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A NEW TOPOLOGY FOR THE TRAJECTORIES OF THE MENISCUS DURING CONTINUOUS STEEL CASTING

NOVA TOPOLOGIJA TRAJEKTORIJ MENISKUSA PRI NEPREKINJENEM LITJU JEKLA

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The theoretical basis for a new computational method is given for the topology of the meniscus trajectories during continuous steel casting. The method is based on the solution of the meniscus equation in $\Re^3$. Here the topology is treated only in the sense of the categorization of trajectories in an orientation space. This suggests a new type of efficient self-adaptive scheme suitable for the solution of the shape of the meniscus. In the present work a new approach is used to overcome previously unknown pathological, non-physical predictions in various constitutive models derived using closure approximations. The generalized meniscus equation as well as its stability is solved. Here it is shown, for the first time, that the cyclic change of the shape of the meniscus depends on the coordinates, while up to now the cyclic change of the meniscus was presented only as a function of the time over the expression of the mould velocity.

Key words and phrases: topology of trajectories, shape of the meniscus, meniscus equation, generalized meniscus equation, meniscus stability.

Dana je teoretična podlaga za nov izračun topologije trajektorij meniskusa pri neprekinjenem litju jekla. Podlaga metode je rešitev enačbe meniskusa $\Re^3$. Topologija je obravnavana le v smislu kategorizacije trajektorij v nekem orientacijskem prostoru. To navaja na novo shemo, ki se sama primerno prilagaja za rešitev oblike meniskusa. V tem delu je uporabljen nov način, da bi se obvladalo preje neznane patološke, nefizikalne napovedi v različnih konstitutivnih modelih, ki so bili razviti z uporabo končnih aproksimacij. Rešeni sta splošna enačba meniskusa in njena stabilnost. Prvič je prikazano, da je ciklična sprememba oblike meniskusa odvisna od koordinat, medtem ko se je dosedaj ciklične spremembe meniskusa prikazalo samo kot funkcijo razmerja čas proti hitrosti kokile.

Ključne besede in stavki: topologija trajektorij, oblika meniskusa, enačba meniskusa, splošna enačba meniskusa, stabilnost meniskusa

1 INTRODUCTION

The study of the changes of the meniscus during continuous steel casting is neither an easy nor a simple task. It is, in fact, very complicated and requires a serious approach and hard work, because the meniscus's appearance depends on many factors of the continuous steel casting process. Generally speaking, this type of multidisciplinary research looks for a sufficient knowledge of steel metallurgy as well as the highest level of mathematics.

Many efforts have been spent to describe the shape of the meniscus during continuous steel casting. For instance, in [1] and [26] the authors considered the shape of the meniscus as a linear function and developed a model based on the Navier-Stokes equation for a hydrodynamic fluid. In [12], by virtue of complex functions, the movement of molten powder between the strand and mould wall is presented as a Newtonian fluid flowing between two parallel plates by neglecting the thermal contact resistance between the solidifying metal and the mould [11]. Particular cases of this complex model appear in the results given in [13,18]. In later research works [14,16] the meniscus is shaped with an exponential function by virtue of the meniscus's dimensions [17].

In [24,25] the authors developed a dimensionless model for the meniscus, introducing Reynolds' lubrication theory. A model closely related to the *free coating problem* [27], is solved numerically and is compared with the published data.

On other hand, in the simulation model [2] a fixed shape of the meniscus is used to calculate the fluid flow and heat transfer. The authors in [5] account for the interdependence of the shape of the flux gape and the fluid flow therein, but still require some parameters to be selected rather arbitrarily, if impossible, to determine the experimental measurements.

Research in [6,7] showed that the movement of molten powder in the flux space may be determined by a *pseudo-transient analytical solution* of the Navier-Stokes equation. The validity of this solution is verified using an explicit finite-difference discretization method and the MATLAB software package. The simulation and behavior of interfacial mould slag layers in the continuous casting of steel are investigated in [8].

In [28,29,30] the authors modified the model for lubrication on the meniscus, given in [14,16], with the difference between steel and flux density and extended it with the heat-transfer phenomena. They do not use the natural logarithm with base *e* for the description of the exponential shape of the meniscus, and use instead the decade logarithm with base 10, without considering the correlation $\ln x \approx 2.303 \lg x$.

The newest research [4,10] is directed to cold model experiment on the infiltration of mould flux during the continuous casting of steel, neglecting the mould oscillations and the infiltration phenomena of molten powder derived from an analysis using the Reynolds equation.

Generally speaking, up to now in the literature the meniscus changes during continuous steel casting were approximately treated with one-dimensional mathematical models by virtue of some real function as a *fixed* shape. The treatment was adopted because the meniscus equation as a function of several variables was not known. In the present work, the introduced meniscus equation as a quasicyclic real function equation, *i.e.*, its solution, can be used for all the possible cases of meniscus changes occurring cyclically during the mould cycle. In this way the mathematical description of the shape of the meniscus is much better, because the shape of the meniscus is closer to its own real shape.

Up to now in relevant references the form of the meniscus was presented as a cyclical change independent of time over the mould speed only, while, in this article, it is shown that the change of the form of the meniscus depends on other coordinates too.

The present work gives a new approach to meniscus vicinity in a sufficiently sophisticated way, which is more complete than previous treatments. With the goal to shed new light on this topic, with this article a new shape of the meniscus is introduced by virtue of the solution of the meniscus equation. With the intention to better understand this approach, emphasis is given to the engineering experience and the theoretical knowledge of mathematical modeling of the continuous steel casting process.

2 PRELIMINARIES

Let $A = [a_{ij}]_{n\times n}$ be a real matrix. Suppose that by elementary transformations the matrix A is transformed into $A = P_1DP_2$, where P_1 and P_2 are regular matrices and D is a diagonal matrix with diagonal entries 0 and 1, such that the number of units is equal to the rank of the matrix A. The matrix $B = P_2^{-1}DP_1^{-1}$ satisfies the equality $ABA = A$. This means that the matrix equation $AXA = A$ has at least one solution for X.

If A satisfies the identity

$$A^r + k_1A^{r-1} + \cdots + k_{r-1}A = O$$

where $k_{r-1} \neq 0$ and O is the zero $n\times n$ matrix, then the matrix

$$X = -(A^{r-2} + k_1A^{r-3} + \cdots + k_{r-2}I)/k_{r-1}$$

where I is the unit $n\times n$ matrix, is also a solution of the equation $AXA = A$.

Now we recall the following theorem proven in [20].

Theorem 2.1. *If B satisfies the condition ABA = A, then*

1° $AX = O \Leftrightarrow X = (I - BA)Q$ (X and Q are $n \times m$ matrices),

2° $XA = O \Leftrightarrow X = Q(I - AB)$ (X and Q are $m \times n$ matrices),

3° $AXA = A \Leftrightarrow X = B + Q - BAQAB$ (*X and Q* are $n \times n$ matrices),

4° $AX = A \Leftrightarrow X = I + (I - BA)Q$,

5° $XA = A \Leftrightarrow X = I + Q(I - AB)$.

Throughout this paper, $\Re$ is a finite-dimensional real vector space. Vectors from $\Re$ will be denoted by $\boldsymbol{X}_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^T$, and also we denote with $\mathbf{O} = (0,0,\ldots,0)^T$ the zero vector in $\Re$. Let $\otimes$ denote the exterior product in $\Re$ and let k $(1 \leq k \leq n)$ be an integer. With respect to the canonical basis in the k-th exterior product space $\otimes^k \Re$, the k-th additive compound matrix $A^{[k]}$ of A is a linear operator on $\otimes^k \Re$ whose definition on a decomposable element $x_1 \otimes \cdots \otimes x_k$ is

$$A^{[k]}(x_1\otimes \cdots \otimes x_k) = \sum_{i=1}^{k} x_1 \otimes \cdots \otimes Ax_i \otimes \cdots \otimes x_k. \qquad (2.1)$$

For any integer $i = 1, 2, \ldots, n!/k!(n-k)!$, let $(i) = (i_1, \ldots, i_k)$ be the i-th member in the lexicographic ordering of integer k-tuples such that $1 \leq i_1 < \cdots < i_k \leq n$. Then the (i, j)-th entry of the matrix $A^{[k]} = [q_{ij}]$ is

$$q_{ij} = a_{i1,i1} + \cdots + a_{ik,ik} \text{ if } (i) = (j)$$
$$q_{ij} = (-1)^{m+s} a_{jm,is} \qquad (2.2)$$

if exactly one entry i_s of (i) does not occur in (j) and j_m does not occur in (i),

$$q_{ij} = 0$$

if (i) differs from (j) in two or more entries.

As special cases, we have $A^{[1]} = A$ and $A^{[n]} = \mathrm{tr}A$ [9].

Let $\sigma(A) = \{\lambda_i, 1 \leq i \leq n\}$ be the spectrum of A. Then the spectrum of $A^{[k]}$ is $\sigma(A^{[k]}) = \{\lambda_{i1} + \cdots + \lambda_{ik}, 1 \leq i_1 < \cdots < i_k \leq n\}$.

Let $|\cdot|$ denote a vector norm in $\Re^n$. The Lozinskiĭ measure μ on $\Re^n$ with respect to $|\cdot|$ is defined by

$$\mu(A) = \lim_{\rho\to 0+} (|I + \rho A| - 1)/\rho \qquad (2.3)$$

The Lozinskiĭ measures of $A = [a_{ij}]_{n\times n}$ with respect to the three common norms

$$|x|_\infty = \sup_i |x_i|$$
$$|x|_1 = \Sigma_i |x_i|$$
$$|x|_2 = (\Sigma_i |x_i|^2)^{1/2}$$

are

$$\mu_\infty(A) = \sup_i (a_{ii} + \Sigma_{k,k\neq i} |a_{ik}|)$$
$$\mu_1(A) = \sup_k (a_{kk} + \Sigma_{i,i\neq k} |a_{ik}|) \qquad (2.4)$$
$$\mu_2(A) = \mathrm{stab}[(A + A^T)/2]$$

where

$$\mathrm{stab}(A) = \max\{\lambda, \lambda\in\sigma(A)\}$$

is the stability modulus of the matrix A, and A^T denotes the transpose of A [3, p.41].

Definition 2. 2. A stable system is that system in which after the transitive action appearance a constant position is achieved [15, p.38].

3 TOPOLOGY OF THE SOLUTIONS OF THE MENISCUS EQUATION

In this section we will give a complete analysis of the meniscus equation represented by a quasicyclic real functional equation for all possible cases. For that purpose we will use techniques for the solution given in [19,21,23].

Let us consider now the equation

$$\begin{aligned} &a_1 f(x_1,x_2,x_3) + a_2 f(x_2,x_3,x_1) + a_3 f(x_3,x_1,x_2) = \\ &= \alpha_1 f(x_1,x_1,x_2) + \alpha_2 f(x_2,x_2,x_3) + \alpha_3 f(x_3,x_3,x_1) \end{aligned} \tag{3.1}$$

f: $\Re^3 \to \Re$, where a_i, α_i $(1 \leq i \leq 3)$ are real constants. For equation (3.1) we suppose that $|a_1| + |a_2| + |a_3| > 0$.

