13$, the factor group H/C_p forms a smooth quotient and so would correspond to an edge-biregular map of the same type as H on a surface with Euler characteristic $\chi = -1$. Applying now Lemma 4.2 one sees that H/C_p is a dihedral group of order 8 or 12. Schur-Zassenhaus now implies that H is a (non-trivial) semi-direct product $H \cong C_p \rtimes D_\nu$. But this is in contradiction to Lemma 4.3, so we conclude that $p \leq 11$. □

As the next step we extend Proposition 4.4 to all odd primes greater than 3, while the case where $p = 3$ is dealt with separately in Proposition 4.6. The results of the following two propositions, and those of Theorem 3.2, could be found using a computer, however here we adopt a more classical approach.

Proposition 4.5. *If $M = (H; x, y, s, t)$ is an edge-biregular map on a surface of Euler characteristic $-p$ for some odd prime $p \geq 5$, then p is not a divisor of $|H|$.*

Proof. The conclusion of Proposition 4.4 leaves us with a few small cases we need to address, namely, those listed in Lemma 4.1 (together with the corresponding types) for $p \in \{3, 5, 7, 11\}$; here we will handle the three larger primes one by one.

- *The prime $p = 5$, type $(4, 8)$ for $|H| = 8 \times 5$ and type $(4, 6)$ for $|H| = 12 \times 5$.*

For type $(4, 8)$ with $|H| = 40$. By Sylow theorems the Sylow 5-subgroup is normal so $H \cong C_5 \rtimes D_8$, contrary to Lemma 4.3.

The second type to consider here, $(4, 6)$, comes with a group $|H|$ of order 60. We know H contains a (face stabiliser) subgroup isomorphic to D_6, and so H is not Abelian. Suppose that H has a unique Sylow 5-subgroup, which (by the Schur-Zassenhaus theorem) implies that $H \cong C_5 \rtimes K$ for some group K of order 12. Out of the five such groups L, however, only D_{12} is generated by involutions, and the possibility $H \cong C_5 \rtimes D_{12}$ contradicts Lemma 4.3. It follows that H contains six Sylow 5-subgroups and so $H \cong A_5$. Now A_5 contains 15 involutions and 5 Sylow 2-subgroups, meaning that the intersection of two copies of V_4 is either trivial or the two groups are the same. Hence $\langle x, y\rangle = \langle y, t\rangle = \langle s, t\rangle \cong V_4$ implies $|H| = 4$, which is a contradiction.

- *The prime $p = 7$, type $(4, 8)$ for $|H| = 8 \times 7$ and type $(4, 6)$ for $|H| = 12 \times 7$.*

For the type $(4, 8)$ we have $|H| = 56$. If the Sylow 7-subgroup of H is not normal then H contains $8 \times 6 = 48$ elements of order 7, leaving only eight other elements in the group H. These eight elements must form the single (and hence normal) Sylow 2-subgroup, which is the face stabiliser, $D_\ell \cong D_8$. But then all the involutions of H must lie in D_ℓ and so H would not be generated by involutions. So the Sylow 7-subgroup of H is normal and hence $H \cong C_7 \rtimes D_8$ which contradicts Lemma 4.3 again.

The type $(4, 6)$ and $|H| = 84$ is excluded by Sylow theorems and the Schur-Zassenhaus as for the same type but in the case $p = 5$: A group of order 84 has a unique (normal) Sylow 7-subgroup, and then the unique possibility is $H \cong C_7 \rtimes D_{12}$, which is impossible by Lemma 4.3.

- *The prime $p = 11$, type $(4, 8)$ for $|H| = 8 \times 11$ and type $(4, 6)$ for $|H| = 12 \times 11$.*

The type $(4, 8)$ with $|H| = 88$ is excluded by observing that H contains a unique Sylow 11-subgroup, leaving only the possibility $H \cong C_{11} \rtimes D_8$ that contradicts Lemma 4.3.

The final case to consider is the one of the type $(4, 6)$ that comes with a group H of order 132. If the Sylow 11-subgroup in H is normal then, by a similar analysis as done for $p = 5$, we would have $H \cong C_{11} \rtimes D_{12}$, which is impossible by Lemma 4.3. So there are 12 Sylow 11-subgroups and there are $12 \times 10 = 120$ elements of order 11. If H contained more than one Sylow 3-subgroup then there would be 8 elements of order 3, leaving room for only one Sylow 2-subgroup. Now, H is generated by involutions and so the Sylow 2-subgroup $D_k \cong V_4$ cannot be unique, otherwise we would have $H = D_k$. So, the Sylow 3-subgroup in H, isomorphic to C_3, must be unique and hence normal. Thus, in our case with $\ell = 6$, the subgroup $\langle sx\rangle \cong C_3$ is normal in H, which implies that $ysxy, tsxt \in \{sx, xs\}$. If we suppose $tsxt = sx$ then $txt = x$ so t commutes with s and x and y. But $\langle x, y\rangle$ is a Sylow 2-subgroup of H and so, by the distinctness of the generators, we would have $t = xy$. Hence $H = \langle sx\rangle \rtimes \langle x, y, t\rangle = \langle sx\rangle \rtimes \langle x, y\rangle$ which has order 12, not 132. By symmetry, supposing that y fixes sx leads to a contradiction too. Hence we have $tsxt = xs$ and $ysxy = xs$. These give, in turn, $txt = sxs$ and $ysy = xsx$ which, when combined, give $txtysy = 1$. However, recalling that y commutes with both x and t, this yields $xs = 1$, a contradiction. This completes the proof. □

We now consider the exceptional case when $p = 3$, which does indeed yield edge-biregular maps, namely those corresponding to the final part of Theorem 3.1.

Proposition 4.6. *If $M = (H; x, y, s, t)$ is an edge-biregular map on a surface of Euler characteristic -3, then up to duality and twinness, H is of type $(4, 6)$ and has a presentation of the form*

$$H = \langle x, y, s, t | x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (yt)^2, (sx)^3, (xyt)^3, (sty)^3, (xyst)^2\rangle \cong D_6 \times D_6$$

Proof. When the prime $p = 3$, we have the following possibilities: type $(6, 12)$ for $|H| = 4 \times 3$; type $(4, 8)$ for $|H| = 8 \times 3$; and type $(4, 6)$ for $|H| = 12 \times 3$.

An edge-biregular map of type $(6, 12)$ on this surface would have $|H| = 4\times3 = 12$ and so the group H must be the face stabiliser $H = D_l \cong D_{12}$ which is dihedral. Referring to the dihedral classification of edge-biregular maps in [12] we can see that such a map does not exist.

For type $(4, 8)$ we have $|H| = 24$ and as H is generated by involutions, the Sylow 2-subgroup $D_\ell \cong D_8$ cannot be normal in H. This leaves only two options: $H \cong C_3 \rtimes D_8$, which is clearly impossible, or $H \cong S_4$. Suppose $H \cong S_4$ is represented as a permutation group on the set $\{1, 2, 3, 4\}$. The non-trivial elements in the three copies of V_4 in S_4 then form the sets $T_1 = \{(12)(34), (12), (34)\}$, $T_2 = \{(13)(24), (13), (24)\}$, $T_3 = \{(14)(23), (14), (23)\}$ and $T_4 = \{(12)(34), (13)(24), (14)(23)\}$. Since $\langle s, x\rangle = D_\ell \cong D_8$, we may assume, without loss of generality, that $\{s, x\} = \{(12)(34), (24)\}$. But then the fact that $\langle y, t\rangle = D_k \cong V_4$ leads us to conclude that for the four *distinct* canonical generators we must have $x, y, t, s \in T_2 \cup T_4$. But these two sets only generate D_8, not the whole of S_4, a contradiction.

We proceed with type $(4, 6)$ for a group H of order 36 and let K be its Sylow 3-subgroup of order 9.

Consider the *non-trivial and transitive* permutation representation π of H on the set H/K of the right cosets of K in H, given as follows: To each $h \in H$ we assign a permutation π_h of the set H/K mapping the coset Kx onto Kxh, for every $x \in H$. Now, as $|H/K| = 4$, the representation π is a group homomorphism from H into S_4, and its kernel is a normal subgroup of H distinct from H; note also that π cannot be surjective (by divisibility of the source and target by 4 and 8). The only proper transitive subgroups of S_4 are A_4 and the unique subgroup isomorphic to V_4, so that $|\mathrm{Ker}(\pi)| \in \{3, 9\}$. But notice that if $|\mathrm{Ker}(\pi)| = 3$, so that H contains a normal subgroup $\mathrm{Ker}(\pi) \cong C_3$, and $\mathrm{Im}(\pi) \cong A_4$ then we have an immediate contradiction since A_4 is not generated by involutions.

It remains to consider the case when $|\mathrm{Ker}(\pi)| = 9$, which means that the kernel coincides with the Sylow 3-subgroup K of order 9 in H, and $H \cong K \rtimes V_4$.

Suppose, for a contradiction, that $K = C_9$. Now, C_3 is characteristic in C_9 and hence normal in H. There is only one copy of C_3 in H, and that is $\langle sx\rangle$. Denoting elements in $H/\langle sx\rangle$ with bar notation, we then have $\bar{s} = \bar{x}$ and hence $H/\langle sx\rangle$ is generated by three commuting involutions $\bar{x}, \bar{y}$ and $\bar{t}$. This has order at most 8, not the required 12, and so is in contradiction to the order of H.

We may consider C_3^2 as a two dimensional vector space over $GF(3)$ with elements written as vectors. The automorphism group for C_3^2 is $GL(2, 3)$ which is known to have just one conjugacy class of subgroups isomorphic to V_4. Thus we let $x = (\underline{x}, x')$,

$y = (\underline{y}, y')$, $t = (\underline{t}, t')$, $s = (\underline{s}, s')$ where the underlined vectors in the first coordinates are elements of C_3^2, and $x', y', t', s' \in V_4 \le GL(2,3)$. Now, since sx has order 3, it is clear that $s'x'$ must be the identity, and so $x' = s'$. Also, since the products xy, yt and ts must all have order 2, we certainly have $x' \ne y' \ne t' \ne x'$, and so $t' = x'y'$.

Suppose the homomorphism $\Phi\colon V_4 \to GL(2,3)$ associated with the semi-direct product is not injective. In any case, the kernel of Φ cannot be the whole of V_4, since this would yield a direct product $H \cong C_3^2 \times V_4$ which certainly cannot be generated by involutions. The only remaining option is for two of the elements in V_4 to be mapped to a given involution $\alpha \in GL(2,3)$, while their product is mapped to the identity. Now $x' \notin \mathrm{Ker}\Phi$, otherwise x and s being involutions would then force $\underline{x} = \underline{s} = \underline{0}$ which is absurd since xs must have order 3. So, up to twinness, we may assume, $x'y' \in \mathrm{Ker}\Phi$. But then this would force $\underline{t} = \underline{0}$, meaning t is central in the group H. Now, the elements x and y are involutions so α acts to invert any non-zero parts of $\underline{x}$ and $\underline{y}$. Also, x and y commute so $\underline{x} + \alpha(\underline{y}) = \underline{y} + \alpha(\underline{x})$, that is $\underline{x} - \underline{y} = \alpha(\underline{x} - \underline{y})$, and this forces $\underline{x} = \underline{y}$. Now we have in fact $t = xy$ and so $H = \langle x, t, s\rangle = \langle x, s\rangle\langle t\rangle \cong D_6 \times C_2$, a contradiction.

We have now shown that the homomorphism mapping from V_4 to the associated actions in $GL(2,3)$ must be injective, since in any other case we could not generate the whole group $H \cong C_3^2 \rtimes V_4$ just with involutions. So we may now identify $\{x', y', t', s'\}$ with their images in $GL(2,3)$, choosing the canonical copy of V_4 in $GL(2,3)$, which consists of the diagonal matrices.

In the interests of generating the whole group $H \cong C_3^2 \rtimes V_4$, it can be checked that $x' = s' = -I$. To satisfy the other known order properties for the products of pairs of generating involutions, up to twinness, we find we are restricted to the set of canonical involutions as follows:

$$x = (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}), \qquad y = (\begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}),$$

$$t = (\begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}), \qquad s = (\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})$$

where $x_1 \ne s_1$ and $x_2 \ne s_2$. Regardless of whichever allowable choices of values for x_i and s_i are made, it is clear that $\langle xyt, sty\rangle \cong C_3^2$. Also we have $\langle y, t\rangle \cong V_4$ as expected, and hence $H = \langle x, y, s, t\rangle = \langle xyt, sty\rangle \rtimes \langle y, t\rangle \cong C_3^2 \rtimes V_4$. Also $H = \langle xyt, xy\rangle \times \langle sty, st\rangle \cong D_6 \times D_6$ and a presentation for the resulting map will be as follows:

$$H = \langle x, y, s, t | x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (yt)^2, (xyt)^3, (sty)^3, (xyst)^2\rangle \cong D_6 \times D_6.$$

By this analysis, the map is unique up to duality and twinness. The above group presentation clearly indicates that this edge-biregular map is isomorphic to its twin, and hence is also a fully regular map. Meanwhile the face length ℓ being 6 is clearer to see when the additional (consistent) relator $(sx)^3$ is incorporated into the presentation . □

The only prime left to be considered in this part is $p = 2$, and by exploring this case we will also establish validity of Theorem 3.2.

Proof of Theorem 3.2.

Proof. We now assume $p = 2$. We recall from the initial workings of this section that the possible types of map which can occur when $\nu \in \{4, 8, 12\}$ giving $|H| = 8$, 16 and

24, respectively, for types $(8, 8)$, $(4, 8)$ and $(4, 6)$. We will also need to address the only remaining case for even k, ℓ such that $4 \mid |H| = 2\nu$, namely when $\nu = 6$ and $|H| = 12$ which occurs when the type (k, ℓ) is $(4, 12)$ or $(6, 6)$. We proceed according to the order of the group H.

The first possibility is dealt with quickly, because in an edge-biregular map of type $(8, 8)$ with $|H| = 8$ we *must* have $H \cong D_8$ (the stabiliser of a (single) vertex, say). The group D_8 contains exactly four non-central involutions, and two subgroups isomorphic to V_4, while the central element cannot be equal to any one of the distinct canonical involutory generators. It may be checked that up to isomorphism (since the resulting map is both self-dual and fully regular) there is just one way to present H canonically, necessarily equivalent to the form given by $H_{2,1}$ in Theorem 3.2.

When we consider $\nu = 6$ for the type $(4, 12)$ we clearly have the group $H = \langle x, s\rangle \cong D_{12}$, and up to twinness and duality there is only one canonical presentation. This is shown as $H_{2,2}$ in Theorem 3.2.

In the case of type $(6, 6)$, we have $\nu = 6$ and the group is again $H \cong D_{12}$. This dihedral group contains two copies of D_6, one of which must be $\langle s, x\rangle$ and the other $\langle t, y\rangle$. There is only one cyclic group of order three contained in D_{12} and so there are only two possibilities for the elements of order 3, namely $sx = yt$ (which results in contradictions) or $sx = ty$ which yields the presentation $H_{2,3}$ in Theorem 3.2.

The next possibility we look at is type $(4, 8)$, with $|H| = 16 = 2\times 8$. We know $\langle s, x\rangle = D_\ell \cong D_8$ is normal in H since it has index two. First note that $H/D_\ell = \langle yD_\ell, tD_\ell\rangle$ has order 2 so either $y \in D_\ell$, or $t \in D_\ell$, or $yD_\ell = tD_\ell \neq D_\ell$ and so the product ty is in D_ℓ. We deal with the latter case after addressing the first two together, since they are equivalent up to twinness. Also $\langle sx\rangle \cong C_4$, being characteristic in the dihedral group, is normal in H. Conjugation by the elements x and s clearly invert sx while each of y and t must either fix or invert sx, resulting in differing canonical presentations for H.

We first suppose, by our choice of the labeling of orbits, that is up to twinness, that $y \in D_\ell$. If y inverts sx then we have $y \in \{sxs, xsx\}$ and so, by the distinctness of generators, $y = sxs$, and the order of ys is four. If t also inverts sx then $tsxt = xs$. But then $y = stsxt = txt$ which implies $y = x$, a contradiction. However, if t fixes sx, which happens if and only if t fixes x, we obtain

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^4, (tx)^2, ysxs\rangle$$

which is the presentation of $H_{2,4}$ in Theorem 3.2. Notice that this map must be supported by an orientable surface since there are no odd length relators in this presentation.

If, on the other hand, y fixes sx then $y = (sx)^2$, the central element in D_ℓ, and hence $y(sx)^2$ is a relator, forcing the supporting surface to be non-orientable. Another consequence of $y = (sx)^2$ is that $(ys)^2$ may also appear as a relator. If t also fixes sx, we may arrive at the presentation

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, y(sx)^2\rangle$$

which is $H_{2,5}$ in Theorem 3.2. Otherwise, t inverts sx, giving

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, y(sx)^2\rangle$$

which is the presentation of the group $H_{2,6}$ in Theorem 3.2.

Now let us suppose that $ty \in D_\ell$. As before, we consider when ty fixes sx, that is in this case $ty = (sx)^2$, the central element in D_ℓ. If one (and hence both) of y and t invert sx then $tsxt = xs$ so $txt = sxs$. This leads to $ty = sxsx = txtx$ which implies $y = xtx$ so $y = t$, a contradiction to the distinctness of the canonical involutions. In the other case both y and t fix sx. This is equivalent to y commuting with s, and t commuting with x, and this gives us the following presentation which yields a group of the right order.

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (tx)^2, ty(sx)^2 \rangle$$

This is listed as $H_{2.7}$ in Theorem 3.2. You may notice this presentation has a redundant relator, which serves to make clear the underlying full regularity of the corresponding map.

On the other hand consider when ty inverts sx, and hence exactly one of y or t must fix sx. Up to twinness we may assume y fixes sx in which case y is central in H and $(ys)^2$ is a relator. In this case then t inverts sx and hence $(tsx)^2$, or equivalently $(stx)^2$, is also a relator.

For $ty \in D_\ell$ to invert sx we must have $ty \in \{sxs, x, xsx, s\}$. The first two options yield a contradiciton, as we shall now see. If $ty = sxs$ then $1 = tysxs = ysys$ so $ytysx = sy$, that is $tsx = sy$ and hence $tx = y$. But then $ty = x = sxs$ which means $xs = sx$, contradicting the order of sx. If $ty = x$ then $1 = tyx = (tsx)^2$ so $y = sxts$, that is $x = yt = sxs$, the same contradiction.

It can be checked that the remaining options, namely ty is either xsx or s, avoid the contradictions which cause the order of the group H to become too small, and they yield these two remaining canonical presentations.

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tyxsx \rangle$$

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^4, (ys)^2, (stx)^2, tys \rangle$$

These are listed respectively as $H_{2.8}$ and $H_{2.9}$ in Theorem 3.2. Careful comparisons of the relators in the above presentations shows that the maps are pairwise non-isomorphic.

The last possibility to address is the type $(4, 6)$ for a group H of order $3 \times 8 = 24$. Suppose the Sylow 3-subgroup is not normal in H. Then $H \in \{SL(2, 3), S_4, A_4 \times C_2\}$. Now, we know H has more than one involution, so it cannot be isomorphic to $SL(2, 3)$. Also, A_4 only contains 3 involutions, all in the subgroup which is isomorphic to V_4, but H is generated by involutions, H so cannot be isomorphic to $A_4 \times C_2$. Thus, if the Sylow 3-subgroup is not normal in H, then $H \cong S_4$.

We will now refer to the standard permutation representation of S_4 on the set $\{1, 2, 3, 4\}$ and remember the sets $T_1 - T_4$ introduced in the first part of the proof of Proposition 4.6 for $p = 3$. Since $\langle s, x\rangle = D_\ell \cong D_6$, we may assume, up to choice of notation and hence without loss of generality, that x and s are (12) and (23), respectively. The condition that $\langle y, t\rangle \cong V_4$ means that $y = (12)(34)$, and $t = (14)(23)$, arriving at the non-orientable map given by the presentation listed as $H_{2,10}$ in Theorem 3.2:

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, t(sy)^2, y(xt)^2 \rangle \cong S_4$$

Now suppose the Sylow 3-subgroup $\langle sx\rangle \cong C_3$ is normal in H. Then conjugation by each of y and t must either fix or invert sx.

Suppose y inverts sx. Then $ysxy = xs$ so $(ysx)^2$ is a relator. If t also inverts sx then $tsxt = xs$. Then $txt = sxs$ and hence $txtx = sxsx$ but y commutes with both t and x

so $txtx = ytxtxy = ysxsxy = xsxs$ is self-inverse. But $sxsx$ has order three so this is a contradiction. If, on the other hand, t fixes sx then we obtain the presentation of the group $H_{2,11}$ from Theorem 3.2:

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ysx)^2, (tx)^2\rangle.$$

Suppose finally that y fixes sx, in which case $ysxy = sx$ so $(ys)^2$ is a relator. In the case where t also fixes sx, this leads to the presentation of the group $H_{2,12}$ from Theorem 3.2:

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ys)^2, (tx)^2\rangle$$

Otherwise, t inverts sx, giving the presentation

$$\langle\, x, y, s, t \mid x^2, y^2, s^2, t^2, (xy)^2, (st)^2, (ty)^2, (sx)^3, (ys)^2, (txs)^2\rangle$$

which represents the twin map of the edge-biregular map determined by the group $H_{2,11}$.

It remains to address isomorphism type, full regularity and orientability of the twelve edge-biregular maps identified above. Obviously, $H_{2,1} \cong D_8$, $H_{2,2} \cong D_{12} \cong H_{2,3}$ and $H_{2,7} \cong S_4$. Observe that the element $t \notin \langle s, x\rangle$ is central in $H_{2,4}$ and $H_{2,5}$ while $st \notin \langle s, x\rangle$ is central in $H_{2,6}$, and $y \notin \langle s, x\rangle$ is central in $H_{2,7}$, $H_{2,8}$ and $H_{2,9}$ and so the six groups are all isomorphic to $D_8 \times C_2$. Further, it follows from the derivation of the remaining two groups that $H_{2,11} = \langle s, x\rangle \rtimes \langle t, y\rangle = \langle s, x\rangle \times \langle t, xy\rangle \cong D_6 \times V_4$ and $H_{2,12} = \langle s, x\rangle \times \langle t, y\rangle \cong D_6 \times V_4$. It is easy to check that the twelve presentations are correct and complete (describing exactly the groups listed, albeit including some redundant relators). The maps defined by $H_{2,1}$, $H_{2,3}$, $H_{2,4}$, $H_{2,7}$, $H_{2,11}$ and $H_{2,12}$ are orientable because all their defining relations have even length in the generators x, y, s, t, which does not apply to the remaining six maps. Finally, the maps associated with $H_{2,1}$, $H_{2,3}$, $H_{2,7}$, $H_{2,10}$, and $H_{2,12}$ are the only fully regular ones out of the above twelve, as their groups admit an automorphism swapping x with s and y with t. The proof of Theorem 3.2 is now complete. □

5 The case when p does not divide the order of H

Let $M = (H; x, y, s, t)$ be a (finite) edge-biregular map of type (k, ℓ) on a surface of Euler characteristic $-p$ for some prime p; in view of Theorem 3.2 we will assume that p is odd and hence the surface is non-orientable. Recall that the group H is assumed to be presented as in (2.5), and its order together with the type of the map and the characteristic of the surface are tied by the equation (3.1). We begin with an auxiliary result and omit a proof since it is almost verbatim the same as the proof of Lemma 3.2 of [4].

Lemma 5.1. *If q is a prime divisor of $|H|$ relatively prime to the Euler characteristic, then Sylow q-subgroups of H are cyclic if q is any odd prime, and dihedral if $q = 2$.* □

From this point on until the end of this section we will assume that p is *not* a divisor of the order of H. Since we are working up to duality, instead of $k \leq \ell$ we henceforth assume that *the* 2*-part of* k *is not smaller than the* 2*-part of* ℓ. Remember also that H contains a subgroup isomorphic to V_4 and so 4 divides $|H|$. Comparing this condition with equation (3.1) allows us to set $k = 4\kappa$ and $\ell = 2\lambda$ for integers κ and λ. The stage will be set by proving solvability of H.

Proposition 5.2. *If p is such that $p \nmid |H|$, the group H is solvable.*

Proof. We start by proving that $|D_k \cap D_\ell| \leq 4$. The group $K = D_k \cap D_\ell$ is obviously cyclic or dihedral. Suppose for a contradiction that $|K| > 4$. Then K contains an element z of order at least 3, so that $z = (sx)^m = (yt)^n$ for some m and n. Clearly, z commutes with sx and also with yt. We also have $(xy)z(xy) = yx(sx)^m xy = y(xs)^m y = y(ty)^n y = (yt)^n = z$, so z commutes with xy as well. Now the map is carried by a non-orientable surface and so $H = \langle xy, sx, yt \rangle$ by the theory explained in section 2. Hence z commutes with all the canonical generators, and thus it is central in H. Specifically, z is central in $D_k \leq H$. It is well-known that the centre of a (non-Abelian) dihedral group is either trivial or has order 2. But, by our assumption, the order of $\langle z \rangle$ is greater than 2, a contradiction.

Next we prove that the group H is a product of D_k and D_ℓ. By (3.1) the assumption $p \nmid |H|$ implies that p must be absorbed by the denominator of $\nu(k,\ell)$, that is, $k\ell - 2(k+\ell) = rp$ for some integer r. Thus, $|H| = 2k\ell/r$, but by the first part of the proof we also have $|H| \geq |D_k||D_\ell|/|D_k \cap D_\ell| \geq k\ell/4$, implying that $r \leq 8$. Using $k = 4\kappa$ and $\ell = 2\lambda$ as agreed before, one has $rp = kl - 2(k+l) = 8\kappa\lambda - 8\kappa - 4\lambda = 4(2\kappa\lambda - 2\kappa - \lambda) = 4cp$. Notice that this, along with the assumption $p \nmid |H|$, also implies $\gcd(2\kappa, \lambda) = c \leq 2$. Hence $|H| = k\ell/(2c)$ where $c = \gcd(2\kappa, \lambda) \in \{1, 2\}$.

If $c = 1$, then $|H| = k\ell/2$ and also λ must be odd, so that $|D_k \cap D_\ell| \leq 2$. We also have $|H| \geq |D_k||D_\ell|/|D_k \cap D_\ell| \geq k\ell/2$ and so equality holds throughout and hence $H = D_k D_\ell$ if $c = 1$. In the case when $c = 2$ we have $(2\kappa\lambda - 2\kappa - \lambda) = 2p$ where p is an odd prime, so λ must be even; also, $\gcd(\kappa, \lambda/2) = 1$. Now, $c = 2$ implies $|H| = k\ell/4$ and $|D_k \cap D_\ell| \leq 4$, and as we also have $|H| \geq |D_k||D_\ell|/|D_k \cap D_\ell| \geq k\ell/4$ we conclude that equality holds throughout and hence $H = D_k D_\ell$.

We may now complete the proof by invoking the result of Huppert [9] that the product of two dihedral groups is solvable. □

The fact that H is solvable yields that it has a non-trivial Fitting subgroup F; recall that F is the largest nilpotent normal subgroup of H. In particular, F is a direct product of its Sylow subgroups. By what we know about the Sylow subgroups of H from Lemma 5.1 we have $F = F_1 \times F_2$, where F_1 is cyclic, of odd order (possibly trivial), and F_2 (if non-trivial) is a cyclic or a dihedral 2-group; we will henceforth split our analysis according to this dichotomy.

As a general remark, observe that we may assume $F \neq H$. Indeed, if $F = H$, then F_1 would have to be trivial (otherwise F could not be generated by involutions) and F_2 would have to be non-cyclic (to contain enough distinct involutions), so that $H = F = F_2$ would have to be dihedral. But edge-biregular maps with dihedral groups H have already been classified in [12]. Without giving details we just state that, as a consequence of the table displaying the classification results in [12], the only edge-biregular maps of Euler characteristic $-p$ for an odd prime p determined by a dihedral group of automorphisms are the first two maps in Theorem 3.1 defined by the groups $H_{p(1)}$ and $H_{p(2)}$.

5.1 The case when the Fitting subgroup is cyclic

From now on we will *assume that F is cyclic*. In such a case F contains either no involution (if $|F|$ is odd) or a unique involution (if F_2 is non-trivial cyclic 2-group). This implies that F can contain at most one of the four involutions s, t, x, y.

We will show that F cannot have index 2 in H. Indeed, suppose $[H : F] = 2$. If F contains no generating involution from the set $\{s, t, x, y\}$, then the index-2 condition implies that $sx, xy, yt \in F$. But since $H = \langle sx, xy, yt\rangle$ we would have $F = H$, a contradiction. Thus, let one of $\{s, t, x, y\}$ be the unique generating involution of H contained in F. We may without loss of generality assume that this element is y, as all the other cases are handled by symmetries in the forthcoming argument. Now $\langle y\rangle$ is characteristic in F and therefore normal in H, and so y, being an involution, is also central in H. Further, from $s, t \notin F$ we have $st \in F$, and by uniqueness of the involution in F it follows that $y = st$.

As now $s, x \notin F$, we have $sx \in F$ and so F contains the cyclic group $\langle sx\rangle$. If its order is odd, then F also contains the cyclic group $K = \langle sx\rangle\langle y\rangle$. But then, recalling centrality of y in H, conjugation by, say, s inverts every element of K and so $L = K \rtimes \langle s\rangle$ is a dihedral group. From $s, xs, y \in L$ and $y = st$ we also have $t, x \in L$ and so $L = H$ is dihedral, the case we have already disposed of. If the order of $\langle sx\rangle$ is even, then $y = (sx)^j$ for j equal to half of the order of sx, since both elements are an involution in the cyclic group F. But we saw earlier that $y = st$, giving $st = (sx)^j$ and hence $t = s(sx)^j$. This, however, with $y = (sx)^j$ shows that $H = \langle s, x\rangle$, which again means that H is dihedral.

Altogether, we have shown that for non-dihedral H and for a cyclic F we must have $[H : F] > 2$. Now, by the available theory H/F embeds in $\mathrm{Aut}(F)$, and as the latter is Abelian if F is cyclic, we conclude that H/F is Abelian. But H/F is generated by four elements of order at most 2, so that $H/F \cong C_2^m$ for some m such that $2 \le m \le 4$. Further, since $H = \langle st, sx, xy\rangle = \langle st, ty, xy\rangle$ it follows that $m \in \{2, 3\}$ and both sxF and tyF have order at most 2, so that $(sx)^2, (ty)^2 \in F$.

From earlier calculations we recall that both k, ℓ are even and greater than 2, and assuming that the 2-part of k is not smaller than the 2-part of ℓ we have the following: *If* $c = \gcd(k/2, \ell/2)$, *then* $c \in \{1, 2\}$, k *is a multiple of* 4, *and* $|H| = k\ell/(2c)$. In particular, note that $8 \nmid \ell$, and $c = 1$ or 2 according as $\ell/2$ is odd or even.

Under these conditions we first show that $ty \notin F$. Indeed, suppose that $ty \in F$. Observe that the order of $(sx)^c$ is $\ell/(2c)$ and hence odd, so that $(sx)^2 \in F$ implies $(sx)^c \in F$ also for $c = 1$. As $ty, (sx)^c \in F$ and the orders of the two elements, $k/2$ and $\ell/(2c)$, are relatively prime, it follows that $(k/2)(\ell/(2c)) \le |F| = |H|/[H : F] \le k\ell/(2c[H : F])$, which yields $[H : F] \le 2$, a contradiction.

It follows that $ty \notin F$. But we know that $(ty)^2 \in F$, and we may use essentially the same chain of inequalities as above, with $k/2$ replaced by $k/4$ (which is the order of $(ty)^2$), to conclude that $[H : F] \le 4$. We saw, however, that $[H : F]$ is a power of 2 and greater than 2, so that $[H : F] = 4$, and we must have equalities in the above chain throughout. In more detail, and using the fact that F is cyclic, we have $F = \langle (sx)^c\rangle\langle (ty)^2\rangle = \langle (sx)^c(ty)^2\rangle \cong C_n$ for $n = k\ell/(8c)$, with $ty \notin F$. Observe that the order of $(sx)^2$ is odd in both cases for $c \in \{1, 2\}$, and is and relatively prime to $k/4$ (the order of $(ty)^2$).

We show that F contains none of the generating involutions s, t, x, y of H, and at most one of the involutions st, xy. For if F contained one of t, y, then this element would have to coincide with the central involution $(ty)^{k/4}$, but such an equality quickly gives $t = 1$ or $y = 1$. If $s \in F$ (and the case $x \in F$ is done similarly), then s would commute with $(sx)^2$, which is equivalent to $(sx)^4 = 1$. Then, $\ell/2$ would have to divide 4 and since $8 \nmid \ell$ we would have $\ell = 4$. But this would give $H = k\ell/4 = k$, so that H would be dihedral.

To address the remaining part, note that by non-orientability we know that

$H = \langle st, xy, yt\rangle$. From $H/F = \langle stF, xyF, ytF\rangle$ and $yt \notin F$ while $(yt)^2 \in F$, together with $[H : F] = 4$, it follows that at most one of st, xy can be contained in F.

In what follows we will without loss of generality assume that $st \notin F$. As F and $\langle s, t\rangle$ now intersect trivially, with the help of the above finding this means that the semi-direct product

$$F \rtimes \langle s, t\rangle = \langle (sx)^c(ty)^2\rangle \rtimes \langle s, t\rangle \cong C_n \rtimes V_4 \tag{5.1}$$

for $n = k\ell/(8c)$ has order $4n = |H|$ and so $H = F \rtimes \langle s, t\rangle \cong C_n \rtimes V_4$. If $n = |F|$ is even, then the unique non-trivial involution $(ty)^4 \in F$ generates a subgroup isomorphic to C_2 that is characteristic in F and hence normal in H, which means that $(ty)^4$ is central in H. But then H would contain the subgroup $\langle (ty)^{k/4}\rangle \times \langle s, t\rangle \cong C_2^3$, contrary to the fact that H has dihedral Sylow 2-subgroups. It follows that n is odd and so is $\kappa = k/4$ (and we know the same about $\ell/(2c) = \lambda/c$); also, both xy and st must lie outside F, and the Sylow 2-subgroups of H are isomorphic to V_4.

In the proof of solvability of H we have encountered the equation $2\kappa\lambda - 2\kappa - \lambda = cp$ for some $c \in \{1, 2\}$. If $c = 2$, then λ is exactly divisible by 2, and then oddness of κ with $2\kappa(\lambda - 1) = 2p + \lambda$ gives a contradiction as the right-hand side is divisible by 4 while the left-hand side is not. It follows that $c = 1$, and so $sx \in F$; observe that then $sy \notin F$ as in the opposite case we would have $(xs)(sy) = xy \in F$ which has already been excluded.

We now let $u = sx$ and $v = ty$; note that our cyclic group F, of order a product of two odd and relatively prime numbers $\ell/2$ and $k/4$ (the orders of u and v^2), is generated e.g. by uv^2. We saw that $y, ys, yt \notin F$, so that by (5.1) for $c = 1$ we must have $y = wst$ for some $w \in F$. The fact that $w = yts$ commutes with $u = sx$ is equivalent to $ty(sx)yt = xs = u^{-1}$, so that conjugation by v inverts u. Similarly, $w = yts$ commutes with $v^2 \in F$, which translates to $[s, v^2] = 1$, and as $u = sx$ commutes with v^2 we also have $[x, v^2] = 1$.

In somewhat more detail, let $w = u^a(v^2)^b = u^av^{2b}$ for uniquely determined integers a, b such that $0 \le a < \ell/2$ and $0 \le b < k/4$. Using the facts that s inverts u and commutes with v^2 and $[u, v^2] = 1$, from $y = wst$ it follows that $v^{-2} = (yt)^2 = (u^av^{2b}s)^2 = v^{4b}$, so that $v^{4b+2} = 1$, and for b in the above range we have $4b + 2 = k/2$ (the order of v) and so $b = (k - 4)/8$. Normality of $\langle u\rangle$ in H (being characteristic in F) further implies that $yuy = u^j$ for for some j, $1 \le j < \ell/2$, such that $j^2 \equiv 1 \bmod \ell/2$. Since v inverts u, we obtain $u^{-1} = tyuyt = tu^jt$, which implies $tut = u^{-j}$. Observing now that conjugation by st maps u^a onto u^{aj} and inverts v^2, from $y = u^av^{2b}st$ we obtain $1 = (u^av^{2b}st)^2 = u^{a(j+1)}$, so that $a(j + 1) \equiv 0 \bmod \ell/2$.

Going one step further and using properties derived above, from $y = u^av^{2b}st$ we have $xy = xu^asv^{2b}t = u^{-a-1}v^{2b}t$, and as conjugation by t inverts v^2 and maps u onto u^{-j} one obtains $1 = (u^{-a-1}v^{2b}t)^2 = u^{(a+1)(j-1)}$, which gives $(a + 1)(j - 1) \equiv 0 \bmod \ell/2$. Subtracting the last congruence from $a(j+1) \equiv 0 \bmod \ell/2$ yields $2a+1 \equiv j \bmod \ell/2$. As $2^{-1} = (\ell+2)/4 \bmod \ell/2$ (which is odd), it follows that $a \equiv (j-1)2^{-1} = (j-1)(\ell+2)/4 \bmod \ell/2$, giving a unique value of $a \in \{0, 1, \ldots, \ell/2 - 1\}$. Observe also that for this value of a and for any j such that $j^2 \equiv 1 \bmod \ell/2$ one has $(j + 1)a = (j^2 - 1)2^{-1} \equiv 0 \bmod \ell/2$, which is the congruence obtained earlier.

The last step will be reintroducing the notation $k = 4\kappa$ and $\ell = 2\lambda$ for odd and relatively prime κ and λ, and observing that $2b + 1 = \kappa$ and so $1 = tyv^{2b}u^as = v^\kappa u^as$. The last relation is equivalent to $v^{\kappa-1}u^a = v^{-1}s$, and from $[u, v^2] = 1$ (a consequence of $[s, v^2] = 1 = [x, v^2]$) and oddness of κ it follows that $1 = [u, v^{\kappa-1}u^a] = [u, v^{-1}s]$ and commutation of u and $v^{-1}s$ is equivalent to u being inverted by conjugation by v. Summing up, the above facts well-define a group $H = H_{p,j}$ of order $k\ell/2$ generated by four involutions s, t, x, y and presented as follows, with $u = sx$ and $v = ty$:

$$H_{p,j} = \langle\, x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, u^\lambda,\, v^{2\kappa},\, [s, v^2],\, [x, v^2],\, tutu^j,\, v^\kappa u^a s \,\rangle \tag{5.2}$$

for a non-negative integer $j < \ell/2$ such that $j^2 \equiv 1 \bmod \lambda$, with $a = (j-1)(\lambda+1)/2$. This is the presentation appearing as a third item in Theorem 3.1. Note that here $F = \langle u, v^2\rangle = \langle uv^2\rangle$ is cyclic, of order $\kappa\lambda = k\ell/8$, and $H_{p,j} = F \rtimes \langle s, t\rangle$, as derived earlier.

Proof of correctness of the presentation (5.2)

Proof. Let $G_0 = \langle s, x \,|\, s^2, x^2, (sx)^\lambda\rangle \cong D_\ell$ and let $u = sx$. Let us introduce a pair of automorphisms θ and τ of G_0 completely defined by letting $\theta(s) = s$ and $\theta(u) = u^{-j}$ for some j such that $j^2 \equiv 1 \bmod \lambda$, and $\tau(u) = u^j$, $\tau(x) = x$. These definitions imply, for example, that $\theta(x) = su^{-j}$ and $\tau(s) = u^j x$. It can be easily verified that θ and τ commute and both are of order two. It follows that we have a well-defined split extension of G_0 by the subgroup $V_4 \cong \langle\theta, \tau\rangle < \mathrm{Aut}(G_0)$; the order of the extension is 4ℓ. By general knowledge on split extensions the new group has an equivalent representation in the form $G_1 = \langle s, x\rangle \rtimes \langle t, y\rangle$, where t, y are two commuting involutions acting on G_0 by conjugation the same way as the two earlier automorphisms do, that is, $\theta(z) = tzt$ and $\tau(z) = yzy$ for every $z \in G_0$. Note that this also implies that the subgroups G_0 and $\langle t, y\rangle$ intersect trivially. Further, using $u = sx$ and $v = ty$ it can be verified that in G_1 one has the relations $uvu = v$; moreover, we have $[s, t] = [x, y] = 1$ by the definition of the two automorphisms and their conversion to conjugations. It follows that the group G_1 has a presentation of the form

$$G_1 = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, u^\lambda,\, v^2,\, tutu^j,\, uvuv^{-1}\rangle \tag{5.3}$$

where, again, $u = sx$, $v = ty$, and j is an integer such that $j^2 \equiv 1 \bmod \lambda$.

Next, let us consider the group G_2 generated by the same involutions as G_1 but with a presentation obtained from that of G_1 by omitting the relator v^2 and adding the conditions of v^2 commuting with s and x:

$$G_2 = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, uvuv^{-1},\, u^\lambda,\, tutu^j,\, [s, v^2],\, [x, v^2]\rangle \tag{5.4}$$

The relators $uvuv^{-1}$ and $tutu^j$ imply $yuy = u^j$ and $txt = su^{-j}$, and these together with $tut = u^{-j}$ and $yxy = x$ show that $\langle u, x\rangle = \langle s, x\rangle$ is a normal subgroup of G_2. By inspection, in the absence of any condition involving the element v the presentation of $G_2/\langle s, x\rangle$ reduces to $\langle t, y \,|\, t^2, y^2\rangle$, which is an infinite dihedral group; hence G_2 is infinite. The subgroup $N = \langle v^2\rangle$ is obviously normal in G_2 and as $G_2/N \cong G_1$ we conclude that N is isomorphic to an infinite cyclic group. The subgroup $\kappa N = \{(v^{2\kappa})^i;\ i \in \mathbb{Z}\}$ is characteristic in N and so normal in G_2. Applying the Third isomorphism theorem we obtain $G_2/N \cong (G_2/\kappa N)/(N/\kappa N)$ and so $|G_2/\kappa N| = \kappa |G_2/N|$. Since $G_2/N \cong G_1$, the new group $G_3 = G_2/\kappa N$ of order $8\kappa\lambda$ has a presentation obtained from that of (5.4) by adding the relator $v^{2\kappa}$:

$$G_3 = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, uvuv^{-1},\, u^\lambda,\, tutu^j,\, [s, v^2],\, [x, v^2],\, v^{2\kappa}\rangle \tag{5.5}$$

The last step is to consider the element $z = v^\kappa u^a s \in G_3$, where $a = 2^{-1}(j-1) \bmod \lambda$ as before. For the calculations that follow it is useful to observe that from the earlier congruence $a(j+1) \equiv 0 \bmod \lambda$ we have $aj \equiv -a \bmod \lambda$, so that $tu^a t = u^a$ and

$yu^ay = u^{-a}$; note also that $u^{2a+1} = u^j$. We begin by showing that z is an involution. Indeed, using $[u, v^2] = 1$, the facts that both s and v invert u, and the properties of u listed before one obtains $(u^a sv^\kappa)^2 = u^{2a} sv^{-1}v^{\kappa+1}sv^\kappa = u^\lambda u^{-1}sv^{-1}sv = u^\lambda yu^{-1}y = 1$, which yields $z^2 = 1$.

We show even more; namely, that z is a central involution of G_3. To show that z commutes with s, note that the above implies that $zs = v^\kappa u^a$ is an involution and so $(sz)^2 = s(v^\kappa u^a)^2 s = 1$, so that $[s, z] = 1$. With the help of the fact that conjugation by t preserves u^a and v^κ (note that $v^\kappa = v^{-\kappa}$) we obtain $[z, t] = v^\kappa u^a st \cdot tv^\kappa u^a s = z^2 = 1$. As κ is odd and so v^κ inverts u, it follows that $[x, z] = xv^\kappa u^a(sx)v^\kappa u^a s = xv^\kappa u^{a+1}v^\kappa u^{a+1}x = 1$. Last, using inversion of u^a and preservation of v^κ by conjugation by y, as well as preservation of u by conjugation by sv, one has $[z, y] = v^\kappa u^a sv^\kappa u^{-a}ysy = v^\kappa u^a v^{\kappa-1}u^{-a}svysy = v^{-1}svysy = 1$.

We have proved that for $z = v^\kappa u^a s$ the subgroup $\langle z\rangle \cong C_2$ is central in G_3. But this means that the group $G_3/\langle z\rangle$, which is isomorphic to $H_{p,j}$, has order $|G_3|/2 = 4\kappa\lambda = k\ell/2$, as claimed. This completes the proof of correctness of the presentation (5.2); the relator $uvuv^{-1}$ can be omitted since it is a consequence of the remaining ones, as shown in the paragraph immediately preceding the presentation (5.2). □

For completeness we show that none of the edge-biregular maps with automorphism groups $H_{p,j}$ presented as in (5.2) are fully regular. Indeed, in the opposite case $H_{p,j}$ would have to admit an automorphism interchanging s with x and t with y. In such a case, however, along with $s = v^\kappa u^a$ the group $H_{p,j}$ would also have to admit the relation $x = v^{-\kappa}u^{-a} = u^a v^\kappa$. This would imply $xs = u^{2a}$ and hence $u^{2a+1} = 1$, and as $2a + 1 \equiv j \mod \lambda$ we would have $u^j = 1$, contrary to $j^2 \equiv 1 \mod \lambda$.

Finally, the Euler-Poincaré formula yields that $p = 2\kappa\lambda - 2\kappa - \lambda$ so $p + 1 = (2\kappa - 1)(\lambda - 1)$. We know κ and λ are both odd so letting $p + 1 = 2^\alpha bd'$, where $b \equiv 1 \mod 4$, and d' is odd, then we have $2\kappa - 1 = b$ and $\lambda - 1 = 2^\alpha d' = d$, which, so long as κ and λ are coprime, will yield an edge-biregular map as described above. Hence, for every factorisation $p + 1 = bd$ such that $b \equiv 1 \mod 4$ and $\gcd(b + 1, d + 1) = 1$ we have such a map of type $(2(b + 1), 2(d + 1))$, completing the analysis related to the third item of Theorem 3.1.

5.2 The case when the 2-part of the Fitting subgroup is dihedral

Recall that we are investigating an edge-biregular map $M = (H; x, y, s, t)$ of type (k, ℓ) with both entries even, on a surface of Euler characteristic $-p$ for some odd prime p; at the beginning of section 5 we also made the assumption that the 2-part of k is not smaller than the 2-part of ℓ and we had $k = 4\kappa$ and $\ell = 2\lambda$. By Proposition 5.2 we know that H is a solvable group and so has a non-trivial Fitting subgroup F, of which we may assume that $F \neq H$. By earlier results and observations we also know that either F is cyclic, or F has a dihedral 2-part, denoted F_2. We have dealt with the first possibility in subsection 5.1 and *from now on we will assume that F_2 is dihedral*, which of course means that $|F_2| \geq 4$. Our next result places a substantial restriction on F_2 and D_k (the vertex-stabilizer in M).

Proposition 5.3. *If F_2 is dihedral, then D_k is a 2-group, $[D_k : F_2] = 2$, and k is a multiple of 8 while $\ell/(2c)$ is odd.*

Proof. We have assumed that the 2-part of k is not smaller than that of ℓ, and we know that $|H| = k\ell/(2c)$ for $c = 1, 2$. Analysis of equation (3.1) with these conditions yields that a Sylow 2-subgroup of H is contained in D_k. As F_2 is the Sylow 2-subgroup of F, normality of F in H implies that F_2 is a subgroup of D_k.

Suppose $F_2 = D_k = \langle t, y\rangle$. Then H/F_2 is generated by sF_2 and xF_2, each of which has order less than or equal to two. But $|H|/|F_2| = |H|/k = \ell/(2c) = \lambda/c$ and since $c = \gcd(2\kappa, \lambda) \leq 2$ it follows that λ/c must be an odd integer and hence the group H/F_2 does not contain any involution. Hence $s, x \in F_2$ and so $H = D_k$, contrary to our assumption that H is not dihedral.

To show that $[D_k : F_2] \leq 2$, up to the choice of labelling of orbits we may assume $F_2 = \langle y, (ty)^\mu\rangle$ for some μ [the alternative option would be $F_2 = \langle t, (ty)^\mu\rangle$]. Now, F_2 is normal in H (because it is characteristic in F) and so $y^{ty} = tyyyt = tyt \in F_2$. But $y \in F_2$ also so this means $(ty)^2 \in F_2$, which implies $|D_k : F_2| \leq 2$. Hence D_k is a 2-group; the conclusions about k and ℓ are obvious from the above. □

Since F_2 is normal in H and hence also in $\langle t, y\rangle \cong D_k$, we have established that there are only three possibilities for a dihedral F_2 if $k \geq 8$: either $F_2 = \langle t, y\rangle$, or F_2 is one of $\langle t, (ty)^2\rangle$, $\langle y, (ty)^2\rangle$; in particular, k must be a power of 2. In the first case we have H/F_2 generated by (at most) four involutions, but oddness of $\ell/(2c) = |H/F_2|$ implies that the generating involutions s, t, x, y all belong to F_2 and hence $H = F_2 \cong D_k$; such maps with a dihedral automorphism group have already been sorted out. In what follows we will assume that $F_2 = \langle z, (ty)^2\rangle$ for some $z \in \{t, y\}$.

If $k \geq 16$, the cyclic subgroup $\langle (ty)^2\rangle$ of F_2, of order $k/4$, is characteristic in F_2 and therefore normal in H. Note that for $k = 8$ and $F_2 \cong C_2 \times C_2$ this need not be valid. For now we will *assume* that $\langle (ty)^2\rangle$ is normal in H also for $k = 8$ and we will return to the opposite case later.

Before proceeding we make a remark about the case $c = 2$. By the proof of solvability of H, for $c = 2$ the subgroups D_k and D_ℓ intersect in a subgroup isomorphic to V_4. For $k \geq 16$ the subgroup $\langle (ty)^{k/4}\rangle \cong C_2$ generated by the centre of F_2 is characteristic in F_2 and hence normal in H, which means that $(ty)^{k/4}$ is a central involution also in H. Note that this is now also valid for $k = 8$ because of the assumption made in the previous paragraph. If $c = 2$ we may also assume that $\ell \geq 12$ (as for $\ell = 4$ we would have H a dihedral group), and so for the central element of $\langle s, x\rangle \cong D_\ell$ we have $(sx)^{\ell/4} = (ty)^{k/4}$.

Normality of $\langle (ty)^2\rangle$ of F_2 implies that $s(ty)^2 s = (ty)^{2i}$ and $x(ty)^2 x = (ty)^{2j}$ for some integers i, j such that $i^2 \equiv j^2 \equiv 1 \bmod k/4$; the exponents 2 in the congruences match the orders of s and x. It follows that $sx(ty)^2xs = s(ty)^{2j}s = (ty)^{2ij}$. But as $(sx)^{\ell/(2c)}$ is either the identity or a central element of H, we must have $(ij)^{\ell/(2c)} \equiv 1 \bmod k/4$. In view of the previous two congruences, oddness of $\ell/(2c)$ implies that $1 \equiv (ij)^{\ell/(2c)} \equiv ij$, so that sx commutes with $(ty)^2$. Since k and $\ell/(2c)$ are relatively prime and $k/4$ is even while $\ell/(2c)$ is odd, it follows that the subgroup $J = \langle sx, (ty)^2\rangle$ of H is cyclic, of order $k\ell/(8c)$, and generated by the product $(sx)(ty)^2$.

From the fact that sx commutes with $(ty)^2$ we have $s(ty)^2 s = x(ty)^2 x$ and this element is in $\langle (ty)^2\rangle$ by normality of this subgroup in H. Note that we cannot have $x \in J$; in the opposite case x would have to be equal to $(ty)^{k/4}$, the unique involution in J, but then x would commute with sx and hence with s, giving $(sx)^2 = 1$, contrary to $\ell \geq 6c$. Thus, the semi-direct product $K = J \rtimes \langle x\rangle$ is a subgroup of H of order $k\ell/(4c)$, and hence normal (of index 2) in H.

The subgroup $\langle sx\rangle < J$ as the unique cyclic subgroup of K of order $\ell/2$ is characteristic in K and hence normal in H. Thus, $y(sx)y = (sx)^a$ and $t(sx)t = (sx)^b$ for some positive integers $a, b < \ell/2$, with both a, b odd if $c = 2$. By $[x, y] = 1 = [s, t]$ we then have $ysy = s(xs)^{a-1}$ and $(sy)^2 = (xs)^{a-1}$; similarly, $txt = (xs)^{b-1}x$ and $(tx)^2 = (xs)^{b-1}$. By normality of F_2 in H, the element sys or xtx (depending on whether $y \in F_2$ or $t \in F_2$) is equal to either the central element $(ty)^{k/4}$ or an involution of the form $(ty)^{2j}z$ for some $j < k/4$. In the first case, either sys or xtx would commute with xs, which is easily seen to be equivalent to $(sy)^2 = 1$ or $(tx)^2 = 1$. In the second case, either $(sy)^2$ or $(tx)^2$ are a power of $(ty)^2$ and so their order is a power of 2. But we have also established that $(sy)^2 = (xs)^{a-1}$ or $(tx)^2 = (xs)^{b-1}$. Both $(xs)^{a-1}$ and $(xs)^{b-1}$ have, however, odd order; this is obvious for $c = 1$ and for $c = 2$ it follows from oddness of a and b. The two order parities can be matched only if $(sy)^2 = 1$ or $(tx)^2 = 1$. In both cases we have established that, depending on whether $y \in F_2$ or $t \in F_2$, we have $(sy)^2 = 1$ or $(tx)^2 = 1$, i.e., $[s, y] = 1$ or $[t, x] = 1$.

