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# Edmonds maps on the Fricke-Macbeath curve

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**Abstract**

In 1985, L. D. James and G. A. Jones proved that the complete graph $K_n$ defines a clean dessin d'enfant (the bipartite graph is given by taking as the black vertices the vertices of $K_n$ and the white vertices as middle points of edges) if and only if $n = p^e$, where $p$ is a prime. Later, in 2010, G. A. Jones, M. Streit and J. Wolfart computed the minimal field of definition of them. The minimal genus $g > 1$ of these types of clean dessins d'enfant is $g = 7$, obtained for $p = 2$ and $e = 3$. In that case, there are exactly two such clean dessins d'enfant (previously known as Edmonds maps), both of them defining the Fricke-Macbeath curve (the only Hurwitz curve of genus 7) and both forming a chiral pair. The uniqueness of the Fricke-Macbeath curve asserts that it is definable over $\mathbb{Q}$, but both Edmonds maps cannot be defined over $\mathbb{Q}$; in fact they have as minimal field of definition the quadratic field $\mathbb{Q}(\sqrt{-7})$. It seems that no explicit models for the Edmonds maps over $\mathbb{Q}(\sqrt{-7})$ are written in the literature. In this paper we start with an explicit model $X$ for the Fricke-Macbeath curve provided by Macbeath, which is defined over $\mathbb{Q}(e^{2\pi i/7})$, and we construct an explicit birational isomorphism $L : X \to Z$, where $Z$ is defined over $\mathbb{Q}(\sqrt{-7})$, so that both Edmonds maps are also defined over that field.

*Keywords: Riemann surface, algebraic curve, dessin d'enfant.*

*Math. Subj. Class.: 30F20, 30F10, 14Q05, 14H45, 14E05*

## 1 Introduction

A dessin d'enfant $D$ on a closed orientable surface is given by a bipartite map on it (vertices will be colored black and white). The dessin d'enfant is called clean if the white vertices have all valence 2.

A Belyi curve is an irreducible non-singular complex projective algebraic curve (i.e. a closed Riemann surface) $S$ admitting a non-constant meromorphic map $\beta : S \to \widehat{\mathbb{C}}$ with

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at most three branch values; which we assume to be inside the set $\{\infty, 0, 1\}$; we say that $(S, \beta)$ is a Belyi pair. Two Belyi pairs $(S_1, \beta_1)$ and $(S_2, \beta_2)$ are called equivalent, denoted this by the symbol $(S_1, \beta_1) \cong (S_2, \beta_2)$, if there is an isomorphism $f : S_1 \to S_2$ so that $\beta_2 \circ f = \beta_1$.

A subfield $\mathbb{K}$ of $\overline{\mathbb{Q}}$ is called a field of definition of a Belyi pair $(S, \beta)$ if there an equivalent Belyi pair $(\widehat{S}, \widehat{\beta})$ where $\widehat{S}$ and $\widehat{\beta}$ are both defined over $\mathbb{K}$. As a consequence of Belyi's theorem [1], the field of algebraic numbers $\overline{\mathbb{Q}}$ is a field of definition of every Belyi pair.

Each Belyi pair $(S, \beta)$ defines a dessin d'enfant on $S$ by taking the edges as the components of $\beta^{-1}((0, 1))$, the black vertices as the points in $\beta^{-1}(0)$ and the white vertices as the points in $\beta^{-1}(1)$. Conversely, as a consequence of the uniformization theorem, every dessin d'enfant on a closed orientable surface induces a (unique up to isomorphisms) Riemann surface structure (being a Belyi curve) and a Belyi map so that the original dessin d'enfant is homotopic to the one associated to the Belyi pair [11, 15].

A field of definition of a dessin d'enfant is a field of definition of the corresponding Belyi pair.

As there is a natural (faithful) action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the collection of Belyi pairs [13], it also provides a (faithful) action on dessins d'enfant. This action may help in the study of the internal structure of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in terms of combinatorial data.

Let us consider a dessin d'enfant $D$, which is defined by the Belyi pair $(S, \beta)$. By Belyi's theorem, we may assume that both $S$ and $\beta$ are defined over $\overline{\mathbb{Q}}$. The field of moduli of $D$ is then defined as the fixed field of the subgroup $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : (S, \beta) \cong (S^\sigma, \beta^\sigma)\}$. The field of moduli of $D$ is always contained in any field of definition of it, but it may be that the field of moduli is not a field of definition of it. Both, the computation of the field of moduli of a dessin d'enfant and to decide if the dessin d'enfant can be defined over it, are in general difficult problems. If the dessin d'enfant is regular, that is, the Belyi map $\beta$ is a Galois branched cover, then J. Wolfart [19] proved that $D$ can be defined over its field of moduli. Also, if the dessin d'enfant has no non-trivial automorphisms, then it is definable over its field of moduli as a consequence of Weil's descent theorem [16]. So, the problem to decide if the field of moduli is a field of definition appears when it has non-trivial automorphisms but it is non-regular.

In 1985, L. D. James and G. A. Jones [10] proved that the complete graph $K_n$ defines a clean dessin d'enfant (the bipartite graph is given by taking as the black vertices the vertices of $K_n$ and the white vertices as middle points of edges) if and only if $n = p^e$, where $p$ is a prime. Later, in 2010, G. A. Jones, M. Streit and J. Wolfart [12] computed the minimal field of definition of such clean dessins d'enfant. The minimal genus $g > 1$ of these types of clean dessins d'enfants is $g = 7$, obtained for $p = 2$ and $e = 3$. In that case, there are exactly two (non-equivalent) such dessins (previously known as Edmonds maps), both of them defining the Fricke-Macbeath curve (the only Hurwitz curve of genus 7) and both forming a chiral pair. The uniqueness of the Fricke-Macbeath curve asserts that it is definable over $\mathbb{Q}$, but each of the two Edmonds maps cannot be defined over $\mathbb{Q}$; they have as minimal field of definition the quadratic field $\mathbb{Q}(\sqrt{-7})$ [12]. No explicit models for the Edmonds maps over $\mathbb{Q}(\sqrt{-7})$ seems to be written in the literature.

In Section 2 we recall an explicit model $X$ for the Fricke-Macbeath curve provided by Macbeath, which is defined over $\mathbb{Q}(e^{2\pi i/7})$, and describe both Edmonds maps. We also provide (as matter of interest for specialists) two different equations, over $\mathbb{Q}$, for the Fricke-Macbeath curve which were independently obtained by Bradley Brock (personal communication) and by Maxim Hendriks in his Ph.D. Thesis [7]. In Section 3 we provide an explicit birational isomorphism $L : X \to Z$, where $Z$ is defined over $\mathbb{Q}(\sqrt{-7})$. In this model we obtain that the two Belyi maps defining the two Edmonds maps are defined over $\mathbb{Q}$; in particular, this provides explicit models for both Edmonds maps over $\mathbb{Q}(\sqrt{-7})$ as desired. In Section 4 we provide an explicit birational isomorphism $\widehat{L} : X \to W$, where $W$ is defined over $\mathbb{Q}$. Unfortunately, the explicit equations over $\mathbb{Q}$ are not very simple (they are long ones) and they can be found in [9]. In Section 5 we construct a generalized Fermat curve $\widehat{S}$ of type $(2, 6)$ [5] that covers the Fricke-Macbeath curve and we provide a description of the three elliptic curves appearing in the equations of $X$ given by Macbeath. Another model of the Fricke-Macbeath curve is also described.

