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Squashing maximum packings of 6-cycles into maximum packings of triples

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Abstract

A 6-cycle is said to be squashed if we identify a pair of opposite vertices and name one of them with the other (and thereby turning the 6-cycle into a pair of triples with a common vertex). The squashing problem for 6-cycle systems was introduced by C. C. Lindner, M. Meszka and A. Rosa and completely solved by determining the spectrum. In this paper, by employing PBD and GDD-constructions and filling techniques, we extend this result by squashing maximum packings of K_n with 6-cycles into maximum packings of K_n with triples. More specifically, we establish that for each $n \geq 6$, there is a max packing of K_n with 6-cycles that can be squashed into a maximum packing of K_n with triples.

Keywords: Maximum packing with triples, maximum packing with 6-cycles.

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1 Introduction

Let G be a graph. A *G-design of order* n is a pair (S, B) where B is a collection of subgraphs (*blocks*), each isomorphic to G, which partitions the edge set of the complete undirected graph K_n with vertex set S. After determining the spectrum for G-designs for different graphs G, many problems have been studied also recently (for example, see [1]-[7]).

A Steiner triple system (more simply, triple system) of order n is a G-design of order n where G is the graph K_3. It is well known that the spectrum for triple systems is precisely the set of all $n \equiv 1$ or 3 (mod 6) [9], and that if (S, T) is a triple system of order n, then $|T| = n(n-1)/6$. Similarly, a 6-cycle system of order n is a G-design of order n where G is 6-cycle. The spectrum for 6-cycle systems is precisely the set of all $n \equiv 1$ or 9 (mod 12) [15], and if (X, C) is a 6-cycle system of order n, then $|C| = n(n-1)/12$. It is worth noting that if (S, T) and (X, C) have order n, then $|T| = 2|C|$.

Given the fact that triple systems and 6-cycle systems coexist for all $n \equiv 1$ or 9 (mod 12), an obvious question to ask is: are there any connections between the two when $n \equiv$ 1 or 9 (mod 12)? The answer, of course, is yes. One much studied connection is that of 2-perfect 6-cycle systems. A 6-cycle system is 2-perfect provided the collection of triples obtained by replacing each 6-cycle (a, b, c, d, e, f) with the two triples (a, c, e) and (b, d, f) is a Steiner triple system. Such systems exist for all $n \equiv 1$ or 9 (mod 12) ≥ 13 [15].

Quite recently a new connection between triple systems and 6-cycle systems has been introduced: the *squashing* of a 6-cycle system into a Steiner triple system. A definition is in order. Let (a, b, c, d, e, f) be a 6-cycle and form the following six bowties (a pair of triples with a common vertex).

If B is any one of the six bowties in Figure 1, we say that we have *squashed* (a, b, c, d, e, f) into B. So there are six different ways to squash a 6-cycle into a bowtie. If (X, C) is a 6-cycle system, $2|C| = 2n(n-1)/12 = n(n-1)/6$ is the number of triples in a Steiner triple system. Therefore it makes sense to ask the following question: what is the spectrum for 6-cycle systems that can be squashed into Steiner triple systems? In [11], a complete solution is given to this problem by constructing for every $n \equiv 1$ or 9 (mod 12) a 6-cycle system that can be squashed into a Steiner triple system.

Example 1.1. (A 6-cycle system of order 9 squashed into a triple system [11].)

(0,1,2,3,4,5)		(0,1,2)(0,4,5)
(3,6,0,2,4,1)		(3,6,0)(3,4,1)
(2,8,4,0,3,7)	SQUASH	(2,8,4)(2,3,7)
(7,0,8,6,5,1)	$\longrightarrow$	(7,0,8)(7,5,1)
(6,1,8,5,7,4)		(6,1,8)(6,7,4)
(5,2,6,7,8,3)		(5,2,6)(5,8,3)

Now if $n \equiv 3$ or 7 (mod 12) there does not exist a 6-cycle system of order n. However, there does exist a maximum packing (max packing) of K_n with 6-cycles with leave a triple (i.e., a pair (X, C) and a set L, the *leave*, where C is a collection of edge disjoint 6-cycles with verteces in X, L is the set of the edges of K_n not belonging to any 6-cycle of C and $|L|$ is as small as possible) and so the following question makes sense. Does there exist for each $n \equiv 3$ or 7 (mod 12) a max packing of K_n with 6-cycles which can be squashed into bowties so that the bowties plus the leave (a triple) form a Steiner triple system?