Next, we show that $ty \notin K$. Indeed, if $ty \in K$, then K would contain the cyclic group $\langle ty \rangle \cong C_{k/2}$ and also the dihedral group $\langle s, x \rangle \cong D_\ell$. By comparing their orders with $|K| = k\ell/(4c)$ it follows that the two groups intersect non-trivially. If $c = 2$ then the two groups would have to intersect in a group of order 4, which would have to be cyclic and dihedral at the same time, a contradiction. It follows that $c = 1$ and the only non-trivial element in their intersection is an involution. Since the only involution contained in the cyclic group is the central one, we would have $(ty)^{k/4} = s(sx)^j$ for some j. But as the central element commutes with sx, the same must hold for $s(sx)^j$ and this is easily seen to be equivalent to $(sx)^2 = 1$, contrary to the bound on ℓ. Thus, $ty \notin K$, as claimed. But then, from $H/K = \langle tK, yK \rangle \cong C_2$ it follows that either $t \in K$ or $y \in K$. This means that K contains a dihedral subgroup of order $k/2$, which is in fact the unique Sylow 2-subgroup F_2 of F. Hence $F_2 < K$ and, moreover, from $[t, x] = 1$ or $[s, y] = 1$ it follows that $K = \langle sx \rangle \cdot F_2 \cong C_{\ell/2} \times D_{k/2}$. (In fact, at this stage it follows that K is the Fitting subgroup F of H; to see this one only needs to see that $F \neq H$ but in the opposite case F/F_2 would be trivial (being generated by involutions) and we would be back in the case $F = D_k$. However, in the light of the conclusion we will arrive at, the fact that $K = F$ will turn out to be irrelevant.)

To finish this part of our argument we explore the fact that $x \in K$. By the above structural information this means that $x \in F_2$, and so x would commute with sx, which is equivalent to $(sx)^2 = 1$, contrary to our bound on ℓ.

It thus remains to investigate the case when $k = 8$ and the subgroup $\langle (ty)^2 \rangle \cong C_2$ of $F_2 = \langle z, (ty)^2 \rangle \cong C_2 \times C_2$ for some $z \in \{t, y\}$ is not a normal subgroup of H, with $|H| = k\ell/(2c) = 4\ell/c$ (note that now 8 exactly divides the order of H). For definiteness we will assume that $t \in F_2$; the case when $y \in F_2$ is done in a completely analogous way by replacing s with x and t with y in the subsequent arguments.

Thus, let $F_2 = \{1, t, yty, (ty)^2\}$; by our assumption $y \notin F_2$. If x was in F_2 then clearly $x \neq 1, t$, and as $x = yty$ implies ty has order two, the only possibility would be $x = (ty)^2$. This would mean that $F_2 = \langle t, x \rangle$; by normality in H, conjugation by s preserves F_2. We cannot have $[s, x] = 1$ as this contradicts our bound on ℓ, and since $[s, t] = 1$ it follows that $sxs = xt$ and hence $(sx)^4 = 1$, contrary to $\ell \geq 6c$. Therefore $x \notin F$, and, as we know, $y \notin F$ either, but note that $xy \in F$. Indeed, in the opposite case, by normality of F_2 and its trivial intersection with $\langle x, y \rangle$, the subgroup $F_2 \rtimes \langle x, y \rangle$ of H would have order 16, contrary to 8 exactly dividing the order of H. A calculation as above shows that the only option for $xy \in F$ is $xy = (ty)^2$ which is is equivalent to $txty = 1$ (hence tx has order 4). Observe that this relation also shows that conjugation of F_2 by x fixes $(ty)^2$, and commutativity of $xy, t \in F_2$ implies $xtx = yty$.

Since F_2 is normal in H, it is preserved by conjugation by s, which fixes t. If s fixed $(ty)^2$, then with x fixing $(ty)^2$ the subgroup $\langle (ty)^2 \rangle$ would be normal in H, contrary to our assumption made in this special case $k = 8$. It follows that conjugation by s induces an automorphism of F_2 fixing t and transposing yty with $(ty)^2$; the relation $s(yty)s = (ty)^2$ is equivalent to $(syty)^2 t = 1$. The composition of the two conjugations, namely, $w \mapsto (xs)w(sx)$ for $w \in F_2 \cong C_2 \times C_2$, induces and automorphism in $\mathrm{Aut}(C_2 \times C_2) \cong S_3$ represented by the cycle $t \mapsto yty \mapsto (ty)^2 \mapsto t$ of length 3. In particular, conjugation by $(sx)^3$ centralizes t. If now $\ell/2 = 3q + r$ for $r = \pm 1$, then $[(sx)^3, t] = 1$ implies $[(sx)^r, t] = 1$, contrary to the fact established earlier that conjugation by sx does not fix t. It follows that $r = 0$ and hence $\ell/2$ is a multiple of 3.

We are now approaching derivation of a presentation of H. Observe first that the relation $xtx = yty$ derived earlier simplifies $(syty)^2 t = 1$ to $(sxy)^2 t = 1$. From $y = txt$ and the fact that $(sx)^3$ centralizes F_2 we obtain $y(sx)^3 y = (xs)^3$, so that $\langle (sx)^3 \rangle$ is a normal cyclic subgroup of H. If $c = 2$, then, by the findings in the previous paragraph, the odd integer $\ell/4$ is a multiple of 3 and so $(sx)^{\ell/4}$ is a central element of H as obviously commutes with s and x, and also with t (because $(sx)^3$ does) and hence also with $y = txt$. But then $\langle (sx)^{\ell/4} \rangle$ must be a subgroup of F_2, otherwise its product with F_2 would be isomorphic to C_2^3, contrary to the Sylow 2-subgroups of H being dihedral. We cannot have $(sx)^{\ell/4}$ equal to t or yty because the two elements do not commute with y, so that the only option is $(sx)^{\ell/4} = (ty)^2$, but while the element on the left commutes with s the one on the right does not. It follows that $c = 1$ and hence $|H| = 4\ell$.

We note that normality of $\langle (sx)^3 \rangle$ cannot be extended to normality of $\langle sx \rangle$, as otherwise we would have $t(sx)t = (sx)^d$ and then $(xt)^2 = (sx)^{d-1}$ for some d, which are elements of orders of different parity (4 versus some (odd) divisor of $\ell/2$). This shows, as an aside, that the Fitting subgroup of H is $F \cong \langle (sx)^3 \rangle \times F_2 \cong C_{\ell/6} \times V_4$. It may also be useful to note that $y = txt$ implies that $sy = stxt = t(sx)t$, showing that the orders of sx and sy are the same, namely, $\ell/2$.

This way, in the case when $k = 8$ and $F_2 = \langle t, yty \rangle$, with $\ell/2 = 3m$ for some odd $m \geq 1$, we have arrived at a presentation of $H = H_p$ of the form

$$H_p = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, (sx)^{3m},\, (ty)^4,\, (sxy)^2 t,\, txty \rangle \tag{5.6}$$

representing the edge-biregular map $M = M_p$ of type $(k, \ell) = (8, 6m)$ that appears in item 4 of Theorem 3.1. The corresponding dual map is obtained by interchanging x with y and s with t, while the twin map is found by interchanging x with s and y with t.

Proof of correctness of the presentation (5.6)

Proof. It remains to prove that the presentation (5.6) indeed defines a group of order $k\ell/2 = 24m$. For this we begin with a 'universal' group U with relators as in (5.6) but with $(sx)^{3m}$ omitted:

$$U = \langle x, y, s, t \,|\, x^2,\, y^2,\, s^2,\, t^2,\, (xy)^2,\, (st)^2,\, (ty)^4,\, (sxy)^2 t,\, txty \rangle \tag{5.7}$$

We show that $N = \langle (sx)^3 \rangle$ is a normal subgroup of U. For this it is sufficient to prove that N is invariant under conjugation by t, as from $y = txt$ and automatic invariance of N under conjugation by s and x it then follows that $yNy = N$. From the last two relators in

(5.7) we have $1 = (sxy)^2t = (s(ty)^2)^2t$, which, using $(ty)^4 = 1$, gives $s(ty)^2s = yty$. It follows that conjugation by s fixes t (by the relation $(st)^2 = 1$) and interchanges yty with $(ty)^2$. On the other hand, as $x = tyt$, one similarly obtains that conjugation by x fixes $(ty)^2$ and interchanges t with yty. It follows that the composition of the two conjugations, that is, the mapping $z \mapsto (xs)z(sx)$, induces a 3-cycle $t \mapsto yty \mapsto (ty)^2 \mapsto t$, so that t commutes with $(sx)^3$ and hence preserves N. (This is what we saw before but now we needed to make sure that it was established solely from the presentation (5.7).)

By the Reidemeister-Schreier theory implemented in MAGMA in the form of its Rewrite command one may check that N is a free group, that is, N is infinite cyclic. Also, by MAGMA one can check that $U/N \cong S_4$, of order 24. Now, for an arbitrary integer $m \geq 1$ let $N_m = \langle (sx)^{3m} \rangle$ be the cyclic subgroup of N of index m. Since N_m is characteristic in N and so normal in U, we may use the Third isomorphism theorem to write $U/N \cong (U/N_m)/(N/N_m)$ and as $U/N_m \cong H$, $N/N_m \cong C_m$ and $U/N \cong S_4$ it follows that $|H| = 24m$, as claimed. This proves correctness of the presentation (5.6). □

We conclude by showing that none of the edge-biregular maps given by the group H_p from (5.6) is regular. For such a map to be regular there would have to be an automorphism of H of order 2 interchanging s with x and t with y. If this is the case then H would also contain the relator $ysyt$. From the relator $txty$ of H we have $y = txt$, which, when substituted into $ysyt = 1$ and canceling terms, gives $txsx = 1$. Combining this with $tysy = 1$ yields $xsx = ysy$, or, equivalently, $(sxy)^2 = 1$. But this in combination with the relator $(sxy)^2t$ of (5.6) gives $t = 1$, a contradiction. This completes both the analysis related to the fourth item of Theorem 3.1 as well as our proofs of the main results from section 3.

6 Concluding remarks

The maps M_p identified in the last part of our proof, those defined by the group H_p presented as in (5.6), deserve particular attention. Their structure is best visualized by considering the associated embedded Cayley graph $\mathrm{Cay}(H_p, X)$ for the group H_p and the generating set $X = \{x, y, s, t\}$. If superimposed onto the map M_p, the associated embedding of $\mathrm{Cay}(H_p, X)$ displays cycles labelled alternately with y and t 'around' each vertex and cycles labelled with x and s 'around' each face, while the 4-cycles labelled $xyxy$ and $stst$ 'surround', respectively, bold and dashed edges. The existence of the relator $txty$ in the presentation (5.6), equivalent to the relation $x = tyt$, demonstrates that all the bold edges are loops. Applying this knowledge to the relator $(sxy)^2t$, which (being of odd length) signals non-orientability, one sees that the edges in the unshaded orbit partition into pairs of double-edges, each forming a 'central cycle' of a Möbius strip in the embedding. The Cayley graph $\mathrm{Cay}(H_p, X)$, part of which is shown in Figure 2, makes this clear. This is also an alternative way of proving non-regularity for this map. Namely, the two edge orbits contain (bold) loops on the one hand, and non-orientable (dashed) 2-cycles on the other, so there will certainly not be an automorphism of the group which interchanges these two orbits of edges.

In section 2 we pointed out that edge-biregular maps of type (k, ℓ) arise as smooth quotients of the subgroup of the full triangle group (2.1) uniquely determined by inclusion of the generators R_0 and R_2 while excluding R_1. In this context it may be interesting to review results for maps on surfaces of negative prime Euler characteristic arising from smooth quotients of all the (up to) seven possible index-two subgroups of the full triangle

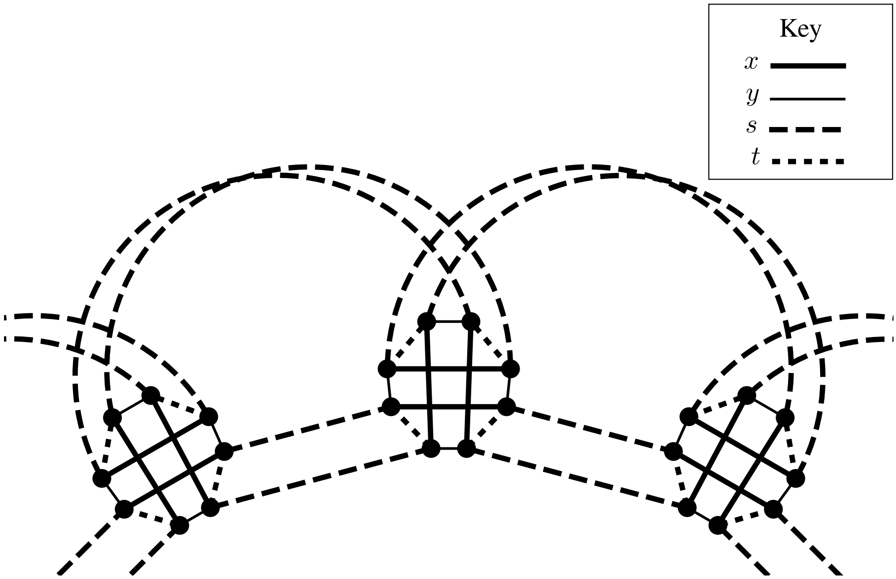

Figure 2: Part of the Cayley graph $\mathrm{Cay}(H_p, X)$ with generating set $X = \{x, y, s, t\}$.

groups. Out of these, the subgroup most frequently referred to is $\langle \overline{R}_0, \overline{R}_1, \overline{R}_2 \rangle$, the smooth quotients of which form the class of orientably-regular maps. A complete classification of orientably-regular maps of Euler characteristic $-p$ for prime p can be found in [5], with earlier results for orientably-regular maps with 'large' automorphism groups available in [1]. The bi-rotary maps of negative prime Euler characteristic, i.e., those arising from smooth quotients of the subgroup $\langle R_0, \overline{R}_1, \overline{R}_2 \rangle$, have been classified in [6]. Since the interchange of the involutory generators R_0 and R_2 induces map duality, analogous classification of maps defined by smooth quotients of the subgroup $\langle \overline{R}_0, \overline{R}_1, R_2 \rangle$ of (2.1) follows from [6] by dualisation.

This way, results in this paper are a further contribution to classification of highly symmetric maps on surfaces as above, this time arising as smooth quotients of the subgroup $\langle R_0, \overline{R}_1, R_2 \rangle$ of index two in the full triangle group. In order to have a complete list of highly symmetric maps on surfaces with Euler characteristic $-p$ for prime p that can be obtained as smooth quotients of index-two subgroups of triangle groups it is therefore sufficient to examine, up to duality, only two more subgroups, namely, $\langle \overline{R}_0, R_1, R_2 \rangle$ and $\langle \overline{R}_0, R_1, \overline{R}_2 \rangle$. For completeness we recall that the fully regular maps on these surfaces, that is those arising as smooth quotients of the full triangle groups, have been classified in [2].

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.09
https://doi.org/10.26493/2590-9770.1405.4ce
(Also available at http://adam-journal.eu)

The string C-group representations of the Suzuki, Rudvalis and O'Nan sporadic groups

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Received 5 March 2021, accepted 2 June 2021, published online 26 July 2022

Abstract

We present new algorithms to classify all string C-group representations of a given group G. We use these algorithms to classify all string C-group representations of the Suzuki, Rudvalis and O'Nan sporadic groups. The new rank three algorithms also permit us to get all string C-group representations of rank three for the Conway group Co_2 and the Fischer group Fi_{22}.

Keywords: Abstract regular polytopes, string C-group representations, sporadic simple groups, algorithms.

Math. Subj. Class.: 52B11, 20D08, 20F05

1 Introduction

Marston Conder has been one of the pioneers in generating all geometric or combinatorial objects of a certain kind, as for instance all regular (hyper)maps of small genus, all abstract regular or chiral polytopes with a sufficiently small number of flags, etc. Conder's website[1] contains an impressive collection of data that greatly helped researchers over the years to get a better insight on the related topics. Most of the data available on Conder's website are in terms of groups and generators, the groups being automorphism groups of the objects considered. A natural way of gaining knowledge on the structure of a group is indeed to search for geometric and combinatorial objects on which it can act. Among those objects, abstract regular polytopes are of great interest as they are highly symmetric combinatorial

*This work was partially supported by an ARC Advanced grant from the Fédération Wallonie-Bruxelles, Belgium. The author would like to thank Martin Macaj for interesting discussions on finding a faster algorithm in the rank three case.

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structures with many singular algebraic and geometric properties. Moreover, due to the well-known one-to-one correspondence between abstract regular polytopes and string C-groups, finding an abstract regular polytope with a given automorphism group G means obtaining a sequence of involutions generating G, satisfying a certain intersection property and such that two nonconsecutive involutions commute. This means that G is a string C-group. Finally, string C-groups are smooth quotients of Coxeter groups, hence any Coxeter diagram found in this process permits to conclude that the group appears as a quotient of the corresponding (infinite) Coxeter group.

In 2006, Michael Hartley published "An atlas of small regular polytopes" [7], where he classified all regular polytopes with automorphism groups of order at most 2000 (not including orders 1024 and 1536). His work has been complemented later on by Conder who computed all regular polytopes with up to 2000 flags, up to isomorphism and duality, excluding those of rank 2 (regular polygons) and the degenerate examples that have a $'2'$ in their Schläfli symbol, using the LowIndexNormalSubgroups function of MAGMA [1]. The complete list is available on Conder's website.

Also in 2006, Dimitri Leemans and Laurence Vauthier published "An atlas of polytopes for almost simple groups" [11], in which they gave classifications of all abstract regular polytopes with automorphism group an almost simple group with socle of order up to $900,000$ elements. More recently in [8], Hartley also classified, together with Alexander Hulpke, all polytopes for the sporadic groups as large as the Held group (of order $4,030,387,200$) and Leemans and Mark Mixer classified, among others, all polytopes for the third Conway group Co_3 (of order $495,766,656,000$) [10]. Leemans, Mixer and Thomas Connor then computed all regular polytopes of rank at least four for the O'Nan-Sims group $\mathsf{O'N}$ [3].

All these computational data lead to many theoretical results. For more details we refer to the recent survey [9].

The recorded computational times and memory usages for the use of previous algorithms on groups among which $\mathsf{O'N}$ and Co_3 provide a motivation to improve the algorithms in order to obtain computational data for groups currently too large.

The challenges involve dealing with groups that have a relatively large smallest permutation representation degree as well as groups with large Sylow 2-subgroups. The largest sporadic group for which a classification of all string C-group representations exists was Co_3 group whose smallest permutation representation degree is on 276 points and whose Sylow 2-subgroups are of order 2^{10}.

We use the technique described in [3] and develop it further to obtain an algorithm that can be used on any permutation group as well as any matrix group to find polytopes of rank at least four.

We also present a new algorithm to find polytopes of rank three and two new algorithms to find polytopes of rank at least four that outperform the previous known algorithms.

With our new algorithms we manage to classify all string C-group representations of the Suzuki sporadic group Suz and the Rudvalis sporadic group Ru. Although these groups have order smaller than the one of Co_3, their smallest permutation representation degrees are respectively 1782 and 4060, and their Sylow 2-subgroups are of respective orders 2^{13} and 2^{14}. We also finish the classification of the string C-group representations for $\mathsf{O'N}$ started in [3] by computing all of its rank three representations. Finally, we classify all string C-group representations of rank three for the Conway group Co_2 and the Fischer group Fi_{22}.

The classifications for Suz, Ru and O′N are made available in the online version of the Atlas [11].

The paper is organised as follows. In Section 2, we give the background needed to understand this paper. In Section 3, we describe the new algorithm to compute rank three string C-group representations of a group. In Section 4, we describe three new algorithms to compute rank three string C-group representations of a group and compare the efficiency of the algorithms. In Section 5, we describe the new results we manage to obtain with the algorithms described in the previous two sections.

2 Abstract regular polytopes and string C-groups

An *abstract n-polytope* $\mathcal{P}$ is a ranked partially ordered set $(\mathcal{P}, \leq)$ with four defining properties **(P1)**, **(P2)**, **(P3)** and **(P4)** as defined below. Here, n is the *rank* of $\mathcal{P}$. We shall name the elements of $\mathcal{P}$ *faces* and differentiate the elements by rank calling an element of rank $i \in \{-1, 0, \cdots, n-1, n\}$ an *i-element*. For any two faces F and G of $\mathcal{P}$ such that $F \leq G$, we also define the *section* G/F to be the collection of all faces H of $\mathcal{P}$ such that $F \leq H \leq G$. Note that any section of a polytope is itself a polytope.

- **(P1)** $\mathcal{P}$ has two *improper* faces: a least face F_{-1} of rank -1 and a greatest face F_n of rank n.
- **(P2)** Each flag (i.e. maximal totally ordered subset) of $\mathcal{P}$ contains $n+2$ faces (including the two improper faces).
- **(P3)** $\mathcal{P}$ is *strongly connected*, that is, each section of $\mathcal{P}$ (including $\mathcal{P}$ itself) is connected (in the sense given below).
- **(P4)** $\mathcal{P}$ satisfies the *diamond condition*, that is, for any two faces F and G of $\mathcal{P}$ such that $F < G$ and $\text{rank}(G) = \text{rank}(F) + 2$, there are exactly two faces H such that $F < H < G$.

A ranked partially ordered set of rank d with properties **(P1)** and **(P2)** is said to be connected if either $d \leq 1$ or $d \geq 2$ and for any two *proper* faces F and G of $\mathcal{P}$ (i.e. any faces other than F_{-1} and F_d) there is a sequence of proper faces $F = H_0, H_1, \cdots, H_{k-1}, H_k = G$ such that $H_i \leq H_{i+1}$ for $i \in \{0, 1, \cdots, k-1\}$.

Two flags of an n-polytope are said to be *adjacent* if they differ by only one face. In particular, two flags are said to be *i-adjacent* if they differ only by their i-face. As a consequence of the diamond condition, any flag Φ of a polytope has exactly one i-adjacent (for any $i \in \{0, 1, ..., n-1\}$) flag that we shall denote by Φ^i. Note that $(\Phi^i)^i = \Phi$ for all i and that $(\Phi^j)^i = (\Phi^i)^j$ for all i and j such that $\mid i - j \mid \geq 2$

The automorphism group of an n-polytope $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$. An n-polytope is said to be *regular* if its automorphism group $\Gamma(\mathcal{P})$ has exactly one orbit on the flags of $\mathcal{P}$, or equivalently, if for any flag $\Phi = \{F_{-1}, F_0, F_1, \cdots, F_n\}$ and each $i \in \{0, 1, ..., n-1\}$, there is a unique (involutory) automorphism ρ_i of $\mathcal{P}$ such that $\rho_i(\Phi) = \Phi^i$. In this case, one usually chooses a flag as a reference flag and calls it *base* flag.

Finally, an n-polytope is called *equivelar* is for any $i \in \{1, 2, ..., n-1\}$, there is an integer p_i such that any section G/F of $\mathcal{P}$ defined by an $(i-2)$-face F and an $(i+1)$-face G is a p_i-gon. If this is the case then we say that $\mathcal{P}$ has *Schläfli type* $\{p_1, p_2, ..., p_{n-1}\}$.

Let G be a group and $S := \{\rho_0, \rho_1, ..., \rho_{n-1}\}$ be a set of elements of G such that $G = \langle S \rangle$. We define the following two properties.

(SP) the *string property*, that is $(\rho_i\rho_j)^2 = 1_G$ for all $i, j \in \{0, 1, ..., n-1\}$ with $\mid i-j \mid \geq 2$;

(IP) the *intersection property*, that is $\langle \rho_i \mid i \in I\rangle \cap \langle \rho_j \mid j \in J\rangle = \langle \rho_k \mid k \in I \cap J\rangle$ for any $I, J \subseteq \{0, 1, ..., n-1\}$.

A pair (G, S) as above that satisfies property **(SP)** is called a *string group generated by involutions* (or sggi in short). A *string C-group* is an sggi that satisfies property **(IP)**. The *rank* of (G, S) is the size of S. For any subset $I \subseteq \{0, \ldots, n-1\}$, we denote $\langle \rho_j : j \in \{0, \ldots, n-1\} \setminus I\rangle$ by G_I. If $I = \{i\}$, we denote $G_{\{i\}}$ by G_i. Similarly, if $I = \{i, j\}$, we denote $G_{\{i,j\}}$ by G_{ij}.

As shown in [12], the automorphism group $\Gamma(\mathcal{P})$ of an abstract regular polytope $\mathcal{P}$, together with the involutions that map a base flag to its adjacent flags is a string C-group representation (see [12, Propositions 2B8, 2B10 and 2B11]). More precisely, if one fixes a base flag Φ of $\mathcal{P}$, $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, ..., \rho_{n-1}\rangle$ with ρ_i the unique involution such that $\rho_i(\Phi) = \Phi^i$ then the pair $(\Gamma(\mathcal{P}), \{\rho_0, \rho_1, ..., \rho_{n-1}\})$ is a string C-group representation. The *Schläfli type* of a string C-group representation $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is the ordered set $\{p_1, p_2, ..., p_{n-1}\}$ where p_i is the order of the element $\rho_{i-1}\rho_i$ for $i = 1, \ldots, n-1$.

Conversely, as shown in [12, Theorem 2E11], a regular n-polytope can be constructed uniquely from a string C-group representation (G, S) with $S := \{\rho_0, ..., \rho_{n-1}\}$. Let $G_i := \langle \rho_j \mid j \neq i\rangle$ for any $i \in \{0, 1, ..., n-1\}$. We also set $G_{-1} = G_n := G$. For $i \in \{-1, 0, 1, ..., n-1, n\}$, we take the set of i-faces of $\mathcal{P}$ to be the set of all right cosets $G_i\varphi$ of G_i in G. We define a partial order on $\mathcal{P}$ as follows : $G_i\varphi \leq G_j\psi$ if and only if $-1 \leq i \leq j \leq n$ and $G_i\varphi \cap G_j\psi \neq \emptyset$.

We say that two string C-group representations (G, S) and (G, S') of G are *isomorphic* if there exists an automorphism of G that maps S onto S'. The *dual* of a string C-group representation $(G, \{\rho_0, \rho_1, ..., \rho_{n-1}\})$ is the string C-group representation $(G, \{\rho_{n-1}, \rho_{n-2}, ..., \rho_0\})$.

Due to the one-to-one correspondence between string C-group representations and automorphism groups of abstract regular polytopes, finding all abstract regular polytopes with a fixed automorphism group amounts to considering all ways of presenting a particular group as a string C-group and verifying whether they yield non-isomorphic abstract regular polytopes. Our algorithms classify string C-group representations of a group G up to isomorphism and duality.

3 The rank three case

Every rank three string C-group representation of a group G is a pair $(G, \{\rho_0, \rho_1, \rho_2\})$ where $\langle \rho_0, \rho_1\rangle$ is a dihedral group and ρ_2 commutes with ρ_0.

The dihedral subgroups of G can be generated from the conjugacy classes of elements of G using the following simple observation.

Lemma 3.1. *Let G be a group. The group G has a dihedral subgroup D if and only if there exists an element τ of G and an involution $\rho \neq \tau$ of G such that $D := \langle \tau, \rho\rangle$ and $\tau^\rho = \tau^{-1}$.*

Proof. If there exists a dihedral subgroup D in G, take two involutions that generate D, say ρ and ρ'. Then $\tau = \rho\rho'$ is such that $D := \langle \tau, \rho\rangle$ and $\tau^\rho = \tau^{-1}$.

On the other hand, suppose there is a subgroup D of G such that $D := \langle \tau, \rho\rangle$ and $\tau^\rho = \tau^{-1}$ for some element τ of G and some involution ρ of G distinct from τ. Then

$\rho' := \rho\tau$ is an involution (as $\rho'\rho' = \rho\tau\rho\tau = \rho^{-1}\tau\rho\tau = \tau^{-1}\tau = 1_G$ and $\rho' \neq 1_G$ since $\rho \neq \tau$) and $\rho\rho' = \rho\rho\tau = \tau$. Therefore D is a dihedral subgroup of G. □

So an easy way to construct all conjugacy classes of dihedral subgroups of a group G is the following:

1. Let $\mathcal{D} := \emptyset$.
2. Compute a set C containing one representative of each conjugacy class of elements of G.
3. For each $\tau \in C$, compute $N(\tau) := N_G(\langle\tau\rangle)$.
4. For each involution $\rho \in N(\tau)$, if $\tau^\rho = \tau^{-1}$ then add $\langle\tau, \rho\rangle$ to $\mathcal{D}$ provided it is not conjugate in G to an element already in $\mathcal{D}$.

At the end of the above process, $\mathcal{D}$ contains one representative of each conjugacy class of dihedral subgroups of G.

In order to generate all rank three string C-group representations of G, we now need to do the following.

1. Let $\mathcal{S} := \emptyset$.
2. For each $D \in \mathcal{D}$
3. For each pair ρ_0, ρ_1 of generating involutions of D, if it is not conjugate in G to a pair already in $\mathcal{S}$, add this pair to $\mathcal{S}$

At the end of the above process, $\mathcal{S}$ contains all the pairs ρ_0, ρ_1 that we try now to extend to a triple of involutions.

Here we can either try to extend the ordered pair $[\rho_0, \rho_1]$ to the right or to the left (which is equivalent to extending the reversed pair $[\rho_1, \rho_0]$ to the right). If we try to extend it to the right, that is by adding a ρ_2, we know that $\rho_2 \in C_G(\rho_0)$. So we can try every involution in $C_G(\rho_0)$, and if G is simple, we also know that ρ_2 does not commute with ρ_1, for otherwise the string C-group $(G, \{\rho_0, \rho_1, \rho_2\})$ is degenerate and G is therefore not simple (e.g. $\langle\rho_1, \rho_2\rangle$ being a non-trivial normal subgroup of G).

This algorithm turns out to be extremely efficient as we will show in Section 5. It permitted us to compute all rank three string C-group representations of Ru, Suz, O′N, Co_2 and Fi_{22}.

4 The ranks four and above

We describe three techniques to find string C-group representations of rank at least four.

4.1 Using centralizers of involutions

This algorithm uses the following observations.

Lemma 4.1. *Let G be a group. Let $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ be a string C-group representation of G. The subgroup G_1 is a subgroup of the centraliser of ρ_0, in particular $G_1 = \langle\rho_0\rangle \times G_{01}$ where $G_{01} \leq C_G(\rho_0)$.*

Proof. This is a direct consequence of Proposition $2B12$ in [12] and of the definition of string C-groups. □

This lemma implies that we can take for ρ_0 a representative of a conjugacy class of involutions of G, compute its centraliser $C_G(\rho_0)$ and then the subgroup G_1 must be a subgroup of $C_G(\rho_0)$.

We then proceed to construct G_1. In order to do so, we produce all string C-group representations of subgroups H of $C_G(\rho_0)$ that have ρ_0 as first generator. As $C_G(\rho_0)$ is substantially smaller than G, one may now apply the former algorithms (from [10]) to $C_G(\rho_0)$ in order to classify all the string C-group representations of all its subgroups. With each string C-group representation $\langle \rho_0, \rho_2, ..., \rho_{n-1}\rangle$ of G_1, we proceed by adding an appropriate involution ρ_1 of G to it, giving a string C-group representation for G. In order to restrict the number of involutions ρ_1 to consider, we use the following observation.

Lemma 4.2. *If $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a string C-group representation of the group G, then $\rho_1 \in C_G(\rho_3) \cap \ldots \cap C_G(\rho_{n-1})$.*

Proof. Since ρ_1 has to commute with every generator ρ_i for $i \geq 3$ by property **(SP)**, we have that $\rho_1 \in C_G(\rho_3) \cap \ldots \cap C_G(\rho_{n-1})$. □

We then check that the resulting pair $(G, \{\rho_0, \rho_1, ..., \rho_{n-1}\})$ is a string C-group representation of G using the following proposition.

Proposition 4.3 ([12, Proposition 2E16(a)]). *Let $(G, \{\rho_0, \rho_1, ..., \rho_{n-1}\})$ be an sggi. If G_0 and G_{n-1} are string C-groups and if $G_{n-1} \cap G_0 = G_{0,n-1}$ then $(G, \{\rho_0, \rho_1, ..., \rho_{n-1}\})$ is a string C-group.*

Finally, let us note that although the former algorithms only allowed to be implemented for groups of permutations, this new algorithm also works for matrix groups. This gives access to groups much larger in size, as long as the centralizers of involutions of G are small enough to be treated as permutation groups.

We give in Table 1 the pseudo-code of our new algorithm.

4.2 Higher ranks from the new rank three algorithm

If in the process of generating the rank three string C-group representations with the algorithm described in Section 3, we keep also the triples that do not generate the group G, we can then try to extend these triples to 4-tuples and so on, getting string C-group representations of all possible ranks.

There are two possible approaches here. We could either decide to keep triples of involutions ρ_0, ρ_1, ρ_2 and try to extend them to rank four and so on, or keep the triples of the shape ρ_0, ρ_1, ρ_3, namely, keeping the triples where the third involution commutes with the first two. We have designed algorithms to try both approaches. We call the first approach the linear approach and the second one the central approach.

4.3 The linear approach

We can easily give an algorithm based on the rank three method presented in Section 3 to produce all string C-group representations of all ranks for a given group. Indeed, if, while computing the rank three representations we keep track of all triples ρ_0, ρ_1, ρ_2 that satisfy

Input: G the group for which we want to compute all string C-group representations.
Output: L a sequence containing the pairwise non-isomorphic string C-group representations.

1. Compute the conjugacy classes $CC(G)$ of elements of G.
2. Initialise L.
3. For each conjugacy class c of elements of order 2 in $CC(G)$:
4. Let r_0 be a representative of that class.
5. Build the centralizer H of r_0 in G.
6. If G is a matrix group,
7. find a permutation representation $P(H)$ of H;
8. reduce the degree of $P(H)$;
9. If G is already a permutation group then
10. only reduce its degree and call it $P(H)$.
11. Using the existing procedures to do so (see [11]),
12. compute the string C-group representations with generators $\{r_0, r_2, \ldots, r_{n-1}\}$ for subgroups of $C_G(r_0)$, forcing r_0 as first generator.
13. For each such representation, try to complete it by inserting an involution r_1 between r_0 and r_2, using the fact that it has to be in the centralisers of the r_i's for $i = 3, \ldots, n-1$.
14. Compute $G_0 = \langle r_1, ..., r_{n-1}\rangle$, $G_{n-1} = \langle r_0, ..., r_{n-2}\rangle$ and $G_{0,n-1} = \langle r_1, ..., r_{n-2}\rangle$.
15. Check the intersection property for G_0 and G_{n-1} and
16. Check that $G_0 \cap G_{n-1} = G_{0,n-1}$.
17. Let $GP = \langle r_0, r_1, ..., r_{n-1}\rangle$ and let $\widetilde{GP} = \langle r_{n-1}, ..., r_1, r_0\rangle$ be its dual.
18. If GP and $\widetilde{GP}$ are non-isomorphic to all the elements of L,
19. add GP to L.
20. At the end, L contains one representative of each isomorphism class of string C-group representation of G.

Table 1: The pseudo-code of the new algorithm for ranks 4 and above.

the intersection property but do not generate the full group G, we can then try to extend these triple to four-tuples and so on in the following way. Suppose $\mathcal{P}$ contains the triples ρ_0, ρ_1, ρ_2 that satisfy the intersection property but do not generate G.

- While $\mathcal{P}$ is not empty do the following.
- Let $\mathcal{Q}$ be an empty set.
- Let r be the number of involutions in a tuple of $\mathcal{P}$.
- For each tuple $\rho_0, \ldots, \rho_{r-1}$ of $\mathcal{P}$, try to extend it on the left by looking for involutions $\rho_{-1} \in C_G(\rho_1) \cap \ldots \cap C_G(\rho_{r-1})$ such that $o(\rho_{-1}\rho_0) > 2$ and such that $\langle \rho_{-1}, \ldots, \rho_{r-1} \rangle = G$ and $\{\rho_{-1}, \ldots, \rho_{r-1}\}$ satisfies the intersection property. All such ρ_{-1} give a string C-group representation of rank $r+1$. In this process, whenever a ρ_{-1} is found such that $\langle \rho_{-1}, \ldots, \rho_{r-1} \rangle < G$ and $\{\rho_{-1}, \ldots, \rho_{r-1}\}$ satisfies the intersection property, add it in $\mathcal{Q}$ provided it is not isomorphic to any element of $\mathcal{Q}$ yet.
- For each tuple $\rho_0, \ldots, \rho_{r-1}$ of $\mathcal{P}$, try to extend it on the right by looking for involutions $\rho_r \in C_G(\rho_0) \cap \ldots, \cap C_G(\rho_{r-2})$ such that $o(\rho_{r-1}\rho_r) > 2$ and such that $\langle \rho_0, \ldots, \rho_r \rangle = G$ and $\{\rho_0, \ldots, \rho_r\}$ satisfies the intersection property. All such ρ_r give a string C-group representation of rank $r+1$. In this process, whenever a ρ_r is found such that $\langle \rho_0, \ldots, \rho_r \rangle < G$ and $\{\rho_0, \ldots, \rho_r\}$ satisfies the intersection property, add it in $\mathcal{Q}$ provided it is not isomorphic to any element of $\mathcal{Q}$ yet.
- At the end of this for loop, we have found all string C-group representations of rank $r+1$ of G and in $\mathcal{Q}$ we have $r+1$ tuples that we can try to extend further.
- Let $\mathcal{P} := \mathcal{Q}$ and continue the while loop.

4.4 The central approach

We can also give an algorithm based on the rank three one to produce all string C-group representations of all ranks for a given group but using the observations of Section 4.1. Indeed, if, while computing the rank three representations we keep track of all ordered triples ρ_0, ρ_1, ρ_2 that satisfy the intersection property but do not generate the full group G, and such that ρ_2 commutes with both ρ_0 and ρ_1, we can then try to extend these triple to four-tuples and so on in the following way. Suppose $\mathcal{P}$ contains these triples ρ_0, ρ_1, ρ_2 described above.

- While $\mathcal{P}$ is not empty do the following.
- Let $\mathcal{Q}$ be an empty set.
- Let r be the number of involutions in a tuple of $\mathcal{P}$.
- For each tuple $\rho_0, \ldots, \rho_{r-1}$ of $\mathcal{P}$, try to extend it on the left by looking for involutions $\rho_{-1} \in C_G(\rho_1) \cap \ldots, \cap C_G(\rho_{r-2})$ such that $o(\rho_{-1}\rho_0) > 2$, $o(\rho_{-1}\rho_{r-1}) > 2$ and such that $\langle \rho_{-1}, \ldots, \rho_{r-1} \rangle = G$ and $\{\rho_{-1}, \ldots, \rho_{r-1}\}$ satisfies the intersection property. All such ρ_{-1} give a string C-group representation $(G, \{\rho_{r-1}, \rho_{-1}, \rho_0, \ldots, \rho_{r-2}\})$ of rank $r+1$. In this process, whenever a ρ_{-1} is found such that $\langle \rho_{-1}, \ldots, \rho_{r-1} \rangle < G$, $o(\rho_{-1}\rho_{r-1}) = 2$ and $\{\rho_{-1}, \ldots, \rho_{r-1}\}$ satisfies the intersection property, add the

ordered tuple $\{\rho_{-1}, \rho_0, \ldots, \rho_{r-2}, \rho_{r-1}\}$ in $\mathcal{Q}$, provided it is not isomorphic to any element of $\mathcal{Q}$ yet.

- For each tuple $\rho_0, \ldots, \rho_{r-1}$ of $\mathcal{P}$, try to extend it on the right by looking for involutions $\rho_r \in C_G(\rho_0) \cap \ldots, \cap C_G(\rho_{r-3})$ such that $o(\rho_{r-1}\rho_r) > 2$, $o(\rho_{r-2}\rho_r) > 2$ and such that $\langle \rho_0, \ldots, \rho_r \rangle = G$ and $\{\rho_0, \ldots, \rho_r\}$ satisfies the intersection property. All such ρ_r give a string C-group representation $(G, \{\rho_0, \ldots, \rho_{r-2}, \rho_r, \rho_{r-1}\})$ of rank $r+1$. In this process, whenever a ρ_r is found such that $\langle \rho_0, \ldots, \rho_r \rangle < G$ and $\{\rho_0, \ldots, \rho_r\}$ satisfies the intersection property, add the ordered tuple $\{\rho_0, \ldots, \rho_{r-2}, \rho_r, \rho_{r-1}\}$ in $\mathcal{Q}$, provided it is not isomorphic to any element of $\mathcal{Q}$ yet, $o(\rho_{r-1}\rho_r) = 2$ and $o(\rho_{r-2}\rho_r) > 2$.

- At the end of this for loop, we have found all string C-group representations of rank $r+1$ of G and in $\mathcal{Q}$ we have $r+1$ tuples that we can try to extend further.

- Let $\mathcal{P} := \mathcal{Q}$ and continue the while loop.

4.5 Computational times

Table 2 gives, for a given group G, the Time[2] it takes (in seconds) to compute all string C-group representations with the central approach, the linear approach, the rank three algorithm and the algorithm described in Section 4.1 for the higher ranks, the old algorithm (when this time is small enough and the old algorithm was capable of getting the whole result), the number of string C-group representations of rank greater than 3 and the number of string C-group representations of rank three. The time in the "High+Rank3" column is given as a sum of two numbers. The first number is the time it takes to compute string C-group representations of rank at least four with the algorithm of Section 4.1. The second number is the time it takes to compute the rank three ones with the new algorithm described in Section 3. We leave a ??? when the computer was unable to finish the computation.

5 New results

Two sporadic groups smaller than Co_3, namely Suz and Ru, had apparently never been investigated. The two main reasons were that these groups have a higher permutation representation degree than Co_3 and that they have larger Sylow 2-subgroups (of respective sizes 2^{13} and 2^{14} while the Sylow 2-subgroups of Co_3 have order 2^{10}). We analysed them with our new set of programs.

Our new algorithm also permitted to complete the classification of string C-group representations for the O'Nan group, and to obtain all string C-group representations of rank three for the Conway group Co_2 and the Fischer group Fi_{22}.

We now summarize our findings.

5.1 The Rudvalis group

The Rudvalis group was discovered by Arunas Rudvalis in 1973 [14]. It has order $145,926,144,000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ and smallest permutation representation degree

[2]The timings presented in this section were obtained using MAGMA [1] running on a computer with 8 cores running at 3.9Ghz and 512Gb of RAM at 2.9Ghz. Note that MAGMA does not do parallel computing, it is using one core at a time.

G	Central	Linear	High+Rank3	Old algo	# pol rk > 3	# pol rk 3
M_{11}	0.11s	0.1s	0.04s + 0.05 = 0.09s	0.76s	0	0
M_{12}	5s	2.7s	2.8s + 0.2s = 3s	191s	14	23
J_1	4.2s	2.4s	0.9s + 1.3s = 2.2s	357s	2	148
M_{22}	1s	0.9s	0.22s + 0.04s = 0.26s	414s	0	0
J_2	11s	7.5s	8.5s + 2.1s = 10.6s		17	137
M_{23}	1.7s	1.8s	0.3s + 0.06s = 0.36s		0	0
HS	102s	74s	68s + 7.5s = 75.5s		59	252
J_3	207s	266s	41.2s + 121.7s = 162.9s		2	303
M_{24}	1622s	384s	1258s + 10s = 1268s = 21m8s		157	490
McL	15s	10s	2.09s+0.56s = 2.65s		0	0
He	20635s=5h43m	6216s=1h43m	16717s + 744s = 17461s		76	1188
Ru	127600s	91338s=25h22m	27739s+73518s = 101257s		227	21594
Suz	40322s=11h12m	43539s	41870s + 10439s = 52309s		270	7119
O'N	1360382s = 15.74 days	1536062s	573962s + 2619817s = 37 days		16	6536
Co_3	18531s=5h09m	27570s	28627s + 5522s = 9.5h		895	10586
Co_2			???s + 1305940s = ???		???	60370
Fi_{22}			???s + 2010594s = ???		???	25052

Table 2: Computing times for sporadic groups.

4060. It has two conjugacy classes of involutions. It has 21594 string C-group representations of rank three, 227 of rank four and none of higher rank.

5.2 The Suzuki sporadic group

The Suzuki sporadic group was discovered by Michio Suzuki in 1968 [15]. It has order $448,345,497,600 = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ and smallest permutation representation degree 1782. It has two conjugacy classes of involutions. It has 7119 string C-group representations of rank three, 257 of rank four, 13 of rank five and none of higher rank.

5.3 The O'Nan group

In 1973, Michael O'Nan provided in [13] strong evidence for the existence of a new sporadic group now called $\mathsf{O'N}$. Later in the seventies, Sims constructed this group with help of a computer (see [6] for a survey of the story of $\mathsf{O'N}$) but his work seems to be unpublished. The group has order $4,608,155,059,20 = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ and smallest permutation representation degree $122,760$. This makes it a difficult group to deal with even though its 2-Sylows are not that large compared to Suz and Ru.

All its string C-group representations of rank at least four were determined by Connor, Leemans and Mixer [3]. In [2], Connor and Leemans computed the number of regular maps having the O'Nan group as automorphism group, therefore bounding the number of string C-group representations of rank three. However, they were unable to compute all string C-group representations of rank three. Thanks to our new algorithm, we finally fill in this gap in the classification of string C-group representations of the O'Nan group. The O'Nan group has 6536 string C-group representations of rank three, 16 of rank four and none of higher rank.

5.4 The Conway group Co_2

The Conway group Co_2 was discovered by John Horton Conway in 1968 as a subgroup of a group he called .0, among with the other two simple sporadic groups Co_1 and Co_3 [4]. As pointed out by Conway in his paper, the simplicity of these groups was proven by John Thompson. The group has order $42,305,421,312,000 = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ and smallest permutation representation degree 2300. It has three conjugacy classes of involutions. It has 60370 string C-group representations of rank three.

5.5 The Fischer group Fi_{22}

The Fischer group Fi_{22} was discovered by Bernd Fischer in 1971 [5] while studying groups generated by 3-transpositions. The group has order $64,561,751,654,400 = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ and smallest permutation representation degree 3510. It has three conjugacy classes of involutions. It has 25052 string C-group representations of rank three.

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.10
https://doi.org/10.26493/2590-9770.1411.6ee
(Also available at http://adam-journal.eu)

On Roli's cube

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Received 6 April 2021, accepted 4 June 2021, published online 26 July 2022

Abstract

First described in 2014, *Roli's cube* $\mathcal{R}$ is a chiral 4-polytope, faithfully realized in Euclidean 4-space (a situation earlier thought to be impossible). Here we describe $\mathcal{R}$ in a new way, determine its minimal regular cover, and reveal connections to the Möbius-Kantor configuration.

Keywords: Regular and chiral polytopes, realizations of polytopes.

Math. Subj. Class.: 51M20, 52B15

1 Introduction

Actually *Roli's cube* $\mathcal{R}$ isn't a cube, although it does share the 1-skeleton of a 4-cube. First described by Javier (Roli) Bracho, Isabel Hubard and Daniel Pellicer in [3], $\mathcal{R}$ is a chiral 4-polytope of type $\{8, 3, 3\}$, faithfully realized in $\mathbb{E}^4$ (a situation earlier thought impossible). Of course, Roli didn't himself name $\mathcal{R}$; but the eponym is pleasing to his colleagues and has taken hold.

Chiral polytopes with realizations of 'full rank' had (incorrectly) been shown not to exist by Peter McMullen in [11, Theorem 11.2]. Mind you, these objects do seem to be elusive. Pellicer has proved in [15] that chiral polytopes of full rank can exist only in ranks 4 or 5.

Roli's cube $\mathcal{R}$ was constructed in [3] as a *colourful polytope*, starting from a hemi-4-cube in projective 3-space. (For more on this, see Section 3.) The construction given here in Section 6 is a bit different, though certainly closely related. In Section 7 we can then easily manufacture the minimal regular cover $\mathcal{T}$ for $\mathcal{R}$, and give both a presentation and faithful

*This work was generously supported at the Universidad Nacional Autónoma de México (Morelia) by PAPIIT-UNAM grant #IN100518. Besides greeting Marston Conder, I also want to thank Daniel Pellicer for generously welcoming me to the Centro de Ciencias Matemáticas at UNAM (Morelia). Finally, thanks are also due the referee, whose useful advice clearly indicated a careful reading of the manuscript.

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representation for its automorphism group. Along the way, we encounter the Möbius-Kantor Configuration 8_3, the regular complex polygon $3\{3\}3$ and the 24-cell $\{3, 4, 3\}$.

In what follows, we can make our way with concrete examples, so we won't need much of the general theory of abstract regular or chiral polytopes and their realizations. We refer the reader to [12, 13] and [16] for more.

This is a good place to offer a thank you to Marston Conder: first, for his wonderfully fresh insights into the world of symmetry; and second, for his humour, which so often has us weeping with laughter!

2 The 4-cube: convex, abstract and colourful

The most familiar of the regular convex polytopes in Euclidean space $\mathbb{E}^4$ is surely the 4-cube $\mathcal{P} = \{4, 3, 3\}$. A projection of $\mathcal{P}$ into $\mathbb{E}^3$ is displayed in Figure 1.

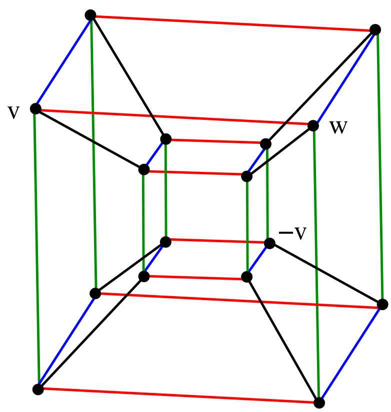

Figure 1: A 2-dimensional look at a 3-dimensional projection of the 4-cube.

Let us equip $\mathbb{E}^4$ with its usual basis $b_1, \dots, b_4$ and inner product. Then we may take the vertices of $\mathcal{P}$ to be the 16 sign change vectors

$$e = (\varepsilon_1, \dots, \varepsilon_4) \in \{\pm 1\}^4. \tag{2.1}$$

At any such vertex there is an edge (of length 2) running in each of the 4 coordinate directions, so that $\mathcal{P}$ has $32 = \frac{16 \cdot 4}{2}$ edges. Similarly we count the 24 squares $\{4\}$ as faces of dimension 2. Finally, $\mathcal{P}$ has 8 facets; these faces of dimension 3 are ordinary cubes $\{4, 3\}$. They lie in four pairs of supporting hyperplanes orthogonal to the coordinate axes. It is enjoyable to hunt for these faces in Figure 2, where the 8 parallel edges in each of the coordinate directions have colours black, red, blue and green, respectively.

Let us turn to the symmetry group G for $\mathcal{P}$. Each symmetry γ is determined by its action on the vertices, which clearly can be permuted with sign changes in all possible ways. Thus G has order $2^4 \cdot 4! = 384$, and we may think of it as being comprised of all 4×4 signed permutation matrices.

In fact, G can be generated by reflections $\rho_0, \rho_1, \rho_2, \rho_3$ in hyperplanes. Here ρ_0 negates the first coordinate x_1 (reflection in the coordinate hyperplane orthogonal to b_1); and, for

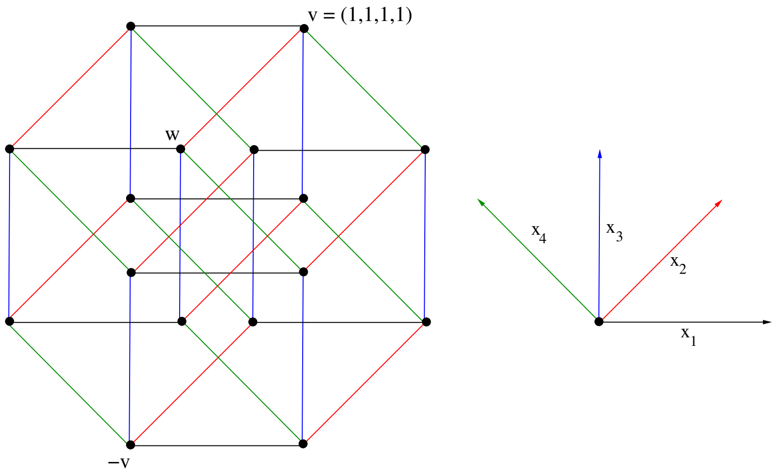

Figure 2: The most symmetric 2-dimensional projection of the 4-cube.

$1 \leq j \leq 3$, ρ_j transposes coordinates x_j, x_{j+1} (reflection in the hyperplane orthogonal to $b_j - b_{j+1}$).