## 2 Macbeath's equations of the Fricke-Macbeath curve and the two Edmonds maps

In this section we recall equations of the Fricke-Macbeath curve, obtained by Macbeath in [14], and we describe both Edmonds maps discovered in [12]. As a matter of interest to specialists, we also describe two different models over $\mathbb{Q}$, one obtained by Bradley Brock (personal communication) and the other by Maxim Hendriks in his Ph.D. Thesis [7].

### 2.1 Hurwitz curves

It is well known that $|\mathrm{Aut}(S)| \leq 84(g-1)$ (Hurwitz upper bound) if $S$ is a closed Riemann surface of genus $g \geq 2$. In the case that $|\mathrm{Aut}(S)| = 84(g-1)$, one says that $S$ is a Hurwitz curve. In this last situation, the quotient orbifold $S/\mathrm{Aut}(S)$ has signature $(0; 2, 3, 7)$, that is, $S = \mathbb{H}^2/\Gamma$, where $\Gamma$ is a torsion free normal subgroup of finite index in the triangular Fuchsian group $\Delta = \langle x, y : x^2 = y^3 = (xy)^7 = 1\rangle$ acting as isometries of the hyperbolic plane $\mathbb{H}^2$.

Wiman [17] noticed that there is no Hurwitz curve in each genera $g \in \{2, 4, 5, 6\}$ and there is exactly one Hurwitz curve (up to isomorphisms) of genus three, this being Klein's quartic $x^3y + y^3z + z^3x = 0$; whose automorphisms group is the simple group $\mathrm{PSL}(2, 7)$ (of order 168).

### 2.2 Macbeath's equations of the Fricke-Macbeath curve

In [14] Macbeath observed that in genus seven there is exactly one (up to isomorphisms) Hurwitz curve, called the Fricke-Macbeath curve; its automorphisms group is the simple group $\mathrm{PSL}(2, 8)$, consisting of 504 symmetries. In the same paper, Macbeath computed the following explicit equations over $\mathbb{Q}(\rho)$, where $\rho = e^{2\pi i/7}$, for the Fricke-Macbeath curve (involving three particular elliptic curves):

$$X = \left\{ \begin{array}{c} y_1^2 = (x-1)(x-\rho^3)(x-\rho^5)(x-\rho^6) \\ \\ y_2^2 = (x-\rho^2)(x-\rho^4)(x-\rho^5)(x-\rho^6) \\ \\ y_4^2 = (x-\rho)(x-\rho^3)(x-\rho^4)(x-\rho^5) \end{array} \right\} \subset \mathbb{C}^4. \tag{2.1}$$

In Section 5 we provide a rough explanation about the elliptic curves in the above equations (different from the approach given in [14]) in geometric terms of the highest regular branched Abelian cover of the orbifold $X/G$ of signature $(0; 2, 2, 2, 2, 2, 2, 2)$.

Another interesting fact on the Fricke-Macbeath curve is that its jacobian variety is isogenous to $E^7$ where $E$ is the elliptic curve with $j$-invariant $j(E) = 1792$ ($E$ does not have complex multiplication); see, for instance, [2]. There are not to many explicit examples of Riemann surfaces whose jacobian variety is isogenous to the product of elliptic curves (see [6]).

### 2.3 Equations over $\mathbb{Q}$ of the Fricke-Macbeath curve

The uniqueness up to isomorphisms of the Fricke-Macbeath curve asserts that its field of moduli is the field of rational numbers $\mathbb{Q}$. As quasiplatonic curves can be defined over their fields of moduli [19] and Hurwitz curve are quasiplatonic curves, it follows that the Fricke-Macbeath curve can be defined over $\mathbb{Q}$. When the author put a first version of this paper in Arxiv [9] we didn't know of explicit equations of the Fricke-Macbeath curve over $\mathbb{Q}$. Later, Bradley Brock sent me an e-mail in which he told me that, using some suitable change of coordinates on the above equations for $X$, he was able to compute a plane equation for $X$ over $\mathbb{Q}$, with some simple nodes as singularities, given as

$$1 + 7xy + 21x^2y^2 + 35x^3y^3 + 28x^4y^4 + 2x^7 + 2y^7 = 0.$$

An automorphism of order 7 is given by $b(x, y) = (\rho x, \rho^{-1}y)$ and one of order two is given by $a_1(x, y) = (y, x)$.

The following model over $\mathbb{Q}$, for the Fricke-Macbeath curve, was recently computed by Maxim Hendriks in his Ph.D. Thesis [7]

$$\left\{\begin{array}{c} -x_1x_2 + x_1x_0 + x_2x_6 + x_3x_4 - x_3x_5 - x_3x_0 - x_4x_6 - x_5x_6 = 0, \\ x_1x_3 + x_1x_6 - x_2^2 + 2x_2x_5 + x_2x_0 - x_3^2 + x_4x_5 - x_4x_0 - x_5^2 = 0, \\ x_1^2 - x_1x_3 + x_2^2 - x_2x_4 - x_2x_5 - x_2x_0 - x_3^2 + x_3x_6 + 2x_5x_0 - x_0^2 = 0, \\ x_1x_4 - 2x_1x_5 + 2x_1x_0 - x_2x_6 - x_3x_4 - x_3x_5 + x_5x_6 + x_6x_0 = 0, \\ x_1^2 - 2x_1x_3 - x_2^2 - x_2x_4 - x_2x_5 + 2x_2x_0 + x_3^2 + x_3x_6 + x_4x_5 + x_5^2 - x_5x_0 - x_6^2 = 0, \\ x_1x_2 - x_1x_5 - 2x_1x_0 + 2x_2x_3 - x_3x_0 - x_5x_6 + 2x_6x_0 = 0, \\ -2x_1x_2 - x_1x_4 - x_1x_5 + 2x_1x_0 + 2x_2x_3 - 2x_3x_0 + 2x_5x_6 - x_6x_0 = 0, \\ 2x_1^2 + x_1x_3 - x_1x_6 + 3x_2x_0 + x_4x_5 - x_4x_0 - x_5^2 + x_6^2 - x_0^2 = 0, \\ 2x_1^2 - x_1x_3 + x_1x_6 + x_2^2 + x_2x_0 + x_3^2 - 2x_3x_6 + x_4x_5 - x_4x_0 + x_5^2 - 2x_5x_0 + x_6^2 + x_0^2 = 0, \\ x_1^2 + x_1x_3 - x_1x_6 + 2x_2x_5 - 3x_2x_0 + 2x_3x_6 + x_4^2 + x_4x_5 - x_4x_0 + x_6^2 + 3x_0^2 = 0 \end{array}\right\} \subset \mathbb{P}^6.$$

In Section 4 we provide an explicit birational isomorphism $\widehat{L} : X \to W$, where $W$ is defined over $\mathbb{Q}$. The explicit form of $\widehat{L}$ may be used to compute explicit equation for $W$; this can be done with MAGMA [3].

