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On the beta distribution, the nonlinear Fourier transform and a combinatorial problem

Pavle Saksida *

Faculty of Mathematics and Physics, University of Ljubljana, Jadranska ulica 21, Ljubljana, Slovenia

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Abstract

The paper describes some probabilistic and combinatorial aspects of the nonlinear Fourier transform associated with the AKNS-ZS problems. We show that the volumes of a family of polytopes that appear in a power expansion of the nonlinear Fourier transform are distributed according to the beta probability distribution. We establish this result by studying an Euler-type discretization of the nonlinear Fourier transform. This approach leads to the combinatorial problem of finding the number of alternating ordered partitions of an integer into a fixed number of distinct parts. We find the explicit formula for these numbers and show that they are essentially distributed according to a novel discretization of the beta distribution for a suitable choice of the shape parameters. We also find the generating functions of the numbers of alternating sums. These functions are expressed in terms of the our discrete nonlinear Fourier transform.

Keywords: Beta distribution, nonlinear Fourier transform, discretisation.

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1 Introduction

As announced in the title, this paper investigates relations between three topics from different parts of mathematics: probability distributions, combinatorics and the theory of nonlinear partial differential equations, more concretely, the nonlinear Fourier transform. Despite the apparent heterogeneity of the topics, the relations between them are rather natural.

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E-mail address: pavle.saksida@fmf.uni-lj.si (Pavle Saksida)

The construction and the study of various versions of the nonlinear Fourier transform stem from the theory of integrable nonlinear partial differential equations. The most famous examples of such equations include the Korteweg-de Vries, nonlinear Schrödinger, and sine-Gordon equations, Heisenberg ferromagnet model, Toda lattices and many others. The role of the nonlinear Fourier transform in the theory of integrable equations is roughly analogous to the role of the linear Fourier transform and, more generally, the Sturm-Liouville expansions in the theory of linear partial differential equations.

The transformation $\mathcal{F}$ used in this text can be thought of as a non-linearization of the usual Fourier transformation. Let $u\colon [0,1] \to \mathbb{R}$ be a function. The nonlinear Fourier transform $\mathcal{F}$ of u that we shall consider in this paper is of the form

$$\mathcal{F}[u](n) = I + \begin{pmatrix} 0 & F[u](n) \\ -F[\overline{u}](-n) & 0 \end{pmatrix} + \sum_{d=2}^{\infty} A_d[u](n),$$

where F is the linear Fourier transform (Fourier series) and $u \mapsto A_d[u]$ are the suitable matrix-valued nonlinear operators.

The beta distribution is one of the oldest and most important probability distributions with a broad spectrum of applications in different areas of probability and statistics, particularly in Bayesian statistical inference. It has been recently mentioned in virtually every book on machine learning and related topics. The beta distribution $\mathrm{Beta}(x; a, b)$ with shape parameters a and b is given by the probability density function

$$p_\beta(x; a, b) = \frac{1}{B(a+1, b+1)} x^a (1-x)^b, \quad x \in [0,1].$$

In this paper, we shall establish a link between the nonlinear Fourier transform and the beta distribution. Let $u_c(x) \equiv u$ be a constant function. The transformation $\mathcal{F}$ is related to a two-parameter family of polytopes $\widehat{D}_d(\lambda)$, where $d \in \mathbb{N}$ and $\lambda \in [0,1]$, given by

$$\widehat{D}_d(\lambda) = \{(x_1, x_2, \ldots, x_d);\ \ 1 \geq x_1 \geq x_2 \ldots \geq x_d \geq 0,\ \ \sum_{i=1}^{d} (-1)^{i-1} x_i = \lambda\}$$

and their projections $D_d(\lambda)$ in the hyperplane $\{(x_1, x_2, \ldots, x_{(d-1)}, 0)\} \subset \mathbb{R}^d$. For the nonlinear Fourier transform $\mathcal{F}[u_c](n)$ of the constant function $u_c \equiv u$ on $[0,1]$, we have

$$\mathcal{F}[u_c](n) = I + \sum_{d=1}^{\infty} u^d \int_0^1 \mathrm{Vol}(D_d(\lambda)) \begin{pmatrix} e^{-2\pi i \lambda n} & 0 \\ 0 & e^{2\pi i \lambda n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \, d\lambda.$$

This formula is proved in Proposition 2.1 on page 7. We shall see that for every fixed d_0, the volumes of the family $\{D_{d_0}(\lambda); \lambda \in [0,1]\}$ are given by the formula for the probability density function of the beta distribution. Theorem 4.3 on page 16 gives the formula

$$\mathrm{Vol}(D_d(\lambda)) = \frac{1}{d!} \begin{cases} p_\beta(\lambda; \frac{d}{2}, \frac{d}{2}+1); & d \text{ even} \\ p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2}); & d \text{ odd.} \end{cases} \tag{1.1}$$

The probabilistic contents of the above formula will be described below.

The statement and the proof of Theorem 4.3 are obtained by considering a suitable discretization $\mathcal{F}_N$ of the nonlinear Fourier transform $\mathcal{F}$. In the expression for $\mathcal{F}_N[u_c]$, the role of the volumes of the polytopes $D_d(\lambda)$ is assumed by the numbers

$$AQ_N(L,d) = \sharp\{(l_1,l_2,\ldots,l_d)\in\mathbb{N};\ \ l_1-l_2+l_3-\ldots+(-1)^{(d-1)}l_d = L\},$$

where $N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0$. So, $AQ_N(L,d)$ is the number of ordered alternating partitions of L into d distinct parts not grater than $N-1$.

The central result of the paper is the explicit formula for the numbers $AQ_N(L,d)$. It is given in Theorem 3.3 on page 10. We show that

$$AQ_N(L,d) = \begin{cases} \binom{L-1}{\lfloor\frac{d-1}{2}\rfloor}\binom{N-L}{\lfloor\frac{d}{2}\rfloor}; & d \text{ even} \\ \binom{L}{\lfloor\frac{d-1}{2}\rfloor}\binom{N-L-1}{\lfloor\frac{d}{2}\rfloor}; & d \text{ odd.}\end{cases} \tag{1.2}$$

The relationship between the numbers $AQ_N(L,d)$ and the nonlinear Fourier transform is best described by the fact that the generating functions for the numbers $AQ_N(L,d)$ are in a natural way expressed in terms of the discrete nonlinear Fourier transform $\mathcal{F}_N$. This is proved in Proposition 3.2 on page 9. We actually get separate generating functions for odd and for even values of d. Understanding the structure of the numbers $AQ_N(L,d)$ was important in the construction of the inverse of $\mathcal{F}_N$ in our recent paper [14].

