On the number of connected and disconnected coverings over a manifold

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Abstract

A general formula is obtained for the number of non-equivalent coverings (possibly disconnected) over a connected manifold with an arbitrary finitely generated fundamental group. Some illustrative examples are considered.

Keywords: Counting homomorphisms, non-equivalent coverings, disconnected coverings, fundamental group; Euler transform.

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1 Introduction

Let $p : \mathcal{U} \to \mathcal{M}$ be an arbitrary (possibly disconnected) covering over a connected manifold $\mathcal{M}$. Then every path $\gamma \in \mathcal{M}$ has a unique lifting $\widetilde{\gamma}$ starting at a given point of $p^{-1}(\gamma(0))$; so we obtain a well-defined mapping $L_\gamma : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ by sending the starting point $\widetilde{\gamma}(0)$ of each lifting $\widetilde{\gamma}$ to its ending point $\widetilde{\gamma}(1)$. By the familiar monodromy theorem, L_γ depends only on the homotopy class of γ. This means that the association $\gamma \mapsto L_\gamma$ gives rise to a homomorphism from $\pi_1(\mathcal{M}, x_0)$ to a permutation group acting on the fiber $F = p^{-1}(x_0)$. We note that this action of $\pi_1(\mathcal{M}, x_0)$ is transitive if and only if the covering

manifold $\mathcal{U}$ is connected. More generally, the number of orbits of the action coincides with the number of connected components of the covering.

Two coverings $p_1 : \mathcal{U}_1 \to \mathcal{M}$ and $p_2 : \mathcal{U}_2 \to \mathcal{M}$ are equivalent if and only if the corresponding actions of $\pi_1(\mathcal{M}, x_0)$ on the fibers F_1 and F_2 over x_0 are isomorphic. This shows that n-sheeted covering spaces over $\mathcal{M}$ are classified by equivalence classes of homomorphisms $\rho : \pi_1(\mathcal{M}, x_0) \to \mathbf{S}_n$, where $\mathbf{S}_n$ is the symmetric group on n symbols and the equivalence relation imposes an equivalence between ρ and all its conjugates $h^{-1}\rho h$ by elements $h \in \mathbf{S}_n$.

All the above-mentioned results remain valid for any path-connected, locally path-connected and semilocally simply connected topological space $\mathcal{M}$; see, for example [2, Ch.1.3: Th.1.38]. In particular, they are valid for finite graphs.

In the present paper we count the number of all non-equivalent coverings (connected or not) over a connected manifold with a finitely generated fundamental group. Then, by making use of the standard Euler transform (see Lemma 2.6 below) we relate the number of connected and disconnected coverings to get a new proof of the main result of [12]. This approach, which makes it possible to calculate the number of connected coverings through the number of disconnected ones, is, essentially, new. Earlier, for the special cases of the free group and the free product of cyclic groups, it has been effectively used in [4] and [15] to enumerate graph coverings and 'unsensed' maps on closed orientable surfaces, respectively.

Related general results for the enumeration of coverings were obtained by H. Tamanoi [18] and by T. Müller and J. Shareshian [13].

Several preceding papers contain formulae for counting non-equivalent connected and disconnected coverings. The numbers of non-equivalent connected and arbitrary coverings of a graph were determined in [7] and [4], respectively. Connected coverings of closed surface were counted in [10]. A general method to enumerate connected coverings over manifolds was given in [11, 12]. In more detail, the enumeration of graphs and manifold coverings (including disconnected ones) is considered in [6]. In [1], a relation between the numbers of connected and disconnected weighted ramified coverings over a torus is derived and used. Some enumerative applications of disconnected structures to chemistry are given in [14].

2   Main results

The following theorem has been proved by the second-named author [11, 12].