If we permute cyclically the variables in the equations (3.1), we obtain

$$\begin{aligned} &a_1 f(x_2,x_3,x_1) + a_2 f(x_3,x_1,x_2) + a_3 f(x_1,x_2,x_3) = \\ &= \alpha_1 f(x_2,x_2,x_3) + \alpha_2 f(x_3,x_3,x_1) + \alpha_3 f(x_1,x_1,x_2) \end{aligned} \tag{3.2}$$

$$\begin{aligned} &a_1 f(x_3,x_1,x_2) + a_2 f(x_1,x_2,x_3) + a_3 f(x_2,x_3,x_1) = \\ &= \alpha_1 f(x_3,x_3,x_1) + \alpha_2 f(x_1,x_1,x_2) + \alpha_3 f(x_2,x_2,x_3) \end{aligned} \tag{3.3}$$

The determinant for the system of the equations (3.1), (3.2) and (3.3) is

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix}$$

Let us note the identity

$$\begin{aligned} \Delta = (a_1 + a_2 + a_3)[(a_1 - a_2)^2 + (a_2 - a_3)^2 + \\ + (a_3 - a_1)^2]/2 \end{aligned} \tag{3.4}$$

First we consider the case

1° Let $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Now the system (3.1), (3.2) and (3.3) takes the form

$$\begin{aligned} a_1 f(x_1,x_2,x_3) + a_2 f(x_2,x_3,x_1) + a_3 f(x_3,x_1,x_2) = 0 \\ a_3 f(x_1,x_2,x_3) + a_1 f(x_2,x_3,x_1) + a_2 f(x_3,x_1,x_2) = 0 \\ a_2 f(x_1,x_2,x_3) + a_3 f(x_2,x_3,x_1) + a_1 f(x_3,x_1,x_2) = 0 \end{aligned} \tag{3.5}$$

If $\Delta \neq 0$, then the system (3.5) implies $f(x_1,x_2,x_3) = 0$.

Now let $\Delta = 0$. According to (3.4), this is possible if the real constants a_1, a_2, a_3 satisfy either $a_1 = a_2 = a_3$ or $a_1 + a_2 + a_3 = 0$.

First, let us suppose that $a_1 = a_2 = a_3$ $(\neq 0)$. Then system (3.5) is equivalent to the equation

$$f(x_1,x_2,x_3) + f(x_2,x_3,x_1) + f(x_3,x_1,x_2) = 0 \tag{3.6}$$

whose general solution according to [22] is

$$f(x_1,x_2,x_3) = F(x_1,x_2,x_3) - F(x_2,x_3,x_1) \tag{3.7}$$

where F: $\Re^3 \to \Re$, is an arbitrary function.

Now let us suppose that $\Delta = 0$, although the real constants are not all equal. Then necessarily $a_1 + a_2 + a_3 = 0$ and we can suppose, without any loss of generality, that $a_1 \neq a_2$. In this case we set $a_3 = -a_1 - a_2$ and system (3.5) can be written in the form

$$\begin{aligned} a_1[f(x_1,x_2,x_3) - f(x_2,x_3,x_1)] = a_2[f(x_3,x_1,x_2) - f(x_1,x_2,x_3)] \\ a_1[f(x_2,x_3,x_1) - f(x_3,x_1,x_2)] = a_2[f(x_1,x_2,x_3) - f(x_2,x_3,x_1)] \\ a_1[f(x_3,x_1,x_2) - f(x_1,x_2,x_3)] = a_2[f(x_2,x_3,x_1) - f(x_3,x_1,x_2)] \end{aligned}$$

From this we derive easily

$$(a_1^3 - a_2^3)[f(x_1,x_2,x_3) - f(x_2,x_3,x_1)] = 0$$

With the assumption that $a_1 \neq a_2$ and they have real values, equation (3.4) reduces to

$$f(x_1,x_2,x_3) - f(x_2,x_3,x_1) = 0 \tag{3.8}$$

According to [20], the general solution of the above functional equation is

$$f(x_1,x_2,x_3) = F(x_1,x_2,x_3) + F(x_2,x_3,x_1) + F(x_3,x_1,x_2) \tag{3.9}$$

where F: $\Re^3 \to \Re$, is an arbitrary function.

The above results concerning the cyclic functional equation

$$a_1 f(x_1,x_2,x_3) + a_2 f(x_2,x_3,x_1) + a_3 f(x_3,x_1,x_2) = 0$$

can be derived from those in [20], where a_1, a_2, a_3 are real constants.

From now we suppose that $|\alpha_1| + |\alpha_2| + |\alpha_3| > 0$. We can distinguish the following two cases:

2° Let $\Delta \neq 0$, from (3.1), (3.2) and (3.3) we obtain

$$f(x_1,x_2,x_3) = \begin{vmatrix} \alpha_1 F(x_1,x_2) + \alpha_2 F(x_2,x_3) + \alpha_3 F(x_3,x_1) & a_2 & a_3 \\ \alpha_1 F(x_2,x_3) + \alpha_2 F(x_3,x_1) + \alpha_3 F(x_1,x_2) & a_1 & a_2 \\ \alpha_1 F(x_3,x_1) + \alpha_2 F(x_1,x_2) + \alpha_3 F(x_2,x_3) & a_3 & a_1 \end{vmatrix}$$

where F: $\Re^2 \to \Re$ is defined by $F(u,v) = f(u,u,v)/\Delta$.

If we introduce the notations

$$\Delta_1 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ a_2 & a_3 & a_1 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad \Delta_3 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \end{vmatrix}$$

then we can write

$$f(x_1,x_2,x_3) = \Delta_1 F(x_1,x_2) + \Delta_2 F(x_2,x_3) + \Delta_3 F(x_3,x_1) \tag{3.10}$$

For (3.10) to be a solution of the functional equation (3.1), the following condition must be satisfied:

$$\begin{aligned} &\alpha_1[(\Delta - \Delta_2)F(x_1,x_2) - \Delta_3 F(x_2,x_1) - \Delta_1 F(x_1,x_1)] + \\ &+ \alpha_2[(\Delta - \Delta_2)F(x_2,x_3) - \Delta_3 F(x_3,x_2) - \Delta_1 F(x_2,x_2)] + \\ &+ \alpha_3[(\Delta - \Delta_2)F(x_3,x_1) - \Delta_3 F(x_1,x_3) - \Delta_1 F(x_3,x_3)] = 0 \end{aligned} \tag{3.11}$$

By a cyclic permutation of the variables x_1, x_2, x_3 in (3.11) we obtain two new equations. The system of these three equations has a nontrivial solution with respect to $(\Delta - \Delta_2)F(x_i,x_{i+1}) - \Delta_3 F(x_{i+1},x_i) - \Delta_1 F(x_i,x_i)$, $i = 1, 2, 3$ (with the convention $x_4 \equiv x_1$) if the following condition is satisfied

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \alpha_2 \end{vmatrix} = 0 \tag{3.12}$$

By virtue of an equality of the type of (3.4) this is true if

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \text{ or } \alpha_1 = \alpha_2 = \alpha_3$$

First, we will consider the case

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{3.13}$$

Since $a_1 + a_2 + a_3 \neq 0$ (because of the assumption $\Delta \neq 0$ and (3.4)), by putting into equation (3.1) $x_1 = x_2 = x_3$ we derive $F(x_1,x_2) = 0$. By using the last equality, the equation (3.11) for $x_3 = x_1$ becomes

$$[\alpha_1(\Delta - \Delta_2) - \alpha_2\Delta_3]F(x_1,x_2) - \\ - [\alpha_1\Delta_3 - \alpha_2(\Delta - \Delta_2)]F(x_2,x_1) = 0 \tag{3.14}$$

If we change the places of x_1 and x_2, the equation (3.14) is transformed into

$$- [\alpha_1\Delta_3 - \alpha_2(\Delta - \Delta_2)]F(x_1,x_2) + \\ + [\alpha_1(\Delta - \Delta_2) - \alpha_2\Delta_3]F(x_2,x_1) = 0 \tag{3.15}$$

Let

$$(\alpha_1^2 - \alpha_2^2)[(\Delta - \Delta_2)^2 - \Delta_3^2] \neq 0$$

then from (3.14) and (3.15) it follows that $F(x_1,x_2) = 0$ and then from (3.10) $f(x_1,x_2,x_3) = 0$.

The condition

$$(\alpha_1^2 - \alpha_2^2)[(\Delta - \Delta_2)^2 - \Delta_3^2] = 0 \tag{3.16}$$

implies $(\Delta - \Delta_2)^2 - \Delta_3^2 = 0$. Let us suppose that the last equality is not true. Now, if we set $x_3 = x_2$ into (3.11), we obtain

$$(\alpha_1^2 - \alpha_3^2)[(\Delta - \Delta_2)^2 - \Delta_3^2] = 0$$

The last equality, with (3.16), gives $\alpha_1^2 = \alpha_2^2 = \alpha_3^2$ which, by virtue of the assumption (3.13), yields $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and this contradicts the hypothesis $|\alpha_1| + |\alpha_2| + |\alpha_3| > 0$.

Thus, we have $\Delta - \Delta_2 = \pm \Delta_3$. For the case $\Delta - \Delta_2 = \Delta_3$ ($\neq 0$), equation (3.14) yields

$$(\alpha_1 - \alpha_2)\Delta_3[F(x_1,x_2) - \Delta_3F(x_2,x_1)] = 0$$

so that, for $\alpha_1 \neq \alpha_2$, we have

$$F(x_1,x_2) = G(x_1,x_2) + G(x_2,x_1) \tag{3.17}$$

where G is an arbitrary function $\Re^2 \to \Re$ such that $G(x_1,x_1) \equiv 0$

If $\alpha_1 = \alpha_2$, then necessarily we must have $\alpha_1 \neq \alpha_3$, (because otherwise we would have $\alpha_1 = \alpha_2 = \alpha_3 = 0$) and by a procedure analogous to the one above we obtain (3.17).

Let $\Delta - \Delta_2 = - \Delta_3$ ($\neq 0$), from (3.14) we obtain

$$(\alpha_1 + \alpha_2)\Delta_3[F(x_1,x_2) + F(x_2,x_1)] = 0 \tag{3.18}$$

For $\alpha_1 + \alpha_2 \neq 0$, the general solution of equation (3.18) is given by

$$F(x_1,x_2) = G(x_2,x_1) - G(x_2,x_1),\ G: \Re^2 \to \Re \tag{3.19}$$

If $\alpha_1 + \alpha_2 = 0$, then from (3.13) we deduce $\alpha_3 = 0$, and then $\alpha_1 + \alpha_3 = \alpha_1 \neq 0$ and we obtain (3.19) by an analogous procedure.

The condition (3.11), for the case $\Delta_3 = \Delta - \Delta_2 = 0$, is satisfied for every function $F(x_1,x_2)$ with the property $F(x_1,x_1) \equiv 0$.

If $(\Delta - \Delta_2)^2 \neq \Delta_3^2$, then, as was mentioned above, equations (3.14) and (3.15) have a trivial solution as a general solution. According to (3.10) we obtain

$$f(x_1,x_2,x_3) \equiv 0$$

Now we suppose that (3.12) is satisfied but (3.13) is not. This means that

$$\alpha_1 = \alpha_2 = \alpha_3 \neq 0 \tag{3.20}$$

It immediately follows that

$$\Delta_1 = \Delta_2 = \Delta_3 \ (\neq 0)$$

The quasicyclic equation (3.11) implies

$$(\Delta - \Delta_1)F(x_1,x_2) - \Delta_1F(x_2,x_1) - \Delta_1F(x_1,x_1) = \\ = \Delta_1P(x_1) - \Delta_1P(x_2) \tag{3.21}$$

where P is an arbitrary function $\Re \to \Re$

For $\alpha_1 = \alpha_2 = \alpha_3$ and $x_1 = x_2 = x_3$ equation (3.1) becomes

$$(a_1 + a_2 + a_3 - 3\alpha_1)F(x_1,x_1) = 0$$

Let $a_1 + a_2 + a_3 = 3\alpha_1$, then $3\Delta_1 = \Delta$ and the equality (3.21) takes the form

$$2F(x_1,x_2) - F(x_2,x_1) = P(x_1) - P(x_2) + R(x_1)$$

where $R(x_1) = F(x_1,x_1)$.

