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Vogalna singularnost torzije kompozitne palice

The Corner Singularity of Composite Bars in Torsion

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Materialna matrika kompozitov je na vsaki materialni komponenti nespremenljiva. Ta nezveznost materialne matrike omejuje regularnost rešitve elastomehanične naloge s kompozitnim materialom. Poleg materialne nezveznosti na regularnost rešitve vpliva še geometrijska oblika stične ploskve med sosednjimi materialnimi komponentami. Vsaka medmaterialna geometrična singularnost je vir singularnosti, ki se praviloma manifestira v obliki koncentracije napetosti. Pomembni podatek za izračun koncentracije napetosti je red vogalne singularnosti. V prispevku je za modelni problem torzije kompozitne prizmatične palice predstavljena metodologija določitve reda vogalne singularnosti. V prvem delu prispevka je podan model torzije s popolno in nepopolno vezjo med materialnimi komponentami. Za model popolne vezi je nato z asimptotičnim razvojem v vrsto dokazan obstoj koncentracije napetosti v vogalu. Izrecno je izračunan red vogalne singularnosti v odvisnosti od vogalnega kota in materialnih lastnosti kompozitov. Pomembna ugotovitev je, da je red singularnosti neodvisen od usmeritve materiala v vogalu. V primeru nepopolne vezi je dokazan obstoj koncentracije napetosti v vogalu za dovolj ohlapno vez. Rezultat je dokazan z regularnim asimptotičnim razvojem v okolici popolne medmaterialne nepovezanosti.

(Ključne besede: kompoziti, palice, problemi moeliranja, analize singularnosti)

The material matrix of composite materials is constant for each individual component. This discontinuity sets regularity bounds upon the solution. Besides material discontinuities, the regularity of the solution is also affected by geometrical singularities along the material interfaces. If the interface has corners, we speak about corner singularities of the composite. Mechanical manifestations of these singularities are stress concentrations. One of the most important pieces of information about the corner singularity is the order of the singularity. In this article a method for determining the order of the singularity for the model problem of a composite bar in torsion is presented. In the first part a model of torsion for a composite bar with perfect and imperfect bonds is given. For a perfect interfacial bond the existence of the corner stress concentration is proved by the asymptotical method. The order of the singularity with respect to the angle of the corner and the material constants is explicitly computed. It is found that the order is independent of the material orientation in the corner. For the case of an imperfect bond the existence of the stress concentration is established for a weak bond. The existence is proved by the regular asymptotic perturbation of the no-material adhesion.

(Keywords: composite, bars, modelling problems, singularity analysis)

0 UVOD

V tehnični praksi so konstrukcijski elementi zaradi različnih razlogov pogosto sestavljeni iz dveh ali več različnih materialov. Takim sestavljenim elementom pravimo kompoziti. Pomemben razlog za uporabe kompozitov je doseči želene materialne lastnosti kompozita s primerno izbiro materialov posameznih komponent [1]. Materialna matrika kompozita je nespremenljiva na vsaki materialni komponenti posebej in je tako stopničasta funkcija. Komponente materialne matrike se pojavljajo kot koeficienti elastične energije.

0 INTRODUCTION

Construction elements are quite often made of composite materials. The reason for using composites is to obtain a product with the desired properties composed of different materials [1]. The material matrix is constant for each material component, but has jump discontinuities across the material interfaces. As a result, the coefficients of the governing partial differential equations have jumps across the interfaces. It is well known [2] that the regularity of the solution depends upon

Znano je, da nezveznost materialne matrike odločilno vpliva na regularnost rešitve [2]. Posebej to pomeni, da moremo pri kompozitih pri prehodu iz ene materialne komponente v drugo pričakovati določeno singularnost. Po drugi strani je znotraj komponent materialna matrika nespremenljiva in je potemtakem rešitev na posamezni komponenti notranje regularna. Poleg nezveznosti materialne matrike pri prehodu iz ene materialne komponente v drugo na regularnost vpliva še geometrijska oblika stične ploskve sosednjih materialnih komponent. Vsaka geometrična singularnost je vir nove singularnosti, ki se praviloma izraža v obliki koncentracije napetosti. Tej singularnosti pravimo vogalna singularnost kompozita.

Znano je, da je poznavanje reda singularnosti rešitve pomembno pri numeričnem modeliranju [3], saj vpliv singularnosti na rešitev ni krajeven, temveč praviloma celovit. Uspešnost neposredne metode omejitve vpliva singularnosti, z zgostitvijo diskretizacije v okolici vira singularnosti, je omejena s povečanjem števila prostostnih stopenj in poslabšanjem numerične pogojenosti naloge. Bolj učinkovita je dekompozicija rešitve na singularni in regularni del ter lepljenje singularnega dela z diskretizacijo regularnega dela. Pomanjkljivost te metode je, da moramo singularni del rešitve poznati dovolj natančno. Preprostejša, zato pa še vedno dovolj učinkovita, je metoda dekompozicije prostora aproksimacije na singularni in regularni del. V metodi končnih elementov to pomeni uporabo singularnih elementov v okolici singularnosti rešitve. Za uporabo te metode je dovolj poznati red singularnosti rešitve [3].

V prispevku bomo določili red vogalne singularnosti za modelni problem torzije prizmatične kompozitne palice. Prispevek je razdeljen v štiri razdelke. Po uvodu sledi formulacija problema torzije kompozitne palice v variacijski obliki in v obliki robne naloge. Formulirana je naloga torzije s popolno vezjo med posameznimi materialnimi komponentami palice in nepopolno vezjo, ki dopušča na medmaterialnem stiku dislokacijo v smeri osi palice. V tretjem razdelku je obravnavana vogalna singularnost s pomočjo asimptotičnega razvoja v okolici vogala. Tu se bomo omejili na torzijo palice s popolno vezjo. Izpeljana je karakteristična enačba za lastne vrednosti in dokazan je obstoj koncentracije napetosti z izrecnim izračunom reda vogalne singularnosti. Torzija palice z nepopolnimi vezmi med materialnimi komponentami je obravnavana v četrtem razdelku. Pokazali bomo, da dislokacija ne sprosti napetosti in da ima tudi v tem primeru rešitev singularnost v vogalu.