Theorem 2.1. *Let $\mathcal{M}$ be a connected manifold with a finitely generated fundamental group Γ. Then the number u_n of non-equivalent* connected *n-fold coverings over $\mathcal{M}$ is given by the formula*

$$u_n = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{K <_m \Gamma} |\mathrm{Epi}(K, \mathbb{Z}_\ell)|, \tag{2.1}$$

where the sum $\sum_{K <_m \Gamma}$ is taken over all subgroups K of index m in the group Γ, $\mathrm{Epi}(K, \mathbb{Z}_\ell)$ is the set of epimorphisms of the group K onto a cyclic group $\mathbb{Z}_\ell$ of order ℓ and $|X|$ denotes the cardinality of a set X.

The aim of the present paper is to extend this theorem to coverings that are not necessarily connected.

Theorem 2.2. *Let $\mathcal{M}$ be a connected manifold with a finitely generated fundamental group Γ. Denote by b_n the number of all non-equivalent* (connected or disconnected) *n-fold coverings over $\mathcal{M}$ and let $B(x) = 1 + b_1\,x + b_2\,x^2 + \cdots$ be the ordinary generating function for these numbers. Then*

$$B(x) = \exp\Big(\sum_{n=1}^{\infty} \sum_{\substack{\ell|n \\ \ell m = n}} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_\ell)| \frac{x^n}{n} \Big), \tag{2.2}$$

where $\mathrm{Hom}(K, \mathbb{Z}_\ell)$ is the set of homomorphisms of the group K into a cyclic group $\mathbb{Z}_\ell$.

Proof. From Burnside's lemma we obtain (reasoning similarly to [8]) the following result.

Lemma 2.3.

$$b_n = \sum_{c_1 + 2c_2 + \cdots + nc_n = n} \prod_{i=1}^{n} \frac{|\mathrm{Hom}(\Gamma, \mathbb{Z}_i \,\mathtt{Wr}\, \mathbb{S}_{c_i})|}{i^{c_i}\, c_i!}, \tag{2.3}$$

where $A \,\mathtt{Wr}\, B$ is the standard wreath product of permutation groups A and B and $\mathbb{Z}_i$ is considered as a regular cyclic permutation group of order i.

Since $n = c_1 + 2c_2 + \cdots + nc_n$, we have from this lemma,

$$b_n x^n = \sum_{c_1 + 2c_2 + \cdots + nc_n = n} \prod_{i=1}^{n} \frac{|\mathrm{Hom}(\Gamma, \mathbb{Z}_i \,\mathtt{Wr}\, \mathbb{S}_{c_i})|}{i^{c_i}\, c_i!}\, x^{ic_i}.$$

Hence, $B(x) = \prod_{i=1}^{\infty} b_i(x)$, where

$$b_i(x) = \sum_{c \geq 0} \frac{|\mathrm{Hom}(\Gamma, \mathbb{Z}_i \,\mathtt{Wr}\, \mathbb{S}_c)|}{i^c\, c!}\, x^{ic}.$$

The next lemma follows directly from formula (5) of [13].

Lemma 2.4. *For any $\ell \geq 1$,*

$$\sum_{m \geq 0} \frac{|\mathrm{Hom}(\Gamma, \mathbb{Z}_\ell \,\mathtt{Wr}\, \mathbb{S}_m)|}{m!}\, x^m = \exp\Big(\sum_{m \geq 1} \ell^{m-1} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_\ell)| \frac{x^m}{m} \Big). \tag{2.4}$$

Now we return to the proof of Theorem 2.2. Taking (2.4) into account we have

$$b_i(x) = \sum_{c \geq 0} \frac{|\mathrm{Hom}(\Gamma, \mathbb{Z}_i \,\mathtt{Wr}\, \mathbb{S}_c)|}{i^c\, c!}\, x^{ic} = \sum_{c \geq 0} \frac{|\mathrm{Hom}(\Gamma, \mathbb{Z}_i \,\mathtt{Wr}\, \mathbb{S}_c)|}{c!} \Big(\frac{x^i}{i} \Big)^c$$

$$= \exp\Big(\sum_{m \geq 1} \frac{i^{m-1} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_i)|}{m} \Big(\frac{x^i}{i} \Big)^m \Big)$$

$$= \exp\Big(\sum_{m \geq 1} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_i)| \frac{x^{im}}{im} \Big).$$