By a permutation of the variables x_1 and x_2 it follows that

$$- F(x_1,x_2) + F(x_2,x_1) = P(x_2) - P(x_1) + R(x_2)$$

From the last two equalities we obtain

$$F(x_1,x_2) = [P(x_1) - P(x_2) + 2R(x_1) + R(x_2)]/3$$

By using the last equality, from (3.10) it follows that

$$f(x_1,x_2,x_3) = Q(x_1) + Q(x_2) + Q(x_3) \tag{3.22}$$

where $Q(x_1) = \Delta_1R(x_1)$.

Let $a_1 + a_2 + a_3 \neq 3\alpha_1$, then $F(x_1,x_1) \equiv 0$ and the formula (2.21) yields

$$(\Delta - \Delta_1)F(x_1,x_2) - \Delta_1F(x_2,x_1) = \Delta_1P(x_1) - \Delta_1P(x_2) \tag{3.23}$$

From the last equality, with a permutation of the variables we obtain

$$- \Delta_1F(x_1,x_2) + (\Delta - \Delta_1)F(x_2,x_1) = \Delta_1P(x_2) - \Delta_1P(x_1)$$

The determinant of the system consisting of the last two equations is $\Delta(\Delta - 2\Delta_1)$. If $\Delta \neq 2\Delta_1$, the solution of the last two equations is

$$F(x_1,x_2) = \Delta_1[P(x_1) - P(x_2)]/\Delta$$

Then the equality (3.10) gives

$$f(x_1,x_2,x_3) \equiv 0$$

Let $\Delta = 2\Delta_1$, then from (3.23) we obtain

$$F(x_1,x_2) - P(x_1) = F(x_2,x_1) - P(x_2)$$

The general solution for the last equation is

$$F(x_1,x_2) = P(x_1) + G(x_1,x_2) + G(x_2,x_1) \tag{3.24}$$

where $P: \Re \to \Re$ and $G: \Re^2 \to \Re$ are arbitrary functions such that

$$G(x_1,x_2) = -P(x_1)/2$$

According to the last relation, the equality (3.24) takes the form

$$F(x_1,x_2) = G(x_1,x_2) + G(x_2,x_1) - 2G(x_1,x_1)$$

and the equality (3.10) becomes

$$f(x_1,x_2,x_3) = G(x_1,x_2) + G(x_2,x_1) - 2G(x_1,x_2) + G(x_2,x_3) + \\ + G(x_3,x_2) - 2G(x_2,x_2) + G(x_3,x_1) + G(x_1,x_3) - 2G(x_3,x_3) \quad (3.25)$$

where we have replaced $G(x_1,x_2)\Delta_1$ by $G(x_1,x_2)$.

For

$$\begin{vmatrix} \alpha_1 \alpha_2 \alpha_3 \\ \alpha_2 \alpha_3 \alpha_1 \\ \alpha_3 \alpha_1 \alpha_2 \end{vmatrix} \neq 0$$

from (3.10) we obtain

$$(\Delta - \Delta_2)F(x_1,x_2) - \Delta_3 F(x_2,x_1) - \Delta_1 F(x_1,x_1) = 0 \quad (3.26)$$

First we suppose that $\Delta - \Delta_1 - \Delta_2 - \Delta_3 \neq 0$. In this case, with the substitution $x_2 = x_1$, equation (3.26) reduces to $F(x_1,x_1) \equiv 0$. On the basis of the last equality, the equation (3.26) becomes

$$(\Delta - \Delta_2)F(x_1,x_2) - \Delta_3 F(x_2,x_1) = 0$$

From the permutation of the variables x_1 and x_2, from the above equation it follows that

$$-\Delta_3 F(x_1,x_2) + (\Delta - \Delta_2)F(x_2,x_1) = 0$$

The system of the last two equations has a nontrivial solution if and only if the following condition $(\Delta - \Delta_2)^2 = \Delta_3^2$ is satisfied.

Let $\Delta - \Delta_2 = \Delta_3$ ($\neq 0$), then we obtain

$$F(x_1,x_2) = G(x_1,x_2) + G(x_2,x_1)$$

where G satisfies $G(x_1,x_1) \equiv 0$.

For the case $\Delta - \Delta_2 = -\Delta_3$ ($\neq 0$) the general solution is

$$F(x_1,x_2) = G(x_1,x_2) - G(x_2,x_1)$$

For $\Delta - \Delta_2 = \Delta_3 = 0$, the unique condition that must be satisfied by the function F is $F(x_1,x_1) \equiv 0$.

The condition $(\Delta - \Delta_2)^2 \neq \Delta_3^2$ gives $F(x_1,x_2) \equiv 0$

Next we will pass on to the case $\Delta - \Delta_1 - \Delta_2 - \Delta_3 = 0$. Now equation (3.26) can be written as

$$(\Delta_1 + \Delta_3)F(x_1,x_2) - \Delta_3 F(x_2,x_1) = \Delta_1 R(x_1)$$

where $R(x_1) = F(x_1,x_1)$. By a permutation of the variables x_1 and x_2 we obtain

$$-\Delta_3 F(x_1,x_2) + (\Delta_1 + \Delta_3)F(x_2,x_1) = \Delta_1 R(x_2)$$

The determinant of this system is $(\Delta_1 + 2\Delta_3)\Delta_1$. If it is not zero, then

$$F(x_1,x_2) = [(\Delta_1 + \Delta_3)R(x_1) + \Delta_3 R(x_2)]/(\Delta_1 + 2\Delta_3)$$

From (3.10) it follows that

$$f(x_1,x_2,x_3) = (\Delta_1^2 + \Delta_1\Delta_3 + \Delta_3^2)Q(x_1) + (\Delta_1\Delta_2 + \Delta_1\Delta_3 + \\ + \Delta_2\Delta_3)Q(x_2) + \Delta_3\Delta Q(x_3)$$

where

$$Q(x_1) = R(x_1)/(\Delta_1 + 2\Delta_3)$$

Let $\Delta_1 = 0$, $\Delta_3 \neq 0$. Then

$$F(x_1,x_2) = F(x_2,x_1)$$

Thus

$$F(x_1,x_2) = G(x_1,x_2) + G(x_2,x_1)$$

where G is an arbitrary function $\Re^2 \to \Re$, and $f(x_1,x_2,x_3)$ is given by the formula (3.10).

Now we suppose that $\Delta_1 = -2\Delta_3 \neq 0$. Then

$$F(x_1,x_2) + F(x_2,x_1) = 2R(x_1) \quad (3.27)$$

By a permutation of the variables x_1 and x_2 we obtain

$$F(x_2,x_1) + F(x_1,x_2) = 2R(x_2) \quad (3.28)$$

From (3.27) and (3.28) we get

$$R(x_1) = R(x_2) = c$$

Thus (3.27) takes on the form

$$[F(x_1,x_2) - c] + [F(x_2,x_1) - c] = 0$$

which implies that

$$F(x_1,x_2) = G(x_1,x_2) - G(x_2,x_1) + c$$

where G: $\Re^2 \to \Re$ is an arbitrary function and c is an arbitrary real constant.

Now from (2.10) we find

$$f(x_1,x_2,x_3) = -2\Delta_3[G(x_1,x_2) - G(x_2,x_1)] + \\ + (\Delta + \Delta_3)[G(x_2,x_3) - G(x_3,x_2)] + \\ + \Delta_3[G(x_3,x_1) - G(x_1,x_3)] + c$$

where c is (another) arbitrary real constant.

In the case $\Delta_1 = \Delta_3 = 0$ equation (3.26) is satisfied for every function F: $\Re^2 \to \Re$.

Now we will use the following result.

Lemma 3. 1. Let $\Delta \neq 0$. Then the system

$$\Delta_1 = 0,\ \Delta_2 - \Delta = 0,\ \Delta_3 = 0 \quad (3.29)$$

implies $\alpha_1 = a_3$, $\alpha_2 = a_1$, $\alpha_3 = a_2$.

Proof. The system (3.29) can be written in the form

$$\begin{aligned} A_{11}(\alpha_1 - a_3) + A_{12}(\alpha_2 - a_1) + A_{13}(\alpha_3 - a_2) = 0 \\ A_{21}(\alpha_1 - a_3) + A_{22}(\alpha_2 - a_1) + A_{23}(\alpha_3 - a_2) = 0 \\ A_{31}(\alpha_1 - a_3) + A_{32}(\alpha_2 - a_1) + A_{33}(\alpha_3 - a_2) = 0 \end{aligned} \quad (3.30)$$

where A_{ij} is the cofactor of the element a_{ij} ($1 \leq i, j \leq 3$) of the determinant Δ. The system (3.30) is a homogeneous linear system with respect to $\alpha_1 - a_3$, $\alpha_2 - a_1$, $\alpha_3 - a_2$. Its determinant is $\Delta^2 \neq 0$, so that it has only the zero solution.

Thus, the equation

$$a_1 f(x_1,x_2,x_3) + a_2 f(x_2,x_3,x_1) + a_3 f(x_3,x_1,x_2) = \\ = a_3 f(x_1,x_1,x_2) + a_1 f(x_2,x_2,x_3) + a_2 f(x_3,x_3,x_1)$$

has the general solution $f(x_1,x_2,x_3) = F(x_2,x_3)$.

3° Let $\Delta = 0$. Then from (3.4) it follows that

$$a_1 + a_2 + a_3 = 0 \text{ or } a_1 = a_2 = a_3 \neq 0.$$

First we will consider the case $a_1 = a_2 = a_3$. From (3.1) and (3.2) we obtain

$$(\alpha_1 - \alpha_3)F(x_1,x_2) + (\alpha_2 - \alpha_1)F(x_2,x_3) + + (\alpha_3 - \alpha_2)F(x_3,x_1) = 0 \quad (3.31)$$

with the notation $f(x_1,x_1,x_2) = F(x_1,x_2)$. If $\alpha_1 = \alpha_2 = \alpha_3$, then the condition (3.31) is satisfied for every function F. For the case $\alpha_1 = \alpha_2 = \alpha_3$ ($\neq 0$), equation (3.1) takes the form

$$a_1 f(x_1,x_2,x_3) - \alpha_1 f(x_1,x_1,x_2) + a_1 f(x_2,x_3,x_1) - - \alpha_1 f(x_2,x_2,x_3) + a_1 f(x_3,x_1,x_2) - \alpha_1 f(x_3,x_3,x_1) = 0 \quad (3.32)$$

This quasicyclic equation has the general solution

$$f(x_1,x_2,x_3) = (\alpha_1/a_1)F(x_1,x_2) + U(x_1,x_2,x_3) - U(x_2,x_3,x_1) \quad (3.33)$$

with the notation $f(x_1,x_1,x_2) = F(x_1,x_2)$.