Results (1.1) and (1.2) can be recast into probabilistic terms. Let our sample space consist of all strictly decreasing d-tuples of integers

$$\Delta_d^D(N) = \{(l_1,l_2,\ldots,l_d);\ \ N-1\geq l_1 > l_2 > \ldots, l_d \geq 0)\},$$

Let all the samples $(l_1,l_2,\ldots,l_d)$ be equally probable and let

$$X_{AS}[N,d]\colon \Delta_d^D(N)\to\mathbb{N}$$

be the random variable which assigns to a randomly chosen point in $\Delta_d^D(N)$ the alternating sum,

$$X_{AS}[N,d](l_1,l_2,\ldots,l_d) = l_1-l_2+l_3-\ldots+(-1)^{(d-1)}l_d.$$

We want to compute the probability $P(X_{AS}[N,d]=L)$ of the event that a randomly chosen d-tuple has the alternating sum equal to L. We shall show that

$$AS[N,d](L) = P(X_{AS}[N,d]=L) = \begin{cases} \dfrac{\binom{L-1}{\lfloor\frac{d-1}{2}\rfloor}\binom{N-L}{\lfloor\frac{d}{2}\rfloor}}{\binom{N}{d}}; & d \text{ even} \\ \dfrac{\binom{L}{\lfloor\frac{d-1}{2}\rfloor}\binom{N-L-1}{\lfloor\frac{d}{2}\rfloor}}{\binom{N}{d}}; & d \text{ odd.}\end{cases} \tag{1.3}$$

The random variable X_{AS} is distributed according to the probability mass function $AS[N,d]$ defined by the right hand side of the above formula.

The question arises: Does this distribution have a sensible limit as N goes to infinity? One possibility is to proceed as follows. Let $\lambda\in[0,1]$ be arbitrary. Let us choose a sequence $\{L_N\}_{N\in\mathbb{N}}$ such that $L_N<N$ and $\lim_{N\to\infty} L_N/N=\lambda$. We shall see that

$$\lim_{N\to\infty} P(X_{AS}[N,d]=L_N)N = \begin{cases} p_\beta(\lambda;\frac{d}{2},\frac{d}{2}+1); & d \text{ even} \\ p_\beta(\lambda;\frac{d+1}{2},\frac{d+1}{2}); & d \text{ odd,}\end{cases} \tag{1.4}$$

where $p_\beta(\lambda; a, b)$ is the beta distribution with shape parameters a and b.

Let our sample space now be the ordered simplex $\Delta_d \subset \mathbb{R}^d$ of the dimension d, given by

$$\Delta_d = \{(x_1, x_2, \ldots, x_d);\ \ 1 \geq x_1 \geq x_2 \ldots \geq x_d \geq 0\},$$

and let all the samples $(x_1, x_2, \ldots, x_d)$ be equally probable. This means that we assigned on Δ_d the uniform distribution $v\colon \Delta_d \to \mathbb{R}$ given by $v(x_1, x_2, \ldots, x_d) \equiv d!$. Let the random variable $X_{as}[d]$ defined on Δ_d be given by

$$X_{as}[d](x_1, x_2, \ldots, x_d) = x_1 - x_2 + x_3 - \ldots + (-1)^{(d-1)} x_d.$$

Formula (1.4) shows that the cumulative probability distribution

$$F_{as}[d]\colon [0,1] \longrightarrow \mathbb{R}, \quad \lambda \to F_{as}[d](\lambda)$$

of the random variable X_{as} is given by

$$F_{as}[d](\lambda) = P(X_{as}[d] \leq \lambda) = \begin{cases} \int_0^\lambda p_\beta(\mu; \frac{d}{2}, \frac{d}{2}+1)\, d\mu\,; & d \ \text{even} \\ \int_0^\lambda p_\beta(\mu; \frac{d+1}{2}, \frac{d+1}{2})\, d\mu\,; & d \ \text{odd}. \end{cases}$$

This result can be recast in geometric terms. Taking into account that the d-dimensional volume of the simplex Δ_d is equal to $\frac{1}{d!}$, we see from the above that the $(d-1)$-dimensional volume of the polytope $D_d(\lambda)$ is indeed given by formula (1.1) explained in Theorem 4.3.

The equality (1.4) suggests a natural generalisation of the probability mass function of $X_{AS}[N, d]$. It can be defined by

$$P_N(L; a, b) = \frac{\binom{L-1}{a}\binom{N-L}{b}}{\binom{N}{a+b+1}},$$

where $L \in \{1, 2, \ldots, N\}$ and are integers such that $a + b < N$. In Proposition 4.2 we show that

$$\lim_{N\to\infty} P_N(\frac{L_N}{N}, a, b)N = p_\beta(\lambda; a, b) = \frac{1}{\beta(a+1, b+1)}\ \lambda^a(\lambda-1)^b.$$

So, the probability mass function $P_N(a, b)$ is a natural discretization of the continuous beta distribution for arbitrary values of shape parameters. But, at the moment, a convincing combinatorial or geometric description of $P_N(a, b)$ remains a task for the future.

Above, we have described a way how the beta distribution emerges as an appropriate limit from a discrete and finite probability distribution. This result is reminiscent to the relation between the Pólya-Eggenberger urn and the beta distribution. Pólya-Eggenberger urn is an urn model with replacement and is tenable - one can draw the balls from the urn infinitely many times. The limit of the quotient $\frac{W_n}{n}$, where W_n is the number of white balls drawn in n draws, is given by

$$\lim_{n\to\infty} P(\frac{W_n}{n} < \lambda) = \int_0^\lambda p_\beta(\mu; \frac{W_0}{s}, \frac{B_0}{s})\, d\mu,$$

where W_0 and B_0 are the initial numbers of white and black balls in the urn and s is the number of the balls added to the urn after drawing and returning a ball of the same colour.

Let $s = 1$. After a finite number n of draws, the probability of $W_n = w$ and $B_n = b$ is equal to

$$P(W_n = w, B_n = b) = \frac{\binom{w-1}{W_0-1}\binom{b-1}{B_0-1}}{\binom{n+\tau_0-1}{\tau_0-1}}, \tag{1.5}$$

where $\tau_0 = W_0 + B_0$. The proofs can be found in the comprehensive treatment of Pólya urn models [9] by H. M. Mahmoud. The values in the (1.5) are related by $w + b = W_0 + B_0 + n = \tau_0 + n$. Our formula (1.3) could therefore be tentatively understood as an outcome of Pólya-Eggenberger process after roughly $N + d$ steps. But the number of steps in constructing an alternating sum $l_1 - l_2 + \ldots + (-1)^{d-1} l_d$ is d. That d is indeed the correct number of steps in our process will become even clearer in the proofs of Theorem 3.3 and Corollary 3.4. These proofs are different from the usual proof of formula (1.5). While the number of steps in the limit of the Pólya-Eggenberger process is infinite, the number of steps in our process remains d even after performing the limit. This comes naturally from the source of our construction which is the nonlinear Fourier transform. The core of our limiting construction is the replacement of the alternating sum of integers: first, by alternating sums of rational numbers and then, in the limit, by the alternating sum of real numbers. This also leads to the geometric expression of our results in terms of the volumes of the polytopes $D_d(\lambda)$, mentioned above.

There are other discretization of the beta distribution with useful properties. One possible approach was studied by A. Punzo in [11]. But, as far as the author is aware, this discretization does not come from some combinatorial source and is given by a very different formula.

The plan of the paper is the following. In section 2, we recall the AKNS-ZS type of the nonlinear Fourier transform and prepare the necessary formulas. We establish the connection between $\mathcal{F}$ and the family of polytopes $\{D_d(\lambda)\}$. We also introduce the discretization $\mathcal{F}_N$ of $\mathcal{F}$. In section 3, we describe and prove the main facts about our central combinatorial problem: the evaluation of the numbers $AQ_N(L, d)$, and the derivation of the generating functions of the numbers $AQ_N(L, d)$ in terms of $\mathcal{F}_N$. In section 4, we prove proposition 4.2 and theorem 4.3 stated above. Section 5 contains graphs illustrating the relation between the beta distribution and our discrete approximation. We conclude the paper by mentioning some problems for further research.