### 2.4 A description of the two Edmonds maps

In the above model $X$ of the Fricke-Macbeath curve it is easy to see a group $\mathbb{Z}_2^3 \cong G = \langle A_1, A_2, A_3 \rangle < \mathrm{Aut}(X)$ where

$$A_1(x, y_1, y_2, y_4) = (x, -y_1, y_2, y_4),$$

$$A_2(x, y_1, y_2, y_4) = (x, y_1, -y_2, y_4),$$

$$A_3(x, y_1, y_2, y_4) = (x, y_1, y_2, -y_4).$$

The quotient Riemann orbifold $X/G$ has signature $(0; 2, 2, 2, 2, 2, 2, 2)$, that is, is the Riemann sphere with exactly 7 cone points of order 2.

An automorphism of order 7 of the Fricke-Macbeath curve is given in such a model by

$$B(x, y_1, y_2, y_4) = \left(\rho x, \rho^2 y_2, \rho^2 y_4, \rho^2 \frac{y_1 y_2}{(x-\rho^5)(x-\rho^6)}\right).$$

The automorphism $B$ normalizes $G$ and it induces, on the orbifold $X/G = \widehat{\mathbb{C}}$, the rotation $T(x) = \rho x$. Moreover, $X/\langle G, B\rangle$ has signature $(0; 2, 7, 7)$, that is, the group $\langle G, B\rangle$ defines a regular dessin d'enfant $(X, \beta)$, where $\beta(x, y_1, y_2, y_4) = x^7$ (this is one of the two Edmonds maps, but is defined over $\mathbb{Q}(\rho)$).

We may also see that $X$ admits the following anticonformal involution

$$J(x, y_1, y_2, y_4) = \left(\frac{1}{\overline{x}}, \frac{\overline{y_1}}{\overline{x}^2}, \frac{\rho^5\overline{y_2}}{\overline{x}^2}, \frac{\rho^3\overline{y_4}}{\overline{x}^2}\right).$$

It can be seen that $JBJ = B$ and $JA_jJ = A_j$, for $j = 1, 2, 3$. In this way, one gets another regular dessin d'enfant $(X, \delta)$, where $\delta(x, y_1, y_2, y_4) = 1/x^7$ (this is the other Edmonds map, again defined over $\mathbb{Q}(\rho)$).

As $\delta = C \circ \beta \circ J$, where $C(x) = \overline{x}$, we have that the two regular dessins d'enfant described above are chirals.

## 3 An explicit model of the Edmonds maps over $\mathbb{Q}(\sqrt{-7})$

In this section we will construct an explicit biregular isomorphism $L : X \to Z$, where $Z$ is defined over $\mathbb{Q}(\sqrt{-7})$, so that both Edmonds maps are defined over such a field.

Note that $\mathbb{Q}(\sqrt{-7}) = \mathbb{Q}(\rho + \rho^2 + \rho^4)$ since $\rho + \rho^2 + \rho^4 = \frac{1}{2}(\sqrt{-7} - 1)$. Most of the computations have been carried out with MAGMA [3] or with MATHEMATICA [20].

### 3.1

Let $N = \mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q}(\sqrt{-7})) = \langle\tau\rangle \cong \mathbb{Z}_3$, where $\tau(\rho) = \rho^2$. If we set

$$\vec{x} = (x_1, x_2, x_3, x_4) = (x, y_1, y_2, y_4),$$

then it is not difficult to check that $\{f_e = I, f_\tau, f_{\tau^2}\}$ is a Weil datum (i.e., they satisfies the Weil co-cycle condition in Weil's descent theorem [16]) with respect to the Galois extension $\mathbb{Q}(\rho)/\mathbb{Q}(\sqrt{-7})$, where $I$ denotes the identity and

$$f_\tau(\vec{x}) = \left(x, y_1, y_4, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}\right),$$

$$f_{\tau^2}(\vec{x}) = \left(x, y_1, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}, y_2\right).$$

### 3.2

Let us consider the rational map

$$\Phi_1 : X \to \mathbb{C}^{12}$$

$$(x, y_1, y_2, y_4) \mapsto (\vec{x}, \vec{w}, \vec{v}),$$

where

$$\vec{w} = (w_1, w_2, w_3, w_4) = f_\tau(\vec{x}),$$

$$\vec{v} = (v_1, v_2, v_3, v_4) = f_{\tau^2}(\vec{x}).$$

We may see that $\Phi_1$ produces a birational isomorphism between $X$ and $\Phi_1(X)$ (its inverse is just the projection on the $\vec{x}$-coordinate). Equations defining the algebraic curve $\Phi_1(X)$ are the following ones

$$\Phi_1(X) = \left\{ \begin{array}{l} x_2^2 = (x_1 - 1)(x_1 - \rho^3)(x_1 - \rho^5)(x_1 - \rho^6) \\ \\ x_3^2 = (x_1 - \rho^2)(x_1 - \rho^4)(x_1 - \rho^5)(x_1 - \rho^6) \\ \\ x_4^2 = (x_1 - \rho)(x_1 - \rho^3)(x_1 - \rho^4)(x_1 - \rho^5) \\ \\ w_1 = x_1,\ w_2 = x_2,\ w_3 = x_4,\ w_4 = \dfrac{x_3 x_4}{(x_1 - \rho^4)(x_1 - \rho^5)}, \\ \\ v_1 = x_1,\ v_2 = x_2,\ v_3 = \dfrac{x_3 x_4}{(x_1 - \rho^4)(x_1 - \rho^5)},\ v_4 = x_3 \end{array} \right\} \tag{3.1}$$

**3.3**

We consider the linear permutation action of $N$ on the coordinates of $\mathbb{C}^{12}$ defined by

$$\Theta_1(\tau)(\vec{x}, \vec{w}, \vec{v}) = (\vec{w}, \vec{v}, \vec{x}).$$

Let us notice that the stabilizer of $\Phi_1(X)$, with respect to the above permutation action, is trivial since

$$\{\eta \in N : \Theta_1(\eta)(\Phi_1(X)) = \Phi_1(X)\} = \{\eta \in N : X^\eta = X\} = \{e\}.$$

**3.4**

Each $\theta \in \mathrm{Gal}(\mathbb{C})$ induces a natural bijection

$$\widehat{\theta} : \mathbb{C}^{12} \to \mathbb{C}^{12} : (y_1, \ldots, y_{12}) \mapsto (\theta(y_1), \ldots, \theta(y_{12})).$$

It is not hard to see that $\widehat{\theta}(X) = X^\theta$.

