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CORNER RINGS OF A CLEAN RING NEED NOT BE CLEAN

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Abstract

We give an example of a clean ring R and an idempotent $e \in R$, such that the corner ring eRe is not clean.

Keywords:
Clean ring, Corner ring, Bergman's example, Exchange ring

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1 Introduction

An element a in a ring R is called *clean* if it can be written as a sum of an idempotent and a unit. A ring R is called *clean* if every element in R is clean. Camillo and Yu [2] showed that every semiperfect ring is clean. In addition, Camillo and Khurana [1] showed that every unit-regular ring is clean. Every clean ring is an exchange ring and every exchange ring with central idempotents is clean [6, Proposition 1.8].

In [3], Han and Nicholson proved that if $e \in R$ is an idempotent and the corner rings eRe and $(1-e)R(1-e)$ are clean, then R is also clean. It is an open question whether the converse holds (see [9] or [7]). Using Bergman's example of an exchange non-clean ring, we show that this need not be true.

In the first part of the paper, we classify all rings R for which the matrices

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

are clean in $M_2(R)$ for every $a \in R$. In the second part, we use this classification to construct a clean ring R and an idempotent $e \in R$ such that the corner ring eRe is not clean.

All our rings will be associative with unity. The set of all units in R will be denoted by $U(R)$, the set of idempotents by $\mathrm{Id}(R)$, and the ring of $n \times n$ matrices over R by $M_n(R)$. If R is any ring, we will denote by $R^{(\mathbb{N})}$ a countably generated free right module over R, and by $M_{\mathbb{N}}(R)$ the ring of all $\mathbb{N} \times \mathbb{N}$ matrices with finite columns over R (which is of course isomorphic to the endomorphism ring $\mathrm{End}(R^{(\mathbb{N})})$).

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2 Rings R for which the matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are clean in $M_2(R)$

We begin with the following lemma.

Lemma 2.1. *Let R be a ring and $a \in R$. Then a is clean in R if and only if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $ua = eu + 1$.*

Proof. Suppose that $a \in R$ is clean. Then we may write $a = e + u$, where $e \in \mathrm{Id}(R)$ and $u \in U(R)$. Multiplying by u^{-1} from the left, we get $u^{-1}a = u^{-1}e + 1 = (u^{-1}eu)u^{-1} + 1$. Hence $u^{-1}eu \in \mathrm{Id}(R)$ and $u^{-1} \in U(R)$ are elements that satisfy the desired property. Conversely, suppose that there exist $e \in \mathrm{Id}(R)$ and $u \in U(R)$ that satisfy $ua = eu + 1$. Then we have $a = u^{-1}eu + u^{-1}$, where $u^{-1}eu$ is an idempotent and u^{-1} is a unit. □

The following proposition gives the characterization of the rings in the title of this section.

Proposition 2.2. *Let R be a ring and $a \in R$ an element. Then the matrix*

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

is clean in $M_2(R)$ if and only if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a - e - u \in (1-e)Ra$.

Proof. Let a be an element of R. We will denote throughout this proof

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R).$$

First, suppose that A is clean in $M_2(R)$. By Lemma 2.1, there exist $E \in \mathrm{Id}(M_2(R))$ and $U \in U(M_2(R))$ such that $UA = EU + 1$. Multiplying by $1 - E$ from the left, we get $(1-E)UA = 1 - E$. Thus $1 - E \in M_2(R)A$ is an idempotent with the second column equal to zero, hence it has the form

$$1 - E = \begin{pmatrix} e & 0 \\ x & 0 \end{pmatrix},$$

where $e \in R$ is an idempotent and $x \in Re$. Write $f = 1 - e$ and

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

From the equation $UA = EU + 1$, we get

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ -x & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + 1,$$

which gives $\alpha a = f\alpha + 1$, $0 = f\beta$, $\gamma a = \gamma - x\alpha$ and $0 = \delta - x\beta + 1$. Since U is invertible,