2 Nonlinear Fourier transforms and its discretization

We review the definition of the nonlinear Fourier transform $\mathcal{F}$ which first appeared in the work of M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur in [1] and [2] and more or less simultaneously in the work of V. Zakharov and A. Shabat in [19]. They studied and solved a certain class of integrable partial differential equations which are now called AKNS-ZS equations. The acronym is also used to denote the nonlinear Fourier transform which figures in the AKNS-ZS theory. In this section, we shall also introduce the Euler-type discretization $\mathcal{F}_N$ of $\mathcal{F}$.

2.1 Nonlinear Fourier transform of AKNS-ZS type

We shall consider the nonlinear Fourier transform $\mathcal{F}$ which appears in the study of the *periodic* AKNS-ZS problems. To every well-behaved function $u(x)\colon [0, 1] \to \mathbb{C}$ it assigns the doubly infinite sequence $\{\mathcal{F}[u](n)\}_{n\in\mathbb{Z}}$ of $SU(2)$ matrices, given by $\mathcal{F}[u](n) =$

$(-1)^n\Phi(x=1,n)$, where $\Phi(x,n)$ is the solution of the linear initial value problem

$$\Phi_x(x,n) = L(x,n)\cdot\Phi(x,n), \quad \Phi(0,n) = I. \tag{2.1}$$

The coefficient matrix $L(x,n)$ is given by

$$L(x,n) = \begin{pmatrix} \pi i\, n & u(x) \\ -\overline{u(x)} & -\pi i\, n \end{pmatrix}.$$

As we mentioned in the Introduction, we will see that $\mathcal{F}$ is of the form

$$\mathcal{F}[u](n) = I + \begin{pmatrix} 0 & F[u](n) \\ -F[\overline{u}](-n] & 0 \end{pmatrix} + \sum_{d=2}^{\infty} A_d[u](n).$$

The amount of literature on various aspects of the inverse scattering method is truly vast, so we shall only mention a few works in which the Fourier analysis aspect is more pronounced. The foundational work was done by Gardner, Greene, Kruskal and Miura in [8] and [7]. The transform, used in this paper was first constructed by Ablowitz, Kaup, Newell and Segur, in [1, 2], and simultaneously by Zakharov and Shabat in [19]. Nonlinear Fourier transforms of functions, defined on $\mathbb{R}$ and $\mathbb{R}^+$, were studied by I. Gelfan'd, A. Fokas and B. Pelloni in [5, 6, 10], and in their other works. A version of transformation, closely related to the one studied in this paper is described by T. Tao and C. Thiele in [15]. Some aspects of the transformation, defined above, were studied in my papers [12, 13] and [14].

Definition of $\mathcal{F}$, given above is the one that is usually found in the texts which study the integrable ANKS-ZS equations. We shall rather represent $\mathcal{F}$ in a different gauge. Let $G(x,n) = \mathrm{diag}(e^{-\pi inx}, e^{\pi inx})$ be the (diagonal) matrix of our gauge transformation. In the new gauge Φ is replaced by $\Phi^G = G\cdot\Phi$ and Φ^G is the solution of the initial-value problem

$$\Phi^G_x(x,n) = L^G(x,z)\cdot\Phi^G(x,n), \quad \Phi^G(0,n) = I. \tag{2.2}$$

The transformed coefficient matrix is then $L^G(x,n) = G_x\cdot G^{-1}(x,n) + G(x,n)\cdot L(x,n)\cdot G^{-1}(x,n)$. Its explicit expression is

$$L^G(x,n) = \begin{pmatrix} 0 & e^{-2\pi inx}u(x) \\ -e^{2\pi inx}\overline{u(x)} & 0 \end{pmatrix}. \tag{2.3}$$

In the new gauge we set $\mathcal{F}^G[u](n) = \Phi^G(x=1,n)$. Since $n\in\mathbb{Z}$, the equation $\Phi^G(1,n) = G(1,n)\cdot\Phi(1,n)$ gives $\mathcal{F}[u](n) = \mathcal{F}^G[u](n)$. The solution to the problem (2.2) can be given in the form of the Dyson series.

$$\Phi^G(x,n) = I + \sum_{d=1}^{\infty}\int_{\Delta_d(x)} L^G(x_1,n)\cdot L^G(x_2,n)\cdots L^G(x_d,n)\,d\vec{x}, \tag{2.4}$$

where $\Delta_d(x)$ is the ordered simplex of dimension d with the edge length equal to x,

$$\Delta_d(x) = \{(x_1,x_2,\ldots,x_d)\in\mathbb{R}^d; x\geq x_1\geq x_2\geq\ldots\geq x_d\geq 0\}.$$

Let us denote

$$E(x,n) = \begin{pmatrix} e^{\pi ixn} & 0 \\ 0 & e^{-\pi ixn} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.5}$$

and let $u(x)$ be real valued. Then we have $L^G(x,n) = u(x)\,E(-2x,n)\cdot J$. Matrices $E(x,n)$ and J do not commute. Instead, they obey the relation

$$E(x,n)\cdot J = J\cdot E(-x,n). \tag{2.6}$$

Recall that $\widehat{D}_d(\lambda)$ denotes the polytope given by

$$\widehat{D}_d(\lambda) = \{(x_1,x_2,\dots x_d)\in\Delta_d(1);\ \sum_{j=1}^{d}(-1)^{j-1}x_j=\lambda\},$$

and $D_d(\lambda)$ is its projection on the hyperplane $x_d = 0$. These are the polytopes, mentioned in the introduction. Denote

$$\mathcal{U}(x_1,x_2,\dots,x_{d-1};\lambda) = u(x_1)\cdots u(x_{d-1})u((-1)^{d-1}(\lambda-\sum_{j=1}^{d-1}(-1)^{j-1}x_j)),$$

and let $d_\lambda\vec{x}$ be the measure on $\widehat{D}_d(\lambda)\subset\mathbb{R}^d$, inherited from the Euclidean measure on $\mathbb{R}^d$. Using (2.6) in the Dyson series and evaluating at $x=1$ gives

$$\mathcal{F}[u](n) = I+\sum_{d=1}^{\infty}\int_{\Delta_d(1)}u(x_1)\,u(x_2)\cdots u(x_d)\,E\Big(-2(\sum_{j=1}^{d}(-1)^{j-1}x_j),n\Big)\cdot J^d\,d\vec{x}$$

which, upon setting $x_1 - x_2 + \ldots + (-1)^{d-1}x_d = \lambda$, can be rewritten as

$$\begin{aligned}\mathcal{F}[u](n) &= I+\sum_{d=1}^{\infty}\int_0^1 E(-2\lambda,n)\,\Big(\int_{\widehat{D}_d(\lambda)}u(x_1)\,u(x_2)\cdots u(x_d)\,d_\lambda\vec{x}\Big)\,\cdot J^d\frac{1}{\sqrt{d}}\,d\lambda\\ &= I+\sum_{d=1}^{\infty}\int_0^1 E(-2\lambda,n)\,\Big(\int_{D_d(\lambda)}\mathcal{U}(x_1,\dots,x_{d-1};\lambda)\,dx_1\cdots dx_{d-1}\Big)J^d\,d\lambda,\end{aligned}$$

Inserting the constant function $u_c(x)\equiv u$ we immediately get the following proposition.