**3.5**

If $\theta \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\sqrt{-7}))$, then we denote by $\theta_N$ is projection in $N$. With this notation, we see that the following diagram commutes (see also [8])

$$\begin{array}{lcl} X & \overset{\Phi_1}{\to} & \Phi_1(X) \\ \downarrow f_{\theta_N} & & \downarrow \Theta_1(\theta_N) \\ X^{\theta_N} & \overset{\Phi_1^{\theta_N}}{\to} & \Theta_1(\theta_N)(\Phi_1(X)) = \Phi_1^{\theta_N}(X^{\theta_N}) = \Phi_1(X)^{\theta_N} \\ \downarrow \widehat{\theta}^{-1} & & \downarrow \widehat{\theta}^{-1} \\ X & \overset{\Phi_1}{\to} & \Phi_1(X) \end{array} \tag{3.2}$$

and, for every $\eta, \theta \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\sqrt{-7}))$, that

$$(*) \quad \Theta_1(\eta_N) \circ \widehat{\theta} = \widehat{\theta} \circ \Theta_1(\eta_N).$$

### 3.6

A generating set of invariant polynomials for the linear action $\Theta_1(N)$ can be obtained with MAGMA as

$$\begin{array}{lcllcl} t_1 & = & x_1+w_1+v_1, & t_2 & = & x_2+w_2+v_2 \\ t_3 & = & x_3+w_3+v_3, & t_4 & = & x_4+w_4+v_4 \\ t_5 & = & x_1^2+w_1^2+v_1^2, & t_6 & = & x_2^2+w_2^2+v_2^2 \\ t_7 & = & x_3^2+w_3^2+v_3^2, & t_8 & = & x_4^2+w_4^2+v_4^2 \\ t_9 & = & x_1^3+w_1^3+v_1^3, & t_{10} & = & x_2^3+w_2^3+v_2^3 \\ t_{11} & = & x_3^3+w_3^3+v_3^3, & t_{12} & = & x_4^3+w_4^3+v_4^3 \end{array}$$

The map

$$\Psi_1 : \mathbb{C}^{12} \to \mathbb{C}^{12}$$
$$(\vec{x}, \vec{w}, \vec{v}) \mapsto (t_1, ..., t_{12})$$

clearly satisfies the following properties:

$$\left\{ \begin{array}{l} \Psi_1^{\tau^j} = \Psi_1, \ j = 0,1,2; \\ \\ \Psi_1 \circ \Theta_1(\tau^j) = \Psi_1, \ j = 0,1,2. \end{array} \right. \tag{3.3}$$

Also (as we have chosen a set of generators of the invariant polynomials for the action of $\Theta_1(N)$), it holds that $\Psi_1$ is a branched regular cover with Galois group $N$. It turns out that, if we set $Z_1 = \Psi_1(\Phi_1(X))$ and $L_1 = \Psi_1 \circ \Phi_1$, then

$$L_1 : X \to Z_1$$

is a birational isomorphism (since the stabilizer of $\Phi_1(X)$ is trivial).

### 3.7

If $\eta \in N$, then

$$Z_1^\eta = L_1(X)^\eta = L_1^\eta(X^\eta) = \Psi_1^\eta \circ \Phi_1^\eta(X^\eta) = \Psi_1 \circ \Theta_1(\eta)(\Phi_1(X)) = \\ \Psi_1 \circ \Phi_1(X) = L_1(X) = Z_1,$$

so $Z_1$ can be defined by polynomials with coefficient over $\mathbb{Q}(\sqrt{-7})$.

### 3.8

Next, we proceed to compute explicit equations for $Z_1$ and the inverse $L_1^{-1} : Z_1 \to X$.

The following equalities hold:

$$x_1 = \frac{t_1}{3}, \quad x_2 = \frac{t_2}{3}, \quad t_4 = t_3$$

$$(*)\ x_4 = \frac{(t_3 - x_3)(\frac{t_1}{3} - \rho^4)(\frac{t_1}{3} - \rho^5)}{x_3 + (\frac{t_1}{3} - \rho^4)(\frac{t_1}{3} - \rho^5)}$$

$$t_5 = \frac{t_1^2}{3}, \quad t_6 = \frac{t_2^2}{3}, \quad t_8 = t_7$$

$$(**)\ x_4^2 = \frac{(t_7 - x_3^2)(\frac{t_1}{3} - \rho^4)^2(\frac{t_1}{3} - \rho^5)^2}{x_3^2 + (\frac{t_1}{3} - \rho^4)^2(\frac{t_1}{3} - \rho^5)^2}$$

$$t_9 = \frac{t_1^3}{9}, \quad t_{10} = \frac{t_2^3}{9}, \quad t_{12} = t_{11}$$

$$(***)\ x_4^3 = \frac{(t_{11} - x_3^3)(\frac{t_1}{3} - \rho^4)^3(\frac{t_1}{3} - \rho^5)^3}{x_3^3 + (\frac{t_1}{3} - \rho^4)^3(\frac{t_1}{3} - \rho^5)^3}$$

Equality $(*)$ permits to obtain $x_4$ uniquely in terms of $t_1$ and $x_3$ and the equation

$$x_2^2 = (x_1 - 1)(x_1 - \rho^3)(x_1 - \rho^5)(x_1 - \rho^6)$$

provides a polynomial equation (relating $t_1$ and $t_2$) given by $P_1(t_1, t_2, t_3, t_7, t_{11}) = 0$, where

$$P_1(t_1, t_2, t_3, t_7, t_{11})$$
$$||$$
$$-81+27(1+(\rho+\rho^2+\rho^4))t_1+9t_1^2-3(\rho+\rho^2+\rho^4)t_1^3-t_1^4+9t_2^2 \in \mathbb{Q}(\sqrt{-7})[t_1, t_2, t_3, t_7, t_{11}].$$

Equation

$$x_3^2 = (x_1 - \rho^2)(x_1 - \rho^4)(x_1 - \rho^5)(x_1 - \rho^6)$$

permits to obtain the new equation

$$(1)\ x_3^2 = (t_1 - 3\rho^2)(t_1 - 3\rho^4)(t_1 - 3\rho^5)(t_1 - 3\rho^6)/81,$$

and the equation

$$x_4^2 = (x_1 - \rho)(x_1 - \rho^3)(x_1 - \rho^4)(x_1 - \rho^5)$$

provides the equation

$$(2)\ x_4^2 = (t_1 - 3\rho)(t_1 - 3\rho^3)(t_1 - 3\rho^4)(t_1 - 3\rho^5)/81.$$

In this way, by replacing the above values for $x_3^2$ and $x_4^2$ (obtained in $(1)$ and $(2)$) in the equality $(**)$, we obtain the polynomial equation $P_2(t_1, t_2, t_3, t_7, t_{11}) = 0$, where

$$P_2(t_1, t_2, t_3, t_7, t_{11})$$
$$||$$
$$27 + 27(\rho + \rho^2 + \rho^4) - 18t_1 - 3(1 + (\rho + \rho^2 + \rho^4))t_1^2 - 2t_1^3 - t_1^4 + 27t_7 \in \mathbb{Q}(\sqrt{-7})[t_1, t_2, t_3, t_7, t_{11}].$$

Now, if we replace, in equality $(***)$, $x_3^3$ by $x_3(x_1-\rho^2)(x_1-\rho^4)(x_1-\rho^5)(x_1-\rho^6)/81$ and $x_4^3$ by $x_4(t_1 - 3\rho)(t_1 - 3\rho^3)(t_1 - 3\rho^4)(t_1 - 3\rho^5)/81$, where $x_4$ is given in $(*)$, then we obtain a polynomial which is of degree one in the variable $x_3$.