By substitution of (3.33) into (3.32) we obtain

$$F(x_1,x_2) - (\alpha_1/a_1)F(x_1,x_1) - U(x_1,x_2,x_3) + U(x_1,x_2,x_1) + F(x_2,x_3) - (\alpha_1/a_1)F(x_2,x_2) - U(x_2,x_2,x_3) + U(x_2,x_3,x_2) + F(x_3,x_1) - (\alpha_1/a_1)F(x_3,x_3) - U(x_3,x_3,x_1) + U(x_3,x_1,x_3) = 0$$

This quasicyclic equation has the general solution

$$F(x_1,x_2) = (\alpha_1/a_1)F(x_1,x_1) + U(x_1,x_1,x_2) - U(x_1,x_2,x_1) + + R(x_1) - R(x_2)$$

where R is an arbitrary function $\Re \rightarrow \Re$.

By using the last equality, for $\alpha_1 = a_1$, the equality (3.33) becomes

$$f(x_1,x_1,x_2) = U(x_1,x_2,x_3) - U(x_2,x_3,x_1) + U(x_1,x_1,x_2) - - U(x_1,x_2,x_1) + S(x_1) - R(x_2) \quad (3.34)$$

where S: $\Re \rightarrow \Re$ is such that $F(x_1,x_1) = S(x_1) - R(x_1)$.

For $\alpha_1 \neq a_1$ it follows from (3.1) that $F(x_1,x_1) = 0$. According to the last identity, the equality (3.33) is transformed into

$$f(x_1,x_2,x_3) = U(x_1,x_2,x_3) - U(x_2,x_3,x_1) + + (\alpha_1/a_1)[U(x_1,x_1,x_2) - U(x_1,x_2,x_1)] + R(x_1) - R(x_2)$$

Now we will suppose that the parameters α_i ($1 \leq i \leq 3$) are not all equal. Let $\alpha_1 \neq \alpha_3$. According to the equality (3.31) for $x_3 = a$ (a real constant) we obtain

$$F(x_1,x_2) = K(x_1) + H(x_1) \quad (3.35)$$

where we used the notations

$$K(x_1) = [(\alpha_2 - \alpha_3)/(\alpha_1 - \alpha_3)]F(a,x_1),\ H(x_2) = = [(\alpha_1 - \alpha_2)/(\alpha_1 - \alpha_3)]F(x_2,a)$$

If we substitute $F(x_1,x_2)$ given by the expression (3.35) into (3.31), and if we set $x_1 = u$, $x_2 = x_3 = b$ (a real constant) and if, on the other hand, we set $x_1 = x_3 = b$, $x_2 = u$, we obtain respectively

$$(\alpha_1 - \alpha_3)[K(u) - K(b)] + (\alpha_3 - \alpha_2)[H(u) - H(b)] = 0 \quad (3.36)$$

$$(\alpha_2 - \alpha_1)[K(u) - K(b)] + (\alpha_1 - \alpha_3)[H(u) - H(b)] = 0 \quad (3.37)$$

The determinant of this system is

$$\begin{vmatrix} \alpha_1 - \alpha_3 & \alpha_3 - \alpha_2 \\ \alpha_2 - \alpha_1 & \alpha_1 - \alpha_3 \end{vmatrix} = [(\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_1)^2]/2$$

According to our assumption its value is not 0, then from (3.36) and (3.37) we find $K(u) = K(b)$ and $H(u) = H(b)$, hence

$$F(x_1,x_2) = m \text{ (a real constant)} \quad (3.38)$$

Now the equation (3. 1) becomes

$$f(x_1,x_2,x_3) - n + f(x_2,x_3,x_1) - n + f(x_3,x_1,x_2) - n = 0 \quad (3.39)$$

where

$$n = (\alpha_1 + \alpha_2 + \alpha_3)m/3a_1$$

The general solution of the cyclic functional equation (3.39) is

$$f(x_1,x_2,x_3) = p(x_1,x_2,x_3) - p(x_2,x_3,x_1) + n \quad (3.40)$$

From (3.38) and (3.40) we find

$$m = F(x_1,x_2) = p(x_1,x_1,x_2) - p(x_1,x_2,x_1) + n$$

If we put into the last equality $x_2 = x_1$, then $m = n$. This is possible if

$$\alpha_1 + \alpha_2 + \alpha_3 = 3a_1 \text{ or } n = 0$$

Moreover,

$$p(x_1,x_1,x_2) - p(x_1,x_2,x_1) = 0 \quad (3.41)$$

Now we will use the following result.

Lemma 3. 2. Let $f(x_1,x_2,x_3)$ be a function of the form

$$f(x_1,x_2,x_3) = p(x_1,x_2,x_3) - p(x_2,x_3,x_1)$$

such that $p(x_1,x_1,x_2) = 0$. Then

$$f(x_1,x_2,x_3) = U(x_1,x_2,x_3) - U(x_2,x_1,x_3) - U(x_2,x_3,x_1) - - U(x_1,x_3,x_2) \quad (3.42)$$

where U: $\Re^3 \rightarrow \Re$ is an arbitrary function.

Proof. Let $p(x_1,x_2,x_3)$ satisfies equation (3.41). We are looking for $p(x_1,x_2,x_3)$ in the form

$$p(x_1,x_2,x_3) = k_1 q(x_1,x_2,x_3) + k_2 q(x_1,x_3,x_2) + k_3 q(x_2,x_1,x_3) + + k_4 q(x_2,x_3,x_1) + k_5 q(x_3,x_1,x_2) + k_6 q(x_3,x_2,x_1)$$

where q: $\Re^3 \rightarrow \Re$ and k_i ($1 \leq i \leq 6$) are real constants.

By a substitution into (3.41) we find

$$k_5 = k_1 - k_2 + k_3,\ k_6 = k_4 - k_1 + k_2$$

Thus

$$f(x_1,x_2,x_3) = k[q(x_2,x_1,x_3) - q(x_1,x_2,x_3)] - \ell[q(x_1,x_3,x_2) - - q(x_2,x_3,x_1)] + (\ell - k)[q(x_3,x_2,x_1) - q(x_3,x_1,x_2)]$$

where k, ℓ are real constants such that $k = k_3 - k_2$, $\ell = k_4 - k_1$

If we denote

$$U(x_1,x_2,x_3) = \ell q(x_2,x_3,x_1) + kq(x_3,x_1,x_2)$$

we obtain (3.42). Conversely, each function of the form (2.42) satisfies $f(x_1,x_1,x_2) = 0$ for arbitrary U: $\Re^3 \rightarrow \Re$. Moreover, $f(x_1,x_2,x_3)$ satisfies

$$f(x_1,x_2,x_3) + f(x_2,x_3,x_1) + f(x_3,x_1,x_2) = 0$$

We note that the representation (3.42) can be obtained just by putting

$$p(x_1,x_2,x_3) = U(x_1,x_2,x_3) - U(x_2,x_1,x_3)$$

Thus the general solution of the equation (3.1) is in this case given by

$$f(x_1,x_2,x_3) = U(x_1,x_2,x_3) - U(x_2,x_1,x_3) - U(x_2,x_3,x_1) + U(x_1,x_3,x_2) + n \tag{3.43}$$

where U is an arbitrary function $\Re^3 \to \Re$ and n is an arbitrary real constant, $n = 0$ if $\alpha_1 + \alpha_2 + \alpha_3 \neq 3a_1$.

Now we will consider the case that a_1, a_2, a_3 are not all equal. Thus we have

$$a_1 + a_2 + a_3 = 0 \tag{3.44}$$

Without any loss of generality, we can assume that $a_1 \neq a_2$. Equation (3.1) can be written as

$$a_1[f(x_1,x_2,x_3) - f(x_3,x_1,x_2)] - a_2[f(x_3,x_1,x_2) - f(x_2,x_3,x_1)] = \alpha_1 F(x_1,x_2) + \alpha_2 F(x_2,x_3) + \alpha_3 F(x_3,x_1) \tag{3.45}$$

where $f(x_1,x_1,x_2) = F(x_1,x_2)$. Also from (3.2) and (3.3) it follows that

$$a_1[f(x_2,x_3,x_1) - f(x_1,x_2,x_3)] - a_2[f(x_1,x_2,x_3) - f(x_1,x_3,x_2)] = \alpha_1 F(x_2,x_3) + \alpha_2 F(x_3,x_1) + \alpha_3 F(x_1,x_2) \tag{3.46}$$

$$a_1[f(x_3,x_1,x_2) - f(x_2,x_3,x_1)] - a_2[f(x_2,x_3,x_1) - f(x_1,x_2,x_3)] = \alpha_1 F(x_3,x_1) + \alpha_2 F(x_1,x_2) + \alpha_3 F(x_2,x_3) \tag{3.47}$$

By adding (3.45), (3.46) and (3.47) we obtain

$$(\alpha_1 + \alpha_2 + \alpha_3)[F(x_1,x_2) + F(x_2,x_3) + F(x_3,x_1)] = 0$$

For $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$, the following condition must be satisfied

$$F(x_1,x_2) + F(x_2,x_3) + F(x_3,x_1) = 0$$

This cyclic functional equation has the general solution

$$F(x_1,x_2) = P(x_1) - P(x_2) \tag{3.48}$$

where P is an arbitrary function $\Re \to \Re$.

From the equation (3.45), (3.46) and (3.47), if we take into account (3.48), we get

$$f(x_1,x_2,x_3) + [1/(a_1^3 - a_2^3)]\{a_1^2[\alpha_1 P(x_1) + \alpha_2 P(x_2) + \alpha_3 P(x_3)] + a_2^2[\alpha_3 P(x_1) + \alpha_1 P(x_2) + \alpha_2 P(x_3)] + a_1 a_2[\alpha_2 P(x_1) + \alpha_3 P(x_2) + \alpha_1 P(x_3)]\} = f(x_1,x_2,x_3) + [1/(a_1^3 - a_2^3)]\{a_1^2[\alpha_1 P(x_2) + \alpha_2 P(x_3) + \alpha_3 P(x_1)] + a_2^2[\alpha_3 P(x_2) + \alpha_1 P(x_3) + \alpha_2 P(x_1)] + a_1 a_2[\alpha_2 P(x_2) + \alpha_3 P(x_3) + \alpha_1 P(x_1)]\}$$

The last equation has the general solution

$$f(x_1,x_2,x_3) + [1/(a_1^3 - a_2^3)]\{a_1^2[\alpha_1 P(x_2) + \alpha_2 P(x_3) + \alpha_3 P(x_1)] + a_2^2[\alpha_3 P(x_2) + \alpha_1 P(x_3) + \alpha_2 P(x_1)] + a_1 a_2[\alpha_2 P(x_2) + \alpha_3 P(x_3) + \alpha_1 P(x_1)]\} = p(x_1,x_2,x_3) + p(x_2,x_3,x_1) + p(x_3,x_1,x_2) \tag{3.49}$$

where p is an arbitrary function $\Re^3 \to \Re$.