Note that the reflection in the j-th coordinate hyperplane is $\rho_0^{\rho_1 \cdots \rho_{j-1}}$ for $1 \leq j \leq 4$. (We use the notation $\gamma^\eta := \eta^{-1}\gamma\eta$.) The product of these 4 special reflections, in any order, is the central element $\zeta : t \mapsto -t$. It is easy to check as well that

$$\zeta = (\rho_0\rho_1\rho_2\rho_3)^4 \,. \tag{2.2}$$

The *Petrie symmetry* $\pi = \rho_0\rho_1\rho_2\rho_3$ therefore has period 8.

For purposes of calculation, we note that $G \simeq C_2^4 \rtimes S_4$ is a semidirect product. Under this isomorphism, each $\gamma \in G$ factors uniquely as $\gamma = e\mu$, where $\mu \in S_n$ is a permutation of $\{1, \ldots, 4\}$ (labelling the coordinates); and e is a sign change vector, as in (2.1). Note that

$$e^\mu = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)^\mu = (\varepsilon_{(1)\mu^{-1}}, \varepsilon_{(2)\mu^{-1}}, \varepsilon_{(3)\mu^{-1}}, \varepsilon_{(4)\mu^{-1}})\,.$$

Now really γ is a signed permutation matrix. But it is convenient to abuse notation, keeping in mind that each e corresponds to a diagonal matrix of signs and each μ to a permutation matrix. Thus we might write

$$\pi = \rho_0\rho_1\rho_2\rho_3 = (-1, 1, 1, 1) \cdot (4, 3, 2, 1) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.3}$$

Next we use the group $G = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ to remanufacture the cube. In this (geometric) version of *Wythoff's construction* [7, §2.4] we choose a *base vertex* v fixed by the subgroup $G_0 := \langle \rho_1, \rho_2, \rho_3 \rangle$ (which permutes the coordinates in all ways). Thus, $v = c(1, 1, 1, 1)$ for some $c \in \mathbb{R}$. To avoid a trivial construction we take $c \neq 0$, so, up to similarity, we may use $c = 1$. Then the orbit of v under G is just the set of 16 points in

(2.1); and their convex hull returns $\mathcal{P}$ to us. Since G_0 is the full stabilizer of v in G, the vertices correspond to right cosets $G_0\gamma$.

The beauty of Wythoff's construction is that all faces of $\mathcal{P}$ can be constructed in a similar way by induction on dimension (see [4, 12] or [13, Section 1B]). For example, the vertices $v = (1, 1, 1, 1)$ and $v\rho_0 = (-1, 1, 1, 1)$ of the *base edge* of $\mathcal{P}$ are just the orbit of v under the subgroup $G_1 := \langle \rho_0, \rho_2, \rho_3 \rangle$; and edges of $\mathcal{P}$ correspond to right cosets of the new subgroup G_1. Furthermore, a more careful look reveals that a vertex is incident with an edge just when the corresponding cosets have non-trivial intersection.

Pursuing this, we see that the face lattice of $\mathcal{P}$ can be recontructed as a coset geometry based on subgroups

$$G_0, G_1, G_2, G_3, \quad \text{where } G_j := \langle \rho_0, \ldots, \rho_{j-1}, \rho_{j+1}, \ldots, \rho_3 \rangle. \tag{2.4}$$

From this point of view, $\mathcal{P}$ becomes an *abstract regular* 4*-polytope*, a partially ordered set whose automorphism group is G. Notice that the distinguished subgroups in (2.4) provide the proper faces in a *flag* in $\mathcal{P}$, namely a mutually incident vertex, edge, square and 3-cube.

The crucial structural property of G is that it should be a string C-group with respect to the generators ρ_j. A *string C-group* is a quotient of a Coxeter group with linear diagram under which an 'intersection condition' on subgroups generated by subsets of generators, such as those in (2.4), is preserved [13, Section 2E].

For the 4-cube $\mathcal{P}$, G is actually isomorphic to the Coxeter group B_4 with diagram

$$\bullet \overset{4}{\text{———}} \bullet \overset{3}{\text{———}} \bullet \overset{3}{\text{———}} \bullet \tag{2.5}$$

Comparing the geometric and abstract points of view, we say that the convex 4-cube is a *realization* of its face lattice (the abstract 4-cube).

When we think of a polytope from the abstract point of view, we often use the term *rank* instead of 'dimension'. An abstract polytope $\mathcal{Q}$ is said to be *regular* if its automorphism group is transitive on flags (maximal chains in $\mathcal{Q}$). Intuitively, regular polytopes have maximal symmetry (by reflections). Next up are *chiral* polytopes, with exactly two flag orbits and such that adjacent flags are always in different orbits (so maximal symmetry by rotations, but without reflections).

We will soon encounter less familiar abstract regular or chiral polytopes, with their realizations. For a first example, suppose that we map (by central projection) the faces of $\mathcal{P}$ onto the 3-sphere $\mathbb{S}^3$ centred at the origin. We can then reinterpret $\mathcal{P}$ as a regular *spherical polytope* (or tessellation), with the same symmetry group G. Now recall that the centre of G is the subgroup $\langle \zeta \rangle$ of order 2. The quotient group $G/\langle \zeta \rangle$ has order 192 and is still a string C-group. The corresponding regular polytope is the *hemi-*4*-cube* $\mathcal{H} = \{4, 3, 3\}_4$, here realized in projective space $\mathbb{P}^3$ [13, Section 6C]; see Figure 3. By (2.2), the product of the four generators of $G/\langle \zeta \rangle$ has order 4; this is recorded as the subscript in the Schläfli symbol for $\mathcal{H}$.

Now we can outline the construction of Roli's cube given in [3].

3 Colourful polyopes

The image in Figure 1 or on the left in Figure 2 can just as well be understood as a graph $\mathcal{G}$, namely the 1-*skeleton* of the 4-cube $\mathcal{P}$. In fact, we can recreate the abstract (or combinatorial) structure of $\mathcal{P}$ from just the edge colouring of $\mathcal{G}$: for $0 \leq j \leq 4$, the j-faces of $\mathcal{P}$

can be identified with the components of those subgraphs obtained by keeping just edges with some selection of the j colours (over all such choices). We therefore say that $\mathcal{P}$ is a *colourful polytope*.

Such polytopes were introduced in [1]. In general, one begins with a finite, connected d-valent graph $\mathcal{G}$ admitting a (proper) edge colouring, say by the symbols $1, \ldots, d$. Thus each of the colours provides a 1-factor for $\mathcal{G}$. The graph $\mathcal{G}$ determines an (abstract) colourful polytope $\mathcal{P}_\mathcal{G}$ as follows. For $0 \leq j \leq d$, a typical j-face (C, v) is identified with the set of all vertices of $\mathcal{G}$ connected to a given vertex v by a path using only colours from some subset C of size j taken from $\{1, \ldots, d\}$. The j-face (C, v) is less than or equal the k-face (D, w) just when $C \subseteq D$ and w can be reached from v by a D-coloured path. (This means that $j \leq k$; and we can just as well replace w by v. The minimal face of rank -1 in $\mathcal{P}_\mathcal{G}$ is formal.) Notice that the 1-skeleton of the d-polytope $\mathcal{P}_\mathcal{G}$ is just $\mathcal{G}$ itself. The polytope is *simple*: each of its vertex-figures is isomorphic to a $(d-1)$-simplex. From [1, Theorem 4.1], the automorphism group of $\mathcal{P}_\mathcal{G}$ is isomorphic to the group of colour-respecting graph automorphisms of $\mathcal{G}$. (Such automorphisms are allowed to permute the 1-factors.)

It is easy to see that the hemi-4-cube $\mathcal{H}$ is also colourful. Its 1-skeleton is the complete bipartite graph $K_{4,4}$ found in Figure 3. We obtain this graph from Figure 1 or Figure 2 by identifying antipodal pairs of points, like v and $-v$.

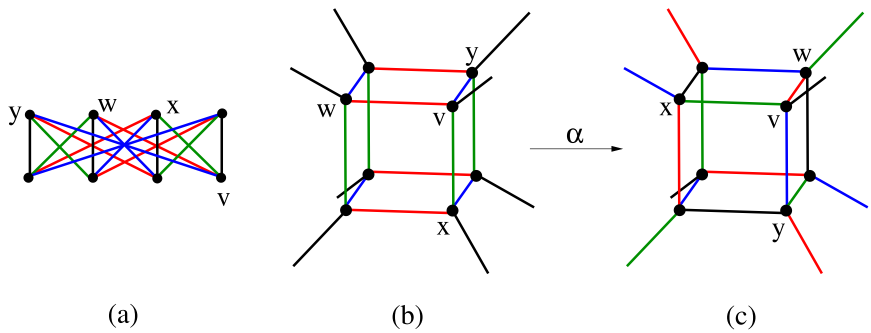

Figure 3: The graph $K_{4,4}$ in (a) is the 1-skeleton of the hemi-4-cube $\{4, 3, 3\}_4$ in (b), (c).

If we lift $\mathcal{H}$, as it is now, to $\mathbb{S}^3$, we regain the coloured 4-cube $\mathcal{P}$. Now keep $K_{4,4}$ embedded in $\mathbb{P}^3$, as in Figure 3. But, following [3], observe that $K_{4,4}$ admits the automorphism α which cyclically permutes, say, the first three vertices y, w, x in the top block, leaving the rest fixed. Clearly, α is a non-colour-respecting automorphism of $K_{4,4}$, so its effect is to recolour 12 of the edges in the embedded graph. On the abstract level nothing has changed for the resulting colourful polytope; it is still the hemi-4-cube $\mathcal{H}$. But faces of ranks 2 and 3 are now differently embedded in $\mathbb{P}^3$. For example, the red-blue 2-face on v, which is planar in Figure 3(b), becomes a skew quadrangle in Figure 3(c) and also acquires an orientation. According to Definition 4.1 below, these skew quadrangles are Petrie polygons for the standard realization of $\mathcal{H}$ in Figure 3(b).

The newly coloured geometric object, which we might label $\mathcal{H}^R$, is a *chiral realization* of the abstract regular polytope $\mathcal{H}$. Comforted by the fact that $\mathbb{P}^3$ is orientable, we could just as well apply α^{-1} to obtain the left-handed version $\mathcal{H}^L$. These two *enantiomorphs* are oppositely embedded in $\mathbb{P}^3$, though both remain isomorphic to $\mathcal{H}$ as partially ordered sets. If we lift either enantiomorph to $\mathbb{S}^3$, we obtain a chiral 4-polytope faithfully realized in $\mathbb{E}^4$ [3, Theorem 2]. This is Roli's cube $\mathcal{R}$.

Next we set the stage for a slightly different construction of $\mathcal{R}$, without the use of $\mathbb{P}^3$.

4 Petrie polygons of the 4-cube

Let us consider the progress of the base vertex $v = (1, 1, 1, 1)$ for $\mathcal{P}$ as we apply successive powers of π in (2.3). We get a centrally symmetric 8-cycle of vertices

$$v \to (1, 1, 1, -1) \to (1, 1, -1, -1) \\ \to (1, -1, -1, -1) \to -v = (-1, -1, -1, -1) \to \dots .$$

Starting from v in Figure 2 we therefore proceed in coordinate directions $4, 3, 2, 1$ (indicated by different colours), then repeat again. This traces out the peripheral octagon $\mathcal{C}$, which in fact is a Petrie polygon for $\mathcal{P}$.

Definition 4.1 (See [6, p. 223] or [13, p. 163])**.** A *Petrie polygon* of a 3-polytope is an edge-path such that any 2 consecutive edges, but no 3, belong to a 2-face. We then say that a *Petrie polygon* of a 4-polytope $\mathcal{Q}$ is an edge-path such that any 3 consecutive edges, but no 4, belong to (a Petrie polygon of) a facet of $\mathcal{Q}$.

For the cube $\mathcal{P}$, the parenthetical condition is actually superfluous; compare [9].

Clearly, we can begin a Petrie polygon at any vertex, taking any of the $4!$ orderings of the colours. But this counts each octagon in 16 ways. We conclude that $\mathcal{P}$ has 24 Petrie polygons. What we really use here is the fact that G is transitive on vertices, and that at any fixed vertex, G permutes the edges in all possible ways. We see that G acts transitively on Petrie polygons.

But the (global) stabilizer of the polygon $\mathcal{C}$ (constructed above with the help of π and v) is the dihedral group K of order 16 generated by $\mu_0 = \rho_0\rho_2\rho_3\rho_2 = (-1, 1, 1, 1) \cdot (2, 4)$, and $\mu_1 = \mu_0\pi = \rho_2\rho_3\rho_2\rho_1\rho_2\rho_3 = (1, 1, 1, 1) \cdot (1, 4)(2, 3)$. (Such calculations are routine using either signed permutation matrices or the decomposition in $C_2^4 \rtimes S_4$. Note that any 4 consecutive vertices of $\mathcal{C}$ form a basis of $\mathbb{E}^4$.) We confirm that $\mathcal{P}$ has $24 = 384/16$ Petrie polygons.

Now we move to the *rotation subgroup*

$$G^+ = \langle \rho_0\rho_1, \rho_1\rho_2, \rho_2\rho_3 \rangle.$$

It has order 192 and consists of the signed permutation matrices of determinant $+1$. Note that $K < G^+$. Thus, under the action of G^+, there are two orbits of Petrie polygons of 12 each. Let's label these two *chiral classes* R and L for right- and left-handed, taking $\mathcal{C}$ in class R.

The two chiral classes must be swapped by any non-rotation, such as any ρ_j. To distinguish them, we could take the determinant of the matrix whose rows are any 4 consecutive vertices on a Petrie polygon. The two chiral classes R and L then have determinants $+8$, respectively, -8. Or starting from a common vertex, the edge-colour sequence along a polygon in one class is an odd permutation of the colour sequence for a polygon in the other class.

The inner octagram $\mathcal{C}^*$ in Figure 2 is another Petrie polygon. Start at the vertex $w = (v)\rho_1\rho_0\rho_1 = (1, -1, 1, 1)$ which is adjacent to v along a red edge; then proceed in directions 4, 1, 2, 3 and repeat. However, the remaining Petrie polygons appear in less symmetrical fashion in Figure 2.

Note that μ_0 actually acts on the diagram in Figure 2 as a reflection in a vertical line, whereas π rotates the octagon $\mathcal{C}$ through one edge-step and the octagram $\mathcal{C}^*$ by three. On the other hand, $\mu_2 = \rho_1\rho_2\rho_0\rho_1 = (1,-1,1,1)\cdot(1,3)$ is an element of G^+ which swaps $\mathcal{C}$ and $\mathcal{C}^*$. Thus there are 6 such unordered pairs like $\mathcal{C}, \mathcal{C}^*$ in class R and another 6 pairs in class L.

Remark 4.2. It can be shown that Figure 2 is the most symmetric orthogonal projection of $\mathcal{P}$ to a plane [6, Section 13.3]. Since all edges after projection have a common length, we may say that this projection is *isometric*.

The Petrie symmetry π is one instance of a *Coxeter element* in the group $G = B_4$, namely the product of the four reflections attached to a simple system of roots. All such Coxeter elements are conjugate [10, Sections 1.3 and 3.16]. Each of them has invariant planes which give rise to the sort of orthogonal projection displayed in Figure 2. A procedure for finding these planes is detailed in [10, Section 3.17]. For π, the two planes are spanned by the rows of

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

These planes are orthogonal complements; and π acts on them by rotations through $45°$ and $135°$, respectively. Figure 2 results from projecting $\mathcal{P}$ onto the first plane. □

5 The map $\mathcal{M}$ and the Möbius-Kantor Configuration 8_3

Look again at the companion Petrie polygons $\mathcal{C}, \mathcal{C}^*$ in Figure 2. Now working around the rim clockwise from v delete the edges coloured blue, red, black, green, and repeat. We are left with the trivalent graph $\mathcal{L}$ displayed in Figure 4.

In fact, $\mathcal{L}$ is the *generalized Petersen graph* $\{8\} + \{\frac{8}{3}\}$, studied in detail by Coxeter in [5, Section 5]. The graph is transitive on 2-arcs but not on 3-arcs, so its automorphism group has order $96 = 16 \cdot 3 \cdot 2^{2-1}$ [2, Chapter 18]. We return to this group later.

We have labelled alternate vertices of $\mathcal{L}$ by the residues $0, 1, 2, 3, 4, 5, 6, 7 \pmod 8$. These will represent the points in a Möbius-Kantor configuration 8_3. The remaining (unlabelled) vertices of $\mathcal{L}$ represent the 8 lines in the configuration. Thus we have lines 013 (represented by the 'north-west' vertex $(-1,-1,1,1)$), 124, 235, and so on, including line 671 represented by v.

Notice that we can interpret the configuration as being comprised of two quadrangles with vertices $0, 2, 4, 6$ and $1, 3, 5, 7$, *each inscribed in the other*: vertex 0 lies on edge 13, vertex 1 lies on edge 24, and so on.

So far this configuration 8_3 is purely abstract. In fact, it can be realized as a point-line configuration in a projective (or affine) plane over any field in which

$$z^2 - z + 1 = 0$$

has a root, certainly over $\mathbb{C}$. However, 8_3 cannot be realized in the real plane.

Coxeter made other observations in [5], including the fact, used above, that the graph $\mathcal{L}$ is a sub-1-skeleton of the 4-cube. Altogether $\mathcal{L}$ contains 3 complementary pairs of Petrie

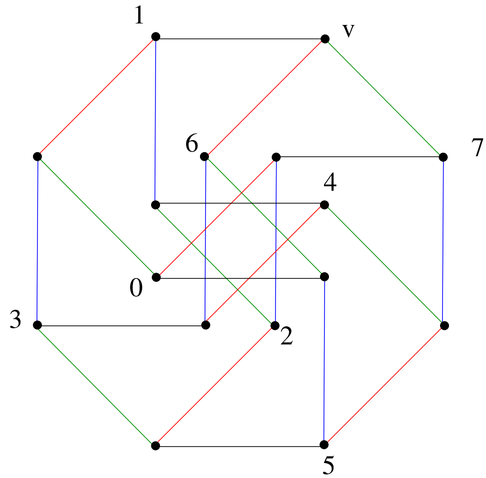

Figure 4: The Levi graph $\mathcal{L} = \{8\} + \{\frac{8}{3}\}$ for the configuration 8_3.

polygons, which we can briefly describe by their alternate vertices:

$$\begin{array}{ccc} 0246(=\mathcal{C}^*) & 0541 & 1256 \\ 1357(=\mathcal{C}) & 2367 & 0743 \end{array}$$

Hence, the configuration can be regarded as a pair of mutually inscribed quadrangles in three ways.

Observe that each edge of $\mathcal{L}$ lies on exactly two of the 6 octagons. For example, the top edge with vertices labelled 1 and v lies on octagons 1357 and 1256. (It does not matter that two such octagons then share a second edge opposite the first.) Furthermore, each vertex lies on the three octagons determined by choices of two edges. We can thereby construct a 3-polytope $\mathcal{M}$ of type $\{8, 3\}$, with 6 octagonal faces, whose 1-skeleton is $\mathcal{L}$. In short, $\mathcal{M}$ is realized by substructures of the 4-cube $\mathcal{P}$.

Moving sideways, we can reinterpret $\mathcal{M}$ in a more familiar topological way as a map on a compact orientable surface of genus 2. Recall that $\mathcal{M}$ is covered by the tessellation $\{8, 3\}$ of the hyperbolic plane, as indicated in Figure 5.

Now return to $\mathbb{E}^4$ where the combinatorial structure of $\mathcal{M}$ is handed to us as faithfully realized. Drawing on [8, Section 8.1], we have that the rotation group $\Gamma(\mathcal{M})^+$ for $\mathcal{M}$ is generated by two special Euclidean symmetries:

$\sigma_1 = \pi = \rho_0\rho_1\rho_2\rho_3 = (-1, 1, 1, 1) \cdot (4, 3, 2, 1)$ (preserving the base octagon $\mathcal{C}$); and

$\sigma_2 = \rho_3\rho_2\rho_1\rho_3 = (1, 1, 1, 1) \cdot (1, 2, 4)$ (preserving the base vertex v on $\mathcal{C}$).

The order of $\Gamma(\mathcal{M})^+$ *must* then be twice the number of edges in $\mathcal{M}$, namely 48. Let us assemble these and further observations in

Theorem 5.1. (a) *The* 3*-polytope* $\mathcal{M}$ *is abstractly regular of type* $\{8, 3\}$*, here realized in* $\mathbb{E}^4$ *in a geometrically chiral way.*

(b) *The rotation subgroup* $\Gamma(\mathcal{M})^+ = \langle\sigma_1, \sigma_2\rangle$ *has order* 48 *and presentation*

$$\langle \sigma_1, \sigma_2 \,|\, \sigma_1^8 = \sigma_2^3 = (\sigma_1\sigma_2)^2 = (\sigma_1^{-3}\sigma_2)^2 = 1 \rangle. \tag{5.1}$$

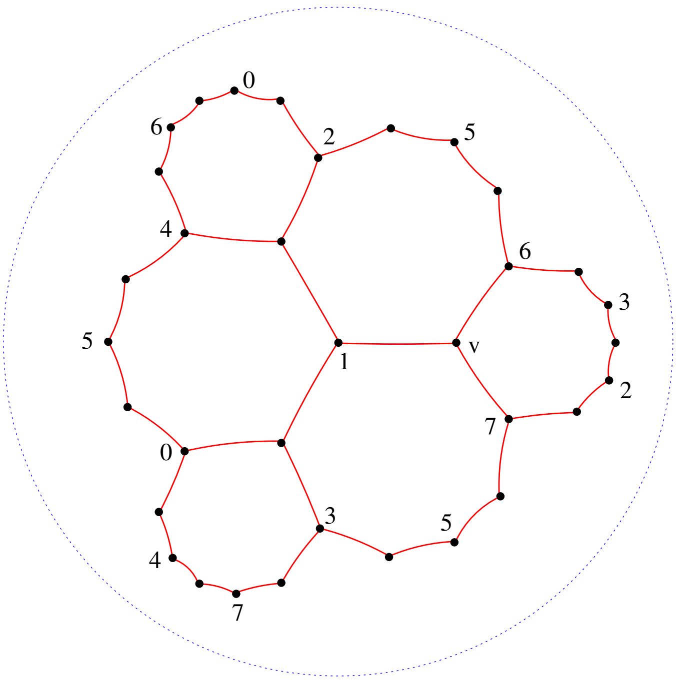

Figure 5: Part of the tessellation $\{8, 3\}$ of the hyperbolic plane.

(c) *The full automorphism group* $\Gamma(\mathcal{M})$ *has order* 96 *and presentation*

$$\langle \tau_0, \tau_1, \tau_2 \,|\, \tau_0^2 = \tau_1^2 = \tau_2^2 = (\tau_0\tau_1)^8 = (\tau_1\tau_2)^3 = (\tau_0\tau_2)^2 = ((\tau_1\tau_0)^3\tau_1\tau_2)^2 = 1\rangle. \tag{5.2}$$

Proof. We begin with (b), where it is easy to check that the relations in (5.1) do hold for the matrix group $\langle \sigma_1, \sigma_2 \rangle$. By a straightforward coset enumeration [8, Chapter 2], we conclude from the presentation in (5.1) that the subgroup $\langle \sigma_1 \rangle$ has the 6 coset representatives

$$1, \sigma_2, \sigma_2^2, \sigma_2\sigma_1^{-1}, \sigma_2^2\sigma_1, \sigma_2\sigma_1^{-1}\sigma_2.$$

(We abuse notation by passing freely between the matrix group and abstract group.) This finishes (b).

We next note that $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{1\}$, since σ_1^j fixes v only for $j \equiv 0 \pmod 8$. Now we are justified in invoking [16, Theorem 1(c)], whereby the 3-polytope $\mathcal{M}$ is regular (rather than just chiral) if and only if the mapping $\sigma_1 \mapsto \sigma_1^{-1}$, $\sigma_2 \mapsto \sigma_1^2\sigma_2$ induces an involutory automorphism τ of $\Gamma(\mathcal{M})^+$. But the new relations induced by applying the mapping to (5.1) are easily verified formally, or even by matrices. For instance, since $\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}$, we have

$$(\sigma_1^2\sigma_2)^3 = (\sigma_1\sigma_2^{-1}\sigma_1^{-1})^3 = \sigma_1\sigma_2^{-3}\sigma_1^{-1} = 1.$$

Thus $\mathcal{M}$ is abstractly regular and $\Gamma(\mathcal{M})$ has order 96. The presentation in (5.2) follows at once by extending $\Gamma(\mathcal{M})^+$ by $\langle \tau \rangle$, then letting $\tau_0 := \tau, \tau_1 := \tau\sigma_1, \tau_0 := \tau\sigma_1\sigma_2$. (In Theorem 7.1 we use the same labels for the first three generators of the linear group T.)

It remains to check that our realization is geometrically chiral [3, Theorem 2]. This means that τ is not represented by a symmetry of $\mathcal{M}$ as realized in $\mathbb{E}^4$. From the combinatorial structure, τ would have to swap vertices 1 and v while preserving the two Petrie polygons on that edge. Thus τ would have to act just like μ_0, that is, just like reflection in a vertical line in Figure 2. But μ_0 does not preserve the set of 8 edges deleted to give $\mathcal{L}$ in Figure 4. □

Remark 5.2. It is helpful to note that the centre of $\Gamma(\mathcal{M})^+$ is generated by σ_1^4. Referring to [8, Section 6.6], we find that $\Gamma(\mathcal{M})^+$ is isomorphic to the group $\langle -3, 4|2\rangle$, which in turn is an extension by C_2 of the binary tetrahedral group $\langle 3, 3, 2\rangle$. Indeed, $a = \sigma_1^{-1}\sigma_2\sigma_1^{-1}, b = \sigma_2\sigma_1^4$ satisfy $a^3 = b^3 = (ab)^2\ (= \zeta)$. Thus, $\langle 3, 3, 2\rangle \triangleleft \Gamma(\mathcal{M})^+$. □

6 Roli's cube – a chiral polytope $\mathcal{R}$ of type $\{8, 3, 3\}$

Under the action of G^+ we expect to find $4 = 192/48$ copies of $\mathcal{M}$. To understand this better, recall that there are 12 Petrie polygons in one chiral class, say R. As with $\mathcal{C}$ and $\mathcal{C}^*$, each polygon $\mathcal{D}$ is paired with a unique polygon $\mathcal{D}^*$ (with the disjoint set of 8 vertices). For each $\mathcal{D}$ there are then *two* ways to remove 8 edges (connecting $\mathcal{D}$ and $\mathcal{D}^*$) so as to get a copy of $\mathcal{L}$ and hence a copy of $\mathcal{M}$. Since $\mathcal{M}$ has six 2-faces like $\mathcal{C}$, we once more find $12 \cdot 2/6 = 4$ copies of $\mathcal{M}$.

Each Petrie polygon lies on 2 copies of $\mathcal{M}$, again from the two ways to remove 8 edges. For example, $\mathcal{C}$ lies on both $\mathcal{M}$ and $(\mathcal{M})\mu_0$. (The same is true for $\mathcal{C}^*$.)

The pointwise stabilizer in G^+ of the base edge joining $v = (1, 1, 1, 1)$ and $(v)\mu_0 = (-1, 1, 1, 1)$ must consist of pure, unsigned even permutations of $\{2, 3, 4\}$. Therefore it is generated by

$$\sigma_3 := \rho_2\rho_3 = (1, 1, 1, 1) \cdot (2, 4, 3).$$

It is easy to check that $G^+ = \langle \sigma_1, \sigma_2, \sigma_3\rangle$.

Since three consecutive edges of a Petrie polygon lie on two adjacent square faces in a cubical facet of $\mathcal{P}$, it must be that every vertex of $\mathcal{R}$ has the same vertex-figure as $\mathcal{P}$, thus of tetrahedral type $\{3, 3\}$.

We have enumerated and (implicitly) assembled the faces of a 4-polytope $\mathcal{R}$, faithfully realized in $\mathbb{E}^4$ and symmetric under the action of G^+. Let's take stock of its proper faces:

rank	stabilizer in G^+	order	number of faces	type
0	$\langle \sigma_2, \sigma_3\rangle$	12	16	vertex of cube $\mathcal{P}$
1	$\langle \sigma_1\sigma_2, \sigma_3\rangle$	6	32	edge of $\mathcal{P}$
2	$\langle \sigma_1, \sigma_2\sigma_3\rangle$	16	12	Petrie polygons of $\mathcal{P}$ in one class R
3	$\langle \sigma_1, \sigma_2\rangle$	48	4	copy of $\mathcal{M}$

It is not hard to see that our 4-polytope $\mathcal{R}$ is isomorphic to Roli's cube, as constructed in [3] and as described in Section 3.

Theorem 6.1. (a) *The 4-polytope $\mathcal{R}$ is abstractly chiral of type $\{8,3,3\}$. Its symmetry group $\Gamma(\mathcal{R}) \simeq G^+$ has order 192 and the presentation*

$$\langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^8 = \sigma_2^3 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \tag{6.1}$$

$$(\sigma_1^{-3}\sigma_2)^2 = 1 \tag{6.2}$$

$$(\sigma_1^{-1}\sigma_3)^4 = 1 \ \rangle. \tag{6.3}$$

(b) *$\mathcal{R}$ is faithfully realized as a geometrically chiral polytope in $\mathbb{E}^4$.*

Proof. The relations in (6.1) are standard for chiral 4-polytopes [16, Theorem 1]; and we have seen that the relation in (6.2) is a special feature of the facet $\mathcal{M}$. Enumerating cosets of the subgroup $\langle \sigma_1, \sigma_2 \rangle$, which still has order 48, we find at most the 8 cosets represented by

$$1, \sigma_3, \sigma_3^2, \sigma_3^2\sigma_1, \sigma_3^2\sigma_1^2, \sigma_3^2\sigma_1^2\sigma_2, \sigma_3^2\sigma_1^2\sigma_2^2, \sigma_3^2\sigma_1^2\sigma_3.$$

Thus the group defined by (6.1) and (6.2) has order at most 384. But G^+, where these relations do hold, has order 192. We require an independent relation. In Section 7, we will see why (6.3) is just what we need.

To show that $\mathcal{R}$ is abstractly chiral we must demonstrate that the mapping $\sigma_1 \mapsto \sigma_1^{-1}, \sigma_2 \mapsto \sigma_1^2\sigma_2, \sigma_3 \mapsto \sigma_3$ does not extend to an automorphism of G^+. This is easy, since

$$(\sigma_1\sigma_3)^4 = \zeta \text{ whereas } (\sigma_1^{-1}\sigma_3)^4 = 1. \tag{6.4}$$

Clearly, $\mathcal{R}$ is realized in a geometrically chiral way in $\mathbb{E}^4$; we have already seen this for its facet $\mathcal{M}$.

Our concrete geometrical arguments should suffice to convince the reader that we really have described here a chiral 4-polytope identical to the original Roli's cube. A skeptic can nail home the proof by applying [16, Theorem 1] to the group G^+, as generated above. □

7 Realizing the minimal regular cover of $\mathcal{R}$

The rotation group G^+ for the cube has order 192 and 'standard' generators $\rho_0\rho_1, \rho_1\rho_2, \rho_2\rho_3$. But for our purposes we use either of two alternate sets of generators. We already have

$$\sigma_1 = (\rho_0\rho_1)(\rho_2\rho_3),\ \sigma_2 = (\rho_3\rho_2)(\rho_1\rho_2)(\rho_2\rho_3),\ \sigma_3 = \rho_2\rho_3. \tag{7.1}$$

Now we also want

$$\overline{\sigma}_1 = \sigma_1^{-1},\ \overline{\sigma}_2 = \sigma_1^2\sigma_2,\ \overline{\sigma}_3 = \sigma_3. \tag{7.2}$$

Recalling our shorthand for such matrices, we have

$$\sigma_1 = (-1,1,1,1)\cdot(4,3,2,1);\ \sigma_2 = (1,1,1,1)\cdot(1,2,4);\ \sigma_3 = (1,1,1,1)\cdot(2,4,3),$$

and

$$\overline{\sigma}_1 = (1,1,1,-1)\cdot(1,2,3,4);\ \overline{\sigma}_2 = (-1,-1,1,1)(1,3,2);\ \overline{\sigma}_3 = (1,1,1,1)\cdot(2,4,3).$$

We have seen that the group $G^+ = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ (with these specified generators) is the rotation (and full automorphism) group of the chiral polytope $\mathcal{R}$ of type $\{8, 3, 3\}$. From [17, Section 3] we have that the (differently generated) group $\overline{G^+} = \langle \overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3 \rangle$ is the automorphism group for the *enantiomorphic* chiral polytope $\overline{\mathcal{R}}$. By generating the common group in these two ways we effectively exhibit right- and left-handed versions of the same polytope.

Our geometrical realization of $\mathcal{R}$ began with the base vertex $v = (1, 1, 1, 1)$ (which also served as base vertex for the 4-cube $\mathcal{P}$). It is crucial here that v does span the subspace fixed by σ_2 and σ_3. By instead taking $\overline{G^+}$ with base vertex $\overline{v} = (-1, 1, 1, 1)$ fixed by $\overline{\sigma}_2$, and $\overline{\sigma}_3\rangle$, we have a faithful geometric realization of $\overline{R}$, still in $\mathbb{E}^4$, of course.

We will soon have good reason to mix G^+ and $\overline{G^+}$ in a geometric way. Each group acts irreducibly on $\mathbb{E}^4$. Construct the block matrices, $\kappa_j = (\sigma_j, \overline{\sigma}_j)$, $j = 1, 2, 3$, now acting on $\mathbb{E}^8$ and preserving two orthogonal subspaces of dimension 4. Obviously we may extend our notation for signed permutation matrices to the cubical group B_8 acting on $\mathbb{E}^8$. Thus, taking the second copy of $\mathbb{E}^4$ to have basis b_5, b_6, b_7, b_8, we may combine our descriptions of $\sigma_j, \overline{\sigma}_j$ to get

$$\begin{aligned}
\kappa_1 &= (-1, 1, 1, 1, 1, 1, 1, -1) \cdot (4, 3, 2, 1)(5, 6, 7, 8), \\
\kappa_2 &= (1, 1, 1, 1, -1, -1, 1, 1) \cdot (1, 2, 4)(5, 7, 6), \\
\kappa_3 &= \quad (1, 1, 1, 1, 1, 1, 1, 1) \cdot (2, 4, 3)(6, 8, 7).
\end{aligned}$$

Now let $T^+ = \langle \kappa_1, \kappa_2, \kappa_3 \rangle$. In slot-wise fashion, $\kappa_1, \kappa_2, \kappa_3$ satisfy relations like those in (6.1) and (6.2). From the proof of Theorem 6.1, we conclude that T^+ has order 384. We even get a presentation for it.

Recall that the centre of G^+ is generated by $\zeta = \sigma_1^4 = \overline{\sigma}_1^4$. Thus the centre of T^+ has order 4, with non-trivial elements

$$(\zeta, 1) = (\kappa_1 \kappa_3)^4,\ (1, \zeta) = (\kappa_1^{-1} \kappa_3)^4, \text{and } (\zeta, \zeta) = \kappa_1^4. \tag{7.3}$$

(This is at the heart of the proof that $\mathcal{R}$ is abstractly chiral.) Looking at (6.4), we see that

$$T^+/\langle (1, \zeta) \rangle \simeq \Gamma(\mathcal{R}),$$

and thus see the reason for the special relation in (6.3). Similarly, $T^+/\langle (\zeta, 1) \rangle \simeq \Gamma(\overline{\mathcal{R}})$. Finally, we have

$$T^+/\langle (\zeta, \zeta) \rangle \simeq \Gamma(\mathcal{P})^+, \tag{7.4}$$

the rotation group of the 4-cube (isomorphic to G^+ generated in the customary way).

Now T^+ is clearly isomorphic to the *mix* $G^+ \diamondsuit \overline{G^+}$ described in [14, Theorem 7.2]. Guided by that result, we seek an isometry τ_0 of $\mathbb{E}^8$ which swaps the two orthogonal subspaces, while conjugating each $(\sigma_j, \overline{\sigma}_j)$ to $(\overline{\sigma}_j, \sigma_j)$. It is easy to check that

$$\tau_0 = (1, 1, 1, 1, 1, 1, 1, 1) \cdot (1, 5)(2, 6)(3, 7)(4, 8)$$

does the job. We find that T^+ is the rotation subgroup of a string C-group $T = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$, where $\tau_1 = \tau_0 \kappa_1, \tau_2 = \tau_0 \kappa_1 \kappa_2, \tau_3 = \tau_0 \kappa_1 \kappa_2 \kappa_3$. The corresponding directly regular 4-polytope has type $\{8, 3, 3\}$ and must be the minimal regular cover of each of the chiral polytopes $\mathcal{R}$ and $\overline{\mathcal{R}}$.

For a little background here, we note that T is an instance of the group H appearing in the proof of [14, Theorem 7.2]. There it was shown that H is isomorphic to the 'connection group' $\mathrm{Mon}(\mathcal{R})$ for the chiral polytope $\mathcal{R}$. From [14, Proposition 3.16], this is just what we need to discuss the regular covers of $\mathcal{R}$. We consolidate all this in

Theorem 7.1. (a) *The group* $T = \langle \tau_0, \tau_1, \tau_2, \tau_3 \rangle$ *is a string C-group of order* 768 *and with the presentation*

$$\langle \tau_0, \tau_1, \tau_2, \tau_3 \mid \tau_j^2 = (\tau_0\tau_1)^8 = (\tau_1\tau_2)^3 = (\tau_3\tau_3)^3 = 1,\ 0 \le j \le 3,$$
$$(\tau_0\tau_2)^2 = (\tau_0\tau_3)^2 = (\tau_1\tau_3)^2 = ((\tau_1\tau_0)^3\tau_1\tau_2)^2 = 1\rangle.$$

(b) *The corresponding regular* 4*-polytope* $\mathcal{T}$ *has type* $\{8, 3, 3\}$ *and is faithfully realized in* $\mathbb{E}^8$*, with base vertex* $(v, \overline{v}) = (1, 1, 1, 1, -1, 1, 1, 1)$*. The polytope* $\mathcal{T}$ *is the minimal regular cover for Roli's cube* $\mathcal{R}$ *and its enantiomorph* $\overline{\mathcal{R}}$*. It is also a double cover of the* 4*-cube* $\mathcal{P}$*.*

(c) $\mathcal{T} \simeq \{\mathcal{M}\,,\,\{3, 3\}\}$ *is the universal regular polytope with facets* $\mathcal{M}$ *and tetrahedal vertex-figures.*

Proof. The centre of T is generated by (ζ, ζ). It is easy to check that $T/\langle(\zeta, \zeta)\rangle \simeq G$, the full symmetry group of the cube; compare (7.4). In other words, the mapping $\tau_j \mapsto \rho_j,\ 0 \le j \le 3$, induces an epimorphism $\varphi\colon T \to G$. Since τ_1, τ_2, τ_3 and ρ_1, ρ_2, ρ_3 both satisfy the defining relations for $\Gamma(\{3, 3\}) \simeq S_4$, φ is one-to-one on $\langle\tau_1, \tau_2, \tau_3\rangle$. By the quotient criterion in [13, 2E17], T really is a string C-group. The remaining details are routine. For background on (c) we refer to [13, 4A]. □

Much as in the proof, the assignment $\kappa_j \mapsto \sigma_j$, $(j = 1, 2, 3)$, induces an epimorphism $\varphi_R\colon T^+ \to G^+$. On the abstract level, this in turn induces a *covering* $\widetilde{\varphi}_R\colon \mathcal{T} \to \mathcal{R}$, in other words, a rank- and adjacency-preserving surjection of polytopes as partially ordered sets. The corresponding covering of geometric polytopes is induced by the projection

$$\mathbb{E}^8 \to \mathbb{E}^4$$
$$(x, y) \mapsto x$$

The projection $(x, y) \mapsto y$ likewise induces the geometrical covering $\widetilde{\varphi}_L : \mathcal{T} \to \overline{\mathcal{R}}$. Both $\widetilde{\varphi}_R$ and $\widetilde{\varphi}_L$ are 3-coverings, meaning here that each acts isomorphically on facets $\mathcal{M}$ and vertex-figures $\{3, 3\}$ [13, page 43]. Notice that each face of $\mathcal{R}$ or $\overline{\mathcal{R}}$ has two preimages in $\mathcal{T}$.

The polytope $\mathcal{T}$ is also a double cover of the 4-cube $\mathcal{P}$. But there is no natural way to embed $\mathcal{P}$ in $\mathbb{E}^8$ to illustrate the geometric covering, since $\kappa_1^4 = -1$ on any subspace of $\mathbb{E}^8$, whereas $(\rho_0\rho_1)^4 = 1$ for $\mathcal{P}$.

8 The Möbius-Kantor configuration again, and the 24-cell

We noted earlier that 8_3 can be 'realized' as a point-line configuration in $\mathbb{C}^2$. We will show this here by first endowing $\mathbb{E}^4$ with a *complex structure*. Thus, we want a suitable orthogonal transformation J on $\mathbb{E}^4$ such that $J^2 = \zeta$. Keeping the addition, we then define

$$(a + \imath b)u = au + b(uJ),\ \text{for } a, b \in \mathbb{R},\ u \in \mathbb{E}^4.$$

Thus $\imath u = uJ$; and over $\mathbb{C}$, $\mathbb{E}^4$ has dimension 2. Our choice for the matrix J is motivated by an orthogonal projection different from that in Figures 2 and 4.

The vectors representing the vertices labelled $0, \ldots, 7$ in Figure 4 are either opposite or perpendicular. Thus, these eight points are the vertices of a cross-polytope $\mathcal{O} = \{3, 3, 4\}$, one of two inscribed in $\mathcal{P}$. (Its vertices are those of $\mathcal{P}$ which have an odd number of entries $+1$.)

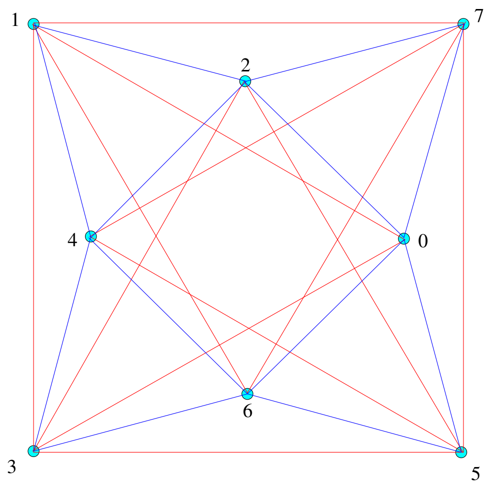

Figure 6: Another projection of the cross-polytope $\mathcal{O}$.

In [7, Figure 4.2A], Coxeter gives another projection of $\mathcal{O}$ which nicely displays certain 2-faces of $\mathcal{O}$. We show this in Figure 6, where each vertex of either of the two concentric squares forms an equilateral triangle with one edge of the other square. These 8 triangles correspond to the unlabelled nodes in Figure 4, and also to the lines of the configuration 8_3. (Each of these real triangles lies on a unique complex line in $\mathbb{C}^2$.) We may take the vertices in Figure 6 to be $(\pm 1, \pm 1)$ and $(\pm r, 0), (0, \pm r)$, where $r = \sqrt{3} - 1$.

But what plane Λ in $\mathbb{E}^4$ actually gives such a projection? Starting with an unknown basis a_1, b_1 for Λ, we can force a lot. For example, the vector $\overline{02}$ from vertex 0 to vertex 2 in Figure 6 is the projection of $(0, 2, -2, 0)$ and is obtained from $\overline{07}$, the projection of $(2, 2, 0, 0)$, by a rotation through $60°$. From such details in the geometry, we soon find that Λ is uniquely determined and get a basis satisfying $a_1 \cdot a_1 = b_1 \cdot b_1$ and $a_1 \cdot b_1 = 0$. But any such basis can still be rescaled or rotated within Λ. Tweaking these finer details, we find it convenient to take a_1, b_1 to be the first two rows of the matrix

$$L = \frac{1}{2\sqrt{3}} \begin{bmatrix} \sqrt{3} & -1 & 1 & -(2+\sqrt{3}) \\ -1 & 2+\sqrt{3} & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & 2+\sqrt{3} & 1 \\ 2+\sqrt{3} & 1 & 1 & \sqrt{3} \end{bmatrix}. \tag{8.1}$$

The last two rows a_2, b_2 of L give a basis for the orthogonal complement $\Lambda^{\perp}$.

Since we want J to induce $90°$ rotations in both Λ and $\Lambda^{\perp}$, we have

$$J = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix} . \tag{8.2}$$

Notice that $a_1 J = b_1$ and $a_2 J = b_2$, so $\{a_1, a_2\}$ is a $\mathbb{C}$-basis for $\mathbb{E}^4$; and the plane Λ in Figure 6 is just $z_2 = 0$ in the resulting complex coordinates. The points in the configuration 8_3 now have these complex coordinates:

Label	(z_1, z_2)
0	$(r, 1-\imath)$
1	$(-1+\imath, r)$
2	$(r\imath, -1-\imath)$
3	$(-1-\imath, -r\imath)$
4	$(-r, -1+\imath)$
5	$(1-\imath, -r)$
6	$(-r\imath, 1+\imath)$
7	$(1+\imath, r\imath)$

(Recall that $r = \sqrt{3} - 1$.) The first coordinates do give the points displayed in Figure 6. The second coordinates describe the projection onto $\Lambda^{\perp}$ ($z_1 = 0$). There labels on the inner and outer squares are suitably swapped.

The line in the configuration 8_3 which contains points $1, 6, 7$ is typical. It has the equation

$$r(1-\imath)z_1 + 2z_2 = 2r(1+\imath) \ .$$

After consulting [7, Sections 10.6 and 11.2], we observe that the eight points are also the vertices of the regular complex polygon $3\{3\}3$. Its symmetry group (of unitary transformations on $\mathbb{C}^2$) is the group $3[3]3$ with the presentation

$$\langle \gamma_1, \gamma_2 \,|\, \gamma_1^3 = 1,\ \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \rangle \ . \tag{8.3}$$

In fact, this group of order 24 is isomorphic to the *binary tetrahedral* group $\langle 3, 3, 2 \rangle$. But in our context, we may identify it with the centralizer in G of the structure matrix J. A bit of computation shows that this subgroup of G is generated by

$$\gamma_1 = \rho_1\rho_2\rho_3\rho_2 = (1, 4, 2) \text{ and } \gamma_2 = \rho_2\rho_0\rho_1\rho_0 = (-1, 1, -1, 1) \cdot (1, 2, 3),$$

which do satisfy the relations in (8.3).

Remark 8.1. The orthogonal projection which underlies Figure 6 has another explanation [7, Section 4.2]. The group G ($\simeq B_4$) can be embedded with index 3 in an isometry group $H = \langle \beta_0, \beta_1, \beta_2, \beta_3 \rangle$ which is isomorphic to the Coxeter group F_4 with diagram

$$\bullet \overset{3}{\text{———}} \bullet \overset{4}{\text{———}} \bullet \overset{3}{\text{———}} \bullet \ . \tag{8.4}$$

To do this we may take β_1 to be reflection in the hyperplane orthogonal to $(1, -1, -1, 1)$, then set $\beta_0 := \rho_0, \beta_2 := \rho_3, \beta_3 := \rho_2$. Indeed, H has order 1152 and $\rho_1 = \beta_2^{\beta_1}$. Notice that the base vertex v is also fixed by $\beta_1, \beta_2, \beta_3$. Applying Wythoff's construction anew, we obtain the 24-cell $\mathcal{Q} = \{3, 4, 3\}$, whose symmetry group is just $H \simeq F_4$. The 24 vertices of this self-dual polytope include the original 16 vertices of the 4-cube $\mathcal{P}$, along with another 8, namely all permutations of $(\pm 2, 0, 0, 0)$. We see that $\mathcal{Q}$ has 3 inscribed cross-polytopes, one of which is the polytope $\mathcal{O}$ whose projection appears in Figure 6. (For more on such regular compounds, see [6].)

As we did for the cube, we can use the procedure in [10, Section 3.17] to find invariant planes for Coxeter elements in H. After considerable tinkering, we find that $\Lambda, \Lambda^{\perp}$ are the invariant planes for the Coxeter element

$$\tilde{\pi} = (\beta_0\beta_1\beta_2\beta_3)^{\beta_2\beta_1\beta_0\beta_1\beta_2\beta_3}$$

in H. We thereby obtain an instance of the most symmetrical projection of the 24-cell $\mathcal{Q}$ onto a plane. In Figure 7, we have altered Figure 6 by showing as well the images of the remaining 16 vertices of $\mathcal{Q}$. (The edges of $\mathcal{O}$ are not edges of $\mathcal{Q}$, so have been removed.) The vertices of $\mathcal{Q}$ appear in two concentric rings of 12. Indeed, Coxeter elements in H, such as $\tilde{\pi}$, have period 12. (The peculiar choice of $\tilde{\pi}$ is an artefact of our pleasant way of first labelling the vertices of the Möbius-Kantor configuration.) □

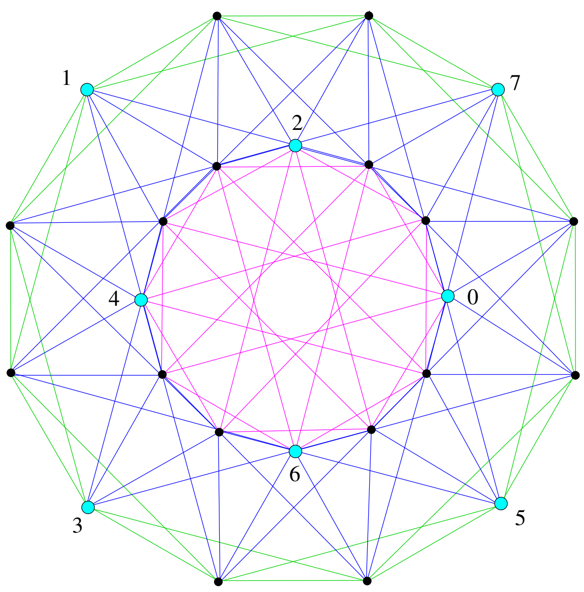

Figure 7: The dodecagonal projection of $\mathcal{Q}$ ={3,4,3}, with vertices of $\mathcal{O}$ labelled.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.11
https://doi.org/10.26493/2590-9770.1406.ec5
(Also available at http://adam-journal.eu)

The interlacing number for alternating semiregular polytopes

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Received 7 March 2021, accepted 20 September 2021, published online 5 August 2022

Abstract

In the classical setting, a convex polytope is said to be semiregular if its facets are regular and its symmetry group is transitive on vertices. This paper continues our study of 'alternating' semiregular abstract polytopes. These structures have abstract regular facets, still with combinatorial automorphism group transitive on vertices and with two kinds of regular facets occurring in an alternating fashion. Here we investigate a parameter called the interlacing number k for a compatible pair of regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$: in what circumstances can we have k copies each of $\mathcal{P}$ and $\mathcal{Q}$ occuring in alternating fashion as the facets of a semiregular $(n+1)$-polytope $\mathcal{S}$?

Keywords: Abstract semiregular polytopes, tail-triangle C-groups.

Math. Subj. Class.: 51M20, 52B15

1 Introduction

The *cuboctahedron* is an example of what we call an *alternating semiregular* 3*-polytope*: its facets are squares and equilateral triangles, two of each occuring in alternate fashion around each vertex. Taking these words as instructions for assembling a convex model, then up to similarity there can be only one end result $\mathcal{S}$ (the cuboctahedron[1]). This sort of alternating behaviour also appears in the familiar tiling $\mathcal{T}$ of Euclidean 3-space by regular

*Supported by the Simons Foundation Award No. 420718

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[1] Polytopes like this are often called *quasi-regular* in the literature. See [1, pages 18 and 69], for example, where the cuboctahedron is denoted by $\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\right\}$. We generalize such notation later. For much more on geometric realizations of quasi-regular polytopes see [10].

octahedra and tetrahedra. Indeed, $\mathcal{T}$ is an infinite alternating semiregular 4-polytope. Our main concern in this paper will be abstract semiregular polytopes like this, with two kinds of regular facets $\mathcal{P}$ and $\mathcal{Q}$ occurring in an 'alternating' fashion.

We began our study of such polytopes in [14]. Here our task is to investigate a parameter called the interlacing number k for a compatible pair of regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$: in what circumstances can we have k copies each of $\mathcal{P}$ and $\mathcal{Q}$ occuring in alternating fashion as the facets of a semiregular $(n+1)$-polytope $\mathcal{S}$?

In fact, when $k = \infty$ such a polytope $\mathcal{S}$ actually does exist. In particular, there is a universal alternating semiregular polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ with infinite interlacing number. We have described its properties with considerable care in [16].

For finite k the situation is less settled. Sometimes a polytope $\mathcal{S}$ with the required structure does exist. In this case, as we show in Theorem 3.3, there exists a universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ of this kind. Its universal covering property is described in Theorem 4.9. (These results help explain, for example, why the cuboctahedron, as constructed above from such simple instructions, *must* be as symmetric as it is!)