1 TORZIJA KOMPOZITNE PALICE

Napetost pri torziji homogene prizmatične palice $\Omega\times[0,l]$, kjer sta Ω prerez palice in l dolžina palice, je dana s Prandtlovo napetostno funkcijo $\hat{\chi}$. V kartezičnem koordinatnem sistemu z osjo z v smeri osi palice sta tako edini neničelni komponenti napetosti $t_{13}= t_{31}=\mu\vartheta(\partial\hat{\chi}/\partial y)$ in $t_{23}= t_{32}=-\mu\vartheta(\partial\hat{\chi}/\partial x)$,

the regularity of the coefficients. On the other hand, coefficients are constant within the components, and the interior regularity is not affected. Besides material discontinuities, there is another possible source of the singularity: the shape of the interface boundary between the material components. Each geometric singularity of the interfacial boundary is the source of another singularity of the solution. The mechanical manifestation of these singularities is through stress concentrations. If the interfacial boundary has corners, we speak about the corner singularities of the composite.

It is well known that accurate numerical modelling depends upon a firm knowledge of the order of the singularities [3], as the numerical solution is globally affected by singularities. The direct approach of the local mesh refinement is hampered by the increase in the number of degrees. Also, the condition number of the problem may be affected by the high ratio of the element sizes. A more effective method is to decompose the solution into the singular and regular parts. However, to do this, the singular part of the solution has to be known in advance. Simpler, and still good enough, is the method of decomposing the discretization space. In the case of the finite-element method this means that singular elements are used around the source of the singularity. To use singular elements one only has to know the order of the singularity [3].

In this paper the discussion is restricted to the model problem of the torsion of a composite bar with perfect and imperfect interfacial bonds. The paper has four parts. After an introduction we have the formulation of the problem. Variational, as well as distributional formulations are given. Attention is given to possible axial dislocations, which arise due to the imperfect bonding. In the next section the corner singularity of bars with a perfect bond is approached by the asymptotic expansion. A characteristic equation is derived and the existence of the stress concentration is established. It is proved that the asymptotic expansion has only one singular term, which gives the stress concentration. In the last section the torsion with the imperfect bond is considered. Stress concentration is proved in the case of the weak bond and thus the axial dislocation does not relax the stress concentrations.

1 TORSION OF COMPOSITE BARS

For a homogenous prismatic bar $\Omega\times[0,l]$, where Ω is the cross section and l is the height of the bar, stress components are given by the Prandtl stress function $\hat{\chi}$. In the Cartesian coordinate system with the z axis aligned with the axis of the bar the only non-zero stress components are $t_{13}= t_{31}=\mu\vartheta(\partial\hat{\chi}/\partial y)$

kjer je μ strižni modul palice, ϑ pa je torzijski zasuk palice na dolžino palice. Potencialna energija torzije palice je vsota elastične energije in potenciala površinskih sil. Elastična energija homogene palice je:

and $t_{23}= t_{32}=-\mu\vartheta(\partial\hat{\chi}/\partial x)$ where μ is the shear modulus of the bar and ϑ is the torsion angle per unit length of the bar. The potential energy of the bar is the sum of the elastic energy and the potential of the surface traction. The elastic energy is:

$$\hat{U}_{e} = \frac{1}{2}\int_{\Omega} \underline{e} : \underline{t}\ d\Omega = l\mu\theta^2 \frac{1}{2}\int_{\Omega}|\nabla\hat{\chi}|^2 d\Omega,$$

potencial površinskih sil pa je:

whereas the potential of the surface traction is:

$$\hat{U}_{t} = -\int_{\Omega}\vec{t}\cdot\vec{u}_{|z=l}d\Omega - \int_{\Sigma}\vec{t}\cdot\vec{u}d\Omega,$$

kjer je $\vec{u}$ vektor pomika, ki ima na osnovni ploskvi z=0 pomik samo v smeri osi palice, Σ je plašč palice. Potencial površinskih sil moremo preoblikovati v:

where $\vec{u}$ is the displacement vector, which is at the base z=0, directed along the z axis. The lateral surface of the bar is denoted by Σ. The potential of the surface traction is rewritten as:

$$\hat{U}_{t} = -2l\mu\theta^2\int_{\Omega}\hat{\chi}d\Omega + l\mu\theta^2\int_{\partial\Omega}\hat{\chi}\vec{r}\cdot\vec{n}d\Gamma - l\mu\theta^2\int_{\partial\Omega}\phi\frac{d\hat{\chi}}{ds}\ d\Gamma.$$

Potencialna energija palice je tako $\hat{U}_{p}=\hat{U}_{e}+\hat{U}_{t}$. Brezrazsežni zapis potencialne energije je $\hat{U}=\hat{U}_{p}/\left(l\mu_0\vartheta^2\right)$, kjer je μ_0 referenčni strižni modul. V nadaljevanju bomo uporabljali izključno brezrazsežni zapis. Da bo pisava enostavnejša, brezrazsežnih in razsežnih strižnih modulov μ_i s pisavo ne bomo ločili.

Potencialna energija $\hat{U}$ kompozitne palice s prerezom $\Omega=\bigcup_{i=1}^{N}\Omega_i$, kjer so Ω_i disjunktni prerezi posameznih materialnih komponent, je vsota potencialnih energij materialnih komponent palice. Potem je potencialna energija torzije enaka:

The potential energy is thus $\hat{U}_{p}=\hat{U}_{e}+\hat{U}_{t}$. The corresponding non-dimensional form is $\hat{U}=\hat{U}_{p}/\left(l\mu_0\vartheta^2\right)$, where μ_0 is a reference shear modulus. In the following, only the non-dimensional form will be used and thus, to simplify the notation, we make no notational distinction between the dimensional and non-dimensional moduli.