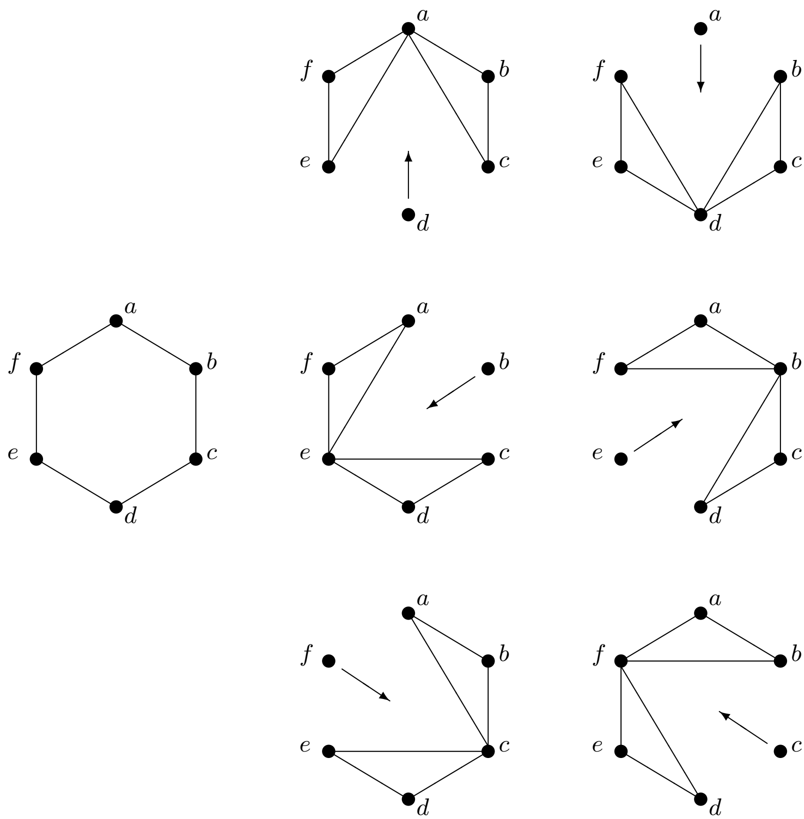


Figure 1

Example 1.2. (A max packing of K_7 squashed into a triple system [11].)

(2,3,4,5,0,1)	SQUASH	(2,1,0)(2,3,4)
(4,6,0,2,5,1)	$\longrightarrow$	(4,1,5)(4,0,6)
(5,6,2,4,0,3)		(5,3,0)(5,6,2)
(1,3,6) leave	$\longrightarrow$	(1,3,6)

The following theorem is proved in [11].

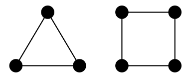
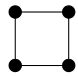
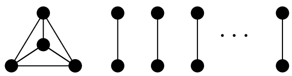
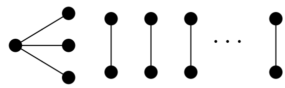
Theorem 1.3. [11] *There exists a* 6*-cycle system of every order* $n \equiv 1$ *or* 9 (mod 12) *that can be squashed into a triple system and a* 6*-cycle maximum packing that can be squashed into a triple system for every* $n \equiv 3$ *or* 7 (mod 12)*,* $n \geq 7$. □

The object of this paper is to finish off the problem of squashing maximum packings of K_n with 6-cycles into maximum packings of K_n with triples. We need to be a bit more precise.

Let (X, C) be a maximum packing of K_n with 6-cycles with leave L. In what follows, to keep the vernacular from getting out of hand, to say that C has been squashed means that the resulting collection $S(C)$ of bowties is a partial triple system.