Proposition 2.1. *In the case where $u_c(x)\equiv u$ is a constant function, we get*

$$\mathcal{F}[u_c](n) = I+\sum_{d=1}^{\infty}u^d\int_0^1 \mathrm{Vol}(D_d(\lambda))\,E(-2\lambda,n)\cdot J^d\,d\lambda. \tag{2.7}$$

□

2.2 Euler-type discretization of $\mathcal{F}$

Many authors studied various discretizations of transformations similar to $\mathcal{F}$, but usually acting on the functions defined on $\mathbb{R}$ or $\mathbb{R}^+$, see e.g. [16, 17, 18]. Important are the discretizations that preserve the integrability of the AKNS-ZS systems. These are constructed in well known works of M. Ablowitz and J. Ladik and also L. Faddeev and A. Yu Volkov, see [3, 4]. A discrete nonlinear Fourier transform, similar to the one studied below, was considered by Tao and Thiele in [15]. In the author's paper [14] an algorithm for evaluating the inverse of the nonlinear Fourier transform, defined below, is constructed. (In [14] a nonlinear Fourier transform of distributions of the form $u(x)=\sum_{n=1}^{N}u_n\,\delta_{x_n}(x)$ is also constructed, together with its inverse.)

We have obtained the nonlinear Fourier transform from an initial value problem for a particular first-order linear differential equation. An obvious approach to construct a discretization is to replace the differential equation with a suitable difference equation. Let $\vec{u} = (u_0, u_1, \dots, u_{N-1}) \in \mathbb{R}^N$ be a vector which plays a role of a function of a discrete variable. Let the L-matrix be given by

$$L_N(k,n) = \begin{pmatrix} 0 & e^{-2\pi i \frac{kn}{N}} u_k \\ -e^{2\pi i \frac{kn}{N}} u_k & 0 \end{pmatrix}.$$

Definition 2.2. Let $k, n \in \{0, 1, \dots, N-1\}$. Discrete nonlinear Fourier transform $\mathcal{F}_N[\vec{u}]$ of $\vec{u}$ is defined by $\mathcal{F}_N[\vec{u}](n) = \Phi_N(k = N-1, n)$, where Φ_N is the solution of the difference initial value problem

$$\frac{\Phi_N(k+1,n) - \Phi_N(k,n)}{\frac{1}{N}} = L_N(k,n) \cdot \Phi_N(k,n), \quad \Phi_N(0,n) = I.$$

Solving the above initial value problem and evaluating at $k = N-1$ gives

$$\mathcal{F}_N[\vec{u}](n) = \prod_{k=N-1}^{0} \Big(I + \frac{1}{N} L_N(k,n)\Big),$$

and this can be expanded into

$$\mathcal{F}_N[\vec{u}](n) = I + \sum_{d=1}^{N} \frac{1}{N^d} \sum_{N-1 \geq l_1 > l_2 > \dots > l_d \geq 0} L_N(l_1,n) \cdot L_N(l_2,n) \cdots L_N(l_d,n). \quad (2.8)$$

This expression is a discrete analogue of Dyson's expansion (2.4).

Let us introduce the notation

$$E_\delta(l,n) = E(l, \frac{n}{N}) = \begin{pmatrix} e^{\pi i l \frac{n}{N}} & 0 \\ 0 & e^{-\pi i l \frac{n}{N}} \end{pmatrix} \quad l, n \in \{0, 1, \dots, N-1\}$$

where E is given by (2.5), and the subscript δ refers to the use in the discretized context. The coefficient matrix L_N can be written in the form

$$L_N(l,n) = u_l \, E_\delta(-2l, n) \cdot J,$$

with J defined in (2.5). By means of relation (2.6), we can collect all the copies of J in (2.8) on the right. Let $\vec{u}_c = (u, \dots, u)$ be a constant vector. We get

$$\mathcal{F}_N[\vec{u}_c](n) = I + \sum_{d=1}^{N} (\frac{u}{N})^d \sum_{N-1 \geq l_1 > l_2 > \dots > l_d \geq 0} E_\delta\Big(-2(l_1 - l_2 + \dots + (-1)^{d-1} l_d), n\Big) \cdot J^d.$$

If we denote $L = l_1 - l_2 + \dots + (-1)^{d-1} l_d$, we can finally write

$$\mathcal{F}_N[\vec{u}_c](n) = I + \sum_{d=1}^{N-1} (\frac{u}{N})^d \sum_{L=0}^{N-1} E_\delta(-2L, n) \sum_{(l_1,\dots,l_d) \in \widehat{D}^{disc}_{d,N}(L)} J^d, \quad (2.9)$$

where

$$\widehat{D}^{disc}_{d,N}(L) = \{(l_1, \dots, l_d) \in \mathbb{N}^d;\ N-1 \geq l_1 > \dots > l_d \geq 0,\ \sum_{j=1}^{d} (-1)^{j-1} l_j = L\}. \quad (2.10)$$

3 Ordered alternating partitions with distinct parts

In this section we introduce the central combinatorial object of the paper, namely the numbers $AQ_N(L,d)$. We establish the connection between the family $\{AQ_N(L,d)\}$ and the discrete nonlinear Fourier transform $\mathcal{F}_N$. The transformation $\mathcal{F}_N$ yields the generating functions for $\{AQ_N(L,d)\}$ separately for even and odd values of d. The main result of the section is the statement and proof of the explicit formula for the numbers $AQ_N(L,d)$ and the evaluation of the probability distribution of these numbers.

Definition 3.1. Let

$$\Delta^D_{N,d} = \{(l_1, l_2, \ldots, l_d) \in \mathbb{N}^d; N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}$$

be the discrete ordered simplex. Denote by $AQ_N(L,d)$ the numbers which count the *ordered alternating partitions of $L \in \mathbb{N}$ into d distinct parts not greater than $N-1$,*

$$AQ_N(L,d) = \sharp\{(l_1, l_2, \ldots, l_d) \in \Delta^D_{N,d};\;\; l_1 - l_2 + l_3 - \ldots (-1)^{d-1} l_d = L\}. \tag{3.1}$$

In other words, $AQ_N(L,d)$ is the number of solutions of the equation

$$l_1 - l_2 + l_3 - \ldots + (-1)^{d-1} l_d = L$$

where $(l_1, l_2, \ldots, l_d)$ is an element of the simplex $\Delta^D_{N,d}$.

The next proposition shows that $\mathcal{F}_N[u_c](n)$ can, roughly speaking, be understood as the discrete *linear* Fourier transform of the generating polynomial of the finite sequence $\{AQ_N(L,d)\}_{d=1}^N$.

Let us denote by $\mathcal{F}_{ev}[u_c](n)$ the upper left entry and by $\mathcal{F}_{odd}[u_c](n)$ the upper right entry of the 2×2 matrix $\mathcal{F}_N[u_c](n)$.