But in general there may be obstructions to assembling $\mathcal{P}$ and $\mathcal{Q}$ in the desired way [15]. In Section 5 we survey some interesting examples. Then in Sections 6 and 7 we describe some families of compatible polytopes $\mathcal{P}$ and $\mathcal{Q}$, together with interlacing number k, such that $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ does exist.

It is our pleasure to dedicate this paper to Marston Conder. We thank him, first of all, for his insights into the world of graphs, maps, polytopes and their groups; and second, for his great good humour!

In addition, we here thank the referee for various helpful suggestions.

2 Abstract polytopes

An abstract n-polytope $\mathcal{P}$ is a partially ordered set rather like the face lattice of a convex n-polytope. In general, however, $\mathcal{P}$ need not be a lattice or finite or at all familiar. See [11, Chapter 2] for a detailed look at these objects or [16, Section 2] for a briefer discussion. The elements F of $\mathcal{P}$ are called its *faces*.

First we recall that $\mathcal{P}$ has a (unique) *minimal* face F_{-1} and *maximal* face F_n such that $F_{-1} \leqslant F \leqslant F_n$, for all $F \in \mathcal{P}$. Every maximal chain, or *flag*, in $\mathcal{P}$ has exactly n (*proper*) faces in addition to the *improper* faces F_{-1}, F_n. We let $\mathcal{F}(\mathcal{P})$ be the set of all flags Φ in $\mathcal{P}$.

It follows that $\mathcal{P}$ has a rank function rk, where $\mathrm{rk}(F)+1$ is the number of faces strictly below F in any flag containing F. An element $F \in \mathcal{P}$ with $\mathrm{rk}(F) = j$ is called a *j-face*; and faces of rank $0, 1, n-2$ or $n-1$ are called *vertices*, *edges*, *ridges* or *facets*, respectively. If $F \leqslant G$ are incident faces in $\mathcal{P}$, the *section* G/F is

$$G/F := \{H \in \mathcal{P} \,|\, F \leqslant H \leqslant G\}.$$

We next recall the *diamond property*, that whenever $F < G$ with $\mathrm{rk}(F) = j-1$ and $\mathrm{rk}(G) = j+1$, there must be exactly two j-faces H with $F < H < G$. For $0 \leqslant j \leqslant n-1$ and any flag $\Phi \in \mathcal{F}(\mathcal{P})$, there thus exists a unique *j-adjacent* flag Φ^j, differing from Φ in just the face of rank j . With this notion of adjacency, $\mathcal{F}(\mathcal{P})$ becomes the *flag graph* for $\mathcal{P}$. Finally we assume that $\mathcal{P}$ is *strongly flag–connected*, that is, the flag graph for each section (including $\mathcal{P}$ itself) is connected.

Suppose $F \leqslant G$ are incident faces in $\mathcal{P}$, with $\mathrm{rk}(F) = j \leqslant k = \mathrm{rk}(G)$. It is easy to prove that the section G/F is itself a $(k-j-1)$-polytope. It is often useful to treat a face

F as if it were the full section F/F_{-1} below it. Likewise, if F is a vertex of $\mathcal{P}$, then the section F_n/F is called the *vertex-figure* at F. More generally, if F is a j-face, then F is said to have *co-rank* $n-j-1$, that being the rank of its *co-face* F_n/F.

The *automorphism group* $\Gamma(\mathcal{P})$ of an n-polytope $\mathcal{P}$ consists of all order-preserving bijections on $\mathcal{P}$. We say $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on the flag set $\mathcal{F}(\mathcal{P})$. In this case we may choose any one flag $\Phi \in \mathcal{F}(\mathcal{P})$ as *base flag*, then define ρ_j to be the (unique) automorphism mapping Φ to Φ^j, for each j in the index set $N := \{0, \ldots, n-1\}$. From [11, Section 2B] we recall that $\Gamma(\mathcal{P})$ is then a *string C-group*. This means first that $\Gamma(\mathcal{P})$ is generated by $\{\rho_j : j \in N\}$. Second, these involutory generators satisfy the commutativity relations typical of a Coxeter group with string diagram, namely

$$(\rho_j \rho_k)^{p_{jk}} = 1, \text{ for } 0 \leqslant j \leqslant k \leqslant n-1, \tag{2.1}$$

where $p_{jj} = 1$ and $p_{jk} = 2$ whenever $|j-k| > 1$. Finally, $\Gamma(\mathcal{P})$ is a *C-group*, meaning that it satisfies the *intersection condition*

$$\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle, \text{ for any } I, J \subseteq N. \tag{2.2}$$

The fact that one can reconstruct a regular polytope in a canonical way from any string C-group Γ is at the heart of the theory [11, Section 2E].

The periods $p_j := p_{j-1,j}$ in (2.1) satisfy $2 \leqslant p_j \leqslant \infty$ and are assembled into the *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$ for the regular polytope $\mathcal{P}$. As a familiar example, we recall that every 2-polytope or *polygon* $\{p\}$ is automatically abstractly regular; its automorphism group is the dihedral group $\mathbb{D}_p$ of order $2p$.

There are many ways to relax symmetry and thereby broaden the class of groups $\Gamma(\mathcal{P})$ [9, page 77]. In [14], for instance, we considered the kind of polytopes described in

Definition 2.1. An abstract polytope $\mathcal{S}$ is *semiregular* if it has regular facets and its automorphism group $\Gamma(\mathcal{S})$ is transitive on vertices. The semiregular polytope $\mathcal{S}$ is *alternating* if its facets (all regular) are of two kinds $\mathcal{P}$ and $\mathcal{Q}$, which further occur in alternating fashion around any face in $\mathcal{S}$ of co-rank 2. (We allow $\mathcal{P} \simeq \mathcal{Q}$.)

Clearly every regular polytope is semiregular. The classical Archimedean solids, prisms and antiprisms are convex semiregular polyhedra.

Remark 2.2. Suppose $\mathcal{S}$ is any polytope with only regular facets. Then all ridges in $\mathcal{S}$ (i.e. facets of facets of $\mathcal{S}$) are isomorphic to a common regular polytope $\mathcal{K}$ [13, Lemma 5.16]. We shall say that regular polytopes are *compatible* if they have the same facets $\mathcal{K}$ up to isomorphism.

One can weaken a little the defining property for a semiregular polytope $\mathcal{S}$ to be alternating. Fix one face F of co-rank 2. For some $k \geqslant 1$, F is surrounded by k copies of $\mathcal{P}$ alternating with k copies of $\mathcal{Q}$, where $\mathcal{P}, \mathcal{Q}$ are compatible regular polytopes. The section $\mathcal{S}/F$ is then a $2k$-gon.

Now, in a polytope $\mathcal{S}$ of rank $n+1$, the $(n-2, n-1)$-incidence graph (whose nodes are the faces of $\mathcal{S}$ of ranks $n-2$ or $n-1$) is connected. From this fact it follows easily that copies of the *same* $\mathcal{P}$ and $\mathcal{Q}$ must occur alternately at each other face of co-rank 2 in $\mathcal{S}$. However, k can vary from one such face to another: □

Example 2.3. Construct a 3-polytope $\mathcal{M}$ by gluing two regular octahedra face-to-face, removing the triangular barrier between. $\mathcal{M}$ has vertices of valency 4 or 6. Now construct the 4-polytope $\mathcal{S} = 2^{\mathcal{M}}$, as described in [18, Section 3]. Then $\mathcal{S}$ is vertex-transitive, with each vertex-figure isomorphic to $\mathcal{M}$. Moreover, all facets of $\mathcal{S}$ are isomorphic to $2^{\{3\}}$, that is, to the cube, which of course is regular. Therefore, $\mathcal{S}$ is a semiregular 4-polytope, but with 4 facets surrounding some edges and 6 facets around certain others. Topologically, this $\mathcal{S}$ is a "cubical" 3-manifold with vertex-links determined by $\mathcal{M}$, and it admits an embedding into the 3-skeleton of the 9-dimensional cube.

Evidently, we could here replace $\mathcal{M}$ by any simplicial convex 3-polytope or indeed by any polytopal map with triangular facets. □

As in [14], we confine our attention to those semiregular polytopes for which the parameter k is constant over all faces F of co-rank 2 in $\mathcal{S}$. In such cases we call k the *interlacing number* for $\mathcal{S}$. This certainly happens when $\mathcal{S}$ has rank 3 (by vertex transitivity) and also, for instance, if $\mathcal{S}$ is *hereditary* [12]. In a hereditary polytope, every automorphism of every facet extends to an automorphism of the whole polytope.

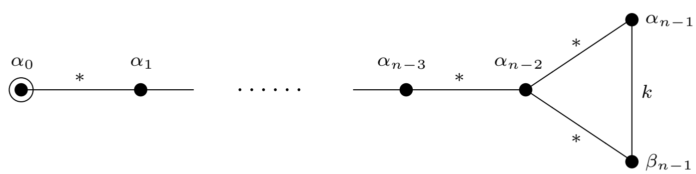
First of all, let us dispose of the somewhat degenerate class of examples which have $k = 1$. Recall that a polytope is said to be (*combinatorially*) *flat* if each of its vertices is incident with each of its facets [11, Section 4E]. For example, any (not necessarily regular) n-polytope $\mathcal{P}$ has a *trivial extension* $\{\mathcal{P}, 2\}$ [19, page 377]. This flat $(n+1)$-polytope has just two facets, each a copy of $\mathcal{P}$.

Lemma 2.4. *Suppose $\mathcal{P}$ is a regular n-polytope. Then $\mathcal{S} = \{\mathcal{P}, 2\}$ is an alternating semiregular $(n+1)$-polytope with interlacing number $k = 1$. In fact, $\mathcal{S}$ is regular.*

Conversely, if each face of co-rank 2 in the polytope $\mathcal{S}$ is incident with just two facets of $\mathcal{S}$, then $\mathcal{S}$ has in total exactly two facets, which are furthermore isomorphic. In other words, $\mathcal{S}$ must be a trivial extension.

Proof. The straightforward argument for the converse has nothing to do with the automorphism group $\Gamma(\mathcal{S})$. □

We are now justified in assuming from here on that $k \geqslant 2$. In fact, we will focus even more narrowly on semiregular polytopes which can be constructed using the combinatorial version of Wythoff's construction described in [14, Section 4] and outlined below. Suppose then that $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ is a group generated by involutions which satisfy the commutativity relations implicit in the *tail-triangle diagram*



(2.3)

The label 'k' indicates that $\alpha_{n-1}\beta_{n-1}$ has period k, for some $k = 2, \ldots, \infty$. However, all other periods of products of two 'adjacent' generators are unspecified for the moment (and indicated by a $*$). The label '2' is possible and indicates the actual absence of the corresponding branch in the diagram. The group Γ is called a *tail-triangle group*. (Anticipating Theorem 2.5, we also say that the group Γ has interlacing number k.) We allow the degenerate (base) case $n = 1$, in which $\Gamma = \langle \alpha_0, \beta_0 \rangle$ is just the dihedral group $\mathbb{D}_k$.

Suppose also that Γ is a C-group, now satisfying the intersection condition (2.2) over a suitable index set N for the $n + 1$ generators of Γ. Then Γ is a *tail-triangle C-group*. It follows that the subgroups

$$\Gamma_n^{\mathcal{P}} := \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle$$

and

$$\Gamma_n^{\mathcal{Q}} := \langle \alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$$

are string C-groups, indeed automorphism groups for regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$, respectively.

As described in Definition 4.2 of [14], the ringed node in (2.3) initiates Wythoff's construction for an $(n+1)$-polytope $\mathcal{S} = \mathcal{S}(\Gamma)$ as a coset geometry over Γ. For later use, we record some details here. First of all, for $0 \leqslant j \leqslant n-2$, we let

$$\Gamma_j^- := \langle \alpha_0, \ldots, \alpha_{j-1} \rangle, \;\; \Gamma_j^+ := \langle \alpha_{j+1}, \ldots, \alpha_{n-1}, \beta_{n-1} \rangle \tag{2.4}$$

and

$$\Gamma_j := \langle \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}, \beta_{n-1} \rangle = \Gamma_j^- \times \Gamma_j^+ . \tag{2.5}$$

The direct product follows easily from (2.2) and the structure of the diagram in (2.3). When $j = n-1$, we also interpret (2.4) and (2.5) as giving $\Gamma_{n-1} = \Gamma_{n-1}^-$, with $\Gamma_{n-1}^+ = \{1\}$.

The j-faces of $\mathcal{S}$ with $j \leqslant n-1$ can be identified with all right cosets of Γ_j in Γ. Likewise, the n-faces of $\mathcal{S}$ are all right cosets of either $\Gamma_n^{\mathcal{P}}$ or $\Gamma_n^{\mathcal{Q}}$. For the improper faces of $\mathcal{S}$ we may take two distinct copies of Γ. Faces of distinct rank are now incident if and only if the corresponding cosets have non-empty intersection. We find that the polytope $\mathcal{S}$ has two flag orbits under the action of Γ, with *base flags*

$$\Phi := [\Gamma_0, \ldots, \Gamma_{n-2}, \Gamma_{n-1}, \Gamma_n^{\mathcal{P}}] \;\text{ and }\; \Psi := [\Gamma_0, \ldots, \Gamma_{n-2}, \Gamma_{n-1}, \Gamma_n^{\mathcal{Q}}]. \tag{2.6}$$

(As usual, we have suppressed the two improper faces.)

We combine the key results of [14, Section 4] about the structure of $\mathcal{S}$ into the following

Theorem 2.5. *Suppose Γ is a tail-triangle C-group corresponding to the diagram* (2.3) *and let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the resulting $(n+1)$-polytope. Then*

(a) *$\mathcal{S}$ is an alternating semiregular polytope. Its facets are isomorphic to $\mathcal{P}$ or $\mathcal{Q}$, with k of each of these occurring alternately around each face $\mathcal{R}$ of co-rank 2. Each 2-section $\mathcal{S}/\mathcal{R}$ is therefore a $2k$-gon; and $\mathcal{S}$ has interlacing number k.*

(b) *The face-wise Γ-stabilizer of any ridge of $\mathcal{S}$ is trivial.*

(c) *Each vertex-figure of $\mathcal{S}$ is isomorphic to the alternating semiregular n-polytope $\widehat{\mathcal{S}}$ defined by deleting the node labelled α_0 in the diagram* (2.3)*, then ringing the node labelled α_1.*

(d) *$\mathcal{S}$ is a regular polytope if and only if Γ admits a group automorphism induced by the diagram symmetry which swaps α_{n-1} and β_{n-1} in* (2.3)*, while fixing the remaining α_j's. In this case $\mathcal{P} \simeq \mathcal{Q}$, say with Schläfli type $\{p_1, \ldots, p_{n-1}\}$, and $\mathcal{S}$ is regular of type $\{p_1, \ldots, p_{n-1}, 2k\}$; moreover, $\Gamma(\mathcal{S}) \simeq \Gamma \rtimes C_2$.*

(e) *If $\mathcal{S}$ is not regular, then $\mathcal{S}$ is a 2-orbit polytope and $\Gamma(\mathcal{S}) \simeq \Gamma$. In particular, this is so if the facets $\mathcal{P}$ and $\mathcal{Q}$ are non-isomorphic.*

Recall that an abstract polytope is called a 2-*orbit polytope* if its automorphism group has precisely two flag orbits. (See [5]; for general structural results about the groups of 2-orbit polytopes, see also [6].)

Typically when working with C-groups it is the verification of the intersection condition (2.2) which causes pain. For relief, we often can make use of an inductive approach, using, for instance, [14, Lemma 4.12]. Here is a refinement of that result.

Lemma 2.6. *Suppose that* $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ *is a tail-triangle group with* $n \geqslant 2$*, and suppose that its subgroups* $\Gamma_n^1 := \Gamma_n^{\mathcal{P}} = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle$*,* $\Gamma_n^2 := \Gamma_n^{\mathcal{Q}} = \langle \alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$ *and* $\Gamma_0 := \langle \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ *are C-groups. Then* Γ *is a tail-triangle C-group if and only if*

(a) $\Gamma_n^1 \cap \Gamma_n^2 = \langle \alpha_0, \ldots, \alpha_{n-2} \rangle$*; and*

(b) $\Gamma_0 \cap \Gamma_n^1 = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$ *and* $\Gamma_0 \cap \Gamma_n^2 = \langle \alpha_1, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$.

Proof. In our original formulation of this in [14, Lemma 4.12], we required a stronger hypothesis in place of (b), namely

(b+) $\Gamma_j^+ \cap \Gamma_n^1 = \langle \alpha_{j+1}, \ldots, \alpha_{n-1} \rangle$ and $\Gamma_j^+ \cap \Gamma_n^2 = \langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$, for $0 \leqslant j \leqslant n-2$.

Recall from (2.4) that $\Gamma_j^+ := \langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$. Notice that $\Gamma_0 = \Gamma_0^+$, so that now just $j = 0$ appears in condition (b). To see that this suffices, suppose in (b+) that $1 \leqslant j \leqslant n-2$. Then $\Gamma_j^+ \subseteq \Gamma_0$, so that

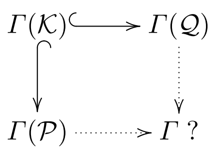
$$\begin{aligned}\Gamma_j^+ \cap \Gamma_n^1 &= (\Gamma_j^+ \cap \Gamma_0) \cap \Gamma_n^1 \\ &= \Gamma_j^+ \cap (\Gamma_0 \cap \Gamma_n^1) \\ &= \Gamma_j^+ \cap \langle \alpha_1, \ldots, \alpha_{n-1} \rangle \quad \text{(by (b))} \\ &= \langle \alpha_{j+1}, \ldots, \alpha_{n-1} \rangle.\end{aligned}$$

The last equality follows from the assumption that Γ_0 is a C-group. The calculation for $\Gamma_j^+ \cap \Gamma_n^2$ is similar. □

We now have a tool for manufacturing lots of examples. For example, we could at least take Γ to be a Coxeter group with a diagram like that displayed in (2.3). (In general, Γ might satisfy independent relations not implied by the structure of the diagram.) Anyway, by [7, Theorem 5.5], any Coxeter group satisfies the intersection condition (2.2), so if Γ is a tail-triangle Coxeter group, then it is a tail-triangle C-group. However, the corresponding regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$ now have a special structure. They are *universal* for their Schläfli type [11, Section 3D]. In finite cases, such polytopes are isomorphic to classical convex (that is, spherical) regular polytopes. What happens, however, if the building blocks $\mathcal{P}$ and $\mathcal{Q}$ are selected from the many other kinds of abstract regular polytopes? We are led to a fundamental

Assembly Problem: suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes with isomorphic facets $\mathcal{K}$; and fix $k \geqslant 2$. Does there exist an alternating semiregular $(n+1)$-polytope $\mathcal{S}$ with facets $\mathcal{P}$ and $\mathcal{Q}$, and interlacing number k (as described in Theorem 2.5)?

We will use $\langle {\mathcal{P} \atop \mathcal{Q}} \rangle^k$ to denote the class of all alternating semiregular $(n+1)$-polytopes with the above data, as constructed from a tail-triangle C-group like that in (2.3). We will recall in Section 5 some situations where this class is empty – somehow there is an obstruction to our alternating assembly of facets. The basic structural problem is summarized in the following diagram, in terms of the relevant automorphism groups:



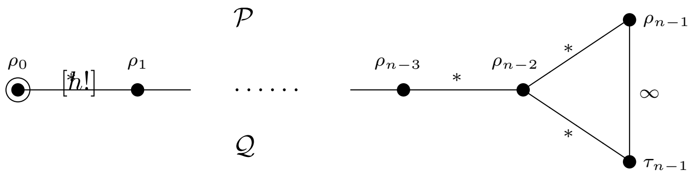
The compatibility of $\mathcal{P}$ and $\mathcal{Q}$ means that natural maps on generators induce the indicated inclusions of $\Gamma(\mathcal{K})$ into $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. We then ask whether there exists a tail-triangle C-group Γ making a commutative diagram in which the dotted maps are also injective. In other words, the class $\langle {\mathcal{P} \atop \mathcal{Q}} \rangle^k$ is non-empty precisely when there is a solution Γ to the diagram. We will prove in such cases (Section 3), that the class then contains a universal polytope, which we might denote $\{ {\mathcal{P} \atop \mathcal{Q}} \}^k$ (in conformity with certain classical notations [1, p. 18]). But for consistency with our earlier papers, we will instead write $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ for this universal polytope.

Remark 2.7. Chapter 4 of [11] concerns the analogous *amalgamation problem* for regular polytopes: given regular n-polytopes $\mathcal{R}, \mathcal{T}$, does there exist a regular $(n+1)$-polytope with facets isomorphic to $\mathcal{R}$ and vertex-figures isomorphic to $\mathcal{T}$? For this to be so, the vertex-figures of $\mathcal{R}$ must be isomorphic to facets of $\mathcal{T}$; however, even given this compatibility condition, the answer can still be 'no'. If the class $\langle \mathcal{R}, \mathcal{T} \rangle$ of such regular $(n+1)$-polytopes is in fact non-empty, then there must be a 'universal polytope', denoted $\{\mathcal{R}, \mathcal{T}\}$, in the class. The trivial extensions mentioned in Lemma 2.4 can be reinterpreted as universal polytopes. □

3 The universal semiregular polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$

In the extreme case $k = \infty$, the interlacing number is never an obstruction to alternating copies of compatible regular polytopes $\mathcal{P}$ and $\mathcal{Q}$. In [16] we cover this situation in depth. On an intuitive level, we explain there how to construct an alternating semiregular $(n+1)$-polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ in a 'free' manner: start with a copy of $\mathcal{P}$; 'attach' copies of $\mathcal{Q}$ to each facet of the $\mathcal{P}$; then attach new copies of $\mathcal{P}$ to each 'exposed' facet of a $\mathcal{Q}$, and so on [16, Theorem 4.10]. To make sense of this we exploit a group Υ which we next describe more carefully.

As always, then, $\mathcal{P}$ and $\mathcal{Q}$ will be regular n-polytopes with isomorphic facets $\mathcal{K}$. Let Υ be the free product of their automorphism groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ amalgamated along $\Gamma(\mathcal{K})$. That is, identify corresponding standard generators for the facet subgroups (copies of $\Gamma(\mathcal{K})$) in each of $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. The result [14, Theorem 5.5] is that Υ is a tail-triangle C-group with a diagram like this:



(3.1)

(For later clarity we use new labels $\rho_0, \ldots, \rho_{n-2}, \rho_{n-1}, \tau_{n-1}$ for the generators.)

From properties of such amalgamated free products, it follows that $\Gamma(\mathcal{P}) \simeq \langle \rho_0, \ldots, \rho_{n-2}, \rho_{n-1} \rangle$, $\Gamma(\mathcal{Q}) \simeq \langle \rho_0, \ldots, \rho_{n-2}, \tau_{n-1} \rangle$ and $\Gamma(\mathcal{K}) \simeq \langle \rho_0, \ldots, \rho_{n-2} \rangle$ are faithfully embedded as subgroups of Υ. Thus we may take $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ simply to be those subgroups of Υ. With this understanding, a presentation for Υ on the given generators is obtained just by taking the union of the various defining relations for $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. This means, for example, that a Coxeter group with a tail-triangle diagram like that in (3.1), and so with $k = \infty$, can be viewed as an amalgamated group Υ of the above kind. (As we remarked earlier, the regular polytopes $\mathcal{P}$ and $\mathcal{Q}$ in this situation are universal for their Schläfli types.)

We recall from Theorem 2.5 that the tail-triangle C-group Υ gives an alternating semiregular $(n+1)$-polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, constructed as a sort of coset geometry. In [16, Theorem 5.7] we show that $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ is 'universal': it covers any $(n+1)$-polytope whose facets alternate between $\mathcal{P}$ and $\mathcal{Q}$. Below we will redescribe $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ as $\mathcal{U}^{\infty}_{\mathcal{P},\mathcal{Q}}$; but before we review how a polytope can have some sort of universal property, we must choose a sensible class of morphisms.

Definition 3.1 ([11, Section 2D])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be polytopes of the same rank. A *covering* is a rank and adjacency preserving homomorphism (*rap-map*) $\eta\colon \mathcal{A} \to \mathcal{B}$. We then say that $\mathcal{A}$ is a *cover* of $\mathcal{B}$ and write $\mathcal{A} \searrow \mathcal{B}$. If also η restricts to an isomorphism on any section of rank at most m in $\mathcal{A}$, then we say that η is an m-covering.

Remark 3.2. It is a useful exercise to check that the flag-connectedness of $\mathcal{B}$ forces the rap-map η to be surjective. It follows that $\mathcal{B}$ is isomorphic, in a natural way, to a quotient of $\mathcal{A}$ [13, Lemmas 2.5 and 2.10]. Let us say that the quotient $\mathcal{B}$ is *induced* by the rap-map η from $\mathcal{A}$ to $\mathcal{B}$. The polytope $\mathcal{A}$ might have other sorts of quotients which we definitely want to avoid here [13, Example 2.13].

Most of our covers will originate from group maps. Suppose, for example, that $\varphi\colon \Theta \to \Gamma$ is an epimorphism of tail-triangle C-groups of the same rank, meaning that φ maps the distinguished generators of Θ to those of Γ, each set indexed by the corresponding diagram. If we refer to Section 2 for the details of the construction of the alternating semiregular polytopes $\mathcal{A}$ and $\mathcal{S}$, from Θ and Γ, respectively, we find that j-faces in the polytopes are right cosets of naturally defined subgroups (see equation (2.5) for Γ); and incidence is given by non-empty intersection. It is quite clear then that application of φ induces a covering map $\eta\colon \mathcal{A} \to \mathcal{S}$. A similar reckoning holds in the regular case for epimorphisms of string C-groups. □

Theorem 3.3. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are compatible regular n-polytopes and $k \geqslant 1$ is an integer for which the class $\langle{\mathcal{P} \atop \mathcal{Q}}\rangle^k \neq \emptyset$. Then there exists a universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}} \in \langle{\mathcal{P} \atop \mathcal{Q}}\rangle^k$ such that for each polytope $\mathcal{S}$ in the class there is a covering*

$$\eta\colon \mathcal{U}^k_{\mathcal{P},\mathcal{Q}} \to \mathcal{S},$$

which acts as an isomorphism on each facet.

Proof. For $k = 1$ we observed in Lemma 2.4 that $\langle{\mathcal{P} \atop \mathcal{Q}}\rangle^1$ is non-empty just when $\mathcal{P} \simeq \mathcal{Q}$ and contains (up to isomorphism) only the trivial extension of $\mathcal{P}$. Thus $\mathcal{U}^1_{\mathcal{P},\mathcal{P}} \simeq \{\mathcal{P}, 2\}$.

Now assume $k \geqslant 2$. The case $n = 0$ is of little interest, since then $\mathcal{P} \simeq \mathcal{Q}$ are single 'vertices' with empty facets. When $n = 1$, $\mathcal{P} \simeq \mathcal{Q}$ are segments, and up to isomorphism $\langle{\mathcal{P} \atop \mathcal{Q}}\rangle^k$ contains just the $(2k)$-gon, so $\mathcal{U}^k_{\mathcal{P},\mathcal{P}} \simeq \{2k\}$.

Suppose that $n \geqslant 1$. Let $\mathcal{S} \in \langle{\mathcal{P} \atop \mathcal{Q}}\rangle^k$ be constructed from a group $\Gamma = \langle\alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1}\rangle$ associated to a diagram like that in (2.3). We explicitly assume that $\mathcal{P}, \mathcal{Q}$ *are* realized as facets of $\mathcal{S}$, so that $\Gamma(\mathcal{P}), \Gamma(\mathcal{Q})$ are embedded as subgroups of Γ. But, as we remarked earlier, such embeddings also hold for $\Upsilon = \Gamma(\mathcal{U}_{\mathcal{P},\mathcal{Q}})$, the free product with amalgamation described above (with diagram (3.1)). Trivially, the subgroups of Υ and Γ corresponding to $\Gamma(\mathcal{P}), \Gamma(\mathcal{Q})$ are isomorphic. By the universal property of a free product with amalgamation, we therefore have an epimorphism

$$\varphi\colon \Upsilon \to \Gamma$$
$$\rho_j, \tau_{n-1} \mapsto \alpha_j, \beta_{n-1},$$

which is bijective on $\Gamma(\mathcal{P}), \Gamma(\mathcal{Q})$.

Let N be the normal closure of $(\rho_{n-1}\tau_{n-1})^k$ in Υ and set $\Upsilon^k := \Upsilon/N$. Thus $N \leqslant \ker\varphi$ and there is an induced map

$$\tilde{\varphi}\colon \Upsilon^k \to \Gamma.$$

We abuse notation by letting ρ_j, τ_{n-1} refer also to their images $N\rho_j, N\tau_{n-1}$ in Υ^k.

We want to show that Υ^k is a tail-triangle C-group. We will use Lemma 2.6 but simplify the notation by letting

$$A := \langle\rho_0, \ldots, \rho_{n-2}, \rho_{n-1}\rangle \ \ (\simeq \Gamma(\mathcal{P})), \ \ B := \langle\rho_0, \ldots, \rho_{n-2}, \tau_{n-1}\rangle \ \ (\simeq \Gamma(\mathcal{Q})), \text{ and}$$

$$C_j := \langle\rho_{n-1-j}, \ldots, \rho_{n-2}, \rho_{n-1}, \tau_{n-1}\rangle, \text{ for } j = 0, \ldots, n-1,$$

be certain crucial subgroups of Υ^k. (Here $C_{n-1} = \Upsilon^k$ and $C_0 = \langle\rho_{n-1}, \tau_{n-1}\rangle$.)

Since $\tilde{\varphi}$ acts bijectively on A and B, these two subgroups are embedded faithfully in Υ^k. Certainly the ρ_j, τ_{n-1} survive as involutions in Υ^k, which does have a diagram like that in (2.3); and $C_0 \simeq \mathbb{D}_k$, since $\tilde{\varphi}$ must also be bijective on C_0.

We will show that C_j is a tail-triangle C-group by induction on $j = 0, \ldots, n-1$. This is trivially the case for the dihedral group C_0, so assume C_j is a tail-triangle C-group for some $j \geqslant 0$. We apply Lemma 2.6 to C_{j+1}, so must show that $A_j := \langle\rho_{n-2-j}, \ldots, \rho_{n-2}, \rho_{n-1}\rangle$ and $B_j := \langle\rho_{n-2-j}, \ldots, \rho_{n-2}, \tau_{n-1}\rangle$ are C-groups and that $A_j \cap B_j \subseteq \langle\rho_{n-2-j}, \ldots, \rho_{n-2}\rangle$; $A_j \cap C_j \subseteq \langle\rho_{n-j-1}, \ldots, \rho_{n-2}, \rho_{n-1}\rangle$; and finally $B_j \cap C_j \subseteq \langle\rho_{n-j-1}, \ldots, \rho_{n-2}, \tau_{n-1}\rangle$.

Now since Γ is a tail-triangle C-group, the corresponding inclusions certainly hold there. But since $\tilde{\varphi}$ is a bijection on A and on B, we can pull-back each such inclusion to Υ^k. (See [11, Theorem 2E17] for this 'quotient criterion' argument in regular cases.) It also follows that A_j and B_j are C-groups with respect to the given generators, since $A \simeq \Gamma(\mathcal{P})$ and $B \simeq \Gamma(\mathcal{Q})$ are themselves string C-groups. By our induction hypothesis, C_j is a C-group. Thus C_{j+1} is a tail-triangle C-group by Lemma 2.6.

We have shown that Υ^k is a tail-triangle C-group. Let $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ be the corresponding alternating semiregular $(n+1)$-polytope. From Remark 3.2 above, we have the desired covering $\eta\colon \mathcal{U}^k_{\mathcal{P},\mathcal{Q}} \to \mathcal{S}$. Furthermore, the construction of $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ is quite independent of any special features of $\mathcal{S}$, apart from the interlacing number k. Thus $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ covers all polytopes in the class. This completes the proof. □

Remark 3.4. Theorem 3.3 implies, of course, that if it exists at all, $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ will be unique to isomorphism. The covering map η is also uniquely specified by a choice of type $\mathcal{P}$ flag in $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ and its image (also of type $\mathcal{P}$) in $\mathcal{S}$. Notice that $\mathcal{U}^\infty_{\mathcal{P},\mathcal{Q}} = \mathcal{U}_{\mathcal{P},\mathcal{Q}}$.

We now see that the Assembly Problem reduces to determining whether or not the universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ even exists. Therefore, let us rethink how we can construct its automorphism group. Again we are guided by the tail-triangle diagram (2.3), whose nodes (for a moment) will index the generators $\alpha_0, \ldots, \alpha_{n-1}, \beta_{n-1}$ of a free group of rank $n+1$. We then impose all relations implied by the diagram, including $(\alpha_{n-1}\beta_{n-1})^k = 1$, along with any relations special to $\Gamma(\mathcal{P}) = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}\rangle$ or to $\Gamma(\mathcal{Q}) = \langle \alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1}\rangle$. (Of course, the ambiguous periods $*$ are now also specified.) The resulting quotient is $\Gamma(\mathcal{U}^k_{\mathcal{P},\mathcal{Q}})$, apart from the extension required for the regular case described in Theorem 2.5(d). Informally speaking, the group of $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ has no defining relations which 'cross' between $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. Note that there could be all sorts of other 'non-universal' tail-triangle C-groups which do support such relations. In Theorem 4.9 below, we prove that $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ must cover each of the corresponding polytopes, indeed other, more general $(n+1)$-polytopes which share the local structure described by $\mathcal{P}$, $\mathcal{Q}$ and k. □

Example 3.5. In Section 5 we recall some situations where even the relations specific to $\Gamma(\mathcal{P})$ and to $\Gamma(\mathcal{Q})$ are somehow made contrary by imposing a particular finite value of k. For example, take $k = 2$, $\mathcal{P}$ the hemicube $\{4,3\}_3$ and $\mathcal{Q}$ the toroidal map $\{4,4\}_{(s,0)}$. (See [15, Section 3] for details.) Then

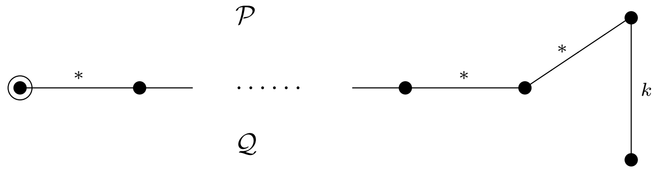
$$\left\langle {\{4,3\}_3 \atop \{4,4\}_{(s,0)}} \right\rangle^2 = \emptyset$$

for odd s. On the other hand, $\mathcal{U}^2_{\{4,3\}_3,\,\{4,4\}_{(s,0)}}$ exists and is finite when s is even. □

Next we observe that the universal polytope must exist when $\mathcal{Q}$, say, is a trivial extension of its facet.

Corollary 3.6. *Suppose $\mathcal{P}$ is a regular n-polytope with facets isomorphic to $\mathcal{K}$; and let $\mathcal{Q}$ be the trivial extension $\{\mathcal{K}, 2\}$. Then for any even $k \geqslant 2$ the universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$, exists.*

Proof. To apply Theorem 3.3 we need only exhibit a polytope $\mathcal{S}$ in the class $\langle {\mathcal{P} \atop \mathcal{Q}} \rangle^k$. But by a dual version of Theorem 5.1 in [17] we do have a *regular* extension $\{\mathcal{P}, k\}$ of $\mathcal{P}$, so long as $k \geqslant 4$ is even. (The case $k = 2$ is covered by Lemma 2.4.) If we merely redraw the corresponding linear diagram as



(3.2)

we see that we have a tail-triangle diagram of the required type. (Compare [16, Example 5.10].) □

What positive results we obtained in [15] suggested the following, seemingly reasonable

Conjecture 1. If k divides m and the universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ exists (equivalently, the class $\langle{\mathcal{P} \atop \mathcal{Q}}\rangle^k$ is non-empty), then also $\mathcal{U}^m_{\mathcal{P},\mathcal{Q}}$ exists.

At first glance it seems that we can use the machinery in the proof of Theorem 3.3 to prove this conjecture. However, we must be cautious and make do with

Proposition 3.7. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are compatible regular n-polytopes; and suppose for $k \geqslant 1$ that $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ exists, corresponding to the tail-triangle C-group* $\Upsilon^k = \langle \rho_0, \ldots, \rho_{n-2}, \rho_{n-1}, \tau_{n-1} \rangle$*, as in the proof of* Theorem 3.3*. Then if $k|m$, the group* $\Upsilon^m = \langle \tilde{\rho}_0, \ldots, \tilde{\rho}_{n-2}, \tilde{\rho}_{n-1}, \tilde{\tau}_{n-1} \rangle$ *is also a tail-triangle C-group which covers Υ^k. However, the supposed period m of $\tilde{\rho}_{n-1}\tilde{\tau}_{n-1}$ conceivably could collapse to some l, with $k|l$ and $l|m$.*

Proof. We use the notation in the proof of Theorem 3.3. Since $k|m$, we clearly have an epimorphism $\psi : \Upsilon^m \to \Upsilon^k$, which is bijective when restricted to the subgroups isomorphic to $\Gamma(\mathcal{P}), \Gamma(\mathcal{Q})$. We may then apply Lemma 2.6, again by induction on the iterated vertex-figure subgroups of Υ^m. This begins with the dihedral group $\langle \tilde{\rho}_{n-1}, \tilde{\tau}_{n-1} \rangle$, which could, in principle, be some $\mathbb{D}_l$, where $k|l|m$. □

Remark 3.8. Of course, we don't actually have an example of the sort of collapse suggested in Proposition 3.7. □

4 The universal covering property of $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$

In order to understand covers of most (that is, less symmetrical) polytopes, we need some new tools. Suppose $\mathcal{A}$ is a polytope of rank $n+1$. For $0 \leqslant j \leqslant n$, let

$$\begin{aligned} r_j \colon \mathcal{F}(\mathcal{A}) &\to \mathcal{F}(\mathcal{A}) \\ \Lambda &\mapsto \Lambda^j \end{aligned}$$

Thus r_j maps a flag Λ of $\mathcal{A}$ to its j-adjacent flag Λ^j in $\mathcal{A}$. Note that each r_j is a fixed-point-free involution on the flag set $\mathcal{F}(\mathcal{A})$.

Definition 4.1. The *connection group* of $\mathcal{A}$ is

$$\mathrm{Mon}(\mathcal{A}) = \langle r_0, \ldots, r_n \rangle.$$

Remark 4.2. The connection group was used in [21] as a tool in the theory of maps. One can interpret relations in the group as describing how an abstract set of flags might be connected together to constitute a polytope. We use the notation $\mathrm{Mon}(\mathcal{A})$ as a reminder that this group is often called the 'monodromy group' in the literature. Etymologically, 'monodromy' does make sense; however, in complex analysis, 'monodromy group' has come to mean something a bit at odds with the intent here. The connection group appears as the 'cartographic group' in [4, page 246]; see also [8]. □

Observe that $\mathrm{Mon}(\mathcal{A})$ is a subgroup of the symmetric group on the flag set. It is always a *string group generated by involutions* (sggi), meaning that relations parallel to those displayed in (2.1) must hold on the specified generators. However, the intersection condition (2.2) can fail if the rank $n+1 \geqslant 4$. See [13, Example 6.8], for instance.

If $g = r_{j_1} \cdots r_{j_l} \in \mathrm{Mon}(\mathcal{A})$ and Λ is any flag of $\mathcal{A}$, we write Λ^g or even $\Lambda^{j_1 \cdots j_l}$ for the image of Λ under g. Notice that $j_1 \dots j_l$ could be any string in the symbols $\{0, \dots, n\}$.

The following simple result is crucial to our calculations.

Lemma 4.3. *If γ is any automorphism of $\mathcal{A}$, Φ is any flag of $\mathcal{A}$ and $g \in \mathrm{Mon}(\mathcal{A})$, then*

$$(\Lambda\gamma)^g = (\Lambda^g)\gamma. \tag{4.1}$$

Proof. Since γ preserves adjacency of flags, the actions of γ and each r_j on $\mathcal{F}(\mathcal{A})$ commute. □

Many covering properties can be rephrased in terms of connection groups. For this, it is, in turn, crucial to understand the flag stabilizer in $\mathrm{Mon}(\mathcal{A})$. (Notice that when $\mathcal{A}$ is a polytope, $\mathrm{Mon}(\mathcal{A})$ acts transitively on the flag set.) For more details we refer to [13], in particular to Propositions 3.11 and 3.13, which we rephrase here as

Proposition 4.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $(n+1)$-polytopes.*

(a) *Any covering $\eta\colon \mathcal{A} \to \mathcal{B}$ induces an epimorphism*

$$\begin{aligned} \overline{\eta}\colon \mathrm{Mon}(\mathcal{A}) &\to \mathrm{Mon}(\mathcal{B}) \\ r_j &\mapsto s_j \end{aligned}$$

(of sggi's). Furthermore, for a given flag Λ of $\mathcal{A}$ with image $\Lambda' = (\Lambda)\eta$ in $\mathcal{B}$, the stabilizer of Λ in $\mathrm{Mon}(\mathcal{A})$ is mapped by $\overline{\eta}$ into the stabilizer of Λ' in $\mathrm{Mon}(\mathcal{B})$.

(b) *Suppose that $\overline{\eta}\colon \mathrm{Mon}(\mathcal{A}) \to \mathrm{Mon}(\mathcal{B})$ is an epimorphism of sggi's and that there are flags Λ of $\mathcal{A}$ and Λ' of $\mathcal{B}$ such that $\overline{\eta}$ maps the stabilizer of Λ in $\mathrm{Mon}(\mathcal{A})$ into the stabilizer of Λ' in $\mathrm{Mon}(\mathcal{B})$. Then there is a covering $\eta\colon \mathcal{A} \to \mathcal{B}$ which maps Λ to Λ'.*

When $\mathcal{S}$ is the alternating semiregular $(n+1)$-polytope constructed from the tail-triangle C-group $\Gamma = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$, with diagram (2.3), we can compare more closely the actions of Γ and $\mathrm{Mon}(\mathcal{S})$ on flags of $\mathcal{S}$. Recall that $\mathcal{S}$ has two flag orbits under the action of Γ, containing in turn the base flags Φ and Ψ described in (2.6).

Lemma 4.5. (a) $\Phi\beta_{n-1} = \Phi^{n,n-1,n}$ *and* $\Psi\alpha_{n-1} = \Psi^{n,n-1,n}$.

(b) $\Phi\alpha_{n-1} = \Phi^{n-1}$ *and* $\Psi\beta_{n-1} = \Psi^{n-1}$.

(c) $\Phi\alpha_j = \Phi^j$ *and* $\Psi\alpha_j = \Psi^j$, *for* $0 \leqslant j \leqslant n-2$.

(d) *The relation* $(r_{n-1}r_n)^{2k} = 1$ *holds in* $\mathrm{Mon}(\mathcal{S})$.

Proof. From the description of the faces of $\mathcal{S}$ in Section 2, as well as the base flags in (2.6), we recall that Φ has n-face $\Gamma_n^{\mathcal{P}}$. Thus Φ^n has n-face $\Gamma_n^{\mathcal{Q}}$; so $\Phi^{n,n-1}$ has $(n-1)$-face $\Gamma_{n-1}\beta_{n-1}$; so finally $\Phi^{n,n-1,n}$ has n-face $\Gamma_n^{\mathcal{P}}\beta_{n-1}$. Since $\beta_{n-1} \in \Gamma_j$, for $0 \leqslant j \leqslant n-2$, we have $\Phi\beta_{n-1} = \Phi^{n,n-1,n}$. The remaining calculations for parts (a), (b), (c) are similar. (For a figure aiding in these calculations see also [14, Equation (9)].) We then have

$$(\Phi)\alpha_{n-1}\beta_{n-1} = (\Phi^{n-1})\beta_{n-1} = (\Phi\beta_{n-1})^{n-1} = \Phi^{n,n-1,n,n-1}.$$

Since $(\alpha_{n-1}\beta_{n-1})^k = 1$, we get $\Phi^{(r_n r_{n-1})^{2k}} = \Phi$. Similarly, $\Psi^{(r_{n-1}r_n)^{2k}} = \Psi$. Applying Lemma 4.3 one last time, we obtain the relation in (d). □

We now focus on the case that $\mathcal{S} = \mathcal{U}_{\mathcal{P},\mathcal{Q}}$ $(= \mathcal{U}^{\infty}_{\mathcal{P},\mathcal{Q}})$ is universal, with ρ_j, τ_{n-1} instead of α_j, β_{n-1}. Let $M := \mathrm{Mon}(\mathcal{U}_{\mathcal{P},\mathcal{Q}}) = \langle r_0, \ldots, r_n\rangle$ and $M^k := \mathrm{Mon}(\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}) = \langle s_0, \ldots, s_n\rangle$. As in the proof of Theorem 3.3, the natural map $\pi\colon \Upsilon \to \Upsilon^k$ induces a covering $\eta\colon \mathcal{U}_{\mathcal{P},\mathcal{Q}} \to \mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$. We recall from Section 2 that a typical j-face of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ is some right coset $\Upsilon_j\mu$, where $\mu \in \Upsilon$. (For facets we have $\Upsilon_n^{\mathcal{P}}\mu$ or $\Upsilon_n^{\mathcal{Q}}\mu$.) The image of such a j-face in $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ is just

$$(\Upsilon_j\mu)\pi = \{N\sigma \,:\, \sigma \in \Upsilon_j\}(N\mu) = \{N\sigma\mu \,:\, \sigma \in \Upsilon_j\},$$

where again $N = \ker\pi$ is the normal closure of $(\rho_{n-1}\tau_{n-1})^k$ in Υ.

The map η in turn induces an epimorphism

$$\begin{aligned}\overline{\eta}\colon M &\to M^k\\ r_j &\mapsto s_j, \quad 0 \leqslant j \leqslant n\end{aligned}$$

(see Proposition 4.4(a) above).

Lemma 4.6. *If two flags of type* $\mathcal{P}$ *(or two of type* $\mathcal{Q}$*) in* $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ *have the same image under* η*, then these flags lie in the same* N*-orbit. No two flags of different types lie in the same* N*-orbit.*

Proof. The base flag Φ of type $\mathcal{P}$ is like that described in equation (2.6), now with group Υ in place of Γ. If $(\Phi\mu)\eta = (\Phi\gamma)\eta$ for $\mu, \gamma \in \Upsilon$, then we have equal cosets $N\Upsilon_j\mu = N\Upsilon_j\gamma$, for $0 \leqslant j \leqslant n-1$. (Also $N\Upsilon_n^{\mathcal{P}}\mu = N\Upsilon_n^{\mathcal{P}}\gamma$.) This means that

$$N\mu\gamma^{-1} \in (\Upsilon_0)\pi \cap \ldots \cap (\Upsilon_{n-1})\pi \cap (\Upsilon_n^{\mathcal{P}})\pi = \{N\},$$

by the interesection condition in Υ^k. Thus $\mu \in N\gamma$. The case of two flags of type $\mathcal{Q}$ is similar.

If two flags of different types had the same image under η, then we would soon get $N\mathcal{U}g_n^{\mathcal{P}} = N\Upsilon_n^{\mathcal{Q}}$. This already contradicts the fact that facets in $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ alternate. But to nail home the argument, note that $\tau_{n-1} \in \Upsilon_n^{\mathcal{Q}}$, so that then $N\Upsilon_n^{\mathcal{P}} = \Upsilon$. This would mean the face $(\Upsilon_{n-1})\eta$ lies on only one facet in $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$, a contradiction. □

We now need two important subgroups of the connection group M. First of all, we let L be the normal closure in M of the element $(r_{n-1}r_n)^{2k}$. Second, we recall from [16, Theorem 5.6], that the stabilizer K in M of the base flag Φ of type $\mathcal{P}$ in $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ is generated by elements of the form w^t, where $t \in M$, $w \in M_n = \langle r_0, \ldots, r_{n-1} \rangle$ and either

(a) w fixes Φ and r_n appears an even number of times in a word for t; or

(b) w fixes Ψ and r_n appears an odd number of times in a word for t.

But remember that for any flag Φ' of type $\mathcal{P}$ in $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ there exists $\gamma \in \Upsilon$ such that $\Phi' = (\Phi)\gamma$. By Lemma 4.3, K therefore fixes *all* flags of type $\mathcal{P}$. Since the base flag Ψ of type $\mathcal{Q}$ is n-adjacent to Φ, all flags of type $\mathcal{Q}$ are stabilized by $K^{r_n} = r_n K r_n$.

Now let us take $\tilde{\Phi} := (\Phi)\eta$ and $\tilde{\Psi} := (\Psi)\eta$ as n-adjacent base flags in $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$.

Lemma 4.7. *If for $g \in M$ we have $\tilde{\Phi}^{(g)\overline{\eta}} = \tilde{\Phi}$, then $g \in KL$. Furthermore,*

$$\ker \overline{\eta} \subseteq KL \cap K^{r_n} L.$$

Proof. Since $\eta \colon \mathcal{U}_{\mathcal{P}\mathcal{Q}} \to \mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ is a rap-map, we have

$$(\Phi^g)\eta = (\Phi\eta)^{(g)\overline{\eta}} = \tilde{\Phi} = (\Phi)\eta.$$

By Lemma 4.6, there exists $\alpha \in N$ such that $\Phi^g = (\Phi)\alpha$. But such an α is a product of various conjugates in Υ of the word $(\rho_{n-1}\tau_{n-1})^k$. We may apply the rewriting rules in Lemma 4.5, with α_j, β_{n-1} replaced by ρ_j, τ_{n-1}. Thus $(\Phi)\alpha = \Phi^h$, where now $h \in M$ is a product of various conjugates of $(r_{n-1}r_nr_{n-1}r_n)^k = (r_{n-1}r_n)^{2k}$. In other words, $h \in L$ and $\Phi^{gh^{-1}} = \Phi$. Thus $gh^{-1} \in K$.

If $g \in \ker \overline{\eta}$, then we also have $\tilde{\Psi}^{(g)\overline{\eta}} = \tilde{\Psi}$, so that we similarly get $g \in K^{r_n} L$. □

Remark 4.8. For any flag Λ of $\mathcal{U}_{\mathcal{P}\mathcal{Q}}$ and $g \in M$ we have $(\Lambda^g)\eta = (\Lambda\eta)^{(g)\overline{\eta}}$. □

Theorem 4.9. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are compatible regular n-polytopes and that k is a fixed positive integer such that $\mathcal{U}^k_{\mathcal{P}\mathcal{Q}}$ exists. Let $\mathcal{B}$ be an $(n+1)$-polytope whose facets are, in alternating fashion, various quotients of $\mathcal{P}$ and $\mathcal{Q}$, all induced by rap-maps. More precisely, suppose that for each face F of co-rank 2 in $\mathcal{B}$, there is a positive integral divisor k_F of k such that F is surrounded by k_F quotients of $\mathcal{P}$ alternating with k_F quotients of $\mathcal{Q}$ (so that $\mathcal{B}/F \simeq \{2k_F\}$). Then there exists a covering*

$$\kappa \colon \mathcal{U}^k_{\mathcal{P},\mathcal{Q}} \to \mathcal{B}.$$

Proof. By [16, Corollary 5.9], there is a covering $\lambda \colon \mathcal{U}_{\mathcal{P},\mathcal{Q}} \to \mathcal{B}$, so there is an epimorphism $\overline{\lambda} : M \to \mathrm{Mon}(\mathcal{B})$, as in

$$\begin{array}{ccc} M & \xrightarrow{\ \overline{\lambda}\ } & \mathrm{Mon}(\mathcal{B}) \\ {\scriptstyle \overline{\eta}}\big\downarrow & \nearrow{\scriptstyle \overline{\kappa}\,?} & \\ M^k & & \end{array}$$

We seek a map $\overline{\kappa}$ (of sggi's) making the diagram commute. But $\ker\overline{\eta} \subseteq KL \cap K^{r_n}L$ from Lemma 4.7, so we want to show that $KL \cap K^{r_n}L \subseteq \ker\overline{\lambda}$. First of all, from the assumption concerning k_F at face F in $\mathcal{B}$, we have $((r_{n-1}r_n)^{2k})\overline{\lambda} = 1$ in $\mathrm{Mon}(\mathcal{B})$. Thus $L \subseteq \ker\overline{\lambda}$.

Next consider the generators w^t of type (a) or (b) for K. Since, as we observed, w^t fixes *all* flags of type $\mathcal{P}$ in $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, and since the corresponding facets of $\mathcal{B}$ are quotients of $\mathcal{P}$ by rap-maps, $(w^t)\overline{\lambda}$ fixes all flags of this kind in $\mathcal{B}$. Likewise, $(K^{r_n})\overline{\lambda}$ fixes the flags in $\mathcal{B}$ whose facets arise as quotients of $\mathcal{Q}$.