$x_3 = (-9\rho^2(-162t_1 - 18t_1^3 + 4t_1^5 - 243(1 + t_{11}) + t_1^2(27 - 54t_3) + 6t_1^4t_3) + 3(729 + 18t_1^4 - 6t_1^5 - 27t_1^3(-6 + t_3) - t_1^6(-2 + t_3) + 243t_1(3 + t_3) + 81t_1^2(2 + t_{11} + t_3)) + \rho^3(2187-$

$t_1^7+27t_1^4(-6+t_3)+9t_1^5(-3+t_3)+486t_1^2t_3+81t_1^3(1+t_3)+729t_1(1+2t_3))+\rho^5(2187+27t_1^4+12t_1^6+t_1^7-729t_1(-1+t_{11}-t_3)+729t_1^2t_3+81t_1^3(5+t_3)+9t_1^5(1+2t_3))+\rho(2916t_1+3t_1^6-t_1^7-81t_1^3(-6+t_3)-2187(-2+t_3)-27t_1^4(-2+t_3)+9t_1^5(2+t_3)+243t_1^2(5+2t_3))+\rho^4(2187+t_1^7-729t_1(-3+t_{11}-2t_3)-81t_1^3(-1+t_3)+27t_1^4(1+t_3)+9t_1^5(-1+2t_3)+243t_1^2(1+3t_3)))/(9(t_1^5-243t_{11}+27t_1^2(-1+t_3)+81t_1t_3+9t_1^3t_3+3t_1^4t_3+\rho(3+t_1)(-81+18t_1^2-9t_1^3+2t_1^4+27t_1t_3)+27\rho^2t_1(3+t_1^2+t_1(3+t_3))+\rho^4t_1(243+3t_1^3+t_1^4+9t_1^2(-1+t_3)+27t_1(3+t_3))+\rho^5(-6t_1^4+t_1^5+243(1+t_3)+81t_1(2+t_3)+9t_1^3(2+t_3)+27t_1^2(3+t_3))+\rho^3t_1(162+36t_1^2+6t_1^3+2t_1^4+27t_1(4+t_3))))$

Then, using $(*)$, we obtain

$x_4=-((3\rho^4-t_1)(3\rho^5-t_1)(-\rho^3(-2187-729t_1+t_1^7+243t_1^2t_3(2+t_3)+9t_1^5(3+t_3)+27t_1^4(6+t_3)+81t_1^3(-1+3t_3))+\rho^4(2187+27t_1^4+t_1^7+9t_1^5(-1+t_3)-729t_1(-3+t_{11}+t_3)-243t_1^2(-1+t_3^2)-81t_1^3(-1+t_3^2))+\rho(4374+486t_1^3+54t_1^4+3t_1^6-t_1^7-9t_1^5(-2+t_3)-243t_1^2(-5+t_3^2)-729t_1(-4-t_3+t_3^2))-3(t_1^6(-2+t_3)+3t_1^5(2+t_3)-729(1+t_{11}t_3)-81t_1^2(2+t_{11}+2t_3-t_3^2)+9t_1^4(-2+t_3^2)+243t_1(-3-t_3+t_3^2)+27t_1^3(-6+t_3+t_3^2))-9\rho^2(4t_1^5-243(1+t_{11})+81t_1(-2+t_3)+6t_1^4t_3+9t_1^3(-2+3t_3)+27t_1^2(1+t_3+t_3^2))+\rho^5(12t_1^6+t_1^7-243t_1^2t_3^2+9t_1^5(1+t_3)+27t_1^4(1+2t_3)-81t_1^3(-5+t_3+t_3^2)-2187(-1+t_3+t_3^2)-729t_1(-1+t_{11}+t_3+t_3^2))))/(9(567t_1^3+6t_1^6+t_1^7+\rho(-3+t_1)(-54t_1^3+t_1^6+9t_1^4(-4+t_3)+729(-2+t_3)+243t_1(-2+t_3)-81t_1^2(-2+t_3))+27t_1^4(-7+t_3)+9t_1^5(-5+t_3)+2187(2+t_3)+243t_1^2(-1+2t_3)+729t_1(1+2t_3)+\rho^5(2187+216t_1^4+3t_1^6+2t_1^7+729t_1t_3+729t_1^2(1+t_3)+18t_1^5(2+t_3)+81t_1^3(16+t_3))+\rho^3t_1(9t_1^5+t_1^6+27t_1^3(-4+t_3)+9t_1^4(3+t_3)+81t_1^2(5+t_3)+729(-5+2t_3)+243t_1(-3+2t_3))+\rho^4(2187+6t_1^6+2t_1^7-81t_1^3(-14+t_3)+18t_1^5(-2+t_3)+1458t_1t_3+27t_1^4(5+t_3)+243t_1^2(1+3t_3))-9\rho^2(-243+243t_1-27t_1^3+t_1^5-54t_1^2(-5+t_3)+t_1^4(-9+6t_3)))).$

Now, using such values for $x_3$ and $x_4$, and replacing them in (1) and (2) above, we obtain another two polynomials identities $P_3(t_1,t_3,t_7,t_{11})=0$ and $P_4(t_1,t_3,t_7,t_{11})=0$, where these two new polynomials are defined over $\mathbb{Q}(\rho)$ (see [9] for these long polynomials). In this way, we have obtained the following equations over $\mathbb{Q}(\rho)$ for $Z_1$:

$$Z_1=\left\{\begin{array}{c} t_4=t_3,\ 3t_5=t_1^2,\ 3t_6=t_2^2,\ t_8=t_9\\ 9t_9=t_1^3,\ 9t_{10}=t_2^3,\ t_{12}=t_{11}\\ P_1(t_1,t_2,t_3,t_7,t_{11})=0\\ P_2(t_1,t_2,t_3,t_7,t_{11})=0\\ P_3(t_1,t_2,t_3,t_7,t_{11})=0\\ P_4(t_1,t_2,t_3,t_7,t_{11})=0 \end{array}\right\}\subset\mathbb{C}^{12}$$

Notice that, by the above computations, we have explicitly the inverse of $L_1$ given as

$$L_1^{-1}:Z_1\to X$$

$$(t_1,...,t_{12})\mapsto(x_1,x_2,x_3,x_4),$$

where $x_1$, $x_2$, $x_3$ and $x_4$ are in terms of $t_1$, $t_2$, $t_3$, $t_7$ and $t_{11}$.