Proposition 3.2. *The power series expansion of $\mathcal{F}_N[u_c]$ around $u=0$ is given by*

$$\mathcal{F}_N[u_c](n) = I + \sum_{d=1}^{N} (\frac{u}{N})^d \sum_{L=0}^{N-1} AQ_N(L,d)\, E_\delta(-2L,n) \cdot J^d. \tag{3.2}$$

For every $L \in \{0,1,\ldots,N-1\}$, the generating polynomials of the numbers

$$\{AQ_N(L,2k)\}_{k=1,\ldots,\lfloor\frac{N}{2}\rfloor} \quad \text{and} \quad \{AQ_N(L,2k-1)\}_{k=1,\ldots,\lfloor\frac{N+1}{2}\rfloor}$$

are given by the equations

$$\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor} (-1)^k (\frac{u}{N})^{2k} AQ_N(L,2k) = \sum_{n=0}^{N-1} e^{2\pi i \frac{Ln}{N}} \cdot \mathcal{F}_{ev}[u_c](n) \tag{3.3}$$

$$\sum_{k=1}^{\lfloor\frac{N+1}{2}\rfloor} (-1)^{k+1} (\frac{u}{N})^{2k-1} AQ_N(L,2k-1) = \sum_{n=0}^{N-1} e^{-2\pi i \frac{Ln}{N}} \cdot \mathcal{F}_{odd}[u_c](n). \tag{3.4}$$

Proof. Recall formula (2.9)

$$\mathcal{F}_N[\vec{u}_c](n) = I + \sum_{d=1}^{N-1} (\frac{u}{N})^d \sum_{L=0}^{N-1} E_\delta(-2L, n) \sum_{(l_1,\ldots,l_d)\in \widehat{D}^{disc}_{d,N}(L)} J^d.$$

The last sum in the formula yields the constant matrix J^d multiplied by the integer $\sharp\widehat{D}^{disc}_{d,N}(L)$. The number $AQ_N(L, d)$ is by its definition the number of elements in $\widehat{D}^{disc}_{d,N}(L)$, so we have

$$\sum_{(l_1,\ldots,l_d)\in \widehat{D}^{disc}_{d,N}(L)} J^d = AQ_N(L, d) \cdot J^d.$$

Let us now take into account

$$J^{2k} = (-1)^k \cdot I = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix}$$

and

$$J^{2k-1} = (-1)^{k+1} \cdot J = \begin{pmatrix} 0 & (-1)^{2k-1} \\ -(-1)^{2k-1} & 0 \end{pmatrix},$$

and consider the diagonal and anti-diagonal parts of $\mathcal{F}_N$ separately. From 3.2, we get two equations, one for each parity of k:

$$\mathcal{F}_{ev}[u_c](n) = \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k (\frac{u}{N})^{2k} \sum_{L=0}^{N-1} e^{-2\pi i \frac{Ln}{N}} \cdot AQ_N(L, 2k)$$

$$\mathcal{F}_{odd}[u_c](n) = \sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} (-1)^{k+1} (\frac{u}{N})^{2k-1} \sum_{L=0}^{N-1} e^{2\pi i \frac{Ln}{N}} AQ_N(L, 2k-1).$$

Now, we perform the inverse discrete linear Fourier transforms on both of the above equations and get the expressions (3.3) and (3.4). □

We now state and prove the explicit formula for the function $AQ_N(L, d)$.

Theorem 3.3. *For any $N \in \mathbb{N}$, $d \leq N$ and $L \in \{0, \ldots, N-1\}$, we have*

$$AQ_N(L, d) = \begin{cases} \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}; & d \text{ even} \\ \binom{L}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor}; & d \text{ odd.} \end{cases} \tag{3.5}$$

Above we use the definition of the binomial symbol for which $\binom{a}{b} = 0$ for negative a.

Proof. Let us define

$$\widehat{AQ}_N(L, d) = \sharp\{(l_1, \ldots, l_d); N \geq l_1 > \ldots > l_d \geq 1, \text{ and } \sum_{j=1}^{d} (-1)^{j-1} l_j = L\}.$$

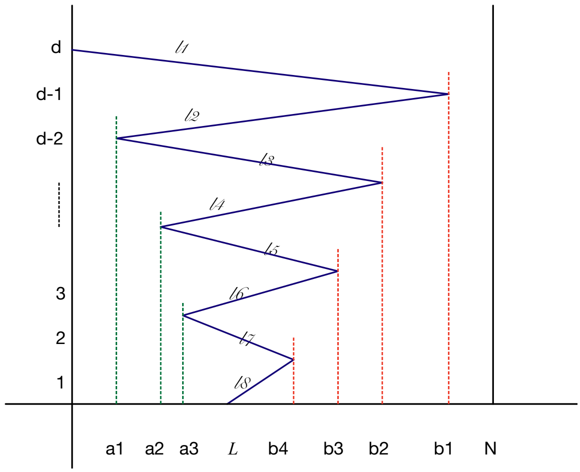


Figure 1: Zigzag path interpretation of an element of $\widehat{AQ}_N(L,d)$ with $d=8$.

We claim that for $\widehat{AQ}_N(L,d)$ we have

$$\widehat{AQ}_N(L,d) = \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}. \tag{3.6}$$

Suppose that $d=2k$ is even. Let us consider the partial sums of the alternating sum

$$\widehat{AQ}_N(L,d) = (l_1-l_2)+(l_3-l_4)+\ldots+(l_{d-1}-l_d)=L, \tag{3.7}$$

namely:

$$\begin{aligned} a_1 &= (l_1-l_2) \\ a_2 &= (l_1-l_2)+(l_3-l_4) \\ \vdots \quad & \quad \quad \vdots \\ a_{k-1} &= (l_1-l_2)+(l_3-l_4)+(l_5-l_6)+\ldots+(l_{d-3}-l_{d-2}). \end{aligned}$$

Let us also introduce the integers b_m, given by

$$\begin{aligned} b_1 &= (N-l_1) \\ b_2 &= (N-l_1)+(l_2-l_3) \\ \vdots \quad & \quad \quad \vdots \\ b_k &= (N-l_1)+(l_2-l_3)+(l_4-l_5)+\ldots+(l_{d-2}-l_{d-1}) \end{aligned}$$

From the above definitions we see that

$$\begin{aligned} l_1 &= N-b_1 \\ l_{2m} &= (N-b_m)-a_m \\ l_{2m-1} &= (N-b_m)-a_{m-1}. \end{aligned}$$

We shall now turn the situation around. Let

$$1 \leq \alpha_1 < \alpha_2 < \ldots, < \alpha_{k-1} \leq L-1 \tag{3.8}$$

be an arbitrary ordered subset of $\{1, 2, 3 \ldots, L-1\}$ and let

$$0 \leq \beta_1 < \beta_2 < \ldots < \beta_k \leq N-L-1 \tag{3.9}$$

be an arbitrary ordered subset of $\{0, 1, 2, \ldots, N-L-1\}$. Let us define

$$\begin{aligned} \lambda_1 &= N - \beta_1 \\ \lambda_{2m} &= (N-\beta_m) - \alpha_m, \qquad m = 1, 2, \ldots, k-1 \\ \lambda_{2m-1} &= (N-\beta_m) - \alpha_{m-1}, \quad m = 2, 3, \ldots, k \end{aligned}$$

From (3.8) and (3.9) we see that

$$N \geq \lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_{d-1} > 1$$

and

$$\lambda_1 - \lambda_2 + \lambda_3 - \ldots + \lambda_{d-1} \geq L+1.$$

Therefore there exists precisely one λ_d such that

$$(\lambda_1 - \lambda_2 + \lambda_3 - \ldots + \lambda_{d-1}) - \lambda_d = L$$

From the construction we also see that $\lambda_d < \lambda_{d-1}$.