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U =$$

$$= \begin{pmatrix} \alpha + \beta(\gamma - x\alpha) & \beta + \beta(\delta - x\beta) \\ \gamma - x\alpha & \delta - x\beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta\gamma a & 0 \\ \gamma a & -1 \end{pmatrix}$$

is invertible, and therefore

$$\begin{pmatrix} \alpha + \beta\gamma a & 0 \\ \gamma a & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma a & 1 \end{pmatrix} = \begin{pmatrix} \alpha + \beta\gamma a & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible. Hence $u = \alpha + \beta\gamma a$ is invertible in R. Substituting $\alpha = u - \beta\gamma a$ in the equation $\alpha a = f\alpha + 1$, and considering $f\beta = 0$, we get $ua - \beta\gamma a^2 = fu + 1$. Multiplying by u^{-1} from the left, we have $a = u^{-1}\beta\gamma a^2 + u^{-1}fu + u^{-1}$. Thus we have an idempotent $u^{-1}fu$ and a unit u^{-1} that satisfy

$$a - u^{-1}fu - u^{-1} = u^{-1}\beta\gamma a^2 = u^{-1}e\beta\gamma a^2 \in u^{-1}euRa = (1 - u^{-1}fu)Ra,$$

as desired.

Conversely, suppose that there exist $e \in \mathrm{Id}(R)$ and $u \in U(R)$ such that $a - e - u \in (1-e)Ra$. Write $f = 1 - e$ and take $x \in R$ with $a - e - u = fxa$. It is easy to see that

$$E = \begin{pmatrix} u^{-1}eu & 0 \\ u^{-1}x(a-1)u^{-1}fu & 1 \end{pmatrix}$$

is an idempotent in $M_2(R)$. The matrix

$$U = \begin{pmatrix} 1 & 0 \\ u^{-1}x(1-a) & 1 \end{pmatrix} \begin{pmatrix} 1 & -u^{-1}fu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ u^{-1}x & -1 \end{pmatrix}$$

is invertible, since it is a product of invertible matrices. A straightforward computation shows that $UA = EU + 1$. Hence by Lemma 2.1, A is clean in $M_2(R)$. This finishes the proof. □

At this point we write down the definition that will be used throughout the rest of this paper.

Definition 2.3. An element a in a ring R is called *weakly clean* if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a - e - u \in (1-e)Ra$. A ring R is called *weakly clean* if every element in R is weakly clean.

Note that the above definition is left-right symmetric after Proposition 2.2.

Remark 2.4. Clearly, every clean ring is weakly clean. It is also not difficult to see that every weakly clean ring is an exchange ring. (Recall that R is an exchange ring if and only if for every $a \in R$ there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$.) Indeed, if we have $a - e - u = (1-e)xa$ with $e \in \mathrm{Id}(R)$, $u \in U(R)$ and $x \in R$, then it is easy to verify that $g = 1 - u^{-1}eu$ is an idempotent that satisfies $g = gu^{-1}(1-x)a \in Ra$ and $1 - g = -(1-g)u^{-1}(1-a) \in R(1-a)$, which concludes the proof. In the next section we give an example of a weakly clean ring that is not clean.

3 The construction of the example

We begin with an example of a non-clean weakly clean ring.

Example 3.1. Let F be any field and $M_{\mathbb{N}}(F)$ the ring of all infinite matrices over F with finite columns. We denote the matrices in $M_{\mathbb{N}}(F)$, as usually, by $A = (a_{ij})_{i,j}$, where $(a_{ij})_i$ are columns and $(a_{ij})_j$ are rows in A. Define

$$\begin{aligned} R = \{ & A = (a_{ij})_{i,j} \in M_{\mathbb{N}}(F) : \\ & \text{there exists } n_A \in \mathbb{N} \text{ such that } a_{ij} = a_{i+1\ j+1} \text{ for every } i \geq n_A, j \geq 1\}. \end{aligned}$$

The ring R consists exactly of the matrices of the form

$$\begin{pmatrix} \vdots & \vdots & \vdots & & & & \\ * & * & * & \dots & & & \\ & & a_1 & a_2 & a_3 & \dots & \\ & & & a_1 & a_2 & a_3 & \dots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