As the variables $t_1$,..., $t_{12}$ are uniquely determined only by the variables $t_1$, $t_2$, $t_3$, $t_7$ and $t_{11}$, if we consider the projection

$$\pi:\mathbb{C}^{12}\to\mathbb{C}^5$$

$$(t_1,...,t_{12})\mapsto(t_1,t_2,t_3,t_7,t_{11}),$$

then

$$L = \pi \circ L_1 : X \to Z$$

$$L_1^*(x, y_1, y_2, y_4)$$

$$||$$

$$\left(3x, 3y_1, y_2 + y_4 + \tfrac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}, y_2^2 + y_4^2 + \tfrac{y_2^2 y_4^2}{(x-\rho^4)^2(x-\rho^5)^2}, y_2^3 + y_4^3 + \tfrac{y_2^3 y_4^3}{(x-\rho^4)^3(x-\rho^5)^3}\right)$$

is a birational isomorphism, where

$$Z = \left\{ \begin{array}{c} P_1(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_2(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_3(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_4(t_1, t_2, t_3, t_7, t_{11}) = 0 \end{array} \right\} \subset \mathbb{C}^5$$

The inverse $L^{-1} : Z \to X$ is given as

$$L^{-1}(t_1, t_2, t_3, t_7, t_{11}) = (x_1, x_2, x_3, x_4).$$

We have obtained equations for $Z$ over $\mathbb{Q}(\rho)$. But, as $Z_1^\eta = Z_1$, for every $\eta \in N$, and $\pi$ is defined over $\mathbb{Q}$, we may see that $Z^\eta = Z$, for every $\eta \in N$, that is, $Z$ can be defined by polynomials over $\mathbb{Q}(\sqrt{-7})$. To obtain such equations over $\mathbb{Q}(\sqrt{-7})$, we replace each polynomial $P_j$ ($j = 3, 4$) by the new polynomials (with coefficients in $\mathbb{Q}(\sqrt{-7})$)

$$Q_{j,1} = \mathrm{Tr}(P_j),\ Q_{j,2} = \mathrm{Tr}(\rho P_j),\ Q_{j,3} = \mathrm{Tr}(\rho^2 P_j)$$

that is

$$Z = \left\{ \begin{array}{c} P_1(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_2(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ P_3(t_1, t_2, t_3, t_7, t_{11}) + P_3(t_1, t_2, t_3, t_7, t_{11})^\tau + P_3(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho P_3(t_1, t_2, t_3, t_7, t_{11}) + \rho^2 P_3(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho^4 P_3(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho^2 P_3(t_1, t_2, t_3, t_7, t_{11}) + \rho^4 P_3(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho P_3(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ P_4(t_1, t_2, t_3, t_7, t_{11}) + P_4(t_1, t_2, t_3, t_7, t_{11})^\tau + P_4(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho P_4(t_1, t_2, t_3, t_7, t_{11}) + \rho^2 P_4(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho^4 P_4(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \\ \rho^2 P_4(t_1, t_2, t_3, t_7, t_{11}) + \rho^4 P_4(t_1, t_2, t_3, t_7, t_{11})^\tau + \rho P_4(t_1, t_2, t_3, t_7, t_{11})^{\tau^2} = 0 \end{array} \right\} \subset \mathbb{C}^5$$

We have obtained an explicit model $Z$ for the Fricke-Macbeath curve over $\mathbb{Q}(\sqrt{-7})$ together explicit birational isomorphisms $L : X \to Z$ and $L^{-1} : Z \to X$.

### 3.9

Finally, notice that the regular dessin d'enfant $(X, \beta)$, given before, is isomorphic to that provided by the pair $(Z, \beta^*)$, where $\beta^*(t_1, t_2, t_3, t_7, t_{11}) = \beta \circ L^{-1}(t_1, t_2, t_3, t_7, t_{11}) = (t_1/3)^7$; that is, the dessin d'enfant is defined over $\mathbb{Q}(\sqrt{-7})$.

## 4 An explicit isomorphism $L : X \to W$ where $W$ is defined over $\mathbb{Q}$

Next we explain how to construct an explicit birational isomorphism $\widehat{L} : X \to W$, where $W$ is known to be defined over $\mathbb{Q}$.

Let us consider the explicit model $Z \subset \mathbb{C}^5$ over $\mathbb{Q}(\sqrt{-7})$ constructed above. Let $M = \mathrm{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q}) = \langle \eta \rangle \cong \mathbb{Z}_2$, where $\eta$ is the complex conjugation. As already noticed, since $X$ admits the anticonformal involution $J$ (defined previously), the curve $Z$ admits the anticonformal involution $T = L \circ J \circ L^{-1}$. It is not difficult to see that by setting $g_e = I$ and $g_\eta = S \circ T$, where $S(t_1, t_2, t_3, t_7, t_{11}) = (\overline{t_1}, \overline{t_2}, \overline{t_3}, \overline{t_7}, \overline{t_{11}})$, we obtain a Weil datum for the Galois extension $\mathbb{Q}(\sqrt{-7})/\mathbb{Q}$. Now, identically as done above, we consider the rational map

$$\Phi_2 : Z \to \mathbb{C}^{10}$$

$$(t_1, t_2, t_3, t_7, t_{11}) \mapsto (t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11})$$

where $g_\eta(t_1, t_2, t_3, t_7, t_{11}) = (s_1, s_2, s_3, s_7, s_{11})$. We may see that $\Phi_2$ induces a birational isomorphism between $Z$ and $\Phi_2(Z)$. In this case,

$$\Phi_2(Z) = \left\{ \begin{array}{c} Q_{1,1}(t_1, t_2, t_3, t_7, t_{11}) = \cdots = Q_{4,3}(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ g_\eta(t_1, t_2, t_3, t_7, t_{11}) = (s_1, s_2, s_3, s_7, s_{11}) \end{array} \right\} \subset \mathbb{C}^{10}.$$

The Galois group $M$ induces the permutation action $\Theta_2(M)$ defined as

$$\Theta_2(\eta)(t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11}) = (s_1, s_2, s_3, s_7, s_{11}, t_1, t_2, t_3, t_7, t_{11})$$

A set of generators for the invariant polynomials (with respect to the previous permutation action) is given by

$$q_1 = t_1 + s_1,\ q_2 = t_2 + s_2,\ q_3 = t_3 + s_3,$$

$$q_4 = t_7 + s_7,\ q_5 = t_{11} + s_{11},\ q_6 = t_1^2 + s_1^2,$$

$$q_7 = t_2^2 + s_2^2,\ q_8 = t_3^2 + s_3^2,\ q_9 = t_7^2 + s_7^2,$$

$$q_{10} = t_{11}^2 + s_{11}^2$$

Then the rational map

$$\Psi_2 : \mathbb{C}^{10} \to \mathbb{C}^{10}$$

$$(t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11}) \mapsto (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10})$$

satisfies the following properties:

$$\left\{ \begin{array}{l} \Psi_2^\eta = \Psi_2; \\ \Psi_2 \circ \Theta_2(\eta) = \Psi_2. \end{array} \right. \tag{4.1}$$

There are two possibilities:

1. $\Phi_2(Z) = \Theta_2(\eta)(\Phi_2(Z))$; in which case $Z^\eta = Z$ and $Z$ will be already defined over $\mathbb{Q}$ (which seems not to be the case); and
2. the stabilizer of $\Phi_2(Z)$ under $\Theta_2(M)$ is trivial.

Under the assumption (2) above, we have that $\Psi_2 : \Phi_2(Z) \to W = \Psi_2(\Phi_2(Z))$ is a biregular isomorphism and that, as before, $W$ is defined over $\mathbb{Q}$. That is, the map $L_1 = \Psi_2 \circ \Phi_2 : Z \to W$ is an explicit biregular isomorphism and $W$ is defined over $\mathbb{Q}$. In this way, $\widehat{L} = L_1 \circ L : X \to W$ is an explicit birational isomorphism as desired.