We have shown that for every choice of a pair (3.8) and (3.9) of subsets of

$$\{1, 2, 3 \ldots, L-1\} \quad \text{and} \quad \{0, 1, 2, \ldots, N-L-1\},$$

respectively, there exists precisely one solution $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ of the equation (3.7). Since the number of such pairs is equal to

$$\binom{L-1}{k-1}\binom{N-L}{k} = \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor},$$

our proposition is proved for even d. The proof for odd d is only a slight variation of the above and we shall omit it.

Proof by induction. Our formula can be proved by induction on N. For $N = 2$, formula (3.6) can be checked by hand. If we divide the alternating sums from $\widehat{AQ}_N(L, d)$ into those, for which $l_1 = N$ and those for which $l_1 < N$, we get the recursion relation

$$\widehat{AQ}_N(L, d) = \widehat{AQ}_{N-1}(L, d) + \widehat{AQ}_{N-1}(N-L, d-1).$$

By the induction hypothesis, the above equation becomes

$$\begin{aligned} \widehat{AQ}_N(L, d) &= \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor} + \binom{N-L-1}{\lfloor \frac{d-2}{2} \rfloor}\binom{L-1}{\lfloor \frac{d-1}{2} \rfloor} \\ &= \binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}, \end{aligned}$$

and this proves (3.6). The second equality above comes from the recurrence relation of the Pascal triangle.

Finally, we observe that

$$AQ_N(L,d) = \widehat{AQ}_N(L,d), \text{ for } d \text{ even}, \quad \text{and} \quad AQ_N(L,d) = \widehat{AQ}_N(L{+}1,d), \text{ for } d \text{ odd}.$$

These relations, together with formula (3.6), prove the proposition. □

Two of the central results of this paper are corollaries of the above theorem.

Corollary 3.4. *Let the random variable*

$$X_{AS}[N,d]\colon \Delta_d^D(N) \longrightarrow \mathbb{R}$$

defined on the discrete ordered simplex

$$\Delta_d^D(N) = \{(l_1, l_2, \ldots, l_d);\;\; N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}$$

be given by

$$X_{AS}[N,d](l_1, l_2, \ldots, l_d) = l_1 - l_2 + l_3 - \ldots + (-1)^{(d-1)} l_d.$$

Then its probability mass function is

$$P(X_{AS}[N,d] = L) = \begin{cases} \dfrac{\binom{L-1}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L}{\lfloor \frac{d}{2} \rfloor}}{\binom{N}{d}} \,; & d \text{ even} \\ \dfrac{\binom{L}{\lfloor \frac{d-1}{2} \rfloor}\binom{N-L-1}{\lfloor \frac{d}{2} \rfloor}}{\binom{N}{d}} \,; & d \text{ odd.} \end{cases}$$

Proof. The number of the favourable events is given by Theorem 3.3, proved above. To evaluate the number of all outcomes it helps to consider Figure 1. We see that the number of all outcomes is equal to the number of the subsets which are composed of all the integer points a_i, all the points b_i and the point L. These are precisely all the subsets with d elements in the set $\{1, 2, \ldots, N\}$. Their number is of course $\binom{N}{d}$. This proves our corollary. □

Inserting the formula (3.5) in the expressions (3.3) and (3.4) yields the following corollary:

Corollary 3.5. *The power series of $\mathcal{F}_N[u_c](n)$ around $u = 0$ is given by*

$$\begin{aligned} \mathcal{F}_N[u_c](n) = I \;&+\; \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k \left(\frac{u}{N}\right)^{2k} \sum_{L=0}^{N-1} \binom{L-1}{k-1}\binom{N-L}{k} \cdot E_\delta(-2L,n) \\ &+\; \sum_{k=0}^{\lfloor \frac{N+1}{2} \rfloor} (-1)^k \left(\frac{u}{N}\right)^{2k+1} \sum_{L=0}^{N-1} \binom{L}{k}\binom{N-L-1}{k} \cdot E_\delta(-2L,n) \cdot J. \end{aligned}$$

4 Beta distribution and polytopes $D_d(\lambda)$

In this section we prove our second theorem which is the expression of the volumes of polytopes $D_d(\lambda)$ in terms of the probability density function of the beta distribution. We obtain this result by taking a suitable limit of the probability mass functions of the random variables $X_{AS}[N, d]$.

4.1 Discretization of beta distribution

The subset $\widehat{D}^{disc}_{d,N}(L)$, given by (2.10) of the discrete ordered simplex

$$\Delta^{disc}_{d,N} = \{(l_1, l_2, \ldots, l_d) \in (\mathbb{N} \cup \{0\})^d; N - 1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}$$

with the edge of size N is given by one equation. The size $AQ_N(L, d)$ of $\widehat{D}^{disc}_{d,N}(L)$ is therefore of the order N^{d-1}.

Lemma 4.1. *Let λ be a real number in $[0, 1]$ and let $\{L_N\}_{N\in\mathbb{N}}$ be a sequence of positive integers such that $L_N < N$ and $\lim_{N\to\infty} \frac{L_N}{N} = \lambda$. Then we have*

$$\lim_{N\to\infty} \frac{AQ_N(L_N, d)}{\binom{N}{d}} N = \begin{cases} p_\beta(\lambda; \frac{d}{2}, \frac{d}{2} + 1)\ ; & d \text{ even} \\ p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2})\ ; & d \text{ odd.} \end{cases} \tag{4.1}$$

where

$$p_\beta(\lambda; a, b) = \frac{1}{B(a + 1, b + 1)} \lambda^a (1 - \lambda)^b$$

is the probability density function of the beta distribution $\mathrm{Beta}(\lambda; a, b)$.

Proof. We shall prove the formula only for even d. The proof for odd d is essentially the same. Consider first the numerator of the quotient under the limit. For $d = 2m$, formula (3.5) gives

$$AQ_N(L_N, d) = \binom{L_N - 1}{m - 1}\binom{N - L_N}{m}.$$

This expression can be expanded into

$$AQ_N(L_N, d) = \frac{1}{(m-1)!\, m!} \prod_{k=0}^{m-2} ((L_N - 1) - k) \prod_{k=0}^{m-1} ((N - L_N) - k). \tag{4.2}$$

Consider the first product above. It is a polynomial of degree $m - 1$ in the variable $(L_N - 1) = (N\frac{L_N}{N} - 1)$. Expanding this polynomial gives

$$(N\frac{L_N}{N} - 1)^{m-1} + \sum_{k=1}^{m-2} (N\frac{L_N}{N} - 1)^k \cdot n(k) = (N\frac{L_N}{N} - 1)^{m-1} + \mathcal{O}(N\frac{L_N}{N})^{m-2}$$

For large N we can replace $\frac{L_N}{N}$ by λ. Taking into account also the second product, (4.2) gives

$$\begin{aligned} AQ_N(L_N, d) &= \frac{1}{(m-1)!\, m!}\Big((N\frac{L_N}{N} - 1)^{m-1} + \mathcal{R}_1\Big)\Big((N - N\frac{L_N}{N})^m + \mathcal{R}_2\Big) \\ &= \frac{1}{(m-1)!\, m!}\Big(N^{m+m-1}(\frac{L_N}{N})^{m-1}(1 - \frac{L_N}{N})^m + \mathcal{R}_3\Big) \end{aligned} \tag{4.3}$$

where

$$\mathcal{R}_1 = \mathcal{R}_3 = \mathcal{O}(\frac{1}{(N\lambda)^{m-2}}) \quad \text{and} \quad \mathcal{R}_2 = \mathcal{O}(\frac{1}{(N\lambda)^{m-1}}).$$