(The first finitely many rows are arbitrary.) It is easy to verify that R is a ring. Let us prove that R is not clean. First define

$$\begin{aligned} I = \{ & A = (a_{ij})_{i,j} \in R : \\ & \text{there exists } n_A \in \mathbb{N} \text{ such that } a_{ij} = 0 \text{ for every } i \geq n_A \text{ and } j \geq 1\} \end{aligned}$$

(the set of all matrices in R with only finitely many nonzero rows). It is clear that I is a two-sided ideal in R. We show that the factor ring R/I is isomorphic to $F((X))$ (the field of formal Laurent series over F). Define a map $\psi : R \to F((X))$ as follows. Take $A = (a_{ij}) \in R$ and choose n large enough so that $n \geq n_A$ and $a_{i1} = 0$ for all $i \geq n+1$. Define

$$\psi(A) = \sum_{j=1}^{\infty} a_{nj} X^{j-n}.$$

Since A satisfies $a_{ij} = a_{i+1\ j+1}$ for all $i \geq n_A$, and $a_{i1} = 0$ for all $i \geq n+1$, the definition is independent of the choice of n. A simple verification shows that

ψ is actually a surjective ring homomorphism, with the kernel I. Therefore we have $R/I \cong F((X))$, as desired.

Now, since R/I is a field, every idempotent in R must be either 0 or 1 modulo I. In particular, every idempotent in R is upper triangular, if we ignore the first finitely many rows. In addition, every unit in R must be upper triangular (ignoring the first finitely many rows), since otherwise its inverse would be strictly upper triangular (ignoring the first finitely many rows) and therefore not injective (as an endomorphism of $F^{(\mathbb{N})}$), which is a contradiction. Therefore every idempotent and every unit in R is upper triangular (ignoring the first finitely many rows). But that means that the matrices that are nonzero below the main diagonal (ignoring the first finitely many rows) cannot be written as a sum of an idempotent and a unit in R. Therefore R is not clean.

Let us prove that R is weakly clean. Take any $A \in R$. First, suppose that A is upper triangular, ignoring the first finitely many rows. Then we shall see that A is actually clean in R. Write a block decomposition

$$A = \begin{pmatrix} A_0 & X \\ 0 & T \end{pmatrix},$$

where A_0 is a finite matrix and $T = (t_{ij})$ upper triangular that satisfies $t_{ij} = t_{i+1\ j+1}$ for every i, j. Since by Han and Nicholson [3] the ring of finite matrices over F is clean, we may write $A_0 = E_0 + U_0$ for an idempotent E_0 and a unit U_0. If the main diagonal of T is nonzero, then T is invertible in R and we can write

$$A = \begin{pmatrix} E_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} U_0 & X \\ 0 & T \end{pmatrix},$$

where the first matrix is an idempotent and the second is invertible in R. If the main diagonal of T is zero, then $T - 1$ is invertible and we can write

$$A = \begin{pmatrix} E_0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} U_0 & X \\ 0 & T-1 \end{pmatrix},$$

with the first matrix an idempotent and the second invertible. Hence A is clean in R.

Now suppose that A is not upper triangular (ignoring the first few rows). As before, write

$$\begin{pmatrix} A_0 & X \\ K & T \end{pmatrix},$$

where $A_0 \in M_n(F)$ is a finite matrix. We choose n large enough so that we have $n \geq n_A$. Since $T = (t_{ij})$ satisfies $t_{ij} = t_{i+1\ j+1}$ for every i, j, and T is not strictly upper triangular, T has a left inverse in R. Write $ST = 1$. Note that S can be chosen such that $SK = 0$. Similarly, since $1-T$ is not strictly upper triangular, we can find $V \in R$ such that $V(1-T) = 1$ and $VK = 0$. By [3], $M_n(F)$ is a clean ring, hence we may write $A_0 = E_0 + U_0$, with E_0 an idempotent and U_0 invertible. Now, with these observations in mind, it is easy to check that

$$E = \begin{pmatrix} E_0 & E_0XV \\ 0 & 0 \end{pmatrix},\ U = \begin{pmatrix} U_0 & XV \\ 0 & -1 \end{pmatrix},\ Z = \begin{pmatrix} E_0 & -XV - E_0XS \\ 0 & 1+S \end{pmatrix}$$

are matrices, with $E \in \mathrm{Id}(R)$ and $U \in U(R)$, that satisfy $A - E - U = (1-E)ZA$. Therefore A is weakly clean in R. This finishes the proof.