We conclude that $\ker\overline{\eta} \subseteq \ker\overline{\lambda}$, so the epimorphism $\overline{\kappa}$ exists and respects specified generators of sggi's. Furthermore, if $\tilde{g} = (g)\overline{\eta}$ fixes the flag $\tilde{\Phi}$ in $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$, then again by Lemma 4.7, $g \in KL$. Once more, on geometrical grounds, $(\tilde{g})\overline{\kappa} = (g)\overline{\lambda}$ fixes the flag $(\Phi)\lambda$ in $\mathcal{B}$. By Proposition 4.4(b), we have a covering $\kappa\colon \mathcal{U}^k_{\mathcal{P},\mathcal{Q}} \to \mathcal{B}$ (sending $\tilde{\Phi}$ to $(\Phi)\lambda$). □

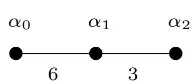
5 Examples with finite interlacing number *k*

To make some sense of the Assembly Problem, it is useful to understand cases in which collapse occurs and others when it does not. In Table 1 we summarize several pertinent examples from [15]. By Theorem 3.3 we are justified in examining only universal polytopes.

$\mathcal{P}$	$\mathcal{Q}$	k	$\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ exists	Comments ($\Gamma = \Gamma(\mathcal{U}^k_{\mathcal{P},\mathcal{Q}})$)
$\{4,3\}_3$ (hemicube)	$\{4,2\}$ or $\{4,4\}_{(2,0)}$	any *odd* $k \geqslant 3$	no	
$\{4,3\}$	$\{4,4\}_{(s,0)}$	2	yes, for any $s \geqslant 2$	finite only for $s = 2$, in which case $\lvert\Gamma\rvert = 384$
$\{4,3\}_3$	$\{4,4\}_{(s,0)}$	2 2	yes, for even $s \geqslant 2$ no, for odd $s \geqslant 1$	finite; $\lvert\Gamma\rvert = 24s^3$
$\{6,3\}_{(1,1)}$	$\{6,3\}$	2	no	$\mathcal{Q}$ forced to collapse to a finite toroid. See Remark 5.1.
$\{6,3\}_{(1,1)}$	$\{6,3\}_{(b,b)}$ $\{6,3\}_{(b,0)}$	2 2	yes, just when $b = 1, 2$ no	
$\{6,3\}_{(1,1)}$	$\{6,3\}_{(2,0)}$	4	no	only known collapse for even $k \geqslant 4$
$\{4,3^{n-3},3\}_n$ (hemi-n-cube)	$\{4,3^{n-3},4\}_{(s,0^{n-2})}$ an n-toroid	2	no, for odd $s \geqslant 3$	collapse in all higher ranks
has FAP w.r.t. facet $\mathcal{K}$	also has FAP w.r.t. facet $\mathcal{K}$	2	yes	[15, Theorem 5.3] explicitly describes an alt. semireg. poly. $\mathcal{S}$, so implying $\mathcal{U}^2_{\mathcal{P},\mathcal{Q}}$ exists. See Remark 5.2.

Table 1: Assembly problems when $k < \infty$.

Remark 5.1. Since the geometry has some strange features, we say more about the case that both $\mathcal{P}$ and $\mathcal{Q}$ have Schläfli type $\{6,3\}$. Here $\{6,3\}$ really indicates the familiar tiling of the Euclidean plane by regular hexagons. Its automorphism group $\Gamma = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is the Coxeter group $[6,3]$ (or $\widetilde{G_2}$) with diagram



(For consistency with notation introduced in [15], we are deviating here from our earlier convention that the letter Γ should denote a tail-triangle group.) The translation subgroup of Γ is generated by $x = (\alpha_0\alpha_1)^2\alpha_2\alpha_1 = (\alpha_0\alpha_1\alpha_2)^2$ and $y = x^{\alpha_1} = (\alpha_1\alpha_0)^2\alpha_1\alpha_2$. These translations, along with mirrors for the generating reflections, are indicated in Figure 5.1. The fundamental region enclosed by these mirrors has been shaded in. It is convenient to use the redundant reflection $\alpha_3 := \alpha_1^{\alpha_0}$. Thus the mirrors for $\alpha_1, \alpha_2, \alpha_3$ enclose an equilateral triangle and furthermore generate the Coxeter group $\widetilde{A}_2$, having index 2 in Γ. Notice that $xy = (\alpha_3\alpha_1\alpha_2)^2$.

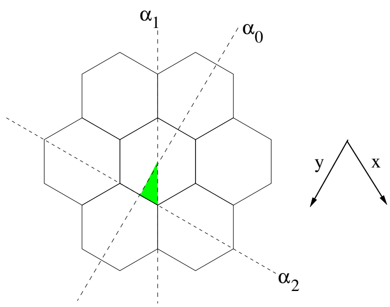
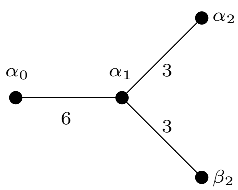


Figure 1: The plane tiling $\{6, 3\}$.

For each $b \geqslant 2$, the group of the finite regular toroid $\{6, 3\}_{(b,0)}$ is obtained by setting $x^b = 1$; this group has order $12b^2$ [2, Section 8.4]. Likewise, for $c \geqslant 1$, the toroid $\{6, 3\}_{(c,c)}$ is obtained by setting $(xy)^c = 1$, or equivalently $(\alpha_1\alpha_2\alpha_3)^{2c} = 1$, giving an automorphism group of order $36c^2$.

Here our concern is really with alternating semiregular polytopes with facets $\mathcal{P} = \{6, 3\}_{(1,1)}$ and $\mathcal{Q} = \{6, 3\}$, so in this instance $\mathcal{Q}$ is the universal cover of $\mathcal{P}$. For interlacing number $k = 2$, we first ask whether the polytope $\mathcal{U}^2_{\mathcal{P},\mathcal{Q}}$ even exists, which means that we must study the group Λ defined by the diagram and subsidiary relation indicated in (5.1).

Λ:  $+ \quad (\alpha_1\alpha_2\alpha_3)^2 = 1$ (5.1)

To force $\mathcal{P}$ to be the toroid $\{6, 3\}_{(1,1)}$ we have imposed the relation $(\alpha_1\alpha_2\alpha_3)^2 = 1$. Remarkably, we find [15, Section 4] that this implies that

$$(\alpha_1\beta_2\alpha_3)^4 = 1 \tag{5.2}$$

must hold in the subgroup $\langle\alpha_0, \alpha_1, \beta_2\rangle$ of Λ. In other words, $\mathcal{Q}$ cannot survive as the infinite, universal polyhedron $\{6, 3\}$. Through its interlacing with $k = 2$ copies of $\mathcal{P} = \{6, 3\}_{(1,1)}$, $\mathcal{Q}$ must collapse *at least* to the toroid $\tilde{\mathcal{Q}} = \{6, 3\}_{(2,2)}$. Any other choice for $\mathcal{Q}$ (still of type $\{6, 3\}$) must result in some sort of loss of structure for $\mathcal{Q}$ or even for $\mathcal{P}$. For example, if $\mathcal{Q} = \{6, 3\}_{(2,0)}$, then with the aid of GAP [3], we find that Λ collapses to a group of order 48. In this case, $\mathcal{Q}$ survives but $\mathcal{P}$ collapses to the non-polytopal map $\{6,3\}_{(1,0)}$.

If $\mathcal{Q} = \mathcal{P} = \{6,3\}_{(1,1)}$ (and still $k = 2$), $\mathcal{U}^2_{\mathcal{P},\mathcal{Q}}$ is the universal regular 4-polytope $\{\{6,3\}_{(1,1)}, \{3,4\}\}$ with automorphism group $S_3 \ltimes [3,4]$, of order 288 (see [11, Theorem 11C13]).

Now increase the interlacing number to $k = 4$, again with the 'worrisome' facets $\mathcal{P} = \{6,3\}_{(1,1)}$ and $\mathcal{Q} = \{6,3\}_{(2,0)}$. We get our first (and so far, only!) instance of collapse for *even* $k > 2$. The group Λ now has order 17280; and $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ survive unscathed. But the intersection condition (2.2) fails. In particular, the vertex-figure subgroup $\langle \alpha_1, \alpha_2, \beta_2 \rangle$ has order 8640, so that the corresponding coset geometry could only have 2 vertices. □

Remark 5.2. This construction yields alternating semiregular polytopes which are unexpectedly 'small', at least compared to the enormous group orders which frequently arise. Informally speaking, a regular polytope is said to have the *flat amalgamation property* (FAP), with respect to its facets, if each facet is the fundamental region, in the entire polytope, for the subgroup generated by the reflections in its own facets (see [11, Section 4E]). We can formalize the FAP in terms of groups. Suppose $n \geqslant 2$ and let $\mathcal{R}$ be a regular n-polytope with facet $\mathcal{L}$ and automorphism group $\Gamma(\mathcal{R}) = \langle \gamma_0, \ldots, \gamma_{n-1} \rangle$. Define

$$N_{\mathcal{R}} := \langle \varphi^{-1}\gamma_{n-1}\varphi \colon \varphi \in \Gamma(\mathcal{R}) \rangle = \langle \varphi^{-1}\gamma_{n-1}\varphi \colon \varphi \in \langle \gamma_0, \ldots, \gamma_{n-2} \rangle \rangle,$$

which is, in fact, the normal closure of γ_{n-1} in $\Gamma(\mathcal{R})$. Then

$$\Gamma(\mathcal{R}) = N_{\mathcal{R}} \cdot \langle \gamma_0, \ldots, \gamma_{n-2} \rangle = \langle \gamma_0, \ldots, \gamma_{n-2} \rangle \cdot N_{\mathcal{R}},$$

a product of subgroups. The polytope $\mathcal{R}$ is said to have the *flat amalgamation property* (*with respect to its facet* $\mathcal{L}$), if this product of subgroups is semi-direct, that is, if

$$\Gamma(\mathcal{R}) \simeq N_{\mathcal{R}} \rtimes \langle \gamma_0, \ldots, \gamma_{n-2} \rangle \simeq N_{\mathcal{R}} \rtimes \Gamma(\mathcal{L}).$$

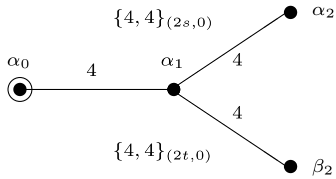
While clearly not every polytope has the FAP, the property nevertheless is not uncommon; for examples of interesting classes of polytopes with the FAP, see [20], where the FAP was referred to as the DAP (for "degenerate amalgamation property"). □

In [15, Section 5] we describe a construction of small alternating semiregular polytopes, with $k = 2$, which employs the FAP for $\mathcal{P}$ and $\mathcal{Q}$, along with a kind of mixing of their groups. We leave the details to [15, Theorem 5.3] and extract what we want here as

Proposition 5.3. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are compatible regular n-polytopes with facets isomorphic to $\mathcal{K}$, and suppose that $\mathcal{P}$ and $\mathcal{Q}$ have the FAP with respect to their facet $\mathcal{K}$. Then there exists an alternating semiregular $(n+1)$-polytope $\mathcal{S}$ with two copies of $\mathcal{P}$ and $\mathcal{Q}$ occurring alternately around each face of co-rank* 2. *(Thus the interlacing number k is equal to* 2.*) The automorphism group of $\mathcal{S}$ is isomorphic to $(N_{\mathcal{P}} \times N_{\mathcal{Q}}) \rtimes \Gamma(\mathcal{K})$ if $\mathcal{P} \not\simeq \mathcal{Q}$, or $((N_{\mathcal{P}} \times N_{\mathcal{Q}}) \rtimes \Gamma(\mathcal{K})) \rtimes C_2$ if $\mathcal{P} \simeq \mathcal{Q}$. In particular, $\mathcal{S}$ is finite if and only if $\mathcal{P}$ and $\mathcal{Q}$ are finite.*

Note that this Proposition does not assert that the constructed polytope $\mathcal{S}$ is universal, although it does follow (Theorem 3.3 again) that $\mathcal{U}^2_{\mathcal{P},\mathcal{Q}}$ exists.

Example 5.4. Among the regular toroidal maps precisely $\{4,4\}_{(2t,0)}$, $\{4,4\}_{(2t,2t)}$, $\{3,6\}_{(t,t)}$, and $\{3,6\}_{(3t,0)}$, with $t \geqslant 1$, have the FAP with respect to their facets (see [20, page 319-320]). For instance, take $\mathcal{P} = \{4,4\}_{(2s,0)}$ and $\mathcal{Q} = \{4,4\}_{(2t,0)}$, with $s, t \geqslant 1$, and consider the alternating semiregular 4-polytope $\mathcal{S}$ obtained from Theorem 5.3 via the diagram



(5.3)

The 4-polytope $\mathcal{S}$ then has s^2t^2 vertices and $4(s^2+t^2)$ facets, namely $4t^2$ facets isomorphic to $\mathcal{P}$ and $4s^2$ facets isomorphic to $\mathcal{Q}$. The polytope is regular if and only if $s = t$. □

6 Prescribing the interlacing number *k*

Recall Conjecture 1, which asserts that if $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ exists and k divides m, then also $\mathcal{U}^m_{\mathcal{P},\mathcal{Q}}$ should exist. Although certain examples in Section 5 urge caution here, it is still true that in many cases a direct construction of an alternating semiregular polytope with interlacing number m from a given alternating semiregular polytope with interlacing number k is available. The new tail-triangle C-group is obtained as a 'mix' of the old tail-triangle C-group and the dihedral group $\mathbb{D}_m$.

Call a regular polytope $\mathcal{R}$ *vertex bipartite* if the edge graph (1-skeleton) of $\mathcal{R}$ is a bipartite graph. Then $\mathcal{R}$ is vertex bipartite if and only if each circuit of the edge graph of $\mathcal{R}$ has even length. The faces of a vertex bipartite regular polytope are also vertex bipartite. Call a regular polytope $\mathcal{R}$ *dually vertex bipartite* if and only if its dual $\mathcal{R}^*$ is vertex bipartite. For example, the toroidal maps $\{4,4\}_{(2t,0)}$ and $\{4,4\}_{(t,t)}$, with $t \geqslant 1$, are all vertex bipartite and dually vertex bipartite.

Now assume, as before, that $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes, with a common facet $\mathcal{K}$, such that an alternating semiregular $(n+1)$-polytope $\mathcal{S}$ with facets $\mathcal{P}$ and $\mathcal{Q}$ and with interlacing number k does exist. Let $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ be the corresponding tail-triangle C-group, so that $\langle \alpha_{n-1}, \beta_{n-1} \rangle = \mathbb{D}_k$. Now suppose k divides m, and let $\mathbb{D}_m = \langle \delta_0, \delta_1 \rangle$, where δ_0, δ_1 are standard involutory generators for $\mathbb{D}_m$. Consider the subgroup Δ of the direct product $\Gamma \times \mathbb{D}_m$ generated by

$$\alpha'_i := \begin{cases} (\alpha_i, 1) & \text{if } \ 0 \leqslant i \leqslant n-2, \\ (\alpha_i, \delta_0) & \text{if } \ i = n-1 \end{cases} \tag{6.1}$$

and

$$\beta'_{n-1} := (\beta_{n-1}, \delta_1). \tag{6.2}$$

Then Δ is a tail-triangle group with $\langle \alpha'_0, \ldots, \alpha'_{n-2} \rangle \simeq \langle \alpha_0, \ldots, \alpha_{n-2} \rangle$ and $\langle \alpha_{n-1}, \beta_{n-1} \rangle = \mathbb{D}_m$. However, in general the subgroups

$$\Delta^{\mathcal{P}}_n := \langle \alpha'_0, \ldots, \alpha'_{n-2}, \alpha'_{n-1} \rangle, \ \Delta^{\mathcal{Q}}_n := \langle \alpha'_0, \ldots, \alpha'_{n-2}, \beta'_{n-1} \rangle$$

will not be isomorphic to $\Gamma(\mathcal{P})$ or $\Gamma(\mathcal{Q})$, respectively. In fact, these subgroups are just the automorphism groups of the polytopes $\mathcal{P}'$ and $\mathcal{Q}'$, respectively, which are the duals, of the mix of the duals $\mathcal{P}^*$ or $\mathcal{Q}^*$ with a 1-polytope (see [11, Theorem 7A7]). It is known from [11, Theorem 7A8] that $\mathcal{P}' \simeq \mathcal{P}$ and $\mathcal{Q}' \simeq \mathcal{Q}$ if all circuits of the edge graph of $\mathcal{P}^*$ or $\mathcal{Q}^*$ are even, that is, if $\mathcal{P}$ and $\mathcal{Q}$ are dually vertex bipartite. (Note that this implies that $\alpha_{n-2}\alpha_{n-1}$ and $\alpha_{n-2}\beta_{n-1}$ have even periods.) We see that dual vertex bipartiteness of $\mathcal{P}$ and $\mathcal{Q}$ is a necessary condition for Δ to be a tail-triangle group of the desired kind.

Theorem 6.1. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are dually vertex bipartite, compatible regular n-polytopes, and k divides m. Suppose there exists an alternating semiregular $(n+1)$-polytope with facets isomorphic to $\mathcal{P}$ or $\mathcal{Q}$, and with interlacing number k, which is derived from a tail-triangle C-group Γ. Then the subgroup Δ of $\Gamma \times \mathbb{D}_m$, with generators as in (6.1) and (6.2), is a tail-triangle C-group. The corresponding alternating semiregular $(n+1)$-polytope still has facets isomorphic to $\mathcal{P}$ or $\mathcal{Q}$, and interlacing number m.*

Proof. It remains to establish that Δ is a C-group. Our proof again appeals to Lemma 2.6 and is similar to the proof of Theorem 3.3. Let π denote the restriction to Δ of the projection of $\Gamma \times \mathbb{D}_m$ onto the first factor Γ. As $\mathcal{P}$ and $\mathcal{Q}$ are dually vertex bipartite, π defines isomorphisms $\pi_{\mathcal{P}}\colon \Delta_n^{\mathcal{P}} \mapsto \Gamma_n^{\mathcal{P}}$ ($\simeq \Gamma(\mathcal{P})$) and $\pi_{\mathcal{Q}}\colon \Delta_n^{\mathcal{Q}} \mapsto \Gamma_n^{\mathcal{Q}}$ ($\simeq \Gamma(\mathcal{Q})$). In particular, $\Delta_n^{\mathcal{P}}$ and $\Delta_n^{\mathcal{Q}}$ are C-groups.

For condition (a) of Lemma 2.6 we must show that $\Delta_n^{\mathcal{P}} \cap \Delta_n^{\mathcal{Q}} = \langle \alpha_0', \ldots, \alpha_{n-2}' \rangle$. However, this follows easily from the fact that $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{Q}}$ are isomorphisms, and that Γ is a C-group, with $\Gamma_n^{\mathcal{P}} \cap \Gamma_n^{\mathcal{Q}} = \langle \alpha_0, \ldots, \alpha_{n-2} \rangle$.

For condition (b) of Lemma 2.6 we argue inductively, using the fact that the vertex-figures of dually vertex bipartite regular polytopes are also dually vertex bipartite.

The case $n = 2$ can be handled directly. The only intersections not yet considered are $\Delta_2^{\mathcal{P}} \cap \langle \alpha_1', \beta_1' \rangle$ and $\Delta_2^{\mathcal{Q}} \cap \langle \alpha_1', \beta_1' \rangle$. As the proofs are similar, we only discuss the first. Consider the two componentwise projections from the direct product $\Gamma \times \mathbb{D}_m$ and again use the fact that Γ is a C-group. This shows that $\Delta_2^{\mathcal{P}} \cap \langle \alpha_1', \beta_1' \rangle$ must necessarily lie in $\langle \alpha_1 \rangle \times \langle \delta_0 \rangle \simeq C_2 \times C_2$. However, recalling our earlier remarks on dual vertex bipartiteness, $\alpha_0\alpha_1$ has even order when $n = 2$. Thus, the element $(1, \delta_0)$ definitely does not lie in $\Delta_2^{\mathcal{P}} = \langle \alpha_0', \alpha_1' \rangle = \langle (\alpha_0, 1), (\alpha_1, \delta_0) \rangle$, so the intersection must just be $\langle \alpha_1' \rangle$, as required. We conclude that Δ is a C-group when $n = 2$.

Now let $n > 2$ and suppose inductively that the subgroup Δ_0 associated with the vertex-figures $\mathcal{P}_0$ of $\mathcal{P}$ and $\mathcal{Q}_0$ of $\mathcal{Q}$ is a C-group. By Lemma 2.6 we must show that

$$\Delta_0 \cap \Delta_n^{\mathcal{P}} = \langle \alpha_1', \ldots, \alpha_{n-1}' \rangle, \quad \Delta_0 \cap \Delta_n^{\mathcal{Q}} = \langle \alpha_1', \ldots, \alpha_{n-2}', \beta_{n-1}' \rangle.$$

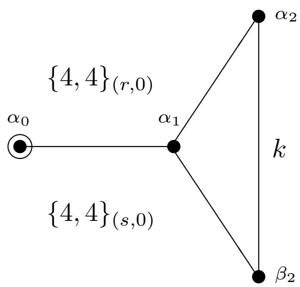
The proofs are similar, so we only verify the first of these conditions. Recall that the projection mapping $\pi_{\mathcal{P}}$ from before is an isomorphism, and that $\Gamma_0 \cap \Gamma_n^{\mathcal{P}} = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$ since Γ is a C-group. Thus the desired property lifts from Γ to Δ itself, and the proof is complete. □

7 Alternating semiregular polytopes with toroidal facets

When k is finite, it generally seems difficult to even decide the existence of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}^k$ for given compatible regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$, let alone pin down the corresponding group explicitly (the group Υ^k in the notation of Section 3). However, in some cases this can be done.

Our approach in this final section is similar to that in [15, Section 3]. We apply a 'twisting operation' of the sort discussed in detail in [11, Chapter 8] to a carefully chosen Coxeter group. We will then show that certain toroidal maps of type $\{4, 4\}$ can be assembled with arbitrary interlacing number $k \geqslant 2$; and we can describe the group for the universal polytope in a very satisfactory way.

To be precise let us take $\mathcal{P} = \{4,4\}_{(r,0)}$ and $\mathcal{Q} = \{4,4\}_{(s,0)}$ with any $r, s \geqslant 2$ and any $k \geqslant 2$ (allowing $k = \infty$). Thus the tail-triangle diagram of Υ^k has the form



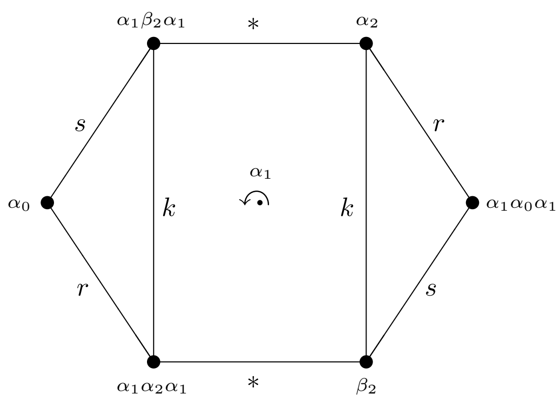
(7.1)

(In Section 3 we had ρ_j, τ_2 in place of α_j, β_2.) As before, when $k = 2$, we regard the vertical branch as missing. The key idea is to investigate how the generator α_1 acts by conjugation on α_0, α_2, β_2 and their conjugates under α_1. In a sense, this is a "reverse twisting" method.

Suppose for a moment that $\mathcal{S}$ is any alternating semiregular 4-polytope with facets $\mathcal{P} = \{4,4\}_{(r,0)}$ and $\mathcal{Q} = \{4,4\}_{(s,0)}$ and with interlacing number k, associated with a tail-triangle C-group $\Gamma = \langle \alpha_0, \alpha_1, \alpha_2, \beta_2 \rangle$ and a diagram as in (7.1). The subgroup

$$W(\mathcal{S}) := \langle \alpha_0, \alpha_2, \beta_2, \alpha_1\alpha_0\alpha_1, \alpha_1\alpha_2\alpha_1, \alpha_1\beta_2\alpha_1 \rangle$$

is invariant under conjugation by α_1, and either $\Gamma = W(\mathcal{S})$ (if $\alpha_1 \in W(\mathcal{S})$) or $\Gamma = W(\mathcal{S}) \rtimes C_2$ (if $\alpha_1 \notin W(\mathcal{S})$). Either way, α_1 permutes the generators of $W(\mathcal{S})$ and acts like the central symmetry on the following diagram for $W(\mathcal{S})$:

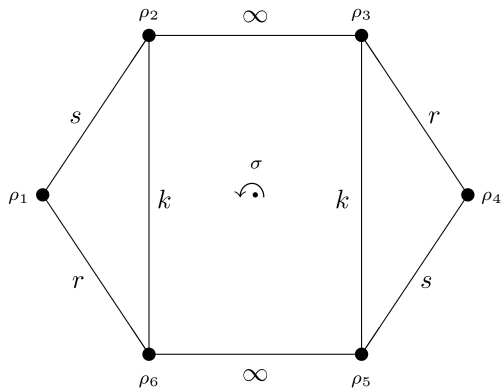


(7.2)

It is straightforward to check that the orders of the pairwise products of generators of $W(\mathcal{S})$, if finite, are just the marks on the branches connecting the corresponding nodes, with order 2 again representing an omitted branch. For example, diametrically opposite nodes represent commuting generators, as both $\mathcal{P}$ and $\mathcal{Q}$ are of type $\{4,4\}$. The vertical branches are marked with the interlacing number k, as $(\alpha_2\beta_2)^k = 1$ in Γ. The marks r and s on the left and right are just the length of the 2-holes of $\mathcal{P}$ and $\mathcal{Q}$, respectively, in accordance with

$(\alpha_0\alpha_1\alpha_2\alpha_1)^r = 1$ and $(\alpha_0\alpha_1\beta_2\alpha_1)^s = 1$ in Γ [11, p. 195]. Finally, the two horizontal branches in (7.2) are marked $*$, since we do not immediately know the actual order of the product $\alpha_1\alpha_2\alpha_1\beta_2$. There are examples of groups Γ where this product has finite order, and others where it has infinite order. Note that $\alpha_1\alpha_2\alpha_1\beta_2$ lies in the subgroup $\langle\alpha_1, \alpha_2, \beta_2\rangle$ of Γ supported on the triangular subdiagram of (7.1).

Now let $W = W(r, s, k)$ denote the Coxeter group with diagram



(7.3)

(Here, of course, ρ_j does not denote a generator of the group Υ^k from Section 3.) Our previous considerations just say that $W(\mathcal{S})$ is a quotient of W. Thus, if $\mathcal{S}$ is any alternating semiregular 4-polytope with facets $\mathcal{P}$ and $\mathcal{Q}$, interlacing number k, and tail-triangle C-group Γ, then either Γ itself is a quotient of W, or Γ is the semi-direct product of a quotient of W by C_2.

We now stand the argument on its head and construct a polytope of the desired kind from a tail-triangle group derived from W itself. This will necessarily be the universal polytope of its kind.

Suppose the parameters r, s and k specifying the facets and interlacing number are given, and consider the Coxeter group $W = W(r, s, k)$ defined by the diagram in (7.3). Then the central diagram symmetry σ shown in (7.3) induces a group automorphism of W permuting the generators $\rho_1, \ldots, \rho_6$, and we can form the semi-direct product of W by C_2. Abusing notation, we set $\Gamma := W \rtimes \langle\sigma\rangle$ and choose the following generators for Γ:

$$\alpha_0 := \rho_1,\ \alpha_1 := \sigma,\ \alpha_2 := \rho_3,\ \beta_2 := \rho_5. \tag{7.4}$$

Then it is straightforward to check that Γ is a tail-triangle group whose generators satisfy the relations implicit in diagram (7.1). For example, $(\alpha_0\alpha_1)^4 = 1$, since

$$(\alpha_0\alpha_1)^2 = \alpha_0(\alpha_1\alpha_0\alpha_1) = \rho_1\rho_4.$$

Similarly, $(\alpha_1\alpha_2)^4 = 1$ and $(\alpha_1\beta_2)^4 = 1$. Further, $(\alpha_2\beta_2)^k = (\rho_3\rho_5)^k = 1$. The facet subgroups of Γ are given by

$$\langle\alpha_0, \alpha_1, \alpha_2\rangle = \langle\rho_1, \rho_3, \rho_4, \rho_6, \sigma\rangle \simeq (\mathbb{D}_r \times \mathbb{D}_r) \rtimes C_2$$

and

$$\langle\alpha_0, \alpha_1, \beta_2\rangle = \langle\rho_1, \rho_2, \rho_4, \rho_5, \sigma\rangle \simeq (\mathbb{D}_s \times \mathbb{D}_s) \rtimes C_2,$$

and have the required form. The vertex-figure subgroup of Γ is

$$\langle\alpha_1, \alpha_2, \beta_2\rangle = \langle\rho_2, \rho_3, \rho_5, \rho_6\rangle \rtimes C_2,$$

where $\langle \rho_2, \rho_3, \rho_5, \rho_6 \rangle$ is a Coxeter group with a rectangular diagram and with branch marks k and ∞ alternating. (Such a subgroup of W is a Coxeter group [7, Theorem 5.5].)

Finally, Γ inherits the intersection property with respect to its generators $\alpha_0, \alpha_1, \alpha_2, \beta_2$ from the intersection property of W with respect to its generators $\rho_1, \ldots, \rho_6$. (Recall that any Coxeter group has the intersection property with respect to its standard generators [7, Theorem 5.5].) For example, $\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cap \langle \alpha_1, \alpha_2, \beta_2 \rangle$ coincides with

$$\langle \rho_1, \rho_3, \rho_4, \rho_6 \rangle \cdot \langle \sigma \rangle \;\cap\; \langle \rho_2, \rho_3, \rho_5, \rho_6 \rangle \cdot \langle \sigma \rangle = \langle \rho_3, \rho_6 \rangle \cdot \langle \sigma \rangle = \langle \alpha_1, \alpha_2 \rangle,$$

as required.

Thus $\Gamma = W \rtimes C_2$ is a tail-triangle group associated with an alternating semiregular 4-polytope with facets $\mathcal{P} := \{4,4\}_{(r,0)}$ and $\mathcal{Q} := \{4,4\}_{(s,0)}$ and interlacing number k. But now it is clear that this must be the universal polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$. In fact, by our earlier considerations, any tail-triangle group associated with an alternating semiregular 4-polytope with these facets and interlacing number must necessarily sit below W or $W \rtimes C_2$, while on the other hand the present tail-triangle group is just $W \ltimes C_2$ itself.

In summary, we have proved the following theorem.

Theorem 7.1. *Let $r, s, k \geqslant 2$, $\mathcal{P} := \{4,4\}_{(r,0)}$ and $\mathcal{Q} := \{4,4\}_{(s,0)}$, and let $W(r,s,k)$ denote the Coxeter group defined by diagram (7.3). Then the universal alternating semiregular 4-polytope $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ exists, and its tail-triangle C-group Υ^k is given by*

$$\Upsilon^k \cong W(r,s,k) \rtimes C_2.$$

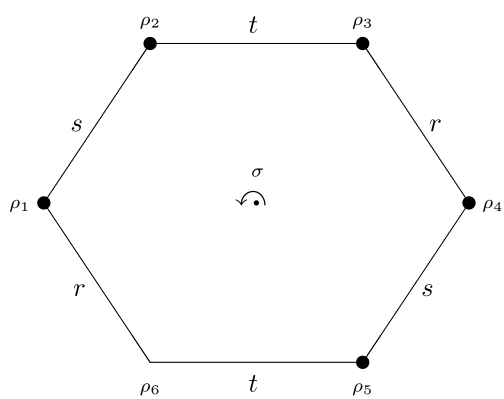
In particular, $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ is infinite, as is its vertex-figure.

If the mark ∞ in diagram (7.3) is replaced by a finite integer $t \geqslant 2$, then the resulting new Coxeter group can similarly be employed to produce an alternating semiregular 4-polytope with facets $\{4,4\}_{(r,0)}$ and $\{4,4\}_{(s,0)}$ and interlacing number k. This polytope is a proper quotient of $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ and is determined by the extra relation

$$(\alpha_1 \alpha_2 \alpha_1 \beta_2)^t = 1$$

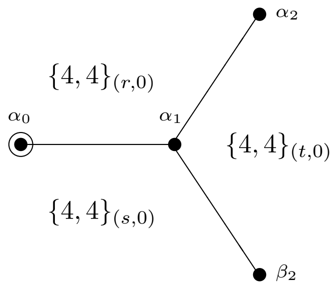
imposed on the generators of Γ (and affecting the vertex-figure group). Thus $\mathcal{U}^k_{\mathcal{P},\mathcal{Q}}$ has infinitely many quotients, each an alternating semiregular 4-polytope with the same facets and interlacing number.

We now look at the case $k = 2$ in more detail. Let $r, s, t \geqslant 2$, allowing $t = \infty$. Consider the Coxeter group $W_t(r,s)$ with Coxeter diagram



(7.5)

Then the same choice of generators as in (7.4) gives a tail-triangle C-group Γ of the form



(7.6)

Now the vertex-figure subgroup of Γ is isomorphic to the automorphism group of the toroidal map $\{4,4\}_{(t,0)}$ (if t is finite) or the square tessellation $\{4,4\}$ (if $t=\infty$). By Theorem 2.5(c), the vertex-figure of the corresponding alternating semiregular 4-polytope itself is isomorphic to the medial polyhedron of $\{4,4\}_{(t,0)}$ or $\{4,4\}$, which is $\{4,4\}_{(t,t)}$ or $\{4,4\}$, respectively. If $t=\infty$, then $W_t(r,s)=W(r,s,2)$ and so we recover the universal polytope $\mathcal{U}^2_{\mathcal{P},\mathcal{Q}}$. Thus we have the following theorem.

Theorem 7.2. *Let $r,s,t \geqslant 2$, $\mathcal{P} := \{4,4\}_{(r,0)}$ and $\mathcal{Q} := \{4,4\}_{(s,0)}$, and let $W_t(r,s)$ denote the Coxeter group defined by diagram (7.5). Then $\mathcal{U}^2_{\mathcal{P},\mathcal{Q}}$ has infinitely many quotients which are alternating semiregular 4-polytopes with the same facets $\mathcal{P}$, $\mathcal{Q}$ and the same interlacing number 2. In particular, there exists such a polytope with tail-triangle C-group isomorphic to $W_t(r,s) \ltimes C_2$ and with vertex-figures $\{4,4\}_{(t,t)}$ (if t is finite); this polytope is finite if and only if $W_t(r,s)$ is finite (that is, $(r,s,t)=(2,2,t)$ with $2 \leqslant t < \infty$ or $(r,s,t)=(2,3,t)$ with $t=3,4,5$, up to permutation of r,s,t).*

Note that the diagram for Γ in (7.6), with the ring around the leftmost node removed, can be interpreted in three different ways as a diagram for a tail-triangle group with 3-generator subgroups corresponding to toroidal maps. Each is determined by ringing one of the outer nodes of the diagram and yields a polytope as in Theorem 7.2.

Here we end, for now, our survey of the interlacing number for compatible regular n-polytopes $\mathcal{P}$, $\mathcal{Q}$. For some fairly large families of such polytopes, assembly into an alternating semiregular $(n+1)$-polytope $\mathcal{S}$ is possible. In other, rather elusive cases, assembly fails for certain interlacing numbers. Our conjectures from [15], including Conjecture 1 in Section 3, remain beyond reach. Intuitively, they should ‘generally’ hold. But it would also be very nice to kill them off with counterexamples!

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.12
https://doi.org/10.26493/2590-9770.1442.cb1
(Also available at http://adam-journal.eu)

On hamiltonian cycles in Cayley graphs of order $pqrs$

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Received 18 August 2021, accepted 29 June 2022, published online 5 August 2022

Abstract

Let G be a finite group. We show that if $|G| = pqrs$, where p, q, r, and s are distinct odd primes, then every connected Cayley graph on G has a hamiltonian cycle.

Keywords: Cayley graph, hamiltonian cycle.

Math. Subj. Class.: 05C25, 05C45

1 Introduction

Definition 1.1 (cf. [4, page 34])**.** If A is a subset of a finite group G, then the corresponding *Cayley graph* $\operatorname{Cay}(G; A)$ is the undirected graph whose vertices are the elements of G, and such that vertices g and h are adjacent if and only if $g^{-1}h \in A \cup A^{-1}$, where $A^{-1} = \{ a^{-1} \mid a \in A \}$.

It is easy to see (and well known) that $\operatorname{Cay}(G; A)$ is connected if and only if A is a generating set of G. Several papers show that all connected Cayley graphs of certain orders are hamiltonian:

Theorem 1.2 (see [6, 7, 9] and references therein)**.** Let A be a generating set of a finite group G. If $|G|$ has any of the following forms (where p, q, and r are distinct primes, and k is a positive integer), then $\operatorname{Cay}(G; A)$ has a hamiltonian cycle:

1. kp, where $k \leq 47$,
2. kpq, where $k \leq 7$,
3. pqr,
4. kp^2, where ≤ 4,
5. kp^3, where $k \leq 2$.
6. p^k.

*I thank an anonymous referee for carefully reading the original version of the paper, and pointing out some serious deficiencies in the exposition.

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The purpose of this note is to add $pqrs$ to the list when it is odd:

Theorem 1.3. If p, q, r, and s are distinct odd primes, then every connected Cayley graph of order $pqrs$ has a hamiltonian cycle.

Remark 1.4.

1. To remove Theorem 1.3's assumption that the primes are odd, it would suffice to show that every connected Cayley graph of order $2pqr$ is hamiltonian (where p, q, and r are distinct odd primes). An important first step in this direction was taken by F. Maghsoudi [7], who handled the case where one of the primes is 3.

2. Theorem 1.3 implies that if p, q, and r are distinct primes, then every connected Cayley graph of order $3pqr$ is hamiltonian. Namely:

   - If $r = 2$, then $3pqr = 6pq$, so the main theorem of [7] applies.
   - If $r = 3$, then $3pqr = 9pq$, so [11, Corollary 1.5] applies.
   - If $\{p, q, r\} \cap \{2, 3\} \neq \emptyset$, then the theorem applies (because 3, p, q, and r are distinct odd primes).

3. Unfortunately, current methods do not seem to be sufficient to remove the assumption that p, q, r, and s are distinct. For example, it is not known that all Cayley graphs of order $9p^2$ or $3p^3$ are hamiltonian (cf. [6]).

2 Preliminaries

We use the following fairly standard notation.

Notation 2.1. Let A be a subset of a finite group G.

1. e is the identity element of G,

2. $|g|$ is the order of an element g of G.

3. $G' = [G, G]$ is the *commutator subgroup* of G.

4. $Z(G) = \{\, z \in G \mid zg = gz \text{ for all } g \in G \,\}$ is the *centre* of G.

5. A sequence $(a_i)_{i=1}^m = (a_1, a_2, \ldots, a_m)$ of elements of $A \cup A^{-1}$ represents the walk in $\operatorname{Cay}(G; A)$ that visits the vertices

$$e, a_1, a_1a_2, \ldots, a_1a_2 \cdots a_m.$$

   Also, we use a^k and a^{-k} to represent the sequences

$$(a, a, \ldots, a) \text{ and } (a^{-1}, a^{-1}, \ldots, a^{-1})$$

   of length k.

Most cases of Theorem 1.3 are easy consequences of three known results that are collected in the following theorem.

Theorem 2.2 ([1, page 257], Durnberger [2], Morris [11])**.** Let A be a generating set of a nontrivial finite group G, and assume that $|G|$ is odd. If either

(0) $|G'| = 1$, or

(1) $|G'|$ is prime, or

(2) $|G'|$ is the product of 2 distinct primes,

then $\mathrm{Cay}(G; A)$ has a hamiltonian cycle.

Remark 2.3. The assumption that G has odd order is not necessary in parts (0) and (1) of Theorem 2.2. Steps toward removing this assumption from part (2) were taken in [3] and [8].

For ease of reference, we record a few other known facts:

Lemma 2.4 ("Factor Group Lemma" [10, §2.2])**.** *Suppose*

- *A is a subset of the group G,*
- *N is a cyclic, normal subgroup of G,*
- *$C = (a_i)_{i=1}^m$ is a hamiltonian cycle in $\mathrm{Cay}(G/N; A)$, and*
- *the voltage $\mathbb{V}(C) = \prod_{i=1}^m a_i$ generates N.*

Then $(a_1, a_2, \ldots, a_m)^{|N|}$ is a hamiltonian cycle in $\mathrm{Cay}(G; A)$.

Lemma 2.5 ([6, Lemma 2.27])**.** *Let A generate a finite group G and let $b \in A$, such that the subgroup generated by b is normal in G. If $\mathrm{Cay}\big(G/\langle b\rangle; A\big)$ has a hamiltonian cycle, and $\langle b\rangle \cap Z(G) = \{e\}$, then $\mathrm{Cay}(G; A)$ has a hamiltonian cycle.*

Lemma 2.6 (cf. [5, Theorem 9.4.3, p. 146] and [7, Lemma 2.16])**.** *Assume G is a finite group of square-free order. Then*

(1) *G' and G/G' are cyclic,*

(2) *$G' \cap Z(G) = \{e\}$, and*

(3) *if $a \in G$, such that $\langle a, G'\rangle = G$, then*

  (a) *a does not commute with any nontrivial element of G', and*

  (b) *$|a| = |G/G'|$.*

Proposition 2.7 (Maghsoudi [7, Proposition 3.1])**.** Assume G is a finite group, such that $|G|$ is a product of four distinct primes. If A is an irredundant generating set of G, such that $|A| \geq 4$, then $\mathrm{Cay}(G; A)$ has a hamiltonian cycle.

3 Proof of Theorem 1.3

Let A be an irredundant generating set of a finite group G, such that $|G| = pqrs$. We wish to show that $\mathrm{Cay}(G; A)$ has a hamiltonian cycle. Although some of the details are new, the line of argument here is quite standard. (See the proofs in [7], for example.)

Let

$$\overline{G} = G/G'.$$

Note that G is solvable by Lemma 2.6(1), so $G' \neq G$. Therefore $|G'|$ has at most 3 prime factors. We may assume it has exactly 3, for otherwise Theorem 2.2 applies. Hence, we may assume $|G'| = qrs$ (so $|\overline{G}| = p$). Since G' is cyclic (see Lemma 2.6(1)), we have

$$G' = \mathcal{C}_q \times \mathcal{C}_r \times \mathcal{C}_s,$$

where $\mathcal{C}_n$ denotes a (multiplicative) cyclic group of order n. Then each element γ of G' can be written uniquely in the form

$$\gamma = \gamma_q\,\gamma_r\,\gamma_s, \text{ where } \gamma_q \in \mathcal{C}_q, \gamma_r \in \mathcal{C}_r, \text{ and } \gamma_s \in \mathcal{C}_s.$$

Let $a \in A$, such that $\langle \overline{a} \rangle = \overline{G}$. Then $|a| = p$ by Lemma 2.6(3)(b). Let b be another element of A, so we may write $b = a^k\gamma$, where $\gamma \in G'$ (and $0 \leq k < p$).

Case 1. *Assume* $A \cap G' \neq \emptyset$. Then we may assume $b \in G'$. Since G' is cyclic (see Lemma 2.6(1)), and it is a standard exercise in undergraduate abstract algebra to show that every subgroup of a cyclic normal subgroup is normal, we know that $\langle b \rangle \triangleleft G$. Let $\widehat{G} = G/\langle b \rangle$. Since $|G| = pqrs$ has only 4 prime factors, we know that $|\widehat{G}|$ has at most 3 prime factors, so we see from part (1), (2), or (3) of Theorem 1.2 that $\mathrm{Cay}(\widehat{G}; A)$ has a hamiltonian cycle. Since

$$\langle b \rangle \cap Z(G) \subseteq G' \cap Z(G) = \{e\}$$

(see Lemma 2.6(2)), we can now conclude from Lemma 2.5 that $\mathrm{Cay}(G; S)$ has a hamiltonian cycle, as desired.

Case 2. *Assume* $|A| = 2$ (*and* $A \cap G' = \emptyset$). This means that $A = \{a, b\}$, so $\langle a, b \rangle = G$. This implies $\langle \gamma \rangle = G'$, so γ_q, γ_r, and γ_s are nontrivial. Also, we have $k \geq 1$, since $A \cap G' = \emptyset$. Let

$$C = (\overline{b}, \overline{a}^{-(k-1)}, \overline{b}, \overline{a}^{p-k-1}),$$

so C is a (well known) hamiltonian cycle in $\mathrm{Cay}(\overline{G}; \overline{A})$. The voltage of C is

$$\mathbb{V}(C) = ba^{-(k-1)}ba^{p-k-1} = a^k\gamma \cdot a^{-(k-1)} \cdot a^k\gamma \cdot a^{p-k-1} = a^k \cdot \gamma_q\gamma_r\gamma_s \cdot a \cdot \gamma_q\gamma_r\gamma_s \cdot a^{-k-1}.$$

Since $|G|$ is odd, we know that a does not invert any nontrivial element of G', so the two occurrences of γ_q in this product do not cancel each other, and similarly for the occurrences of γ_r and γ_s. Therefore, this voltage projects nontrivially to $\mathcal{C}_q$, $\mathcal{C}_r$, and $\mathcal{C}_s$, so it generates G'. Hence, the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\mathrm{Cay}(G; A)$.

Case 3. *Assume* $|A| \geq 3$ (*and* $A \cap G' = \emptyset$). Let c be a third element of A. We may assume $|A| < 4$ (for otherwise Proposition 2.7 applies), so $A = \{a, b, c\}$. Write $c = a^\ell\gamma'$ (with $\gamma' \in G'$). Since $\langle a, b, c \rangle = G$, we must have $\langle \gamma, \gamma' \rangle = G'$, but the fact that A is irredundant implies $\langle \gamma \rangle \neq G'$ and $\langle \gamma' \rangle \neq G'$. If $|\gamma| = q$, then we must have $|\gamma'| = rs$. However, this implies $\langle b, c \rangle = G$, which contradicts the fact that the generating set A is irredundant. So $|\gamma|$ and $|\gamma'|$ must each have precisely two prime factors. Therefore, we may assume without loss of generality that $|\gamma| = qr$ and $|\gamma'| = rs$. Then

$$\gamma = \gamma_q\,\gamma_r \text{ and } \gamma' = \gamma_r'\,\gamma_s',$$

and each of γ_q, γ_r, γ_r', and γ_s' is nontrivial.

We may assume $k, \ell \leq (p-1)/2$ (by replacing some generators with their inverses, if necessary). We may also assume, without loss of generality, that $\ell \leq k$. (If $k \neq \ell$, this implies $\ell \leq (p-3)/2$.) Also note that $k, \ell \geq 1$, since $A \cap G' = \emptyset$.

Subcase 3.1. *Assume* $p = 3$. This implies $\overline{a} = \overline{b} = \overline{c}$, so we have the following two hamiltonian cycles in $\mathrm{Cay}(\overline{G}; A)$:

$$C_1 = (\overline{a}, \overline{b}, \overline{c}) \text{ and } C_2 = (\overline{a}, \overline{c}, \overline{b}).$$

Their voltages are

$$\mathbb{V}(C_1) = abc = a \cdot a\gamma \cdot a\gamma' = a^2\, \gamma_q\, \gamma_r\, a\, \gamma_r'\, \gamma_s'$$

and

$$\mathbb{V}(C_2) = acb = a \cdot a\gamma' \cdot a\gamma = a^2\, \gamma_r'\, \gamma_s'\, a\, \gamma_q\, \gamma_r.$$

In each of these voltages, there is nothing that could cancel the factor γ_q or the factor γ_s'. So both voltages project nontrivially to $\mathcal{C}_q$ and $\mathcal{C}_s$.

Therefore, we may assume that both voltages project trivially to $\mathcal{C}_r$, for otherwise we have a voltage that projects nontrivially to all three factors, and therefore generates G', so the Factor Group Lemma 2.4 applies. Hence

$$a^2\gamma_r a\gamma_r' = \mathbb{V}(C_1)_r = e = \mathbb{V}(C_2)_r = a^2\gamma_r' a\gamma_r,$$

so, letting $x = \gamma_r'\, \gamma_r^{-1}$, we have $ax = xa$, which means that x commutes with a. However, we also know from Lemma 2.6(3)(a) that a does not commute with any nontrivial element of G'. Therefore, we must have $x = e$, which means $\gamma_r' = \gamma_r$. Also, since a does not invert any nontrivial element of G' (since a has odd order), we know that $\gamma_r a \gamma_r \neq a$. Therefore

$$\mathbb{V}(C_1)_r = a^2 \cdot \gamma_r a \gamma_r' = a^2 \cdot \gamma_r a \gamma_r \neq a^2 \cdot a = e.$$

This voltage therefore projects nontrivially to all three factors of G', so it generates G'. Hence, the Factor Group Lemma 2.4 applies.

Subcase 3.2. *Assume* $\overline{a}$*,* $\overline{b}$*, and* $\overline{c}$ *are not all distinct.* We may assume $\overline{a} = \overline{c}$ (which means $\ell = 1$). We may also assume $p \geq 5$, for otherwise Subcase 3.1 applies. Therefore

$$p - k \geq p - (p-1)/2 = (p+1)/2 \geq (5+1)/2 = 3,$$

so we have the following two hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{A})$:

$$C_1 = (\overline{b}, \overline{a}^{-(k-1)}, \overline{b}, \overline{c}, \overline{a}^{p-k-2}) \text{ and } C_2 = (\overline{b}, \overline{a}^{-(k-1)}, \overline{b}, \overline{a}, \overline{c}, \overline{a}^{p-k-3}).$$

Their voltages are

$$\begin{aligned} \mathbb{V}(C_1) &= ba^{-(k-1)}bca^{p-k-2} \\ &= a^k\gamma \cdot a^{-(k-1)} \cdot a^k\gamma \cdot a\gamma' \cdot a^{p-k-2} \\ &= a^k\, \gamma_q\gamma_r\, a\, \gamma_q\gamma_r\, a\, \gamma_r'\gamma_s'\, a^{-k-2} \end{aligned}$$

and

$$\begin{aligned}\mathbb{V}(C_2) &= ba^{-(k-1)}baca^{p-k-3} \\ &= a^k\gamma \cdot a^{-(k-1)} \cdot a^k\gamma \cdot a \cdot a\gamma' \cdot a^{p-k-3} \\ &= a^k\, \gamma_q\gamma_r\, a\, \gamma_q\gamma_r\, a^2\, \gamma'_r\gamma'_s\, a^{-k-3}.\end{aligned}$$

Since a does not invert γ_q (because a has odd order), we see that the two occurrences of γ_q in these voltages cannot cancel each other. Hence, both voltages project nontrivially to $\mathcal{C}_q$. Also, there is nothing in either voltage that could cancel the single occurrence of γ'_s, so both voltages also project nontrivially to $\mathcal{C}_s$.

Therefore, in order to apply the Factor Group Lemma 2.4, it suffices to show that at least one of these voltages projects nontrivially to $\mathcal{C}_r$. If not, then both projections are trivial, so

$$\begin{aligned}a^k\gamma_r a\gamma_r a \cdot \gamma'_r a \cdot a^{-k-3} &= a^k\gamma_r a\gamma_r a\gamma'_r a^{-k-2} \\ &= \mathbb{V}(C_1)_r \\ &= e \\ &= \mathbb{V}(C_2)_r \\ &= a^k\gamma_r a\gamma_r a^2\gamma'_r a^{-k-3} \\ &= a^k\gamma_r a\gamma_r a \cdot a\gamma'_r \cdot a^{-k-3},\end{aligned}$$

which implies $\gamma'_r a = a\gamma'_r$. This contradicts the fact that a does not centralize any nontrivial element of G' (see Lemma 2.6(3)(a)).

Subcase 3.3. *Assume* $\ell \neq (p-3)/2$. We may assume that the preceding case does not apply. In particular, then $\overline{b} \neq \overline{c}$, so $k \neq \ell$, so we have $\ell \leq (p-5)/2$, which implies $k + \ell \leq p - 3$. Therefore, we have the following two hamiltonian cycles in $\mathrm{Cay}(\overline{G}; \overline{A})$:

$$C_1 = (\overline{b}, \overline{a}^{-(k-1)}, \overline{b}, \overline{c}, \overline{a}^{-(\ell-1)}, \overline{c}, \overline{a}^{p-k-\ell-2}) \tag{A}$$

and

$$C_2 = (\overline{b}, \overline{a}^{-(k-1)}, \overline{b}, \overline{a}, \overline{c}, \overline{a}^{-(\ell-1)}, \overline{c}, \overline{a}^{p-k-\ell-3}).$$

Their voltages are

$$\begin{aligned}\mathbb{V}(C_1) &= ba^{-(k-1)}bca^{-(\ell-1)}ca^{p-k-\ell-2} \\ &= a^k\gamma \cdot a^{-(k-1)} \cdot a^k\gamma \cdot a^\ell\gamma' \cdot a^{-(\ell-1)} \cdot a^\ell\gamma' \cdot a^{p-k-\ell-2} \\ &= a^k \cdot \gamma_q\gamma_r a\gamma_q\gamma_r \cdot a^\ell \cdot \gamma'_r\gamma'_s a\gamma'_r\gamma'_s \cdot a^{p-k-\ell-2}\end{aligned} \tag{B}$$

and

$$\begin{aligned}\mathbb{V}(C_2) &= ba^{-(k-1)}baca^{-(\ell-1)}ca^{p-k-\ell-3} \\ &= a^k\gamma \cdot a^{-(k-1)} \cdot a^k\gamma \cdot a \cdot a^\ell\gamma' \cdot a^{-(\ell-1)} \cdot a^\ell\gamma' \cdot a^{p-k-\ell-3} \\ &= a^k \cdot \gamma_q\gamma_r a\gamma_q\gamma_r \cdot a^{\ell+1} \cdot \gamma'_r\gamma'_s a\gamma'_r\gamma'_s \cdot a^{p-k-\ell-3}.\end{aligned}$$

In each product, the two occurrences of γ_q cannot cancel each other, and the two occurrences of γ_s' cannot cancel each other (because a does not invert any nontrivial element of G'), so both voltages project nontrivially to $\mathcal{C}_q$ and $\mathcal{C}_s$.