For the denominator $\binom{N}{d}$ we have

$$\binom{N}{d} = \frac{1}{d!}\Big(N(N-1)\cdots(N-(d-1))\Big) = \frac{N^d}{d!} + \mathcal{O}(\frac{1}{N^{(d-2)}}). \tag{4.4}$$

Because $d-1 = m + (m-1)$ and $\lim_{N\to\infty} \frac{L_N}{N} = \lambda$ formulas (4.3) and (4.4) yield

$$\lim_{N\to\infty} \frac{AQ_N(L_N, d)}{\binom{N}{d}} N = \frac{d!}{(m-1)!m!}\ \lambda^{m-1}\lambda^m.$$

The definition of the Euler beta function for positive integers gives $\frac{d!}{(m-1)!\,m!} = \frac{1}{B(m,m+1)}$, and this proves formula (4.1) for even d. □

The above calculation suggests the definition of a discrete version $\mathrm{Beta}_N(a, b)$ of beta distribution for arbitrary choice of the shape parameters. Let a, b and N be integers. Let the probability mass function of $\mathrm{Beta}_N(a, b)$ be defined by

$$P_N(L; a, b) = \frac{\binom{L-1}{a}\binom{N-L}{b}}{\binom{N}{a+b+1}}$$

for $L \in \{1, 2, \ldots, N\}$.

Proposition 4.2. *Let λ be an arbitrary real number in the unit interval $[0, 1]$ and let $\{L_N\}_{N\in\mathbb{N}}$ be a sequence, such that $L_N < N$ and $\lim_{N\to\infty} \frac{L_N}{N} = \lambda$. Then*

$$\lim_{N\to\infty} P_N(\frac{L_N}{N}, a, b)N = p_\beta(\lambda; a, b) = \frac{1}{\beta(a+1, b+1)}\ \lambda^a(\lambda-1)^b$$

Proof. The proof is an obvious adaptation of the proof of Lemma 4.1. We only have to replace the particular values $m-1$ and m of the shape parameters by an arbitrary pair a and b of positive integers. Then the same calculations as those performed in the proof of Lemma 4.1 yield the proof of the proposition. □

4.2 Volumes of polytopes $D_d(\lambda)$

Recall formula (2.7):

$$\mathcal{F}[u_c](n) = I + \sum_{d=1}^{\infty} u^d \int_0^1 \mathrm{Vol}(D_d(\lambda)) \begin{pmatrix} e^{-2\pi i\lambda n} & 0 \\ 0 & -e^{2\pi i\lambda n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^d d\lambda.$$

We have the following theorem.

Theorem 4.3. *For every dimension d, the volumes of polytopes $D_d(\lambda)$ are essentially distributed according to the beta distribution with the shape parameters $(\frac{d}{2}, \frac{d}{2}+1)$, if d is even, and $(\frac{d+1}{2}, \frac{d+1}{2})$, if d is odd. More concretely, we have the following expression:*

$$\mathrm{Vol}(D_d(\lambda)) = \frac{1}{d!}\begin{cases} \frac{1}{B(\frac{d}{2},\frac{d}{2}+1)}\ \lambda^{\frac{d}{2}-1}(1-\lambda)^{\frac{d}{2}} & = p_\beta(\lambda; \frac{d}{2}, \frac{d}{2}+1);\ d \text{ even} \\ \frac{1}{B(\frac{d+1}{2},\frac{d+1}{2})}\ \lambda^{\frac{d-1}{2}}(1-\lambda)^{\frac{d-1}{2}} & = p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2});\ d \text{ odd}, \end{cases} \tag{4.5}$$

where $p_\beta(\lambda; a, b)$ is the probability density function of the distribution with shape parameters a and b.

Proof. Recall the set $\widehat{D}^{disc}_{d,N}(L)$, given by (2.10). Rescaling it by the factor $1/N$ gives the set

$$\widehat{D}^{disc}_d(\frac{L}{N}) = \{(\frac{l_1}{N}, \frac{l_2}{N}\dots, \frac{l_d}{N}); \frac{N-1}{N} \geq \frac{l_1}{N} > \dots > \frac{l_d}{N} \geq 0, \sum_{j=1}^{d}(-1)^{j-1}l_j = L\}$$

which contains the same number of points as $\widehat{D}^{disc}_{d,N}(L)$, but lies in the polytope $\widehat{D}_d(\frac{L}{N})$. Let $D^{disc}_d(\frac{L}{N})$ denote the orthogonal projection of $\widehat{D}^{disc}_d(\frac{L}{N})$ on the hyperplane

$$\{(x_1, \dots, x_{d-1}, 0)\} \subset \mathbb{R}^d.$$

The number $\sharp D^{disc}_d(\frac{L}{N})$ of points in $D^{disc}_d(\frac{L}{N})$ is clearly equal to the number of points in $\widehat{D}^{disc}_d(\frac{L}{N})$. The value $\frac{1}{N^{d-1}}\sharp D^{disc}_d(\frac{L}{N})$ is approximately equal to the volume $\mathrm{Vol}(D_d(\frac{L}{N}))$ of the projection $D_d(\frac{L}{N})$ of $\widehat{D}_d(\frac{L}{N})$ on the hyperplane $x_d = 0$ in $\mathbb{R}^d$. So, on the one hand, the number $\sharp D^{disc}_d(\frac{L}{N})$ is equal to $AQ_N(L, d)$, while on the other, the value $\frac{1}{N^{d-1}}\sharp D^{disc}_d(\frac{L}{N})$ is an approximation of $\mathrm{Vol}(D_d(L))$. Let now $\{\lambda_N\}_{N\in\mathbb{N}}$ be a sequence of rationals $\frac{L_N}{N}$ converging to $\lambda \in [0, 1]$. We have

$$\lim_{N\to\infty} \frac{1}{N^{d-1}}\, AQ_N(NL_N, d) = \mathrm{Vol}(D_d(\lambda)).$$

In the proof of Lemma 4.1 we have seen that

$$\binom{N}{d} = \frac{N^d}{d!} + \mathcal{O}(\frac{1}{N^{(d-2)}}),$$

so

$$\frac{\binom{N}{d}}{N} = \frac{N^{(d-1)}}{d!} + \mathcal{O}(\frac{1}{N^{(d-3)}}).$$

Therefore

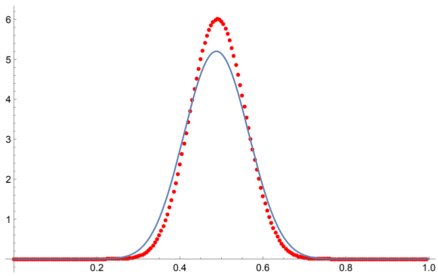
$$\lim_{N\to\infty} \frac{1}{N^{d-1}}\, AQ_N(NL_N, d) = \lim_{N\to\infty} \frac{AQ_N(L_N, d)}{\binom{N}{d}} N.$$

This equality, together with Lemma 4.1, proves our theorem. □

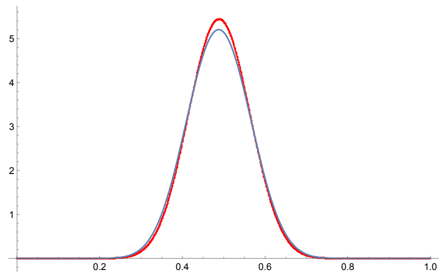
5 Quantitative comparisons

In this section we shall investigate by experimental means the comparison between the probability density function of $\mathrm{Beta}(l; a, b)$ distribution and its approximations, given by the probability mass functions $P_N(l; a, b)$. For the sake of brevity we shall concentrate on the shape parameters $(a, b) = (m - 1, m)$ which appear in connection with the nonlinear Fourier transform. It is now clear that absolute value of the difference $p_\beta(l; a, b) - P_N(l; a, b)$ decreases for every $l = \frac{L_N}{N}$ as N increases. But the quality of the approximation also depends crucially on the choice of the shape parameters. We shall see that, roughly speaking, the value $|p_\beta(l; a, b) - P_N(l; a, b)|$ at a fixed N, increases with increasing of $a + b$. Explicit formula for this difference can be deduced from formulas (4.1) and (4.2), but it is quite complicated. The images will provide a better illustration of the relations between $p_\beta(l; a, b)$ and $P_N(l; a, b)$.