As $R_2$ and $Z$ are explicitly given, one may compute explicit equations for $W$ over $\mathbb{Q}(\sqrt{-7})$, say by the polynomials $A_1, ..., A_m \in \mathbb{Q}(\sqrt{-7})[q_1, ..., q_{10}]$ (this may be done with MAGMA [3] or by hands, but computations are heavy and very long). To obtain equations over $\mathbb{Q}$ we replace each $A_j$ (which is not already defined over $\mathbb{Q}$) by the traces $A_j + A_j^{\eta}$ and $iA_j - iA_j^{\eta}$.

## 5 A remark on the elliptic curves in the model $X$

### 5.1 A connection to homology covers

Let us set $\lambda_1 = 1$, $\lambda_2 = \rho$, $\lambda_3 = \rho^2$, $\lambda_4 = \rho^3$, $\lambda_5 = \rho^4$, $\lambda_6 = \rho^5$ and $\lambda_7 = \rho^6$, where $\rho = e^{2\pi i/7}$. If $S$ is the Fricke-Macbeath curve, then there is a regular branched cover $Q : S \to \widehat{\mathbb{C}}$ having deck group $G \cong \mathbb{Z}_2^3$ and whose branch locus is the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$.

Let us consider a Fuchsian group

$$\Gamma = \langle \alpha_1, ..., \alpha_7 : \alpha_1^2 = \cdots = \alpha_7^2 = \alpha_1\alpha_2\cdots\alpha_7 = 1 \rangle$$

acting on the hyperbolic plane $\mathbb{H}^2$ uniformizing the orbifold $S/G$.

If $\Gamma'$ denotes the derived subgroup of $\Gamma$, then $\Gamma'$ acts freely and $\widehat{S} = \mathbb{H}^2/\Gamma'$ is a closed Riemann surface. Let $H = \Gamma/\Gamma' \cong \mathbb{Z}_2^6$; a group of conformal automorphisms of $\widehat{S}$. Then there exists a set of generators of $H$, say $a_1$,..., $a_6$, so that the only elements of $H$ acting with fixed points are these and $a_7 = a_1a_2a_3a_4a_5a_6$. In [4, 5] it was noted that $\widehat{S}$ corresponds to the generalized Fermat curve of type $(2, 6)$ (also called the homology cover of $S/H$)

$$\widehat{S} = \left\{ \begin{array}{rcl} x_1^2 + x_2^2 + x_3^2 & = & 0 \\ \left(\dfrac{\lambda_3 - 1}{\lambda_4 - 1}\right) x_1^2 + x_2^2 + x_4^2 & = & 0 \\ \left(\dfrac{\lambda_4 - 1}{\lambda_5 - 1}\right) x_1^2 + x_2^2 + x_5^2 & = & 0 \\ \left(\dfrac{\lambda_5 - 1}{\lambda_6 - 1}\right) x_1^2 + x_2^2 + x_6^2 & = & 0 \\ \left(\dfrac{\lambda_6 - 1}{\lambda_7 - 1}\right) x_1^2 + x_2^2 + x_7^2 & = & 0 \end{array} \right\} \subset \mathbb{P}^6_{\mathbb{C}},$$

that $a_j$ is just multiplication by $-1$ at the coordinate $x_j$ and that the regular branched cover $P : \widehat{S} \to \widehat{\mathbb{C}}$ given by

$$P([x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7]) = \frac{x_2^2 + x_1^2}{x_2^2 + \lambda_7 x_1^2} = z$$

has $H$ has its deck group and branch locus given by the set of the $7th$-roots of unity $\{\lambda_1, ..., \lambda_7\}$.

By classical covering theory, there should be a subgroup $K < H$, $K \cong \mathbb{Z}_2^3$, acting freely on $\widehat{S}$ so that there is an isomorphism $\phi : S \to \widehat{S}/K$ with $\phi G \phi^{-1} = H/K$.

Let us also observe that the rotation $R(z) = \rho z$ lifts under $P$ to an automorphism $T$ of $\widehat{S}$ of order 7 of the form

$$T([x_1 : \cdots : x_7]) = [c_1 x_7 : c_2 x_1 : c_3 x_2 : c_4 x_3 : c_5 x_4 : c_6 x_5 : c_7 x_6]$$

for suitable comples numbers $c_j$. One has that $T a_j T^{-1} = a_{j+1}$, for $j = 1, ..., 6$ and $T a_7 T^{-1} = a_1$. The subgroup $K$ above must satisfy that $TKT^{-1} = K$ as $R$ also lifts to the Fricke-Macbeath curve (as noticed in the Introduction).

## 5.2 About the elliptic curves in the Fricke-Macbeath curve

The subgroup

$$K^* = \langle a_1 a_3 a_7, a_2 a_3 a_5, a_1 a_2 a_4 \rangle \cong \mathbb{Z}_2^3$$

acts freely on $\widehat{S}$ and it is normalized by $T$. In particular, $S^* = \widehat{S}/K^*$ is a closed Riemann surface of genus 7 admitting the group $L = H/K^* = \{e, a_1^*, ..., a_7^*\} \cong \mathbb{Z}_2^3$ (where $a_j^*$ is the involution induced by $a_j$) as a group of automorphisms and it also has an automorphism $T^*$ of order 7 (induced by $T$) permuting cyclically the involutions $a_j^*$. As $S^*/\langle L, T^* \rangle = \widehat{S}/\langle H, T \rangle$ has signature $(0; 2, 7, 7)$, we may see that $S = S^*$ and $K = K^*$.

We may see that $L = \langle a_1^*, a_2^*, a_3^* \rangle$ and $a_4^* = a_1^* a_2^*$, $a_5^* = a_2^* a_3^*$, $a_6^* = a_1^* a_2^* a_3^*$ and $a_7^* = a_1^* a_3^*$. In this way, we may see that every involution of $H/K$ is induced by one of the involutions (and only one) with fixed points; so every involution in $L$ acts with $4$ fixed points on $S$.

Let $a_i^*, a_j^* \in H/K$ be any two different involutions, so $\langle a_i^*, a_j^* \rangle \cong \mathbb{Z}_2^2$. Then, by the Riemann-Hurwitz formula, the quotient surface $S/\langle a_i^*, a_j^* \rangle$ is a closed Riemann surface of genus 1 with six cone points of order 2. These six cone points are projected onto three of the cone points of $S/H$. These points are $\lambda_i$, $\lambda_j$ and $\lambda_r$, where $a_r^* = a_i^* a_j^*$. In this way, the corresponding genus one surface is given by the elliptic curve

$$y^2 = \prod_{k \notin \{i,j,r\}} (x - \lambda_k)$$

So, for instance, if we consider $i = 2$ and $j = 3$, then $r = 5$ and the elliptic curve is

$$y_1^2 = (x - 1)(x - \rho^3)(x - \rho^5)(x - \rho^6).$$

If $i = 1$ and $j = 2$, then $r = 4$ and the elliptic curve is

$$y_2^2 = (x - \rho^2)(x - \rho^4)(x - \rho^5)(x - \rho^6).$$

If $i = 1$ and $j = 3$, then $r = 7$ and the elliptic curve is

$$y_4^2 = (x - \rho)(x - \rho^3)(x - \rho^4)(x - \rho^5).$$

We have obtained the three elliptic curves appearing in the Fricke-Macbeath equation (2.1).