The potential energy $\hat{U}$ of the composite bar with the cross section $\Omega=\bigcup_{i=1}^{N}\Omega_i$, where Ω_i are cross sections of the individual material components, is the sum of the potential energies of the individual material components. Thus:

$$\hat{U}=\sum_{i=1}^{N}\mu_i\int_{\Omega_i}\left(\frac{1}{2}|\nabla\hat{\chi}_i|^2-2\hat{\chi}_i\right)d\Omega+\sum_{i=1}^{N}\mu_i\int_{\partial\Omega_i}\hat{\chi}_i\vec{r}\cdot\vec{n}d\Gamma-\sum_{i=1}^{N}\mu_i\int_{\partial\Omega_i}\phi_i\frac{d\hat{\chi}_i}{ds}d\Gamma, \tag{1}$$

kjer sta μ_i ter $\hat{\chi}_i$ strižni modul in napetostna funkcija i-te komponente, ϕ_i pa je pomik i-te komponente v smeri osi palice. Rob $\partial\Omega_i$ materialne komponente je vsota robov do sosednjih materialnih komponent in zunanjega roba. Tu smo vzeli, da je prerez kompozitne palice enostavno povezano območje. V primeru, da ima prerez luknjo, lahko luknjo obravnavamo kot materialno komponento s strižnim modulom, ki limitira proti nič [4]. Zunanji rob prereza je prost, zato imajo napetostne funkcije tistih komponent, ki sestavljajo ovoj na zunanjem robu, nespremenljivo vrednost. Napetostna funkcija je določena do stalnice natančno, zato moremo te stalnice izbrati tako, da imajo napetostne funkcije na zunanjem robu vrednost nič. Na robu med dvema materialnima komponentama velja ravnovesni pogoj recipročne Cauchyjeve relacije. To v zapisu z napetostno funkcijo na skupnem robu i-te in j-te materialne komponente pomeni enakost $\mu_i(d\hat{\chi}_i/ds)=\mu_j(d\hat{\chi}_j/ds)$. Funkciji $\hat{\chi}_i$ in $\hat{\chi}_j$ se torej na skupnem robu razlikujeta le za stalnico. Pri predpostavki enostavno povezanega prereza Ω_i in izbire vrednosti napetostnih funkcij na zunanjem robu potem sledi enakost izrazov $\mu_i\hat{\chi}_i$ in $\mu_j\hat{\chi}_j$ na

where μ_i and $\hat{\chi}_i$ are the shear modulus and the stress function of the cross section Ω_i, and ϕ_i is the dislocation of the i-th component in the direction of the z axis. The boundary $\partial\Omega_i$ of the i-th material component is the union of the boundaries between the material components and the part of the outer boundary. We assume here that the cross section of the bar is simply connected. In the opposite case, where the cross section has a hole, the hole can be treated as the limit of the material, with the shear modulus vanishing, [4]. The outer boundary is traction free, and thus the stress functions along the outer boundary are constant. The stress function is determined up to a constant factor, and thus the constants can be arranged such that the stress functions along the outer boundary are all vanishing. Across the interfacial boundary the Couchy reciprocal relation holds. In particular, along the interfacial boundary between i-th and j-th component we have $\mu_i(d\hat{\chi}_i/ds)=\mu_j(d\hat{\chi}_j/ds)$. Functions $\hat{\chi}_i$ and $\hat{\chi}_j$ thus differ along the common boundary for a constant. Due to the arrangement of the constants along the outer boundary it follows then that $\mu_i\hat{\chi}_i$ and $\mu_j\hat{\chi}_j$ are equal along the common boundary. Therefore, skupnem robu. To pomeni, da napetostna funkcija $\hat{\chi}$, katere zožitev na Ω_i je enaka $\hat{\chi}_i$, ni zvezna na Ω.

Pri ravnovesnem pogoju na meji med različnima materialoma je druga vsota v (1) enaka nič. Potem je:

function $\hat{\chi}$, which equals $\hat{\chi}_i$ on Ω_i is not continuous on Ω.

It follows from the equilibrium that the second sum in (1) vanishes. Hence:

$$\hat{U} = \hat{U}(\hat{\chi}) = \sum_{i=1}^{N} \mu_i \int_{\Omega_i} \left(\frac{1}{2} \left|\nabla \hat{\chi}_i\right|^2 - 2\hat{\chi}_i \right) d\Omega - \sum_{i<j} \mu_i \int_{\Gamma_{ij}} [\![\phi]\!] \frac{d\hat{\chi}_i}{ds} d\Gamma \qquad (2),$$

kjer je Γ_{ij} skupni rob komponent Ω_i in Ω_j, in

where Γ_{ij} is the common boundary of Ω_i and Ω_j, and:

$$[\![\phi]\!]_{|\Gamma_{ij}} = \left(\phi_i - \phi_j\right)_{|\Gamma_{ij}}$$

skok osnega pomika na meji med dvema materialoma. V primeru popolne vezi med materialnimi komponentami je ta skok enak nič. Točneje, velja $[\![\phi]\!] = 0$. Potemtakem je potencialna energija za popolno kompozitno palico enaka:

is the axial dislocation at the interfacial boundary. In the case of perfect bonding there are no dislocations and thus $[\![\phi]\!] = 0$. Therefore, the potential energy of the composite bar with perfect bonding is:

$$\hat{U} = \hat{U}(\hat{\chi}) = \sum_{i=1}^{N} \mu_i \int_{\Omega_i} \left(\frac{1}{2} \left|\nabla \hat{\chi}_i\right|^2 - 2\hat{\chi}_i \right) d\Omega. \qquad (3).$$