Finally, we obtain

$$B(x) = \prod_{i=1}^{\infty} b_i(x) = \exp\Big(\sum_{i=1}^{\infty} \sum_{m \geq 1} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_i)| \frac{x^{im}}{im} \Big)$$

$$= \exp\Big(\sum_{n=1}^{\infty} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_\ell)| \frac{x^n}{n} \Big).$$

$\square$

As a consequence of Theorem 2.2 we can give a new proof of Theorem 2.1. To that end we will make use of the following preliminary results [3]:

Lemma 2.5.

$$|\mathrm{Epi}(K, \mathbb{Z}_\ell)| = \sum_{d \mid \ell} \mu\Big(\frac{\ell}{d} \Big) |\mathrm{Hom}(K, \mathbb{Z}_\ell)|,$$

where $\mu(n)$ is the number-theoretic Möbius function.

In enumerative combinatorics, there is a simple general formula[1] (see, e.g., [19, Th. 3.14.1]) called the *Euler transform* [17, p. 20] that relates the generating function of arbitrary, possibly disconnected, *unlabeled* objects of a certain kind with that of connected objects of the same kind (serving as their independent "building blocks"). In terms of coverings it is formulated as follows.

Lemma 2.6. *Let u_i and $B(x)$ be defined as in Theorems 2.1 and 2.2, respectively. Then the following identity is valid:*

$$B(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-u_i}.$$

Proof. Denote by $\mathfrak{S}$ the set of equivalence classes of all connected coverings of finite multiplicity over the manifold $\mathcal{M}$. Let $\mathfrak{D} = \mathbb{N}\mathfrak{S}$ be the set of all finite linear combinations of the type $m_1 C_1 + m_2 C_2 + \ldots + m_k C_k$, where $m_1, m_2, \ldots, m_k \in \mathbb{N}$ and $C_1, C_2, \ldots, C_k \in \mathfrak{S}$. We identify $\mathfrak{D}$ with the set of equivalence classes of all coverings of finite multiplicity over the manifold $\mathcal{M}$.

Define $\hat{\mu} : \mathfrak{S} \rightarrow \mathbb{N}$ to be the multiplicity of the covering. For any $d = m_1 C_1 + m_2 C_2 + \ldots + m_k C_k \in \mathfrak{D}$, we set $\hat{\mu}(d) = m_1 \cdot \hat{\mu}(C_1) + m_2 \cdot \hat{\mu}(C_2) + \ldots + m_k \cdot \hat{\mu}(C_k)$. Then $u_n = |(c \in \mathfrak{S} : \hat{\mu}(c) = n)|$ and $b_n = |(d \in \mathfrak{D} : \hat{\mu}(d) = n)|$ are the numbers of equivalence classes of connected and all n-fold coverings, respectively.

Let $C_1^i, C_2^i, \ldots, C_{a_i}^i \in \mathfrak{S}$ be the list of equivalence classes of all i-fold connected coverings for any $i = 1, 2, \ldots$. Then any n-fold covering, up to equivalence, can be uniquely represented as $C = \sum_{i=1}^{n} \sum_{j=1}^{a_i} m_j^i C_j^i$ for some non-negative integers m_j^i satisfying $\hat{\mu}(C) = n$.

Since $\hat{\mu}(C_j^i) = i$, the last condition is equivalent to $\sum_{i=1}^{n} \sum_{j=1}^{a_i} i \cdot m_j^i = n$. Hence, b_n coincides with the number of non-negative solution of above equation. We set also $b_0 = 1$. Then we have

$$b_n x^n = \sum_{\sum_{i=1}^{n} \sum_{j=1}^{u_i} i \cdot m_j^i = n} \prod_{i=1}^{n} \prod_{j=1}^{a_i} x^{i \cdot m_j^i}$$

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[1]In applications to unlabeled graphs it is also known as Riddell's formula.

and, consequently,

$$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + \ldots)^{u_i} = \prod_{i=1}^{\infty} (1 - x^i)^{-u_i}.$$

$\square$

Now we are ready to prove Theorem 2.1 as a consequence of Theorem 2.2.