Therefore, in order to apply the Factor Group Lemma 2.4, it suffices to show that at least one of these voltages projects nontrivially to $\mathcal{C}_r$. If not, then both projections are trivial, so, much as in Subcase 3.2, we have

$$\begin{aligned} a^k\gamma_r a\gamma_r a^\ell \cdot \gamma_r' a \gamma_r' a \cdot a^{p-k-\ell-3} &= a^k \cdot \gamma_r a \gamma_r \cdot a^\ell \cdot \gamma_r' a \gamma_r' \cdot a^{p-k-\ell-2} \\ &= \mathbb{V}(C_1)_r \\ &= e \\ &= \mathbb{V}(C_2)_r \\ &= a^k \cdot \gamma_r a \gamma_r \cdot a^{\ell+1} \cdot \gamma_r' a \gamma_r' \cdot a^{p-k-\ell-3} \\ &= a^k\gamma_r a \gamma_r a^\ell \cdot a\gamma_r' a \gamma_r' \cdot a^{p-k-\ell-3}, \end{aligned}$$

so $\gamma_r' a \gamma_r' a = a\gamma_r' a \gamma_r'$.

Now, note that $\langle \gamma_r' \rangle \triangleleft G$ (since every subgroup of the cyclic normal subgroup G' is normal), so there is some $\widehat{\gamma} \in \langle \gamma_r' \rangle$, such that $a\gamma_r' = \widehat{\gamma} a$. With this notation, the conclusion of the preceding paragraph tells us that $\gamma_r' \widehat{\gamma} a^2 = \widehat{\gamma} a^2 \gamma_r'$. Since γ_r' and $\widehat{\gamma}$ are in the abelian group $\langle \gamma_r' \rangle$, we know that they commute with each other, so this implies $\gamma_r' a^2 = a^2 \gamma_r'$.

However, since $\overline{a}$ generates $\overline{G}$, and $|\overline{G}| = p$ is odd, we know that $\overline{a}^2$ also generates $\overline{G}$. Hence, we see from Lemma 2.6(3)(a) that a^2 does not centralize any nontrivial element of G'. This contradicts the conclusion of the preceding paragraph.

Subcase 3.4. *Assume* $p = 7$. We may assume Subcase 3.2 does not apply, so $1 < \ell < k \leq (p-1)/2$. Since $(p-1)/2 = (7-1)/2 = 3$, we conclude that $\ell = 2$ and $k = 3$. Thus, we have

$$p = 7, \;\; \overline{c} = \overline{a}^2, \;\text{ and }\; \overline{b} = \overline{a}^3.$$

Now, the Cayley graph $\operatorname{Cay}(\overline{G}; \overline{a}, \overline{b}, \overline{c})$ is the complete graph K_7, so it has *many* hamiltonian cycles. In particular, we have the hamiltonian cycle

$$C = (\overline{c}, \overline{a}^2, \overline{c}, \overline{a}^{-1}, \overline{b}, \overline{a}^{-1}).$$

Since G' is a cyclic, normal subgroup of G (and $|a| = p = 7$), there is some $\alpha \in \mathbb{Z}^+$, such that $\alpha^7 \equiv 1 \pmod{qrs}$ and

$$a\,g = g^\alpha\, a \;\text{ for all } g \in G'.$$

Also, we may write $\gamma_r' = \gamma_r^x$ for some $x \in \mathbb{Z}^+$. With this notation, the voltage of C is

$$\begin{aligned} \mathbb{V}(C) &= ca^2ca^{-1}ba^{-1} \\ &= a^2\gamma' \cdot a^2 \cdot a^2\gamma' \cdot a^{-1} \cdot a^3\gamma \cdot a^{-1} \\ &= a^2\gamma' a^4 \gamma' a^2 \gamma a^{-1} \\ &= (\gamma')^{\alpha^2+\alpha^6} \gamma^\alpha \\ &= (\gamma_r'\gamma_s')^{\alpha^2+\alpha^6} (\gamma_q\,\gamma_r)^\alpha \\ &= (\gamma_r^x\gamma_s')^{\alpha^2+\alpha^6} (\gamma_q\,\gamma_r)^\alpha \\ &= \gamma_q^\alpha\, \gamma_r^{(\alpha^2+\alpha^6)x+\alpha}\, (\gamma_s')^{\alpha^2+\alpha^6}. \end{aligned}$$

Since $\alpha^7 \equiv 1 \pmod{qrs}$, we know that $\alpha^7 \equiv 1 \pmod{q}$ and $\alpha^7 \equiv 1 \pmod{s}$, so it is easy to see that $\alpha \not\equiv 0 \pmod{q}$ and $\alpha^2 + \alpha^6 \not\equiv 0 \pmod{s}$. Hence, it is clear that $\mathbb{V}(C)$ projects nontrivially to $\mathcal{C}_q$ and $\mathcal{C}_s$. Therefore, if Theorem 2.4 does not apply, then this voltage projects trivially to $\mathcal{C}_r$, which means

$$(\alpha^2 + \alpha^6)x + \alpha \equiv 0 \pmod{r}. \tag{C}$$

We also have the hamiltonian cycle

$$C' = (\overline{b}, \overline{a}^{-2}, \overline{b}, \overline{c}, \overline{a}^{-1}, \overline{c}).$$

It is easy to see that the voltage of this hamiltonian cycle projects nontrivially to $\mathcal{C}_q$ and $\mathcal{C}_s$. (As in Subcase 3.3, this is because the only occurrences of γ_q come from the two occurrences of $\overline{b}$, and the only occurrences of γ_s come from the two occurrences of $\overline{c}$. None of these can cancel each other, because no element of G inverts γ_q or γ_s.) Therefore, if $\mathbb{V}(C')$ does not generate G', then this voltage must project trivially to $\mathcal{C}_r$.

Note that the hamiltonian cycle C' can be obtained from the hamiltonian cycle C_1 of Equation (A), with $p = 7$, $k = 3$, and $\ell = 2$, by deleting the final term $\overline{a}^{p-k-\ell-2} = \overline{a}^{7-3-2-2} = \overline{a}^0$ of C_1. Therefore, the formula (B) for the voltage of C_1 also yields the voltage of C' (if we delete the irrelevant final term $a^{p-k-\ell-2} = a^0 = e$), so we have

$$\begin{aligned} e &= \mathbb{V}(C_1)_r \\ &= a^k \cdot \gamma_r a \gamma_r \cdot a^\ell \cdot \gamma_r' a \gamma_r' \\ &= a^3 \cdot \gamma_r a \gamma_r \cdot a^2 \cdot \gamma_r^x a \gamma_r^x \\ &= \gamma_r^{\alpha^3 + \alpha^4 + \alpha^6 x + \alpha^7 x}, \end{aligned}$$

so

$$\alpha^3 + \alpha^4 + \alpha^6 x + \alpha^7 x \equiv 0 \pmod{r}.$$

Multiplying by α^4, and recalling that $\alpha^7 \equiv 1 \pmod{r}$, we conclude that

$$(1 + \alpha) + \alpha^3(1 + \alpha)x \equiv 0 \pmod{r}.$$

Dividing this equation by $1 + \alpha$ yields $1 + \alpha^3 x = 0$ (in $\mathbb{Z}_r$), so $x = -1/\alpha^3$. Plugging this into Equation (C) yields $-(\alpha^2 + \alpha^6)/\alpha^3 + \alpha = 0$ in $\mathbb{Z}_r$, so (multiplying by $-\alpha$) we have $1 + \alpha^4 - \alpha^2 \equiv 0 \pmod{r}$. Recall that we also know $\alpha^7 - 1 \equiv 0 \pmod{r}$. Since the polynomial $1 + x^4 - x^2$ is relatively prime to $x^7 - 1$, this is impossible. More concretely, we have

$$\begin{aligned} 0 &= (\alpha^6 - \alpha^5 + \alpha^4 - \alpha^3 - 1) \cdot 0 \;-\; (\alpha^3 - \alpha^2) \cdot 0 \\ &\equiv (\alpha^6 - \alpha^5 + \alpha^4 - \alpha^3 - 1)(1 + \alpha^4 - \alpha^2) - (\alpha^3 - \alpha^2)(\alpha^7 - 1) \qquad \pmod{r} \\ &= -1, \end{aligned}$$

which is an obvious contradiction.

Subcase 3.5. *Assume that the preceding cases do not apply.* Since Subcase 3.3 does not apply, we must have $\ell = (p-3)/2$. Since $\ell < k \leq (p-1)/2$, this implies

$$k = \frac{p-1}{2} = 1 + \frac{p-3}{2} = 1 + \ell.$$

Therefore

$$\overline{a}\,\overline{c} = \overline{a}\,\overline{a}^{\ell} = \overline{a}^{1+\ell} = \overline{a}^{k} = \overline{b}.$$

Putting c into the role of a will usually give us new values for k and ℓ. Indeed, if Subcase 3.3 does not apply after this change (that is, after replacing the triple (a,b,c) with the appropriate triple $(c, a^{\epsilon_1}, b^{\epsilon_2})$ or $(c, b^{\epsilon_1}, a^{\epsilon_2})$, with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$), then either

$$\overline{a}^{\pm 1} = \overline{c}^{(p-3)/2} \text{ and } \overline{b}^{\pm 1} = \overline{c}^{(p-1)/2} = \overline{a}^{\pm 1}\,\overline{c}$$

or

$$\overline{b}^{\pm 1} = \overline{c}^{(p-3)/2} \text{ and } \overline{a}^{\pm 1} = \overline{c}^{(p-1)/2} = \overline{b}^{\pm 1}\,\overline{c}.$$

However, we have

$$\overline{a}^{-1}\,\overline{c} = \overline{a}^{\ell-1} = \overline{a}^{(p-5)/2} \notin \{\overline{a}^{\pm(p-1)/2}\} = \{\overline{b}^{\pm 1}\}$$

and

$$\overline{b}\,\overline{c} = \overline{a}^{k+1} = \overline{a}^{(p+1)/2} \notin \{\overline{a}^{\pm 1}\}.$$

Therefore, $\overline{c}^{(p-3)/2}$ must be either $\overline{a}$ or $\overline{b}^{-1}$. Noting that (since $\overline{a}\,\overline{c} = \overline{b}$) we have $\overline{b} = \overline{a}\,\overline{c}$ and $\overline{a}^{-1} = \overline{b}^{-1}\,\overline{c}$, this implies that the only possibilities are that either

$$\overline{a} = \overline{c}^{(p-3)/2} \text{ and } \overline{b} = \overline{a}\,\overline{c} = \overline{c}^{(p-1)/2} \tag{D}$$

or

$$\overline{b}^{-1} = \overline{c}^{(p-3)/2} \text{ and } \overline{a}^{-1} = \overline{b}^{-1}\,\overline{c} = \overline{c}^{(p-1)/2}. \tag{E}$$

If (D) holds, then

$$\overline{a} = \overline{c}^{(p-3)/2} = \left(\overline{a}^{\ell}\right)^{(p-3)/2} = \left(\overline{a}^{(p-3)/2}\right)^{(p-3)/2} = \overline{a}^{(p-3)^2/4},$$

so $(p-3)^2/4 \equiv 1 \pmod{p}$, so $9 \equiv 4 \pmod{p}$, which implies $p = 5$. However, since Subcase 3.2 does not apply, we know that $\overline{c} \neq \overline{a}$, so $\ell > 1$. This means $(p-3)/2 \geq 2$, so $p \geq 7$. This contradicts our recent previous conclusion that $p = 5$.

So (E) must hold. Then $\overline{b}^{-1} = \overline{c}^{(p-3)/2}$. Since $\overline{b} = \overline{a}^{k} = \overline{a}^{(p-1)/2}$ and $\overline{c} = \overline{a}^{\ell} = \overline{a}^{(p-3)/2}$, this means

$$-(p-1)/2 \equiv (p-3)^2/4 \pmod{p}.$$

so $2 \equiv 9 \pmod{p}$, which implies $p = 7$. So Subcase 3.4 applies.

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.13
https://doi.org/10.26493/2590-9770.1535.8ad
(Also available at http://adam-journal.eu)

Regular Leonardo polyhedra: Mathematics and art

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Received 22 March 2022, accepted 24 March 2022, published online 12 August 2022

Abstract

Five known combinatorial regular polyhedra with genus $g \geqslant 2$ and with the rotation group of a Platonic solid are described, along with their history. They are close (hyperbolic) analogues of the Platonic solids. The protagonists of their discovery are Leonardo da Vinci and two mathematicians: Alicia Boole Stott and Harold Scott MacDonald Coxeter. For a sixth candidate we discovered that there is so far only a Kepler-Poinsot type available. This is a correction of a result from 1986.

Keywords: Polyhedral manifold, Leonardo polyhedra, regular map, Klein's quartic

Math. Subj. Class.: 52B70

Leonardo da Vinci

In 1498 Leonardo was invited to illustrate the mathematician's book *De Divina Proportione* of Luca Pacioli, [16]. Leonardo gladly accepted this invitation and then spent four years, of course with interruptions, busy with it. One reason for this was for sure that he wanted to perfect his knowledge of the central perspective. Another reason may have been to impress his colleagues and competitors, and certainly also potential customers.

Leonardo's work proceeded in three phases. First he drew the five Platonic solids and the 13 Archimedian bodies in the well-known way. In the second and decisive phase, he

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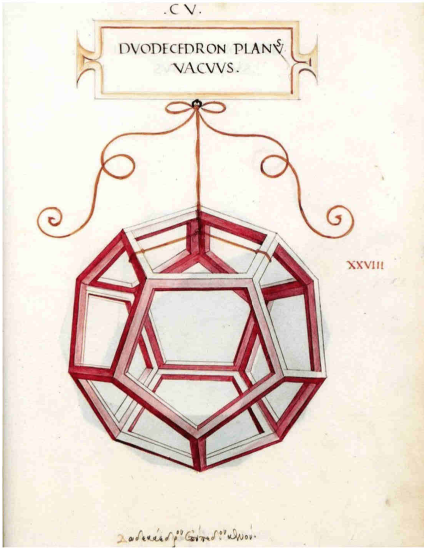

Figure 1: Drawing by Leonardo da Vinci for the work of Luca Pacioli.

only drew the edge skeleton and replaced the edges with slender struts formed by convex polygons, predominantly by quadrilaterals (vacuus). With this he had created the first polyhedra (with genus $g \geqslant 2$).

In the third phase he made the polyhedron even more complicated and on each *hole* he placed the edge skeleton of a pyramid (elevatus vacuus). It is known that these constructions have made a great impression on his contemporaries and later generations. The best known is probably Kepler's *Harmonici Mundi*, a nice but wrong explanation of the planetary system via the Platonic solids. For mathematicians, Leonardo's polyhedrons were of little interest because they are not regular. The first two combinatorially regular polyhedra in the style of Leonardo only were found four hundred years later by the mathematician Alicia Boole Stott. At least now it's time for some basic definitions.

Definitions

For this work we understand a *polyhedron* to be a compact topological 2-manifold without boundary, embedded in the three-dimensional Euclidean space E^3 and made up of convex polygons.

The embedding means that no self-intersections are allowed. The convex polygons of the cells of the manifold exclude general topological embeddings, maintain a close relationship with the Platonic solids and form the faces, edges, and vertices of the polyhedron.

A polyhedron with the rotational symmetry of a Platonic solid and with genus $g \geqslant 2$ is called *Leonardo polyhedron*. This definition, of course, includes polyhedra with the full symmetry group of a Platonic solid.

The numbers of vertices, edges, and faces f_0, f_1, and f_2 of a polyhedron are given by Euler's polyhedron formula $f_2 - f_1 + f_0 = 2 - 2g = \chi$. Here g is the genus and χ is the Euler characteristic.

If all faces of a polyhedron are p-gons and all vertices are q-valent, we call the polyhedron *locally regular* or *equivelar*. We use the Schläfli symbol $\{p, q\}$ to describe this. A triple consisting of a face, an edge of this face, and a vertex of this edge is called a *flag*. Local regularity is a rather weak restriction. For example there are infinitely many locally regular Leonardo polyhedra. So we do not try to survey them all here. We consider the much stronger condition of regularity.

A polyhedron is called *regular* if its group of automorphisms acts transitively on its flags. This elegant definition can be rather complicated to use in practice. In the next sections we use some equivalent methods to prove regularity.

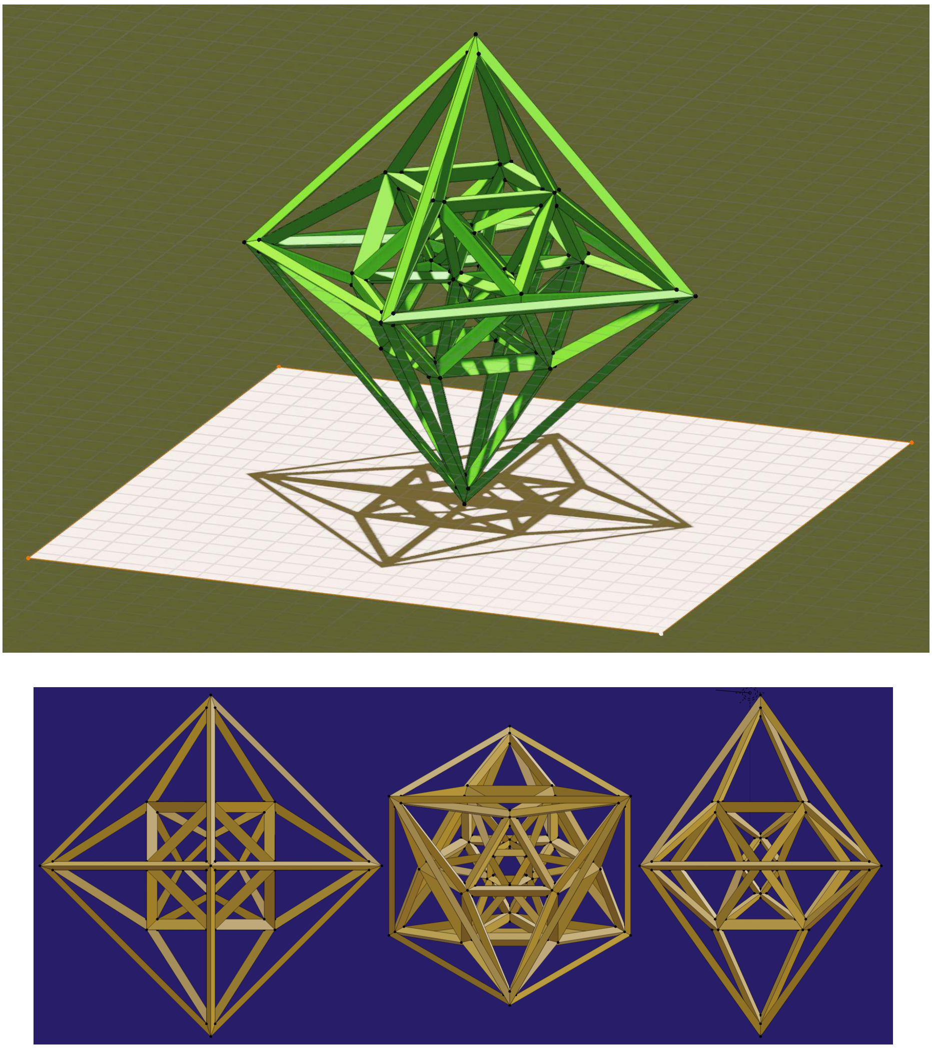

Figure 2: Regular Leonardo Polyeder of Alicia Boole Stott with symmetrical orthogonal projections.

Alicia Boole Stott

Alicia Boole Stott, 1860 – 1940, was the daughter of George Boole, one of the pioneers of set theory. She received her mathematics education from her parents and grandparents, since for women in those years studying mathematics was very difficult or completely impossible.

In 1910 she wrote a paper presenting the first two regular Leonardo polyhedra, [2]. Construction and proof were as simple as they were ingenious: just like Leonardo, she started from regular convex polytopes, not in dimension 3 but instead in dimension 4. More precisely, she used projections into the 3-dimensional space, the so-called Schlegel diagrams. Like Leonardo, she only took the edge skeleton and replaced it with slender struts bounded by convex quadrilaterals. Furthermore, she examined only the two self-dual polytopes among the six regular ones in E^4, i.e. the 4-simplex and the 24-cell. Obviously the two resulting polyhedra are locally regular. All of their faces are quadrilaterals, and $q = 6$ for the smaller of these, depicted in Figure 3, whereas $q = 8$ for the larger polyhedron of Figure 2.

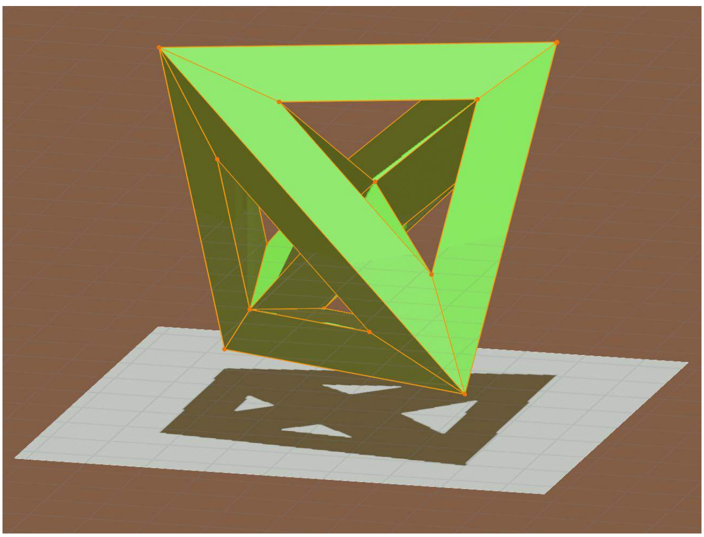

Figure 3: Regular Leonardo polyhedron of Alicia Boole Stott derived from the 4-simplex.

A short YouTube video of the regular Leonardo polyhedron of genus 73 shown in Figure 2 can be seen at https://youtu.be/d3Im8BLxVoM

The proof of regularity is also so simple that we can sketch it here. Consider first the smaller polyhedron. It has the same symmetry group (of order 24) as the tetrahedron. The five possible Schlegel diagrams are isomorphic. And because of self-duality, the automorphism group of the polyhedron doubles in size. Thus one sees that the automorphism group of the polyhedron has order $24 \times 5 \times 2 = 240$, which is exactly the number of flags of the polyhedron. (Note that only the identity can fix a particular flag.) Likewise, but with larger numbers, for the other polyhedron, we get a group of order $48 \times 24 \times 2 = 2304$ and that is the number of flags. In this way, the first two regular Leonardo polyhedra were found. In

1914, Alicia Boole Stott received an honorary doctorate from the University of Groningen for her life's work. And in 1930 she met H.S.M. Coxeter. She was then 70 years old and Coxeter a 23 year old college student.

Harold Scott MacDonald Coxeter

H.S.M. Coxeter, 1907 – 2003, was one of the most important geometers of the 20th century. He published countless works on geometry, algebra, and related fields, he wrote quite a few books and had many students. For the two Leonardo polyhedra by Alicia Boole-Stott he found (using standard methods) the two dual polyhedra [5].

For both polyhedra, Coxeter initially specified only 4-dimensional embeddings. However, the illustration in Figure 4 shows an embedding in 3-dimensional space, which was given by Schulte and Wills [18] based on Coxeter's result. In contrast, for the second polyhedron it turned out during our recent investigation that a 3-dimensional version through an obvious construction with a Schlegel diagram leads in the end to self-intersections. A 3-dimensional model without self-intersections has not yet been found.

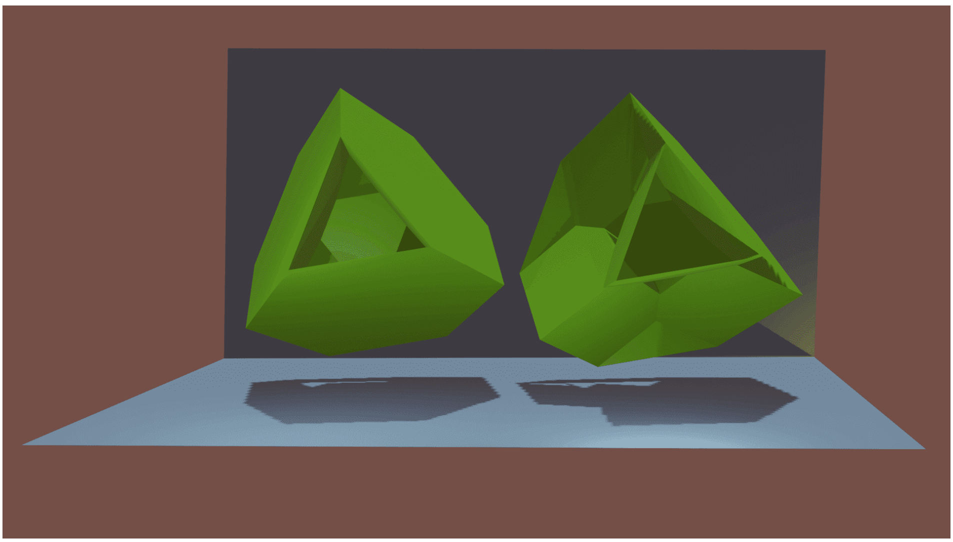

Figure 4: Coxeter's regular Leonardo Polyhedron of type $\{6, 4|3\}$. On the right two hexagons have been removed to show the interior.

We describe an idea for a construction of Coxeter's $\{8, 4|3\}$ starting with Coxeter's 4-dimensional version. We use the edge skeleton of the 24-cell represented within a Schlegel diagram. We mark the midpoints of the 96 edges. Eight edges are incident to each of the 24 vertices of the 24-cell. We connect their centers appropriately and get an (affine) cube in each case that contains a vertex in the interior. The 24 cubes touch each other in pairs at the corners. When we enlarge these cubes simultaneously, and when we leave out the intersections, we have that polyhedra $\{8.4|3\}$. Or do we?

Initially, everything works fine. However, in the end the six outer cubes cause self intersections. We see from the pictures that a simultaneous magnification of six squares (those that would cause self-intersections) would not lead to octagons.

Thus we have so far three regular Leonardo polyhedra. These polyhedra also have the nice and easily controlled property that all geodesics, i.e., the shortest paths around a hole or around a strut, are of equal (combinatorial) length, that is, they are all triangles. Coxeter has already shown that this is sufficient for showing the regularity of the four examples. This theorem is part of a much more general method, the beginnings of which go back to Felix Klein (the Eightfold way) and which is used today to characterize regular maps and polyhedra. This exhausted Leonardo's approach. And it was almost 50 years before the next regular polyhedra (with $g \geqslant 2$) were found.

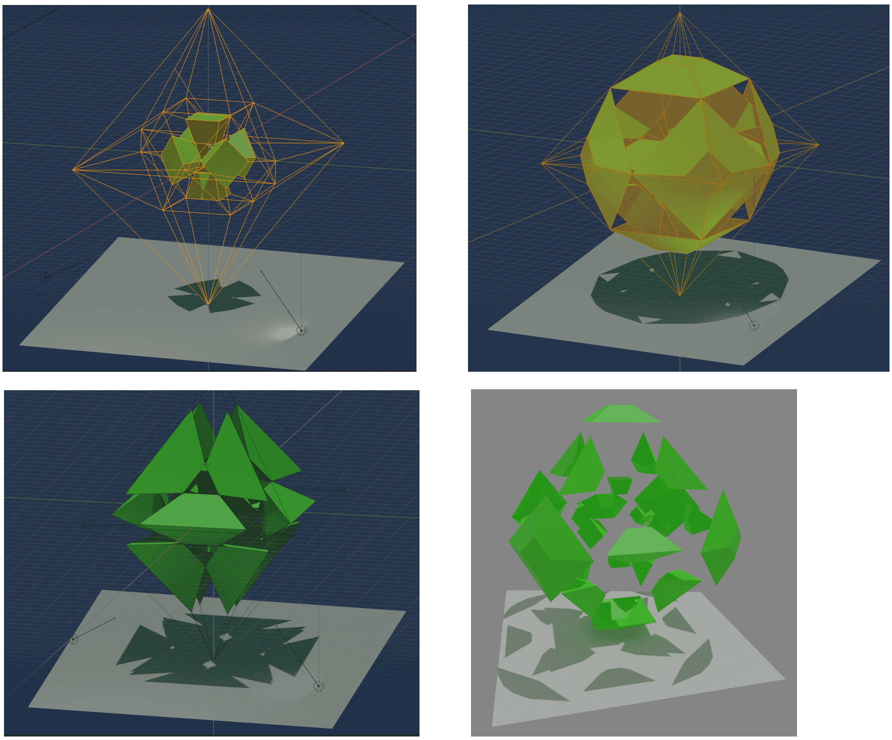

Figure 5: Futile steps for constructing Coxeter's regular Leonardo Polyhedron $\{8, 4|3\}$ in E^3.

New developments

It would be beyond the scope of this article to describe the more recent developments even halfway completely. We therefore limit ourselves to the essential facts. Starting in 1982, an infinite number of regular polyhedra were discovered, including two mutually dual infinite series. But only two regular Leonardo polyhedra were among them. All these regular polyhedra were derived from regular maps (Riemann surfaces, algebraic curves). Regular maps are the main generalization of the Platonic solids.

The first (and most famous) regular maps were made by the Munich mathematicians Felix Klein (1879), Walter von Dyck (1880), [7, 8], Adolf Hurwitz (1893), [11], and Robert Fricke (1890). Throughout the 20th century countless new regular maps were added, especially by Coxeter, [6]. In recent decades, Marston Conder, using a globally networked computer program Magma, has enumerated all regular orientable maps of genus up to $g = 100$. For these group theory contributions, see [3, 4].

This provides a huge arsenal of regular maps to generate corresponding regular polyhedra. But it can be very difficult to generate a polyhedron embedded in E^3 given only a combinatorial description of a particular regular map.

For a difficult example see [1]. And there are theorems excluding entire classes of polyhedra, such as the Ringel-Youngs theorem.

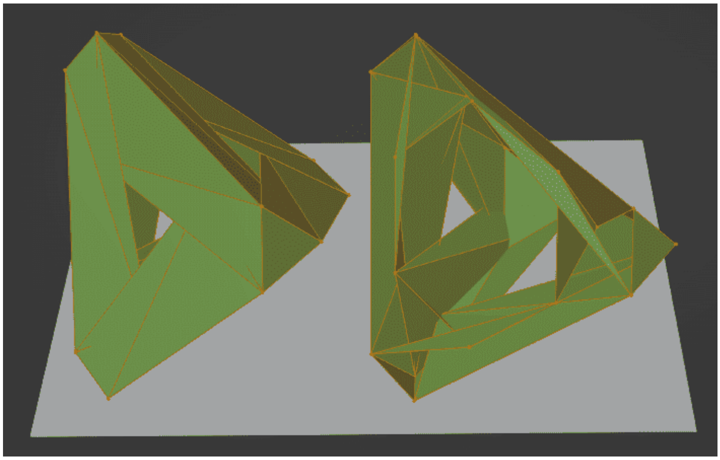

Figure 6: Regular Leonardo Polyeder due to Schulte and Wills [17], based on Klein's quartric.

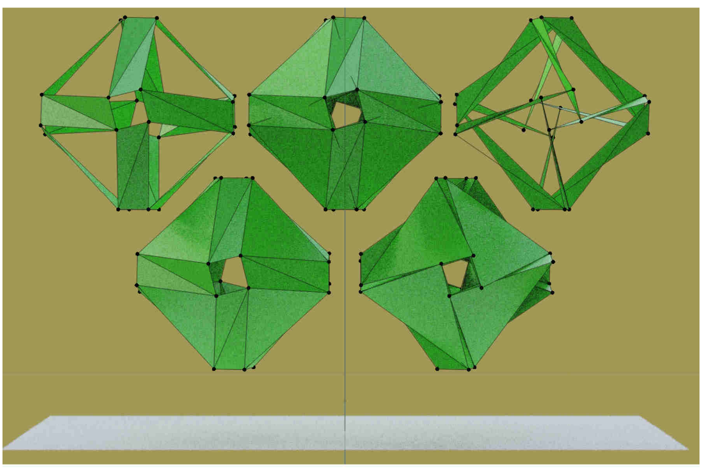

Figure 7: Regular Leonardo Polyeder based on Fricke-Klein, by Grünbaum et al., see [9, 10].

The only two new regular Leonardo polyhedra are the Klein polyhedron, [12, 13], $\{3,7\}_8$ with $g = 3$ and the rotation group of the tetrahedron as well as the Fricke-Klein polyhedron, [14], $\{3,8\}_{12}$ with $g = 5$ and the rotation group of the octahedron. Both polyhedra have geodesics of length 4.

Also, both are triangulations, and therefore their duals do not exist by a theorem of Cauchy. Because the Klein map is so famous, one nevertheless can see the dual as a curved surface constructed, not as a polyhedron but from Carrara marble, on the Berkeley campus, [15].

And how does it continue? There are certainly many more regular polyhedra, maybe infinite series, too. Whether there is a sixth Leonardo polyhedron, or even more, is an open problem.

There does exist a sixth regular Leonardo polyhedron of type $\{8,4|3\}$ in 3-space without self-intersections. This will be explained in detail in a forthcoming paper in the ADAM journal of these two authors.

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.14
https://doi.org/10.26493/2590-9770.1468.a2e
(Also available at http://adam-journal.eu)

On polynomials of small degree over finite fields representing quadratic residues*

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Dedicated to Marston Conder for his 65th birthday

Received 20 September 2021, accepted 13 June 2022, published online 12 August 2022

Abstract

Let F be a finite field of odd characteristic. It was proved by Madden and Vélez in [*Pacific J. Math.* **98** (1982), 123–137] that except for finitely many exceptions of F, for every polynomial $f(x) \in F[x]$ which is not of the form $\alpha g(x)^2$ or $\alpha x g(x)^2$, where $\alpha \in F$, there exists a primitive root $\beta \in F$ such that $f(\beta)$ is a nonzero square in F. When this theoretical result is concretely used, it is necessary to determine such finitely many exceptions. In this paper, the possible exceptions are determined, provided the degree of $f(x)$ is less than 9. Remark that our arguments do not hold when $f(x)$ is of degree 9 and also the capacity at this stage of a computer is not enough to do it. Moreover, such kinds of results have been used to solve some combinatorial problems.

Keywords: Finite field, polynomial, quadratic residues.

Math. Subj. Class.: 12E99

1 Introduction

In early eighties, motivated by a question posed by Alspach, Heinrich and Rosenfeld [1] in the context of decompositions of complete symmetric digraphs, Madden and Vélez [5] investigated polynomials that represent quadratic residues at primitive roots. They proved that, with finitely many exceptions, for any finite field F of odd characteristic, for every

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polynomial $f(x) \in F[x]$ of degree $r \geq 1$ which has a nonzero constant term and is not of the form $\alpha g(x)^2$ or $\alpha x g(x)^2$, there exists a primitive root β such that $f(\beta)$ is a nonzero square in F. It is the purpose of this paper to refine their result for polynomials of degree r where $2 \leq r \leq 8$ and find such possible exceptions. Moreover, we refine the result in [3] when F is of prime order and $f(x)$ is of degree 4.

Our main results is the following theorem.

Theorem 1.1. Let F be a finite field of prime power order p^n, where p^n is odd and not given in the sets S_r listed in Table 1. Then for every polynomial $f(x) \in F[x]$ of degree r that is not of the form $\alpha g(x)^2$ or $\alpha x g(x)^2$, there exists a primitive root $\beta \in F$ such that $f(\beta)$ is a nonzero square in F.

2 Preliminary results

Let $\mathcal{J} = [q_1 = 2, q_2, \ldots, q_m]$ be an increasing sequence of prime numbers with $q_i < q_j$ for $1 \leq i < j \leq m$. As in [5], we define the following functions with $\mathcal{J}$:

$$d(u,m) = 2(1 - \frac{1}{q_u})(1 - \frac{1}{q_{u+1}}) \cdots (1 - \frac{1}{q_m}), \tag{2.1}$$

$$c_r(u,m) = 2r\sqrt{(\frac{q_1 q_2 \cdots q_{u-1}}{q_u q_{u+1} \cdots q_m})}. \tag{2.2}$$

Let $k(m)$ be the unique integer such that $d(k(m)-1, m) \leq 1 < d(k(m), m)$. Then $k(m) \geq 2$.

The following propositions is our starting point.

Proposition 2.1 ([5, Corollary 1])**.** *Let F be a finite field with p^n elements. If s and t are integers such that*

(i) *s and t are coprime,*

(ii) *a prime q divides $p^n - 1$ if and only if q divides st, and*

(iii) $2\phi(t)/t > 1 + (rs-2)p^{n/2}/(p^n-1) + (rs+2)/(p^n-1)$,

then, given any polynomial $f(x) \in F[x]$ of degree r, square-free and with nonzero constant term, there exists a primitive root $\gamma \in F$ such that $f(\gamma)$ is a nonzero square in F.

Proposition 2.2 ([5, Lemma 5])**.** *Let $[2 = q_1, q_2, \ldots, q_m]$ be a finite increasing sequence of primes satisfying $m \leq 2k(m)+1$. Then $m \leq 9$ and $q_{k(m)-1} \leq 5$. In fact, the sequence must satisfy one of the followings:*

(i) *$k(m) = 4$, $q_{k(m)-1} = 5$ and $m = 9$,*

(ii) *$k(m) = 3$, $q_{k(m)-1} = 5$ and $m \leq 7$,*

(iii) *$k(m) = 3$, $q_{k(m)-1} = 3$ and $m \leq 7$, or*

(iv) *$k(m) = 2$, $q_{k(m)-1} = 2$ and $m \leq 5$.*

Set $\mathcal{T} = \{$ 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 97, 101, 103, 109, 127, 131, 139, 151, 157, 163, 181, 193, 199, 211, 223, 229, 241, 271, 277, 281, 283, 307, 313, 331, 337, 349, 367, 373, 379, 397, 409, 421, 457, 463, 541, 547, 571, 601, 613, 631, 661, 673, 691, 727, 751, 757, 811, 859, 883, 967, 991, 1009, 1021, 1051, 1093, 1171, 1201, 1231, 1291, 1321, 1381, 1471, 1531, 1951, 2311, 2521, 2731, 3361, 3571, 4621, 4831 $\}$.

Proposition 2.3 ([3]). *Let F be a finite field of prime order p, where p is an odd prime not given in $\mathcal{T}$. Then for every polynomial $f(x) \in F[x]$ of degree 4 that has a nonzero constant term and is not of the form $\alpha g(x)^2$, there exists a primitive root $\beta \in F$ such that $f(\beta)$ is a square in F.*

3 The proof of Theorem 1.1

In the following lemma, we give a generalization of [5, Lemma 3] when $r \le 8$. Moreover, Lemma 3.1 plays an indispensable role in programming and computing.

Lemma 3.1. *Let $\mathcal{J}_q = [q_1 = 2, q_2, q_3, \ldots, q_{k(m)} = q, q_{k(m)+1}, \ldots, q_m]$ be a finite increasing sequence of primes satisfying $m \ge 2k(m) + 2$, and let $r \le 8$. Then*

$$d(k(m)+1, m) - c_r(k(m)+1, m) > 1. \tag{3.1}$$

Proof. It suffices to show that this is true for $r = 8$. Since $2 \le k(m) \le \frac{m}{2} - 1$, we have $m \ge 6$. Since

$$d(k(m)+1, m) = (1 + \frac{1}{q_{k(m)} - 1})d(k(m), m) > 1 + \frac{1}{q_{k(m)} - 1},$$

(3.1) holds if

$$1 + \frac{1}{q_{k(m)} - 1} - 2r(\frac{q_1 q_2 \cdots q_{k(m)}}{q_{k(m)+1} q_{k(m)+2} \cdots q_m})^{\frac{1}{2}} > 1,$$

which may be rewritten in the following form

$$q_2 q_3 \cdots q_{k(m)}(q_{k(m)} - 1)^2 < \frac{1}{512} q_{k(m)+1} \cdots q_{m-1} q_m, \tag{3.2}$$

in view of the fact that $r = 8$ and $q_1 = 2$.

We divide the proof into two cases, depending on whether $(k(m), m) \in T = \{(2,6), (3,8), (4,10), (5,12)\}$ or not.

CASE 1. $(k(m), m) \notin T$.
We shall in fact prove a more general result:

$$q_2 q_3 \cdots q_l (q_l - 1)^2 < \frac{1}{512} q_{l+1} \cdots q_{m-1} q_m,$$

where $m \ge 7, (l, m) \notin T$ and $l \le \frac{m}{2} - 1$ is any integer. To show this, let Ω be the increasing sequence of all prime numbers. For the above prime q_l, let $\mathcal{I}_{q_l} = [w_1 = 2, w_2, w_3, \ldots, w_l = q_l, w_{l+1}, \ldots, w_m]$ be a subsequence of Ω not missing any prime in Ω from w_2 to w_m. Then we may get the desired result if the following (3.3) holds:

$$w_2 w_3 \cdots w_l (w_l - 1)^2 < \frac{1}{512} w_{l+1} \cdots w_{m-1} w_m, \tag{3.3}$$

where $m \geq 7, (l, m) \notin T$ and $l \leq \frac{m}{2} - 1$ is any integer. If $w_m \geq 512$, then (3.3) is clearly true. So we only need to consider primes that are smaller than 512. Note that $m \geq 2l + 2$, it suffices to show that (3.3) is true for $(l, m) \in \{(2, 7), (6, 14)\}$.

A computer search shows that (3.3) holds for all primes $w_m \leq 512$ with $(l, m) \in \{(2, 7), (6, 14)\}$. Suppose now that (3.3) is true for (l, m). Then we have

$$\begin{aligned} w_2 w_3 w_4 \cdots w_l w_{l+1} (w_{l+1} - 1)^2 &= w_2 (w_3 \cdots w_l w_{l+1} (w_{l+1} - 1)^2) \\ &< w_2 (w_{l+2} w_{l+3} \cdots w_m w_{m+1}) \\ &< (w_{l+2} w_{l+3} \cdots w_m w_{m+1}) w_{m+2}. \end{aligned}$$

Therefore, (3.3) is true for $(l, m) \notin T$ with $m \geq 2l + 2$. Hence (3.2) holds, and so does (3.1).

CASE 2. $(k(m), m) \in T$.
Let Ω be the increasing sequence of all prime numbers. Let $\mathcal{I}_q = [w_1 = 2, w_2, w_3, \ldots, w_{k(m)} = q, w_{k(m)+1}, \ldots, w_m]$ be a subsequence of Ω not missing any prime in Ω from w_2 to w_m. Also, let $d(u, m)'$ and $c_r(u, m)'$ be the corresponding values for $\mathcal{I}_q$ as defined by functions d and c_r in (2.1) and (2.2). Then one can easily see that $d(k(m) + 1, m)' \leq d(k(m) + 1, m)$ and that $c_r(k(m) + 1, m)' \geq c_r(k(m) + 1, m)$, and so (3.1) holds for $\mathcal{J}_q$ if it holds for $\mathcal{I}_q$.

From (3.3), we assume $w_m \leq 512$. MAGMA[2] searching for all the primes less than 512 shows that (3.1) holds for $\mathcal{I}_q$ with $(k(m), m) \in T$. This completes the proof of Lemma 3.1. □

Remark 3.2. For $r = 9$, (3.1) may not hold. For example, let $p = 180181$. Then p is prime and the increasing sequence of prime divisors of $p - 1 = 4 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ is $J = [2, 3, 5, 7, 11, 13]$. For $k(m) = 2$, (3.1) does not hold for J.

Let $sp(r)$ denote the smallest prime number that is greater than $8r^2$. Then $sp(2) = 37, sp(3) = 73, sp(4) = 131, sp(5) = 211, sp(6) = 293, sp(7) = 397$ and $sp(8) = 521$. With a similar proof to [5, Lemma 7], we have the following lemma.

Lemma 3.3. *Suppose $r \in \{3, 4, 5, 6, 7, 8\}$ and p is an odd prime. Let $[2 = q_1, q_2, \ldots, q_m]$ be a finite increasing sequence of primes satisfying $m \leq 2k(m) + 1$, and let $p^n - 1 = q_1^{i_1} q_2^{i_2} \cdots q_m^{i_m}$ with $q_m \geq sp(r)$. Then there exist s and t such that*

(i) *s and t are coprime,*

(ii) *a prime q divides $p^n - 1$ if and only if q divides st, and*

(iii) $2\phi(t)/t > 1 + (rs - 2)\sqrt{p^n}/(p^n - 1) + (rs + 2)/(p^n - 1)$.

In order to proceed with the proof of Theorem 1.1 we now need to identify all those increasing sequences $[2 = q_1, q_2, \ldots, q_m]$ with $q_m < 8r^2$ for which one cannot choose $s = q_1 q_2 \cdots q_u$ and $t = q_{u+1} q_{u+2} \cdots q_m$ so as to satisfy Lemma 3.3(iii). Since Lemma 3.1 holds for each q_m, we can assume that for each of these sequences Proposition 2.2 applies. With the help of MAGMA[2], we have the following proposition.

Proposition 3.4. *Let* $[2 = q_1, q_2, \ldots, q_m]$ *be a finite increasing sequence of primes satisfying* $m \leq 2k(m) + 1$, *and let* $p^n - 1 = q_1^{i_1} q_2^{i_2} \cdots q_m^{i_m}$ *with* $q_m < 8r^2$ *where* $r \in \{2, 3, 4, 5, 6, 7, 8\}$. *If* p^n *is not listed in the set*

$$S_2' = \{3, 5, 7, 9, 11, 13, 19, 25, 31, 37, 43, 49, 61, 67, 79, 103, 121, 127, 151, 169, 181, 211, 241, 271, 331, 421, 631\}$$

or S_r *where* $3 \leq r \leq 8$ *in* Table 1, *then there exist* s *and* t *such that*

(i) s *and* t *are coprime,*

(ii) *a prime* q *divides* $p^n - 1$ *if and only if* q *divides* st, *and*

(iii) $2\phi(t)/t > 1 + (rs - 2)\sqrt{p^n}/(p^n - 1) + (rs + 2)/(p^n - 1)$.

Remark 3.5. Let $S_4' = \{9, 25, 27, 49, 81, 121, 169, 289, 343, 361, 529, 625, 841, 961, 1681\}$. For $r = 4$, the numbers in S_4' were missed to discuss in [3], but their result is still true.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We may assume without loss of generality that $f(x)$ is square free, since leaving out a square factor does not affect the validity of the conclusion. It follows by Proposition 2.1 that a polynomial $f(x)$ represents a nonzero square at some primitive root in F if there exist s and t satisfying the following three conditions:

(i) s and t are coprime,

(ii) a prime q divides $p^n - 1$ if and only if q divides st, and

(iii) $2\phi(t)/t > 1 + (rs - 2)\sqrt{p^n}/(p^n - 1) + (rs + 2)/(p^n - 1)$.

Now, we show that such s and t exist for all odd primes p that are not listed in the set S_r.

Let $[q_1 = 2, q_2, \ldots, q_m]$ be an increasing sequence of prime divisors of $p^n - 1$. If $m \leq 2k(m) + 1$ then Lemma 3.3 applies for $q_m \geq 8r^2$, and Proposition 3.4 applies for $q_m < 8r^2$.

Suppose now that $m \geq 2k(m) + 2$. Then, by Lemma 3.1, we have

$$d(k(m) + 1, m) > 1 + c_r(k(m) + 1, m).$$

Let $s = q_1 q_2 \cdots q_{k(m)}$ and $t = q_{k(m)+1} \cdots q_m$. Then we have $2\phi(t)/t = d(k(m) + 1, m)$, and

$$c_r(k(m) + 1, m) = 2r \cdot \sqrt{\frac{q_1 q_2 \cdots q_{k(m)}}{q_{k(m)+1} q_{k(m)+2} \cdots q_m}} = \frac{2rs}{\sqrt{q_1 q_2 \cdots q_m}} \geq \frac{2rs}{\sqrt{p^n - 1}}.$$

Since s is even and $r(p - 1) \geq rs \geq 3$, we may apply [5, Lemma 6] to see that

$$\frac{(rs - 2)\sqrt{p}}{p - 1} \leq \frac{rs}{\sqrt{p - 1}}.$$

It follows that

$$\frac{2\phi(t)}{t} = d(k(m)+1,m) \geq 1 + c_r(k(m)+1,m) \geq 1 + \frac{2rs}{\sqrt{p^n}-1}$$
$$\geq 1 + \frac{(rs-2)\sqrt{p^n}}{p^n-1} + \frac{rs+2}{p^n-1}.$$

(Note that the last inequality holds since $p^n \geq 7$.) □

Remark 3.6. For the case $r = 2$, the number 103 was missed to discuss in [5, Theorem 3], but the result is still true. With the help of a computer, S_2' may be restricted to S_2, see below.

Remark 3.7. We are sure that Theorem 1.1 has lots of applications. For instance, in a paper of Kutnar, Marusic and the first author [4], the case for $r = 4$ in Theorem 1.1 (with an independent proof) is used to prove the existence of Hamilton cycles in vertex-transitive graphs of order a product of two primes.

Table 1: The Sets S_r.