Further, if t is a triple belonging to L and $S(C) \cup \{t\}$ is a maximum packing of K_n with triples (or a triple system), we will say that we have squashed (X, C) into a maximum packing of K_n with triples. So, for example, Example 1.2 is the squashing of a maximum packing of K_7 with 6-cycles into a triple system of order 7.

The following easy to read table gives the leaves for max packings for both 6-cycles and triples not covered by Theorem 1.3. (See [8] and [13].)

K_n	6-cycles leave	triples leave
$n \equiv 0, 2, 6, 8$ (mod 12)	1-factor	1-factor
$n \equiv 5$ (mod 12)	4-cycle	4-cycle
$n \equiv 11$ (mod 12)	 4 leaves are possible	
$n \equiv 4$ or 10 (mod 12)	 22 leaves are possible for $n \geq 16$	 tripole [13]

We remark that if $n \equiv$ 0, 2, 6, 8 or 5 (mod 12) and a 6-cycle maximum packing can be squashed, there are no triples to be added; i.e., the resulting collection of bowties is a maximum packing of K_n with triples. If $n \equiv 4$, 10 or 11 (mod 12) and a 6-cycle maximum packing can be squashed, then a triple is taken from the 6-cycle leave in order to obtain a maximum packing of K_n with triples.

2 Preliminaries

From now on to say that the 6-cycle (a, b, c, d, e, f) is squashed we will always mean that it has been squashed into the bowtie $(a, b, c)(a, e, f)$; see Figure 2.

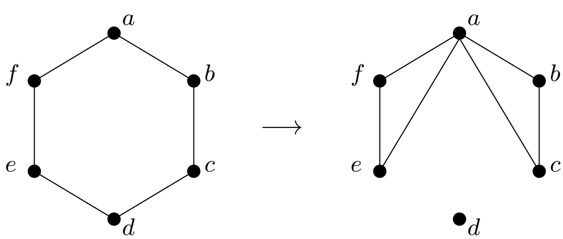


Figure 2

So, for example, in Example 1.1 we can simply list the 6-cycles (without listing the bowties they have been squashed into) and say they can be squashed into a triple system.

The following three examples are used repeatedly in what follows.

Example 2.1. (A max packing of K_6 with 6-cycles squashed into a max packing of K_6 with triples.)

$C = \{(5, 0, 1, 2, 4, 3), (2, 3, 1, 5, 4, 0)\}$, leave $L = \{(0, 3), (1, 4), (2, 5)\}$. (There are no triples in the leave.)

Example 2.2. (A max packing of K_8 with 6-cycles squashed into a max packing of K_8 with triples.)

$X = Z_4 \times Z_2$; $C = \{(0_0, 3_0, 1_1, 2_0, 3_1, 0_1), (1_0, 0_0, 2_1, 3_0, 0_1, 1_1), (2_0, 1_0, 3_1, 0_0, 1_1, 2_1), (3_0, 2_0, 0_1, 1_0, 2_1, 3_1)\}$, leave $L = \{(0_0, 2_0), (1_0, 3_0), (0_1, 2_1), (1_1, 3_1)\}$. (There are no triples in the leave.)

Example 2.3. (Decomposition of $K_{4,4,4}$ into 6-cycles squashed into triples.) (An obvious definition.)

$X = Z_4 \times \{1, 2, 3\}$; $C = \{(1_2, 1_3, 0_1, 0_2, 0_3, 1_1), (0_2, 2_3, 0_1, 1_2, 0_3, 2_1), (1_1, 0_2, 3_3, 0_1, 2_2, 1_3), (0_1, 3_2, 3_3, 1_1, 2_2, 0_3), (3_2, 1_1, 2_3, 1_2, 3_1, 0_3), (1_2, 2_1, 2_3, 3_2, 3_1, 3_3), (3_1, 0_2, 1_3, 2_1, 2_2, 2_3), (2_1, 3_2, 1_3, 3_1, 2_2, 3_3)\}$. (There is no leave.)