Če ima pomik ϕ skok na meji med dvema materialoma, je vez med materialoma nepopolna. Pri nepopolni vezi se materialne komponente dislocirajo v osni smeri. Najpreprostejši model, glej [5], nepopolne vezi temelji na predpostavki, da je dislokacija v osni smeri sorazmerna napetosti. Z enačbo je $[\![\phi]\!] = -\alpha\mu(d\hat{\chi}/ds)$, kjer je α pozitivna stalnica. Pripadajoča potencialna energija je:

In the case of the axial dislocations we speak of the imperfect bonding and ϕ has a jump across the interface. The most simple dislocation model is based on the assumption that axial dislocations are proportional to the axial stresses, [5]. Thus $[\![\phi]\!] = -\alpha\mu(d\hat{\chi}/ds)$, where α is a positive proportional factor. The potential energy for the imperfect bonding is thus:

$$\hat{U} = \hat{U}(\hat{\chi}) = \sum_{i=1}^{N} \mu_i \int_{\Omega_i} \left(\frac{1}{2} \left|\nabla \hat{\chi}_i\right|^2 - 2\hat{\chi}_i \right) d\Omega + \alpha \sum_{i<j} \mu_i^2 \int_{\Gamma_{ij}} \left| \frac{d\hat{\chi}_i}{ds} \right|^2 d\Gamma \qquad (4).$$

Očitno se (4) za α=0 reducira v potencial popolne vezi (3). Potencial v (4) je definiran na množici:

Evidently, for α=0, (4) reduces to (3). The potential (4) is defined on the functional space:

$$\hat{V} = \left\{ \hat{\chi} : \hat{\chi}_{|\Omega_i} \in H^1(\Omega_i) \text{ in } \frac{d\hat{\chi}}{ds}_{|\Gamma_{ij}} \in L^2(\Gamma_{ij}) \right\},$$

kjer je $H^1(\Omega_i)$ prostor Soboljeva prvega reda. Tu smo privzeli, da so robovi Γ_{ij} odsekoma regularni.

Napetostna funkcija χ ima skok na prehodu iz enega v drugi material. Ta skok moremo s preprosto preslikavo $\chi_i = \mu_i \hat{\chi}_i$ regularizirati. Pripadajoči regulariziran potencial je:

where $H^1(\Omega_i)$ is the Sobolev space of the first order. In the above it was assumed that the interfacial boundaries Γ_{ij} are piecewise regular.

The stress function χ has jumps across the material interfaces. These jumps are eliminated with a simple transformation $\chi_i = \mu_i \hat{\chi}_i$. The corresponding potential has the form:

$$U = \sum_{i=1}^{N} \int_{\Omega_i} \left(\frac{1}{2\mu_i} \left|\nabla \hat{\chi}\right|^2 - 2\hat{\chi} \right) d\Omega + \alpha \sum_{i<j} \int_{\Gamma_{ij}} \left| \frac{d\hat{\chi}_i}{ds} \right|^2 d\Gamma \qquad (5),$$

prostor pa:

with the corresponding space:

$$V = \left\{ \chi : \chi_{|\Omega_i} \in H^1(\Omega_i) \text{ in } \frac{d\chi}{ds}_{|\Gamma_{ij}} \in L^2(\Gamma_{ij}) \right\}.$$

Prostor V je s skalarnim zmnožkom:

The functional space V, equipped with the scalar product:

$$\langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega + \sum_{i<j} \int_{\Gamma_{ij}} \frac{du}{ds} \frac{dv}{ds} \, d\Gamma$$

Hilbertov prostor. Omenimo, da moremo pri privzetku regularnosti robov potencial definirati namesto na V na množici: $H^1(\Omega) \bigcap \left\{ \chi : \chi_{i|\Gamma_{ij}} \in H^{3/2}(\Omega_i) \right\}$.

Rešitev torzijske naloge kompozitne palice je minimum potenciala torzije na dopustni množici. Rešitev je torej dana z minimizacijsko nalogo:

is a Hilbert space. It should be noted that assuming some additional regularity upon Γ_{ij} the space V can be replaced with $H^1(\Omega) \bigcap \left\{ \chi : \chi_{i|\Gamma_{ij}} \in H^{3/2}(\Omega_i) \right\}$.

A solution of the torsional problem is a solution of the minimization problem for the potential energy. Thus we have:

$$\min_{\chi\in V} U(\chi).$$

Eksistenca in enoličnost rešitve izhaja iz šibke napol zveznosti in koerkivnosti potenciala na Hilbertovem prostoru V, glej [6]. Minimizacijski nalogi pripada variacijska naloga: poišči $\chi\in V$, tako da je:

The existence and the uniqueness of the solution follows from the fact that the potential energy is a coercive, weakly lower semicontinuous function defined on the Hilbert space V, see [6]. To the minimization problem is associated a variational problem: find $\chi\in V$, such that:

$$a(\chi,w)=b(w),\quad \text{za vsak/for all } w\in V \tag{6}$$

kjer je: / where

$$a(\chi,w)=\sum_{i=1}^{N}\frac{1}{\mu_i}\int_{\Omega_i}\nabla\chi\cdot\nabla w d\Omega+\alpha\sum_{i<j}\int_{\Gamma_{ij}}\frac{d\hat{\chi}}{ds}\frac{dw}{ds}d\Gamma$$

in / and

$$b(w)=2\int_{\Omega} w\, d\Omega.$$

Variacijski nalogi je v porazdelitvenem pomenu enakovredna robna naloga: najdi $\chi\in V(\Delta)=V\bigcap\left\{\chi:\Delta\chi_{|\Omega_i}\in L^2(\Omega_i)\right\}$, tako da velja:

The variational problem is equivalent to the following boundary value problem in the distributional sense: find $\chi\in V(\Delta)=V\bigcap\left\{\chi:\Delta\chi_{|\Omega_i}\in L^2(\Omega_i)\right\}$ such that:

$$-\frac{1}{\mu_i}\Delta\chi=2 \quad \text{v/in } \Omega_i \quad \text{za vsak/for all } i=1,\ldots,N \quad \text{in/and}$$
$$\left[\!\left[\frac{1}{\mu}\frac{\partial\chi}{\partial\vec{n}}\right]\!\right]-\alpha\frac{d^2\chi}{ds^2}=0 \quad \text{na/on } \Gamma_{ij} \quad \text{za vsak/for all } 1\le i<j\le N.$$