Remark 3.2. The ring R constructed above is isomorphic to the opposite ring of a well-known example due to Bergman (see [4, Example 1]). The isomorphism between the rings is given by

$$A = (a_{ij})_{i,j} \in R \ \mapsto \ \Big(\sum_{i=0}^{\infty} \lambda_i X^i \mapsto \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} \lambda_j a_{j+1\ i+1})X^i\Big).$$

Note that the mapping on the right is an endomorphism of the ring of formal power series $F[[X]]$, as an F-module. It could be checked that the above mapping is well-defined and that it is indeed a ring isomorphism (see [4] for the exact definition of Bergman's ring). However, our construction offers a different approach and also provides a very simple argument that the ring R is not clean, even for fields F with $\mathrm{char}(F) = 2$. Previously, this fact was only known when $\mathrm{char}(F) \neq 2$ (see [2]).

We need the following proposition, which, roughly speaking, states that the weakly clean property behaves nicely under taking corners and matrix extensions.

Proposition 3.3. *Let R be a ring.*

(i) *If e is an idempotent in R such that eRe and $(1-e)R(1-e)$ are weakly clean rings, then R is weakly clean. In particular, if R is a weakly clean ring, then $M_n(R)$ is weakly clean for every $n \in \mathbb{N}$.*

(ii) *If R is weakly clean, then eRe is weakly clean for every $e \in \mathrm{Id}(R)$.*

Proof. (i) In the proof we use techniques similar to those in [3]. We take an idempotent $e \in R$ and write $f = 1 - e$. Suppose that eRe and fRf are weakly clean rings. We use the Pierce decomposition of the ring R:

$$R = \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.$$

Since a is weakly clean in eRe, we may write $a - g - u = (e - g)xa$, where $g \in \mathrm{Id}(eRe)$, $u \in U(eRe)$ and $x \in eRe$. Let u' be the inverse of u in eRe, so that we have $uu' = u'u = e$. Since $d - cu'(e - x + gx)b$ is weakly clean in fRf, we may write $d - cu'(e - x + gx)b - h - v = (f - h)y(d - cu'(e - x + gx)b)$, where $h \in \mathrm{Id}(fRf)$, $v \in U(fRf)$ and $y \in fRf$. It follows that $(f - y + hy)cu'(e - x + gx)b = (f - y + hy)d - h - v$. Now it is easy to verify that

$$E = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, \; U = \begin{pmatrix} e & 0 \\ (f - y + hy)cu' & f \end{pmatrix} \begin{pmatrix} u & (e - x + gx)b \\ 0 & v \end{pmatrix} =$$

$$= \begin{pmatrix} u & (e - x + gx)b \\ (f - y + hy)c & (f - y + hy)d - h \end{pmatrix} \text{ and } X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

are matrices, with E an idempotent and U a unit, that satisfy $A - E - U = (1 - E)XA$. Hence A is weakly clean in R.

(ii) The proof of this part uses ideas similar to those in the proof of Proposition 2.2 ($\Rightarrow$). There, it is assumed that the matrix A is clean, but this assumption can be relaxed to assume that A is only weakly clean. Let us present the proof this time without the use of Pierce decomposition.

Suppose that R is weakly clean and $e \in R$ is an idempotent. Take $a \in eRe$. Since a is weakly clean in R, there exist $g \in \mathrm{Id}(R)$ and $u \in U(R)$ such that $a - g - u \in (1 - g)Ra$. Write $h = 1 - g$ and $f = 1 - e$. Since $af = 0$, we have $gf + uf = 0$, so $eu^{-1}gf = -eu^{-1}uf = 0$. Hence we have $eu^{-1}g \in eRe$. From the equality $eu^{-1}geue = eu^{-1}gue = eu^{-1}g(u - uf) = eu^{-1}g(u + gf) = eu^{-1}gu + eu^{-1}gf = eu^{-1}gu$ it follows that $(eueu^{-1}g)^2 = eueu^{-1}geueu^{-1}g = eueu^{-1}guu^{-1}g = eueu^{-1}g$. Hence $\epsilon = eueu^{-1}g$ is an idempotent in eRe.