Set	Degree	orders of the finite fields
S_2	2	3, 5, 7, 9, 11, 13, 19, 25, 31
S_3	3	3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 49, 61, 67, 71, 73, 79, 97, 103, 109, 121, 127, 139, 151, 157, 163, 169, 181, 199, 211, 223, 229, 241, 271, 307, 313, 331, 337, 361, 379, 397, 421, 463, 541, 547, 571, 601, 631, 661, 691, 751, 841, 991, 1021, 1051, 1171, 1321, 1471, 1681, 2311, 2731
S_4	4	3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 81, 97, 101, 103, 109, 121, 127, 131, 139, 151, 157, 163, 169, 181, 193, 199, 211, 223, 229, 241, 271, 277, 281, 283, 289, 307, 313, 331, 337, 343, 349, 361, 367, 373, 379, 397, 409, 421, 457, 463, 529, 541, 547, 571, 601, 613, 625, 631, 661, 673, 691, 727, 751, 757, 811, 841, 859, 883, 961, 967, 991, 1009, 1021, 1051, 1093, 1171, 1201, 1231, 1291, 1321, 1381, 1471, 1531, 1681, 1951, 2311, 2521, 2731, 3361, 3571, 4621, 4831
S_5	5	3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 81, 83, 89, 97, 101, 103, 109, 113, 121, 127, 131, 139, 151, 157, 163, 169, 181, 191, 193, 199, 211, 223, 229, 241, 271, 277, 281, 283, 289, 307, 313, 331, 337, 343, 349, 361, 367, 373, 379, 397, 409, 421, 433, 439, 457, 463, 487, 491, 499, 523, 529, 541, 547, 571, 577, 601, 607, 613, 619, 625, 631, 643, 661, 673, 691, 709, 727, 733, 739, 751, 757, 811, 829, 841, 859, 883, 911, 919, 937, 961, 967, 991, 1009, 1021, 1051, 1093, 1123, 1171, 1201, 1231, 1291, 1303, 1321, 1327, 1381, 1429, 1471, 1483, 1531, 1597, 1621, 1681, 1723, 1741, 1801, 1831, 1849, 1861, 1933, 1951, 2011, 2131, 2143, 2161, 2221, 2251, 2281, 2311, 2341, 2371, 2521, 2551, 2731, 2851, 2971, 3061, 3121, 3301, 3361, 3511, 3571, 3631, 4201, 4621, 4831, 5041, 6091, 8191, 9241, 11551

S_6	6	3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 81, 83, 89, 97, 101, 103, 107, 109, 113, 121, 125, 127, 131, 137, 139, 151, 157, 163, 169, 181, 191, 193, 197, 199, 211, 223, 229, 239, 241, 251, 271, 277, 281, 283, 289, 307, 311, 313, 331, 337, 343, 349, 361, 367, 373, 379, 397, 409, 421, 433, 439, 457, 463, 487, 491, 499, 523, 529, 541, 547, 571, 577, 601, 607, 613, 619, 625, 631, 643, 661, 673, 691, 701, 709, 727, 733, 739, 751, 757, 769, 787, 811, 823, 829, 841, 853, 859, 877, 883, 907, 911, 919, 937, 961, 967, 991, 997, 1009, 1021, 1033, 1051, 1063, 1069, 1093, 1117, 1123, 1129, 1171, 1201, 1231, 1249, 1291, 1303, 1321, 1327, 1369, 1381, 1429, 1453, 1471, 1483, 1531, 1597, 1621, 1681, 1723, 1741, 1783, 1801, 1831, 1849, 1861, 1933, 1951, 2011, 2017, 2131, 2143, 2161, 2221, 2251, 2269, 2281, 2311, 2341, 2371, 2401, 2437, 2521, 2551, 2671, 2731, 2791, 2851, 2857, 2971, 3001, 3061, 3121, 3181, 3271, 3301, 3331, 3361, 3391, 3481, 3511, 3541, 3571, 3631, 3691, 3697, 3721, 4201, 4591, 4621, 4831, 4951, 5041, 5281, 5881, 6007, 6091, 6271, 6301, 7351, 7411, 7561, 7591, 8191, 8581, 9241, 9661, 9871, 10711, 11551, 11971, 16381, 18481
S_7	7	3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 81, 83, 89, 97, 101, 103, 107, 109, 113, 121, 125, 127, 131, 137, 139, 149, 151, 157, 163, 167, 169, 173, 179, 181, 191, 193, 197, 199, 211, 223, 229, 239, 241, 243, 251, 271, 277, 281, 283, 289, 307, 311, 313, 331, 337, 343, 349, 361, 367, 373, 379, 397, 401, 409, 421, 431, 433, 439, 457, 461, 463, 487, 491, 499, 521, 523, 529, 541, 547, 571, 577, 601, 607, 613, 619, 625, 631, 643, 661, 673, 691, 701, 709, 727, 733, 739, 751, 757, 769, 787, 811, 823, 829, 841, 853, 859, 877, 883, 907, 911, 919, 937, 961, 967, 991, 997, 1009, 1021, 1033, 1039, 1051, 1063, 1069, 1087, 1093, 1117, 1123, 1129, 1153, 1171, 1201, 1213, 1231, 1237, 1249, 1279, 1291, 1297, 1303, 1321, 1327, 1331, 1369, 1381, 1399, 1423, 1429, 1453, 1471, 1483, 1489, 1531, 1549, 1567, 1597, 1621, 1657, 1681, 1723, 1741, 1783, 1801, 1831, 1849, 1861, 1873, 1933, 1951, 2011, 2017, 2029, 2053, 2113, 2131, 2143, 2161, 2179, 2221, 2251, 2269, 2281, 2311, 2341, 2347, 2371, 2377, 2401, 2437, 2521, 2551, 2647, 2671, 2689, 2707, 2731, 2791, 2851, 2857, 2971, 3001, 3037, 3061, 3067, 3109, 3121, 3181, 3271, 3301, 3319, 3331, 3361, 3391, 3433, 3481, 3511, 3541, 3571, 3613, 3631, 3691, 3697, 3721, 3823, 3931, 4021, 4051, 4111, 4159, 4201, 4231, 4261, 4441, 4561, 4591, 4621, 4651, 4831, 4951, 5041, 5101, 5281, 5521, 5701, 5851, 5881, 6007, 6091, 6121, 6241, 6271, 6301, 7351, 7411, 7561, 7591, 7921, 8191, 8581, 8821, 8971, 9241, 9661, 9871, 10501, 10711, 11131, 11551, 11971, 12391, 14281, 16381, 17851, 18481, 21841, 25411, 43891

S_8	8	3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 81, 83, 89, 97, 101, 103, 107, 109, 113, 121, 125, 127, 131, 137, 139, 149, 151, 157, 163, 167, 169, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 243, 251, 271, 277, 281, 283, 289, 307, 311, 313, 331, 337, 343, 349, 361, 367, 373, 379, 397, 401, 409, 419, 421, 431, 433, 439, 457, 461, 463, 487, 491, 499, 521, 523, 529, 541, 547, 571, 577, 601, 607, 613, 617, 619, 625, 631, 643, 661, 673, 691, 701, 709, 727, 733, 739, 751, 757, 761, 769, 787, 811, 823, 829, 841, 853, 859, 877, 881, 883, 907, 911, 919, 937, 961, 967, 991, 997, 1009, 1021, 1033, 1039, 1051, 1063, 1069, 1087, 1093, 1117, 1123, 1129, 1153, 1171, 1201, 1213, 1231, 1237, 1249, 1279, 1291, 1297, 1303, 1321, 1327, 1331, 1369, 1381, 1399, 1423, 1429, 1447, 1453, 1459, 1471, 1483, 1489, 1531, 1543, 1549, 1567, 1579, 1597, 1609, 1621, 1627, 1657, 1663, 1669, 1681, 1693, 1699, 1723, 1741, 1747, 1753, 1759, 1777, 1783, 1789, 1801, 1831, 1849, 1861, 1867, 1873, 1879, 1933, 1951, 1993, 1999, 2011, 2017, 2029, 2053, 2089, 2113, 2131, 2143, 2161, 2179, 2209, 2221, 2251, 2269, 2281, 2311, 2341, 2347, 2371, 2377, 2381, 2401, 2437, 2521, 2551, 2647, 2671, 2689, 2707, 2731, 2791, 2809, 2851, 2857, 2887, 2971, 3001, 3037, 3061, 3067, 3109, 3121, 3163, 3169, 3181, 3271, 3301, 3319, 3331, 3361, 3391, 3433, 3481, 3499, 3511, 3529, 3541, 3571, 3613, 3631, 3691, 3697, 3721, 3739, 3823, 3907, 3931, 4021, 4051, 4111, 4159, 4201, 4231, 4243, 4261, 4327, 4441, 4561, 4591, 4621, 4651, 4663, 4789, 4801, 4831, 4861, 4951, 4999, 5011, 5041, 5101, 5281, 5431, 5521, 5581, 5641, 5701, 5791, 5821, 5851, 5881, 6007, 6091, 6121, 6151, 6211, 6241, 6271, 6301, 6361, 6451, 6469, 6661, 6841, 6961, 7351, 7411, 7561, 7591, 7921, 8161, 8191, 8581, 8779, 8821, 8971, 9241, 9661, 9871, 9901, 10501, 10711, 11131, 11311, 11551, 11971, 12211, 12391, 12541, 12601, 13441, 14071, 14281, 15121, 15331, 15541, 16381, 17161, 17851, 18061, 18481, 19321, 19531, 21841, 24571, 25411, 32341, 34651, 43891, 46411, 51871

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 5 (2022) #P3.15
https://doi.org/10.26493/2590-9770.1395.a37
(Also available at http://adam-journal.eu)

Finite and infinite vertex-transitive cubic graphs and their distinguishing cost and density*

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Dedicated to Marston Conder on the occasion of his 65th birthday.

Received 3 December 2020, accepted 20 May 2022, published online 30 August 2022

Abstract

A set S of vertices in a graph G with nontrivial automorphism group is *distinguishing* if the identity mapping is the only automorphism that preserves S as a set. If such sets exist, then their minimum cardinality is the *distinguishing cost* $\rho(G)$ of G. A closely related concept is the *distinguishing density* $\delta(G)$. For finite G it is the quotient of $\rho(G)$ by the order of G.

We consider connected, vertex-transitive, cubic graphs G and show that either $\rho(G) \leq 5$ or $\rho(G) = \infty$ and $\delta(G) = 0$ if G has one or three arc-orbits, or two arc-orbits and vertex-stabilisers of order at most 2.

For the case of two arc-orbits and vertex stabilizers of order > 2 we show the existence of finite graphs with $\rho(G) > 5$ and infinite graphs with $\delta(G) > 0$.

*We are deeply indebted to the reviewers for their insightful comments that led to numerous expository and mathematical changes. To one reviewer with a particularly detailed report we also owe a major mathematical correction. Likewise we thank Gabriel Verret for having drawn our attention to Split Praeger–Xu graphs, and the Austrian Science Fund for support of the second and fourth author (FWF: F5505-N26 and Y-901 for the second and FWF: W1230 for the fourth).

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We also prove that two well known results about finite, vertex-transitive, cubic graphs hold without the finiteness condition and construct infinitely many cubic GRRs.

Keywords: Distinguishing number, distinguishing cost, vertex-transitive cubic graphs, automorphisms, infinite graphs, asymmetrizing set.

Math. Subj. Class.: 05C15, 05C10, 05C25, 05C63, 03E10.

1 Introduction

This paper is concerned with automorphism breaking of graphs by vertex colorings. Following Albertson and Collins [1], we call a vertex coloring of a graph G *distinguishing* if the identity is the only automorphism of G that preserves it. The smallest number of colors needed is the *distinguishing number* $\mathrm{D}(G)$ of G. One says such a coloring *breaks* the automorphisms of G. When $\mathrm{D}(G) = 2$ each of the two colors induces a set of vertices which is preserved only by the identity automorphism. We call such sets *distinguishing*, but also use the term *asymmetrizing*, which was introduced in 1977 by Babai [2], when referring to results predating Albertson and Collins' paper [1].

The cardinality of a smallest distinguishing set of a graph G is the 2-*distinguishing cost*. It was introduced by Boutin [4] in 2008 and denoted $\rho(G)$. Clearly $0 < \rho(G) \leq |V(G)|/2$. Although we cannot talk of the cost unless we already know that $\mathrm{D}(G) = 2$, when it is clear from the context, we refer to $\rho(G)$ as the *distinguishing cost*, or simply as the *cost*, without adding that $\mathrm{D}(G) = 2$.

For 2-distinguishable graphs G the *distinguishing density* $\delta(G)$, or simply the *density*, is defined in Section 3. For finite G it is the quotient $\rho(G)/|V(G)|$.

Note that $\mathrm{D}(G)$ is 1 for asymmetric graphs and 2 for almost all other finite graphs, because almost all finite graphs that are not asymmetric have just one automorphism, a transposition of two vertices. Such graphs can be distinguished by coloring one of the two vertices black, and all other vertices white. This means that $\rho(G) = 1$ for almost all finite graphs that are not asymmetric.

Here we investigate the distinguishing cost and density of graphs of maximum valence 3, which we call *subcubic*. This is a rich class of graphs that is fully classified with respect to the distinguishing number, see [16], but little is known about its cost and density.

In Section 2 we construct finite, connected, subcubic graphs, whose costs take on numerous values between 1 and $|V(G)|/2$, and in Section 3 infinite, connected graphs of maximum valence 3 with various densities between 0 and $1/2$. With the exception of the obvious bounds $1 \leq \rho \leq |V(G)|/2$ for the cost, and $0 \leq \delta \leq 1/2$ for the density, there seem to be few restrictions on the values.

Beginning with Section 4 we focus on connected, vertex-transitive, cubic graphs. For them the situation is completely different and the main topic of interest. Hence, after Section 3 all graphs will be connected, vertex-transitive and cubic, unless it is otherwise stated or clear from the context.

For their investigation we classify them by the number of arc-orbits and treat the classes separately. *Arcs* are ordered pairs (u, v) of adjacent vertices u, v in G, and the *orbit of an*

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arc (u, v) under the action of $\mathrm{Aut}(G)$ is the set

$$O((u, v)) = \{(\alpha(u), \alpha(v)) \mid \alpha \in \mathrm{Aut}(G)\}.$$

Recall that to any two vertices u, w of a vertex-transitive graphs there is an $\alpha \in \mathrm{Aut}(G)$ such that $w = \alpha(u)$. Therefore any arc of the form (w, z) is in the orbit of $(u, \alpha^{-1}(z))$. If G is cubic, then there are only three possibilities for $\alpha^{-1}(z)$, and thus the number of arc-orbits in a vertex-transitive, cubic graph is $1, 2$ or 3.

A closely related, but weaker concept, is that of *edge-orbits*. One says two edges uv and xy are in the same edge-orbit if there exists an automorphism α that maps the unordered pair $\{u, v\}$ into the unordered pair $\{x, y\}$. The relationship between arc- and edge-orbits is described in Section 4.

Graphs with only one arc-orbit are called *arc-transitive*. Four of them are not 2-distinguishable. All other arc-transitive graphs G are 2-distinguishable with cost ≤ 5 unless G is the infinite 3-valent tree, which has infinite cost and zero density; see Theorem 6.1. If G has three arc-orbits, then each automorphism of G that fixes a vertex is the identity, so $\rho(G) = 1$.

For graphs with two arc-orbits we will show in Corollary 4.2 that one of the orbits consists of pairs (u, v) and (v, u), where the edges uv are independent and meet each vertex of G. Because *complete matchings* of a graph G are defined as sets of independent edges that meet each vertex of G, we call this orbit the *matching orbit* and its edges *matching edges*.

If one removes the matching edges, but not their endpoints from G, then the vertices in the remaining graph have valence 2. This implies that its connected components are cycles or 2-sided infinite paths. Because G is vertex-transitive, all components must be either 2-sided infinite paths or cycles of the same lengths. We call this orbit the *cycle-orbit*.

Let uv be a matching edge in a graph with two arc-orbits. Then u, v have no common neighbors unless G is the K_4, which is not 2-distinguishable. If x, y are the neighbors of u different from v, and w, z the neighbors of v different from u, then Figure 1 depicts the subgraph of G consisting of the edges ux, yx, uv, vw and vz. It need not be induced unless G has neither 3- nor 4-cycles. If there are 3-cycles, then they must be in the cycle-orbit and then both uxy and vwz are 3-cycles. If there are 4-cycles, then there may be a pair of independent edges between the sets $\{x, y\}$ and $\{w, z\}$. It is also possible that u and v are in the same cycle of the cycle-orbit.

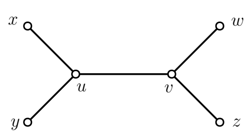

Figure 1: Edges incident with a matching edge uv in a graph with two arc-orbits.

The pairs of arcs $(u, x), (u, y)$ and $(v, w), (v, z)$ are in the cycle-orbit. Hence there exists an $\alpha \in \mathrm{Aut}(G)$ that fixes u and interchanges x, y, and a $\beta \in \mathrm{Aut}(G)$ that fixes v and interchanges w, z. If each automorphism that fixes u and interchanges x, y also interchanges w, z, then we call G *rigid*, and otherwise *flexible*. Note that G is rigid exactly when the order of the group induced by $\mathrm{Aut}(G)$ on the subgraph formed by a matching edge and its four incident edges is 2.

Other important concepts are the *girth* and the *motion* of a graph G. The girth $g(G)$ is the size of a smallest cycle of G, and the motion $\mathrm{m}(G)$ is the smallest number of vertices moved by any non-trivial automorphism of G.

Our results pertaining to cost and density and some of the ensuing questions can be summarized as follows:

Let G be a connected, vertex-transitive, cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph.

1. *If G has three arc-orbits, then G has trivial vertex stabilizers and $\rho(G) = 1$.*

2. *If G is arc-transitive, then $\rho(G) \leq 5$, except for the infinite cubic tree, which has infinite distinguishing cost and density 0. See* Theorem 6.1.

   For the existence of certain infinite, arc-transitive cubic graphs of girth 6 see Question 6.9.

3. *If G is rigid with two arc-orbits, then $\rho(G) \leq 3$, see* Theorem 7.1.

4. *If G is finite, flexible with two arc-orbits and of girth 3, then $\rho(G) \leq 3$. See* Theorem 8.1.

   For the existence of such graphs compare Proposition 8.2.

5. *For two arc-orbit, flexible graphs G of girth 4 we construct families of connected finite graphs with cost > 5 and infinite graphs of density $\delta(G) = 1/\mathrm{m}(G) > 0$.*

   For both families we have $\delta(G) \leq 1/4$. See Theorems 9.10 *and* 9.11.

   We do not know whether other connected, two arc-orbit, flexible graphs G of girth ≥ 4 with positive density or finite cost > 5 exist. See Question 9.12 and Lemma 9.3.

   If they do not exist, then all finite or infinite, vertex-transitive, cubic graphs on at least 20 vertices have density $\leq 1/4$. See Questions 2.1 and 3.5.

6. *We also construct infinite, two arc-orbit, flexible, cubic graphs G of all girths with $\rho(G) = \infty$ and $\delta(G) = 0$. See* Theorem 7.6 *and* Lemma 9.3.

   We do not know whether two arc-orbit, flexible, cubic graphs G, whose cycle-orbits consist of 2-sided infinite paths, exist. See Question 7.5.

Furthermore, we prove a number of other results. In Section 2 we provide examples of infinite graphs with finite and infinite distinguishing cost. In Section 3 we define and give examples of the distinguishing density of infinite graphs. In these sections the graphs are not necessarily cubic or vertex-transitive. In Section 4 we show that edge-transitivity implies arc-transitivity not only for finite, but also for infinite, vertex-transitive, cubic graphs. Section 5 treats vertex-transitive, cubic graphs with three arc-orbits, that is, vertex-transitive graphs with trivial vertex stabilizers. Such graphs are called Graphical Regular Representations, GRRs for short. In Section 6 Theorem 6.2 extends a result about the existence of connected, vertex-transitive, cubic graphs of girth at most 5 to infinite graphs. Section 9 is concerned with flexible cubic graphs of girth 4. As a byproduct we construct a class of infinitely many cubic GRRs with two edge-orbits. The paper ends with a new characterization of Split Praeger–Xu graphs.

We conclude the introduction with the remark that Babai's paper [2] was not the only one on automorphism breaking that predates Albertson and Collins [1]. After Babai, it was notably Polat and Sabidussi [24, 25] and Polat [22, 23], who published deep and interesting results on the asymmetrization of graphs of any cardinality.

Moreover, the concept of asymmetrization, or distinguishing, extends to groups of permutations on a set. As most results on asymmetrizing sets of groups are also relevant for graphs, let us mention that Gluck [13] showed in 1983 that permutation groups of odd order can be asymmetrized by two colors, and that Cameron, Neumann, and Saxl [6] proved in 1984 that all but finitely many primitive permutation groups other than A_n, S_n can be asymmetrized by two colors. In 1997 Seress [28] classified the remaining ones, taking recourse to the classification of finite simple groups.

2 Cost

We continue with examples on bounds on the distinguishing cost for finite graphs and show that infinite graphs can have finite or infinite cost.

It is important to keep in mind that the distinguishing cost is only defined for graphs that are 2-distinguishable. In other words, the results on cost and density only hold for such graphs. For cubic graphs this is not a strong restriction, because the only vertex-transitive cubic graphs that are not 2-distinguishable are the K_4, the $K_{3,3}$, the cube and the Petersen graph, where the K_4 has distinguishing number 4, and the others 3.

If the distinguishing number of a graph G is 2, then $V(G)$ can be partitioned into two sets V_1, V_2, where the stabilizer of either one is the trivial subgroup. This means, if $\alpha \in \mathrm{Aut}(G)$ and $\alpha(V_i) = V_i$ for $i = 1$ or $i = 2$, then $\alpha = \mathrm{id}$. Recall from the introduction that either of the sets V_1 or V_2 is called distinguishing, and that the smallest possible size of such a set is the distinguishing cost $\rho(G)$ of G.

To fix ideas we shall always choose a minimum distinguishing set of G, color its vertices black and all others white. Thus $\rho(G)$ is the minimal number of black vertices needed to break all automorphisms.

Clearly $0 < \rho(G) \leq |V(G)|/2$. The upper bound can easily be attained. The simplest example is a single edge. To construct larger graphs with $\rho(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ we consider binary trees:

Let B_k be the binary tree of height k. B_k is defined as a rooted tree, whose root has valence 2, whose vertices of distance k from the root are leaves, and where all other vertices have valence 3. B_k has $2^{k+1} - 1$ vertices, $\rho(B_k) = 2^k - 1$ and $\delta(B_k) = 1/2 - 1/(2^{k+1} - 1) = \lfloor |V(G)|/2 \rfloor$. By attaching a path to the root one obtains a graph with smaller density. It can be made arbitrarily small.

For cost $|V(G)|/2$ we consider the graph G_k consisting of two binary trees B_k, whose roots are connected by an edge. Clearly $\rho(G_k) = 2^{k+1} - 1 = |V(G_k)|/2$.

If the graphs are vertex-transitive, then we have stricter bounds. In Theorem 6.1 we show that $\rho(G) \leq 5$ for finite, connected, arc-transitive, cubic graphs that are different from K_4, $K_{3,3}$, the cube and the Petersen graph. Hence, for finite arc-transitive graphs G on at least 20 vertices $\delta(G) \leq 1/4$.

There are 24 connected, vertex-transitive graphs of order < 20, see [27]. As a brief application of results of the paper we show that exactly five of these 24 graphs have density $> 1/4$: two are Möbius ladders, two circular ladders, and the fifth is the Heawood graph. A *k-Möbius ladder* consists of a cycle of length $2k$, together with k diagonals that connect opposite vertices of the cycle. The k diagonals are the rungs of the ladder.

A *circular ladder* or *k-ladder* is a prism over a k-cycle, that is, the Cartesian product of C_k by K_2. For $k \leq 9$ they have < 20 vertices. If they are 2-distinguishable, their cost is 3 by Theorem 7.1, and thus the density is $3/(2k)$. Since we are interested in density $> 1/4$ we can restrict attention to $k \leq 5$.

The 2-Möbius ladder is the K_4, the 3-Möbius ladder the $K_{3,3}$, the 4-Möbius ladder has density $3/8$ and the 5-Möbius ladder density $3/10$. The 2-ladder does not exist, the 3-ladder is the prism over a triangle and has density $1/2$, the 4-ladder is the cube and the 5-ladder has density $3/10$.

For the Heawood graph, which has girth 6 and 14 vertices we show in Section 6 that its cost is at most five. However, it is relatively easy to see that its cost is at least 4, and hence its density $> 1/4$.

The Petersen graph, the Heawood graph, the eight Möbius ladders, and the seven prisms of order < 20 account for 17 vertex-transitive cubic graphs of order < 20.

Two of the remaining seven graphs have girth 3. If one contracts their triangles to single vertices one obtains the K_4 and the $K_{3,3}$. By Theorem 8.1 the costs of the uncontracted graphs are 3 and the densities $1/4$ and $1/6$. Two other graphs are the crossed 3- and 4-ladders of densities $1/4$, see Theorem 9.11. The remaining three graphs have girth 6. One is the Pappus graph. It has 18 vertices, and its cost is 3 by the remark after Lemma 6.8. Another graph on 18 vertices has cost 1. It is the smallest GRR, i.e. graph with three arc-orbits, see Section 5. The last graph has 16 vertices, and thus cannot have three arc-orbits. By Corollary 4.2 it also cannot have two arc-orbits. Therefore it is arc-transitive. By Lemma 6.8 and the following remarks it has cost ≤ 3, because it is not the Heawood graph.

Question 2.1. Are there finite, connected, vertex-transitive, cubic graphs on at least 20 vertices with distinguishing densities $> 1/4$?

If such graphs exist, then they must be flexible by Lemma 5.1, Theorem 6.1 and Theorem 7.1.

The cost can be finite for infinite graphs. This was studied in [3], where it was shown that the cost $\rho(G)$ of connected, locally finite, infinite graphs G is finite if and only if $\mathrm{Aut}(G)$ is countable. (In this statement it is not needed to explicitly assume 2-distinguishability, because all locally finite graphs with countable automorphism group are 2-distinguishable, see [18].)

An example for an infinite graph with countable automorphism group and finite cost is the infinite ladder of Figure 2. It has distinguishing cost 3, as is easily verified.

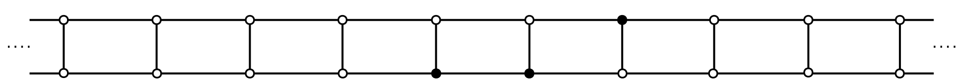

Figure 2: The infinite ladder with a distinguishing coloring.

Contrariwise, the closely related *chain of quadrangles* Q of Figure 3, also called *infinite crossed ladder*, is an example for an infinite, connected graph with uncountable automorphism group, and hence infinite distinguishing cost. Q can be formally defined as a graph consisting of the 2-sided infinite paths $\dots u_{-2}u_{-1}u_0u_1u_2 \dots$ and $\dots v_{-2}v_{-1}v_0v_1v_2 \dots$, with the edges $u_{2n}v_{2n+1}, v_{2n}u_{2n+1}$, where $n \in \mathbb{Z}$.

To see that $\mathrm{Aut}(Q)$ is uncountable, observe that we obtain an automorphism of Q for any integer n by simultaneously interchanging u_{2n-1} with v_{2n-1}, and u_{2n-1} with u_{2n}, while fixing all other vertices. This automorphism interchanges the pair of matching edges $u_{2n-1}u_{2n}$ and $v_{2n-1}v_{2n}$. Because Q has infinitely many such pairs of matching edges, and because the edges in each pair can be interchanged independently of the other pairs, $\mathrm{Aut}(Q)$ has at least $2^{\aleph_0}$ elements, and is thus uncountable.

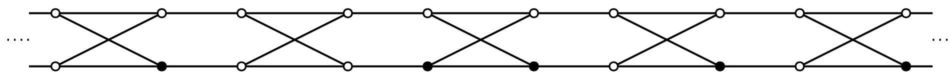

Figure 3: The chain of quadrangles Q with a distinguishing coloring.

A distinguishing 2-coloring is displayed in the figure. The black vertices are u_0 and u_{2n+1}, where n is an integer different from -1. Because u_0, u_1 are the only adjacent black vertices each automorphism must map the set $\{u_0, u_1\}$ into itself. As there is a black vertex of distance 2 from u_1, namely u_3, but no black vertex of distance 2 from u_0, the vertices u_0, u_1, u_3 are fixed individually. It is easy to see that this implies that all four-cycles are fixed setwise, and thus also all pairs of matching edges $u_{2n-1}u_{2n}$ and $v_{2n-1}v_{2n}$. Each such pair has exactly one black vertex, hence all of its vertices must be fixed. Because $V(Q)$ is partitioned by the sets of endpoints of these pairs of matching edges the coloring is distinguishing.

3 Density

To define the *distinguishing density* we follow [17] and begin with the density of sets of vertices. Let S be a set of vertices of a graph G, $v \in G$, and $B_G(v,n) = \{w \in G : d(v,w) \leq n\}$ be the ball of radius n with center v. If no ambiguity arises, we also write $B(v,n)$ for $B_G(v,n)$. Then

$$\delta_v(S) = \limsup_{n\to\infty} \frac{\mid B(v,n) \cap S \mid}{\mid B(v,n) \mid}$$

is the *density* of S at v. If $\delta_v(S)$ exists for all vertices, which is the case for locally finite graphs, then

$$\delta(S) = \sup\{\delta_v(S) : v \in V(G)\}$$

is the density of S in G. Finally,

$$\delta(G) = \inf\{\delta(S) \mid S \text{ is a distinguishing set of } G\}$$

is the *distinguishing density* of G.

Note that $\delta(G) = \rho(G)/|V(G)|$ for finite graphs and zero for infinite graphs with finite distinguishing cost.

Under certain conditions all values of $\delta_v(G)$, $v \in V(G)$, are equal. Then $\delta(G)$ is equal to $\delta_v(G)$ for arbitrarily chosen $v \in V(G)$. We have two such conditions, one for density zero, and a stricter one for positive density. First the condition for density zero see [17, Lemma 1].

Lemma 3.1. *Let G be a connected graph and $v, w \in V(G)$. Suppose there is a constant c such that for all $n \in \mathbb{N}$ we have $|B(w, n+1)| < c|B(w, n)|$. If G has distinguishing density zero at v, then G has distinguishing density zero.*

For an example of a 2-distinguishable graph that does not have distinguishing density zero, but nevertheless distinguishing density zero at some vertex v, see [17]. That graph does not satisfy the conditions of the lemma.

The infinite tree T_k, where each vertex has degree k, is an example of a graph with distinguishing density zero. For the proof and for many other examples of graphs with distinguishing density zero we refer to [17].

For positive distinguishing density we have the following lemma.

Lemma 3.2. *Let G be a connected graph, $S \subseteq V(G)$, $v \in V(G)$, and*

$$\limsup_{n\to\infty} \frac{|B(v,n)|}{|B(v,n+1)|} = 1. \tag{3.1}$$

If there is a vertex u in G with $\delta_u(S) = a \geq 0$, then $\delta_w(S) = a$ for all $w \in V(G)$.

Proof. We first observe that (3.1) implies that

$$\limsup_{n\to\infty} \frac{|B(v,n)|}{|B(v,n+d)|} = 1, \tag{3.2}$$

for all natural numbers d.

Let $\delta_v(S) = a$, $w \in G$, and $d = d(v, w)$. Clearly

$$|B(v,n)| \leq |B(w,n+d)| \leq |B(v,n+2d)|,$$

and hence

$$\limsup_{n\to\infty} \frac{|B(v,n)|}{|B(v,n+2d)|} \leq \limsup_{n\to\infty} \frac{|B(w,n+d)|}{|B(v,n+2d)|} \leq \limsup_{n\to\infty} \frac{|B(v,n+2d)|}{|B(v,n+2d)|} = 1.$$

By (3.2)

$$\limsup_{n\to\infty} \frac{|B(w,n+d)|}{|B(v,n+2d)|} = 1. \tag{3.3}$$

Clearly

$$|B(v,n) \cap S| \leq |B(w,d+n) \cap S| \leq |B(v,n+2d) \cap S|,$$

and hence

$$\frac{|B(v,n) \cap S|}{|B(v,n)|} \frac{|B(v,n)|}{|B(v,n+2d)|} \leq \frac{|B(w,n+d) \cap S|}{|B(w,n+d)|} \frac{|B(w,n+d)|}{|B(v,n+2d)|} \leq \frac{|B(v,n+2d) \cap S|}{|B(v,n+2d)|}.$$

By (3.2) the $\limsup$ of the term on the left is equal to the $\limsup$ of the term on the right, and by (3.3)

$$\delta_w(S) = \limsup_{n\to\infty} \frac{|B(w,n+d) \cap S|}{|B(w,n+d)|} = \limsup_{n\to\infty} \frac{|B(v,n) \cap S|}{|B(v,n)|} = \delta_v(S)$$

for all $w \in V(G)$. As we assume the existence of a vertex $u \in V(G)$ with $\delta_u(S) = a$, we infer that $\delta_w(S) = a$ for all $w \in V(G)$, i.e. $\delta(G) = a$. □

Lemma 3.2 will be useful in Subsection 3.1. Interestingly we do not need it for the chain of quadrangles.

Lemma 3.3. *The distinguishing density of the chain of quadrangles Q is $1/4$.*

Proof. In Section 2 we already observed that any distinguishing set S of Q must contain at least one vertex in each pair of matching edges connecting two quadrangles. It is easy to see that this implies that $\delta_v(S) \geq 1/4$ for arbitrary S and v. Hence $\delta(G) \geq 1/4$. If we choose for S the set of black vertices in Figure 3, then $\delta_v(S) = 1/4$ for all v, and therefore the distinguishing density of Q is indeed $1/4$. □

Let us note nonetheless that Q satisfies the conditions of Lemma 3.2, because

$$|B_Q(v, n+1)| = |B_{Q)}(v, n)| + 4$$

for $n \geq 4$.

We wish to add that it is relatively easy to construct graphs with non-zero distinguishing density that are not vertex-transitive, see [17].

3.1 Upper bounds for the density

For infinite graphs the situation is similar. If we take a one-sided infinite path P and connect each of its vertices by an edge to the root of a copy of the binary tree B_k, then we obtain a graph, say H_k, of distinguishing cost $\rho(H_k) = 1/2 - 1/2^{k+1}$. In this way we can reach densities that are arbitrarily close to $1/2$, but not $1/2$.

To reach density $1/2$ we have to be more careful. Let $P_{1/2}$ be the graph obtained from the one-sided infinite path $P = v_1 v_1 v_2 \dots$ as follows: Connect v_1 to the root of B_1 by an edge, then each of v_2, v_3 by an edge to the root of distinct copies of B_2. Continue by connecting v_4, v_5, v_6 to the roots of distinct copies of B_3, and so on.

Lemma 3.4. $\delta(P_{1/2}) = 1/2$.

Proof. Each minimal 2-distinguishing coloring of $P_{1/2}$ leaves P and its neighbors white. The remaining vertices come in pairs $\{u, v\}$ of vertices of equal distance from v_1, where one vertex has to be colored black, and the other white. Let S be the set of black vertices of such a distinguishing coloring. Then $|B(v_1, n) \cap S| = (|B(v_1, n)| - 2n - 1)/2$, and $|B(v, n) \cap S|/|B(v_1, n)| = 1/2 - (2n-1)/(2 \cdot |B(v, n)|)$. The supremum of the latter expression is $1/2$ if the growth of $B(v_1, n)$ is more than linear. This is achieved by our construction. It is not hard to see that $P_{1/2}$ satisfies the conditions of Lemma 3.2. We conclude that $\rho(P_{1/2}) = 1/2$. □

It seems that one can construct connected, subcubic infinite graphs with arbitrary density between 0 and $1/2$ in similar ways.

But we know of no infinite, connected, vertex-transitive, cubic graph with density larger than $1/4$.

Question 3.5. Are there infinite, connected, vertex-transitive, cubic graphs with distinguishing densities larger than $1/4$?

As we already observed after Question 2.1, any examples of such graphs must be flexible.

4 Arc- versus edge-orbits

This section describes the relationship between arc- and edge-orbits, respectively between arc- and edge-transitivity.

Edge-transitive graphs need not be vertex-transitive, as the $K_{2,3}$ shows. But even a vertex-transitive graph G that is also edge-transitive need not be arc-transitive. The existence of such graphs was shown in 1966 by Tutte [33]. He also proved that any finite graph of this type must be regular of even degree. The first examples were given by Bouwer [5], and the smallest graph of this type is the Doyle graph. It is quartic and has 27 vertices, see [15] and [26] .

By Tutte's result each finite, vertex-transitive, cubic graph that is edge-transitive is arc-transitive. In 1989 Thomassen and Watkins [30] extended Tutte's result to infinite graphs of subexponential growth. They proved that each vertex- and edge-transitive graph of odd valence with subexponential growth is 1-transitive. For infinite cubic graphs we do not need the growth condition:

Theorem 4.1. *Let G be a finite or infinite, connected, vertex-transitive, cubic graph. If it is edge-transitive, then it is also arc-transitive.*

Proof. Let G satisfy the assumptions of the Theorem. If it is not arc-transitive, then it has two or three arc-orbits.

Let u be an arbitrary vertex of G with the incident edges uv, uw, ux. Suppose first that G has three arc-orbits. By edge-transitivity there is an $\alpha \in \mathrm{Aut}(G)$ that maps uv into uw and a $\beta \in \mathrm{Aut}(G)$ that maps uw into ux. Because G has three arc-orbits $\alpha(u) = w$, $\alpha(v) = u$, and $\beta(w) = u$, $\beta(u) = x$. Hence $\beta\alpha$ maps the arc (u, v) into the arc (u, x), a contradiction.

Suppose now that G has two arc-orbits, where (u, w) and (u, x) are in the same arc-orbit. Let O_1 be the arc-orbit of (u, v), O_2 the orbit of (u, w), and D be the digraph on $V(G)$ whose arcs are the arcs of O_2. Because G is edge-transitive at least one of the arcs (u, v) or (v, u) must be in O_2. By assumption $(u, v) \notin O_2$, hence $(v, u) \in O_2$. Therefore, in D, the vertex u has one incoming arc and two outgoing arcs. By vertex-transitivity this holds for all vertices of G. But then G cannot be finite, because the total number of incoming arcs in D has to be the same as the total number of outgoing arcs.

Hence G is infinite. It cannot be a tree, otherwise it would be the infinite cubic tree T_3, which has only one arc-orbit. Therefore G has a cycle, say C. By vertex-transitivity we can assume that u is in C. Because, u has two outgoing arcs in D, one of them, say (u, w), must be in C. Continuing this argument one sees that C is a directed cycle in D. By edge-transitivity each arc is in directed cycle in D.

Let C' be a cycle containing (u, x). Clearly both C and C' contain the arc (v, u). Let P be the longest path in $C \cap C'$ that contains (v, u). One of its endpoints is u, let the other one be z. Clearly z has one outgoing arcs in D, and two incoming arcs, which is not possible. □

Corollary 4.2. *Let G be a finite or infinite, connected, vertex-transitive, cubic graph. If it has two arc-orbits, then one of the arc-orbits consists of pairs (u, v), (v, u), where the edges uv are independent and meet each vertex of G.*

Proof. Let G be a finite or infinite, connected, vertex-transitive, cubic graph with two arc-orbits and u be an arbitrary vertex of G with the incident edges uv, uw, ux. As in the proof of Theorem 4.1 we assume that (u, v) is in O_1 and that $(u, w), (u, x)$ are in O_2. In the proof of the theorem the assumption that (v, u) was in O_2 led to a contradiction to the number of arc-orbits of G. Hence, $(v, u) \in O_1$. By vertex-transitivity each vertex $y \in V(G)$ meets exactly two arcs of the form $(y, z), (z, y) \in O_1$. Clearly the corresponding edges yz are independent and meet each vertex of G. □

Corollary 4.3. *Let G be a finite or infinite, connected, vertex-transitive, cubic graph. If it has two arc-orbits, then it has two edge-orbits, but if it has three arc-orbits, it may have two or three edge-orbits.*

Proof. By definition each edge-orbit consists of the undirected arcs of the union of one or more arc-orbits. We have seen that G has only one arc-orbit if G has only one edge-orbit. Hence, if G has two arc-orbits, they must be in different edge-orbits, and thus G has two edge-orbits.

If G has three arc-orbits, then it has two or three edge-orbits, because the case of one edge-orbit is not possible by Theorem 4.1. □

Recall from the introduction that we called O_1 the matching orbit. The other orbit was called cycle-orbit. By definition O_1 and O_2 are arc-orbits, but by the corollary they correspond to edge-orbits. By abuse of language we do not distinguish between O_1 or O_2 as and arc- or edge-orbits. In this sense O_2 is the disjoint union of cycles of the same length or of 2-sided infinite paths.

5 Three arc-orbits

Let G be a connected, vertex-transitive, cubic graph with three arc-orbits. If we fix a vertex v, then all neighbors of v are also fixed. As G is connected, this implies that all vertices of G are fixed if one is fixed. To break its automorphisms it suffices to color one vertex black and leave all others white. The distinguishing cost is 1.

Lemma 5.1. *The distinguishing cost of connected, vertex-transitive, cubic graphs with three arc-orbits is* 1.

Recall that Graphical Regular Representations, or GRRs, are vertex-transitive graphs with trivial vertex stabilizers. They have been widely investigated and although GRRs are abundant, it is interesting to explicitly describe special classes.

From Corollary 4.3 we know that the case of three arc-obits allows two or three edge-orbits. The smallest example of a GRR has 18 vertices. It has two edge-orbits and girth 6. The smallest example for the case of three edge-orbits is the truncated cuboctahedron. It is the skeleton of an Archimedean solid that was already described by Kepler, where each vertex is in one cube, one hexagon and one octagon. For both graphs we refer to a list of cubic GRRs with at most 120 vertices from 1981 by Coxeter, Frucht and Powers, see [8].

A series of infinitely many such graphs was constructed by Godsil in 1983 [14], the smallest of order $19!/2$. In Section 9 we also construct infinitely many such graphs, see Corollary 9.5. The smallest has 48 vertices and is also listed in [8].

Let us note in passing that the same publication lists two GRRs of girth 5, both of them on 110 vertices, whereas there is only one finite 2-distinguishable arc-transitive cubic graph of girth 5, the dodecahedron, and no infinite one. See Theorem 6.2.

We continue with graphs with one and two arc-orbits. Clearly the cost in these cases is at least 2.

6 One arc-orbit

By definition such graphs are arc-transitive. Tutte [31] also calls them *symmetric*, but the notation is not uniform and *symmetric* is also used for edge-transitive graphs that are not arc-transitive. To avoid confusion we only speak of arc-transitive graphs. But we need a refinement of the concept.

Following Tutte [31] we call a sequence of distinct vertices $v_0, v_1, \ldots, v_s \in V(G)$ an *s-arc* if $v_i v_{i-1} \in E(G)$ for $1 \leq i \leq s$, but $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. Then G is *s-arc-transitive* if $\mathrm{Aut}(G)$ is transitive on the set of all s-arcs on G. Moreover, we call G *s-arc-regular* if for any two s-arcs $v_0 v_1 \ldots v_s$ and $w_0 w_1 \ldots w_s$ there is a unique automorphism φ that maps $v_0 v_1 \ldots v_s$ into $w_0 w_1 \ldots w_s$ and respects the order of the vertices.

In this section we shall prove the following theorem.

Theorem 6.1. *Let G be an arc-transitive, cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph. If it has finite girth, then $\rho(G) \leq 5$, otherwise it is the infinite cubic tree T_3, which has infinite distinguishing cost and distinguishing density* 0.

We will break up the proof of Theorem 6.1 into several parts, but begin with the remark that there is only one connected acyclic cubic graph. It is the infinite cubic tree T_3, of which we already mentioned that it has distinguishing density zero.

It thus remains to prove the theorem for graphs with finite girth. We begin with a structure theorem about graphs of girth at most 5. For girth 3 and 4 the theorem is folklore, for finite graphs of graphs of girth 5 it was shown by Glover and Marušić [12]. Here is a concise proof for all cases.

Theorem 6.2. *The only connected, arc-transitive, cubic graphs of girth at most 5 are K_4, $K_{3,3}$ the cube, the Petersen graph and the dodecahedron.*

Proof. Let G be a connected, arc-transitive, cubic graphs of girth at most 5. Because we forbid multiple edges the smallest girth is 3 and the arcs incident to an arbitrary vertex induce a $K_{1,3}$. Let a, b, c be the arcs incident with a vertex v of G. We show first that there is an automorphism α of G that rotates a, b, c. In other words, there is an automorphism α whose cycle representation of its action on $\{a, b, c\}$ is (abc) or (acb). If this were not the case, there would exist automorphisms whose actions on $\{a, b, c\}$ are of the type (ab), (ac), because of arc-transitivity. Then their product $(ab)(ac) = (abc)$ is the desired rotation.

Let x, y, z be the endpoints of a, b, c different from v, and let the notation be chosen such that there exists an automorphism α of G that rotates the arcs a, b, c and their endpoints.

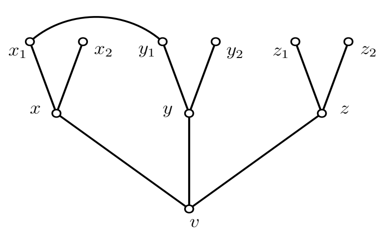

Figure 4: Basic structure for girth 5.

Girth 3. If G has a triangle we can assume without loss of generality that it is vxy. By applying α twice we see that G also contains the edges yz and zx. Hence $G = K_4$.

Girth 5. We first observe that the neighbors of x, y, z that are different from v are distinct. Let them be x_1, x_2, y_1, y_2, z_1, z_2 and vxx_1y_1y a pentagon, see Figure 4.

Rotating clockwise it is seen that there is a pentagon $vy\alpha(x_1)z_1z$, where $\alpha(x_1) \in \{y_1, y_2\}$. Suppose $\alpha(x_1) = y_1$. Notice that we have two pentagons that share two edges. The rotation moves z_1 into a neighbor of x. It must be x_2, because we have no triangles. By further rotations we obtain the 6-cycle $x_1y_1z_1x_2y_2z_2$, see Figure 5. Clearly G is the Petersen graph.

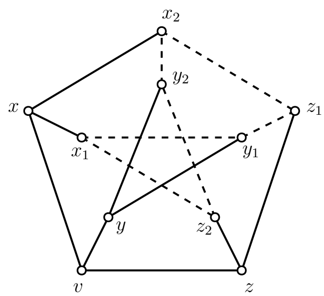

Figure 5: The Petersen graph.

Suppose now that no two pentagons share two edges. Then $\alpha(x_1) = y_2$, which implies $\alpha(x_2) = y_1$. Furthermore, we can choose the notation such that $\alpha(y_1) = z_2$, and $\alpha(y_2) = z_1$. Thus $z_2 = \alpha(y_1) = \alpha^2(x_2)$, and hence $\alpha(z_2) = x_2$ and $\alpha(z_1) = x_1$. By rotation we obtain the pentagons vyy_2z_2z and vzz_1x_2x from vxx_1y_1y. Let H be the union of these pentagons, see Figure 6. Note that v is in three pentagons, hence, by vertex-transitivity, this holds for all vertices. More important, any two incident arcs determine exactly one pentagon.

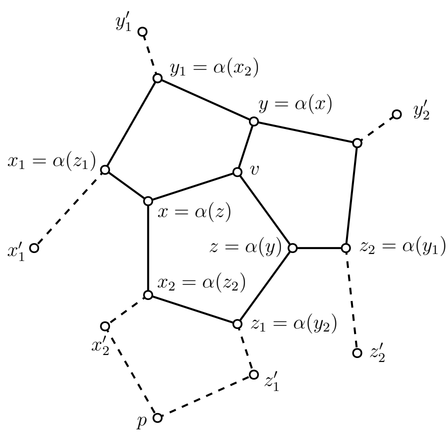

Figure 6: The subgraph H (solid lines).

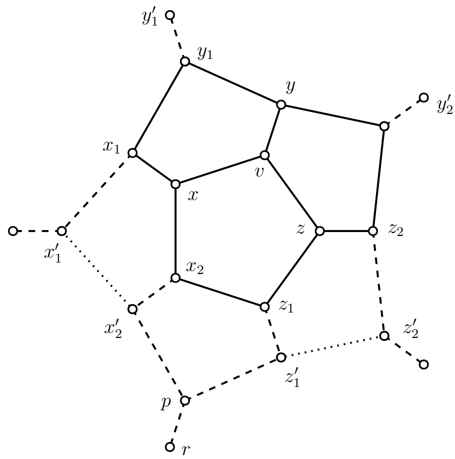

Figure 7: The graph H'.

The vertices of valence 2 in H form the set $S = \{x_1, y_1, y_2, z_2, z_1, x_2\}$. If $u \in S$, let u' be the third neighbor of u. Let $S' = \{x_1', y_1', y_2', z_2', z_1', x_2'\}$. By the girth condition and the condition that two pentagons share only one edge we infer that the sets S' and H are disjoint and that any two vertices of S' distinct.

Now consider x. It is in two pentagons in H, but has to be in a third in G. Neither x_1x_1' nor x_2x_2' can be in the first two pentagons, hence $x_1x_2' \in E(G)$. Similarly we find that $z_1'z_2'$ and $y_1'y_2'$ are in $E(G)$. The new graph, say H', is depicted in Figure 7.

Consider the path $x_2'x_2z_1z_1'$. Its edges must be in a pentagon that is not in H', which is only possible if there exists a vertex, say p, that is adjacent to x_2' and z_1'. Because G has girth 5 the vertex p cannot be a neighbor of x_1', nor of z_2', and because G is cubic it cannot be any vertex of H'.

Now consider the neighbors of x_1', p and z_2' that are not in $V(H') \cup \{p\}$, and denote them by q, r, s. Because of the girth condition and because G is cubic they are pairwise distinct and not in $V(H') \cup \{p\}$.

Finally, because all vertices have to be in three pentagons we find four additional edges that complete the graph to a dodecahedron. □

Corollary 6.3. *The only* 2*-distinguishable, finite or infinite, connected, arc-transitive, cubic graph of girth at most* 5 *is the dodecahedron. Its distinguishing cost is* 3.

Proof. We already know that $K_4, K_{3,3}$, the cube and the Petersen graph are not 2-distinguishable. Hence, by the theorem, the only 2-distinguishable graph of girth at most five is the dodecahedron. It is easily seen that its distinguishing cost is 3. □

For girth > 5 we will invoke a result of Tutte for finite graphs and its extension to infinite graphs by Djokovic and Miller. First the result from [32] for finite graphs.

Theorem 6.4. *Let G be a connected, finite, arc-transitive, cubic graph. Then G is s-arc-regular for some $s \leq 5$.*

And now the extension to infinite graphs from [9].

Theorem 6.5. *Every connected, infinite, arc-transitive, cubic graph is s-arc-regular for some $s \leq 5$ with the exception of the infinite cubic tree.*

We begin with a result for 1-arc-regular graphs.

Lemma 6.6. *Each connected, 1-arc-regular cubic graph has distinguishing cost 2.*

Proof. Let G be a connected, 1-arc-regular cubic graph. Then the action of $\mathrm{Aut}(G)$ on the arcs incident with an arbitrary vertex v cannot have involutions. To see this, let a, b, c be the arcs incident with v, and $\alpha \in \mathrm{Aut}(G)$ induce the involution (cb) on $\{a, b, c\}$. Then $\alpha(a) = a$, which contradicts 1-regularity.

Furthermore, by Theorem 6.2 the girth of G must be at least 5, because the graphs K_4, $K_{3,3}$, the cube, the Petersen graph and the dodecahedron are not 1-arc-regular. We now choose two vertices u, v of distance 2 in G and color them black. Because $g(G) > 4$ there is a unique vertex w that is adjacent to both u and v. Therefore any color preserving automorphism α fixes w. If it interchanges u with v, then it interchanges two arcs incident with w and fixes the third, and thus induces an involution on the arcs incident with w. If α fixes u, then it fixes the arc uw and is the identity by 1-arc-regularity. □

The existence of such graphs is guaranteed by a result of Frucht [11], who provided an example of a 1-arc-regular cubic graph of girth 12 with 432 vertices. It can be embedded in a surface of genus 55 so as to form a map of 108 dodecagons.

For the girth of s-arc-regular cubic graphs we will use the bound

$$2s \leq g(G) + 2 \tag{6.1}$$

from [31].

We first consider girth 7 and then girth 6.

Lemma 6.7. *Let G be a connected, s-arc-regular, cubic graph of girth at least 7. If $s = 1$, then $\rho(G) = 2$, otherwise $\rho(G) \leq 3$, unless $s = 4$ and $g(G) = 7$, then $\rho(G) \leq 4$.*

Proof. Because of arc-transitivity we can invoke Theorems 6.4 and 6.5. They imply that our graphs are s-arc-regular for some $s \leq 5$.

By Lemma 6.6 we can assume that $s > 1$. For $s = 2$ or 3 we choose a path $uxvw$ in G. This is possible because the girth is > 6. We color u, v, w black as visualized in Figure 8. Each color preserving automorphism φ fixes u because it is the only black vertex without black neighbors. As v and w have different distances from u, they are also fixed. Hence φ fixes the s-arcs ux, uxv and $uxvw$, where $s = 2, 3$, respectively. By s-arc-regularity φ is the identity.

Now, let $s = 4$. For girth $g > 7$ we choose a path $uxyvw$ and color u, v and w black as in Figure 8. Then we argue as before to prove that the 4-arc $uxyvw$ is fixed by all color preserving automorphisms. If the girth is 7 this coloring allows that both v and w have distance 3 from u. In this case it suffices to color y black to fix the 4-arc $uxyvw$ by all color preserving automorphisms.

For $s = 5$ we first observe that the girth is at least 8 by Equation (6.1). We choose a path $uxyzvw$ of length 5 and color u, v and w black. If the girth is different from 9 this coloring fixes the 5-arc $uxyzvw$. If the girth is 9, then v, w could be interchanged by color preserving automorphisms. □

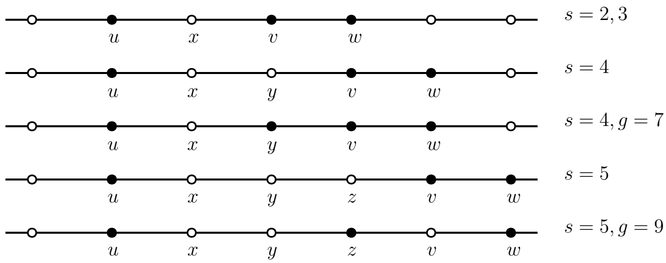

Figure 8: Colorings of s-arcs.

Lemma 6.8. *Let G be a connected, arc-transitive, cubic graph of girth 6. Then G is at most 4-arc-regular. If G is 1-arc-regular, then $\rho(g) = 2$, if G is 2 or 3-arc-regular, then $\rho(G) \leq 3$, otherwise $\rho(G) \leq 5$.*

Proof. That G is at most 4-arc-regular follows from Equation 6.1. When G is 1-arc-regular we invoke Lemma 6.6 and when G is 2- or 3-arc-regular we can use the same coloring as in the proof of Lemma 6.7.

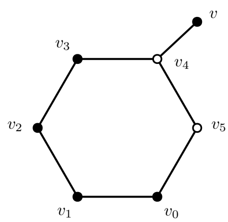

Figure 9: Coloring for $s = 4$ and girth 6.

If G is 4-arc-regular we choose an arbitrary 6-gon $v_0 \cdots v_5$ and color $v_0, \ldots, v_3$ black as well as the neighbor of v_4 that is not in the cycle. Let this neighbor be v, see Figure 9. Clearly v cannot be adjacent to any vertex of the cycle, because G has girth 6. Hence v is fixed by the coloring. Furthermore, v cannot have distance 2 from v_0, otherwise G would have a cycle of length 5. As v has distance 2 from v_3, this implies that the coloring fixes v_3. Invoking the girth condition we see that v_4 is also fixed, because it is on a path of length 2 between the fixed vertices v_3 and v. This fixes the entire 4-arc $v_0v_1v_2v_3v_4$. By 4-arc-regularity this coloring is distinguishing. □

Together with Corollary 6.3, Lemma 6.7 and Theorem 6.5 this also completes the proof of Theorem 6.1.