3 Basic Lemmas

With the examples of Section 2 in hand we can go to the general constructions, where we shall make use of GDDs. Let H be a set of integers and X be a set of size n; a GDD(n, H, k) is a triple (X, G, B) where G is a partition of X into subsets called *groups* of size in H, B is a set of subsets of X (called *blocks*) of size k, such that a group and a block contain at most one common point and every pair of points from distinct groups occurs in exactly one block. A PBD is a GDD$(n, \{1\}, k)$.

We break the constructions into the eight cases: 2, 6, 8; 0; 11; 4, 10; 5 (mod 12).

3.1 $n \equiv 2, 6$ and 8 (mod 12)

These are the easiest cases, so a good place to start.

$n \equiv 2$ (mod 12) Write $12k + 2 = 2(6k + 1)$ and let (X, T) be a Steiner triple system of order $6k + 1$. Let $S = X \times \{1, 2\}$ and define a collection C of 6-cycles as follows: For each triple $t = \{a, b, c\} \in T$ define a copy of Example 2.1 on $\{a, b, c\} \times \{1, 2\}$ with leave $L_t = \{(a_1, a_2), (b_1, b_2), (c_1, c_2)\}$ and put these 6-cycles in C. Then C is a max packing of K_{12k+2} with 6-cycles with leave $L = \{L_t |\ t \in T\}$. Trivially, C can be squashed into a max packing of K_{12k+2} with triples with leave L.

$n \equiv 6$ (mod 12) The case for $n = 6$ is handled with Example 2.1. So now write $12k+6 = 2(6k + 3)$ and proceed exactly as in the case $n \equiv 2$ (mod 12).

$n \equiv 8$ (mod 12) Write $12k + 8 = 2(6k + 4)$. The case $n = 8$ is handled by Example 2.2. So let $12k + 8 \geq 20$. It is well kown that there is a PBD with block sizes 3 and 4 for every $n \equiv 4$ (mod 6) [13]. Let (X, B) be such a PBD, $|X| \equiv 4$ (mod 6), and proceed exactly as in the cases for $n \equiv 2$ or 6 (mod 12), using Example 2.2 as well as Example 2.1.

Lemma 3.1. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 2, 6, 8 \pmod{12} \geq 6$.* □

3.2 $n \equiv 0$ (mod 12)

We begin with an example.

Example 3.2. ($n = 12$)

Let $X = \{\infty_1, \infty_2\} \cup Z_{10}$ and define a collection of 6-cycles C as follows:

$(0,\infty_1,2,1,3,6)$, $(2,\infty_2,4,3,5,8)$, $(4,\infty_1,6,5,7,0)$, $(6,\infty_2,8,7,9,2)$
$(8,\infty_1,9,0,1,4)$, $(1,\infty_1,3,0,2,7)$, $(3,\infty_2,5,2,4,9)$, $(5,\infty_1,7,4,6,1)$,
$(7,\infty_2,9,6,8,3)$, $(0,\infty_2,1,8,9,5)$,

with leave $L = \{(0,8),(1,9),(2,3),(4,5),(6,7),(\infty_1,\infty_2)\}$. Then (X,C) is a max packing of K_{12} with 6-cycles and can be squashed into a max packing of K_{12} with triples with leave L.

We will need two constructions for $12k \geq 24$: one when k is even and one when k is odd.

$12k$, k even Write $12k = 4(3k)$ and let (P,G,B) be a GDD$(3k,\{2\},3)$, set $X = P \times \{1,2,3,4\}$ and define a collection of 6-cycles C as follows:

(i) For each group $g \in G$ place Example 2.2 on $g \times \{1,2,3,4\}$ with leave $L_g = \{g \times \{1\}, g \times \{2\}, g \times \{3\}, g \times \{4\}\}$ and place these 6-cycles in C.

(ii) For each triple $t = \{a,b,c\} \in B$ place a copy of Example 2.3 on $K_{4,4,4}$ with parts $\{a\} \times \{1,2,3,4\}$, $\{b\} \times \{1,2,3,4\}$, and $\{c\} \times \{1,2,3,4\}$ and place these 6-cycles in C.