By virtue of (3.48) $f(x_1,x_2,x_3) = P(x_1) - P(x_2)$, then from (3.49) it follows that

$$P(x_1) - P(x_2) + [1/(a_1^3 - a_2^3)]\{a_1^2[(\alpha_1 + \alpha_3)P(x_1) + \alpha_2 P(x_2)] + a_2^2[(\alpha_2 + \alpha_3)P(x_1) + \alpha_1 P(x_2)] + a_1 a_2[(\alpha_1 + \alpha_2)P(x_1) + \alpha_3 P(x_2)]\} = p(x_1,x_1,x_3) + p(x_1,x_2,x_1) + p(x_2,x_1,x_1) \tag{3.50}$$

For $x_2 = x_1$ this equality takes the form

$$[(\alpha_1 + \alpha_2 + \alpha_3)(a_1^2 + a_2^2 + a_1 a_2)/(a_1^3 - a_2^3)]P(x_1) = 3p(x_1,x_1,x_1)$$

which implies

$$P(x_1) = [3(a_1 - a_2)/(\alpha_1 + \alpha_2 + \alpha_3)]p(x_1,x_1,x_1)$$

Now from (3.49) we find the general solution in the form

$$f(x_1,x_2,x_3) = p(x_1,x_2,x_3) + p(x_2,x_3,x_1) + p(x_3,x_1,x_2) - [3/(a_1^2 + a_2^3 + a_1 a_2)(\alpha_1 + \alpha_2 + \alpha_3)]\{a_1^2[\alpha_1 p(x_2,x_2,x_2) + \alpha_2 p(x_3,x_3,x_3) + \alpha_3 p(x_1,x_1,x_1)] + a_2^2[\alpha_1 p(x_3,x_3,x_3) + \alpha_2 p(x_1,x_1,x_1) + \alpha_3 p(x_2,x_2,x_2)] + a_1 a_2[\alpha_1 p(x_1,x_1,x_1) + \alpha_2 p(x_2,x_2,x_2) + \alpha_3 p(x_3,x_3,x_3)]\} \tag{3.51}$$

where p: $\Re^3 \to \Re$ must satisfy the following condition derived from (3.50)

$$[3(a_1 - a_2)/(\alpha_1 + \alpha_2 + \alpha_3)][p(x_1,x_1,x_1) - p(x_2,x_2,x_2)] + [3/(a_1^2 + a_2^3 + a_1 a_2)(\alpha_1 + \alpha_2 + \alpha_3)]\{a_1^2[(\alpha_1 + \alpha_2)p(x_1,x_1,x_1) + \alpha_2 p(x_2,x_2,x_2)] + a_2^2[(\alpha_2 + \alpha_3)p(x_1,x_1,x_1) + \alpha_1 p(x_2,x_2,x_2)] + a_1 a_2[(\alpha_1 + \alpha_2)p(x_1,x_1,x_1) + \alpha_3 p(x_2,x_2,x_2)]\} = p(x_1,x_1,x_2) + p(x_1,x_2,x_1) + p(x_2,x_1,x_1) \tag{3.52}$$

It is easy to see that (3.52) is an equation of the form

$$p(x_1,x_1,x_2) + p(x_1,x_2,x_1) + p(x_2,x_1,x_1) = (3 - \gamma)p(x_1,x_1,x_1) + \gamma p(x_2,x_2,x_2) \tag{3.53}$$

where the real constant γ is given by

$$\gamma = -3(a_1 - a_2)/(\alpha_1 + \alpha_2 + \alpha_3) + 3(a_1^2\alpha_2 + a_2^2\alpha_1 + a_1 a_2 \alpha_3)/[(a_1^2 + a_2^2 + a_1 a_2)(\alpha_1 + \alpha_2 + \alpha_3)].$$

Lemma 3. 3. Let $f(x_1,x_2,x_3)$ be a function of the form

$$f(x_1,x_2,x_3) = p(x_1,x_2,x_3) + p(x_2,x_3,x_1) + p(x_3,x_1,x_2) \tag{3.54}$$

such that $f(x_1,x_2,x_3) = 0$. Then

$$f(x_1,x_2,x_3) = U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1) \tag{3.55}$$

where U: $\Re^3 \to \Re$ is an arbitrary function.

Proof. We are looking for a function of the form

$$p(x_1,x_2,x_3) = k_1 q(x_1,x_2,x_3) + k_2 q(x_2,x_3,x_1) + k_3 q(x_3,x_1,x_3) + k_4 q(x_2,x_1,x_3) + k_5 q(x_1,x_3,x_2) + k_6 q(x_3,x_2,x_1)$$

where k_i ($1 \leq i \leq 6$) are real constants, satisfying

$$p(x_1,x_1,x_2) + p(x_1,x_2,x_1) + p(x_2,x_1,x_1) = 0 \tag{3.56}$$

for any function q: $\Re^3 \to \Re$. By a substitution into (3.56) we find

$$k_1 + k_2 + k_3 = k_4 + k_5 + k_6$$

Thus

$$f(x_1,x_2,x_3) = (k_1 + k_2 + k_3)[q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2)] - (k_1 + k_2 + k_3)[q(x_2,x_1,x_3) + q(x_1,x_3,x_2) + q(x_3,x_2,x_1)]$$

If we put

$$U(x_1,x_2,x_3) = (k_1 + k_2 + k_3)\, q(x_1,x_2,x_3)$$

we obtain the representation (3.55).

We note that the representation (3.55) can be obtained from (3.54) by putting

$$p(x_1,x_2,x_3) = U(x_1,x_2,x_3) - U(x_2,x_1,x_3)$$

Let us suppose that $p(x_1,x_2,x_3) = S(x_1) \neq 0$. The equation

$$p(x_1,x_1,x_2) + p(x_1,x_2,x_1) + p(x_2,x_1,x_1) = (3-\gamma)S(x_1) + \gamma S(x_2)$$

has a no constant solution of the form

$$p(x_1,x_1,x_2) = S(x_1)$$

or, more generally,

$$p(x_1,x_2,x_3) = m_1 S(x_1) + m_2 S(x_2) + (1 - m_1 - m_2)S(x_3)$$

only if $\gamma = 1$. Indeed, we have

$$(\gamma - 1)[S(x_1) - S(x_2)] = 0$$

On the other hand, any $S(x_1) \equiv a$, where a is a real constant, satisfies the last equality.

Let us put

$$p(x_1,x_2,x_3) = \tilde{U}(x_1,x_2,x_3) + S(x_1)$$

Then $\tilde{U}(x_1,x_2,x_3)$ satisfies an equation of the form (3.56) and we have proved this result.

Corollary 3. 4. Let $f(x_1,x_2,x_3)$ be a function of form (3.54) such that $p(x_1,x_2,x_3)$ satisfies (3.53). Then

$$f(x_1,x_2,x_3) = U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_1,x_2) + S(x_1) + S(x_2) + S(x_3)$$

where U: $\Re^3 \to \Re$ is an arbitrary function and S is an arbitrary function $\Re \to \Re$ for $\gamma = 1$, S is equal to a constant $a \in \Re$ otherwise.

Thus from (3.51) we find that $f(x_1,x_2,x_3)$ is given by (3.55) if $\gamma \neq 1$

$$f(x_1,x_2,x_3) = S(x_1) + S(x_2) + S(x_3) - [3/(a_1^2 + a_2^2 + a_1a_2)(\alpha_1 + \alpha_2 + \alpha_3)]\{a_1^2[\alpha_3 S(x_1) + \alpha_1 S(x_2) + \alpha_2 S(x_3)] + a_2^2[\alpha_2 S(x_1) + \alpha_3 S(x_2) + \alpha_1 S(x_3)] + a_1a_2[\alpha_1 S(x_1) + \alpha_2 S(x_2) + \alpha_3 S(x_3)]\} + U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1).$$

Now we pass on the case $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Then from (3.45), (3.46) and (3.47) we obtain

$$f(x_3,x_1,x_2) - [1/(a_1^3 - a_2^3)]\{a_1^2[\alpha_1 F(x_3,x_1) - \alpha_2 F(x_2,x_3)] + a_2^2[\alpha_1 F(x_1,x_2) - \alpha_2 F(x_3,x_1)] + a_1a_2[\alpha_1 F(x_2,x_3) - \alpha_2 F(x_1,x_2)]\} = f(x_1,x_2,x_3) - [1/(a_1^3 - a_2^3)]\{a_1^2 [\alpha_1 F(x_1,x_2) - \alpha_2 F(x_3,x_1)] + a_2^2[\alpha_1 F(x_2,x_3) - \alpha_2 F(x_1,x_2)] + a_1a_2[\alpha_1 F(x_3,x_1) - \alpha_2 F(x_2,x_3)]\} \quad (3.57)$$

The general solution of equation (3.57) is given as

$$f(x_1,x_2,x_3) = [1/(a_1^3 - a_2^3)]\{a_1^2[\alpha_1 F(x_1,x_2) - \alpha_2 F(x_3,x_1)] + a_2^2[\alpha_1 F(x_2,x_3) - \alpha_2 F(x_1,x_2)] + a_1a_2[\alpha_1 F(x_3,x_1) - \alpha_2 F(x_2,x_3)]\} + q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2) \quad (3.58)$$

where q: $\Re^3 \to \Re$ is an arbitrary function.

From the equality (3.58) we obtain

$$F(x_1,x_2) = [1/(a_1^3 - a_2^3)]\{a_1^2[\alpha_1 F(x_1,x_1) - \alpha_2 F(x_2,x_1)] - + a_2^2[\alpha_1 F(x_1,x_2) - \alpha_2 F(x_2,x_1)] + a_1a_2[\alpha_1 F(x_2,x_1) - \alpha_2 F(x_1,x_2)]\} + q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) \quad (3.59)$$

For $x_2 = x_1$ (3.59) yields

$$F(x_1,x_2) = [(\alpha_1 - \alpha_2)/(a_1 - a_2)]F(x_1,x_1) + 3q(x_1,x_1,x_1) \quad (3.60)$$

If $\alpha_1 - \alpha_2 = a_1 - a_2$, then $q(x_1,x_1,x_1) = 0$ and $F(x_1,x_1) = P(x_1)$, where P is an arbitrary function $\Re \to \Re$. Now we have

$$[1 + a_2(\alpha_1 - a_1)/(a_1^2 + a_2^2 + a_1a_2)]F(x_1,x_2) + [a_1(\alpha_1 - a_1)/(a_1^2 + a_2^2 + a_1a_2)]F(x_2,x_1) = [\alpha_1(a_1 + a_2) + a_2^2]P(x_1)/(a_1^2 + a_2^2 + a_1a_2) + q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) \quad (3.61)$$

First we consider the particular case, $\alpha_1 = a_1$ (then $\alpha_2 = a_2$, $\alpha_3 = a_3 = -(a_1 + a_2)$)

Now

$$F(x_1,x_2) = P(x_1) + q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1)$$

Thus, we find that the functional equation

$$a_1 f(x_1,x_2,x_3) + a_2 f(x_2,x_3,x_1) - (a_1 + a_2)f(x_3,x_1,x_2) = a_1 f(x_1,x_1,x_2) + a_2 f(x_2,x_2,x_3) - (a_1 + a_2)f(x_3,x_3,x_1)$$

has the general solution

$$f(x_1,x_2,x_3) = P(x_1) + q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) + q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2)$$

where P: $\Re \to \Re$ is an arbitrary function and q: $\Re^3 \to \Re$ is an arbitrary function satisfying $q(x_1,x_1,x_1) \equiv 0$.