The two images in Figure 2 show the comparison between $p_\beta(l; 21, 22)$ and $P_N(l; 21, 22)$ for $N = 200$ and $N = 1000$.



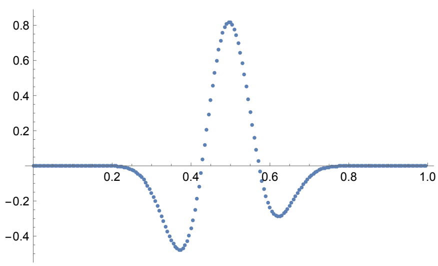
(a) $N = 200$, $a = 21$, $b = 22$



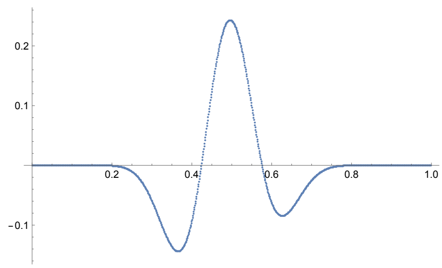
(b) $N = 1000$, $a = 21$, $b = 22$

Figure 2: Comparison of graphs.

Figure 3 shows that for any choice of the shape parameters the difference $P_N(l; a, b) - p_\beta(l; a, b))$ has three local extrema. For the cases, related to the number of alternating partitions of integers where $a = b - 1$ or $a = b$, the maximum is located roughly at the center of the interval $[0, 1]$.



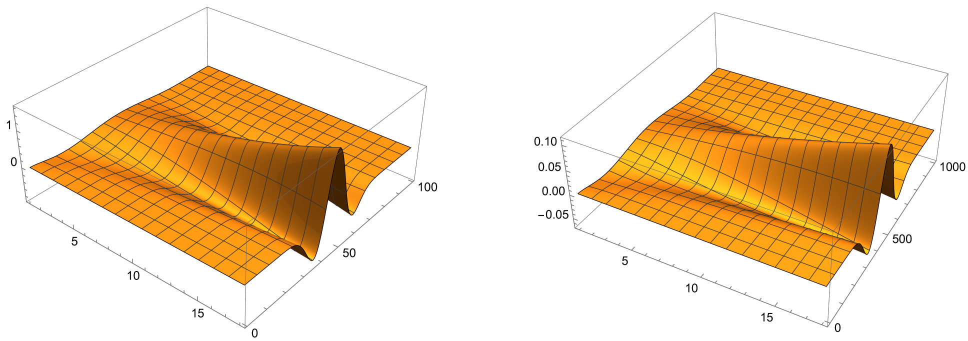
(a) $N = 200$, $a = 21$, $b = 22$



(b) $N = 1000$, $a = 21$, $b = 22$

Figure 3: The shape of the difference.

The two images in Figure 4 illustrate the dependence of the difference $p_\beta(l;a,b) - P_N(l;a,b)$ on the size of the shape parameters. Again we consider $(a,b) = (a,a+1)$. We see, that at a fixed N the difference increases with increasing of the shape parameter a.



(a) $N = 100$, $a \in [3,20]$ (b) $N = 1000$, $a \in [3,20]$

Figure 4: Dependence of the difference on the size of the shape parameter.

Even if the shape parameters a and b are very different, the corresponding graphs are of similar shapes to the above. The only difference is that, in case where the shape parameters a and b are significantly different the peaks of the graphs are shifted away from the center. This is clear from the following fact. Suppose that a is considerably larger than b. Then the left zero of limit function $p_\beta(l;a,b) = \frac{1}{B(a+1,b+1)} l^a(1-l)^b$ is of higher degree than the right one. The function is therefore flatter and closer to zero in the vicinity of 0 and the peak of the graph is pushed towards the right. Qualitatively the shape of the difference does not change.

6 Conclusions and outlook

In the paper we arrived at the construction of a discrete probability distribution with probability mass function $P_N(l;a,b)$ which converges to the probability density function $p_\beta(l;a,b)$ as $N \to \infty$. The result is precisely stated in Proposition 4.2. Crucial in the construction is the connection of $P_N(l;a,a)$ and $P_N(l;a-1,a)$ to the following combinatorial problem: find the number $AQ_N(L,d)$ of alternating ordered partitions of the positive integer $L < N$ into d distinct parts, not greater than $N-1$. The number $AQ_N(L,d)$ can also be represented by the number of the zig-zag paths, drawn in Figure (1). This combinatorial problem naturally appeared in the context of the discretization $\mathcal{F}_N$ of the nonlinear Fourier transform $\mathcal{F}$, described in Section 2. The essential connection between the numbers $AQ_N(L,d)$ and $\mathcal{F}_N$ is given by Proposition 3.2 where we show that the inverse linear Fourier transform of the entries of $\mathcal{F}_N$ yields the generating polynomials of the numbers $AQ_N(L,d)$.

The formula for distribution $P_N(l;a,b)$ can also be interpreted as the distribution describing the Pólya-Eggenberger urn, but this interpretation is different from ours. We have the connection of $P_N(l;a,b)$ to the combinatorial problem and the nonlinear Fourier transform only for the shape parameters of the form $(a,b) = (a,a)$ or $(a,b) = (a-1,a)$. The natural question arises: can we find a combinatorial problem whose relation with $P_N(l;a,b)$ for an arbitrary choice of a and b would be analogous to the relation between $P_N(l; a-1, a)$ and $P_N(l; a, a)$ and the problem of alternating ordered partitions $AQ_N(L, d)$? Does there exist a meaningful generalisation $\mathcal{F}_{a,b}$ of the nonlinear Fourier transform $\mathcal{F}$, whose relation with $p_\beta(x; a, b)$ would be analogous to the relation between $\mathcal{F}$ and $p_\beta(x; a, a)$ and $p_\beta(x; a-1, a)$, described in theorem 4.3. These are the natural problems for further investigation, based on this paper. Finding answers to these questions would importantly improve understanding the nonlinear Fourier transform and its structure.

In this paper, we considered the nonlinear Fourier transform $\mathcal{F}[u]$ evaluated on the simplest of functions, namely, the constant function $u \equiv c$. An obvious direction of further research is to try to extend the approach used in this paper, to the context, where $\mathcal{F}[u]$ is evaluated on some more interesting class of functions u.

ORCID iDs

Pavle Saksida https://orcid.org/0000-0003-3093-9863

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