Regularnost rešitve robne naloge je na vsaki materialni komponenti enaka regularnosti roba komponente. Če je rob $\partial\Omega_i$ razreda C^s, leži zožitev rešitve na Ω_i v prostoru $H^s(\Omega_i)$.Vsak vogal na robu $\partial\Omega_i$ vpliva na regularnost rešitve. Znano je, da leži rešitev Poissonove naloge z regularno desno stranjo na konveksni ravninski množici v prostoru H^2 in da za nekonveksno poligonsko množico ta regularnost ni dosežena [7]. V primeru kompozitne palice situacija ni tako preprosta. Če je komponenta Ω_i konveksna, vsaj ena komponenta, ki ima z Ω_i skupen rob, ni konveksna, zato podobnega rezultata ne moremo pričakovati. Kako je dejansko z regularnostjo v okolici medmaterialnega vogala, bomo raziskali v naslednjem razdelku.

On each material component the regularity of the solution is determined by the regularity of the boundary. If the boundary $\partial\Omega_i$ is of class C^s, the restriction of the global solution to the cross section Ω_i belongs to the space $H^s(\Omega_i)$. Each corner of the boundary $\partial\Omega_i$ affects the regularity of the solution. For example, it is well known that for the Poisson boundary value problem stated on a convex polygonal plane domain the solution belongs to the space H^2, and that for a non-convex domain this regularity is not achieved [7]. In the case of the composite bar the situation is more involved. If a cross section Ω_i is convex, then at least one component that bonds to Ω_i is not convex. How exactly the regularity depends upon the interfacial corner singularities will be studied in the next section.

2 ASIMPTOTIČNI RAZVOJ V OKOLICI MEDMATERIALNEGA VOGALA

2 ASYMPTOTIC EXPANSION IN THE NEIGHBOURHOOD OF THE INTERFACIAL CORNER

Najprej si bomo pogledali primer popolne vezi α=0. Pri tem se bomo zaradi enostavnosti omejili na vogal na meji med dvema materialoma. Enako lahko obravnavamo tudi vogal na meji med več materiali, s to razliko, da je računska pot težja. Naj bosta Ω_1 in Ω_2 dve materialni komponenti s skupnim robom Γ, ki ima v točki O vogal z odprtjem $\kappa\pi$, $\kappa\in(0,2)$. Privzeli bomo, da je v okolici O vogala rob odsekoma raven. V okolici O uvedemo polarni koordinati r, φ s središčem v O, tako da se rob Γ ujema s poltrakoma φ=0 in $\varphi=\kappa\pi$. Na O iščemo rešitev nepopolne robne naloge:

In this section the case of a perfect bonding α=0 will be considered. To simplify the notation the discussion is restricted to the bimaterial interface. With the same method, but with more evolved computations, a more general case of the multimaterial corner singularity can be studied. Let Ω_1 and Ω_2 be two materials with the common boundary Γ, which has a corner at O with the angle $\kappa\pi$, $\kappa\in(0,2)$. It is assumed that the boundary is locally flat in the neighbourhood of O. In the neighbourhood of O the polar coordinates r, φ are introduced such that the boundary Γ coincides with the rays φ=0 and $\varphi=\kappa\pi$. In the neighbourhood a solution of the incomplete boundary value problem:

$$-\Delta\chi_i = 2\mu_i \quad \text{v/in } \Omega_i,\ i=1,2,\dots,N, \tag{7a}$$

$$\chi_{1|\varphi=0} = \chi_{2|\varphi=2\pi} \quad \text{in/and} \quad \chi_{1|\varphi=\kappa\pi} = \chi_{2|\varphi=\kappa\pi}, \tag{7b}$$

$$\mu_2 \frac{\partial\chi_1}{\partial\vec{n}}_{|\varphi=0} = \mu_1 \frac{\partial\chi_2}{\partial\vec{n}}_{|\varphi=2\pi} \quad \text{in/and} \quad \mu_2 \frac{\partial\chi_1}{\partial\vec{n}}_{|\varphi=\kappa\pi} = \mu_1 \frac{\partial\chi_2}{\partial\vec{n}}_{|\varphi=\kappa\pi} \tag{7c}.$$

Rešitev iščemo v obliki $\chi_i = \chi_i^p + \chi_i^0$, kjer je χ_i^0 harmonična funkcija, ki zadošča pogojem (7b) in (7c) in χ_i^p posebna rešitev (7a), ki prav tako zadošča pogojem (7b) in (7c). Znano je, da moremo pri obravnavi vogalne singularnosti Poissonove naloge ločiti primera $\kappa \neq 1/2$ in $\kappa = 1/2$. Podobno velja tudi sedaj. Za $\kappa \neq 1/2$ je posebna rešitev:

is sought. The solution has the form $\chi_i = \chi_i^p + \chi_i^0$, where χ_i^0 is a harmonic function with the boundary conditions (7b) and (7c), and χ_i^p is a particular solution of (7a), which also satisfies the boundary conditions (7b) and (7c). For the asymptotic expansion of the Laplace equation it is well known that the cases $\kappa \neq 1/2$ and $\kappa = 1/2$ must be treated separately. The same is also true for the Poisson equation. For $\kappa \neq 1/2$ a particular solution is:

$$\chi_i^p = \frac{1}{2}\mu_i r^2 \left(-1 + \cos 2\varphi + \sin 2\varphi \tan \kappa\pi\right).$$

Za $\kappa = 1/2$ rešitev ni tako preprosta in je:

For $\kappa = 1/2$ the solution is more involved and it is:

$$\chi_1^p = -\frac{1}{2}r^2\mu_1 + \frac{r^2(\mu_1-\mu_2)\left(\left(-4\vartheta\mu_1 + \pi(\mu_1 - 5\mu_2)\right)\cos 2\varphi - 4\mu_1 \log r \sin 2\varphi\right)}{2\pi(\mu_1 + 3\mu_2)},$$
$$\chi_2^p = -\frac{1}{2}r^2\mu_2 + \frac{2r^2\mu_2(-\mu_1+\mu_2)(\varphi\cos 2\varphi + \log r \sin 2\varphi)}{\pi(\mu_1 + 3\mu_2)}.$$

Rešitev vsebuje logaritemski člen, ki pa je pomnožen z r^2, tako da je logaritemska singularnost navzoča šele v drugem odvodu χ_i^p po r in potemtakem ni vir napetostne koncentracije. Harmonično rešitev zapišemo z razvojem:

The solution has a logarithmic term, which is multiplied by r^2. Thus the logarithmic singularity shows up only after the second derivative of χ_i^p with respect to r, and consequently it is not a source of the stress concentration. The harmonic solution is considered in the form:

$$\chi_i^0 = \chi_i^0(r,\varphi) = \sum_{k=0}^{\infty} r^{\lambda_k}\Phi_{i|k}(\varphi),$$

kjer so λ_k pozitivne lastne vrednosti in $\Phi_{i|k}$ lastne funkcije problema:

where λ_k are positive eigenvalues and $\Phi_{i|k}$ are corresponding eigenvectors of the problem:

$$\frac{d^2\Phi_{i|k}}{d\varphi^2} + \lambda_k^2\Phi_{i|k} = 0 \quad \text{v/in } \Omega_i$$

z robnimi pogoji (7b) in (7c), kjer je normalni odvod zamenjan z odvodom po φ. Pri razvoju harmoničnega dela rešitve ni treba ločiti primera $\kappa \neq 1/2$ in $\kappa = 1/2$. Lastne vrednosti so ničle karakterističnega polinoma:

with the boundary conditions (7b) and (7c). Here, the normal derivative is rewritten as the derivative with respect to φ. The above form of the asymptotic expansion is valid for both cases $\kappa \neq 1/2$ and $\kappa = 1/2$. The eigenvalues are zeros of the characteristic polynomial:

$$\begin{aligned} p &= p(\lambda;\kappa,\mu_1,\mu_2) = \\ &= \lambda^2\left(-4\mu_1\mu_2 + (\mu_1+\mu_2)^2\cos 2\pi\lambda - (\mu_1-\mu_2)^2\cos 2\pi(1-\kappa)\lambda\right) = \\ &= \frac{1}{2}\mu_1^2(1+\beta)^2\lambda^2\left[\left(\frac{1-\beta}{1+\beta}\right)^2(1-\cos 2\pi(1-\kappa)\lambda) - (1-\cos 2\pi\lambda)\right] \end{aligned} \tag{8},$$

kjer je $\beta = \mu_2/\mu_1$. Očitno velja $p(\lambda;\kappa,\mu_1,\mu_2) = p(\lambda;\kappa,\mu_2,\mu_1)$. Z drugimi besedami za red singularnosti ni pomembno, kateri material je znotraj kota $\kappa\pi$. Poleg tega velja tudi $p(\lambda;\kappa,\mu_1,\mu_2) = p(\lambda;2-\kappa,\mu_1,\mu_2)$. To pomeni, da se moremo v nadaljevanju omejiti na $\kappa \in (0,1]$. Vidimo tudi, da so za $\kappa=1$ ničle karakterističnega polinoma celoštevilčne in da potemtakem rešitev nima singularnosti. V nadaljevanju bomo zato privzeli $\kappa \in (0,1)$. Singularnosti prav tako ni v primeru $\beta=1$, zato v nadaljevanju $\beta \neq 1$.

where $\beta = \mu_2/\mu_1$. Evidently $p(\lambda;\kappa,\mu_1,\mu_2) = p(\lambda;\kappa,\mu_2,\mu_1)$. In other words, for the order of the singularity it is not important which material is within the angle $\kappa\pi$ and which within the angle $(2-\kappa)\pi$. Also, it follows $p(\lambda;\kappa,\mu_1,\mu_2) = p(\lambda;2-\kappa,\mu_1,\mu_2)$, and thus the further discussion is restricted to $\kappa \in (0,1]$. In the case of $\kappa=1$, zeros of the characteristic polynomial are integer, and thus there are no singularities. In the following we assume $\kappa \in (0,1)$. There are also no singularities if $\beta=1$. Hence, we assume $\beta \neq 1$. Let us

Poglejmo, kako je z ničlami izraza v oglatem oklepaju v (8) v razponu $\lambda \in [k, k+1], k \in \mathbb{N}$. Z zapisom $\lambda \in k+x, x \in [0,1)$ se moremo pri iskanju ničel omejiti na iskanje presečišč funkcij $\gamma\left(1-\cos\left(2\pi(1-\kappa)x-\delta\right)\right)$ in 1-cos2πx v razponu $x\in[0,1)$, kjer je $\gamma\in(0,1)$ in $\delta=\delta(k)\in[0,2\pi)$. Funkciji imata za $\delta\neq0$ na $(0,2\pi)$ natanko dve presečišči. Za $\delta=0$ imata funkciji eno presečišče $x\in$(1-arccos(1-2γ),1) in dotikališče v x=0. Fazni pomik δ je enak nič za k=0. V primeru $\kappa \in \mathbb{Q}$ je δ=0 tudi za $k \in \left[\mathbb{Z}^+/1-\kappa\right]$, kjer je oglati oklepaj celi del argumenta. Najmanjša pozitivna lastna vrednost je torej manjša od 1 in je potemtakem vir koncentracije napetosti. Ta lastna vrednost je edini vir koncentracije napetosti, saj je naslednja lastna vrednost po velikosti že večja od 1. Tako smo dokazali:

IZREK *Vodilni člen asimptotičnega razvoja napetostne funkcije bikompozitne palice v vogalu meje med dvema materialoma je r^λ, kjer je λ rešitev karakteristčne enačbe (8). Velja ocena:*

now consider (8) for $\lambda \in [k, k+1], k \in \mathbb{N}$. Denoting $\lambda \in k+x, x \in [0,1)$, searching for zeros of (8) is the same as finding intersections of functions $\gamma\left(1-\cos\left(2\pi(1-\kappa)x-\delta\right)\right)$ and 1-cos2πx on the interval $x\in[0,1)$, where $\gamma\in(0,1)$ and $\delta=\delta(k)\in[0,2\pi)$. The functions have, for $\delta\neq0$, exactly two intersections on $(0,2\pi)$. For δ=0 there is only one intersection $x\in$(1-arccos(1-2γ),1), and a contact at x=0. The phase shift δ equals zero for k=0. In the case $\kappa \in \mathbb{Q}$ it follows that δ=0 also for $k \in \left[\mathbb{Z}^+/1-\kappa\right]$, where the bracket denotes the integer part. The smallest intersection is less than 1, and thus it is the source of the stress concentration. It is the only source of the stress concentration since the next intersection is always greater than 1. Thus it was proved:

THEOREM *The leading term of the asymptotic expansion of the stress function of the bicomposite bar in the neighbourhood of the interfacial corner is r^λ, where λ is the solution of the characteristic equation (8). For λ the following estimate holds:*

$$\lambda \in (1-\frac{1}{2\pi}\arccos(1-2\gamma),1) \quad \text{za/for} \quad \gamma=\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^2$$

Izrek postavlja zgornjo mejo singularnosti rešitve v okolici vogala. Ta meja je neodvisna od kota vogala. Sam red singularnosti, ki je rešitev transcendentne enačbe, je odvisen od kota. V primeru $\kappa=1/2$ lahko ničle brez težav izračunamo. Ničle so:

With the theorem the upper bound on the singularity is established. The bond is independent of the angle. However, the order of the singularity, which is given by the solution of the transcendent equation, depends upon the angle. In the particular case $\kappa=1/2$ the characteristic equation is readily solved. The zeros are:

$$\bigcup_{k=0}^{\infty}\left\{2k, 2k+\lambda_1, 2(k+1)-\lambda_1\right\}, \quad \text{kjer je/where} \quad \lambda_1=\frac{1}{\pi}\arccos\frac{1+6\beta+\beta^2}{2(1+\beta)^2}.$$

Lastne funkcije so:

The eigenvectors are:

$$\Phi_{i|k}(\varphi)=a_{i|k}\cos\lambda_k\varphi+b_{i|k}\sin\lambda_k\varphi,$$

kjer so $a_{i|k}$ in $b_{i|k}$ stalnice, določene z robnimi pogoji (7b) in (7c). V primeru enojne ničle λ_k je lastni podprostor enorazsežen, v primeru dvojne pa dvorazsežen. Lastne funkcije, ki pripadajo različnim lastnim vrednostim, so paroma ortogonalne v skalarnem zmnožku:

where $a_{i|k}$ and $b_{i|k}$ are constants given by the boundary conditions (7b) and (7c). In the case of a single zero the associated eigenvector space is one dimensional and if the zero is double, the eigenspace is two dimensional. Eigenvectors that belong to different eigenvalues are orthogonal with respect to the scalar product:

$$\left\langle\Phi,\Psi\right\rangle=\mu_1\int_0^{\kappa\pi}\Phi\Psi\,d\varphi+\mu_2\int_{\kappa\pi}^{2\pi}\Phi\Psi\,d\varphi$$

in sestavljajo poln sistem funkcij prostora periodičnih funkcij $L^2([0,2\pi])$.

Dokazali smo, da je rešitev torzije bikompozitne palice singularna v medmaterialnem vogalu in izračunali red singularnosti. V posebnem primeru krožnega prereza moremo rešitev torzije dobiti z razvojem po polnem sistemu funkcij $\Phi_{i|k}$. V splošnem primeru rešitev dobimo z metodo končnih elementov. Ker poznamo red singularnosti, moremo pri tem uporabiti v okolici vogala singularne elemente, recimo Sternove elemente.

and constitute a complete set in the subspace of the periodic functions of $L^2([0,2\pi])$.

It was proved that the solution of the torsional problem of the bimaterial composite bars has a corner singularity. In the particular case of a circular cross section of the bar the solution can be found by the series expansion over the complete set of eigenfunctions $\Phi_{i|k}$. The general case solution can be found using the finite-element method. Since the order of the singularity is known, the singularity of the problem can be modelled with singular finite elements, for example, with Stern elements.

Metodo, uporabljeno za dvokompozitno palico, lahko uporabimo tudi za večkompozitno palico. V nasprotju z dvokompozitno palico pri večkompozitni palici najmanjše pozitivne lastne vrednosti ne moremo oceniti navzdol z oceno, ki je neodvisna od strižnih modulov materialnih komponent. Na primer, pri štirikompozitni palici s krožnim prerezom z enakima materialnima komponentama v prvem in tretjem ter drugem in četrtem kvadrantu gre najmanjša pozitivna vrednost z limito $\beta \to 0$ proti nič.

As already noted, the same method also works for multicomposite bars. However, a similar estimation upon the order of the singularity, as in the Theorem, does not exist. The estimation now also depends upon the shear modulo. For example, for a bar made of two materials arranged in a check-board pattern the smallest positive zero of the characteristic equation goes to zero with $\beta \to 0$.