Then (X,C) is a max packing of K_{12k} with 6-cycles with leave $L = \{g\times\{1\}, g\times\{2\}, g\times\{3\}, g \times \{4\} \mid g \in G\}$. It is straightforward to see that the 6-cycles in (i) and (ii) can be squashed into a max packing of K_{12k} with triples with leave L.

$12k$, k odd Write $12k = 4(3k)$. Since k is odd, $3k$ is the order of a Kirkman triple system (P,T). Let $X = P \times \{1,2,3,4\}$, π a parallel class in T, and define a collection of 6-cycles C as follows:

(i) For each triple $t = \{a,b,c\} \in \pi$, place a copy of Example 3.2 on $\{a,b,c\} \times \{1,2,3,4\}$ with leave L_t and place these 6-cycles in C.

(ii) For each triple $\{a,b,c\} \in T \setminus \pi$, place a copy of Example 2.3 on $K_{4,4,4}$ with parts $\{a\} \times \{1,2,3,4\}$, $\{b\} \times \{1,2,3,4\}$, and $\{c\} \times \{1,2,3,4\}$, and place these 6-cycles in C.

Then (X,C) is a max packing of K_{12k} with 6-cycles with leave $L = \{L_t \mid t \in \pi\}$. Squashing these 6-cycles produces a max packing of K_{12k} with triples with leave L.

Lemma 3.3. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 0 \pmod{12}$.* □

3.3 $n \equiv 11 \pmod{12}$

We begin with an example.

Example 3.4. (n = 11)

Let $X = Z_9 \cup \{\infty_1,\infty_2\}$ and define a collection of 6-cycles C as follows:

$(4,8,\infty_2,7,\infty_1,0)$, $(5,0,\infty_2,4,\infty_1,1)$, $(6,1,\infty_2,5,\infty_1,2)$, $(7,2,\infty_2,6,\infty_1,3)$
$(3,2,5,7,0,1)$, $(7,4,5,3,0,6)$, $(4,3,6,8,1,2)$, $(8,5,6,4,1,7)$

with leave $L = \{(\infty_1, \infty_2, 3, 8), (0, 2, 8)\}$. Then (X, C) is a max packing of K_{11} with 6-cycles with leave L. Squashing these 6-cycles and adding (0, 2, 8) from the leave L gives a max packing of K_{11} with triples with leave the 4-cycle $(\infty_1, \infty_2, 3, 8)$.

We can now give a general construction for 11 (mod 12) ≥ 23.

$\underline{12k + 11 \geq 23}$ Write $12k + 11 = 2(6k + 4) + 3$. Let (P, G, B) be a GDD$(6k + 4, \{4^*, 2\}, 3)$ [13], set $X = \{\infty_1, \infty_2, \infty_3\} \cup (P \times \{1, 2\})$, and define a collection of 6-cycles C as follows:

(i) Let b^* be the unique group of size 4 and define a copy of Example 3.4 on $\{\infty_1, \infty_2, \infty_3\} \cup (b^* \times \{1, 2\})$ with leave $L = \{(\infty_1, \infty_2, \infty_3), (x, y, z, w)\}$, where $\{x, y, z, w\} \subseteq b^* \times \{1, 2\}$ and place these 6-cycles in C.

(ii) For each group $g \in G$ of size 2, define a copy of a max packing of K_7 with 6-cycles, with vertex set $\{\infty_1, \infty_2, \infty_3\} \cup (b \times \{1, 2\})$, that can be squashed into 6-triples with leave $(\infty_1, \infty_2, \infty_3)$ [11]. Add these 6-cycles to C.

(iii) For each triple $t = \{a, b, c\} \in B$, place a copy of Example 2.1 on $t \times \{1, 2\}$ with leave $\{a\} \times \{1, 2\}$, $\{b\} \times \{1, 2\}$, and $\{c\} \times \{1, 2\}$ and place these 6-cycles in C.

Then (X, C) is a max packing of K_{12k+11} with 6-cycles with leave L in (i). If we squash these 6-cycles and add the triple $(\infty_1, \infty_2, \infty_3)$ from the leave L in (i) we have a max packing of K_{12k+11} with triples with leave (x, y, z, w) in (i).