Now we consider equation (3.61) in the general case. With a permutation of the variables x_1 and x_2 we derive the equation

$$[a_1(\alpha_1 - a_1)/(a_1^2 + a_2^2 + a_1a_2)]F(x_1,x_2) + [1 + a_2(\alpha_1 - a_1)/(a_1^2 + a_2^2 + a_1a_2)]F(x_2,x_1) = [\alpha_1(a_1 + a_2) + a_2^2]P(x_2)/(a_1^2 + a_2^2 + a_1a_2) + q(x_2,x_2,x_1) + q(x_2,x_1,x_2) + q(x_1,x_2,x_2) \quad (3.62)$$

The determinant of the system (3.61), (3.62) is

$$(a_2^2 + a_1\alpha_1 + a_2\alpha_1)(2a_1^2 + a_2^2 - a_1\alpha_1 + a_2\alpha_1)/(a_1^2 + a_2^2 + a_1a_2)^2 \quad (3.63)$$

If this expression is not 0, then the solution of this system is

$$F(x_1,x_2) = [(a_1^2 + a_2^2 + a_2\alpha_1)P(x_1) - a_1(\alpha_1 - a_1)P(x_2)]/(2a_1^2 + a_2^2 - a_1\alpha_1 + a_2\alpha_1) + [(a_1^2 + a_2^2 + a_1a_2)/(a_2^2 + a_1\alpha_1 + a_2\alpha_1)(2a_1^2 + a_2^2 - a_1\alpha_1 + a_2\alpha_1)] \times \{(a_1^2 + a_2^2 + a_2\alpha_1)[q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1)] - a_1(\alpha_1 - a_1)[q(x_2,x_2,x_1) + q(x_2,x_1,x_2) + q(x_1,x_2,x_2)]\}$$

Now let us suppose that the expression (3.63) is 0. First let

$$a_2^2 + \alpha_1(a_1 + a_2) = 0$$

If $a_1 + a_2 = 0$, then $a_1 = a_2 = a_3 = 0$, which is contradiction. Thus

$$\alpha_1 = -a_2^2/(a_1 + a_2),\ \alpha_2 = -a_1^2/(a_1 + a_2)$$

Now (3.58) takes the form

$$f(x_1,x_2,x_3) = [a_1F(x_3,x_1) + a_2F(x_2,x_3)]/(a_1 + a_2) + + q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2)$$

while (3.61) becomes

$$[a_1/(a_1 + a_2)][F(x_1,x_2) - F(x_2,x_1)] = = q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) \quad (3.64)$$

Equation (3.64) implies

$$q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) = 0$$

and

$$F(x_1,x_2) - F(x_2,x_1) = 0$$

i.e.,

$$q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2) = = U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

where U: $\Re^3 \to \Re$ is an arbitrary function, and

$$F(x_1,x_2) = G(x_1,x_2) + G(x_2,x_1)$$

where G: $\Re^2 \to \Re$ is an arbitrary function.

Thus the general solution of the functional equation

$$a_1f(x_1,x_2,x_3) + a_2f(x_2,x_3,x_1) - (a_1 + a_2)f(x_3,x_1,x_2) = = - [a_2^2/(a_1 + a_2)]f(x_1,x_1,x_2) - [a_1^2/(a_1 + a_2)]f(x_2,x_2,x_3) - - [(a_1^2 + a_2^2)/(a_1 + a_2)]f(x_3,x_3,x_1)$$

is given by the relation

$$f(x_1,x_2,x_3) = \{a_1[G(x_1,x_3) + G(x_3,x_1) + a_2[G(x_2,x_3) + + G(x_3,x_2)]\}/(a_1 + a_2) + U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

Next we suppose that

$$2a_1^2 + a_2^2 - (a_1 - a_2)\alpha_1 = 0$$

Since $a_1 \neq a_2$, we have

$$\alpha_1 = (2a_1^2 + a_2^2)/(a_1 - a_2), \alpha_2 = (a_1^2 + 2a_1a_2)/(a_1 - a_2)$$

Now (3.58) takes the form

$$f(x_1,x_2,x_3) = [2a_1F(x_1,x_2) - a_1F(x_3,x_{11}) - a_2F(x_2,x_3)]/ /(a_1 - a_2) + q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2)$$

while (3.61) becomes

$$[a_1/(a_1 - a_2)][F(x_1,x_2) + F(x_2,x_1) - 2F(x_1,x_1)] = = q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) \quad (3.65)$$

Equation (3.65) implies

$$q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) = 0$$

and

$$F(x_1,x_2) + F(x_2,x_1) - 2F(x_1,x_1) = 0$$

i.e.,

$$q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) = U(x_1,x_2,x_3) + + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

where U: $\Re^3 \to \Re$ is an arbitrary function, and

$$F(x_1,x_2) = G(x_1,x_2) - G(x_2,x_1) + c$$

where G: $\Re^2 \to \Re$ is an arbitrary function and $c \in \Re$ is an arbitrary constant.

Thus, the general solution of the functional equation

$$a_1f(x_1,x_2,x_3) + a_2f(x_2,x_3,x_1) - (a_1 + a_2)f(x_3,x_1,x_2) = = [(2a_1^2 + a_2^2)/(a_1 - a_2)]\, f(x_1,x_1,x_2) + + [(a_1^2 + 2a_1a_2)/(a_1 - a_2)]\, f(x_2,x_2,x_3) - - [(3a_1^2 + 2a_1a_2 - a_2^2)/(a_1 - a_2)]\, f(x_3,x_3,x_1)$$

is given by the relation

$$f(x_1,x_2,x_3) = [1/(a_1 - a_2)]\{2a_1[G(x_1,x_2) - G(x_2,x_1)] + + a_1[G(x_1,x_3) - G(x_3,x_1)] - a_2[G(x_2,x_3) - G(x_3,x_2)]\} + c + + U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

Now let $[(\alpha_1 - \alpha_2)/(a_1 - a_2)] = \gamma \neq 1$. Then from (2.60) we find

$$F(x_1,x_1) = [3/(1 - \gamma)]\, q(x_1,x_1,x_1) \quad (3.66)$$

Now we have

$$[1 + a_2(\alpha_1 - a_1\gamma)/(a_1^2 + a_2^2 + a_1a_2)]F(x_1,x_2) + + [a_1(\alpha_1 - a_1\gamma)/(a_1^2 + a_2^2 + a_1a_2)]F(x_2,x_1) = = 3[\alpha_1(a_1 + a_2) + a_2^2\gamma]q(x_1,x_1,x_1)/(a_1^2 + a_2^2 + a_1a_2) \times \times (1 - \gamma) + q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) \quad (3.67)$$

By a permutation of the variables x_1 and x_2 we derive the equation

$$[a_1(\alpha_1 - a_1\gamma)/(a_1^2 + a_2^2 + a_1a_2)]F(x_1,x_2) + + [1 + a_2(\alpha_1 - a_1\gamma)/(a_1^2 + a_2^2 + a_1a_2)]F(x_2,x_1) = = 3[\alpha_1(a_1 + a_2) + a_2^2\gamma]q(x_2,x_2,x_2)/(a_1^2 + a_2^2 + a_1a_2) \times \times (1 - \gamma) + q(x_2,x_2,x_1) + q(x_2,x_1,x_2) + q(x_1,x_2,x_2) \quad (3.68)$$

The determinant of the system (3.67) and (3.68) is

$$[(a_1^2 + a_1a_2)(1 - \gamma) + a_2^2 + (a_1 + a_2)\alpha_1] \times \times[a_1^2(1 + \gamma) + a_1a_2(1 - \gamma) + a_2^2 - (a_1 - a_2)\alpha_1]/ /(a_1^2 + a_2^2 + a_1a_2)^2 \quad (3.69)$$

If the above expression is not 0, then

$$F(x_1,x_2) = 3[\alpha_1(a_1 + a_2) + a_2^2\gamma]/[(a_1^2 + a_1a_2)(1 - \gamma) + + a_2^2 + (a_1 + a_2)\alpha_1]\ \{[a_1^2 + a_2^2 + a_1a_2(1 - \gamma) + a_2\alpha_1] \times \times q(x_1,x_1,x_1) - a_1(\alpha_1 - a_1\gamma)q(x_2,x_2,x_2)\}/ /[a_1^2(1 + \gamma) + a_1a_2(1 - \gamma) + a_2^2 - (a_1 - a_2)\alpha_1] + + (a_1^2 + a_2^2 + a_1a_2)/[(a_1^2 + a_1a_2)(1 - \gamma) + a_2^2 + + (a_1 + a_2)\alpha_1]\ \{[a_1^2 + a_2^2 + a_1a_2(1 - \gamma) + a_2\alpha_1] \times \times [q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1)] - - a_1(\alpha_1 - a_1\gamma)[q(x_2,x_2,x_1) + q(x_2,x_1,x_2) + q(x_1,x_2,x_2)]\}/ /[a_1^2(1 + \gamma) + a_1a_2(1 - \gamma) + a_2^2 - (a_1 - a_2)\alpha_1]$$

Now let us suppose that the expression (3.69) is 0. First let

$$(a_1^2 + a_1a_2)(1 - \gamma) + a_2^2 + (a_1 + a_2)\alpha_1 = 0$$

Then

$$\alpha_1 = - [a_2^2 + (a_1^2 + a_1a_2)(1 - \gamma)]/(a_1 + a_2), \alpha_2 = = - [a_1^2 + (a_2^2 + a_1a_2)(1 - \gamma)]/(a_1 + a_2)$$

Now (3.58) takes the form

$$f(x_1,x_2,x_3) = (1 - \gamma)F(x_1,x_2) + [a_1F(x_3,x_1) + a_2F(x_2,x_3)]/ /(a_1 + a_2) + q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2),$$

while (3.67) becomes

$$a_1[F(x_1,x_2) - F(x_2,x_1)]/(a_1 + a_2) = q(x_1,x_1,x_2) + + q(x_1,x_2,x_1) + q(x_2,x_1,x_1) - 3q(x_1,x_1,x_1)$$

The last equation implies

$$q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2) = U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1) + P(x_1) + P(x_2) + P(x_3)$$

$$F(x_1,x_2) = G(x_1,x_2) + G(x_2,x_1) - (a_1 + a_2)P(x_1)/a_1$$

where U: $\Re^3 \to \Re$, G: $\Re^2 \to \Re$ and P: $\Re \to \Re$ are arbitrary functions. The condition (3.66) yields

$$2G(x_1,x_1) = [(a_1 + a_2)/a_1 + 3/(1 - \gamma)]P(x_1)$$

Next we suppose that

$$a_1^2(1 + \gamma) + a_1a_2(1 - \gamma) + a_2^2 - (a_1 - a_2)\alpha_1 = 0$$

In this case we have

$$\alpha_1 = [a_1^2 + a_1^2(1 + \gamma) + a_1a_2(1 - \gamma)]/(a_1 - a_2)$$
$$\alpha_2 = [a_1^2 + a_2^2(1 - \gamma) + a_1a_2(1 + \gamma)]/(a_1 - a_2)$$

Now (3.58) takes the form

$$f(x_1,x_2,x_3) = \{[a_1(1 + \gamma) + a_2(1 - \gamma)]F(x_1,x_2) - a_1F(x_3,x_1) - a_2F(x_2,x_3)\}/(a_1 - a_2) + q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2),$$

while (3.67) becomes

$$[a_1/(a_1 - a_2)][F(x_1,x_2) + F(x_2,x_1)] = [3/(a_1 - a_2)][(1 + \gamma)a_1/(1 - \gamma) + a_2]q(x_1,x_1,x_1) + q(x_1,x_1,x_2) + q(x_1,x_2,x_1) + q(x_2,x_1,x_1)$$

If

$$[3/(a_1 - a_2)][(1 + \gamma)a_1/(1 - \gamma) + a_2] \neq -1$$

i.e.,

$$(2 + \gamma)a_1 + (1 - \gamma)a_2 \neq 0,$$

as above we find that

$$q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2) = U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

$$F(x_1,x_2) = G(x_1,x_2) - G(x_2,x_1)$$

and

$$f(x_1,x_2,x_3) = \{[a_1(1 + \gamma) + a_2(1 - \gamma)][G(x_1,x_2) - G(x_2,x_1)] + a_1[G(x_1,x_3) - G(x_3,x_1)] - a_2[G(x_2,x_3) - G(x_3,x_1)]\}/(a_1 - a_2) + U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

where G: $\Re^2 \to \Re$ and U: $\Re^3 \to \Re$ are arbitrary functions.