Since $hfu^{-1}g$ is nilpotent, $v = (1 - hfu^{-1}g)u$ is invertible in R. We have $vf = uf - hfu^{-1}guf = -gf + hfu^{-1}gf = -gf - hfu^{-1}uf = -gf - hf = -f$, which also implies $v^{-1}f = -f$. Therefore $evev^{-1}e = e(v - vf)v^{-1}e = e(v + f)v^{-1}e = e$, and similarly $ev^{-1}eve = e$. Hence $\nu = eve$ is a unit in eRe.

Applying $ga = g + gu$, we have

$$\epsilon(a - \epsilon - \nu) = \epsilon a - \epsilon - \epsilon ve = eueu^{-1}(g + gu) - eueu^{-1}g - eueu^{-1}g(1 - hfu^{-1}g)ue = eueu^{-1}gu - eueu^{-1}gue = eueu^{-1}guf = -eueu^{-1}gf = 0.$$

Therefore $a - \epsilon - \nu \in (e - \epsilon)R$. Applying $\nu = e(1 - (1 - g)fu^{-1}g)ue = e(1 + gfu^{-1}g)ue = eue - eufu^{-1}gue = eue - eufu^{-1}(1 - h)ue = eue + eufu^{-1}hue$, we have

$$\epsilon + \nu = eueu^{-1}g + eue + eufu^{-1}hue = eueu^{-1}(g + u) + eufu^{-1}h(ue + ge).$$

Since $ge + ue = g + u \in Ra$, it follows that $\epsilon + \nu \in Ra$, and therefore $a - \epsilon - \nu \in Ra$. Hence we have $a - \epsilon - \nu \in (e - \epsilon)R \cap Ra = (e - \epsilon)Ra$. Thus a is weakly clean in eRe, which concludes the proof. $\square$

Now we are ready to give an example of a clean ring S and an idempotent $e \in S$ such that eSe is not clean.

Example 3.4. Let R be any non-clean weakly clean ring with $\mathrm{char}(R) = 2$. For instance, take R from Example 3.1 with $\mathrm{char}(F) = 2$. Let $M_{\mathbb{N}}(R)$ denote the ring of column-finite matrices over R, and let T be a (non-unital) subring of all matrices that have only finitely many nonzero entries. Since $\mathbb{Z}_2$ is a subring of $M_{\mathbb{N}}(R)$, we may define

$$S = T + \mathbb{Z}_2.$$

Clearly, S is a (unital) subring of $M_{\mathbb{N}}(R)$.

The ring S is clean. To see this, it suffices to check that every matrix in T is clean. (Indeed, since an element x is clean if and only if $1 - x$ is clean.) Thus let $A \in T$. Choose n such that A has the block decomposition $A = \left(\begin{smallmatrix} A_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with $A_0 \in M_n(R)$. Since R is weakly clean, $M_n(R)$ is weakly clean by Proposition 3.3 (i). Hence by Proposition 2.2,

$$A' = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$$

is clean in $M_2(M_n(R)) = M_{2n}(R)$. Thus we have $E' \in \mathrm{Id}(M_{2n}(R))$ and $U' \in U(M_{2n}(R))$ such that $A' = E' + U'$. But that means that A can be written as

$$A = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E' & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} U' & 0 \\ 0 & 1 \end{pmatrix},$$

with the first matrix an idempotent and the second invertible. Therefore A is clean in S, concluding the proof.

Now we have a clean ring S. Taking the idempotent

$$e = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix} \in S,$$

we have the corner ring $eSe \cong R$, which is not clean.