Lemma 3.5. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 11$ (mod 12).* □

3.4 $n \equiv 4$ or 10 (mod 12)

The following three examples are necessary for the constructions in this section.

Example 3.6. **($n = 10$)**

Let $X = \{\infty\} \cup (Z_3 \times Z_3)$ and define a collection of 6-cycles C as follows: $(0_1, 1_1, 0_0, 1_2, 0_2, \infty)$, $(1_2, 2_1, 0_0, 0_1, 0_2, 1_0)$, $(1_1, 2_1, 1_0, 2_2, 1_2, \infty)$, $(2_2, 0_1, 1_0, 1_1, 1_2, 2_0)$, $(2_1, 0_1, 2_0, 0_2, 2_2, \infty)$, $(0_2, 1_1, 2_0, 2_1, 2_2, 0_0)$ with leave $L = \{\{\infty, 2_0, 1_0, 0_0\}, (0_1, 1_2), (1_1, 2_2), (2_1, 0_2)\}$. (We remark that $\{\infty, 2_0, 1_0, 0_0\}$ is a copy of K_4 and not a 4-cycle.) Then (X, C) is a max packing of K_{10} with 6-cycles with leave L. If we squash these 6-cycles and remove a triple from $\{\infty, 2_0, 1_0, 0_0\}$, the result is a max packing of K_{10} with triples with leave the tripole $K_{1,3} \cup \{(0_1, 1_2), (1_1, 2_2), (2_1, 0_2)\}$.

Example 3.7. **($n = 16$)**

Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{i_j \mid i \in Z_6, j \in \{0, 1\}\}$. Further, for each $i \in Z_6$, define $\alpha(i) = \infty_1$ if i is odd and ∞_2 if i is even. For each $i \in Z_6$ define a collection of 6-cycles C as follows: $(i_1, i_0, (4+i)_1, (2+i)_1, (1+i)_1, \alpha(i))$, $(i_0, (1+i)_1, (4+i)_0, (2+i)_0, (1+i)_0, \alpha(i))$, and $(i_0, (2+i)_1, \infty_3, (1+i)_0, \infty_4, (5+i)_1)$ with leave $L = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{(i_j, (3+i)_j) \mid i \in \{0, 1, 2\}, j \in \{0, 1\}\}$. (Once again we remark that $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ is a copy of K_4.) Then (X, C) is a max packing of K_{16} with 6-cycles with leave L. If we squash these 6-cycles and remove a triple from $\{\infty_1, \infty_2, \infty_3, \infty_4\}$, the result is a max packing of K_{16} with triples, with leave the tripole $K_{1,3} \cup \{(i_j, (3+i)_j) \mid i \in \{0, 1, 2\}, j \in \{0, 1\}\}$.

Example 3.8. **($n = 28$)**

Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (Z_{12} \times \{1, 2\})$, and let (P, G, B) be a GDD$(12, \{3\}, 4)$ (equivalent to a pair of orthogonal quasigroups of order 3) and define a collection of 6-cycles C as follows:

(i) For each group $g \in G$, place a copy of Example 3.6 on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2\})$ with leave $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{(x_1, x_2) \mid x \in g\}$ (with K_4 based on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$).

(ii) For each block $b \in B$ place a copy of Example 2.2 on $b \times \{1, 2\}$ with leave $\{\{x_1, x_2\} \mid x \in b\}$.

Then (X, C) is a max packing of K_{28} with 6-cycles with the leave in (i). Now squashing the 6-cycles in (i) and (ii) and removing a triple from K_4 gives a max packing of K_{28} with triples with leave a tripole.

We can now go to the general constructions for $n \equiv 10 \pmod{12}$, $n \geq 22$ and $n \equiv 4 \pmod{12}$, $n \geq 40$.