If, however

$$(2 + \gamma)a_1 + (1 - \gamma)a_2 = 0$$

then

$$q(x_1,x_2,x_3) + q(x_2,x_3,x_1) + q(x_3,x_1,x_2) = U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1) + a_1[P(x_1) + P(x_2) + P(x_3)]$$

$$F(x_1,x_2) = G(x_1,x_2) - G(x_2,x_1) + (a_1 - a_2)P(x_1)$$

where P: $\Re \to \Re$ and U: $\Re^3 \to \Re$ are arbitrary functions, and

$$f(x_1,x_2,x_3) = \{a_1[- G(x_1,x_2) + G(x_2,x_1) + G(x_1,x_3) - G(x_3,x_1)] - a_2[G(x_2,x_3) - G(x_3,x_2)]\}/(a_1 - a_2) + (a_1 - a_2)P(x_2) + U(x_1,x_2,x_3) + U(x_2,x_3,x_1) + U(x_3,x_1,x_2) - U(x_2,x_1,x_3) - U(x_1,x_3,x_2) - U(x_3,x_2,x_1)$$

Example 3. 5. Now we will assume as a meniscus the relation (3.6). Let assume further as arbitrary its general solution (3.7) is the function

$$F(x_1,x_2,x_3) \equiv (x_1/\zeta_1)^{2/3} + (x_2/\zeta_2)^{2/3} + (x_3/\zeta_3)^{2/3} = 1$$

where ζ_i ($1 \leq i \leq 3$) are real constants, then the shape of the meniscus will be given by the expression

$$f(x_1,x_2,x_3) = (\zeta_1^{-2/3} - \zeta_3^{-2/3})x_1^{2/3} + (\zeta_2^{-2/3} - \zeta_1^{-2/3})x_2^{2/3} + (\zeta_3^{-2/3} - \zeta_2^{-2/3})x_3^{2/3}$$

This shows that the shape of the meniscus changes cyclically during the mould cycle.

Remark 3. 6. Also, the above results hold for the vector extension of equation (3.1) of the form

$$a_1f(\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3) + a_2f(\boldsymbol{X}_2,\boldsymbol{X}_3,\boldsymbol{X}_1) + a_3f(\boldsymbol{X}_3,\boldsymbol{X}_1,\boldsymbol{X}_2) = \alpha_1f(\boldsymbol{X}_1,\boldsymbol{X}_1,\boldsymbol{X}_2) + \alpha_2f(\boldsymbol{X}_2,\boldsymbol{X}_2,\boldsymbol{X}_3) + \alpha_3f(\boldsymbol{X}_3,\boldsymbol{X}_3,\boldsymbol{X}_1)$$

f: $\Re^3 \to \Re$, where $\boldsymbol{X}_i = (x_{1i},x_{2i},x_{3i})^{\mathrm{T}}$ are real vectors and a_i, α_i ($1 \leq i \leq 3$) are real constants.

4 GENERALIZED RESULTS

As a natural consequence of the previous considered meniscus equation, we will give the following more general result.

Theorem 4. 1. The generalized meniscus equation

$$\mathrm{E}(f) \equiv \sum_{i=1}^{n} a_i f(x_i,x_{i+1},\ldots,x_{i+n-1}) = \tag{4.1}$$
$$= \sum_{i=1}^{n} a_i f(x_i,x_i,x_{i+1},\ldots,x_{i+n-2}) \quad (x_{n+i} \equiv x_i,\ n > 1)$$

where a_i, α_i ($1 \leq i \leq n$) are real constants, has a solution if the right-hand side of (4.1) satisfies

$$(AC + I)\Lambda[g(x_1,x_2,\ldots,x_{n-1}), g(x_2,x_3,\ldots,x_n),\ldots,g(x_n,x_1,\ldots,x_{n-2})]^{\mathrm{T}} = \mathbf{O} \tag{4.2}$$

where $A = \mathrm{cycl}(a_1,a_2,\ldots,a_n)$, $\Lambda = \mathrm{cycl}(\alpha_1,\alpha_2,\ldots,\alpha_n)$, $g(x_1,x_2,\ldots,x_{n-1}) = f(x_1,x_1,x_2,\ldots,x_{n-1})$, C is any non-zero $n\times n$ cyclic matrix with constant real entries satisfying $ACA + A = O$, O is the $n\times n$ zero matrix and I is the $n\times n$ unit matrix.

If the equality (4.2) holds for some C, then the general solution of equation (4.1) is given by the following formula

$$[f(x_1,x_2,\ldots,x_n),f(x_2,x_3,\ldots,x_n,x_1),\ldots,f(x_n,x_1,\ldots,x_{n-1})]^{\mathrm{T}} = \mathrm{B}[h(x_1,x_2,\ldots,x_n),h(x_2,x_3,\ldots,x_n,x_1),\ldots,h(x_n,x_1,\ldots,x_{n-1})]^{\mathrm{T}} - \Lambda[g(x_1,x_2,\ldots,x_{n-1}),g(x_2,x_3,\ldots,x_n),\ldots,g(x_n,x_1,\ldots,x_{n-2})]^{\mathrm{T}} \tag{4.3}$$

where the non-zero $n\times n$ cyclic matrix B given by

$$B = \mathrm{cycl}(b_1,b_2,\ldots,b_n)$$

satisfies the condition

$$AB = \mathrm{O}$$

and h is an arbitrary real function $\Re^n \to \Re$.

Proof. By a cyclic permutation of the variables in (4.1) we get

$$a_1 f(x_1,x_2,\ldots,x_n) + a_2 f(x_2,x_3,\ldots,x_n,x_1) + \cdots + a_n f(x_n,x_1,\ldots,x_{n-1}) = \alpha_1 g(x_1,x_2,\ldots,x_{n-1}) + \alpha_2 g(x_2,x_3,\ldots,x_n) + \cdots + \alpha_n g(x_n,x_1,\ldots,x_{n-2})$$

$$a_n f(x_1,x_2,\ldots,x_n) + a_1 f(x_2,x_3,\ldots,x_n,x_1) + \cdots + a_{n-1} f(x_n,x_1,\ldots,x_{n-1}) = \alpha_n g(x_1,x_2,\ldots,x_{n-1}) + \alpha_1 g(x_2,x_3,\ldots,x_n) + \cdots + \alpha_{n-1} g(x_n,x_1,\ldots,x_{n-2})$$

$$\vdots$$

$$a_2 f(x_1,x_2,\ldots,x_n) + a_3 f(x_2,x_3,\ldots,x_n,x_1) + \cdots + a_1 f(x_n,x_1,\ldots,x_{n-1}) = \alpha_2 g(x_1,x_2,\ldots,x_{n-1}) + \alpha_3 g(x_2,x_3,\ldots,x_n) + \cdots + \alpha_1 g(x_n,x_1,\ldots,x_{n-2})$$

i.e., in matrix form

$$AF = \Lambda G \tag{4.4}$$

where

$$F = [f(x_1,x_2,\ldots,x_n), f(x_2,x_3,\ldots,x_n,x_1),\ldots,f(x_n,x_1,\ldots,x_{n-1})]^T$$

and

$$G = [g(x_1,x_2,\ldots,x_{n-1}), g(x_2,x_3,\ldots,x_n),\ldots,g(x_n,x_1,\ldots,x_{n-2})]^T$$

We suppose that equation (4.4) has a solution F and C satisfies $ACA + A = O$. Then

$$(AC + I)\Lambda G = (AC + I)AF = (ACA + A)F = \boldsymbol{O}$$

i.e., equation (4.2) must be satisfied. Conversely, let equation (4.2) hold for a cyclic matrix C. Then $-C\Lambda G$ is easily seen to be a solution of equation (4.4):

$$A(-C\Lambda G) = -(AC + I)\Lambda G + I\Lambda G = I\Lambda G = \Lambda G$$

Now let us prove that equality (4.3) gives the general solution of equation (4.1).

Let f be a solution of equation (4.1), which we will write in the form

$$E(f) = L(g) \tag{4.5}$$

We denote by f_h the general solution of the equation $E(f) = 0$, and by f_p we denote a particular solution of equation (4.5).

Then $f = f_h + f_p$ is the general solution of equation (4.5). Indeed

$$E(f_h + f_p) = E(f_h) + E(f_p) = E(g)$$

On the other hand, let f be an arbitrary solution of equation (4.5). Then

$$E(f - f_p) = E(f) - E(f_p) = L(g) - L(g) = 0$$

i.e., $f - f_p$ is a solution of the associated homogeneous equation. So there exists a specialization f_h^* of the expression f_h such that

$$f - f_p = f_h^*,\ i.e.,\ f = f_h^* + f_p$$

Thus $f_h + f_p$ includes all the solutions of equation (4.5).

The general solution of the homogeneous equation $E(f) = 0$ given in matrix form is BH, where

$$H = [h(x_1,x_2,\ldots,x_n), h(x_2,x_3,\ldots,x_n,x_1),\ldots,h(x_n,x_1,\ldots,x_{n-1})]^T$$

and a particular solution of the equation $E(f) = L(g)$ in matrix form is $-C\Lambda G$, then $F = BH - C\Lambda G$ includes all the solutions of the nonhomogeneous equation.

On the other hand, every function of the form (4.3) satisfies the functional equation (4.1).

Remark 4. 2. The same results hold for the vector extension of equation (4.1) of the form

$$E(f) \equiv \sum_{i=1}^{n} a_i f(\boldsymbol{X}_i,\boldsymbol{X}_{i+1},\ldots,\boldsymbol{X}_{i+n-1}) = \sum_{i=1}^{n} a_i f(\boldsymbol{X}_i,\boldsymbol{X}_i,\boldsymbol{X}_{i+1},\ldots,\boldsymbol{X}_{i+n-2})\ (\boldsymbol{X}_{n+i} \equiv \boldsymbol{X}_i,\ n > 1)$$

f: $\Re^n \to \Re$, where $\boldsymbol{X}_i = (x_{1i},x_{2i},\ldots,x_{ni})^T$ are real vectors and a_i, α_i ($1 \le i \le n$) are real constants.

5 MENISCUS STABILITY

The meniscus stability was considered for the first time in [15,16], but only according to definition 2.2 for the solution of the Navier-Stokes equation, including the pressure gradient. Here, we will give a completely new matrix approach to the solution of the meniscus stability problem.

Now we will derive a necessary and sufficient condition for the stability of the meniscus given by the quasicyclic functional equation (4.1), *i.e.,* its matrix form (4.4) using a simple spectral property of compound matrices.