Remark 3.5. In the above construction, it is essential that the idempotent e is 'small' in S, i. e. $SeS \neq S$. In [3], a stronger form of the question was stated, whether there exists a clean ring and an idempotent $e \in S$, such that the corner ring eSe is not clean and $SeS = S$. We do not know the answer to that question.

4 Final remarks

Although we were able to provide an example of a non-clean weakly clean ring, we do not know whether there exists an exchange ring that is not weakly clean. Since there seems to be a lack of examples of non-clean exchange rings (so far the only known example is Bergman's ring), it would be interesting to provide such an example. The following lemma shows that there, indeed, might be a substancial 'difference' between exchange rings and weakly clean rings.

Lemma 4.1. *A ring R is weakly clean if and only if for every $a \in R$ there exist an idempotent $e \in Ra$ and a unit $u \in U(R)$ such that $1-e = (1-e)u(1-a)$.*

Proof. Direction ($\Rightarrow$) follows from Remark 2.4. To see the converse, suppose that we have $e \in \mathrm{Id}(R)$, $u \in U(R)$ and $r \in R$, such that $e = ra$ and $1-e = (1-e)u(1-a)$. Then an easy computation shows that $a-(1-u^{-1}eu)+u^{-1} = u^{-1}e(u+r)a \in u^{-1}euRa$, as desired. □

In [5], Bergman's ring was used to show that the extension of a clean ring by another clean ring need not be clean. With our understanding of Bergman's ring, we are able to give a shorter and more elementary proof. In conclusion, we write down this proof.

Following [7], a ring I (possibly without unity) is called a *clean general ring* if every element $a \in I$ can be written as $a = e + q$, where e is an idempotent in I and q is an element of the set

$$\{q \in I : \text{there exists } p \in I \text{ such that } q + p + qp = q + p + pq = 0\}.$$

For unital rings, this definition coincides with the standard definition of clean rings (see [7] or [5] for details).

If I is a right or left ideal of a ring R, then we say that idempotents *lift strongly* modulo I if for every $a \in R$ that satisfies $a^2 - a \in I$, there exists $e \in \mathrm{Id}(R)$ such that $e - a \in I$ and $e \in Ra$ (see [8]).

Proposition 4.2 ([5, Example 4]). *There exist a ring R and an ideal I of R, such that R/I and I are both clean rings and idempotents lift strongly modulo I, but R is not clean.*

Proof. Let R and I be as in Example 3.1. Since the factor ring R/I is isomorphic to the field $F((X))$, R/I is a clean ring.

To show that idempotents lift strongly modulo I, take $A \in R$ with $A^2 - A \in I$. Since $A + I$ is an idempotent in the field R/I, we have either $A \in I$ or $A - 1 \in I$. In the first case, the idempotent $E = 0$ will satisfy $A - E \in I$ and $E \in RA$. In the second case, writing in the block decomposition, we have $A = \left(\begin{smallmatrix} A_0 & X \\ 0 & 1 \end{smallmatrix}\right)$. Then $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an idempotent in RA that satisfies $E - A \in I$. Therefore idempotents lift strongly modulo I.

To finish the proof, we show that I is a clean general ring. Following the definition, we take any $A \in I$. In the block decomposition, we have $A = \left(\begin{smallmatrix} A_0 & X \\ 0 & 0 \end{smallmatrix}\right)$, with $A_0 \in M_n(F)$. Since $M_n(F)$ is a clean ring, we may write

$1 + A_0 = E_0 + U_0$, where $E_0 \in \mathrm{Id}(M_n(F))$ and $U_0 \in U(M_n(F))$. Now, an easy computation shows that

$$E = \begin{pmatrix} E_0 & 0 \\ 0 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} U_0 - 1 & X \\ 0 & 0 \end{pmatrix}, \; P = \begin{pmatrix} {U_0}^{-1} - 1 & -{U_0}^{-1}X \\ 0 & 0 \end{pmatrix}$$

are matrices in I, with $E \in \mathrm{Id}(R)$, that satisfy $A = E+Q$ and $Q+P+QP = Q+P+PQ = 0$. Therefore I is a clean general ring. $\square$

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