$n \equiv 10 \pmod{12}, n \geq 22$ Write $12k + 10 = 2(6k + 5)$ and let (P, B) be a PBD$(6k + 5, \{5^*, 3\})$ [13]. Set $X = P \times \{1, 2\}$ and define a collection C of 6-cycles as follows:

(i) Let b^* be the unique block of size 5 and define a copy of Example 3.6 on $b^* \times \{1, 2\}$ and place these 6-cycles in C. (The leave is $K_4 \cup \{$1-factor$\}$.)

(ii) For each triple $t = \{a, b, c\} \in B$, define a copy of Example 2.1 on $t \times \{1, 2\}$ with leave $\{(a_1, a_2), (b_1, b_2), (c_1, c_2)\}$ and place these 6-cycles in C.

Then (X, C) is a max packing of K_{12k+10} with 6-cycles with leave $K_4 \cup \{$1-factor$\}$. Squashing the 6-cycles in C and removing a triple from the leave in (i) produces a max packing of K_{12k+10} with triples with leave a tripole.

$n \equiv 4 \pmod{12}, n \geq 40$ Write $12k+4 = 4+2(6k)$ and let (P, G, B) be a GDD$(6k, \{6\}, 3)$ [13]. Set $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (P \times \{1, 2\})$ and define a collection of 6-cycles as follows:

(i) For each group $g \in G$ define a copy of Example 3.7 on$\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2\})$ with leave $K_4 \cup \{(x_1, x_2) \mid x \in G\}$ (K_4 is based on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$).

(ii) For each triple $t = \{a, b, c\} \in B$, define a copy of Example 2.1 on $t \times \{1, 2\}$ with leave $\{(a_1, a_2), (b_1, b_2), (c_1, c_2)\}$ and place these 6-cycles in C.

Then (X, C) is a max packing of K_{12k+4} with 6-cycles. Squashing the 6-cycles in C and removing a triple from the leave K_4 in (i) produces a max packing of K_{12k+4} with triples with leave a tripole.

Lemma 3.9. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 4$ or $10 \pmod{12} \geq 6$.* □

3.5 $n \equiv 5 \pmod{12}$

This case requires three examples.

Example 3.10. $(n = 17)$

Let $X = \{\infty_1, \infty_2\} \cup Z_{15}$ and define a collection of 6-cycles C as follows: $\{(0,9,4,5,1,3)+i \mid i \in Z_{15}\} \cup \{(7,14,\infty_2,13,\infty_1,0), (8,0,\infty_2,7,\infty_1,1), (9,1,\infty_2,8,\infty_1,2), (10,2,\infty_2,9,\infty_1,3), (11,3,\infty_2,10,\infty_1,4), (12,4,\infty_2,11,\infty_1,5), (13,5,\infty_2,12,\infty_1,6)\}$ with leave the 4-cycle $(\infty_1,\infty_2,6,14)$. Squashing all of the 6-cycles in C produces a max packing of K_{17} with triples with leave the 4-cycle $(\infty_1,\infty_2,6,14)$.

Example 3.11. $(\boldsymbol{n = 29})$

Let $X = \{\infty_1, \infty_2\} \cup Z_{27}$ and define a collection of 6-cycles C as follows: $\{(0,3,1,5,4,9)+i, (0,16,10,17,7,15)+i \mid i \in Z_{27}\} \cup \{(14,0,\infty_2,13,\infty_1,1)+j \in \{0,1,2,...,11\}, (13,26,\infty_2,25,\infty_1,0)\}$ with leave the 4- cycle $(\infty_1,\infty_2,12,26)$. Squashing the 6-cycles in C gives a max packing of K_{29} with triples with leave the 4-cycle $(\infty_1,\infty_2,12,26)$.

Example 3.12. $(\boldsymbol{n = 53})$

Let $X = \{\infty_1, \infty_2\} \cup Z_{51}$ and define a collection of 6-cycles C as follows: $\{(0,3,1,5,4,21)+i, (0,19,9,20,11,23)+i, (0,24,8,15,7,22)+i, (0,20,6,11,5,18)+i \mid i \in Z_{51}\} \cup \{(26,0,\infty_2,25,\infty_1,1)+j \mid j \in \{0,1,2,\ldots,23\}\} \cup \{(25,50,\infty_2,49,\infty_1,0)\}$ with leave the 4-cycle $(\infty_1,\infty_2,24,50)$. Squashing these 6-cycles gives a max packing of K_{53} with triples with leave the 4-cycle $(\infty_1,\infty_2,24,50)$.