Let $\det A \neq 0$, then relation (4.4) takes the form

$$F = A^{-1}\Lambda G \equiv SG \tag{5.1}$$

where S is also a cyclic matrix.

Definition 5. 1. The quasicyclic functional equation (5.1) is stable if stab(S) < 0.

Proposition 5. 2. For any cyclic matrix $S \in \Re$ it holds

$$\text{stab}(S) = \inf\{\mu(S),\ \mu \text{ is a Lozinskiĭ measure on } \Re^n\} \tag{5.2}$$

Proof. The relation (5.2) obviously holds for diagonizable matrices in view of

$$\mu_T(S) = \mu(TST^{-1})\ (T \text{ is an invertible matrix}) \tag{5.3}$$

and the first two relations in (2.4). Furthermore, the infimum in (5.2) can be achieved if S is diagonizable. The general case can be shown based on this observation, the fact that S can be approximated by diagonizable matrices in $\Re$ and the continuity of $\mu(\cdot)$, which is implied by the property

$$|\mu(A) - \mu(B)| \le |A - B|$$

Remark 5. 3. From the above proof it follows that

$$\text{stab}(S) = \inf\{\mu_\infty(TST^{-1}),\ T \text{ is invertible}\}.$$

The same relation holds if μ_∞ is replaced by μ_1.

Corollary 5. 4. Let $S \in \Re$. Then $\text{stab}(S) < 0 \Leftrightarrow \mu(S) < 0$ for some Lozinskiĭ measure μ on $\Re^n$.

Theorem 5. 5. For stab(S) < 0 it is sufficient and necessary that stab($S^{[2]}$) < 0 and $(-1)^n \det(S) > 0$.

Proof. Using the spectral property of $S^{[2]}$, the condition stab($S^{[2]}$) < 0 implies that at least one eigenvalue of S can be nonnegative. We may thus suppose that all eigenvalues are real. It is then simple to see that the existence of one and only one non-negative eigenvalue is precluded by the condition $(-1)^n\det(S) > 0$.

Theorem 5.5 and Corollary 5.4 lead to the following result.

Theorem 5. 6. Suppose that $(-1)^n\det(S) > 0$. Then S is stable if and only if $\mu(S^{[2]}) < 0$ for some Lozinskiĭ measure μ on $\Re^N$, $N = n!/2!(n-2)!$.

Theorem 5. 7. If stab($S^{[2]}(\beta)$) < 0 for $\beta \in (a, b)$, then (a, b) contains no Hopf bifurcation points of $S(\beta)$.

Proof. Let $\beta \to S(\beta) \in \Re$ be a function that is continuous for $\beta \in (a, b)$. A point $\beta_0 \in (a, b)$ is said to be a Hopf bifurcation point for $S(\beta)$ if $S(\beta)$ is stable for $\beta < \beta_0$, and there exists an eigenvalue $\lambda(\beta)$ of $S(\beta)$ such that $\lambda(\beta) > 0$, while the rest of the eigenvalues of $S(\beta)$ are nonzero for $\beta > \beta_0$. From the proof of Theorem 5.5 we see that stab($S^{[2]}$) ≤ 0 precludes the existence of a non-negative eigenvalue of S.

Let S and P be $n \times n$ real cyclic matrices. A subspace $\Omega \in \Re$ is invariant under S if $S(\Omega) \subset \Omega$. S is said to be stable with respect to an invariant subspace Ω if the restriction of S to Ω, $S|_\Omega$: $\Omega \to \Omega$ is stable. Let the matrix P be such that rank $P = r$ $(1 < r < n)$ and

$$PS = O \qquad (5.4)$$

Then Ker$P = \{x \in \Re, Px = 0\}$ satisfies $S(\Re) \subset$ Ker$P$. In particular, Ker$P$ is an $(n-r)$-dimensional invariant space of S. It is of interest to study the stability of S with respect to Ker$P$ when (5.4) holds.

Lemma 5. 8. Let $\Omega \subset \Re$ be a subspace such that $S(\Re) \subset \Omega$ and $\dim\Omega = k < n$. Then 0 is an eigenvalue of S, and there exist $n - k$ null eigenvectors that do not belong to Ω.

Proof. Let $\Im$ be the quotient space $\Re/\Omega$. Then $\Re \cong \Omega \oplus \Im$ and $S(\Im) = \{0\}$ since $S(\Re) \subset \Omega$. This establishes the lemma.

Theorem 5. 9. Suppose that P and S satisfy (5.4) and rank$P = r$ $(1 < r < n)$. Then for S to be stable with respect to Ker$P$, it is necessary and sufficient that

1° $\quad$ stab($S^{[r+2]}$) < 0

and

2° $\quad \limsup_{\varepsilon \to 0+}$ sign $[\det(\varepsilon I + S)] = (-1)^{n-r}$

Proof. Let λ_i $(1 \leq i \leq n - r)$ be eigenvalues of $S|_{\mathrm{Ker}P}$. By Lemma 5.8, the eigenvalues of S can be written as

$$\lambda_1, \lambda_2, \ldots, \lambda_{n-r}, 0, \ldots, 0, (r \text{ zeros})$$

and thus $\{\lambda_i + \lambda_j, 1 \leq i < j \leq n - r\} \subset \sigma(S^{[r+2]})$ by the spectral property of additive compound matrices discussed in Section 2. It follows that stab($S^{[r+2]}$) < 0 precludes the possibility of more than one non-negative λ_i $(1 \leq i \leq n - r)$. For $\varepsilon > 0$ sufficiently small

$$\text{sign}[\det(\varepsilon I + S)] = \text{sign}(\varepsilon^r \lambda_1 \cdots \lambda_{n-r})$$

The theorem can be proved using the same arguments as in the proof of Theorem 5.5.

Remark 5. 10. If $r = n$ in (5.4), then P is of full rank and hence $S = O$. If $r = n - 1$, then Ker$P$ is of dimension 1 and thus the eigenvalues of S are λ_1 and 0 of multiplicity $n - 1$. From the above proof we know that Theorem 5.9 still holds in this case, if condition 1° is replaced by tr(S) < 0.

Corollary 5. 11. Suppose that S and P_1 satisfy

$$P_1 S = \beta P_1 \qquad (5.5)$$

and rank$P_1 = r$ $(1 < r < n)$. Thus S is stable with respect to KerP_1 if and only if the following conditions hold:

1° $\quad$ stab($S^{[r+2]}$) < $(r + 2)\beta$

and

2° $\quad (\text{sign } \beta)^r(-1)^{n-r}\det(S) > 0$

Proof. Let the matrix P_1 be such that rank$P_1 = r$ $(1 < r < n)$ and (5.5) holds for some scalar $\beta \neq 0$. Then KerP_1 is an invariant subspace of S. Noting that (5.5) is equivalent to $P_1(S - \beta I) = O$, one can apply Theorem 5.9 to $S - \beta I$ and obtain the proof.

6 DISCUSSION

The previous results may be extended in two different ways.

1° A first possible generalization of equation (3.1) is the following functional equation

$$a_1 f_1(x_1,x_2,x_3) + a_2 f_2(x_2,x_3,x_1) + a_3 f_3(x_3,x_1,x_2) = \alpha_1 f_1(x_1,x_1,x_2) + \alpha_2 f_2(x_2,x_2,x_3) + \alpha_3 f_3(x_3,x_3,x_1)$$

f_i: $\Re^3 \to \Re$, where a_i, α_i $(1 \leq i \leq 3)$ are real constants.

In other words, it means to extend the representation of the meniscus with three unknown functions f_i $(1 \leq i \leq 3)$ instead of by one unknown function f, as was shown in Section 3. This kind of meniscus representation is really much better, but this problem is very hard and it requires a new method of solvability.

If we continue in this way, it will hold for the generalization of equation (4.1) given by the formula

$$E(f) \equiv \sum_{i=1}^{n} a_i f_i(x_i,x_{i+1},\ldots,x_{i+n-1}) =$$
$$= \sum_{i=1}^{n} a_i f_i(x_i,x_i,x_{i+1},\ldots,x_{i+n-2}) \; (x_{n+i} \equiv x_i, n > 1)$$

f_i: $\Re^n \to \Re$, where a_i, α_i $(1 \leq i \leq n)$ are real constants. This equation is extremely hard and its solution is unknown up to now.

2° A second generalization of equation (3.1) is the vector equation

$$a_1 f_1(\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3) + a_2 f_2(\boldsymbol{X}_2,\boldsymbol{X}_3,\boldsymbol{X}_1) + a_3 f_3(\boldsymbol{X}_3,\boldsymbol{X}_1,\boldsymbol{X}_2) = \\ = \alpha_1 f_1(\boldsymbol{X}_1,\boldsymbol{X}_1,\boldsymbol{X}_2) + \alpha_2 f_2(\boldsymbol{X}_2,\boldsymbol{X}_2,\boldsymbol{X}_3) + \alpha_3 f_3(\boldsymbol{X}_3,\boldsymbol{X}_3,\boldsymbol{X}_1)$$

f_i: $\mathfrak{R}^3 \to \mathfrak{R}$, where $\boldsymbol{X}_i = (x_{1i},x_{2i},x_{3i})^{\mathrm{T}}$ are real vectors and a_i, α_i $(1 \le i \le 3)$ are real constants.

For equation (4.1) its generalized form is given by the equation

$$E(f) \equiv \sum_{i=1}^{n} a_i f_i(\boldsymbol{X}_i,\boldsymbol{X}_{i+1},\ldots,\boldsymbol{X}_{i+n-1}) = \\ = \sum_{i=1}^{n} a_i f_i(\boldsymbol{X}_i,\boldsymbol{X}_i,\boldsymbol{X}_{i+1},\ldots,\boldsymbol{X}_{i+n-2})\ (\boldsymbol{X}_{n+i} \equiv \boldsymbol{X}_i,\ n > 1)$$

f_i: $\mathfrak{R}^n \to \mathfrak{R}$, where $\boldsymbol{X}_i = (x_{1i},x_{2i},\ldots,x_{ni})^{\mathrm{T}}$ are real vectors and a_i, α_i $(1 \le i \le n)$ are real constants.

Really, in this section the considered equations are more sophisticated than the solved equations in previous sections, but their solutions are extremely difficult and up to now unknown to the author. In any case, their solutions will describe the meniscus form in a way that will be much closer to reality.

7 CONCLUSION

In this work the analyzed meniscus equation shows that it is possible to interpret the meniscus shape with a quasicyclic real functional equation. During continuous steel casting, the shape of the meniscus changes according to the mould cycle. The derived results are appropriate for use in a huge mathematical model for a description of all the appearances on the meniscus considering the technical characteristics of the process.

The mathematical results are summarized as follows:

1) The meniscus equation is completely solved in $\mathfrak{R}^3$, for all possible cases. The induced topology only categorizes the trajectories in an orientation space.
2) The extended form of the meniscus equation is derived too in $\mathfrak{R}^n$ by a compact matrix approach, different from the method used for the solution of the meniscus equation in $\mathfrak{R}^3$.
3) The meniscus stability problem is solved by using a simple spectral property of the compound matrices and Lozinskiĭ measure on $\mathfrak{R}^n$.

In this work the shape of the meniscus during continuous steel casting is considered for the first time as a non-fixed characteristic according to the cyclic operation of the mould. The analysis has shown that it is more appropriate to use this kind of meniscus modeling than some approximate form. It is also shown that the old one-dimensional interpretations of the meniscus may only be used as an approximation.

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