Proof. By Theorem 2.2 we have

$$B(x) = \exp\Big(\sum_{n=1}^{\infty} v_n \frac{x^n}{n} \Big), \tag{2.5}$$

where

$$v_n = \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_\ell)|.$$

By Lemma 2.6 we also have

$$B(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-u_i}. \tag{2.6}$$

Then, combining (2.5) and (2.6) and taking the logarithm we obtain

$$\sum_{n=1}^{\infty} v_n \frac{x^n}{n} = -\sum_{i=1}^{\infty} u_i \, \log(1 - x^i). \tag{2.7}$$

After differentiating (2.7) and multiplying by x we get

$$\sum_{n=1}^{\infty} v_n \, x^n = \sum_{i=1}^{\infty} \frac{i \, u_i x^i}{1 - x^i}.$$

Comparing the coefficients of the Taylor and Lambert series we have $v_n = \sum_{j | n} j \, u_j$. Now, by the Möbius inversion formula we obtain

$$n u_n = \sum_{j | n} \mu\Big(\frac{n}{j}\Big) v_j = \sum_{j | n} \mu\Big(\frac{n}{j}\Big) \sum_{m | j} \sum_{K <_m \Gamma} \Big|\mathrm{Hom}\big(K, \mathbb{Z}_{\frac{j}{m}}\big)\Big|$$

$$= \sum_{j | n} \sum_{m | j} \sum_{K <_m \Gamma} \mu\Big(\frac{n}{j}\Big) \Big|\mathrm{Hom}\big(K, \mathbb{Z}_{\frac{j}{m}}\big)\Big|$$

$$= \sum_{m | n} \sum_{m | j | n} \sum_{K <_m \Gamma} \mu\Big(\frac{n}{j}\Big) \Big|\mathrm{Hom}\big(K, \mathbb{Z}_{\frac{j}{m}}\big)\Big| = \sum_{m | n} \sum_{K <_m \Gamma} \sum_{m | j | n} \mu\Big(\frac{n}{j}\Big) \Big|\mathrm{Hom}\big(K, \mathbb{Z}_{\frac{j}{m}}\big)\Big|$$

$$= \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K <_m \Gamma} \sum_{d | \ell} \mu\Big(\frac{\ell}{d}\Big) \, |\mathrm{Hom}(K, \mathbb{Z}_d)| = \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K <_m \Gamma} |\mathrm{Epi}(K, \mathbb{Z}_\ell)|$$

(the last equality follows from Lemma 2.5). This is formula (2.1). $\square$

3   Examples

3.1   The number of not necessarily connected coverings over a circle

In this case, $\Gamma = \pi_1(S^1) = \mathbb{Z}$, an infinite cyclic group. The total number of n-fold coverings over the circle S^1 is given by the standard partition function $p(n)$, where

$$p(0) + p(1)\, x + p(2)\, x^2 + \cdots = \prod_{i=1}^{\infty}(1 - x^i)^{-1}$$

(so that, $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11, \ldots$). This is the well-known Euler equation. To derive it in the framework of our approach we note that the group $\mathbb{Z}$ has only one subgroup of index n. Hence, $u_n = 1$ for every $n \geq 1$, and the result follows from Lemma 2.6.

3.2   The number of coverings over a graph

Let G be a finite connected graph with Betti number $r = \beta(G)$. Then $\Gamma = \pi_1(G) = F_r$ is a free group of rank r. By [7, 8] we have $|\mathrm{Hom}(\Gamma, \mathbb{Z}_i \, \mathrm{Wr} \, \mathbb{S}_{c_i})| = (i^{c_i}\, c_i!)^r$. Hence, by Lemma 2.3,

$$b_n = \sum_{c_1 + 2c_2 + \cdots + nc_n = n} \prod_{i=1}^{n}(i^{c_i}\, c_i!)^{r-1}. \tag{3.1}$$

This is a result by J. H. Kwak and Y. Lee [4]. It has been used to calculate the number of connected coverings over a graph G. The latter coincides with the number of conjugacy classes of subgroups of index n in the free group F_r obtained earlier in [7]. Thus, Theorem 2.2 gives an affirmative answer to the question posed by J. H. Kwak [5] as to whether (and how) it is possible to relate the enumerative results of papers [4] and [7].