Let us mention that, except for the interchange of black and white, our coloring for the 4-arc-regular case is the same as that for a graph described in [16, Theorem 7.3].

We also observe that there exists only one finite, connected, 4-arc-regular, cubic graph of girth 6. It is the Heawood graph, see [7, 10], also known as Tutte's 6-cage [31]. It is not know whether there exist infinite, connected, 4-arc-regular, cubic graphs of girth 6.

Similarly, by [7] there are only two finite, connected, 3-arc-regular cubic graphs of girth 6, namely the Pappus graph and the generalized Petersen graph $GP(10, 3)$.

Question 6.9. Are there any infinite, 3-arc- or 4-arc-regular, cubic graphs of girth 6?

7 Two arc-orbits

Now we turn to connected, vertex-transitive, cubic graphs G with two arc-orbits, say O_1 and O_2. We choose the notation such that O_1 is the matching orbit and O_2 the cycle-orbit. By the *elements* of O_1 and O_2 we mean their connected components, and recall that, by Lemma 4.2, the case of two arc-orbits coincides with the case of two edge-orbits. Hence, in this case we need not distinguish between arc- and edge-orbits, but we need the following observation about the action $\mathrm{Aut}(G)$ on the elements of $O1$ and O_2.

The group of automorphisms of G acts on the elements of O_2 by reflections and vertex-transitively, that is, $\mathrm{Aut}(G)$ induces the full group of automorphisms on each element of O_2. The full groups of automorphisms of finite cycles are called *dihedral*, and the full group of 2-sides infinite paths is called *infinite dihedral*. It acts by reflections and translations. On the elements of O_1 the group $\mathrm{Aut}(G)$ acts by reflections.

First a Theorem about connected, vertex-transitive, cubic graphs with two arc-orbits that are rigid. Recall that graphs with two arc-orbits are rigid if the order of the group induced by $\mathrm{Aut}(G)$ on the subgraph formed by a matching edge and its four incident edges is 2.

Theorem 7.1. *The distinguishing cost of connected, vertex-transitive, rigid cubic graphs G with two arc-orbits is 2, unless G is an infinite ladder, a k-ladder or a k-Möbius ladder with $k > 3$. Then $\rho(G) = 3$.*

Proof. Let G be a connected, vertex-transitive, rigid cubic graph with matching orbit O_1 and cycle-orbit O_2. Suppose there is a quadrangle $uvwz$, where uv and wz are in O_1. If we color u, v, w black and leave all other vertices white, then w is fixed by all color preserving automorphism α, and hence also u, because uv is a matching edge. Hence α also fixes w, and therefore the element of the cycle-orbit that contains uw, say A.

If an automorphism α of a rigid graph fixes an element A of O_2, be it a cycle or a 2-sided infinite path, then α fixes all neighboring elements of A in O_2, and because G is connected all elements of O_2, and thus the entire graph G.

It is easily seen that the only graphs that satisfy the assumptions of the lemma and that contain such a quadrangle $uvwz$ are an infinite ladder, or a k-ladder, resp. a k-Möbius ladder. As the 3-ladder is the cube and the 3-Möbius ladder the $K_{3,3}$, both of which are not 2-distinguishable, we have to require that $k > 3$.

In all other cases one considers a matching edge uv together with a neighbor w of v, and colors u and w black. As there is no quadrangle $uvwz$ in G, the vertex v is the only common neighbor of u and w, and hence fixed. Now we conclude as before that the coloring is distinguishing. □

We have two corollaries, one about the order of the vertex-stabilizers of rigid and of flexible graphs, and one about planar graphs.

Corollary 7.2. *Let G be a connected, vertex-transitive, cubic graph with two arc-orbits. When G is rigid, then the order of its vertex-stabilizers is 2, otherwise, that is when G is flexible, it is at least 4.*

Proof. The validity of the assertion for rigid G is clear by the proof of the theorem, and for flexible G it follows directly from the definition. □

Corollary 7.3. *Let G be a 3-connected, planar, vertex-transitive, cubic graph with two arc-orbits. Then G is rigid and $\rho(G) = 2$ unless G is a k-ladder or a k-Möbius ladder with $k > 3$.*

Proof. Let G satisfy the assumptions of the corollary. By Whitney's theorem, which by [29] also holds for infinite graphs, G is uniquely embeddable into the plane in the sense that any automorphism that reverses the order of the edges incident with an arbitrary vertex reverses this order for all vertices. Hence G is rigid, and $\rho(G) = 2$ by Theorem 7.1. □

For flexible cubic graphs we have the following lemma.

Lemma 7.4. *Let G be a connected, vertex-transitive, flexible cubic graph with two arc-orbits, say O_1 and O_2, where O_2 consists of finite cycles. Then:*

(i) *One arc-orbit of G, say O_1, consists of arcs (u,v) and (v,u), where the edges uv form a complete matching.*

(ii) *The number of cycles in O_2 is a least 2.*

(iii) *O_2 consists of finite cycles of equal lengths without diagonals.*

(iv) *The number of edges between pairs of adjacent cycles is either 1 for all pairs, or 2 for all pairs.*

(v) *If the number of edges between adjacent cycles is 2, then the edges between neighboring cycles connect pairs of opposite vertices.*

(vi) *If the length ℓ of the cycles in the cycle-orbit is at least 6, then $g(G) = \ell$.*

Proof. Item (i) is Lemma 4.2. As usual we let O_1 be the matching orbit.

For (ii) we begin with the case when O_2 has only one connected component. Suppose it is the cycle $C_n = u_0u_1 \dots u_{n-1}$ of length n. By (i) each vertex of C_n is incident with exactly one edge of O_1. Let u_0u_j be the edge of O_1 that is incident with u_0, and α be the reflection of O_2 that fixes u_0. Clearly $\alpha(u_j) = u_{n-j}$. Because G is cubic, $u_0u_j = u_0u_{n-j}$, which is only possible if n is even, say $n = 2k$, and $j = k$. By vertex-transitivity the other edges of O_1 are of the form u_iu_{i+k}. The resulting graph is the k-Möbius ladder. As we have already seen, for $k = 3$ it is the $K_{3,3}$, which is not 2-distinguishable, and for $k > 3$ it is rigid. This proves (ii).

So we turn to the case when O_2 has at least two cycles, and all cycles of O_2 have the same length, say ℓ. If a cycle has a diagonal, then this diagonal must be in O_1. By vertex-transitivity this implies all elements of O_1 are diagonals, but then G is disconnected. This proves (iii).

Now, let k be the number of edges between neighboring cycles. Clearly k is a divisor of ℓ. When $k = \ell$, then G is the prism over the k-cycle and rigid.

When $3 \le k < \ell$, consider a cycle $u_0u_1 \dots u_\ell$ in O_2 and a neighboring cycle $u'_0u'_1 \dots u'_\ell$ in O_2, and then the induced subgraph of G that contains the paths from $u_{\ell-\ell/k}$ to u_ℓ and from $u'_{\ell-\ell/k}$ to u'_ℓ. It consists of two cycles of length $4\ell/k + 2$ with the common edge $u_0u'_0$, which is in O_1. When u_0 is fixed and $u_{\ell-\ell/k}$ is interchanged with u_ℓ, then $u'_{\ell-\ell/k}$ is interchanged with u'_ℓ. In this case G is rigid. This means that k is 1 or 2 if G is flexible, which proves (iv).

If $k = 2$, then ℓ is even. Furthermore, the matching edges between two neighboring cycles must connect pairs of opposite edges in the cycles. To see this, let $C = v_0 v_1 \dots v_\ell$ be an element of O_2, C' a neighboring cycle, and v_0, v_j be the origins of the matching edges form C to C'. Because the edges $v_0 v_{\ell-1}$ and $v_0 v_1$ are in the same arc-orbit, there is an automorphism α that fixes v_0 and interchanges $v_{\ell-1}$ with v_1. Clearly α preserves C' and maps v_j into $v_{\ell-j}$, which is only possible if $v_{\ell-j} = v_j$, because there are only two edges between C and C'. Hence $\ell = 2j$. This proves (v).

It is easy to check (vi). □

Lemma 7.4 excludes graphs whose cycle-orbits O_2 consist of 2-sided infinite paths. If such a graph exists, then it is easily seen that O_2 contains at least two connected components, and that there can be at most one matching edge between any two components, otherwise G would be rigid. Furthermore G cannot be G acyclic, because then it is the infinite cubic tree T_3, which has only one arc-orbit. This leads to the following question:

Question 7.5. Are there any two arc-orbit, flexible, cubic graphs G whose cycle-orbits consist of 2-sided infinite paths?

It is not hard to show that the girth of any such graph, if it exists, must be at least 9.

We conclude this part with a theorem on tree-like graphs. We call an infinite, flexible cubic graph G *tree-like* if its cycle-orbit consist of finite cycles, and if the graph G^* that is obtained from G by contraction of the cycles in the cycle-orbit to single vertices and by replacement of double edges, if they occur, by single edges, is a tree of valence at least 3.

Theorem 7.6. *Let G be a tree-like infinite, flexible cubic graph. Then $\rho(G) = \infty$ and $\delta(G) = 0$.*

Proof. Suppose G satisfies the assumptions of the theorem, and let k be the length of the cycles in the cycle-orbit. Then G^* is an infinite tree T_r of valence $r = k$ or $r = k/2$, where $r \geq 3$. By [17] T_r is 2-distinguishable, has uncountable automorphism group, infinite distinguishing cost and distinguishing density zero. It is easy to see that the coloring c of G that is obtained from a distinguishing coloring c^* of G^* by coloring an arbitrary preimage of each black vertex of G^* black is distinguishing, has infinitely many black vertices, and distinguishing density 0.

To complete the proof we have to show that $\rho(G)$ is infinite. We know that G^* has uncountable group and any automorphism of G^* is induced by one of G. If α and β are two automorphisms of G that induce different automorphisms of G^* then they cannot leave all k-cycles invariant and must be distinct. Hence $\mathrm{Aut}(G)$ is uncountable. □

Tree like graphs are the basis for the construction of connected, vertex-transitive, flexible cubic graphs of girth four with two arc-orbits, infinite cost and zero density. See Lemma 9.3.

8 Flexible graphs of girth 3

Let G be a connected, vertex-transitive, cubic graph with two arc-orbits and girth 3. Then the cycle-orbits of G are triangles. Let G^* be the graph obtained by contracting each such triangle to a single vertex, making the matching edges of G the edges of G^*.

This implies that $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G^*)$ are isomorphic. To see this, let α be an automorphism of a graph H. It induces an incidence preserving permutation on $E(H)$ that maps each edge uv of H into $\alpha(u)\alpha(v)$. We denote it by α_E and the group of incidence preserving permutations of $E(H)$ by $\mathrm{Aut}_E(H)$. If H is a nontrivial graph, then $\mathrm{Aut}(H)$ and $\mathrm{Aut}_E(H)$ are isomorphic if and only if H has at most one isolated vertex and K_2 is not a component. It is not hard to prove directly, but actually it is a theorem about the automorphism group of the line graph of a graph, see e.g. [19, Theorem 1.2].

This means that $\mathrm{Aut}(G^*) \cong \mathrm{Aut}_E(G^*)$. Furthermore, each $\varphi \in \mathrm{Aut}(G)$ induces an incidence preserving permutation φ^*_E of $E(G^*)$. Clearly the mapping $\varphi \mapsto \varphi^*_E$ is bijective and thus $\mathrm{Aut}(G)$ and $\mathrm{Aut}_E(G^*)$ are isomorphic. By the above this implies that $\mathrm{Aut}(G) \cong \mathrm{Aut}(G^*)$. We also observe:

1. G^* is arc-transitive. By Theorems 6.4 and 6.5 this implies that it is s-arc-regular for some $s \leq 5$, unless it is the infinite cubic tree.

2. If $G^* \neq T_3$, then G^* is s-arc-regular for $s > 2$ if and only if G is flexible. In particular, if G^* is 3-connected and planar, then $s = 2$ and G is rigid. Observe that the assertion about s for G^* is a consequence of the unique embeddability of 3-connected planar graphs into the plane. Compare the proof of Corollary 7.3.

3. $\rho(G) \leq \rho(G^*)$, because if coloring the vertices of a set $U \subseteq V(G^*)$ black distinguishes G^*, then coloring one vertex black in each triangle corresponding to a vertex in U distinguishes G. A similar argument was used in the proof of Theorem 7.6.

4. In some cases one can show that $\rho(G) < \rho(G^*)$ by using the extra freedom of three choices for vertices in each triangle. Our approach is as follows. Let c be a coloring of the vertices of G and $\mathrm{Aut}(G)_c$ be the group of the color preserving automorphisms of G. Recall that $\mathrm{Aut}(G^*) \cong \mathrm{Aut}(G)$. So $\mathrm{Aut}(G)_c$ is isomorphic to a subgroup, say $(\mathrm{Aut}(G)_c)^*$, of $\mathrm{Aut}(G^*)$. If G^* is s-regular and if $(\mathrm{Aut}(G)_c)^*$ fixes an s-arc, then $(\mathrm{Aut}(G)_c)^*$ must be the trivial group, and thus also $\mathrm{Aut}(G)_c$.

   Therefore, if c is a 2-coloring with n black vertices, where the other vertices are white, and $n < \rho(G^*)$, then $\rho(G) < \rho(G^*)$ if $(\mathrm{Aut}(G)_c)^*$ is trivial.

Theorem 8.1. *Let G be a finite, vertex-transitive, flexible cubic graph with two arc-orbits and girth* 3*. Then $\rho(G) = 2$, unless $g(G^*) = 4, 6, 8$. In these cases G^* is the $K_{3,3}$, the Heawood graph or the Tutte-Coxeter graph and $\rho(G) = 3$.*

Proof. Suppose $g(G^*) \leq 5$, so G^* is K_4, the cube, the dodecahedron, the Petersen graph, or $K_{3,3}$. The first three are 3-connected planar and thus G is rigid, contrary to assumption. When G^* is the Petersen graph or $K_{3,3}$, then G is flexible, because both graphs are 3-regular by [7].

By Theorems 6.4 and 6.5 G^* is s-arc-regular with $s \leq 5$. Given a G^* with $s \leq 5$, we choose an s-arc whose arcs $a_1^*, \dots a_s^*$ in G^* correspond to matching arcs $a_1, \dots a_s$ in G. Set $a_i = u_i u_i'$, and color u_1, u_s black. Let u_1^*, u_s^* be the vertices in G^* corresponding to u_1, u_s in G.

If the girth of G^* is at least $2s - 1$ then $u_1 u_1' u_2 u_2' \cdots u_s$ is the unique shortest u_1, u_s-path in G. Because $u_1 u_1'$ is a matching edge, but not $u_{s-1}' u_s$ each color preserving automorphism of G must fix u_1 and u_s. But if u_s is fixed, then u_s' is also fixed. Let $\varphi \in \mathrm{Aut}(G)$ fix u_1 and u_s'. Then it fixes $a_1, \dots a_s$ in G and φ^* fixes $a_1^*, \dots a_s^*$ in G^*. Because G^* is s-arc regular φ^* is the identity, and thus also φ. Hence the coloring of G is distinguishing and $\rho(G) = 2$, see Figure 10 for $s = 5$.

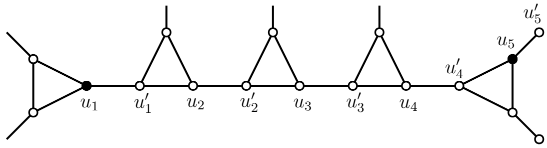

Figure 10: Distinguishing coloring of G for $s = 5$, $g(G) \geq 9$.

Therefore, cost 2 is assured for $s = 3, 4, 5$ when $g(G^*) \geq 5, 7, 9$, respectively. This leaves the cases $g(G^*) = 4, 6, 8$ for $s = 3, 4$ and 5.

If $s = 3$, then $g(G^*) = K_{3,3}$, which is the only remaining graph $g(G^*)$ of girth at most 5. It is Tutte's 4-cage. Furthermore, at the end of Section 6 we noted that the only finite, 4-arc-regular, cubic graph of girth 6 is the Heawood graph, also known as Tutte's 6-cage. Finally, in 1991 M. J. Morton [21] showed that the Tutte-Coxeter graph, also called Tutte's 8-cage, is the only finite graph with $s = 5$ and girth 8.

Using the same notation for all three cages, we color u_1, u_s and u_s' black. As $d_G(u_1, u_s) = 2s - 2$ and $d_G(u_1, u_s') = 2s - 3$, all black vertices have to be fixed by any color preserving $\varphi \in \mathrm{Aut}(G)$. As before we conclude that φ^* fixes $a_1^*, \ldots a_s^*$ in G^*. Hence the coloring of G is distinguishing, see Figure 11 for $s = 4$.

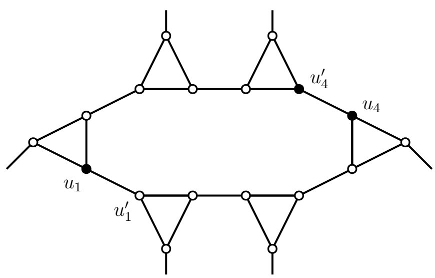

Figure 11: Distinguishing coloring of G for $s = 4$, $g(G) = 6$.

It remains to show that $\rho(G) > 2$ when G^* is a Tutte $4, 6$ or 8-cage, that is when $s = 3, 4$ or 5. We first show that each path of length $s - 1$ is in two cycles of length $2s - 2$. Let C be a cycle of length $2s - 2$ in G^* and $a_1^*, \ldots a_s^*$ be the arcs of an s-arc in C. By arc-transitivity there is $\alpha \in \mathrm{Aut}(G^*)$ that fixes $a_1^*, \ldots a_{s-1}^*$, but not a_s^*. It maps C into another cycle $\alpha(C)$. Clearly the $C \cap \alpha(C) = a_1^*, \ldots a_{s-1}^*$. By arc-transitivity this implies that any path of length $s - 1$ is in two cycles of length $2s - 2$.

To show that $\rho(G) > 2$ it suffices to show that to any vertices u,v in G there is an automorphism of G that interchanges them or a nontrivial automorphism that fixes them. If u,v are in the same triangle, then there clearly is an automorphism that interchanges them. Because the diameter of the cages is $s-1$ we can thus assume that $1 < d_{G^*}(u^*, v^*) \leq s-1$.

Suppose $d_{G^*}(u^*, v^*) = s - 1$ and that u^* and v^* are the endpoints of an $s - 1$-arc $a_1^*, \ldots a_{s-1}^*$. Clearly there are at least three automorphisms of G^* that interchange u^* and v^*: one that reverses the $s - 1$-arc $a_1^*, \ldots a_{s-1}^*$, another one that rotates C, and one that rotates $\alpha(C)$. Therefore, if $d_G(u, v) = 2s - 2$ or $2s$, then an inflection interchanges u with v, and if $d_G(u, v) = 2s - 1$ a rotation interchanges them.

This leaves the case $d = d_{G^*}(u^*, v^*) \leq s-2$. If $d_G(u,v) = 2d-1$ or $2d+1$ then we can interchange u, v by an inflection. Hence we can assume that $d_G(u,v) = 2d$. Then each shortest u, v-path either begins or ends with a matching edge. We choose the notation such that it begins with a matching edge. In G^* there clearly is an s-arc $a_1^*, \dots a_s^*$ where u^* is the origin of a_2^* and v^* the origin of a_{d+2}^*. Note that $d+2 \leq s$. As before we let a_i denote the matching edge in G that corresponds to a_i^* and set $a_i = u_i u_i'$. By our assumptions $u = u_2$ and $v = u_{d+2}$. There is an $\alpha^* \in \mathrm{Aut}(G^*)$ that moves a_1^*, but fixes $a_2^*, \dots, a_s^*$. The automorphism α^* uniquely extends to an $\alpha \in \mathrm{Aut}(G)$ that fixes $u_2, u_2', \dots u_{d+2}, u_{d+2}'$. Figure 12 depicts the case when d is maximal, i.e. when $v = u_{d+2} = u_s$. Observe that α interchanges the neighbors of u_2 that are different from u_2', fixes all other depicted vertices, but the non-matching edges incident with u_s' can be interchanged. □

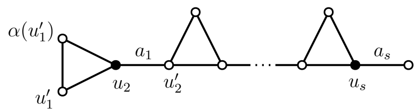

Figure 12: Nontrivial α fixing $u = u_2$ and $v = u_s$.

We conclude the section with a remark about the construction of connected, vertex-transitive, flexible cubic graph with two arc-orbits and girth 3. The process of contracting triangles in G to get G^* can be reversed by *truncation*, namely replacing each vertex by a triangle. In this case, we begin with G^* and construct G by truncating G^*. This way every cubic graph is a G^* for some cubic graph G. For example, in the proof of Theorem 8.1, the graph G with $G^* = K_{3,3}$ is flexible since $K_{3,3}$ is. In general, we have

Proposition 8.2. *Let G^* be an s-arc regular cubic graph, where $s \geq 3$. Then its truncation G is flexible of girth* 3.

Proof. Clearly, G has two arc-orbits, one orbit consisting of arcs in the triangles and a matching orbit corresponding to arcs in G^*. □

9 Flexible graphs of girth 4

In this section we mainly consider connected, flexible, vertex-transitive, cubic graphs G with two arc-orbits and girth 4. We show that this class contains finite graphs of arbitrarily large distinguishing cost and infinite graphs with positive distinguishing densities.

For our constructions we need two operations on G, one we call *folding*, and the other *unfolding*. The first folds the 4-cycles of a graph G into edges of a new graph $F(G)$, and the second unfolds the matching edges of a graph G into 4-cycles of a new graph $U(G)$.

Figure 13 shows how a matching edge uv of the graph on the left is unfolded into a 4-cycle in the graph on the right, respectively it shows how a 4-cycle in the graph on the right is folded into an edge of the graph on the left.

To be more precise, let G be a cubic graph with a complete matching M and C be the subgraph of G with the edge-set $E(G) - M$. The edges not in M form a subgraph C in which each edge has valence 2. Hence, each component of C is a finite cycle or a 2-sided infinite path. When G is vertex-transitive only one of these options is possible, and if there are finite cycles, then all have the same length.

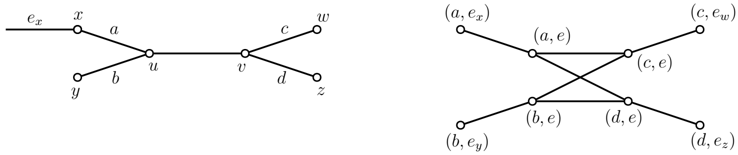

Figure 13: Unfolding and folding.

For a formal description of the new operations we denote incidence of edges a, b by $a|b$, and non-incidence by $a \nmid b$. The *unfolded graph* $U(G)$ is then defined by:

$$V(U(G)) = \{(a, e) : a \in E(C), e \in M, a|e\},$$
$$E(U(G)) = \{(a, e)(c, f) : a = c, a|e, a|f, e \neq f \text{ or } e = f, e|a, e|c, a \neq c, a \nmid c\}.$$

Each vertex u of G is incident with exactly one matching edge, say e, and two edges from C, say a,b. It thus gives rise to two vertices of $U(G)$, we say each vertex is split into two vertices. Furthermore, if $(a, e)(b, f) \in E(U(G))$, then either a connects an endpoint of e with one of f, or e connects an endpoint of a with one of b. Hence, identification of the two vertices in each pair of split vertices is a homomorphism from $U(G)$ onto G.

Clearly unfolding is well defined for all cubic graphs G with a matching M. The folding operation, however, is only defined for cubic graphs G with a set of disjoint of 4-cycles. Given such a graph G, the *folded* graph $F(G)$ is obtained by identification of opposite vertices in the 4-cycles of G, and replacement of the ensuing multiple edges from the 4-cycles by single edges.

Observe that we did not require vertex-transitivity for our operations.

Now we recall that any $\alpha \in \mathrm{Aut}(G)$ induces an incidence preserving permutation on $E(G)$ that maps each edge uv of G into $\alpha(u)\alpha(v)$. We denote it by α_E and, as before, we let $\mathrm{Aut}_E(G)$ be the group of incidence preserving permutations of $E(G)$. We already observed that $\mathrm{Aut}(G) \cong \mathrm{Aut}_E(G)$ if and only if G has at most one isolated vertex and K_2 is not a component.

Lemma 9.1. *Let G be a connected, vertex-transitive, cubic graph with a complete matching M, and let C denote the subgraph of G consisting of the edges not in M. Then*

(i) *$U(G)$ is a connected, cubic graph that consists of disjoint quadrangles, which form a spanning subgraph, and a set of independent edges that form a complete matching. The quadrangles arise from the edges in M, and the matching edges of $U(G)$ are in one-one correspondence with the edges of C.*

(ii) *If all quadrangles of G are components of C, then any two adjacent quadrangles in $U(G)$ are connected by exactly one edge.*

(iii) *There is an isomorphism of $\mathrm{Aut}_E(G)$ into $\mathrm{Aut}(U(G))$. If G has three arc-orbits, then $U(G)$ is not vertex-transitive.*

(iv) *If G is vertex-transitive with two arc-orbits, then $U(G)$ is also vertex-transitive, but may have three arc-orbits, see* Corollary 9.5.

(v) *If G is vertex-transitive and flexible with two arc-orbits, then $U(G)$ is vertex-transitive with two arc-orbits, but may be rigid. See* Corollary 9.8 *and the remark after its proof.*

Proof. Let $e = uv \in M$, and $a, b, c, d \in V(C)$, where a, b are incident with u, and c, d incident with v. Then $(a, e)(c, e)(b, e)(d, e)$ is a quadrangle in $U(G)$. Note that (a, e) is adjacent to (c, e) and (d, e) in $U(G)$, but not to (b, e).

If $a = ux \in M$ and e_x is the matching edge incident with x, then $(a, e)(a, e_x) \in E(U(G))$. Hence the the neighbors of (a, e) are (c, e), (d, e) and (a, e_x). This proves (i).

For (ii) let A, B be two neighboring quadrangles in $U(G)$ that are joined by (at least) two edges. By (i) they are matching edges, say e, f. Let a_e, a_f be the endpoints of e, resp. f, in A and b_e,b_f be the endpoints in B. If a_e and a_f arise from the same vertex in G, then G has a double edge or a triangle, depending on whether b_e and b_f arise from the same vertex in G or not. Hence a_e, a_f, b_e, b_f arise from different vertices in G. It also means that $a_e a_f$ and $b_e b_f$ arise from matching edges, while e, f arise from non-matching edges. But then G has a quadrangle that is not in C, contrary to assumption. This proves (ii).

To prove (iii) we observe that the mapping $(a, e) \mapsto (\alpha_E(a), \alpha_E(e))$ is an automorphism of $U(G)$, say α^*. It is then easy to see that the mapping $\alpha_E \mapsto \alpha^*$ from $\mathrm{Aut}_E(G)$ into $\mathrm{Aut}(U(G))$ is an injective isomorphism. If G has three arc-orbits and if a, b are the non-matching edges that are incident with a vertex v and if e is the matching edge incident with v, then there is automorphism of G that fixes v and interchanges a with b, and thus no automorphism of $U(G)$ that maps (a, e) into (b, e). This proves (iii).

To prove (iv) consider two vertices (a, e), (b, f) of $U(G)$. Let u be the common vertex of a and e, and v the common vertex of b and f. If G is vertex-transitive with two arc-orbits, then there exists an automorphism α that maps u into v, where $\alpha_E(e) = f$ and $\alpha_E(a)$ is incident with v. If $\alpha_E(a) \neq b$, then there is a $\beta \in \mathrm{Aut}(G)$ that fixes v and maps $\alpha_E(a)$ into b. Then $(\beta\alpha)^*(a, e) = (b, f)$, and $U(G)$ is vertex-transitive. This proves (iv).

Finally, consider a vertex (a, e) and its two non-matching neighbors, say (c, e), (d, e). If $e = uv$, then c, d are non-matching edges of G that are incident with v. Because G is flexible with two arc-orbits, there is an automorphism α that fixes a, u, e and v and where α_E interchanges c, d. Clearly α^* fixes (a, e) and interchanges (c, e) with (d, e). □

Lemma 9.2. *Let G be a cubic graph with a complete matching M and a set C of quadrangles, where any two neighboring quadrangles of C are connected by one edge of M. Then there is a natural isomorphism between* $\mathrm{Aut}(G)$ *and* $\mathrm{Aut}(U(G))$.

Proof. Suppose G satisfies the conditions of the lemma. Then all quadrangles of G must be in the set C. If not there must be a quadrangle $uvwx$ that contains a matching edge, say $e = uv$. Vertices u and v are in different quadrangles from C, say A and B. Then ux is in A, vw in B, and wx is a second edge that connects A with B.

By Lemma 9.1 this implies that any two neighboring quadrangles in $U(G)$ are connected by exactly one edge, and thus the partition of $E(U(G))$ into a set of matching edges and a set of quadrangles is unique. Hence the set of pairs of split vertices arising in the construction of $U(G)$ from G is exactly the set of pairs of opposite vertices of the quadrangles in $U(G)$. Thus there is only one way to fold $U(G)$, and $G = F(U(G))$.

Each automorphism α of $U(G)$ preserves the pairs of opposite vertices of the quadrangles in $U(G)$ and thus acts as a permutation α' on $V(G)$. As adjacences are preserved α' is an automorphism. Moreover, $(\phi\psi)' = \phi'\psi'$, and thus $\phi \mapsto \phi'$ is a homomorphism.

If there were two distinct automorphisms ϕ and ψ of $U(G)$ that induce the same action on G, they would have to coincide on these pairs as sets, but in at least one pair $\{u, v\}$ they would have to act differently. Then $\psi^{-1}\phi(u)$ is not the identity on $\{u, v\}$, and thus interchanges u and v. But then the matching edge e_u incident with u has to be interchanged with the matching edge e_v incident with v. As e_u and e_v lead to different quadrangles, not all pairs of opposite vertices in the quadrangles can be preserved. Hence $\phi \mapsto \phi'$ is injective, and hence also the mapping $\phi \mapsto (\phi')_E$

The observation that $\mathrm{Aut}(G) \cong \mathrm{Aut}_E(G)$ and that there is an injective isomorphism of $\mathrm{Aut}_E(G)$ into $\mathrm{Aut}(U(G))$ completes the proof. □

As an application of Lemma 9.1 we have the following lemma.

Lemma 9.3. *There exist infinitely many connected, vertex-transitive, flexible cubic graphs of girth four with cost* ≤ 5 *or with infinite cost and zero density.*

Proof. Let G be the truncation of an s-transitive cubic graph, where $s \geq 3$. By Lemmas 8.2 and 8.1 it is flexible with cost ≤ 5 or zero density. Then $U(G)$ has girth 4 and it is easy to see that it is flexible with cost ≤ 5 or zero density.

Similarly, if G is one of the tree-like graphs of Theorem 7.6, then $U(G)$ is flexible of girth 4 and its density is zero. □

9.1 The graphs $Q(n)$ and $Q_k(n)$

In Section 2 we defined the chain of quadrangles Q, which we also called the infinite crossed ladder, as a graph with vertex set $V(Q) = \{u_i, v_i : i \in \mathbb{Z}\}$ and edge set

$$E(Q) = \{u_i u_{i+1}, v_i v_{i+1}, u_{2i} v_{2i+1}, v_{2i} u_{2i+1} : i \in \mathbb{Z}\}.$$

If we replace $\mathbb{Z}$ in the definition of Q by $\mathbb{Z}_{2k}$ and take indices modulo $2k$, then we obtain the *crossed* $2k$*-ladder*, which we will denote by Q_k. As the crossed 4-ladder is the cube, which is not 2-distinguishable, we only consider crossed ladders Q_k for $k \geq 3$.

Q and Q_k for $k \geq 3$ consist of a set M of matching edges and a set C of disjoint quadrangles, where adjacent quadrangles are connected by two matching edges. Note that M consists of the edges of G that are not in the quadrangles and that no matching edge is in a 4-cycle of G. Hence, if G is Q or a crossed $2k$-ladder for $k \geq 3$, then $U(G)$ is well-defined if we unfold with respect to M. By Lemma 9.1 $U(G)$ also consists of a set of matching edges and a set of quadrangles. This property is retained by all graphs that are obtained by iterations of the unfolding process if we unfold with respect to the edges that are not in the quadrangles.

We set $Q(1) = Q$, $Q_k(1) = Q_k$, and, for $k \geq 3$, $n > 1$, iteratively define $Q_k(n)$ by $Q_k(n) = U(Q_k(n-1))$, and $Q(n) = U(Q(n-1))$. We always unfold with respect to the edges that are not in the quadrangles. They are uniquely defined, and hence so are also $Q(n)$ and $Q_k(n)$.

$Q(1)$ and $Q_k(1)$, $k \geq 3$, are vertex-transitive and flexible with two arc-orbits. As we unfold with respect to the matching orbit we infer by Lemma 9.1 that $Q(n)$ and $Q_k(n)$, $k \geq 3$, $2 \leq n < k-1$, are connected, vertex-transitive, flexible cubic graphs with two arc-orbits, where the cycle orbit consists of 4-cycles, and where any two adjacent 4-cycles are connected by exactly one edge in the matching orbit.

Furthermore, by Corollary 9.8 we shall see that $Q_k(k-1)$ is rigid with two arc-orbits.

As there is a unique matching orbit in $Q_k(k-1)$ the graph $Q_k(k) = U(Q_k(k-1))$ is vertex-transitive by Lemma 9.1(v). That it has three arc-obits is asserted in Corollary 9.5.

Because we keep unfolding with respect to the edges that are not in quadrangles, the graphs $Q_k(n)$ are still uniquely defined for $n > k$, but not vertex-transitive any more.

In the definition of $F(G)$ we only merged multiple edges that arose from folded 4-cycles, but not multiple edges that arise from matching edges. Such cases may occur, for example when folding the chain of quadrangles Q, which we now call $Q(1)$, see Figure 3. The folding process maps $Q(1)$ into a chain of alternating single and double edges and $Q_k(1)$ into a cycle of alternating single and double edges. We set $Q(0) = F(Q(1))$ and $Q_k(0) = F(Q_k(1))$. For $Q(0)$ see Figure 14, and for a part of $Q(0)$ and how it is unfolded, see Figure 15.

Figure 14: $Q(0)$.

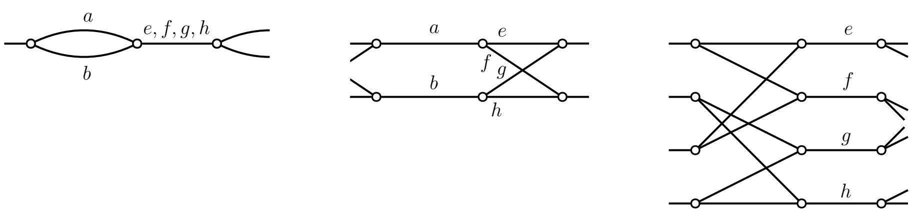

Figure 15: A part G of $Q(0)$ and how it is unfolded.

Clearly all unfolded graphs of $Q(0)$ are isomorphic up to $\mathrm{Aut}_F(Q(0))$, and the graphs $Q_k(0)$ are isomorphic up to $F(Q_k(0))$. Moreover, $\mathrm{Aut}(Q(1)) \cong \mathrm{Aut}_F(Q(0))$, and $\mathrm{Aut}(Q_k(1)) \cong \mathrm{Aut}_F(Q_k(0))$. Recall that $Q_2(1) = U(Q_2(0))$ is the cube.

Theorem 9.4. *The order of* $\mathrm{Aut}(Q_k(n))$*,* $k \geq 3$*,* $1 \leq n \leq k$*, is* $2k \cdot 2^k$*.*

Proof. Clearly the order of $\mathrm{Aut}_E(Q_k(0))$ is $2k \cdot 2^k$. By Lemma 9.1(iii) this is also the order of all groups $\mathrm{Aut}(Q_k(n))$, $k \geq 3$, $1 \leq n \leq k$. □

Corollary 9.5. *For* $k \geq 3$ *the graphs* $Q_k(k)$ *are GRRs.*

Proof. By the lemma $|\mathrm{Aut}(Q_k(k))| = 2k \cdot 2^k$, which equals $|V(Q_k(k))|$. Because $Q_k(k)$ is vertex-transitive this implies that its vertex stabilizers are trivial. Hence $Q_k(k)$ has three arc-orbits and is a GRR. □

Note that the smallest such graph is $Q_3(3)$ and has 48 vertices. The graph $Q_4(4)$ already has 128 vertices.

Finally, observe that the graphs $Q_k(n)$ for $n > k$ are not vertex-transitive by Lemma 9.1(iii).

One calls a partition of the vertex set of a vertex-transitive graph a *system of imprimitivity* if it is preserved by all automorphisms. The partition of $V(G)$ into sets of size one, $\{\{v\} : v \in V(G)\}$ and into just one set of size $|V(G)|$ are called *trivial partitions*. We have seen that the sets of pairs of opposite vertices in the quadrangles of $Q(1)$, resp. $Q_k(1), k \geq 3$, form sets of imprimitivity. Each such pair is the preimage of a single vertex in $Q(0)$, resp. $Q_k(0), k \geq 3$.

Lemma 9.6. *Let $n \in \mathbb{N}$ and G be one of the graphs $Q(n)$ or $Q_k(n)$, $k \geq 3$. Furthermore, let φ denote the mapping from $V(G)$ into $V(F(G))$. Then the preimages of the vertices of $Q(0)$, resp. $Q_k(0)$, $k \geq 3$, with respect to φ^n, are a system of imprimitivity in $Q(n)$, resp. $Q_k(n)$, $k \geq 3$.*

Proof. The assertion of the lemma is true for $n = 1$. Suppose it is true for $n \geq 1$. Let G be $Q(0)$ or $Q_k(0)$, $k \geq 3$, and $\{\varphi^{-n}(v) : v \in V(G)\}$, where $\varphi^{-n}(v)$ is the preimage of v with respect to φ^n. By the induction hypothesis $\{\varphi^{-(n-1)}(v) : v \in V(G)\}$ is a system of imprimitivity in $U^{(n-1)}(G)$. The set $\varphi^{-n}(v)$ arises from $\varphi^{-(n-1)}(v)$ by splitting each of its vertices into a pair of opposite vertices in the quadrangles of $U(U^{(n-1)}(G))$. The observation that $\mathrm{Aut}(U(U^{(n-1)}(G)))$ preserves the set of these pairs concludes the proof. □

Let G be $Q(0)$ or $Q_k(0)$, $3 \leq k$, and $n \geq 1$. Then we call the sets $\varphi^{-n}(v)$, $v \in V(G)$, the *columns* of $U^n(G)$, and denote them by c_v^n. Setting $V(Q(0)) = \mathbb{Z}$ and $V(Q_k(0)) = \mathbb{Z}_{2k}$, $k \geq 3$, the columns of $U^n(G)$ are thus c_i^n, $i \in \mathbb{Z}$ or $\mathbb{Z}_{2k}$.

Lemma 9.7. *Let $n \in \mathbb{N}$ and $c^n_{-n+2}, \dots c^n_{n+1}$ be $2n$ columns in $Q(n)$ or $Q_k(n)$, where $k - 1 \geq n$. Then there exists an $\alpha \in \mathrm{Aut}(Q(n))$, respectively $\alpha \in \mathrm{Aut}(Q_k(n))$, that moves all vertices in columns $c^n_{-n+2}, \dots, c^n_{n+1}$ and fixes all other vertices.*

Proof. We proceed by induction with respect to n and note that the assertion of the lemma is true for $n = 1$. Let G be $\mathrm{Aut}(Q(n))$ or $\mathrm{Aut}(Q_k(n))$ and suppose the assertion is true for $n \geq 1$. Then there exists an $\alpha \in \mathrm{Aut}(G)$ that moves all vertices in $c^n_{-n+2}, \dots, c^n_{n+1}$ and fixes all other vertices.

Recall from Lemma 9.1, if G is a connected, vertex-transitive cubic graph with a complete matching M, then $\mathrm{Aut}(G) \cong \mathrm{Aut}_E(G) \cong \mathrm{Aut}(U(G))$. Furthermore, a vertex u of G that is incident with a matching edge e and the non-matching edges f, g gives rise to the vertices (f, e) and (g, e) in $U(G)$. Each $\alpha \in \mathrm{Aut}(G)$ then gives rise to an automorphism α^* of $U(G)$ defined by $\alpha^*(f, e) = (\alpha_E(f), \alpha_E(e))$. This means that all elements in $c^{n+1}_{-n+2}, \dots, c^{n+1}_{n+1}$ are moved by α^*. As the edges between c^{n+1}_{-n+1} and c^{n+1}_{-n+2} are matching edges, as well as the edges between c^{n+1}_{n+1} and c^{n+1}_{n+2}, the vertices in c^{n+1}_{-n+1} and c^{n+1}_{n+2} are also moved.

As α fixes all vertices in c^n_{-n} and c^n_{n+3}, as well as the vertices in the neighboring columns, all edges incident with vertices in c^n_{-n} and c^n_{n+3} are fixed, and thus α^* fixes all vertices in c^{n+1}_{-n} and c^{n+1}_{n+3}.

It is easily seen that all other vertices of $U(G)$ that are not in $c^{n+1}_{-n+1}, \dots, c^{n+1}_{n+2}$ are also fixed. But, we wish to point out that columns c^{n+1}_{-n} and c^{n+1}_{n+3} can be adjacent. In this case $n = k - 1$, c^{n+1}_{-n}, c^{n+1}_{n+3} are connected by matching edges and are the only columns whose elements are fixed by α^*. □

Corollary 9.8. *The graphs $Q(n)$ and $Q_k(n)$, $n \in \mathbb{N}$, $k-2 \leq n$, are flexible, but $Q_k(k-1)$ is rigid.*

Proof. Let $\alpha \in \mathrm{Aut}(Q(n))$ or $\mathrm{Aut}(Q_k(n))$ be the automorphism that moves all vertices in the columns $c^n_{-n+2}, \ldots, c^n_{n+1}$ and fixes all other vertices. Observe that the edges between c^n_{-n+2} and c^n_{-n+3} are matching edges, and hence also the edges between c^n_{-n} and c^n_{-n+1}, which are fixed. Let v be a vertex in c^n_{-n+1}. It is fixed and has two neighbors, say w,z in c^n_{-n+2} that are both moved. The other neighbor of v, say u is in c^n_{-n}, and uv is a matching edge. Clearly the other two neighbors of u, say x,y are in c^n_{-n-1} and fixed, unless $k-1 = n$, because then $c_{-n-1} = c_{-n+2k} = c_{n+1}$, all of whose elements are moved, which means that $Q_k(k-1)$ is rigid. □

The fact that $Q_k(n)$ is flexible for $1 \leq n \leq k-2$ and rigid for $n = k-1$ is also a consequence of Theorem 9.4 and Corollary 7.2. To see this, observe that $|V(Q(k,n))| = 2k{\cdot}2^n$. The graphs are vertex-transitive and the order of their automorphism groups is $2k{\cdot}2^k$ by Theorem 9.4. Hence the order of their vertex-stabilizers is at least 4 when $n \leq k-2$ and 2 when $n = k-1$. Now an application of Corollary 7.2 completes the argument.

Theorem 9.9. *Let G be $Q(n)$ or $Q_k(n), k-1 \geq n$. Then the motion $\mathrm{m}(G)$ of G is $n \cdot 2^{n+1}$, and to each automorphism α of G that stabilizes the columns of G there is a row of $2n$ columns, all of whose vertices are moved by α.*

Proof. Let G be $Q(n)$ or $Q_k(n), n \leq k-1$. By Lemma 9.7 there exists an automorphism that moves all vertices in the $2n$ columns $c^n_{-n+2}, \ldots, c^n_{n+1}$ of G. Because each column contains 2^n vertices, $\mathrm{m}(G) \leq n \cdot 2^{n+1}$.

If an automorphism of G moves a column into another one, then it has to move all columns. This is easily seen, because after n foldings G maps onto $Q(0)$ or $Q_k(0)$, whose automorphisms move all vertices if they move at least one vertex. Hence, if not all columns are stabilized, then at least $2k$ columns are moved, and thus at least $2k{\cdot}2^n > 2n{\cdot}2^n$ vertices.

We can thus restrict attention to automorphisms that stabilize the columns. We shall show the existence of a row of $2n$ columns to each automorphism α of G, where α moves all vertices in the columns, and where the edges between the outermost pairs of columns are matching.

This is true for $n = 1$. We wish to show that it holds for n under the assumption that it holds for $n-1 \geq 1$. Each automorphism of G is of the form α^*, where $\alpha \in F(G)$, where $F(G)$ is $Q(n-1)$ or $Q_k(n-1), n \leq k-1$. Hence there are $2n-2$ columns,all of whose vertices are moved by α. By vertex-transitivity we can assume that the columns are $c^{n-1}_{-n+3}, \ldots, c^{n-1}_{n}$. (This also assures that the edges between the outermost pairs of columns are matching.)

Then α^* moves all vertices in $c^n_{-n+3}, \ldots, c^n_n$. As the edges between columns c^n_{-n+2}, c^n_{-n+3}, and c^n_n, c^n_{n+1} are matching, all vertices in the columns $c^n_{-n+2}, \ldots, c^n_{n+1}$ are moved. □

Theorem 9.10. *The graphs $Q(n)$, $n \in \mathbb{N}$, are connected, vertex-transitive, flexible cubic graphs with two arc-orbits, girth 4, and motion $\mathrm{m}(Q(n)) = n \cdot 2^{n+1}$. Furthermore, $\rho(Q(n)) = \infty$ and $\delta(Q(n)) = 1/\mathrm{m}(Q(n))$.*

Proof. By Lemma 9.7, Theorem 9.9 and Corollary 9.8 we only have to find a distinguishing coloring with density $1/(n \cdot 2^{n+1})$. We choose the columns c^n_{2nj}, $j \in \mathbb{Z} - \{0\}$ and color one vertex in each column black. Then we choose a vertex in c^n_0, say u. It is incident with a matching edge e, say uv. Instead of coloring u, we color v black. Let c be this coloring. Clearly the projection of c into $Q(0)$ distinguishes $\mathrm{Aut}(Q(0))$, where $\mathrm{Aut}(Q(0))$

is the group of permutations of $V(Q(0))$ that preserve double and single edges. Hence any color preserving automorphism of $Q(n)$ stabilizes the columns, and thus fixes all black vertices. But, because e is a matching edge, it also fixes u, and thus any color preserving automorphism α fixes at least one vertex in any row of $2n$ columns. By Theorem 9.9 α must be the identity. □

Theorem 9.11. *The graphs $Q_k(n)$, $1 \leq n \leq k-2$, are connected, vertex-transitive, flexible cubic graphs with two arc-orbits, girth 4 and motion $\mathrm{m}(Q(n)) = n \cdot 2^{n+1}$. Their distinguishing cost is $\rho(G) = \lceil \frac{k}{n} \rceil$, unless $\frac{k}{n} = 2$; then $\rho(G) = 3$. In all cases $\delta(Q_k(n)) \leq 1/4$.*

Proof. We choose the columns c^n_{2nj}, $1 \leq j \leq \lceil \frac{k}{n} \rceil$, and color one vertex in each column black. For c^n_0 we proceed differently. We choose a $u \in c^n_0$. It is incident with a matching edge $e = uv$, and instead of coloring u, we color v black. Let c be this coloring.

If $k/n \neq 2$ the projection of c into $Q_k(0)$ distinguishes $\mathrm{Aut}(Q_k(0))$, hence any color preserving automorphism of $Q_k(n)$ stabilizes the columns, and thus fixes all black vertices. But, because e is a matching edge, it also fixes u, and thus any color preserving automorphism α fixes at least one vertex in any row of $2n$ columns. By Theorem 9.9 α must be the identity.

If $k/n = 2$ we need three black vertices in $Q_k(0)$ to break $\mathrm{Aut}(Q_k(0))$. It is easily seen, that it suffices to color one vertex in each of the columns c^n_0, c^n_1, and c^n_n black, in order to obtain a distinguishing coloring. Hence $\rho(G) \leq 3$.

But we still have to show that two black vertices do not suffice. Suppose two black vertices suffice, say u,v. When $k/n = 2$ we have $4n$ columns, and because there are automorphisms that move $2n$ contiguous columns and fix all vertices of the other columns, we have to place the black vertices in columns of distance $2n$, say $u \in c^n_1$ and $v \in c^n_{2n+1}$. We do not need consider the case that the black vertices are in c^n_0 and c^n_{2n}, because there is a reflection that interchanges the pair c^n_1, c^n_{2n+1} with c^n_0, c^n_{2n}.

By vertex-transitivity there is an automorphism β with $\beta(u) = v$. Set $w = \beta(v)$. If $u = w$, then our coloring is not distinguishing, hence $w \neq u$. If we can find an automorphism that ψ that fixed v and maps w into u, then $\psi\beta(u) = v$ and $\psi\beta(v) = \psi(w) = u$, and the coloring is not distinguishing.

Let n be fixed and $1 \leq m \leq n$. If $m = 1$, then c^m_1 consists of just two vertices, u and w and one sees directly that one can interchange them while fixing the vertices in c^m_ℓ, for $2m+1 \leq \ell \leq 4n-2m-1$. i.e. also v. We continue by induction with respect to m and assume that for $m-1 \geq 1$ and any two vertices $x, y \in c^{m-1}_1$ there is an automorphism ψ, such that $x = \psi(y)$ and where ψ fixes all vertices in c^m_ℓ, $2m+1 \leq \ell \leq 4n-2m-1$. Given $u, w \in c^m_1$ we consider their images under the folding homomorphism φ. Let $x = \varphi(u)$ and $y = \varphi(w)$. There is an automorphism ψ of $Q(m-1, k)$ that maps y into x and fixes c^m_ℓ, $2m-1 \leq \ell \leq 4n-2m+1$ pointwise. Then ψ^* moves the preimage of y into the preimage of x and fixes all vertices in the columns c^m_ℓ for $2m+1 \leq \ell \leq 4n-2m-1$. If ψ^* does not move w into u, then it moves it into a vertex u', where u, u' are opposite vertices in a quadrangle. But then there is an automorphism that moves u' into u and fixes all vertices in c^m_ℓ for $2m+1 \leq \ell \leq 4n-2m-1$. □

Question 9.12. We wonder whether there exist any infinite, connected, vertex-transitive cubic graphs with positive density other than $Q(n)$, where $1 \leq n$, or finite connected, vertex-transitive cubic graphs with cost > 5 other than $Q_k(n)$, where $1 \leq n \leq k-2$ and $\lceil \frac{k}{n} \rceil > 5$. If they exist, they must be flexible with two arc-orbits.

9.1.1 Split Praeger–Xu graphs

The graphs $Q_k(n)$, $1 \le n \le k-1$, $k \ge 3$, are also know as Split Praeger–Xu graphs $\mathrm{SPX}(2,k,n)$, see [20]. Their vertex sets are $\mathbb{Z}_2^n \times \mathbb{Z}_k \times \{+,-\}$ and the edge-sets consists of the pairs

$$(i_0,\dots,i_{n-1},x,+)(i_1,\dots,i_n,x+1,-) \text{ and } (i_0,\dots,i_{n-1},x,+)(i_0,\dots,i_{n-1},x,-)$$

for $i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_k$.

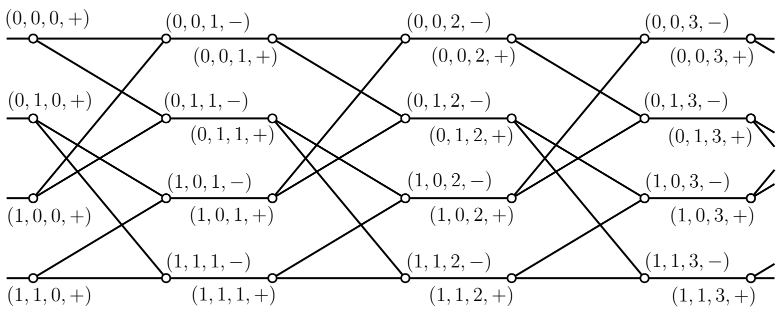

Figure 16: Part of $\mathrm{SPX}(2,k,2)$ for large k.

For $\mathrm{SPX}(2,k,2)$, where k is large, compare Figure 16. We leave it to the reader to verify that $Q_k(n) = \mathrm{SPX}(2,k,n)$. Note that the graphs $\mathrm{SPX}(2,k,n)$, $k \ge 3$, are flexible for $1 \le n \le k-2$, rigid for $n = k-1$ by Corollary 9.8, and not defined as SPX-graphs for $n = k$.

By our preceding results on $Q_k(n)$ the following theorem characterizes the Split Praeger–Xu graphs.

Theorem 9.13. *Let* $1 \le n \le k-1$ *and* $k \ge 3$. *Then the Split Praeger–Xu graphs* $\mathrm{SPX}(2,k,n)$ *are exactly those cubic graphs G that have a spanning subgraph consisting of disjoint* 4*-cycles and can be folded onto* $Q_k(0)$ *by n foldings.*

We could have based our presentation of $Q_k(n)$ on that of [20], but preferred the more graph theoretic approach. It allowed us to directly treat the graphs $Q(n)$, which are infinite, and to illustrate the role of motion. Besides, it also led to a new series of GRRs.

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