### 5.3 Another model for the Fricke-Macbeath curve

The above description of the Fricke-Macbeath curve in terms of the homology cover $\widehat{S}$ permits to obtain an explicit model. Let us consider now an affine model of $\widehat{S}$, say by taking $x_7 = 1$, which we denote by $\widehat{S}^0$. In this case the involution $a_7$ is multiplication of all coordinates by $-1$. A set of generators for the algebra of invariant polynomials in $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ under the natural linear action induced by $K$ is

$$t_1 = x_1^2, t_2 = x_2^2, t_3 = x_3^2, t_4 = x_4^2, t_5 = x_5^2, t_6 = x_6^2, t_7 = x_1x_2x_5, t_8 = x_1x_2, x_3x_6$$

$$t_9 = x_1x_4x_6, t_{10} = x_1x_3x_4x_5, t_{11} = x_2x_4x_5x_6, t_{12} = x_2x_3x_4, t_{13} = x_3x_5x_6.$$

If we set

$$F : \widehat{S}^0 \to \mathbb{C}^{13}$$

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}),$$

then $F(\widehat{S}^0)$ will provide a model for the Fricke-Macbeath (affine) curve $S$. Equations for such an affine model of $S$ are

$$\left\{\begin{array}{c}
t_1 + t_2 + t_3 = 0\\
\left(\frac{\lambda_3-1}{\lambda_4-1}\right) t_1 + t_2 + t_4 = 0\\
\left(\frac{\lambda_4-1}{\lambda_5-1}\right) t_1 + t_2 + t_5 = 0\\
\left(\frac{\lambda_5-1}{\lambda_6-1}\right) t_1 + t_2 + t_6 = 0\\
\left(\frac{\lambda_6-1}{\lambda_7-1}\right) t_1 + t_2 + 1 = 0\\
t_6t_{10} = t_9t_{13},\; t_6t_7t_{12} = t_8t_{11},\; t_5t_9t_{12} = t_{10}t_{11}\\
t_5t_8 = t_7t_{13},\; t_5t_6t_{12} = t_{11}t_{13},\; t_4t_8 = t_9t_{12}\\
t_4t_7t_{13} = t_{10}t_{11},\; t_4t_6t_7 = t_9t_{11},\; t_3t_{11} = t_{12}t_{13}\\
t_3t_6t_7 = t_8t_{13},\; t_3t_5t_9 = t_{10}t_{13},\; t_3t_5t_6 = t_{13}^2\\
t_3t_4t_7 = t_{10}t_{12},\; t_2t_{10} = t_7t_{12},\; t_2t_9t_{13} = t_8t_{11}\\
t_2t_5t_9 = t_7t_{11},\; t_2t_4t_{13} = t_{11}t_{12},\; t_2t_4t_5t_6 = t_{11}^2\\
t_2t_3t_9 = t_8t_{12},\; t_2t_3t_4 = t_{12}^2,\; t_1t_{12}t_{13} = t_8t_{10}\\
t_1t_{11} = t_7t_9,\; t_1t_6t_{12} = t_8t_9,\; t_1t_5t_{12} = t_7t_{10}\\
t_1t_4t_{13} = t_9t_{10},\; t_1t_4t_6 = t_9^2,\; t_1t_3t_4t_5 = t_{10}^2\\
t_1t_2t_{13} = t_7t_8,\; t_1t_2t_5 = t_7^2,\; t_1t_2t_3t_6 = t_8^2
\end{array}\right\} \subset \mathbb{C}^{13}$$

Of course, one may see that the variables $t_2$, $t_3$, $t_4$, $t_5$ and $t_6$ are uniquely determined by the variable $t_1$. Other variables can also be determined in order to get a lower dimensional model.


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## References


[1] G. V. Belyĭ, On Galois extensions of a maximal cyclotomic field, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), 267–276, 479.

[2] K. Berry and M. Tretkoff, The period matrix of Macbeath's curve of genus seven, Curves, Jacobians, and abelian varieties (Amherst, MA, 1990), 31–40, *Contemp. Math.*, **136**, Amer. Math. Soc., Providence, RI, 1992.

[3] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system, I. The user language, Computational algebra and number theory (London, 1993), J. Symbolic Comput. **24** (1997), 235–265, `http://dx.doi.org/10.1006/jsco.1996.0125`

[4] A. Carocca, V. González-Aguilera, R. A. Hidalgo and R. E. Rodríguez, Generalized Humbert curves, *Israel J. Math.* **164** (2008), 165–192.

[5] G. González-Diez, R.A. Hidalgo and M. Leyton, Generalized Fermat curves, *Journal of Algebra* **321** (2009), 1643–1660.

[6] T. Ekedahl and J.-P. Serre, Exemples de courbes algébriques à jacobienne complètement décomposable, *C. R. Acad. Sci. Pari Sér. I Math.* **317** (1993), 509–513.

[7] M. Hendriks, Platonic maps of low genus, Ph. D. Thesis, Technische Universiteit Eindhoven, 2013

[8] R. A. Hidalgo and S. Reyes-Carocca, Towards a constructive proof of Weil's descent theorem, `http://arxiv.org/abs/1203.6294`

[9] R. A. Hidalgo. A computational note about Fricke-Macbeath's curve. `http://arxiv.org/abs/1203.6314`

[10] L. D. James and G. A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory Ser. B* **39** (1985), 353–367.

[11] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* **37** (1978), 273–307.

[12] G. A. Jones, M. Streit and J. Wolfart, Wilson's map operations on regular dessins and cyclotomic fields of definition, *Proc. Lond. Math. Soc.* **100** (2010), 510–532.

[13] S. K. Lando and A. Zvonkin, Graphs on surfaces and their applications, with an appendix by Don B. Zagier, Encyclopaedia of Mathematical Sciences **141**, Low-Dimensional Topology, II. Springer-Verlag, Berlin, 2004.

[14] A. Macbeath, On a curve of genus 7, *Proc. London Math. Soc.* **15** (1965), 527–542.

[15] D. Singerman. Automorphisms of maps, permutation groups and Riemann surfaces. *Bull. London Math. Soc.* **8** (1976), 65–68.

[16] A. Weil, The field of definition of a variety, *Amer. J. Math.* **78** (1956), 509–524.

[17] A. Wiman, "Ueber die hyperelliptischen Curven und diejenigen vom Geschlechte $p = 3$, welche eindeutigen Transformationen in sich zulassen" and "Ueber die algebraischen Curven von den Geschlechtern $p = 4, 5$ und 6 welche eindeutigen Transformationen in sich besitzen", Bihang Till Kongl. Svenska Vetenskaps-Akademiens Hadlingar, Stockholm, 1895–96.

[18] J. Wolfart (joint work with G. Jones and M. Streit), Wilson's operations on regular dessins and cyclotomic fields of definition, Talk in Ankara, March 2011.

[19] J. Wolfart, ABC for polynomials, dessins d'enfants and uniformization – a survey. Elementare und analytische Zahlentheorie, 313–345, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006.

[20] Wolfram Research, Inc., Mathematica, Version 8.0, Champaign, IL (2010).