3 VOGALNA SINGULARNOST TORZIJE PALICE Z NEPOPOLNO VEZJO

3 CORNER SINGULARITY FOR IMPERFECT BOND

Podobno kakor prejšnjem razdelku definiramo okolico O in vse preostalo s to razliko, da imamo sedaj namesto pogoja (7c) pogoja:

As in the previous section we define the neighbourhood O and all the others with the exception that instead of (7c) we now apply the conditions:

$$\frac{1}{\mu_1}\frac{\partial \chi_1}{\partial \varphi}_{|\varphi=0} - \frac{1}{\mu_2}\frac{\partial \chi_2}{\partial \varphi}_{|\varphi=2\pi} = \alpha \frac{d^2\chi}{dr^2}_{|\varphi=0} \tag{9a}$$

$$\frac{1}{\mu_1}\frac{\partial \chi_1}{\partial \varphi}_{|\varphi=\kappa\pi} - \frac{1}{\mu_2}\frac{\partial \chi_2}{\partial \varphi}_{|\varphi=\kappa\pi} = \alpha \frac{d^2\chi}{dr^2}_{|\varphi=\kappa\pi} \tag{9b}$$

Robna pogoja (9a, b) v nasprotju s pogojem (7c) ne ločita posameznih potenc r^{λ}, zato namesto sistema štirih enačb za vsako lastno vrednost dobimo neskončni sistem. Naloge se zato lotimo z asimptotičnim razvojem rešitve po $\varepsilon = 1/\alpha$ v okolici $\alpha=\infty$. Iz zapisa (5) potencialne energije izhaja, da limitna vrednost $\alpha=\infty$ ustreza popolni nepovezanosti materialnih komponent, kjer vsaka komponenta pomeni palico, ki je v torziji neodvisno od sosednjih komponent. To pomeni, da ima napetostna funkcija χ na vsakem robu $\partial\Omega_i$ nespremenljivo vrednost. Velja omeniti, da je v nasprotju z regularnim razvojem v okolici neskončnosti, razvoj v okolici $\alpha=0$ singularen. Kot medmaterialnega vogala je za eno materialno komponento ostri, za drugo pa topi kot. Znano je [7], da ima rešitev Poissonove naloge s homogenim robnim pogojem v vogalu s topim kotom singularnost. Vodilni člen asimptotičnega razvoja je $r^{1/\kappa}$, kjer je $\kappa\in(1,2)$. Torej je rešitev za $\varepsilon=0$ singularna. Nadaljnje člene asimptotičnega razvoja po ε dobimo iz robne naloge, kjer je na levi strani (9a,b) skok normalnega odvoda predhodnega člena v asimptotičnem razvoju. Na desni (9a, b) je drugi odvod po r, zato ima vodilni člen vsakega nadaljnjega člena v asimptotičnem razvoju vodilno potenco, ki je za dve večja od predhodnega člena. To pomeni, da so vsi nadaljnji členi nesingularni in da je tako vsa singularnost v ničtem členu razvoja po ε. Ostaja pa odprto vprašanje, kako je s konvergenčnim polmerom razvoja po ε. To vprašanje bo tema nadaljnjih raziskav.

The boundary conditions (9a, b), in contrast to (7c), do not separate the powers r^{λ}, and thus instead of a linear system with four equations for four unknowns, for each eigenvalue an infinite linear system is obtained. The problem is thus rather approached with the asymptotic expansion with respect to $\varepsilon = 1/\alpha$ in the neighbourhood of $\alpha=\infty$. From (5) it follows that the limiting value $\alpha=\infty$ corresponds to the no-adhesion case, where each component is in a separate torsion. Therefore, the stress function χ has a constant value on the boundary $\partial\Omega_i$ of the each component. It should be noted that the asymptotic expansion with respect to α is singular in the neighbourhood of $\alpha=0$. The interfacial corner is acute with respect to one material and obtuse with respect to the other. It is well known that a solution of the Poisson boundary-value problem has a corner singularity, [7]. The leading term of the singularity is $r^{1/\kappa}$, where $\kappa\in(1,2)$. Therefore, the solution is singular for $\varepsilon=0$. Further terms of the expansion are obtained from (9a, b). For a given order of ε there is a jump of the normal derivative of the previous term of the expansion on the left-hand side, and the second derivative of a new term on the right-hand side. It follows then that all further terms of the expansion are nonsingular and hence that all the singularity is in the first term. However, it is an open question as to what is the convergence radius of the expansion. This will be the object of further research.

4 SKLEP

4 CONCLUSION

V primeru torzije kompozitne palice s popolno vezjo je z asimptotičnim razvojem v okolici vogala dokazano, da je vir koncentracije napetosti v vodilnem členu asimptotičnega razvoja. Izpeljana je

In the case of the perfect bonding it is shown that the source of stress concentration is in the leading term of the asymptotic expansion in the neighbourhood of the corner. A characteristic enačba za določitev reda vogalne singularnosti v odvisnosti od kota vogala in materialnih lastnosti kompozita. Dokazano je, da je red vogalne singularnosti neodvisen od usmeritve vogala. Iz teh ugotovitev izhaja, da moremo vogalno singularnost torzije kompozitne palice s popolno vezjo brez težav numerično modelirati z uporabo singularnih elementov. V primeru nepopolne vezi analiza reda vogalne singularnosti ni popolna. Dokazano je, da je v primeru ohlapne vezi tudi v tem primeru opazna koncentracija napetosti v vogalu.

Na koncu je na mestu še opomba, da iz povedanega izhaja, da je torzija vogalov v območju plastičnosti. Omenimo, da moramo v modelu elastoplastične torzije energijski funkcional zožiti na množico:

equation, which determines the order of the singularity in terms of the corner angle and material properties, is derived. It is shown that the order of the singularity is independent of the material orientation with respect to the corner. It follows then that the corner singularity can be numerically modelled by using singular elements. For the case of imperfect bonding the conclusions are not complete. The existence and the order of the singularity is proved only for weak bonding.

At the end it should be remarked that from the above discussion it follows that the interfacial corners of the composite bar in torsion are actually plastic. Note, that in the case of elastoplastic torsion the appropriate function space is:

$$V \cap \left\{ \chi : |\nabla \chi|_{|\Omega_i} \leq \sigma_i \right\},$$

kjer so σ_i meje plastičnih napetosti.

where σ_i are the yields stresses.

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5 REFERENCES

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