3.3   The number of all coverings over a closed orientable surface

In this case we have

$$B(x) = \exp\left(\sum_{n=1}^{\infty} \sum_{\substack{\ell \mid n \\ \ell m = n}} \ell^{(2g-2)m+2}\, M_{\Phi_g}(m)\, \frac{x^n}{n} \right), \tag{3.2}$$

where Φ_g is the fundamental group of a closed orientable surface of genus g and $M_{\Phi_g}(n)$ is the number of subgroups of index n in Φ_g. Indeed, by the Riemann-Hurwitz formula, any subgroup K of index m in the group Φ_g is isomorphic to $\Phi_{g'}$, where $(2g'-2) = (2g-2)\, m$. Now, the first homology group $H_1(\Phi_{g'}) = \Phi_{g'}/[\Phi_{g'},\, \Phi_{g'}]$ is equal to $\mathbb{Z}^{2g'} = \mathbb{Z}^{(2g-2)\, m+2}$ and $|\mathrm{Hom}(K, \mathbb{Z}_\ell)| = |\mathrm{Hom}(H_1(K), \mathbb{Z}_\ell)| = \ell^{(2g-2)m+2}$. Therefore,

$$\sum_{K <_m \Gamma} |\mathrm{Hom}(K, \mathbb{Z}_\ell)| = \ell^{(2g-2)m+2}\, M_{\Phi_g}(m).$$

Substituting this equality into the statement of Theorem 2.2 we obtain (3.2).

Recall [9] that $M_{\Phi_g}(m)$ can be computed by the following linear recursion formula:

$$M_{\Phi_g}(m) = m\, \beta_m - \sum_{j=1}^{m-1} \beta_{m-j}\, M_{\Phi_g}(j), \quad M_{\Phi_g}(1) = 1, \tag{3.3}$$

where

$$\beta_k = \sum_{\chi \in D_k} \left(\frac{k!}{f\chi} \right)^{2g-2}, \tag{3.4}$$

D_k is the set of all irreducible representations of the symmetric group $\mathbf{S}_k$ and f^χ is the degree of a representation χ.

Instead of (3.2), by the Euler transform (Lemma 2.6), we have immediately,

$$B(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-u_i}, \tag{3.5}$$

where $u_n = N_{\Phi_g}(n)$ denotes the number of connected n-fold coverings of a closed orientable surface of genus g, which is equal to the number of conjugacy classes of subgroups of index n in Φ_g found in [9, 10]. Equating the right-hand sides of (3.2) and (3.5), we obtain the following equation for $N_{\Phi_g}(n)$ in terms of generating functions:

$$\prod_{i=1}^{\infty} (1 - x^i)^{-N_{\Phi_g}(i)} = \exp\Big(\sum_{n=1}^{\infty} \sum_{\substack{\ell|n \\ \ell m = n}} \ell^{(2g-2)m+2} M_{\Phi_g}(m) \frac{x^n}{n} \Big), \quad g \geq 0, \tag{3.6}$$

which is equivalent to the explicit formula obtained in [9].

Let $b_n = DC(n, \nu)$ be the total number of all non-equivalent n-fold coverings (connected or disconnected) over a closed orientable surface of genus g, where $\nu = 2g - 2$ is its *characteristic*. Then by formula (3.2) we calculated for $n = 1, \ldots, 7$:

$$DC(1, \nu) = 1$$
$$DC(2, \nu) = 4 \cdot 2^\nu$$
$$DC(3, \nu) = 2 \cdot 6^\nu + 4 \cdot 3^\nu + 2 \cdot 2^\nu$$
$$DC(4, \nu) = 2 \cdot 24^\nu + 12^\nu + 6 \cdot 8^\nu + 9 \cdot 4^\nu + 3 \cdot 3^\nu$$
$$DC(5, \nu) = 2 \cdot 120^\nu + 2 \cdot 30^\nu + 2 \cdot 24^\nu + 20^\nu + 4 \cdot 12^\nu + 4 \cdot 8^\nu + 14 \cdot 6^\nu$$
$$\qquad + 5 \cdot 5^\nu + 5 \cdot 4^\nu$$
$$DC(6, \nu) = 2 \cdot 720^\nu + 4 \cdot 144^\nu + 2 \cdot 80^\nu + 2 \cdot 72^\nu + 8 \cdot 48^\nu + 45^\nu + 4 \cdot 24^\nu$$
$$\qquad + 12 \cdot 18^\nu + 16 \cdot 16^\nu + 6 \cdot 9^\nu + 18 \cdot 8^\nu + 12 \cdot 6^\nu + 5 \cdot 5^\nu$$
$$DC(7, \nu) = 2 \cdot 5040^\nu + 2 \cdot 840^\nu + 4 \cdot 360^\nu + 2 \cdot 336^\nu + 252^\nu + 6 \cdot 240^\nu + 2 \cdot 144^\nu$$
$$\qquad + 6 \cdot 72^\nu + 4 \cdot 60^\nu + 16 \cdot 48^\nu + 2 \cdot 40^\nu + 3 \cdot 36^\nu + 34 \cdot 24^\nu + 6 \cdot 18^\nu$$
$$\qquad + 4 \cdot 16^\nu + 32 \cdot 12^\nu + 20 \cdot 10^\nu + 3 \cdot 9^\nu + 8 \cdot 8^\nu + 7 \cdot 7^\nu + 6 \cdot 6^\nu.$$

Table 1 contains initial numerical values of the function $DC(n, \nu)$, $\nu \geq -2$. For completeness we included values for odd ν as well, in particular, fractional values for $\nu = -1$. The latter have a simple combinatorial meaning: $DC(n, -1) = r^2(n)/n!$, where $r(n)$ is the sum of the degrees of the irreducible representations of the symmetric group $\mathbf{S}_n$ (cf. formula (3.4)) [2].

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[2]This is the sequence A000085 [16]. The generating function for $r^2(n)/n!$ is $\exp\big(x/(1-x)\big)/\sqrt{1-x^2}$, and the values of $r^2(n)$ form the sequence A111883.

ν	g	1	2	3	4	5	6	7
-2	0	1	1	1	1	1	1	1
-1	–	1	2	16/3!	100/4!	676/5!	5776/6!	53824/7!
0	1	1	4	8	21	39	92	170
1	–	1	8	28	153	577	3612	19228
2	2	1	16	116	1851	33693	1175852	53529098
3	–	1	32	556	33105	3558577	761393916	257606692684
4	3	1	64	2948	711411	417285429	539383049612	
5	–	1	128	16588	16380633	49835381257	387243914607612	
6	4	1	256	96356	386803851	5973885950253		

Table 1: Values of $DC(n,\nu)$ for $\nu \leq 6$ and $n \leq 7$.

3.4 The number of coverings over a torus

In conclusion we address the particular case $\nu = 0$. By [10] we have $M_{\Phi_1}(n) = \sigma(n)$, where $\sigma(n) = \sum_{d|n} d$ is the sum of positive divisors of n. Setting $g = 1$ in formula (3.2) we obtain

$$B(x) = \exp\Big(\sum_{n=1}^{\infty} \sum_{\substack{\ell|n \\ \ell m = n}} \ell^{\,2}\,\sigma(m)\,\frac{x^n}{n} \Big). \tag{3.7}$$

Again, instead of (3.7), by the Euler transform we have

$$B(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-\sigma(i)}. \tag{3.8}$$

Accordingly, $b_n = DC(n,0)$, $n \geq 1$, is the sequence A061256 [16], which starts with $1, 4, 8, 21, 39, 92, 170, 360, 667, 1316, 2393, 4541, 8100, 14824, 26071, 46422$.

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