We can now give two general constructions to finish off the case $n \equiv 5 \pmod{12}$.

$\underline{12k+5,\ k \text{ odd}}$ Write $12k+5 = 1+4(3k+1)$. Since k is odd, $1+4(3k+1) = 1+4(6t+4)$. Let (P,G,B) be a GDD$(6t,\{4^*,2\},3)$, set $X = \{\infty\} \cup (P \times \{1,2,3,4\})$ and define a collection of 6-cycles as follows:

(i) For the unique group b^* of size 4, define a copy of Example 3.10 on $\{\infty\} \cup (b^* \times \{1,2,3,4\})$ (the leave is a 4-cycle) and place these 6-cycles in C.

(ii) For each group g of size 2, define a copy of Example 1.1 on $\{\infty\} \cup (g \times \{1,2\})$ and place these 6-cycles in C. (There is no leave.)

(iii) For each triple $\{a,b,c\} \in B$, place a copy of Example 2.3 on $\{a,b,c\} \times \{1,2,3,4\}$ with parts $\{a\} \times \{1,2,3,4\}$, $\{b\} \times \{1,2,3,4\}$, $\{c\} \times \{1,2,3,4\}$, and place these 6-cycles in C. (There is no leave.)

Then (X,C) is a max packing of K_{12k+5} with 6-cycles with leave a 4-cycle. If we squash the 6-cycles in (i), (ii) and (iii), we have a max packing of K_{12k+5} with triples with leave a 4-cycle.

$\underline{12k+5,\ k \text{ even}}$ Write $12k+5 = 1+4(3k+1)$. Since k is even, $1+4(3k+1) = 1+4(6t+1)$. Since $12k+5 \geq 77$, $6t+1 \geq 19$, and there exists a GDD$(6t+1,\{7^*,3\},3)$ [11] (P,G,B). Define a collection C of 6-cycles on $X = \{\infty\} \cup (P \times \{1,2,3,4\})$ as follows:

(i) For the unique group b^* of size 7, define a copy of Example 3.11 on $\{\infty\} \cup (b \times \{1,2,3,4\})$ (with leave a 4-cycle) and place these 6-cycles in C.

(ii) For each group g of size 3, place a copy of a 6-cycle system of order 13 which can be squashed into a triple system [11] (no leave) on $\{\infty\} \cup (g \times \{1,2,3,4\})$ and place these 6-cycles in C.

(iii) For each triple $\{a,b,c\} \in B$, place a copy of Example 2.3 on $\{a,b,c\} \times \{1,2,3,4\}$ with parts $\{a\} \times \{1,2,3,4\}$, $\{b\} \times \{1,2,3,4\}$, $\{c\} \times \{1,2,3,4\}$, and place these 6-cycles in C. (There is no leave.)

Then (X, C) is a max packing of K_{12k+5} with 6-cycles with leave the 4-cycle in (i). Squashing the 6-cycles in (i), (ii) and (iii) produces a max packing of K_{12k+5} with triples with leave the 4-cycle in (i).

Lemma 3.13. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 5 \pmod{12} \geq 17$.* □

4 Main result and further developments

Putting together the results in Section 3 gives the following theorem.

Theorem 4.1. *For each $n \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$, $n \geq 6$, there exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples. There are no exceptions.* □

Since a complete solution is also a max packing, we can combine Theorem 1.3 and Theorem 8.1 into the following corollary (giving a complete solution to the squashing of max packings of 6-cycles into max packings with triples).

Corollary 4.2. *For each $n \geq 6$, there is a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples.* □

In this paper we give a complete solution to the problem of squashing maximum packings of K_n with 6-cycles into maximum packings of K_n with triples. An open problem is to solve the general case, i.e. squashing a maximum packing of K_n with $2m$-cycles into a maximum packing of K_n with m-cycles; the case $m = 4$ is completely solved in [10] .

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